A.1.21

$$\vec{a} = \vec{e}_x + \vec{e}_y + \vec{e}_z$$
$$\vec{b} = \vec{e}_x - \vec{e}_y$$
$$\vec{c} = \vec{e}_x + 2\vec{e}_y - 2\vec{e}_z$$

$$\vec{a} + \vec{b} = 2\vec{e}_x + \vec{e}_z$$

$$\vec{a} + \vec{c} = 2\vec{e}_x + 3\vec{e}_y - \vec{e}_z$$

$$|\vec{a} - \vec{c}| = |-\vec{e}_y + 3\vec{e}_z| = \left|\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}\right| = \sqrt{10}$$

$$\vec{a} \cdot \vec{b} = 1 - 1 = 0$$

$$\vec{a} \cdot \vec{c} = 1 + 2 - 2 = 1$$

$$\cos(\sphericalangle(\vec{a}, \vec{c})) = \frac{\vec{a} \cdot \vec{c}}{|\vec{a}||\vec{c}|} = \frac{1}{\sqrt{3}\sqrt{9}} = \frac{1}{3\sqrt{3}}$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \vec{e}_x + \vec{e}_y - 2\vec{e}_z$$

$$\sin(\sphericalangle(\vec{a}, \vec{b})) = \pm 1 \qquad |\vec{a} \times \vec{b}| = 2 \cdot \frac{|\vec{a}||\vec{b}|}{2} \sin(\sphericalangle(\vec{a}, \vec{b}))$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) =$$

$$= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1 + 2 + 4 = 7$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) =$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} 1 - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} 0 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \vec{b}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2+4 \\ +2-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2\vec{e}_x + \vec{e}_z$$

A.1.2.2.

Zylinderkoordinaten $(\rho, \alpha, z) \rightarrow (x, y, z)$

$P(5, \frac{3\pi}{2}, 0)$

$Q(5, \frac{\pi}{2}, 10)$

$x = \rho \cdot \cos(\alpha) = 5 \cdot \cos(\frac{3\pi}{2}) = 0$

$y = \rho \cdot \sin(\alpha) = 5 \cdot \sin(\frac{3\pi}{2}) = -5$

$z = z$

$P(0 \mid -5 \mid 0)$

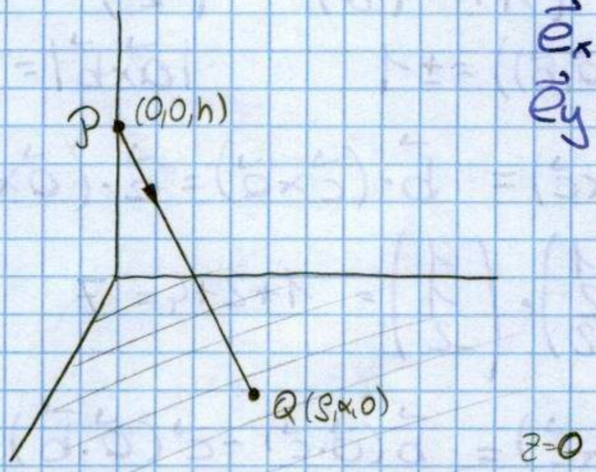
$Q(0 \mid 5 \mid 10)$

$$\vec{PQ} = Q - P = \begin{pmatrix} 0 \\ 10 \\ 10 \end{pmatrix}$$

$$|\vec{PQ}| = \sqrt{100+100} = \sqrt{200} = 10\sqrt{2}$$

A.1.2.3.

Zylinderkoordinaten (ρ, α, z)



$\vec{e}_x = \cos(\alpha)\vec{e}_\rho - \sin(\alpha)\vec{e}_\alpha$

$\vec{e}_y = \cos(\alpha)\vec{e}_\alpha + \sin(\alpha)\vec{e}_\rho$

$Q(\rho\cos(\alpha), \rho\sin(\alpha), 0)$

$$\vec{PQ} = (\rho\cos(\alpha), \rho\sin(\alpha), -h)$$

$$|\vec{PQ}| = \sqrt{\rho^2\cos^2(\alpha) + \rho^2\sin^2(\alpha) + h^2} = \sqrt{\rho^2+h^2}$$

$$\vec{e}_{PQ} = \frac{1}{\sqrt{\rho^2+h^2}}(\rho\cos(\alpha), \rho\sin(\alpha), -h) =$$

$$= \frac{1}{\sqrt{\rho^2+h^2}}\left(\rho\cos(\alpha)\vec{e}_x + \rho\sin(\alpha)\vec{e}_y - h\vec{e}_z\right) =$$

$$= \frac{1}{\sqrt{\rho^2+h^2}}\left(\rho\cos(\alpha)(\cos(\alpha)\vec{e}_\rho - \sin(\alpha)\vec{e}_\alpha) + \rho\sin(\alpha)(\sin(\alpha)\vec{e}_\rho + \cos(\alpha)\vec{e}_\alpha) - h\vec{e}_z\right)$$

$$= \# \left(\rho\cos^2(\alpha)\vec{e}_\rho - \cancel{\rho\cos(\alpha)\sin(\alpha)\vec{e}_\alpha} + \rho\sin^2(\alpha)\vec{e}_\rho + \cancel{\rho\cos(\alpha)\sin(\alpha)\vec{e}_\alpha} - h\vec{e}_z\right) =$$

$$= \# \; \rho\vec{e}_\rho - h\vec{e}_z$$

A.1.2.4.

$$\vec{f}(\vec{r}) = y\vec{e}_x + x\vec{e}_y + \frac{x^2}{\sqrt{x^2+y^2}}\vec{e}_z$$

$$\vec{e}_x = \cos(\alpha)\vec{e}_\rho - \sin(\alpha)\vec{e}_\alpha$$
$$\vec{e}_y = \sin(\alpha)\vec{e}_\rho + \cos(\alpha)\vec{e}_\alpha$$

$$\vec{f}(\vec{r}) = y\left(\cos(\alpha)\vec{e}_\rho - \sin(\alpha)\vec{e}_\alpha\right) + x\left(\sin(\alpha)\vec{e}_\rho + \cos(\alpha)\vec{e}_\alpha\right) +$$
$$+ \frac{x^2}{\sqrt{x^2+y^2}}\vec{e}_z = \left(y\cos(\alpha) + x\sin(\alpha)\right)\vec{e}_\rho +$$
$$+\left(x\cos(\alpha) - y\sin(\alpha)\right)\vec{e}_\alpha + \frac{x^2}{\sqrt{x^2+y^2}}\vec{e}_z$$

$$x = \rho\cdot\cos(\alpha) \qquad y = \rho\cdot\sin(\alpha)$$

$$\vec{f}(\vec{r}) = \left(\rho\sin(\alpha)\cos(\alpha) + \rho\cos(\alpha)\sin(\alpha)\right)\vec{e}_\rho +$$
$$+ \left(\rho\cos^2(\alpha) - \rho\sin^2(\alpha)\right)\vec{e}_\alpha + \frac{\rho^{\cancel{2}}\cos^2(\alpha)}{\cancel{\rho}}\vec{e}_z =$$
$$= 2\rho\sin(\alpha)\cos(\alpha)\vec{e}_\rho + \cos(2\alpha)\vec{e}_\alpha + \rho\cos^2(\alpha)\vec{e}_z =$$
$$= \rho\sin(2\alpha)\vec{e}_\rho + \cos(2\alpha)\vec{e}_\alpha + \rho\cos^2(\alpha)\vec{e}_z$$

A.1.2.5.

$$\underset{\sim}{T}(\vec{r}) = r^2\sin^2(\theta)\left[\cos(\theta)\vec{e}_r - \sin(\theta)\vec{e}_\theta\right] \otimes \vec{e}_\alpha$$

$$x = \rho\sin(\theta)\cos(\alpha) \qquad y = \rho\sin(\theta)\sin(\alpha) \qquad z = \rho\cos(\theta)$$

$$\vec{e}_r = \sin(\theta)\cos(\alpha)\vec{e}_x + \sin(\theta)\sin(\alpha)\vec{e}_y + \cos(\theta)\vec{e}_z$$
$$\vec{e}_\theta = \cos(\theta)\cos(\alpha)\vec{e}_x + \cos(\theta)\sin(\alpha)\vec{e}_y - \sin(\theta)\vec{e}_z$$
$$\vec{e}_\alpha = -\sin(\alpha)\vec{e}_x + \cos(\alpha)\vec{e}_y$$

$$\underset{\sim}{T}(\vec{r}) = r^2\sin^2(\theta)\left[\cancel{\cos(\theta)\sin(\theta)\cos(\alpha)\vec{e}_x} + \cancel{\cos(\theta)\sin(\theta)\sin(\alpha)\vec{e}_y} + \cos^2(\theta)\vec{e}_z -\right.$$
$$\left.-\cancel{\sin(\theta)\cos(\theta)\cos(\alpha)\vec{e}_x} - \cancel{\sin(\theta)\cos(\theta)\sin(\alpha)\vec{e}_y} + \sin^2(\theta)\vec{e}_z\right] \otimes$$
$$\otimes\left(-\sin(\alpha)\vec{e}_x + \cos(\alpha)\vec{e}_y\right) = r^2\sin^2\theta\,\vec{e}_z \otimes \left(-\sin(\alpha)\vec{e}_x + \cos(\alpha)\vec{e}_y\right) =$$

$$\underset{\sim}{T}(\vec{r}) = \left(r^2 \sin^2(\theta)\, \vec{e}_z \right) \otimes \left(-\sin(\alpha)\, \vec{e}_x + \cos(\alpha)\, \vec{e}_y \right)$$

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \qquad r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2 = \rho^2 + z^2$$

$$r^2 \sin^2(\theta) = r^2 - r^2\cos^2(\theta) = r^2 - \rho^2$$

$$\Rightarrow \underset{\sim}{T}(\vec{r}) = \rho^2 \vec{e}_z \otimes \left(-\frac{y}{\rho}\, \vec{e}_x + \frac{x}{\rho}\, \vec{e}_y \right) =$$

$$= \sqrt{x^2+y^2}\, \left(-y\, \vec{e}_z \otimes \vec{e}_x + x\, \vec{e}_z \otimes \vec{e}_y \right).$$

A.1.2.6

$$f(\vec r) = K\cdot(3xy^3 + y^2z^2 - z^3x) \qquad K = \text{const.}$$

$$\vec r_0 = (\vec e_x + 3\vec e_y - 2\vec e_z) \qquad \text{... Punkt}$$
$$\vec a = 3\vec e_x + 2\vec e_y - \vec e_z \qquad \text{... Richtung}$$

$$|\vec a| = \sqrt{9+4+1} = \sqrt{14}$$

$$\vec e_a = \frac{\vec a}{|\vec a|} = \frac{1}{\sqrt{14}}(3\vec e_x + 2\vec e_y - \vec e_z)$$

$$\vec\nabla f = K\cdot\left((3y^3 - z^3)\vec e_x + (9xy^2 + 2yz^2)\vec e_y + (2y^2z - 3z^2x)\vec e_z\right)$$

Richtungsableitung:

$$\vec e_a\cdot\vec\nabla f = \frac{K}{\sqrt{14}}(3\vec e_x + 2\vec e_y - \vec e_z)\cdot\left([3y^3 - z^3]\vec e_x + (9xy^2 + 2yz^2)\vec e_y + (2y^2z - 3z^2x)\vec e_z\right) =$$

$$= \frac{K}{\sqrt{14}}\left(3(3y^3 - z^3) + 2(9xy^2 + 2yz^2) - (2y^2z - 3z^2x)\right)$$

$$\cancel{(\vec e_a\cdot\vec\nabla f)(\vec r_0) = \frac{K}{\sqrt{14}}[3(3\,3^2 + 2^3) + 2(9\cdot 3\cdot 4 - 2\cdot 3\cdot 4) - (2\cdot 9\cdot(-2) - 3\cdot 4\cdot 1)] =}$$
$$\cancel{= \frac{K}{\sqrt{14}}[3\cdot(27+8) + 2(27+24) - (-36-12)] =}$$
$$\cancel{= \frac{K}{\sqrt{14}}[3\cdot 35 + 2\cdot 51 + 48 = 105 + 102 + 48] =}$$
$$\cancel{= 255\cdot\frac{K}{\sqrt{14}} =}$$

$$\vec e_a\cdot\vec\nabla f = \frac{K}{\sqrt{14}}\left(9y^3 - 3z^3 + 18xy^2 + 4yz^2 - 2y^2z + 3xz^2\right)$$

$$r_0 = (1, 3, -2)$$

$$(\vec e_a\cdot\vec\nabla f)(\vec r_0) = \frac{K}{\sqrt{14}}\left(9\cdot(3)^3 - 3(-2)^3 + 18\cdot 1\cdot 3^2 + 4\cdot 3\cdot(-2)^2 - 2\,3^2(-2) + 3\cdot 1\cdot(-2)^2\right) =$$

$$= K\cdot 140{,}31$$

A. 1.2.7.

(i) $\operatorname{div}\operatorname{rot}\vec{f} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{f})$

(ii) $\operatorname{rot}\operatorname{grad} g = \vec{\nabla}\times(\vec{\nabla}g)$

} schon gezeigt in Theorietut

(iii) $\operatorname{rot}\operatorname{rot}\vec{f} = \operatorname{grad}\operatorname{div}\vec{f} - \Delta\vec{f}$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{f}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{f}) - \Delta\vec{f}$

$$(\partial_x\vec{e}_x + \partial_y\vec{e}_y + \partial_z\vec{e}_z)\times\left[(\partial_x\vec{e}_x + \partial_y\vec{e}_y + \partial_z\vec{e}_z)\times(f_x\vec{e}_x + f_y\vec{e}_y + f_z\vec{e}_z)\right] =$$

$$= (\partial_x\vec{e}_x + \partial_y\vec{e}_y + \partial_z\vec{e}_z)\times(\partial_x f_y\vec{e}_z - \partial_x f_z\vec{e}_y - \partial_y f_x\vec{e}_z + \partial_y f_z\vec{e}_x + \partial_z f_x\vec{e}_y - \partial_z f_y\vec{e}_x) =$$

$$= (\partial_x\vec{e}_x + \partial_y\vec{e}_y + \partial_z\vec{e}_z)\times((\partial_x f_y - \partial_y f_x)\vec{e}_z + (\partial_z f_x - \partial_x f_z)\vec{e}_y + (\partial_y f_z - \partial_z f_y)\vec{e}_x) =$$

$$= -(\partial_{xx}f_y - \partial_{xy}f_x)\vec{e}_y + (\partial_{xz}f_x - \partial_{xx}f_z)\vec{e}_z + (\partial_{xy}f_y - \partial_{yy}f_x)\vec{e}_x - (\partial_{yy}f_z - \partial_{yz}f_y)\vec{e}_z - (\partial_{zz}f_x - \partial_{xz}f_z)\vec{e}_x + (\partial_{yz}f_z - \partial_{zz}f_y)\vec{e}_y$$

$$= (\partial_{xy}f_y - \partial_{yy}f_x - \partial_{zz}f_x + \partial_{xz}f_z)\vec{e}_x + (\partial_{yz}f_z - \partial_{zz}f_y - \partial_{xx}f_y + \partial_{xy}f_x)\vec{e}_y + (\partial_{xz}f_x - \partial_{xx}f_z - \partial_{yy}f_z + \partial_{yz}f_y)\vec{e}_z =$$

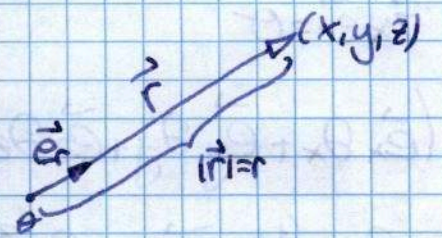
$$= (\partial_{xy}f_y - \partial_{yy}f_x - \partial_{zz}f_x - \partial_{xx}f_x + \partial_{xx}f_x + \partial_{xz}f_z)\vec{e}_x + (\partial_{yz}f_z - \partial_{zz}f_y - \partial_{xx}f_y - \partial_{yy}f_y + \partial_{yy}f_y + \partial_{xy}f_x)\vec{e}_y + (\partial_{xz}f_x - \partial_{xx}f_z - \partial_{yy}f_z - \partial_{zz}f_z + \partial_{zz}f_z + \partial_{yz}f_y)\vec{e}_z =$$

$$= \left[\partial_x(\partial_y f_y + \partial_x f_x + \partial_z f_z) - (\partial_{xx}f_x + \partial_{yy}f_x + \partial_{zz}f_x)\right]\vec{e}_x + \left[\partial_y(\partial_z f_z + \partial_y f_y + \partial_x f_x) - (\partial_{xx}f_y + \partial_{yy}f_y + \partial_{zz}f_y)\right]\vec{e}_y + \left[\partial_z(\partial_x f_x + \partial_y f_y + \partial_z f_z) - (\partial_{xx}f_z + \partial_{yy}f_z + \partial_{zz}f_z)\right]\vec{e}_z =$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{f}) - \Delta\vec{f} \quad \square$$

A.1.2.8. Ortsvektor

$\vec{r} = r\,\vec{e}_r \qquad r = |\vec{r}|$



$\vec{e}_r = \frac{1}{r}\cdot\vec{r} \qquad \vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$

$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

z.z. $\vec{\nabla} r = \vec{e}_r$

$\vec{\nabla} r = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\sqrt{x^2+y^2+z^2} =$

$= \frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y + \frac{z}{r}\vec{e}_z = \frac{1}{r}(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z) =$

$= \frac{1}{r}\vec{r} = \vec{e}_r$

z.z. $\vec{\nabla}\cdot\vec{r} = 3$

$\vec{\nabla}\cdot\vec{r} = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z) =$

$= 1+1+1 = 3$

z.z. $\vec{\nabla}\times\vec{r} = \vec{0}$

$(\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\times(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z) =$

$= \cancel{\partial_x y}\,\vec{e}_z - \cancel{\partial_x z}\,\vec{e}_y - \partial_y x\,\vec{e}_z + \partial_y z\,\vec{e}_x - \cancel{\partial_z x}\,\vec{e}_y - \partial_z y\,\vec{e}_x =$

$= \vec{0}$

z.z. $\vec{\nabla}\cdot\vec{e}_r = \frac{2}{r}$

nach (1) gilt:

~~$\vec{\nabla}\cdot\vec{\nabla}\vec{e}_r = \Delta\vec{e}_r = (\partial_{xx}+\partial_{yy}+\partial_{zz})\frac{1}{r}\vec{r} = (\partial_{xx}+\partial_{yy}+\partial_{zz})\frac{x\vec{e}_x+y\vec{e}_y+z\vec{e}_z}{\sqrt{x^2+y^2+z^2}}$~~

~~$\vec{\nabla}\cdot\vec{\nabla} r = \Delta r = \Delta\sqrt{x^2+y^2+z^2} =$~~

$\vec{\nabla}\cdot\frac{\vec{r}}{r} = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\cdot\frac{1}{r}(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z) =$

$= \partial_x\frac{x}{r} + \partial_y\frac{y}{r} + \partial_z\frac{z}{r} = (*)$

NR.: $\partial_x\frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{1\sqrt{x^2+y^2+z^2} - x\frac{x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} = \frac{r-\frac{x^2}{r}}{r^2} = \frac{1}{r} - \frac{x^2}{r^3}$

$(*) = \frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3} + \frac{1}{r} - \frac{z^2}{r^3} = \frac{3}{r} - \frac{x^2+y^2+z^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$

z.z. $\vec\nabla \times \vec{e}_r = \vec{0}$

$$(\vec{e}_x \partial_x + \vec{e}_y \partial_y + \vec{e}_z \partial_z) \times \left(\frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y + \frac{z}{r}\vec{e}_z\right) =$$

$$= \partial_x \frac{y}{r}\vec{e}_z - \partial_x \frac{z}{r}\vec{e}_y - \partial_y \frac{x}{r}\vec{e}_z + \partial_y \frac{z}{r}\vec{e}_x + \partial_z \frac{x}{r}\vec{e}_y -$$
$$- \partial_z \frac{y}{r}\vec{e}_x = (*)$$

N.R.: $\partial_x \frac{y}{r} = \partial_x \frac{y}{\sqrt{x^2+y^2+z^2}} = \partial_x\, y\,(x^2+y^2+z^2)^{-\frac{1}{2}} =$

$$= y \cdot -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x = \frac{-xy}{r^3}$$

$$(*) = -\cancel{\frac{xy}{r^3}\vec{e}_z} + \cancel{\frac{xz}{r^3}\vec{e}_y} + \cancel{\frac{xy}{r^3}\vec{e}_z} - \cancel{\frac{yz}{r^3}\vec{e}_x} - \cancel{\frac{xz}{r^3}\vec{e}_y} +$$
$$+ \cancel{\frac{yz}{r^3}\vec{e}_x} = \vec{0}$$

z.z. $\vec{f}\cdot\vec\nabla\vec{e}_r = \frac{1}{r}\vec{f}_\perp \quad$ mit $\quad \vec{f}_\perp = \vec{f} - \vec{e}_r\,\vec{e}_r\cdot\vec{f}$

$$\vec\nabla\vec{e}_r = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\left(\frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y + \frac{z}{r}\vec{e}_z\right) =$$
$$= \partial_x\frac{x}{r}\vec{e}_x\otimes\vec{e}_x + \partial_x\frac{y}{r}\vec{e}_x\otimes\vec{e}_y + \partial_x\frac{z}{r}\vec{e}_x\otimes\vec{e}_z +$$
$$+ \partial_y\frac{x}{r}\vec{e}_y\otimes\vec{e}_x + \partial_y\frac{y}{r}\vec{e}_y\otimes\vec{e}_y + \partial_y\frac{z}{r}\vec{e}_y\otimes\vec{e}_z +$$
$$+ \partial_z\frac{x}{r}\vec{e}_z\otimes\vec{e}_x + \partial_z\frac{y}{r}\vec{e}_z\otimes\vec{e}_y + \partial_z\frac{z}{r}\vec{e}_z\otimes\vec{e}_z =$$
$$= \left(\frac{1}{r}-\frac{x^2}{r^3}\right)\vec{e}_x\otimes\vec{e}_x - \frac{xy}{r^3}\vec{e}_x\otimes\vec{e}_y - \frac{xz}{r^3}\vec{e}_x\otimes\vec{e}_z -$$
$$- \frac{yx}{r^3}\vec{e}_y\otimes\vec{e}_x + \left(\frac{1}{r}-\frac{y^2}{r^3}\right)\vec{e}_y\otimes\vec{e}_y - \frac{yz}{r^3}\vec{e}_y\otimes\vec{e}_z -$$
$$- \frac{xz}{r^3}\vec{e}_z\otimes\vec{e}_x - \frac{yz}{r^3}\vec{e}_z\otimes\vec{e}_y + \left(\frac{1}{r}-\frac{z^2}{r^3}\right)\vec{e}_z\otimes\vec{e}_z$$

$\vec{f} = (f_x\vec{e}_x + f_y\vec{e}_y + f_z\vec{e}_z) \quad$ und $\quad \vec{e}_{x_i}\cdot\vec{e}_{x_j}\otimes\vec{e}_{x_k} = \delta_{ij}\otimes\vec{e}_{x_k} = \delta_{ij}\,\vec{e}_{x_k}$

$$\Rightarrow \vec{f}\cdot\vec\nabla\vec{e}_r = f_x\left(\frac{1}{r}-\frac{x^2}{r^3}\right)\vec{e}_x - \frac{1}{r^3}f_x\,xy\,\vec{e}_y - \frac{1}{r^3}f_x\,xz\,\vec{e}_z -$$
$$- f_y\frac{xy}{r^3}\vec{e}_x + f_y\left(\frac{1}{r}-\frac{y^2}{r^3}\right)\vec{e}_y - f_y\frac{xz}{r^3}\vec{e}_z -$$
$$- f_z\frac{xz}{r^3}\vec{e}_x - f_z\frac{yz}{r^3}\vec{e}_y + f_z\left(\frac{1}{r}-\frac{z^2}{r^3}\right)\vec{e}_z =$$
$$= \left(f_x\left(\frac{1}{r}-\frac{x^2}{r^3}\right) - f_y\frac{xy}{r^3} - f_z\frac{xz}{r^3}\right)\vec{e}_x +$$
$$+ \left(f_y\left(\frac{1}{r}-\frac{y^2}{r^3}\right) - f_x\frac{xy}{r^3} - f_z\frac{yz}{r^3}\right)\vec{e}_y +$$
$$+ \left(f_x\frac{xz}{r^3} - f_y\frac{xz}{r^3} + f_z\left(\frac{1}{r}-\frac{z^2}{r^3}\right)\right)\vec{e}_z$$

A.1.28 (Forts.)

$$= \left(\frac{f_x}{r} - f_x \frac{x^2}{r^3} - f_y \frac{xy}{r^3} - f_z \frac{xz}{r^3}\right) \vec{e}_x +$$
$$+ \left(f_y \frac{1}{r} - f_y \frac{y^2}{r^3} - f_x \frac{xy}{r^3} - f_z \frac{yz}{r^3}\right) \vec{e}_y +$$
$$+ \left(f_x \frac{xz}{r^3} - f_y \frac{xz}{r^3} + f_z \frac{1}{r} - f_z \frac{z^2}{r^3}\right) \vec{e}_z =$$
$$= \left(f_x \frac{1}{r} - \frac{x}{r^3}\left(f_x x + f_y y + f_z z\right)\right) \vec{e}_x +$$
$$+ \left(f_y \frac{1}{r} - \frac{y}{r^3}\left(f_x x + f_y y + f_z z\right)\right) \vec{e}_y +$$
$$+ \left(f_z \frac{1}{r} - \frac{z}{r^3}\left(f_x x + f_y y + f_z z\right)\right) \vec{e}_y$$
$$= \frac{1}{r}\left(f_x \vec{e}_x + f_y \vec{e}_y + f_z \vec{e}_z\right) - \frac{1}{r^3}\left(f_x x + f_y y + f_z z\right)\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right) =$$
$$= \frac{1}{r} \vec{f} - \frac{1}{r^3} \left(f_x x + f_y y + f_z z\right) \vec{r} =$$
$$= \frac{1}{r} \vec{f} - \frac{1}{r^2} \underbrace{\left(f_x x + f_y y + f_z z\right)}_{\vec{f}\cdot\vec{r}} \vec{e}_r = \frac{1}{r}\vec{f} - \frac{1}{r} \vec{f}\cdot\vec{e}_r \vec{e}_r =$$
$$= \frac{1}{r}\left(\vec{f} - \vec{f}\cdot\vec{e}_r \vec{e}_r\right)$$

$\square$

A.1.2.9.

(i) $\operatorname{div}\vec{f} = \vec{\nabla}\cdot\vec{f} = \partial_x f_x + \partial_y f_y + \partial_z f_z$

$\operatorname{rot}\vec{f} = \vec{\nabla}\times\vec{f} = (\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z)\times(f_x\vec{e_x} + f_y\vec{e_y} + f_z\vec{e_z}) =$

$= \partial_x f_y \vec{e_z} - \partial_x f_z \vec{e_y} - \partial_y f_x \vec{e_z} + \partial_y f_z \vec{e_x} +$

$+ \partial_z f_x \vec{e_y} - \partial_z f_y \vec{e_x} =$

$= (\partial_y f_z - \partial_z f_y)\vec{e_x} + (\partial_z f_x - \partial_x f_z)\vec{e_y} + (\partial_x f_y - \partial_y f_x)\vec{e_z}.$

(ii) $\vec{e_\rho} = \cos\alpha\,\vec{e_x} + \sin\alpha\,\vec{e_y}$ $\qquad \vec{e_x} = \cos\alpha\,\vec{e_\rho} - \sin(\alpha)\vec{e_\alpha}$

$\vec{e_\alpha} = -\sin\alpha\,\vec{e_x} + \cos\alpha\,\vec{e_y}$ $\qquad \vec{e_y} = \sin(\alpha)\vec{e_\rho} + \cos\alpha\,\vec{e_\alpha}$

$\vec{r} = x\vec{e_x} + y\vec{e_y} + z\vec{e_z} = r_\rho\vec{e_\rho} + r_\alpha\vec{e_\alpha} + z\vec{e_z}$

$\partial_x\vec{r} = 1\vec{e_x} = \cos\alpha\,\vec{e_\rho} - \sin\alpha\,\vec{e_\alpha} =$

$= \partial_x(r_\rho\vec{e_\rho} + r_\alpha\vec{e_\alpha}) = \partial_x(r_\rho\cos\alpha\,\vec{e_x} + r_\rho\sin\alpha\,\vec{e_y}) +$

$+ \partial_x(-r_\alpha\sin\alpha\,\vec{e_x} + r_\alpha\cos\alpha\,\vec{e_y}) =$

$= \partial_x(r_\rho\cos\alpha - r_\alpha\sin\alpha)\vec{e_x} + \partial_x(r_\rho\sin\alpha + r_\alpha\cos\alpha)\vec{e_y}$

$x = \rho\cos\alpha \qquad y = \rho\sin\alpha \qquad \rho = \rho(x,y) \quad \alpha = \alpha(x,y)$

$\vec{f} = f_\rho\vec{e_\rho} + f_\alpha\vec{e_\alpha} + f_z\vec{e_z}$

$\partial_\alpha f_\rho$

$\partial_x\vec{f} = \partial_x f_\rho$

A. 1.2.10.

$$(x,y) \longmapsto (u,\beta)$$

$$x = x(u,\beta) = \frac{a \cdot \sinh(u)}{\cosh(u)+\cos(\beta)} \qquad -\infty < u < \infty$$

$$y = y(u,\beta) = \frac{a \sin(\beta)}{\cosh(u)+\cos(\beta)} \qquad -\pi < \beta < \pi$$

(i) Koordinatenlinien.

β fest, u var.

$$x = a \cdot \frac{\sinh(u)}{\cosh(u)+c} \qquad c := \cos(\beta)$$

$$y = a \cdot \frac{s}{\cosh(u)+c} \qquad s := \sin(\beta)$$

$$\sinh(u) = \frac{1}{2}\left(e^{u} - e^{-u}\right) \qquad \cosh(u) = \frac{1}{2}\left(e^{u} + e^{-u}\right)$$

$\beta = \frac{\pi}{2}$:
$$x = \frac{a \sinh(u)}{\cosh(u)} = a \cdot \tanh(u)$$
$$y = \frac{a}{\cosh(u)}$$

A.1.2.11.

(i) $\vec{\nabla}\{\vec{\nabla}\cdot[f(z)\vec{r}]\}$

$\vec{\nabla}\cdot(f(z)\vec{r}) = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\cdot(f(z)x\vec{e}_x + f(z)y\vec{e}_y + f(z)z\vec{e}_z) =$

$= f(z)\partial_x x + f(z)\partial_y y + \partial_z f(z)z = 2f(z) + \partial_z f(z)z + f(z) =$

$= 3f(z) + z\partial_z f(z)$

$\vec{\nabla}\{3f(z) + z\partial_z f(z)\} = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)(3f(z) + z\partial_z f(z)) =$

$= 3\vec{e}_z\partial_z f(z) + \vec{e}_z(\partial_z f(z) + z\partial_{zz} f(z)) =$

$= 4\vec{e}_z\partial_z f(z) + z\partial_{zz} f(z)\vec{e}_z = (4\partial_z f(z) + z\partial_{zz} f(z))\vec{e}_z$

(ii) $\vec{r}\times\{\vec{\nabla}\times[f(z)\vec{r}]\}$

$\vec{\nabla}\times f(z)\vec{r} = (\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\times(f(z)x\vec{e}_x + f(z)y\vec{e}_y + f(z)z\vec{e}_z)$

$= x\partial_z f(z)\vec{e}_y - y\partial_z f(z)\vec{e}_x$

$(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)\times(x\partial_z f(z)\vec{e}_y - y\partial_z f(z)\vec{e}_x) =$

$= x^2\partial_z f(z)\vec{e}_z + y^2\partial_z f(z)\vec{e}_z - xz\partial_z f(z)\vec{e}_x - yz\partial_z f(z)\vec{e}_y =$

$= -xz\partial_z f(z)\vec{e}_x - yz\partial_z f(z)\vec{e}_y + (x^2+y^2)\partial_z f(z)\vec{e}_z$

(iii) $\vec{r}\times[\vec{r}\times\vec{\nabla}f(z)]$

$\vec{\nabla}f(z) = \partial_z f(z)\vec{e}_z$

$(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)\times(\partial_z f(z)\vec{e}_z) = -x\partial_z f(z)\vec{e}_y + y\partial_z f(z)\vec{e}_x$

$(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)\times(-x\partial_z f(z)\vec{e}_y + y\partial_z f(z)\vec{e}_x) =$

$= -x^2\partial_z f(z)\vec{e}_z - y^2\partial_z f(z)\vec{e}_z + xz\partial_z f(z)\vec{e}_x + yz\partial_z f(z)\vec{e}_y =$

$= +xz\partial_z f(z)\vec{e}_x + yz\partial_z f(z)\vec{e}_y - (x^2+y^2)\partial_z f(z)\vec{e}_z$

A.1.2.12 $\{\varsigma, \alpha, z\}$

$$f(\varsigma,\alpha,z) = z\cos(\alpha)$$

$$\vec{r} \times \vec{\nabla}\left(\frac{1}{\varsigma} f\right) \qquad \vec{r} = \varsigma \vec{e}_\varsigma + z\vec{e}_z$$

$$\vec{\nabla} = \vec{e}_\varsigma \partial_\varsigma + \vec{e}_\alpha \frac{1}{\varsigma}\partial_\alpha + \vec{e}_z \partial_z$$

$$\vec{\nabla}\left(\frac{1}{\varsigma} f\right) = \left(\vec{e}_\varsigma \partial_\varsigma + \vec{e}_\alpha \frac{1}{\varsigma}\partial_\alpha + \vec{e}_z \partial_z\right)\left(\frac{1}{\varsigma}\cdot z \cdot \cos(\alpha)\right) =$$
$$= -\frac{1}{\varsigma^2} z\cos(\alpha)\vec{e}_\varsigma - \frac{1}{\varsigma^2} z\sin(\alpha)\vec{e}_\alpha + \frac{1}{\varsigma}\cos(\alpha)\vec{e}_z$$

$$\vec{r} \times \vec{\nabla}\left(\frac{1}{\varsigma} f\right) = \left(\varsigma\vec{e}_\varsigma + z\vec{e}_z\right) \times \left(-\frac{1}{\varsigma^2} z\cos(\alpha)\vec{e}_\varsigma - \frac{1}{\varsigma^2} z\sin(\alpha)\vec{e}_\alpha + \right.$$
$$\left. + \frac{1}{\varsigma}\cos(\alpha)\vec{e}_z\right) =$$

$$= -\varsigma \cdot \frac{1}{\varsigma^2} z\sin(\alpha)\vec{e}_z - \varsigma\frac{1}{\varsigma}\cos(\alpha)\vec{e}_\alpha - \frac{1}{\varsigma^2} z^2\cos(\alpha)\vec{e}_\alpha +$$
$$+ \frac{1}{\varsigma^2} z^2 \sin(\alpha)\vec{e}_\varsigma = -\frac{1}{\varsigma} z\sin(\alpha)\vec{e}_z - \cos(\alpha)\vec{e}_\alpha - \frac{z^2}{\varsigma^2}\cos(\alpha)\vec{e}_\alpha +$$
$$+ \frac{z^2}{\varsigma^2}\sin(\alpha)\vec{e}_\varsigma =$$

$$= \frac{z^2}{\varsigma^2}\sin(\alpha)\vec{e}_\varsigma - \left(1 + \frac{z^2}{\varsigma^2}\right)\cos(\alpha)\vec{e}_\alpha - \frac{z}{\varsigma}\sin(\alpha)\vec{e}_z$$

A.1.2.13.

