

The SPDE Method for Source Inference

Colin Okasaki, Mevin Hooten, Andrew Berdahl

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- Pollution

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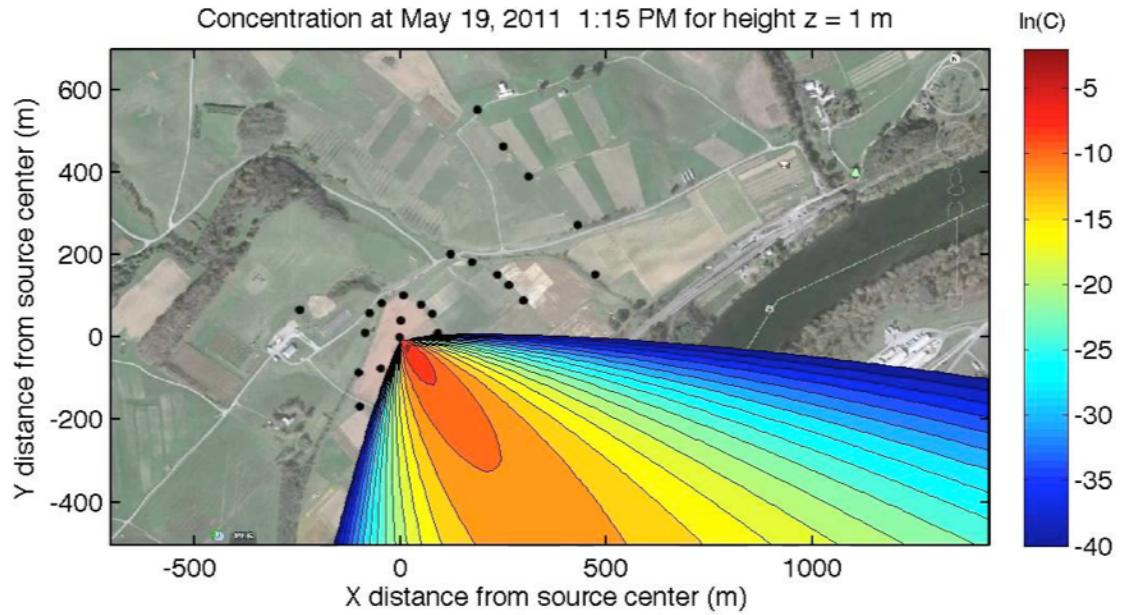
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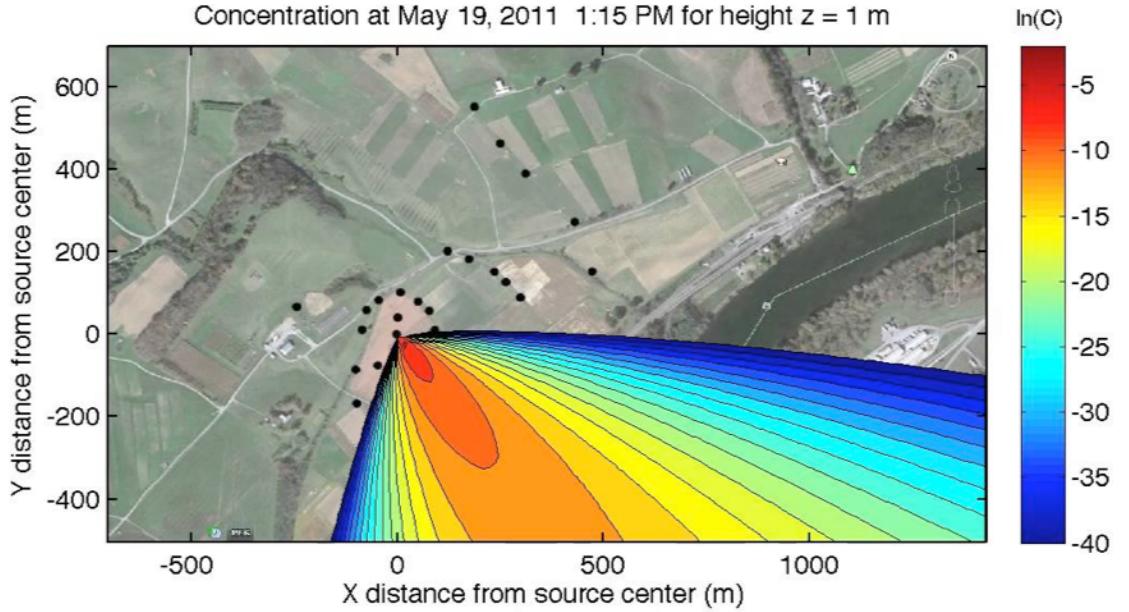
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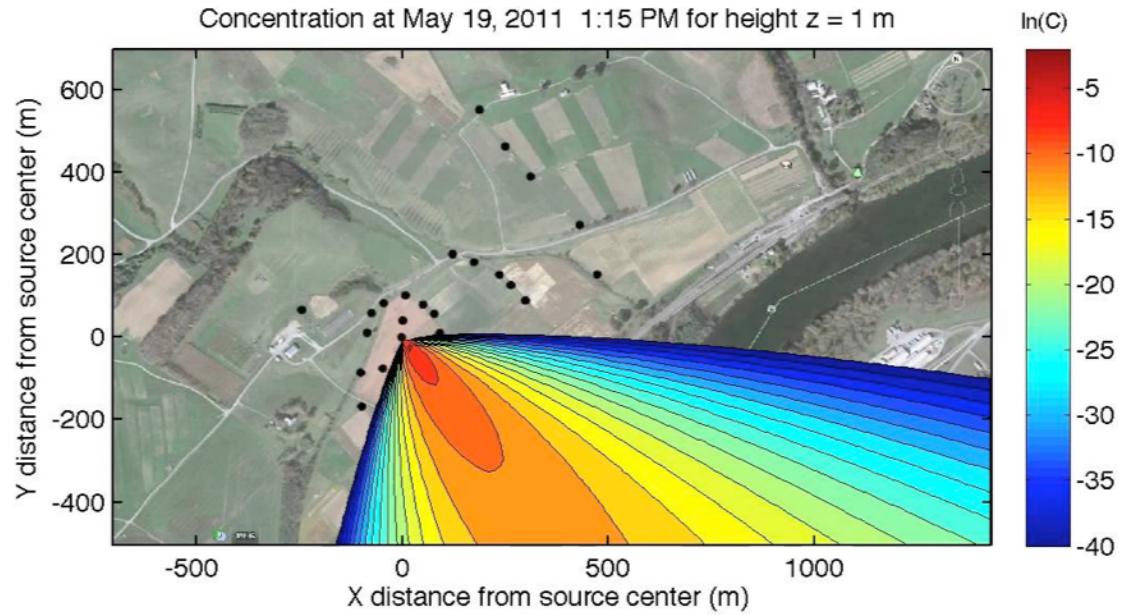