$\{r, \theta, \alpha\}$

$f(r,\theta,\alpha) = r^2 \cos(\theta)$

$\vec{\nabla} \times (\vec{\nabla} \times (\vec{r} f))$ $\qquad \vec{r} = r\vec{e_r}$

$\vec{\nabla} = \vec{e_r}\partial_r + \vec{e_\theta}\frac{1}{r}\partial_\theta + \vec{e_\alpha}\frac{1}{r\sin\theta}\partial_\alpha$

$\vec{\nabla} \times (\vec{r} f) = (\vec{e_r}\partial_r + \vec{e_\theta}\frac{1}{r}\partial_\theta + \vec{e_\alpha}\frac{1}{r\sin\theta}\partial_\alpha) \times (r^3\cos(\theta)\vec{e_r}) =$

$= +r^2\sin(\theta)\,\vec{e_\alpha}$

$\vec{\nabla} \times (\vec{\nabla} \times (\vec{r} f)) = (\vec{e_r}\partial_r + \vec{e_\theta}\frac{1}{r}\partial_\theta + \vec{e_\alpha}\frac{1}{r\sin\theta}\partial_\alpha) \times (r^2\sin(\theta)\vec{e_\alpha}) =$

$\cancel{= -2r\sin(\theta)\,\vec{e_\theta} + r\cos(\theta)\,\vec{e_r} =}$

$\cancel{= r\cos(\theta)\vec{e_r} - 2r\sin(\theta)\vec{e_\theta}.}$

$\vec{e_r} \times (2r\sin\theta\,\vec{e_\alpha}) + \vec{e_\theta} \times (\frac{1}{r}r^2\cos\theta\,\vec{e_\alpha}) +$

$+ \vec{e_\alpha} \times \frac{1}{r\sin\theta}\partial_\alpha(r^2\sin(\theta)\vec{e_\alpha}) =$

$-2r\sin(\theta)\,\vec{e_\theta} + r\cos(\theta)\,\vec{e_r} +$

$+ \vec{e_\alpha} \times \frac{1}{\cancel{r\sin\theta}} \overset{r}{\cancel{r^2\sin(\theta)}} (\overset{-\sin\theta\,\vec{e_r}}{-\cos\theta\,\vec{e_\theta}}) =$

$= -2r\sin\theta\,\vec{e_\theta} + r\cos\theta\,\vec{e_r} + r\cos\theta\,\vec{e_r} - r\sin\theta\,\vec{e_\theta}$

$= 2r\cos\theta\,\vec{e_r} - 3r\sin\theta\,\vec{e_\theta}$

A.1.2.14.

$$\cancel{\int_0^{2\pi}\int_{\theta_1}^{\theta_2} a \, d\theta \, d\alpha = 2\pi a^2(\theta_2-\theta_1)}$$

Kugel: $(x-x_m)^2+(y-y_m)^2+(z-z_m)^2=r^2$

$x^2+y^2+z^2=a^2$

$x=r\sin(\theta)\cos(\alpha)$ $\quad y=r\cdot\sin(\theta)\sin(\alpha)$ $\quad z=r\cos(\theta)$

$$A=\int_0^{2\pi}\int_{\theta_1}^{\theta_2} a^2\sin\theta \, d\theta \, d\alpha = a^2\cdot 2\pi\int_{\theta_1}^{\theta_2}\sin(\theta)\,d\theta =$$

$$= -2\pi a^2\cos(\theta)\Big|_{\theta=\theta_1}^{\theta=\theta_2} = -2\pi a^2\left[\cos(\theta_2)-\cos(\theta_1)\right] =$$

$$= 2\pi a^2\left[\cos(\theta_1)-\cos(\theta_2)\right].$$

A.1.2.15

$$\vec{F}(\vec{r}) = K r^n \vec{e}_r \qquad K = \text{const.},\ n = \text{konst.},\ r > 0$$

$$\cancel{\oint_{\mathscr{C}} \vec{F}(\vec{x})\, d\vec{x}} \qquad \cancel{\mathscr{C} := \{(x,y,z) \mid x^2 + y^2 + z^2 \overset{!}{=} a,\ z = \text{const}\}}$$

$$\cancel{\int_0^{2\pi} K r^n \vec{e}_r\, r^2 \sin\theta\, d\theta = -K r^{n+2} \cos(\theta)\Big|_{\theta=0}^{\theta=2\pi} = 0}$$

Konservativität $\hat{=}$ lokaler Wirbelfreiheit, d.h. $\vec{\nabla} \times \vec{F} = \vec{0}$.

$\hat{=}$ $\exists$ eines Skalarpotentials $\vec{F} = \vec{\nabla} G$

$$\vec{\nabla} \times \vec{F} = \left(\vec{e}_r \partial_r + \vec{e}_\theta \frac{1}{r} \partial_\theta + \vec{e}_\alpha \frac{1}{r \sin\theta} \partial_\alpha\right) \times \left(K r^n \vec{e}_r\right) =$$

$$= \vec{e}_r \times K \partial_r (r^n \vec{e}_r) + \vec{e}_\theta \times \frac{1}{r} \partial_\theta (K r^n \vec{e}_r) + \vec{e}_\alpha \times \frac{1}{r \sin\theta} \partial_\alpha (K r^n \vec{e}_r)$$

$$= \vec{e}_r \times K n r^{n-1} \vec{e}_r + \vec{e}_\theta \times \frac{1}{r} K r^n \vec{e}_\theta + \vec{e}_\alpha \times \frac{1}{r \sin\theta} K r^n \sin\theta\, \vec{e}_\alpha$$

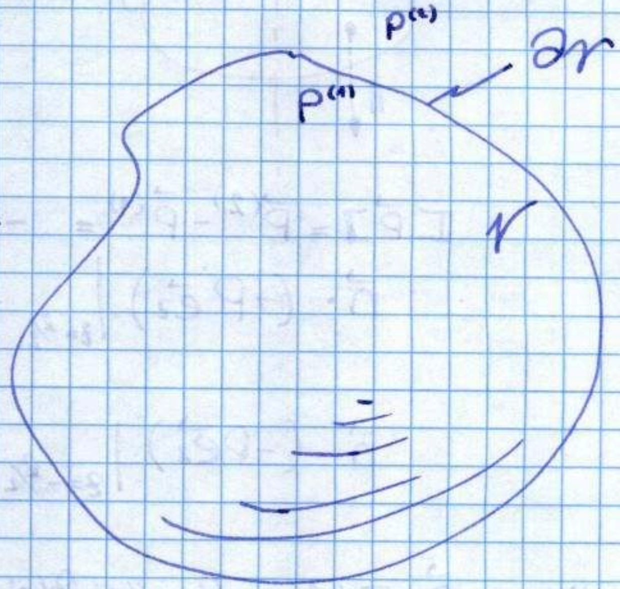
$$= \vec{0}$$

A.2.11. Polarisationsladungen

Ges. fiktive L.: $\int\limits_{\mathcal{V}} \varrho^f \, dV + \int\limits_{\partial\mathcal{V}} \sigma^f \, dA$

$$\varrho^f = -\vec{\nabla}\cdot\vec{P} \qquad \sigma^f = -\vec{n}\cdot[\![\vec{P}]\!]$$

$$\Rightarrow -\int\limits_{\mathcal{V}} \vec{\nabla}\cdot\vec{P} \, dV - \int\limits_{\partial\mathcal{V}} \vec{n}\cdot[\![\vec{P}]\!] \, dA$$



$[\![\vec{P}]\!] = \vec{P}^{(2)} - \vec{P}^{(1)}$

$P^{(2)} = 0$ ... da leere Schicht

$[\![\vec{P}]\!] = -\vec{P}^{(1)}$; $\vec{P} = \vec{P}^{(1)}$

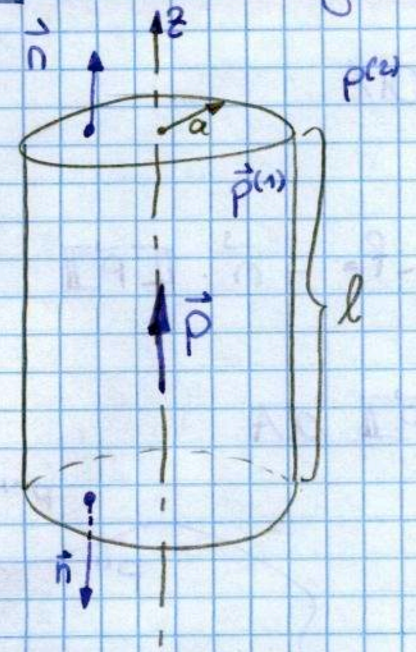
Satz v. Gauß:

$$\int\limits_{\mathcal{V}} \vec{\nabla}\cdot\vec{f} \, dV = \int\limits_{\partial\mathcal{V}} \vec{n}\cdot\vec{f} \, dA$$

$$\Rightarrow -\int\limits_{\partial\mathcal{V}} \vec{\nabla}\cdot\vec{P}^{(1)} \, dA - \int \vec{n}\cdot(-\vec{P}^{(1)}) \, dA = 0$$

$\square$

A.2.1.2. Starr homogen elektrisch polarisierter Kreiszylinder



(i) effektive Ladungsverteilung

$$\vec{P} = P\,\vec{e}_z \,, \quad P = \text{const.}$$

$$\rho^e = \rho^f \qquad \sigma^e = \sigma^f$$

da Körper nicht geladen

$$\rho^e = \rho^f = -\vec{\nabla}\cdot\vec{P} = -(\vec{e}_x\partial_x + \vec{e}_y\partial_y + \vec{e}_z\partial_z)\cdot(P\vec{e}_z) = 0$$

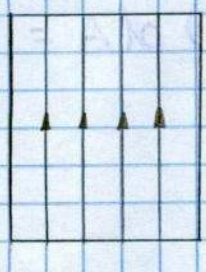
$$\sigma^e = \sigma^f = -\vec{n}\cdot[\![\vec{P}]\!] = \begin{cases} P & z = \ell/2 \\ -P & z = -\ell/2 \end{cases}$$

$$[\![\vec{P}]\!] = \vec{P}^{(2)} - \vec{P}^{(1)} = -\vec{P}^{(1)} = -P\vec{e}_z$$

$$\vec{n}\cdot(-P\vec{e}_z)\Big|_{z=\ell/2} = -P\;\vec{n}\cdot\vec{e}_z\Big|_{z=\ell/2} = -P$$

$$\vec{n}\cdot(-P\vec{e}_z)\Big|_{z=-\ell/2} = -P\;\vec{n}\cdot\vec{e}_z\Big|_{z=\ell/2} = P$$

(ii) $\vec{P}$-Feld: Parallelfeld im Körperinneren



$\vec{E}$-Feld:

A. 2.1.3. Starr homogen magnetisierter Kreiszylinder

$$\vec{M} = M\,\vec{e_z} \qquad M = \text{const.}$$

$$\vec{J}^e = \vec{J}^f \qquad \vec{K}^e = \vec{K}^f$$

$$\vec{J}^f = \underbrace{\partial_t \vec{P}}_{0} + \vec{\nabla} \times \vec{M} \qquad \vec{K}^f = \vec{n} \times [\![\vec{M}]\!]$$

$$\vec{J}^e = \vec{\nabla} \times \vec{M} = \vec{0}$$

$$\vec{K}^e = M\,\vec{e_\alpha}$$

A. 2.1.4. Wahre und fiktive Ladungsdichte

$$\rho^e = \rho + \rho^f$$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 (1-\chi) \vec{E} = \varepsilon_0 \vec{E} - \varepsilon_0 \chi \vec{E}$$

$$\vec{D} = \varepsilon_0 \vec{E} - \varepsilon_0 \chi \vec{E}$$

$$\vec{P} = \varepsilon_0 (\varepsilon_r - 1) \vec{E} = \cancel{\vec{D}(\varepsilon_r - 1)} = \left(1 - \frac{1}{\varepsilon_r}\right) \vec{D}$$

$$\varepsilon_0 (\varepsilon_r - 1) \vec{E} = \left(1 - \frac{1}{\varepsilon_r}\right) \vec{D}$$

$$\varepsilon_0 \vec{E} = \frac{1 - \frac{1}{\varepsilon_r}}{\varepsilon_r - 1} \vec{D} = \frac{\varepsilon_r - 1}{\varepsilon_r^2 - \varepsilon_r} \vec{D} = \frac{\cancel{\varepsilon_r - 1}}{\varepsilon_r \cancel{(\varepsilon_r - 1)}} \vec{D}$$

$$\boxed{\varepsilon_0 \varepsilon_r \vec{E} = \vec{D}} \qquad \boxed{\vec{P} = \varepsilon_0 \chi \vec{E} = \varepsilon_0 (\varepsilon_r - 1) \vec{E}}$$

$$\vec{P} = \varepsilon_0 \chi \vec{E} = \varepsilon_0 (\varepsilon_r - 1) \vec{E} = \cancel{\varepsilon_0} (\varepsilon_r - 1) \frac{\vec{D}}{\cancel{\varepsilon_0} \varepsilon_r} = \left(1 - \frac{1}{\varepsilon_r}\right) \vec{D}$$

$$\rho^f = -\nabla \cdot \vec{P} = - \left(1 - \frac{1}{\varepsilon_r}\right) \nabla \cdot \vec{D} = -\left(1 - \frac{1}{\varepsilon_r}\right) \rho = \left(\frac{1}{\varepsilon_r} - 1\right) \rho$$

A.2.1.5.

$\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}$ $\qquad$ $\mu_r = 1 + \chi \Rightarrow \chi = \mu_r - 1$

$\vec{j}^f = \underbrace{\partial_t \vec{P}}_{0 \text{ ... da } \vec{P} = \vec{0}} + \vec{\nabla} \times \vec{M}$ $\qquad$ $\vec{M} = \chi \vec{H} = (\mu_r - 1)\vec{H}$

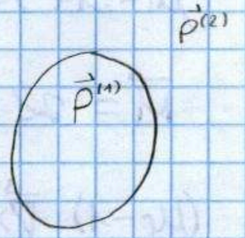
$\vec{j}^f = \vec{\nabla} \times ((\mu_r - 1)\vec{H}) = (\mu_r - 1)\, \vec{\nabla} \times \vec{H}$

$\vec{\nabla} \times \vec{H} = \vec{j} + \underbrace{\partial_t \vec{D}}_{=\vec{0}}$ ... Verschiebungsstrom

$\vec{j}^f = (\mu_r - 1)\vec{j}$

A. 2.1.6 Homogen elektrisch polarisierte Kugel.

$$\varrho^f = -\vec{\nabla} \cdot \vec{P}$$



$$\vec{P} = \text{const} = \vec{P}^{(1)}$$

$$[\![\vec{P}]\!] = \vec{P}^{(2)} - \vec{P}^{(1)} = -P$$

$$\vec{P} = P\,\vec{e}_z$$

$$\varrho^f = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot P\,\vec{e}_z = 0$$

$$\sigma^f = -\vec{n} \cdot [\![\vec{P}]\!] = \vec{e}_r \cdot \vec{e}_z \cdot P = P \cdot \cos(\theta)$$

wegen: $\cos(\theta) = \vec{e}_r \cdot \vec{e}_z$

A. 2.1.7. $\{x, y, z\}$

$$\vec{M} = M\vec{e_z} \quad , \quad M = \text{const.}$$

$$\vec{J}^f = \partial_t \vec{P} + \vec{\nabla} \times \vec{M} = \vec{0}$$

$$\vec{K}^f = \vec{n} \times [\![\vec{M}]\!]$$

$$[\![\vec{M}]\!] = \vec{M}^{(2)} - \vec{M}^{(1)} = -M\vec{e_z}$$

$$\vec{K}^f = -(\vec{e_r} \times \vec{e_z})M = -M(\vec{e_r} \times \vec{e_z})$$

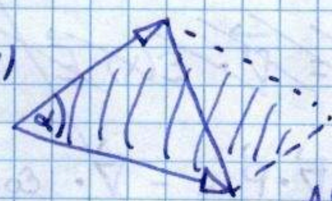
$$\vec{r} = x\vec{e_x} + y\vec{e_y} + z\vec{e_z} = r\vec{e_r} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\vec{e_r} = \frac{1}{r}(x\vec{e_x} + y\vec{e_y} + z\vec{e_z})$$

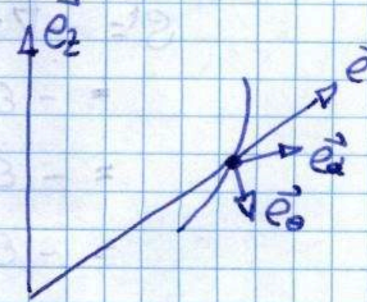
$$\vec{K}^f = -\frac{M}{r}(x\vec{e_x} + y\vec{e_y} + z\vec{e_z}) \times \vec{e_z} =$$

$$= -\frac{M}{r}(-x\vec{e_y} + y\vec{e_x}) = -\frac{M}{r}(y\vec{e_x} - x\vec{e_y}) =$$

! $$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{a \cdot b}{2} \cdot \sin(\alpha)$$



$$\vec{e_a} \times \vec{e_b} = \sin(\alpha)$$



$$\vec{e_r} \times \vec{e_z} = -\sin(\alpha)\,\vec{e_\alpha}$$
$$\vec{e_z} \times \vec{e_r} = \sin(\alpha)\,\vec{e_\alpha}$$

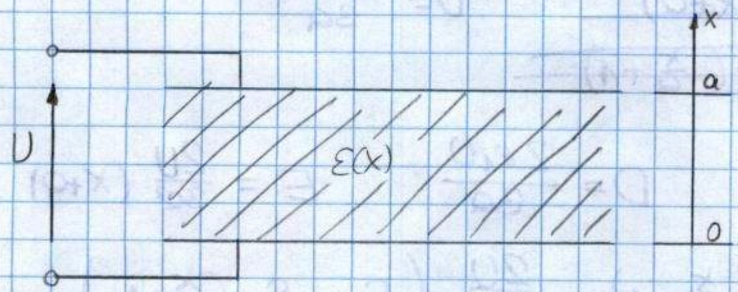
$$= -\frac{M}{r}\Big(y\left[\sin(\theta)\cos(\alpha)\vec{e_r} + \cos(\theta)\cos(\alpha)\vec{e_\theta} - \sin(\alpha)\vec{e_\alpha}\right] -$$

$$-x\left[\sin(\theta)\sin(\alpha)\vec{e_r} + \cos(\theta)\sin(\alpha)\vec{e_\theta} - \cos(\alpha)\vec{e_\alpha}\right]\Big) =$$

$$= -\frac{M}{r}\Big\{\sin(\theta)\left[y\cos(\alpha) - x\sin(\alpha)\right]\vec{e_r} + \cos(\theta)\left[y\cos(\alpha) - x\sin(\alpha)\right]\vec{e_\theta} -$$

$$- \left[y\sin(\alpha) + x\cos(\alpha)\right]\vec{e_\alpha}$$

A 2.1.8.



$$\varepsilon(x) = \varepsilon_1 \frac{a}{x+a}$$

$\vec{D} = \varepsilon(x)\vec{E}$ $\quad \chi = \varepsilon_r - 1$ $\quad \vec{P} = \varepsilon_0 \chi \vec{E}$

$U = \int_{\mathcal{S}} \vec{s}\cdot\vec{E}\, ds = \int_0^a \vec{E}\, dx$ $\quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$\varrho^P = -\vec{\nabla}\cdot\vec{P}$ $\quad \sigma^P = -\vec{n}\cdot [\![\vec{P}]\!]$

! Im Inneren eines perfekt nichtleitenden Dielektrikum können sich keine wahren Ladungen ansammeln.

$\vec{D} = D(x)\vec{e_x}$ $\quad \vec{E} = E(x)\vec{e_x}$ $\quad \vec{P} = P(x)\vec{e_x}$

$\vec{\nabla}\cdot\vec{D} = (\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z)\cdot(D(x)\vec{e_x}) = \partial_x D(x)$

$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \varrho - \vec{\nabla}\cdot\vec{P}$

$\varepsilon_0 \vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = \varrho$

$\vec{\nabla}\cdot\underbrace{(\varepsilon_0\vec{E}+\vec{P})}_{\vec{D}} = \varrho$

$\vec{\nabla}\cdot\vec{D} = \varrho \overset{!}{=} 0$

daher $\quad \partial_x D(x) = 0$

$\Rightarrow D(x) = \text{const.} = D$

$\vec{D} = \varepsilon\vec{E} \Rightarrow \vec{E} = \frac{D}{\varepsilon(x)}$

$U = D\int_0^a \frac{1}{\varepsilon(x)}\, dx = \frac{D}{\varepsilon_1}\int_0^a \frac{x+a}{a}\, dx =$

$= \frac{D}{\varepsilon_1}\int_0^a \frac{x}{a}+1\, dx = \frac{D}{\varepsilon_1}\left(\frac{x^2}{2a}+x\right)\Big|_{x=0}^{x=a} =$

$= \frac{D}{\varepsilon_1}\left(\frac{a^2}{2a}+a\right) = \frac{D}{\varepsilon_1}\left(\frac{a}{2}+a\right) = \frac{D}{\varepsilon_1}\frac{3a}{2}$

$U = \frac{D}{\varepsilon_1}\frac{3a}{2} \Rightarrow D = \frac{2\varepsilon_1 U}{3a}$

$\vec{D} = \varepsilon(x)\vec{E} = D\vec{e_x}$

$E = \frac{1}{\varepsilon(x)}\cdot D = \frac{x+a}{\varepsilon_1 a}\,\frac{2\varepsilon_1 U}{3a} =$

$= \frac{2U}{3a^2}(x+a) = E(x)$

$[\![\vec{P}]\!] = -\vec{P}$

$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \vec{P} = \vec{D} - \varepsilon_0\vec{E}$ $\cancel{\Rightarrow P = \frac{D}{\varepsilon_1}\frac{3a}{2} - \varepsilon_0\frac{2U}{3a^2}(x+a)}$

$$\cancel{\vec{P} = \frac{D}{\varepsilon_1}\frac{3a}{2} - \varepsilon_0 \frac{8U}{8a^2}(x+a)} \qquad D = \frac{2\varepsilon_1 U}{3a}$$

$$\cancel{= \frac{2U}{3a}\frac{3a}{2} - \varepsilon_0 \frac{U}{a}\left(\frac{x}{a}+1\right)}$$

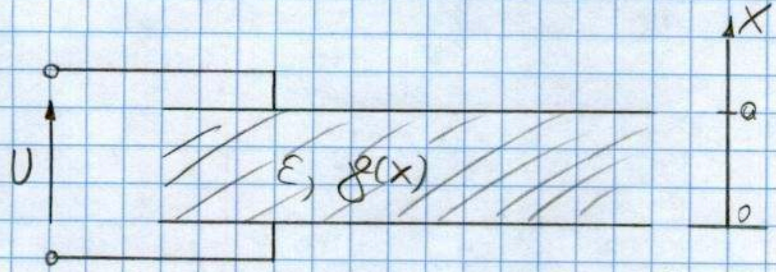
$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} \qquad D = \frac{2\varepsilon_1 U}{3a} \qquad E = \frac{2U}{3a^2}(x+a)$$

$$P = \frac{2\varepsilon_1 U}{3a} - \varepsilon_0 \frac{2U}{3a}\left(\frac{x}{a}+1\right) = \frac{2U}{3a}\left(\varepsilon_1 - \varepsilon_0\left(\frac{x}{a}+1\right)\right)$$

$$\sigma^P = -\vec{n}\ [\![\vec{P}]\!] = \begin{cases} -P(x)\ , x=a & = \frac{2U}{3a}(\varepsilon_1 - 2\varepsilon_0) \\ P(x)\ , x=0 & = \frac{2U}{3a}(\varepsilon_1 - \varepsilon_0) \end{cases}$$

$$\sigma = \vec{n}\cdot[\![\vec{D}]\!] = \begin{cases} -\frac{2\varepsilon_1 U}{3a} & x=a \\ +\frac{2\varepsilon_1 U}{3a} & x=0 \end{cases}$$

A 2.1.9.



$\varrho(x) = \varrho_0 \frac{a}{x+a}$

$\vec{J} = \varrho(x)\,\vec{E}$ $\qquad U = \int_0^a E(x)\,dx$

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$\vec{J} = J(x)\,\vec{e}_x \qquad \vec{E} = E(x)\,\vec{e}_x \qquad \vec{D} = D(x)\,\vec{e}_x \qquad \vec{P} = P(x)\,\vec{e}_x$

$\vec{\nabla}\cdot\vec{J} = 0$ ... da stationärer (zeitunabhängiger) Fall

$\partial_x J(x) = 0 \Rightarrow J(x) = const. = J$

$\Rightarrow \quad J = \varrho(x)\,E(x) \Rightarrow E(x) = \frac{J}{\varrho(x)} = \frac{J}{\varrho_0}\left(\frac{x}{a}+1\right)$

$U = \int_0^a \frac{J}{\varrho_0}\left(\frac{x}{a}+1\right)dx = \frac{J}{\varrho_0}\left(\frac{x^2}{2a}+x\right)\Big|_{x=0}^{x=a} = \frac{J}{\varrho_0}\left(\frac{a}{2}+a\right) = \frac{J}{\varrho_0}\frac{3a}{2}$

$J = \frac{2\varrho_0 U}{3a} \Rightarrow E(x) = \frac{J}{\varrho_0}\left(\frac{x}{a}+1\right) \frac{2\varrho_0 U}{3a\varrho_0}\left(\frac{x}{a}+1\right) = \frac{2U}{3a}\left(\frac{x}{a}+1\right)$

$D = \varepsilon E = \qquad P = D - \varepsilon_0 E = (\varepsilon - \varepsilon_0) E$

A.2.110. $\{x, y, z\}$

$$\vec{J}^f = \underbrace{\partial_t \vec{P}}_{0} + \vec{\nabla} \times \vec{M} \qquad \vec{K}^f = \vec{n} \times [\![\vec{M}]\!]$$

$$\vec{\nabla} \times \vec{M} = M_0 (\vec{e_x}\,\partial_x + \vec{e_y}\,\partial_y + \vec{e_z}\,\partial_z) \times (-\sin(kx)\vec{e_x} + \cos(kx)\vec{e_z}) =$$

$$= M_0 \cdot (-\partial_x(\cos(kx))\vec{e_y} + \partial_y(\cancel{\sin(kx)})\,\vec{e_z} + \partial_y(\cancel{\cos(kx)})\vec{e_x} -$$

$$- \partial_z\,\cancel{(\sin(kx))}\,\vec{e_y}) = M_0 k \sin(kx)\,\vec{e_y}$$

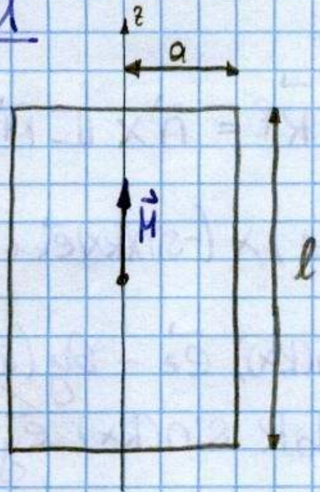
$$[\![\vec{M}]\!] = -\vec{M} \qquad \vec{n}\big|_{z=d} = \vec{e_z} \qquad \vec{n}\big|_{z=0} = -\vec{e_z}$$

$$\vec{K}^f\big|_{z=0} = -\vec{e_z} \times (\sin(kx)\,\vec{e_x} - \cos(kx)\,\vec{e_z})\,M_0 =$$

$$= -M_0 \sin(kx)\,\vec{e_y}$$

$$\vec{K}^f\big|_{z=d} = \vec{e_z} \times [\![\vec{M}]\!] = M_0 \sin(kx)\,\vec{e_y}$$

A. 2.1.11



$$\vec j^f = \vec\nabla\times\vec M$$

$$\vec M = \left(1-\frac{\varrho}{2a}\right)M_0\,\vec e_z$$

$$\vec K^f = \vec n\times [\![\vec M]\!]$$

$$\vec\nabla\times\vec f = \vec e_\varrho\left(\frac{1}{\varrho}\partial_\alpha f_z - \cancel{\partial_z f_\alpha}\right) + \vec e_\alpha\left(\cancel{\partial_z f_\varrho} - \partial_\varrho f_z\right) + \vec e_z\left[\cancel{\partial_\varrho(\varrho f_\alpha)} - \cancel{\partial_\alpha f_\varrho}\right]$$

hier:

$$\vec\nabla\times\vec f = \vec e_\varrho\left(\frac{1}{\varrho}\partial_\alpha f_z\right) - \vec e_\alpha\,\partial_\varrho f_z$$

$$\vec j^f = \cancel{\vec e_\varrho\left(\frac{1}{\varrho}\partial_\alpha\left(1-\frac{\varrho}{2a}\right)M_0\right)} - \vec e_\alpha\,\partial_\varrho\left(1-\frac{\varrho}{2a}\right)M_0 =$$

$$= -\vec e_\alpha M_0\left(-\frac{1}{2a}\right) = \frac{M_0}{2a}\vec e_\alpha$$

$$\vec n\big|_{\text{Mantel}} = \vec e_\varrho \qquad \vec n\big|_{z=\ell/2} = \vec e_z \qquad \vec n\big|_{z=-\ell/2} = -\vec e_z$$

$$[\![\vec M]\!] = -\vec M = M_0\left(\frac{\varrho}{2a}-1\right)\vec e_z \qquad \{\varrho,\alpha,z\}$$

Mantel: $\vec n\times[\![\vec M]\!] = \vec e_\varrho\times M_0\left(\frac{a}{2a}-1\right)\vec e_z = M_0\left(1-\frac{1}{2}\right)\vec e_\alpha = \frac{M_0}{2}\vec e_\alpha$

$z=\ell/2$: — " — $= \vec e_z\times$ — " — $\vec e_z = \vec 0$

$z=-\ell/2$: — " — $= -\vec e_z\times$ — " — $\vec e_z = \vec 0$

A. 2.1.12

$$\underset{\sim}{\varepsilon} = \left(2{,}6\, \vec{e_x} \otimes \vec{e_x} + 1{,}2\, \vec{e_y} \otimes \vec{e_y} + 1{,}7\, \vec{e_z} \otimes \vec{e_z}\right) \varepsilon_0$$

$$\vec{D} = \underset{\sim}{\varepsilon} \cdot \vec{E} = \frac{\varepsilon_0 E_0}{\sqrt{21}} \left(2{,}6\, \vec{e_x} \otimes \vec{e_x} + 1{,}2\, \vec{e_y} \otimes \vec{e_y} + 1{,}7\, \vec{e_z} \otimes \vec{e_z}\right) \cdot$$
$$\cdot \left(\vec{e_x} - 4\vec{e_y} + 2\vec{e_z}\right) =$$

$$= \frac{\varepsilon_0 E_0}{\sqrt{21}} \left(2{,}6\, \vec{e_x} \otimes \vec{e_x} \cdot \vec{e_x} - 4{,}8\, \vec{e_y} \otimes \vec{e_y} \cdot \vec{e_y} + 3{,}4\, \vec{e_z} \otimes \vec{e_z} \cdot \vec{e_z}\right) =$$

$$= \frac{\varepsilon_0 E_0}{\sqrt{21}} \left(2{,}6\, \vec{e_x} - 4{,}8\, \vec{e_y} + 3{,}4\, \vec{e_z}\right) = \vec{D}$$

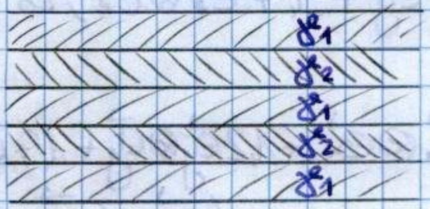
$$\vec{E} = \frac{E_0}{\sqrt{21}} \left(\vec{e_x} - 4\vec{e_y} + 2\vec{e_z}\right)$$

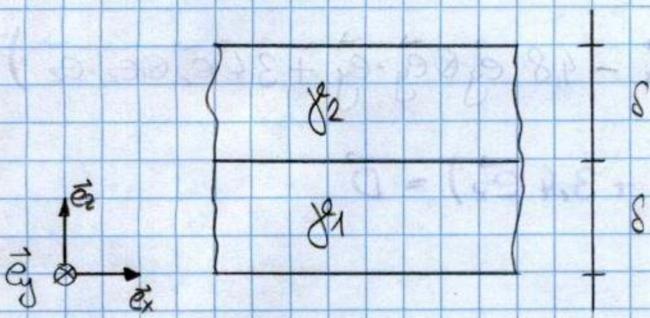
$$\cos\varphi = \frac{(2{,}6\, \vec{e_x} - 4{,}8\, \vec{e_y} + 3{,}4\, \vec{e_z}) \cdot (\vec{e_x} - 4\vec{e_y} + 2\vec{e_z})}{\sqrt{2{,}6^2 + 4{,}8^2 + 3{,}4^2}\ \sqrt{1 + 4^2 + 2^2}} =$$

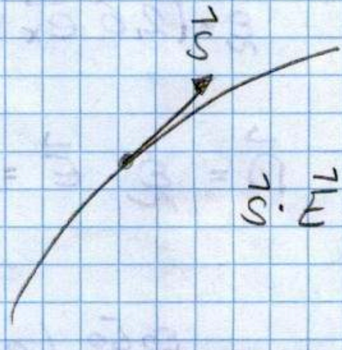
$$= \frac{2{,}6 + 4{,}8 \cdot 4 + 2 \cdot 3{,}4}{\sqrt{2{,}6^2 + 4{,}8^2 + 3{,}4^2}\sqrt{1 + 4^2 + 2^2}} = \frac{28{,}6}{6{,}43\ldots \cdot \sqrt{21}} = 0{,}970434$$

$$\Rightarrow \varphi = 0{,}24377\ \text{rad} = 13{,}96715\,°$$

A 2.1.13







$\vec{s} \cdot \vec{E} = E_t$

$\vec{J} = J_x \vec{e_x} + J_y \vec{e_y} + J_z \vec{e_z}$

$\vec{E} = E_x \vec{e_x} + E_y \vec{e_y} + E_z \vec{e_z}$

$\vec{E}, \vec{J}$ in jeder Schicht konstant.