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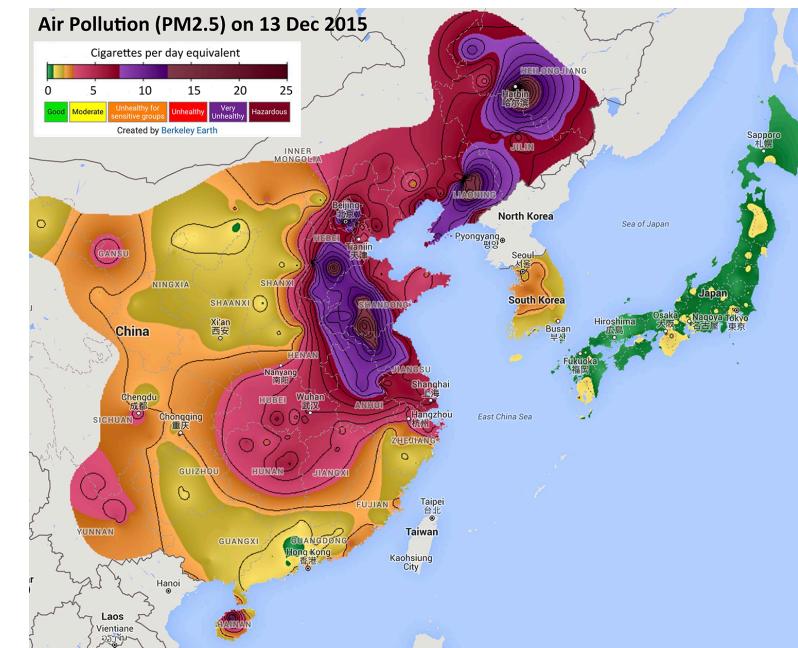
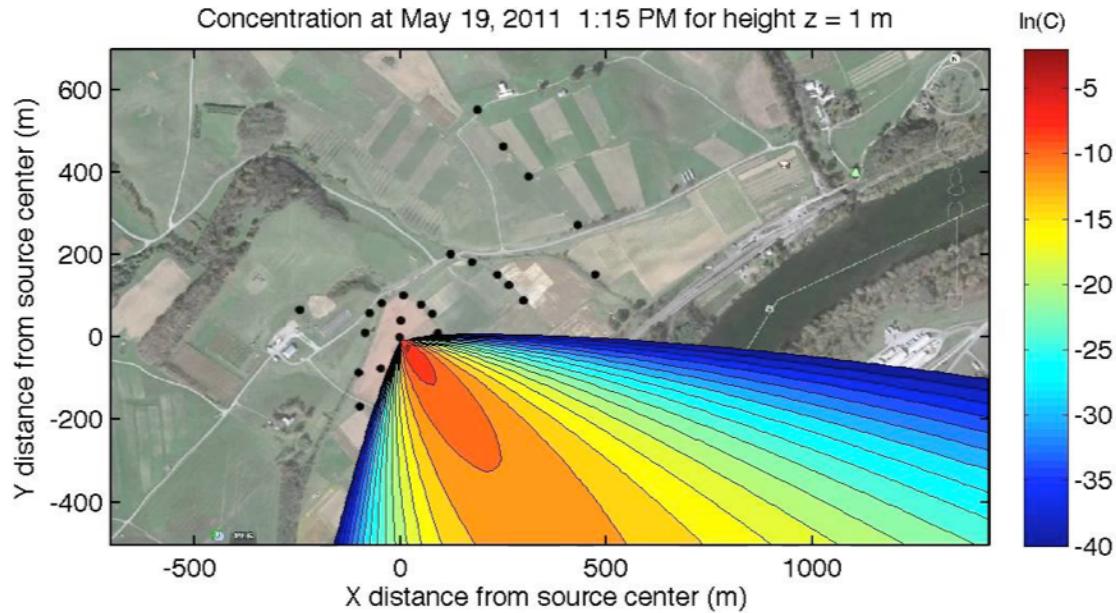
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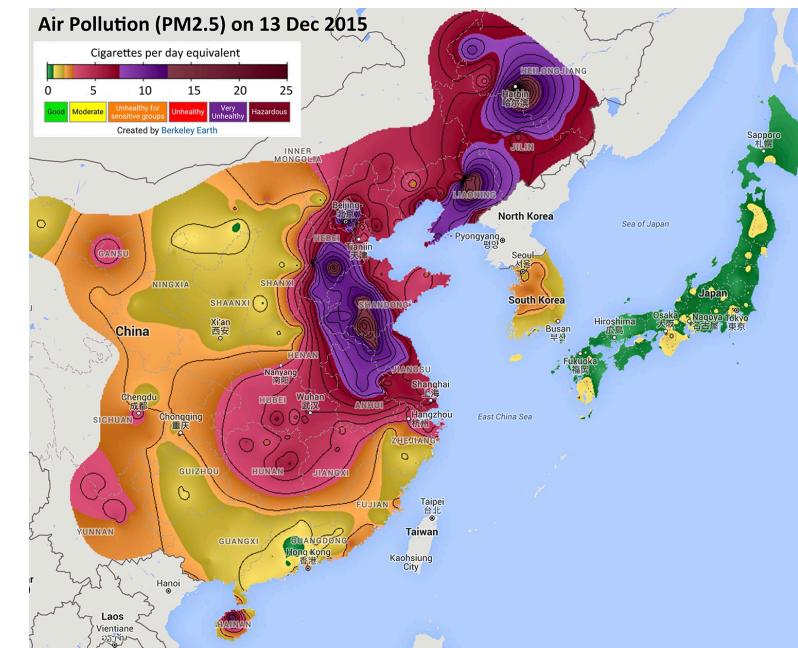
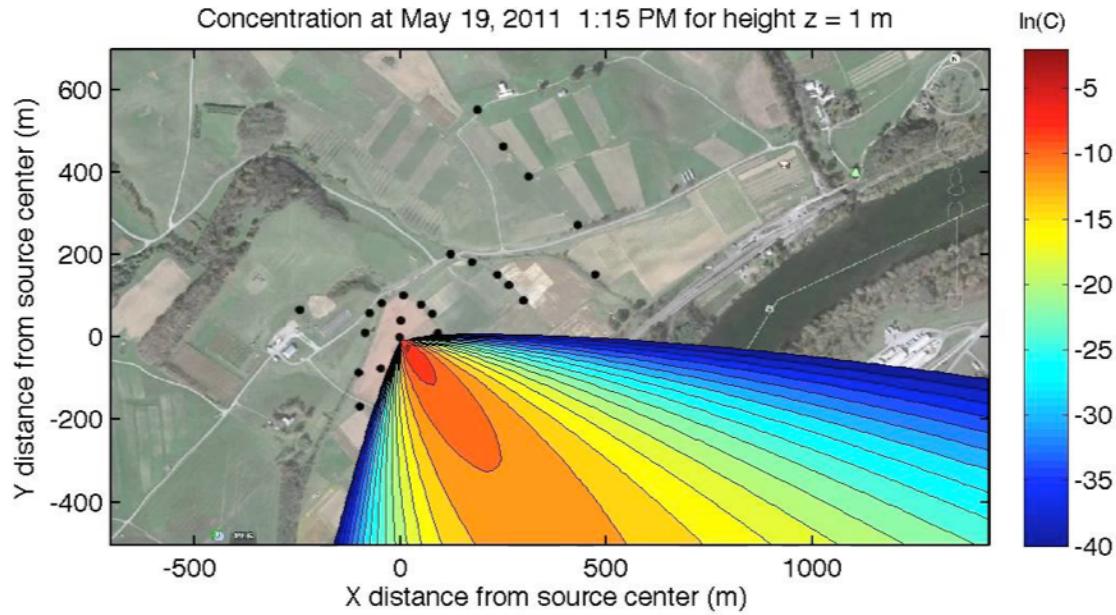
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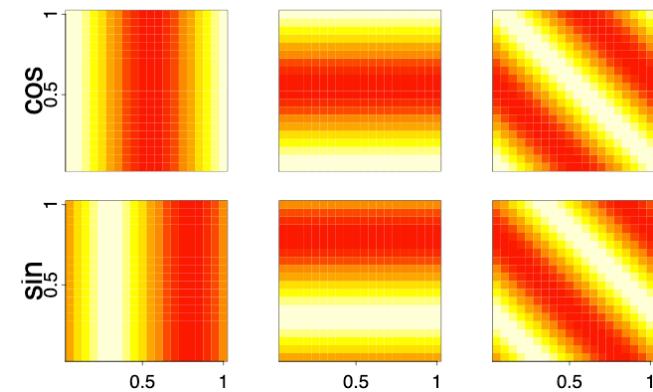
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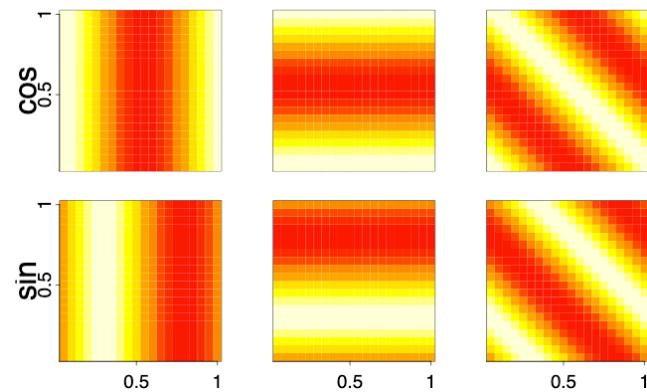