A.2.2.1. Wasserstoffatom

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\frac{2r}{a}}}{r} \left(1 + \frac{r}{a}\right)$$

grad in Kugelkoordinaten:

$$\vec{\nabla} f = \vec{e_r}\, \partial_r f + \vec{e_\theta} \frac{\partial_\theta f}{r} + \vec{e_\alpha} \frac{\partial_\alpha f}{r \sin(\theta)}$$

$$\vec{\nabla}\varphi = \vec{e_r}\, \partial_r \varphi = \vec{e_r} \frac{q}{4\pi\varepsilon_0} \partial_r \left[e^{-\frac{2r}{a}} \left(\frac{1}{r} + \frac{1}{a} \right) \right] =$$

$$= \vec{e_r} \frac{q}{4\pi\varepsilon_0} \left[-\frac{2}{a} e^{-\frac{2r}{a}} \left(\frac{1}{r} + \frac{1}{a} \right) + e^{-\frac{2r}{a}} \left(-\frac{1}{r^2} \right) \right] =$$

$$= \vec{e_r} \frac{q}{4\pi\varepsilon_0} e^{-\frac{2r}{a}} \left(-\frac{2}{ar} - \frac{2}{a^2} - \frac{1}{r^2} \right)$$

$$\vec{E} = -\vec{\nabla}\varphi = \frac{q}{4\pi\varepsilon_0} e^{-\frac{2r}{a}} \underbrace{\left(\frac{1}{r^2} + \frac{2}{ar} + \frac{2}{a^2} \right)}_{\left(\frac{1}{a}+\frac{1}{r}\right)^2 + \frac{1}{a}} \vec{e_r} =: \varphi_r \vec{e_r}$$

div in Kugelkoordinaten:

$$\vec{\nabla} \cdot \vec{f} = \frac{\partial_r (r^2 f_r)}{r^2} + \frac{\partial_\theta [\sin(\theta) f_\theta]}{r \sin(\theta)} + \frac{\partial_\alpha f_\alpha}{r \sin(\theta)}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial_r (r^2 \varphi_r)}{r^2} = \frac{1}{r^2} \partial_r (r^2 \varphi_r)$$

NR: $\partial_r \left\{ r^2 e^{-\frac{2r}{a}} \left[\left(\frac{1}{r} + \frac{1}{a} \right)^2 + \frac{1}{a} \right] \right\} = \partial_r \left\{ e^{-\frac{2r}{a}} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right\} =$

$$= -\frac{2}{a} e^{-\frac{2r}{a}} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) + e^{-\frac{2r}{a}} \left(\frac{2}{a} + \frac{4r}{a^2} \right) =$$

$$= e^{-\frac{2r}{a}} \left(-\cancel{\frac{2}{a}} - \cancel{\frac{4r}{a^2}} - \frac{4r^2}{a^3} + \cancel{\frac{2}{a}} + \cancel{\frac{4r}{a^2}} \right) = -e^{-\frac{2r}{a}} \frac{4r^2}{a^3}$$

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{r^2} \cdot \frac{q}{4\pi\varepsilon_0} \frac{4r^2}{a^3} = -\frac{q}{4\pi\varepsilon_0} \cdot \frac{4}{a^3} e^{-\frac{2r}{a}}$$

$$\rho = \varepsilon_0 \cdot \vec{\nabla} \cdot \vec{E} = -\frac{q}{\pi a^3} e^{-\frac{2r}{a}}$$

$$Q(\mathcal{V}) = \Psi(\partial \mathcal{V})$$

$$Q(\mathcal{V}) = \int_{\mathcal{V}} \rho \, dV$$

A 2.2.2

(i) Gesamtladung

$$Q(r) = \int_r \rho \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_0 \left(1-\left(\frac{r}{a}\right)^2\right) r^2 \cdot \sin\theta \quad dr\, d\theta\, d\alpha =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_0 \left(r^2 - \frac{r^4}{a^2}\right) \sin\theta \quad dr\, d\theta\, d\alpha =$$

$$= \rho_0 \int_0^{2\pi}\int_0^{\pi} \left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right)\Big|_{r=0}^{r=a} \sin\theta \quad d\theta\, d\alpha =$$

$$= 2\rho_0 \pi \int_0^{\pi} \left(\frac{a^3}{3} - \frac{a^5}{5a^2}\right) \sin\theta \quad d\theta = \rho_0 \pi \frac{4a^3}{15} \int_0^{\pi} \sin\theta \, d\theta =$$

$$= \frac{4a^3\rho_0\pi}{15} \quad -\cos(\theta)\Big|_0^{\pi} = \frac{4a^3\rho_0\pi}{15} \cdot \left[-\left((-1)-1\right)\right] = \underline{\underline{\frac{8a^3\rho_0\pi}{15}}}$$

(ii) $\vec{\nabla}\cdot\vec{D} = \rho$

$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho$

Div. in Kugelkoordinaten:

$$\vec{\nabla}\cdot\vec{f} = \frac{\partial_r(r^2 f_r)}{r^2} + \frac{\partial_\theta[\sin(\theta) f_\theta]}{r\sin(\theta)} + \frac{\partial_\alpha f_\alpha}{r\sin(\theta)}$$

alternative Lösung siehe Skript

Hier: Kugelsymmetrie $\Rightarrow$ Änderung nur in $\vec{e_r}$

$$\varepsilon_0 \frac{\partial_r(r^2 E_r)}{r^2} = \rho_0 \left(1-\left(\frac{r}{a}\right)^2\right)$$

$$\partial_r(r^2 E_r) = \frac{\rho_0}{\varepsilon_0} r^2\left(1-\frac{r^2}{a^2}\right)$$

$$r^2 E_r = \frac{\rho_0}{\varepsilon_0}\int r^2\left(1-\frac{r^2}{a^2}\right) dr = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right)$$

$$E_r = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5a^2}\right) \quad \dots \quad r<a$$

$$E_r = \frac{c}{r^2} \quad \dots \quad r>a$$

Max.: $E_r' = 0$

$$\frac{1}{3} - \frac{3r^2}{5a^2} = 0$$

$$\frac{3r^2}{5a^2} = \frac{1}{3}$$

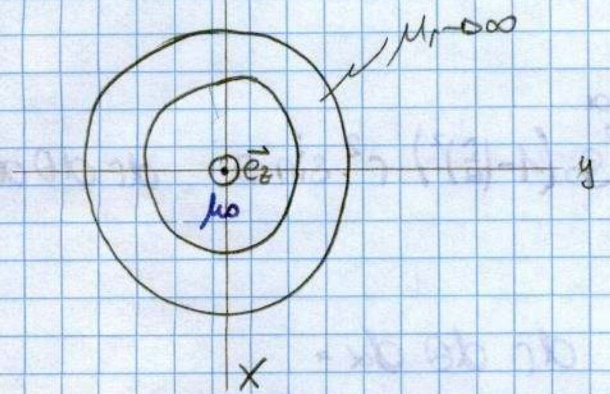
$$r^2 = \frac{5a^2}{9}$$

$$\underline{r = \frac{a}{3}\sqrt{5}}$$

$$E_r\Big|_{r=\frac{a}{3}\sqrt{5}} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a\sqrt{5}}{9} - \frac{a^3\sqrt{5}^3}{27\cdot 5a^2}\right) =$$

$$= \frac{\rho_0}{\varepsilon_0}\left(\frac{a\sqrt{5}}{9} - \frac{a\sqrt{5}}{9}\cdot\frac{1}{3}\right) = \frac{\rho_0}{\varepsilon_0}\frac{2a\sqrt{5}}{27}$$

A.2.2.3



$\vec{e}_r = \vec{e}_\varsigma$

$r = \varsigma$

hoch permeabel (id. magnetisierbar) $\Longrightarrow$ $\vec{H} = \vec{0}$

$\vec{n} \times [\![\vec{H}]\!] = \vec{K}$

$r < \frac{d}{2}$: $\quad \vec{B} = \mu_0 \vec{H}$

$\frac{1}{\mu_0} \vec{n} \times [\![\vec{B}]\!] = \vec{K}$

$r > \frac{d}{2}$: $\quad \vec{H} = \vec{0}$

$[\![\vec{H}]\!] = 0 - \frac{\vec{B}}{\mu_0}$ $\qquad \vec{n} = -\vec{e}_r$

$\vec{B} = -B \cdot \vec{e}_x \qquad B > 0 \qquad [\![\vec{H}]\!] = \frac{B}{\mu_0} \vec{e}_x$

$\vec{e}_x = \cos(\alpha)\, \vec{e}_\varsigma - \sin(\alpha)\, \vec{e}_\alpha$

$\vec{n} \times [\![\vec{H}]\!] = -\frac{B}{\mu_0} \vec{e}_r \times (\cos(\alpha)\vec{e}_\varsigma - \sin(\alpha)\vec{e}_\alpha) =$

$= -\frac{B}{\mu_0} \cos(\alpha) \cancel{\vec{e}_r \times \vec{e}_\varsigma} + \frac{B}{\mu_0} \sin(\alpha)\, \vec{e}_r \times \vec{e}_\alpha =$

$= \frac{B}{\mu_0} \sin(\alpha)\, \vec{e}_z = \vec{K}$ $\square$

A 2.2.4. $\{x, y, z\}$

(i) $\vec{\nabla} \times \vec{E} = (\vec{e_x} \partial_x + \vec{e_y} \partial_y + \vec{e_z} \partial_z) \times \left(E_0 \sin\left(\frac{\pi y}{a}\right) \cos\left(\frac{\pi c t}{a}\right) \right) \vec{e_x} =$

$= -\partial_y \left(E_0 \sin\left(\frac{\pi y}{a}\right) \cos\left(\frac{\pi c t}{a}\right) \right) \vec{e_z} + \cancel{\partial_z \left(E_0 \sin\left(\frac{\pi y}{a}\right) \cos\left(\frac{\pi c t}{a}\right) \right)} \vec{e_y} =$

$= E_0 \cos\left(\frac{\pi c t}{a}\right) \left[-\frac{\pi}{a} \cos\left(\frac{\pi y}{a}\right) \vec{e_z} \right] =$

$= -\frac{E_0 \pi}{a} \cos\left(\frac{\pi c t}{a}\right) \cos\left(\frac{\pi y}{a}\right) \vec{e_z}$

$\vec{\nabla} \cdot \vec{E} = 0$

$\vec{E} = -\vec{\nabla} \varphi$ geht nicht, nur in Elektrostatik.

(ii) $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$

$+ \frac{E_0 \pi}{a} \cos\left(\frac{\pi c t}{a}\right) \cos\left(\frac{\pi y}{a}\right) = + \partial_t B$

$B = \frac{E_0 \pi}{a} \cos\left(\frac{\pi y}{a}\right) \int \cos\left(\frac{\pi c t}{a}\right) dt =$

$B = \frac{E_0}{c} \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi c t}{a}\right) + C$

$\vec{B} = B \vec{e_z} + \vec{B_0}(\vec{r})$

(iii) $\vec{n} \cdot [\![\vec{D}]\!] = \sigma$ $\qquad$ $\vec{n} \cdot [\![\vec{J}]\!] = -\partial_t \sigma$

$y=0$: $\vec{n} = \vec{e_y}$ $\qquad$ $[\![\vec{E}]\!] = -\vec{E}$

$y=a$: $\vec{n} = -\vec{e_y}$ $\qquad$ $\varepsilon_0 \, \vec{n} \cdot [\![\vec{E}]\!] = \sigma$

$\vec{e_y} \cdot \vec{e_x} = 0 \Rightarrow \sigma = 0$

$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \vec{D}$ $\qquad$ $\vec{D} = \varepsilon_0 \vec{E}$

$\vec{B} = \mu_0 \vec{H} \Rightarrow \vec{H} = \vec{B} \frac{1}{\mu_0}$

$\vec{D} = \varepsilon_0 E_0 \sin\left(\frac{y\pi}{a}\right) \cos\left(\frac{\pi c t}{a}\right) \vec{e_x}$

$\partial_t \vec{D} = -\frac{\varepsilon_0 E_0 \pi c}{a} \sin\left(\frac{y\pi}{a}\right) \sin\left(\frac{\pi c t}{a}\right) \vec{e_x}$

$$\vec{B} = \frac{E_0}{c_0} \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi c_0 t}{a}\right) \vec{e_z}$$

$$\vec{H} = \frac{E_0}{\mu_0 c_0} \cos(\dots) \sin(\dots) \vec{e_z}$$

$$\vec{\nabla} \times \vec{H} = (\vec{e_x}\partial_x + \vec{e_y}\partial_y + \cancel{\vec{e_z}\partial_z}) \times \left(\frac{E_0}{\mu_0 c_0}\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\vec{e_z}\right) =$$

$$= \partial_y\left(\frac{E_0}{\mu_0 c_0} \cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\right) \vec{e_y} \times \vec{e_z} =$$

$$= -\frac{E_0}{\mu_0 c_0}\frac{\pi}{a} \sin\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\vec{e_x}$$

$$\vec{J} = \vec{\nabla}\times\vec{H} - \partial_t \vec{D} = -\frac{E_0 \pi}{\mu_0 c_0 a}\sin\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\vec{e_x} +$$

$$+ \frac{\varepsilon_0 E_0 \pi c_0}{a}\sin\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\vec{e_x}$$

$\varepsilon_0 \mu_0 c_0^2 = 1$

$$\vec{J} = \left(\frac{\varepsilon_0 E_0 \pi c_0^2 \mu_0}{\mu_0 c_0 a} - \frac{E_0 \pi}{\mu_0 c_0 a}\right)\sin\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\vec{e_x} = \vec{0}$$

$$\vec{n} \times [\![\vec{H}]\!] = \vec{K} \qquad [\![\vec{H}]\!] = -\vec{H}$$

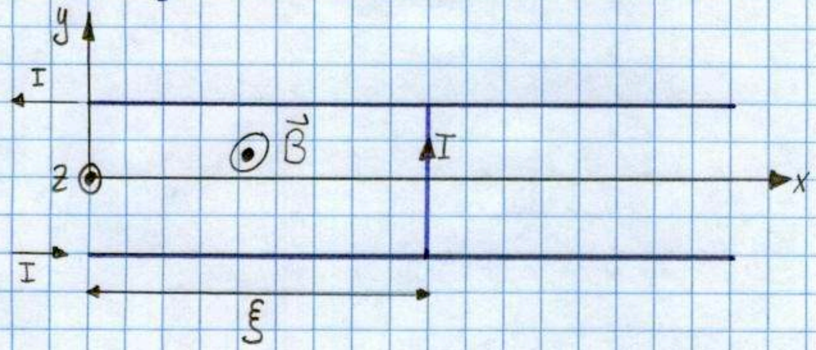
$y=0$: $$\underbrace{\vec{e_y}\times\vec{e_z}}_{\vec{e_x}}\left(-\frac{E_0}{\mu_0 c_0}\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\right)\Big|_{y=0} = -\frac{E_0}{\mu_0 c_0}\sin(\dots)\vec{e_x}$$

$y=a$: $$\frac{E_0}{\mu_0 c_0}\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi c_0 t}{a}\right)\vec{e_x}\Big|_{y=0} = -\frac{E_0}{\mu_0 c_0}\sin(\dots)\vec{e_x}$$

A. 2.3.2. Bewegter Kurzschlussbügel

dominant magnetisches Feldsystem

$\vec{v} = v\,\vec{e}_x$, $v = \text{const.}$



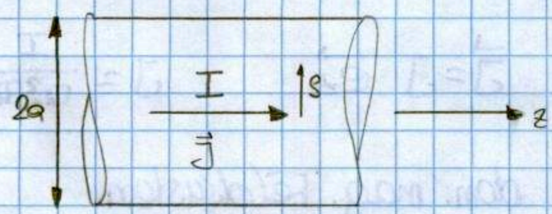
Durchflutungssatz: $\nabla \times \vec{H} = \vec{J} + \cancel{\partial_t \vec{D}}$ da dom. mag. $\hat{=}$ Ampere-Maxwell-Satz im dom. mag. Syst. ohne $\partial_t \vec{D}$

Induktionsgesetz: $\nabla \times \vec{E} = -\partial_t \vec{B}$

$U_0 + I \cdot R = U$

$\vec{B} = B\,\vec{e}_z$, $x < \xi$; da $\vec{e}_z$ rechtswendig zu I zugeordnet ist.

A. 2.3.3. Poynting-Fluß in einem Leiter



$$\vec{S} = \vec{E} \times \vec{H}$$

$\vec{J} = J\,\vec{e_z}$ $\qquad J = \frac{I}{A} = \frac{I}{a^2\pi}$ $\qquad$ gleichförmig $\Rightarrow I = \text{const.}$

$\vec{J} = \gamma\,\vec{E}$ $\qquad \vec{E} = \frac{I}{\gamma a^2\pi}\,\vec{e_z}$

$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$

$\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \vec{B} = \text{const.} \Rightarrow \vec{H} = \text{const.}$

$\vec{\nabla} \times \vec{H} = \vec{J}$

$J \cdot \varrho^2\pi$ ... Strom im Zylinder mit Radius ϱ

$H = \frac{J \cdot \varrho^2\pi}{2\pi\varrho}$ ... zugehörige Feldstärke

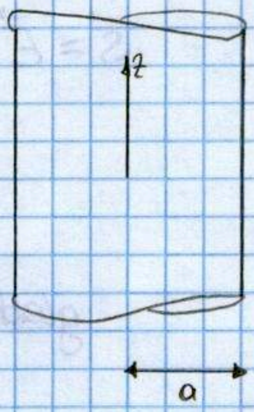
$\vec{H} = H\,\vec{e_\alpha}$ $\qquad H = \frac{\frac{I}{\pi a^2} \cdot \varrho^2\pi}{2\pi\varrho} = \frac{I}{2\pi a^2} \cdot \varrho$

Poynting-Vektor:

$$\vec{S} = \vec{E} \times \vec{H} = \left(\frac{I}{\gamma a^2\pi}\,\vec{e_z}\right) \times \left(\frac{I}{2\pi a^2}\,\varrho\,\vec{e_\alpha}\right) = -\frac{I^2}{2\pi^2 a^4 \gamma}\,\varrho\,\vec{e_\varrho}$$

Sein Fluß deckt genau die lokalen Joule-Verluste im Leiter.

A. 2.3.4. Elektromagnetischer Impulsfluß



$$\vec{J} = J\,\vec{e}_z \qquad J = \frac{I}{a^2\pi}$$

dom. mag. Feldsystem.

$$H = \frac{J\varrho^2\pi}{2\pi\varrho} \qquad \vec{B} = \mu_0 \cdot \frac{I}{2\pi a^2}\varrho\,\vec{e}_\alpha$$

... $\varrho < a$

Impulsflußdichte $\underset{\sim}{P}^e = \frac{1}{2\mu_0} B^2 \underset{\sim}{\delta} - \frac{1}{\mu_0}\vec{B}\otimes\vec{B}$

$$B^2 = \vec{B}\cdot\vec{B} = \left(\frac{\mu_0 I \varrho}{2\pi a^2}\right)^2$$

$$\vec{B}\otimes\vec{B} = B^2\,\vec{e}_\alpha\otimes\vec{e}_\alpha = \left(\frac{\mu_0 I\varrho}{2\pi a^2}\right)^2 \vec{e}_\alpha\otimes\vec{e}_\alpha$$

$$\underset{\sim}{P} = \frac{1}{2\mu_0}\left(\frac{\mu_0 I\varrho}{2\pi a^2}\right)^2 \underset{\sim}{\delta} - \frac{1}{\mu_0}\left(\frac{\mu_0 I\varrho}{2\pi a^2}\right)^2 \vec{e}_\alpha\otimes\vec{e}_\alpha$$

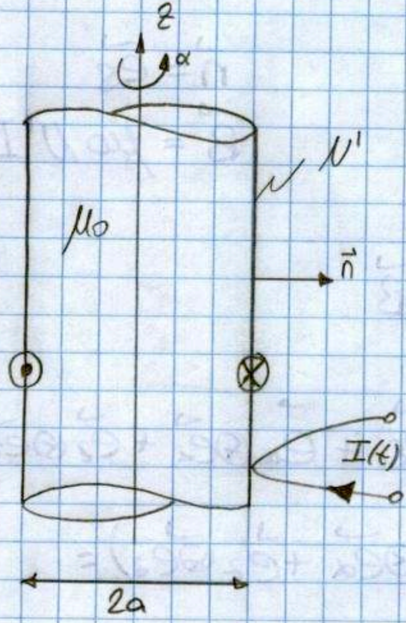
$\varrho > a$: $\qquad H = \frac{I}{l} = \frac{I}{2\pi\varrho} \qquad B = \mu_0\frac{I}{2\pi\varrho}$

analoge Rechnung.