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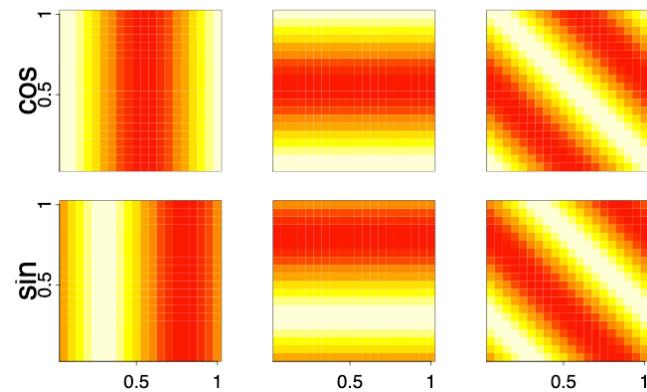
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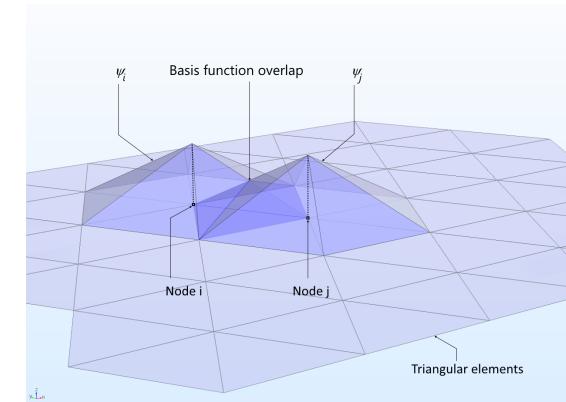
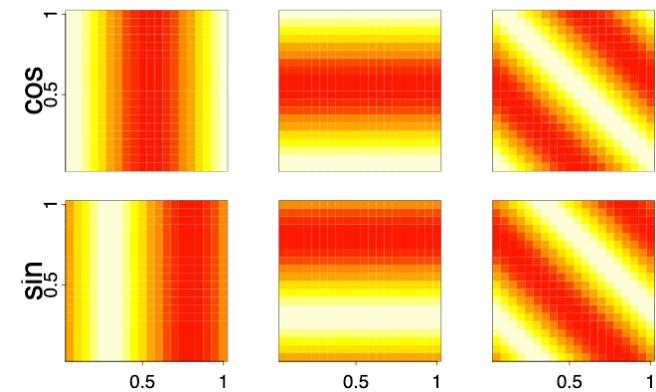
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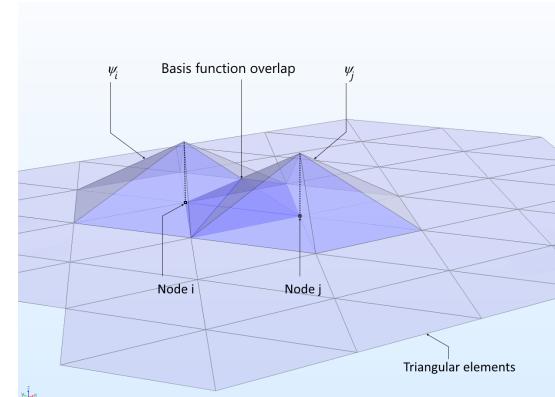
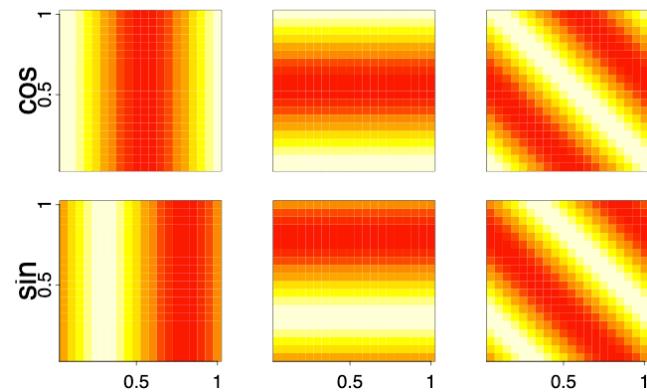
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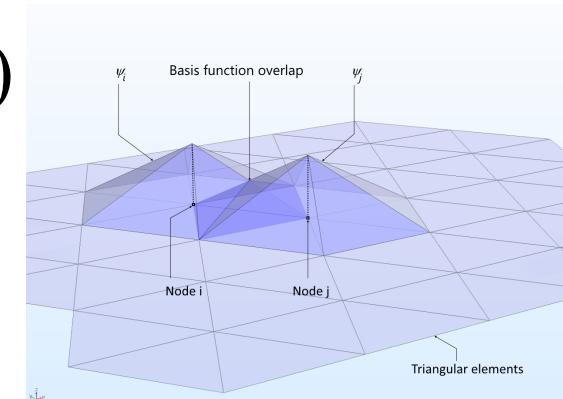
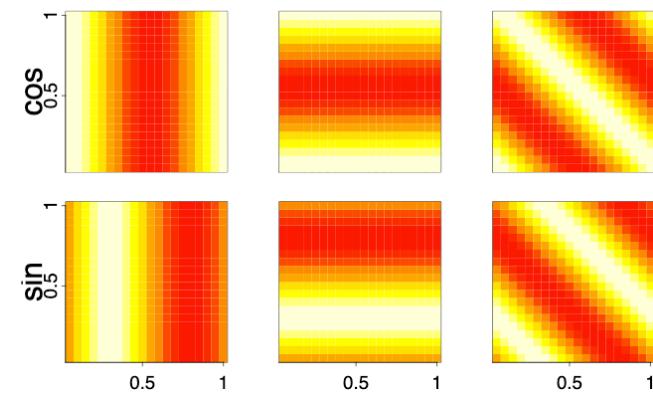
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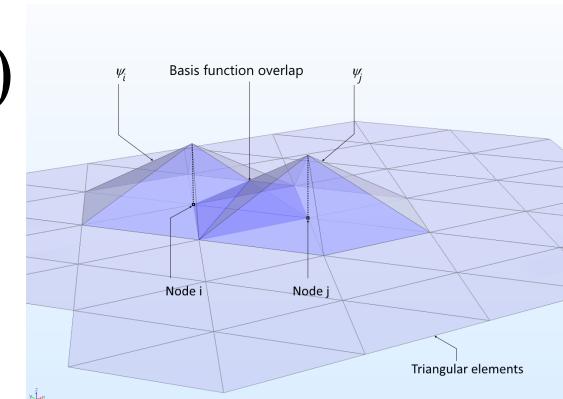
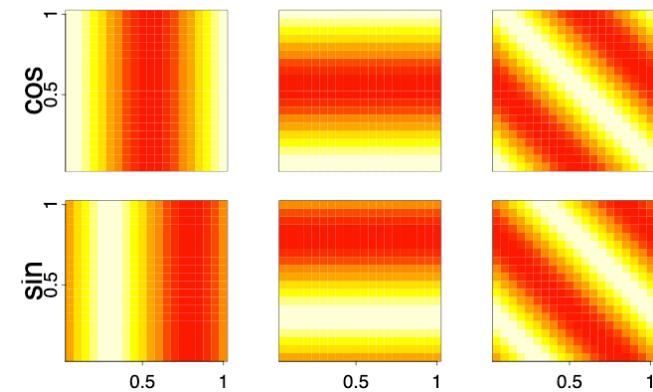
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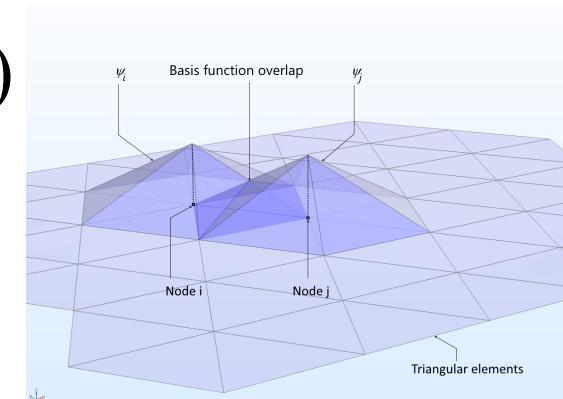
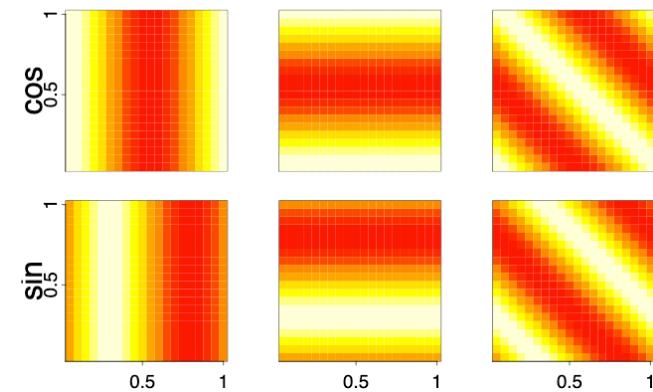
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 - Basically a fancy kernel trick