Insgesamt mit $\underset{\sim}{\delta} = \vec{e}_\varrho\otimes\vec{e}_\varrho + \vec{e}_\alpha\otimes\vec{e}_\alpha + \vec{e}_z\otimes\vec{e}_z$ erhält man den Ausdruck:

$$\underset{\sim}{P}^e = \frac{\mu_0 I^2}{8\pi^2 a^2}\left(\vec{e}_\varrho\otimes\vec{e}_\varrho - \vec{e}_\alpha\otimes\vec{e}_\alpha + \vec{e}_z\otimes\vec{e}_z\right)\cdot\begin{cases}\left(\frac{\varrho}{a}\right)^2, & 0\le\varrho\le a\\ \left(\frac{a}{\varrho}\right)^2, & \varrho > a.\end{cases}$$

A. 2.3.5. Energiefluß in dünnwandiger Kreiszylinderspule



N'... längenbez. Windungszahl $\left[\frac{\text{Wdg}}{\text{m}}\right]$

dominant magnetisch

$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$

$\vec{\nabla}\times\vec{H} = \vec{J}$

$\vec{J} = J\,\vec{e_\alpha}$; $\vec{B} = B\,\vec{e_z}$

(i) Poynting-Vektor an der Innenwand der Spule.

$\vec{S} = \vec{E}\times\vec{H}$ $\qquad I = I(t)$!

$\vec{K} = I\cdot N'\,\vec{e_\alpha}$ $\qquad \vec{n} = \vec{e_S}$

$\vec{n}\times[\![\vec{H}]\!] = \vec{K}$

$\vec{e_S}\times[\![\vec{H}]\!] = I\cdot N'\,\vec{e_\alpha}$ $\qquad \{S,\alpha,z\}$

$\Rightarrow [\![\vec{H}]\!] = -I\cdot N'\,\vec{e_z}$

$\vec{H}^{(2)} - \vec{H}^{(1)} = [\![\vec{H}]\!]$ $\qquad H^{(2)} = 0$

$-\vec{H}^{(1)} = -I N'\vec{e_z}$

$\vec{H}^{(1)} = I N'\vec{e_z}$ $\qquad \vec{B} = \mu I N'\,\vec{e_z}$

$\Phi = B\cdot A = B\cdot a^2\pi$

$U = -\dot{\Phi} = -(\dot{B}\cdot a^2\pi) = -\mu_0 N' a^2\pi\,\dot{I}(t)$

$E = \frac{U}{2\pi a} = -\frac{1}{2}\mu_0 a N'\,\dot{I}(t)$ $\qquad \vec{E} = E\,\vec{e_\alpha}$

$\vec{S} = \vec{E}\times\vec{H} = E\cdot H\;\vec{e_\alpha}\times\vec{e_z} = -\frac{\mu_0}{2} a N'\dot{I} N' I\,\vec{e_S} = -\frac{\mu_0}{2} a N'^2 I\dot{I}\,\vec{e_S}$

A. 2. 3. 6. Kräfte in dünnwandiger Zylinderspule

$$P(\mathcal{A}) = \int_{\mathcal{A}} \vec{n} \cdot \underset{\sim}{\sigma} \, dA \qquad \vec{n} = \vec{e_s}$$

$$\vec{B} = \mu_0 N' I \, \vec{e_z}$$

$$\underset{\sim}{\sigma}^e = \frac{1}{2\mu_0} B^2 \underset{\sim}{\delta} - \frac{1}{\mu_0} \vec{B} \otimes \vec{B}$$

$$\underset{\sim}{\sigma}^e = \frac{1}{2\mu_0} (\mu_0 N' I)^2 (\vec{e_s} \otimes \vec{e_s} + \vec{e_\alpha} \otimes \vec{e_\alpha} + \vec{e_z} \otimes \vec{e_z}) - \frac{1}{\mu_0} (\mu_0 N' I)^2 \vec{e_z} \otimes \vec{e_z}$$

$$\vec{n} \cdot \underset{\sim}{\delta} = \vec{e_s} \cdot (\vec{e_s} \otimes \vec{e_s} + \vec{e_\alpha} \otimes \vec{e_\alpha} + \vec{e_z} \otimes \vec{e_z}) =$$

$$= \vec{e_s}$$

$$\vec{e_s} \cdot (\vec{e_z} \otimes \vec{e_z}) = \vec{0}$$

$$\vec{n} \cdot \underset{\sim}{\sigma}^e = \frac{\mu_0 N'^2 I^2}{2} \vec{e_s}$$

$$P(\mathcal{A}) = \int_{\mathcal{A}} \frac{\mu_0 N'^2 I^2}{2} \vec{e_s} \, dA$$

A 2.3.7. Kraft und Drehmoment an el. Punktdipol

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} \qquad \vec{p} \ldots \text{el. Moment}$$

$$\vec{T} = \vec{p} \times \vec{E}$$

Einheit:

$$[\vec{F}] = \frac{kg\,m}{s^2} \qquad [\vec{\nabla}] = \frac{1}{m} \qquad [\vec{E}] = \frac{V}{m} = \frac{kg\,m}{A s^3} \Rightarrow [\vec{p}] = \cancel{\frac{kg\,m}{s^2}} \cdot \frac{A s^3}{\cancel{kg\,m}} = A s\, m^{-1}$$

$$U \cdot I \cdot t = \frac{mv^2}{2} \Rightarrow [U] = \frac{kg\,m^2}{s^2} \cdot \frac{1}{As} = \frac{kg\,m^2}{A s^3}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\,\vec{e_r} \qquad \vec{e_r} = \frac{\vec{r}}{r}$$

$$= \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{r}}{r^3} = \underbrace{\frac{Q}{4\pi\varepsilon_0 r_0^2}}_{=: E_0} \cdot r_0^2 \frac{\vec{r}}{r^3} = E_0 r_0^2 \frac{\vec{r}}{r^3}$$

$$\vec{r} = x\vec{e_x} + y\vec{e_y} + z\vec{e_z}$$

$$\cancel{\vec{\nabla}\vec{E} = \vec{\nabla}\left(E_0 r_0^2 \frac{\vec{r}}{r^3}\right) = E_0 r_0^2\, \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) =}$$

$$\cancel{\vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) = (\partial_x \vec{e_x} + \partial_y \vec{e_y} + \partial_z \vec{e_z})\left(\frac{x\vec{e_x} + y\vec{e_y} + z\vec{e_z}}{\sqrt{x^2+y^2+z^2}^3}\right)}$$

$$\vec{p} \cdot \vec{\nabla} = (p_x \vec{e_x} + p_y \vec{e_y} + p_z \vec{e_z}) \cdot (\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z) =$$

$$p_x \partial_x + p_y \partial_y + p_z \partial_z$$

$$\vec{F} = \vec{p} \cdot \vec{\nabla}\vec{E} = (p_x \partial_x + p_y \partial_y + p_z \partial_z)(E$$

$$\vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) = \vec{\nabla}\left(\vec{r} \cdot \frac{1}{r^3}\right) = \frac{1}{r^3}\vec{\nabla}(\vec{r}) + \vec{r}\,\vec{\nabla}\left(\frac{1}{r^3}\right)$$

$$\vec{p} \cdot \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right)$$

$$\vec{F} = \vec{p} \cdot \vec{\nabla} \vec{E} = \alpha \cdot \vec{p} \cdot \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right)$$

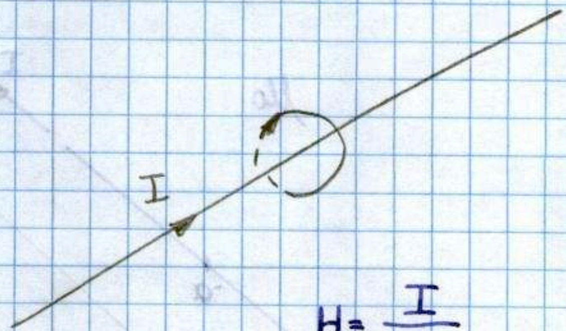
$$\frac{1}{r^3} \vec{\nabla}(\vec{r}) + \vec{r}\, \vec{\nabla}\left(\frac{1}{r^3}\right)$$

$$\frac{1}{r^3} \vec{p} \cdot \vec{\nabla} \vec{r} + \vec{p} \cdot \vec{r}\, \nabla\left(\frac{1}{r^3}\right)$$

A.2.3.8. Kraft und Drehmoment an mag. Punktdipol

$\vec{F} = (\vec{m} \times \vec{\nabla}) \times \vec{B}$

$\vec{T} = \vec{m} \times \vec{B}$



$H = \frac{I}{2\pi\varsigma}$

$\vec{B} = B\,\vec{e_\alpha} = B_0 \frac{r_0}{\varsigma}\vec{e_\alpha} \qquad B = \mu_0 \cdot H = \frac{\mu_0 I}{2\pi \varsigma}$

$B = \underbrace{\frac{\mu_0 I}{2\pi r_0}}_{B_0} \frac{r_0}{\varsigma}$

$\begin{pmatrix} \vec{e_\varsigma} \\ \vec{e_\alpha} \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \vec{e_x} \\ \vec{e_y} \end{pmatrix} \qquad \vec{e_\alpha} = -\sin(\alpha)\vec{e_x} + \cos(\alpha)\vec{e_y}$

$\vec{B} = B_0 \frac{r_0}{\varsigma}\left(-\sin(\alpha)\vec{e_x} + \cos(\alpha)\vec{e_y}\right) \qquad \{\varsigma, \alpha, z\}$

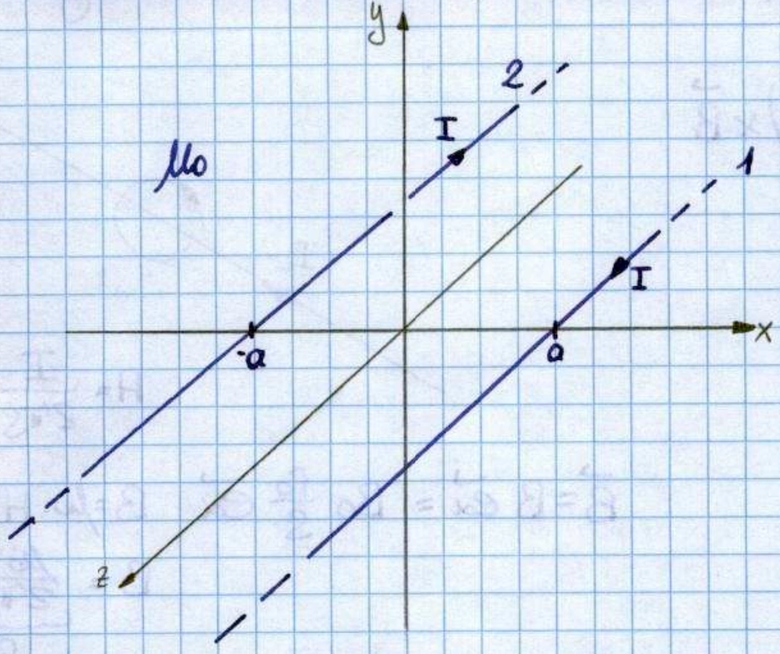
$(\Theta A\,\vec{n}) \times \left(\vec{e_\varsigma}\,\partial_\varsigma + \vec{e_\alpha}\frac{1}{\varsigma}\partial_\alpha + \vec{e_z}\,\partial_z\right) \qquad \vec{n} = \vec{e_\varsigma}$

$\Theta A\left\{\vec{e_\varsigma} \times \left(\vec{e_\varsigma}\,\partial_\varsigma + \vec{e_\alpha}\frac{1}{\varsigma}\partial_\alpha + \vec{e_z}\,\partial_z\right)\right\}$

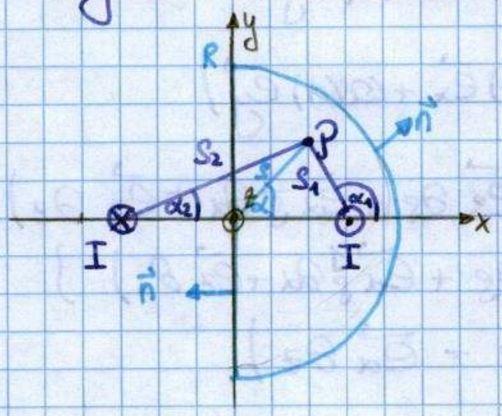
$\Theta A\left(\vec{e_z}\,\frac{1}{\varsigma}\partial_\alpha - \vec{e_\alpha}\,\partial_z\right)$

$\left(\vec{e_z}\,\frac{1}{\varsigma}\partial_\alpha - \vec{e_\alpha}\,\partial_z\right) \times \left(B_0 \frac{r_0}{\varsigma}\vec{e_\alpha}\right) = -\frac{r_0}{\varsigma^2}\vec{e_\varsigma}$

A.2.3.9. par. Linienleiter



Längenbezogene Kraft auf Leiter 1 durch Integration des Maxwell-Spannungen.



$$\vec{B}(P) = \frac{\mu_0 I}{2\pi S}\vec{e_\alpha} \quad \text{... pro Leiter.}$$

$$\vec{B}(P) = \frac{\mu_0 I}{2\pi}\left\{ \frac{1}{S_1}\Big[\overbrace{-\sin(\alpha_1)\vec{e_x} + \cos(\alpha_1)\vec{e_y}}^{\hat{=}\,+\vec{e_{\alpha_1}}}\Big] + \right.$$

$$\left. + \frac{1}{S_2}\Big[\overbrace{\sin(\alpha_2)\vec{e_x} - \cos(\alpha_2)\vec{e_y}}^{\hat{=}\,-\vec{e_{\alpha_2}}}\Big] \right\} =$$

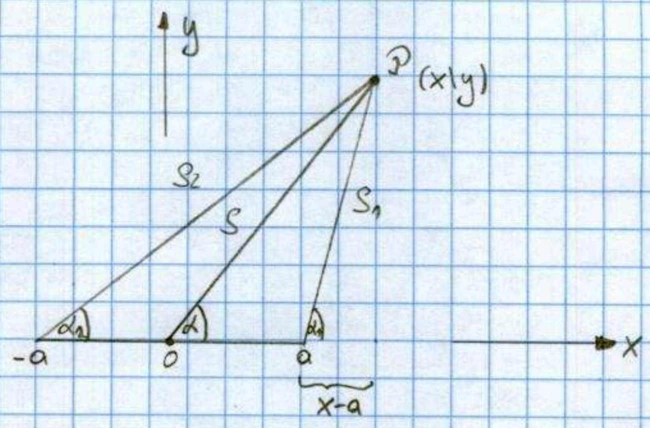
$$= \frac{\mu_0 I}{2\pi}\left\{ \Big[-\frac{1}{S_1}\sin(\alpha_1) + \frac{1}{S_2}\sin(\alpha_2)\Big]\vec{e_x} + \Big[\frac{1}{S_1}\cos(\alpha_1) - \frac{1}{S_2}\cos(\alpha_2)\Big]\vec{e_y} \right\}$$

Zylinder Koordinaten: $\quad x = S_i \cdot \cos(\alpha_i) \qquad y = S_i \cdot \sin(\alpha_i) \qquad i = 1,2$

$$\cos(\alpha_i) = \frac{x}{S_i} \qquad \sin(\alpha_i) = \frac{y}{S_i}$$

$$\vec{B}(P) = \frac{\mu_0 I}{2\pi}\left\{ \Big(-\frac{y}{S_1^2} + \frac{y}{S_2^2}\Big)\vec{e_x} + \Big(\frac{x}{S_1^2} - \frac{x}{S_2^2}\Big)\vec{e_y} \right\}$$

ad A. 2.3.9.



$$s_1^2 = (x-a)^2 + y^2$$
$$s_2^2 = (x+a)^2 + y^2$$

$$\sin(\alpha_1) = \frac{y}{s_1} \qquad \cos(\alpha_1) = \frac{x-a}{s_1} \qquad \cos(\alpha) = \frac{x}{s}$$

$$\sin(\alpha_2) = \frac{y}{s_2} \qquad \cos(\alpha_2) = \frac{x+a}{s_2}$$

$$\vec{B}(P) = \frac{\mu_0 I}{2\pi} \left\{ \left(-\frac{y}{s_1^2} + \frac{y}{s_2^2} \right) \vec{e_x} + \left(\frac{x-a}{s_1^2} - \frac{x+a}{s_2^2} \right) \vec{e_y} \right\} =$$

$$= \frac{\mu_0 I}{2\pi} \left\{ -\left(\frac{1}{s_1^2} - \frac{1}{s_2^2} \right) y \, \vec{e_x} + \left[\left(\frac{1}{s_1^2} - \frac{1}{s_2^2} \right) x - \left(\frac{1}{s_1^2} + \frac{1}{s_2^2} \right) a \right] \vec{e_y} \right\}$$

$x = 0$: $(s_1 = s_2)$

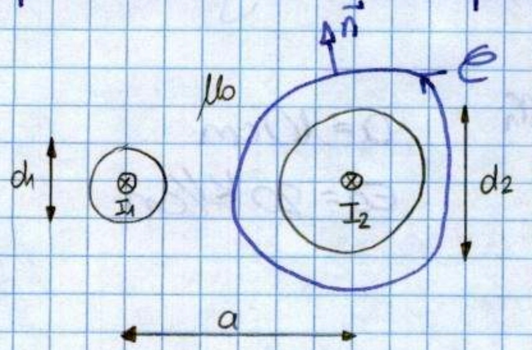
$$\vec{B}(P) = \frac{\mu_0 I}{2\pi} \left\{ - \underbrace{\left(\frac{1}{s_1^2} - \frac{1}{s_2^2} \right)}_{0} y \vec{e_x} - \left(\frac{1}{s_1^2} + \frac{1}{s_2^2} \right) a \vec{e_y} \right\} =$$

$$= -\frac{\mu_0 I}{2\pi} \left(\frac{1}{s_1^2} + \frac{1}{s_2^2} \right) a \, \vec{e_y} \qquad s_1 = s_2 = \sqrt{a^2 + y^2}$$

$$\vec{B}(P) = -\frac{\mu_0 I}{2\pi} \frac{2}{s_1^2} a \vec{e_y} = -\frac{\mu_0 I}{\pi} \frac{a}{a^2 + y^2} \vec{e_y}$$

$s^2 = x^2 + y^2 \gg a$:

A. 2.3.10 par. Leiter mit Kreisquerschnitt

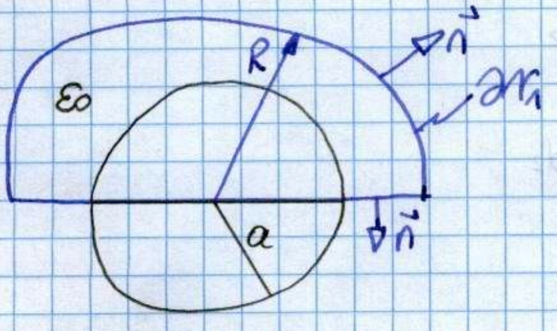


$$F' = \mu_0 \frac{I_1 I_2}{2\pi a}$$

$$F_R' = \int_C \frac{1}{\mu_0}\left(\vec{n}\cdot\vec{B}\vec{B} - \frac{1}{2}B^2\vec{n}\right) ds$$

über Kreisquerschnitt gleichförmige Stromverteilung
$\Rightarrow$ gleiche Feldverteilung im Außenraum wie bei Linienleiter

A. 2.3.11 Kraft zwischen zwei Halbkugeln



$a = 10\,cm$

$E_0 = 20\,kV/cm$

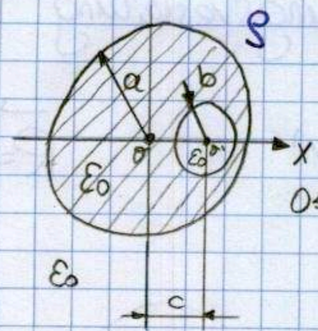
$$\vec{F} = \int_{\star} \varepsilon_0 \left(\vec{n} \cdot \vec{E}\vec{E} - \frac{1}{2} E^2 \vec{n} \right) dA$$

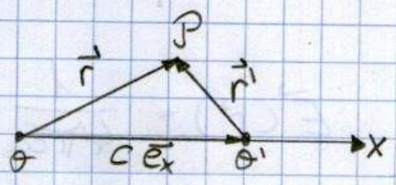
$$\vec{E} = E_0 \frac{a^2}{r^2} \vec{e}_r \qquad r > a$$

$$\vec{E} = \vec{0} \qquad r < a$$

$$\vec{F}_R = \int_{\partial V_1} \varepsilon_0 \left(\vec{n} \cdot \vec{E}\vec{E} - \frac{1}{2} E^2 \vec{n} \right) dA$$

A 3.1.1 Geladene Vollkugel mit Hohlraum





$\vec{r} - \vec{r}' = c\vec{e}_x$

$$\left(\varphi_P = \frac{1}{4\pi\varepsilon} \int_V \frac{\varrho(\vec{r}')}{|\vec{r}-\vec{r}'|} \, dV' \right)$$

$$Q = \varrho \cdot V \qquad V = \frac{4\pi r^3}{3}$$

$$\vec{E}_1 = \frac{Q}{4\pi\varepsilon_0 r^2} \vec{e}_r = \frac{\varrho \cdot \cancel{\frac{4\pi r^3}{3}}}{\cancel{4\pi\varepsilon_0 r^2}} \vec{e}_r = \frac{\varrho}{3\varepsilon_0} \underbrace{r\vec{e}_r}_{\vec{r}} \qquad r<a$$

$$\vec{E}_2 = -\frac{\varrho}{3\varepsilon_0} \vec{r}' \qquad r'<b$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\varrho}{3\varepsilon_0} \vec{r} - \frac{\varrho}{3\varepsilon_0} \vec{r}' = \frac{\varrho}{3\varepsilon_0} (\vec{r} - \vec{r}') = \underline{\underline{\frac{\varrho}{3\varepsilon_0} c\vec{e}_x}}$$

A.3.1.2. Integralformeln für die elektrische Feldstärke

$\varrho(\vec{r})$ ... quasielektrostatische Ladungsverteilung

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\vec{R}}{R^3} \varrho(\vec{r}\,') \, dV' \qquad \vec{R} = \vec{r} - \vec{r}\,'$$

äquivalente Form:

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\varepsilon_0} \int_V \frac{\vec{\nabla}'\varrho(\vec{r}\,')}{R} \, dV' \qquad \text{gilt es zu zeigen.}$$

Es gilt: $\vec{R} = R\vec{\nabla}R = -R\vec{\nabla}'R$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{R\vec{\nabla}R}{R^{3}} \varrho(\vec{r}\,') \, dV' = -\frac{1}{4\pi\varepsilon_0} \int_V \frac{\vec{\nabla}'R}{R^2} \varrho(\vec{r}\,') \, dV'$$

$$\frac{\vec{R}}{R^3}\varrho' = -\frac{\varrho'}{R^2}\vec{\nabla}'R = \varrho'\vec{\nabla}'\left(\frac{1}{R}\right) = \vec{\nabla}'\left(\frac{\varrho'}{R}\right) - \frac{\vec{\nabla}'(\varrho')}{R}$$

$$\vec{\nabla}'\left(\frac{1}{R}\right) = \vec{\nabla}'(R^{-1}) = (-1)R^{-2}\vec{\nabla}'(R) = -\frac{1}{R^2}\vec{\nabla}'(R)$$

$$\vec{\nabla}'\left(\frac{\varrho'}{R}\right) = \vec{\nabla}'\left(\frac{1}{R}\varrho'\right) = \vec{\nabla}'\left(\frac{1}{R}\right)\varrho' + \frac{1}{R}\vec{\nabla}'(\varrho')$$

$$\varrho'\vec{\nabla}'\left(\frac{1}{R}\right) = \vec{\nabla}'\left(\frac{\varrho'}{R}\right) - \frac{\vec{\nabla}'(\varrho')}{R}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \left(\vec{\nabla}'\left(\frac{\varrho'}{R}\right) - \frac{\vec{\nabla}'(\varrho')}{R}\right) dV' =$$

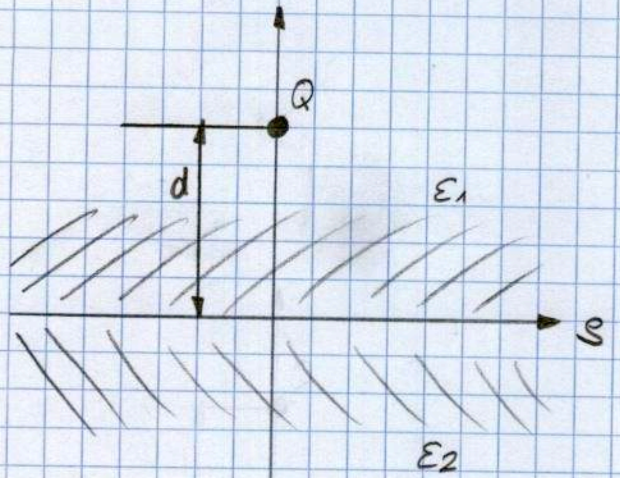
$$= \frac{1}{4\pi\varepsilon_0} \int_V \vec{\nabla}'\left(\frac{\varrho'}{R}\right) dV' - \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{\nabla}'(\varrho')}{R} \, dV'$$

Green-Transformation: $\int_V \vec{\nabla} \otimes \underline{F} \, dV = \int_{\partial V} \vec{n} \otimes \underline{F} \, dA$

$$\int_V \vec{\nabla}'\left(\frac{\varrho'}{R}\right) dV' = \int_{\partial V} \vec{n}\,\frac{\varrho'}{R} \, dV'$$

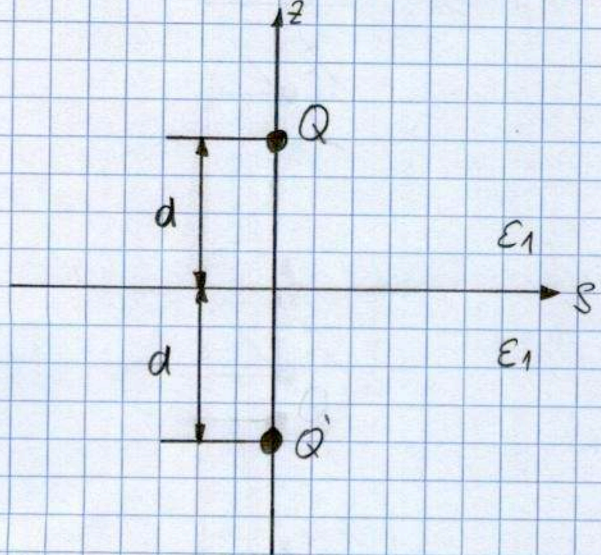
$$\Rightarrow \varrho = 0 \text{ auf } \partial V$$

A.3.1.3 Erweiterung der elektrostatischen Spiegelungsmethode



$$\vec{n} \times [\![\vec{E}]\!] = \vec{0}$$
$$\vec{n} \cdot [\![\vec{D}]\!] = \sigma \stackrel{!}{=} 0$$

(i)

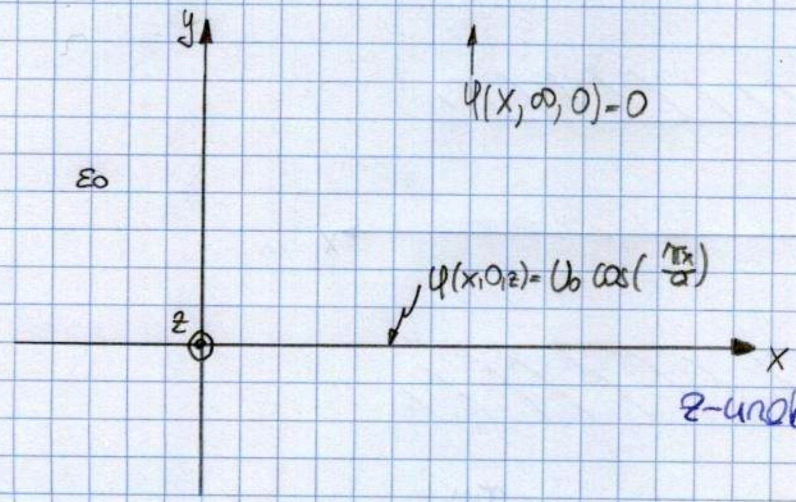


$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^3} (\vec{r} - d\vec{e}_z)$$
$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0} \frac{Q'}{r^3} (\vec{r} + d\vec{e}_z)$$

$$\vec{E}(P) = \vec{E}_1 + \vec{E}_2$$

(lt. Skript)

A.3.2.1. Randfeldstärke



z-unabhängig

$\vec{E} = -\vec{\nabla}\varphi$

$\Delta\varphi = 0$

φ auf ∂A

$A = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

$\partial A = \{y = 0\} \cup \{y = \text{"}\infty\text{"}\}$

$$\varphi(x,y) = \varphi = C\, e^{-kx} \sin(ky)$$

Lösung:

$$\varphi(x,y) = C \cdot e^{-ky} \cos(kx)$$

$$\varphi(x,0) = C \cdot e^{0} \cdot \cos(kx) = C \cos(kx) \overset{!}{=} U_0 \cdot \cos\left(\frac{\pi x}{a}\right)$$

$$\Rightarrow \quad C = U_0$$

$$k = \frac{\pi}{a} + 2\pi n, \; n \in \mathbb{Z}$$

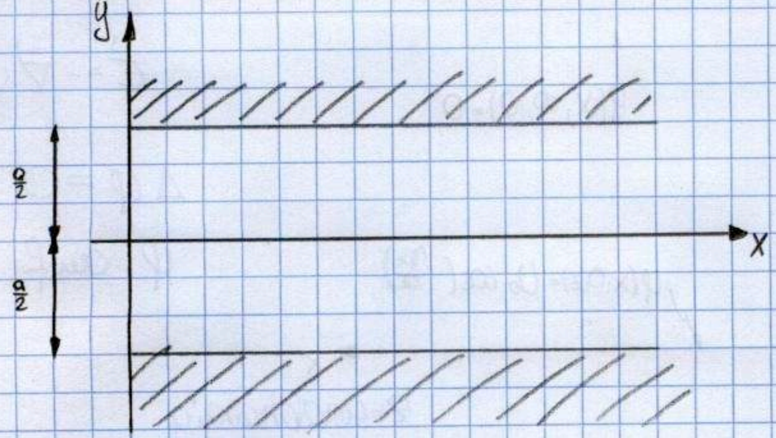
$$\varphi(x,y) = U_0\, e^{-\left(\frac{\pi}{a} + 2\pi n\right) y} \cos\left(\left(\frac{\pi}{a} + 2\pi n\right) x\right)$$

$$\varphi(x,y) = U_0\, e^{-\frac{\pi y}{a}} \cos\left(\frac{\pi x}{a}\right)$$

$$\vec{E} = -\vec{\nabla}\varphi = -\left(-U_0 e^{-\frac{\pi y}{a}} \frac{\pi}{a} \sin\left(\frac{\pi}{a} x\right) \vec{e_x} + U_0 \left(-\frac{\pi}{a}\right) e^{-\frac{\pi y}{a}} \cos\left(\frac{\pi x}{a}\right) \vec{e_y}\right)$$

$$= \frac{\pi}{a} U_0 e^{-\frac{\pi y}{a}} \left(\sin\left(\frac{\pi}{a} x\right) \vec{e_x} + \cos\left(\frac{\pi x}{a}\right) \vec{e_y} \right)$$

A.3.2.2. Ebenes Dirichlet-Problem für einen Spalt



$x=0$: $\quad \varphi = U_0 \cos\left(\frac{\pi y}{a}\right)$

$y=\pm\frac{a}{2}$: $\quad \varphi = 0$

$$\varphi(x,y) = C\, e^{-kx} \cos(ky)$$

$$\varphi\left(x, \pm\frac{a}{2}\right) = 0 = C\, e^{-kx} \cos\left(\pm k \frac{a}{2}\right) \stackrel{!}{=} 0$$

$$k\frac{a}{2} = \frac{\pi}{2} + n\pi$$

$$C = U_0 \qquad \underline{\underline{k = \frac{\pi}{a}}} + \frac{2\pi n}{a} \qquad n \in \mathbb{Z}$$

$$\varphi = (x,y) = U_0 e^{-\frac{\pi x}{a}} \cos\left(\frac{\pi y}{a}\right)$$

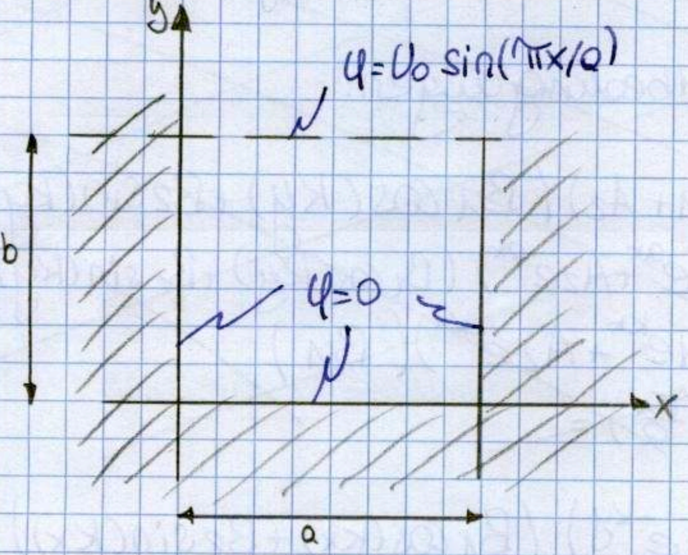
$$\vec{E}(x,y) = -\vec{\nabla}\varphi = -\left(U_0 \left(-\frac{\pi}{a}\right) e^{-\frac{\pi x}{a}} \cdot \cos\left(\frac{\pi y}{a}\right)\vec{e_x} + \right.$$

$$\left. + U_0\, e^{-\frac{\pi x}{a}} \left(-\frac{\pi}{a}\right) \sin\left(\frac{\pi y}{a}\right) \vec{e_y} \right) =$$

$$= U_0 \frac{\pi}{a} e^{-\frac{\pi x}{a}} \left(\cos\left(\frac{\pi y}{a}\right)\vec{e_x} + \sin\left(\frac{\pi y}{a}\right)\vec{e_y} \right)$$

Feldbild $\rightarrow$ siehe Lösung.