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- Assume that \mathcal{L} is a linear operator
- Discretize this as $Ku = Lf$
- We model $f \sim X\beta + Z\gamma + \epsilon$
- If everything is normal then

$$[f|d] \sim \text{MVN}(\mu_{\text{post}}, Q_{\text{post}}^{-1})$$

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 - What is the limiting spatial distribution?

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$$u(x, \infty)$$

Space-time limiting distributions

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- We already have

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Space-time limiting distributions

- We already have $u_{t+1} = (I - Kdt)u_t + Lf_t dt$
- Then taking some limits we get two possibilities

$$K\Sigma_u + \Sigma_u K^T = L\Sigma_f L^T$$

or

$$K_f \Sigma_{uf} + \Sigma_{uf} K_f^T = L_u \Sigma_{ff}$$

$$K_u \Sigma_{uu} + \Sigma_{uu} K_u^T = L_u \Sigma_{fu} + \Sigma_{uf} L_u^T$$

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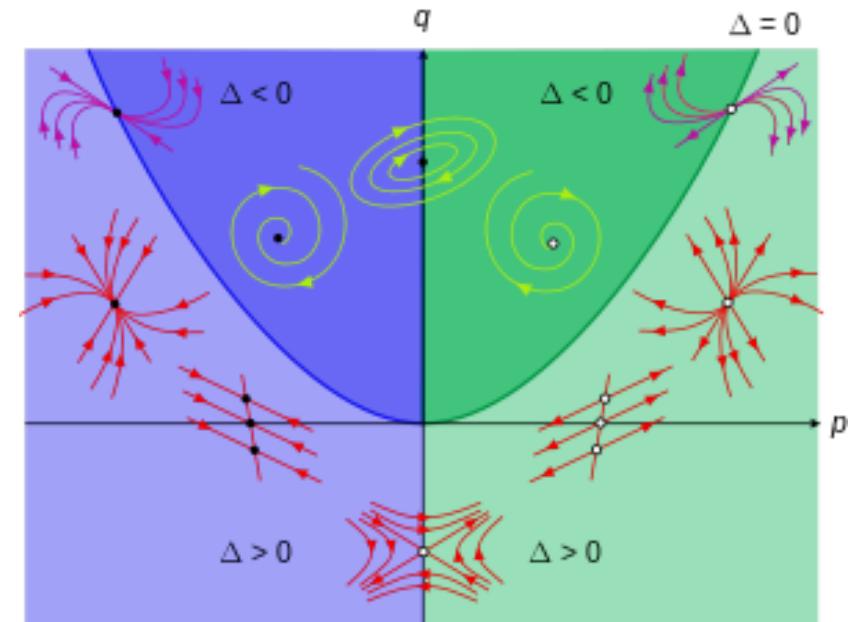
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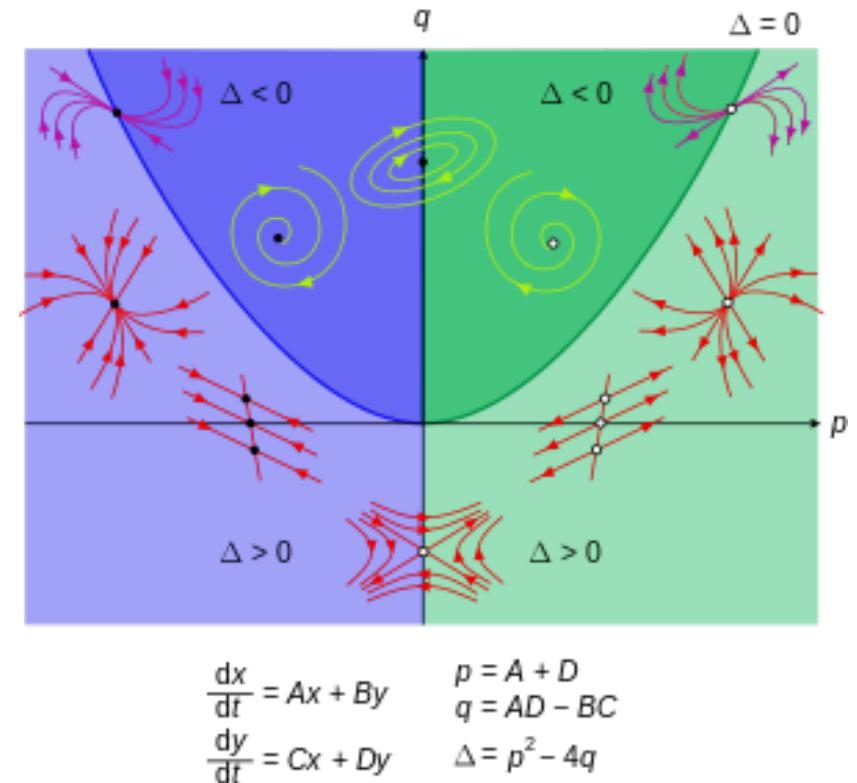
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$$\begin{aligned}\frac{dx}{dt} &= Ax + By & p &= A + D \\ \frac{dy}{dt} &= Cx + Dy & q &= AD - BC \\ && \Delta &= p^2 - 4q\end{aligned}$$

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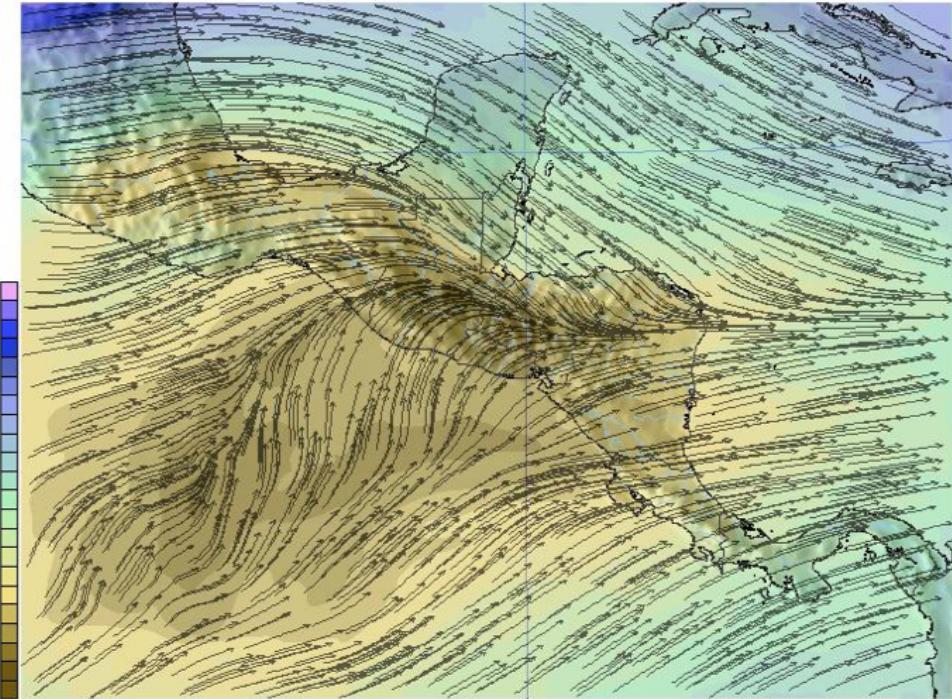
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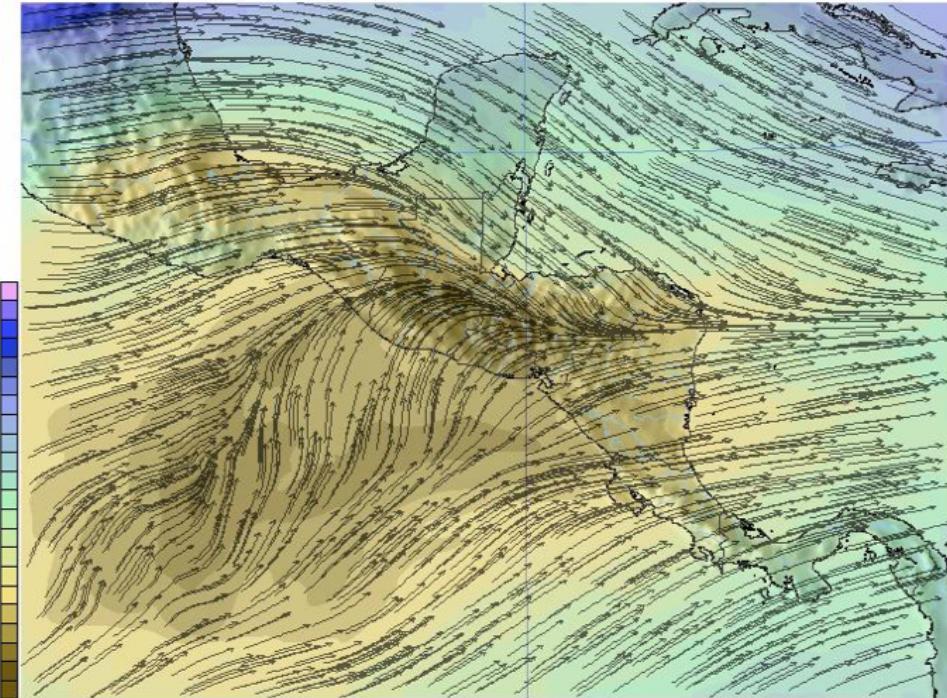
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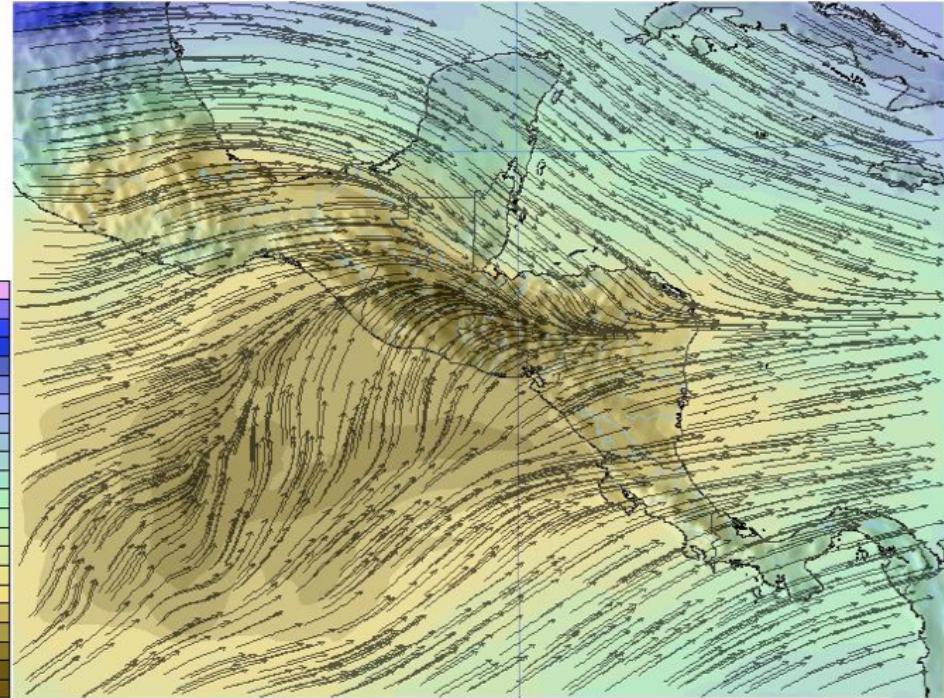
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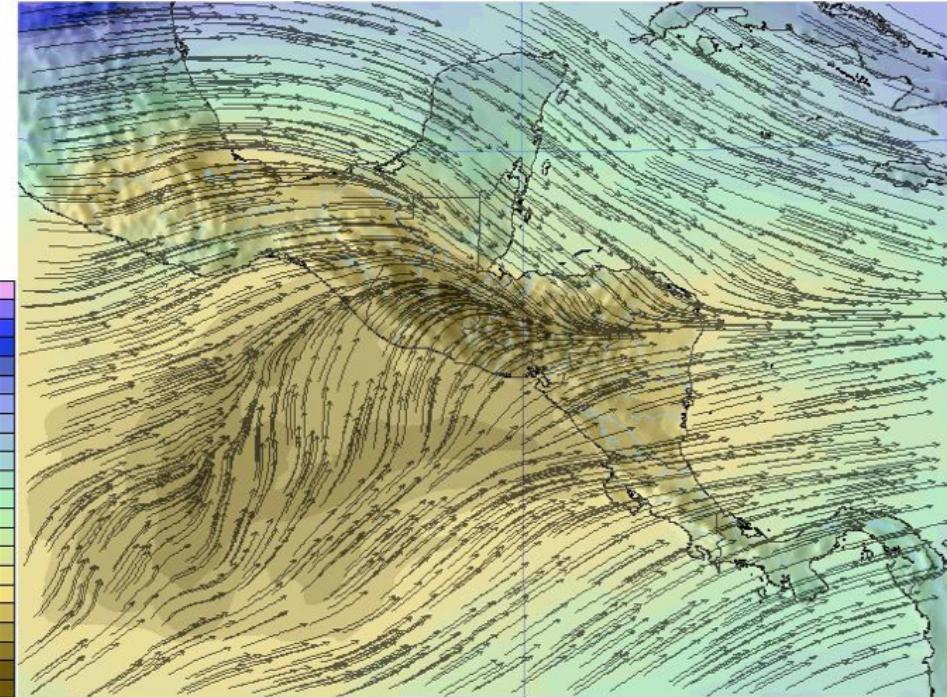
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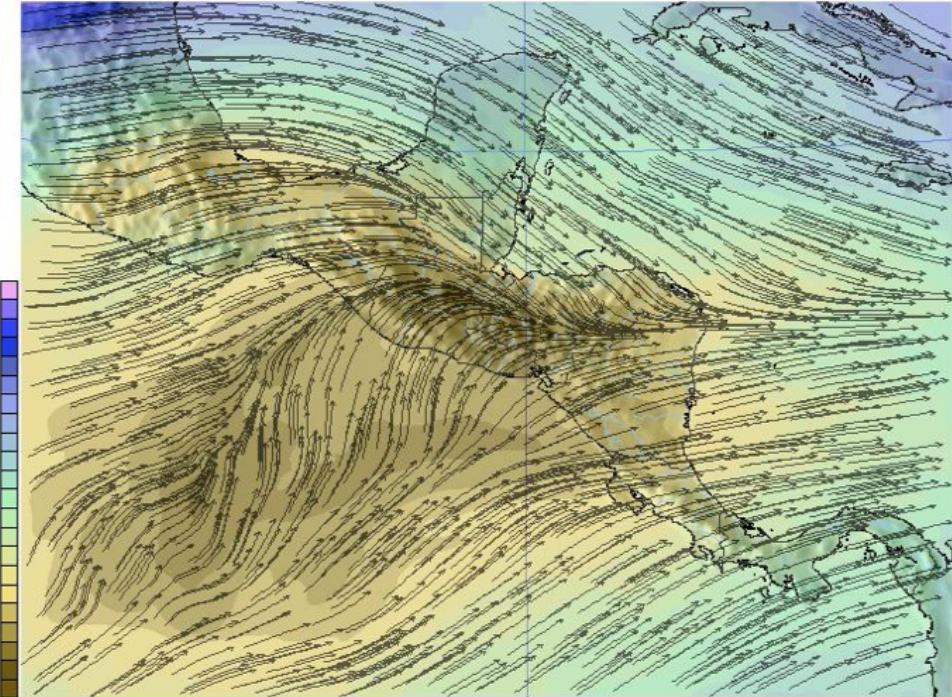
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- But suppose we have enough data for a really good guess
 - Just a little variation around it



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- Then we conduct inference as usual, get GP u^*
- Then we might model small errors in θ as $\mathcal{N}(\theta^* + \Delta_\theta) = \eta$

$$\mathcal{L}[\theta^* + \Delta_\theta](u^* + \Delta_u) = f$$

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- This gives us a fully linear system of PDEs
- However: since u^* and Δ_θ are both GPs, their product is not
- Thus there is no simple analytical solution for Δ_u

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$$\Delta_u \sim \text{MVN}(0, \Sigma_{\Delta_\theta} \circ (\Sigma_{u^*} + \mu_{u^*} \mu_{u^*}^T))$$
- Challenge:
 - Hadamard (element-wise) product for precision matrices?

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 - Solution: interpolate with perturbation, then simulate

Applications

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 - Advection-diffusion

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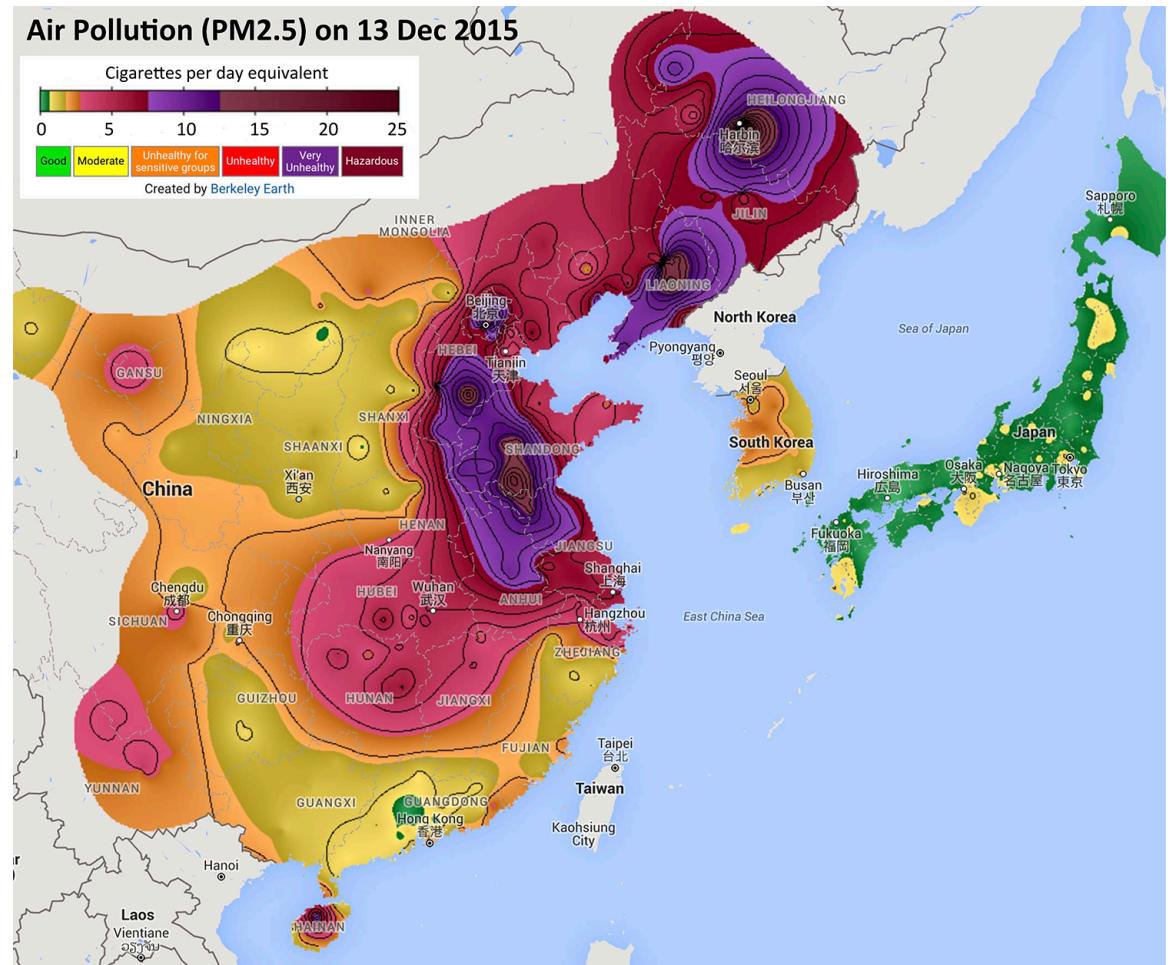
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