A.3.2.3 Ebenes Dirichlet-Problem für eine Nut



$$\Delta\varphi = 0 \qquad \varphi(x,b) = U_0 \sin\left(\frac{\pi x}{a}\right)$$
$$\varphi(x,0) = 0 = \varphi(0,y) = \varphi(a,y)$$

$$\varphi(x,y) = C e^{-ky} \sin(kx)$$

$$\varphi(0,y) = C e^{-ky} \sin(0) = 0$$

$$\varphi(a,y) = C e^{-ky} \sin(\pi) = 0$$

$$\varphi(x,0) = C e^{0} \cdot \sin(kx) = 0 \quad \lightning$$

$$\varphi(x,y) = X(x)\, Y(y)$$

$$\varphi_{xx} = X_{xx} \cdot Y \qquad \varphi_{yy} = X\, Y_{yy}$$

$$\varphi_{xx} + \varphi_{yy} = \Delta\varphi = 0$$

$$Y X_{xx} + X Y_{yy} = 0$$

$$\frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = 0 = k^2$$

$$\Rightarrow \quad \frac{X_{xx}}{X} = k^2 \qquad \frac{Y_{yy}}{Y} = -k^2$$

$$X_{xx} - k^2 X = 0 \qquad Y_{yy} + k^2 Y = 0$$

$$X(x) = \ \ldots$$

$$\cancel{\varphi(x,y) = (A_1 e^{kx} + A_2 e^{-kx})(B_1 \cos(ky) + B_2 \sin(ky))}$$

~~anpassen an Randbedingungen:~~

$$\cancel{\varphi(0,y) = 0 = (A_1 + A_2)(B_1 \cos(ky) + B_2 \sin(ky))}$$
$$\cancel{\varphi(a,y) = 0 = (A_1 e^{ax} + A_2 e^{-ax})(B_1 \cos(ky) + B_2 \sin(ky))}$$
$$\cancel{\varphi(x,0) = 0 = (A_1 e^{kx} + A_2 e^{-kx})(B_1)}$$
$$\cancel{\varphi(x,b) = U_0 \sin\left(\frac{\pi x}{a}\right) =}$$

$$\varphi(x,y) = (A_1 e^{ky} + A_2 e^{-ky})(B_1 \cos(kx) + B_2 \sin(kx))$$

anpassen an Randbedingungen:

$$\text{I}: \varphi(0,y) = 0 = (A_1 e^{ky} + A_2 e^{-ky})(B_1)$$
$$\text{II}: \varphi(a,y) = 0 = (A_1 e^{ky} + A_2 e^{-ky})(\cancel{B_1 \cos(ka) +} B_2 \sin(ka))$$
$$\text{III}: \varphi(x,0) = 0 = (A_1 + A_2)(B_1 \cos(kx) + B_2 \sin(kx))$$
$$\text{IV}: \varphi(x,b) = U_0 \sin\left(\frac{\pi x}{a}\right) = (A_1 e^{kb} + A_2 e^{-kb})(B_1 \cos(kx) + B_2 \sin(kx))$$

$B_1 = 0$: I ✓ IV: $A_1 e^{kb} + A_2 e^{-kb} = 1$, $B_2 = U_0$, $k = \frac{\pi}{a}$

III: $A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$$\Rightarrow A_1 (\underbrace{e^{kb} - e^{-kb}}_{2\sinh(kb)}) = 1$$

$$\Rightarrow A_1 = \frac{1}{2\sinh(kb)} \qquad B_1 = 0$$

$$k = \frac{\pi}{a} \qquad A_2 = -\frac{1}{2\sinh(kb)} \qquad B_2 = U_0$$

$$\varphi(x,y) = \frac{1}{2\sinh(kb)} (e^{ky} - e^{-ky})\, U_0 \sin(kx) =$$

$$= U_0 \frac{\sinh(ky)}{\sinh(kb)} \sin(kx)$$

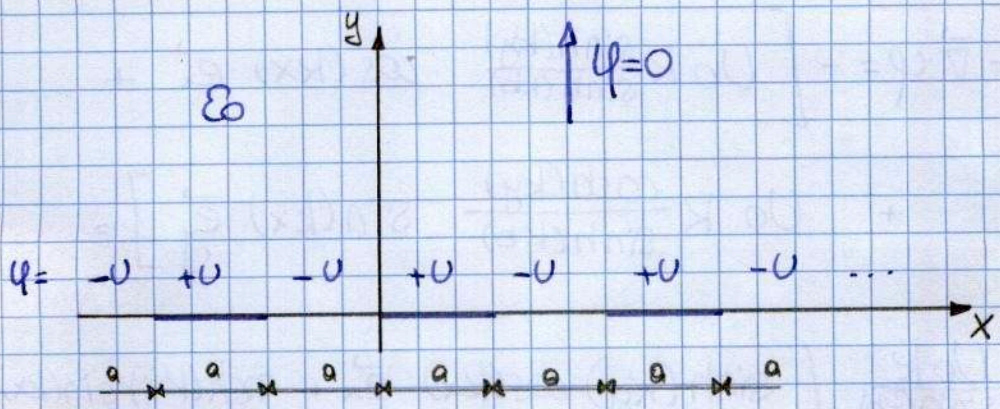
ad 3.2.3

$$\vec{E} = -\vec{\nabla}\varphi = -\left[U_0 k \frac{\sinh(ky)}{\sinh(kb)} \cos(kx)\,\vec{e}_x + \right.$$

$$\left. + \; U_0 k \frac{\cosh(ky)}{\sinh(kb)} \sin(kx)\,\vec{e}_y \right] =$$

$$= -\frac{U_0 k}{\sinh(kb)} \left(\sinh(ky)\cos(kx)\,\vec{e}_x + \cosh(ky)\sin(kx)\,\vec{e}_y \right)$$

A. 3.2.4 Halbraum mit periodischem Randpotential

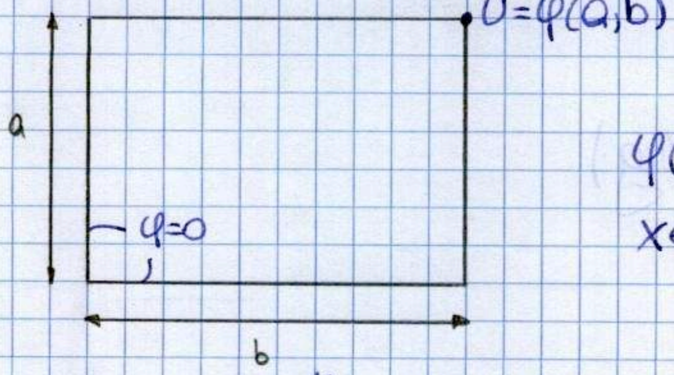


$$\varphi(x,0) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4U}{n\pi} \sin\left(\frac{n\pi x}{a}\right)$$

(i) $$\varphi(x,y) = e^{-\frac{n\pi}{a}y} \cdot \varphi(x,0)$$

(ii) (lt. Lösung)

A. 3.2.5. Strom durchflossene Platte



$$\varphi(x,0)=0=\varphi(0,y)$$
$$x\in(0,b) \quad y\in(0,a)$$

$$\varphi(b,y)=\frac{U}{a}y$$

$$\varphi(x,a)=\frac{U}{b}x$$

$$\vec{J}=\gamma\vec{E}$$

Lösung erraten: $\varphi(x,y)=U\frac{xy}{ab}$

$$\vec{E}=-\vec{\nabla}\varphi$$

A. 3. 2. 7. Additive Seperationslösung

$$\varphi(x,y,z) = F_1(x) + F_2(y) + F_3(z)$$

$$\Delta\varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$$

$$\varphi_{xx} = F_{1,xx} \qquad \varphi_{yy} = F_{2,yy} \qquad \varphi_{zz} = F_{3,zz}$$

$$\Delta\varphi = F_{1,xx} + F_{2,yy} + F_{3,zz} = 0$$

$$\Rightarrow \quad F_{1,xx} = -\left(F_{2,yy} + F_{3,zz}\right) = A_1 = \text{const.}$$

$$F_{2,yy} = -\left(A_1 + F_{3,zz}\right) = A_2 = \text{const.}$$

$$F_{3,zz} = -\left(A_1 + A_2\right) = A_3$$

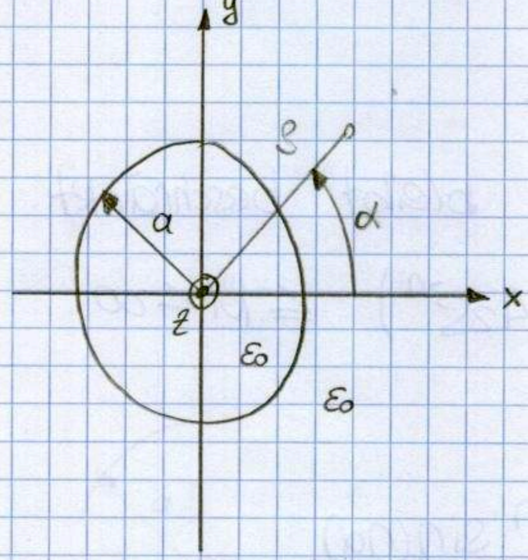
Mögliche Lösungen:

$$F_1 = A_1 x^2 + B_1 x + C_1$$
$$F_2 = A_2 y^2 + B_2 y + C_2$$
$$F_3 = A_3 z^2 + B_3 z + C_3$$

einsetzen in φ $\Rightarrow$ allgem. Lsg.

A. 3.2.9. Ein inneres und ein äußeres Dirichlet-Problem

$\varphi(a,\alpha) = \varphi_0 \sin(n\alpha) \quad n \in \mathbb{N}$



$$\varphi(\varrho,\alpha) = (A_1 \varrho^{k} + A_2 \varrho^{-k})(B_1 \cos(k\alpha) + B_2 \sin(k\alpha))$$

$$\vec{\nabla}^2 \varphi = \frac{1}{\varrho} \partial_\varrho (\varrho \partial_\varrho \varphi) + \frac{1}{\varrho^2} \partial_\alpha^2 \varphi = 0$$

Ansatz: $\varphi(\varrho,\alpha) = R(\varrho) \sin(n\alpha)$

$\partial_\varrho \varphi = \partial_\varrho R \sin(n\alpha)$

$\partial_\alpha \varphi = R(\varrho)\, n \cos(n\alpha) \qquad \partial_\alpha^2 \varphi = -R(\varrho)\, n^2 \sin(n\alpha)$

$$\frac{1}{\varrho} \partial_\varrho (\varrho \cdot \partial_\varrho R(\varrho) \sin(n\alpha)) + \frac{1}{\varrho^2} (-R(\varrho) n^2 \sin(n\alpha)) = 0$$

$$\frac{\sin(n\alpha)}{\varrho} \partial_\varrho (\varrho \cdot \partial_\varrho R(\varrho)) - \frac{n^2}{\varrho^2} R(\varrho) \sin(n\alpha) = 0$$

$$\frac{\cancel{\sin(n\alpha)}}{\cancel{\varrho}} (\partial_\varrho R(\varrho) + \varrho\, \partial_\varrho^2 R(\varrho)) - \frac{n^2}{\varrho^{\cancel{2}}} R(\varrho) \cancel{\sin(n\alpha)} = 0$$

$$\varrho\, \partial_\varrho R(\varrho) + \varrho^2 \partial_\varrho^2 R(\varrho) - n^2 R(\varrho) = 0$$

$R(\varrho) = A \varrho^{k} \qquad R'(\varrho) = A k \varrho^{k-1} \qquad R''(\varrho) = A k(k-1) \varrho^{k-2}$

$$\varrho \cancel{A} k \varrho^{k-1} + \varrho^2 \cancel{A} k(k-1) \varrho^{k-2} - n^2 \cancel{A} \varrho^{k} = 0$$

$$\varrho^{k} k + \varrho^{k} k(k-1) - n^2 \varrho^{k} = 0 \qquad \varrho^{k} = 0 \text{ ↯}$$

$$k + k^2 - k - n^2 = 0 \qquad k^2 = n^2 \qquad k = \pm n$$

$$R(\rho) = A_1 \rho^n + A_2 \rho^{-n}$$

$$\varphi(\rho,\alpha) = (A_1 \rho^n + A_2 \rho^{-n}) \sin(n\alpha)$$

(i) Innenraum $0 \leq \rho < a$

Bedingung: Potential bleibt beschränkt

$$\lim_{\rho \to 0} (A_1 \rho^n + A_2 \rho^{-n}) \leq M < \infty$$

$$\Rightarrow \quad A_2 = 0$$

$$\varphi(\rho,\alpha) = A_1 \rho^n \sin(n\alpha)$$

$$\varphi(a,\alpha) = \varphi_0 \sin(n\alpha) = A_1 a^n \sin(n\alpha)$$

$$A_1 a^n = \varphi_0$$
$$A_1 = \varphi_0 a^{-n}$$

$$\varphi(\rho,\alpha) = \varphi_0 a^{-n} \rho^n \sin(n\alpha)$$

$$\vec{E} = -\vec{\nabla}\varphi = \vec{e}_\rho \partial_\rho f + \vec{e}_\alpha \frac{1}{\rho} \partial_\alpha f = -\Big(\vec{e}_\rho \varphi_0 a^{-n} n \rho^{n-1} \sin(n\alpha) +$$
$$+ \vec{e}_\alpha \frac{1}{\rho} \varphi_0 a^{-n} \rho^n n \cos(n\alpha)\Big) = -\varphi_0 a^{-n} n \rho^{n-1} \sin(n\alpha) \vec{e}_\rho -$$
$$- \varphi_0 a^{-n} \rho^{n-1} n \cos(n\alpha) \vec{e}_\alpha$$

(ii) Außenraum $\rho > a$

$$\lim_{\rho \to \infty} (A_1 \rho^n + A_2 \rho^{-n}) \leq M < \infty$$

$$\Rightarrow \quad A_1 = 0$$

$$\varphi(\rho,\alpha) = A_2 \rho^{-n} \sin(n\alpha)$$

$$\varphi(a,\alpha) = A_2 a^{-n} \sin(n\alpha) = \varphi_0 \sin(n\alpha) \Rightarrow A_2 = a^n \varphi_0$$

$$\varphi(\rho,\alpha) = \varphi_0 a^n \rho^{-n} \sin(n\alpha)$$

$$\vec{E} = -\vec{\nabla}\varphi = -\Big(\varphi_0 a^n (-n) \rho^{-n-1} \sin(n\alpha) \vec{e}_\rho + \frac{1}{\rho} \varphi_0 a^n \rho^{-n} n \cos(n\alpha) \vec{e}_\alpha\Big)$$

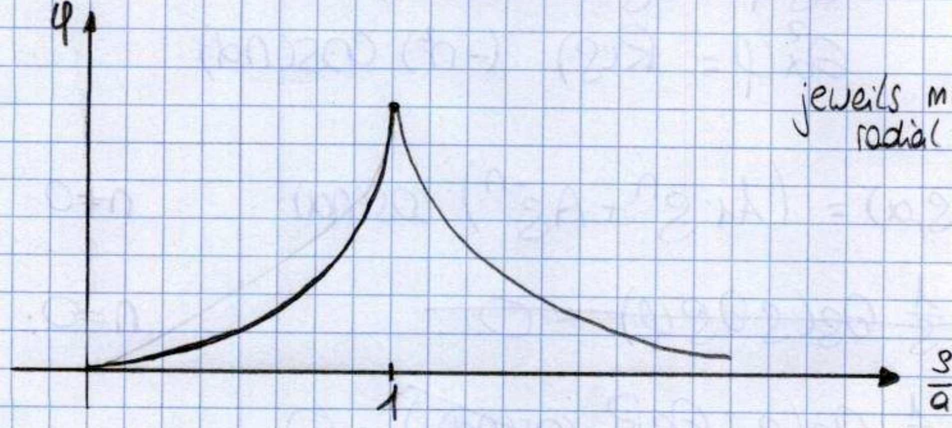
$$\vec{E} = \varphi_0 \, a^n \, \varrho^{-n-1} n \sin(n\alpha) \, \vec{e_\varrho} + \varphi_0 a^n \varrho^{-n-1} n \cos(n\alpha) \, \vec{e_\alpha}$$

$n=3$: Innen: $\varphi(\varrho, \alpha) = \varphi_0 \, a^{-3} \varrho^3 \sin(n\alpha) = \varphi_0 \left(\frac{\varrho}{a}\right)^3 \sin(3\alpha)$

$$\vec{E} = -\varphi_0 \frac{3}{\varrho} \left(\frac{a}{\varrho}\right)^3 \sin(3\alpha) \vec{e_\varrho} - \varphi_0 \frac{3}{\varrho} \left(\frac{a}{\varrho}\right)^3 \cos(3\alpha) \vec{e_\alpha}$$

Außen: $\varphi(\varrho, \alpha) = \varphi_0 \, a^3 \varrho^{-3} \sin(3\alpha) = \varphi_0 \left(\frac{a}{\varrho}\right)^3 \sin(3\alpha)$

$$\vec{E} = \varphi_0 \frac{3}{\varrho} \left(\frac{a}{\varrho}\right)^3 \sin(3\alpha) \vec{e_\varrho} + \varphi_0 \frac{3}{\varrho} \left(\frac{a}{\varrho}\right)^3 \cos(3\alpha) \vec{e_\alpha}$$



jeweils mit $\sin(3\alpha)$ radial gewichtet

A 3.2.10. Dickwandiger Kreiszylinder

$$\varphi(a,\alpha) = U_0 \cos(n\alpha) \qquad n \in \mathbb{N}_0$$

Ansatz: $R(\rho)\cos(n\alpha) = \varphi(\rho,\alpha)$

$$\vec{\nabla}^2 \varphi = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho\varphi) + \frac{1}{\rho^2}\partial_\alpha^2 \varphi = 0$$

$$\partial_\rho \varphi = \partial_\rho R \cos(n\alpha)$$
$$\partial_\alpha^2 \varphi = R(\rho)\,(-n^2)\cos(n\alpha)$$

$$\varphi(\rho,\alpha) = (A_1 \rho^n + A\rho^{-n})\cos(n\alpha) \qquad n \neq 0$$

$$\cancel{\frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho\varphi) = 0} \qquad n=0:$$

$$\cancel{\frac{1}{\rho}}\partial_\rho(\rho\;\partial_\rho R\;\cancel{\cos(n\alpha)}) = 0$$

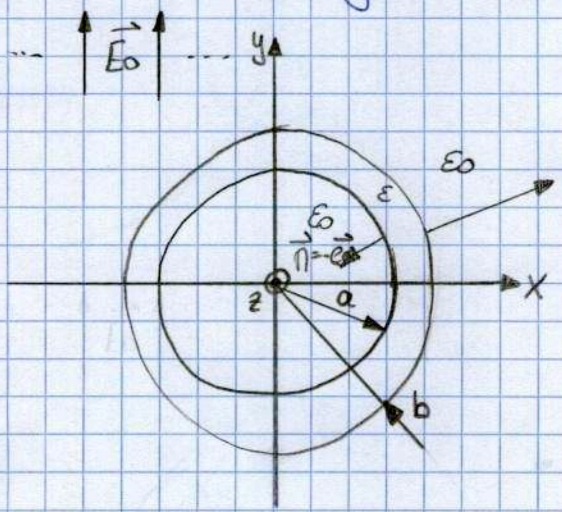
$$\partial_\rho(\rho\,\partial_\rho R(\rho)) = 0$$
$$\partial_\rho R(\rho) = \frac{c}{\rho}$$

$$R(\rho) = \int \frac{c}{\rho}\,d\rho = c\ln(\rho) + \bar{c}$$

$$\varphi(\rho,\alpha) = c\ln(\rho) + \bar{c}$$
$$\varphi(a,\alpha) = U_0 = c\ln(a) + \bar{c}$$

$$\partial_\rho R + \rho\,\partial_\rho^2 R = 0 \qquad R(\rho) = B\rho^\lambda$$
$$\cancel{B}\lambda\cancel{\rho^{\lambda-1}} + \cancel{\rho}\cancel{B}\lambda(\lambda-1)\cancel{\rho^{\lambda-2}} = 0 \qquad \partial_\rho R(\rho) = B\lambda\rho^{\lambda-1}$$
$$\lambda + \lambda^2 - \lambda = 0 \qquad \partial_\rho^2 R(\rho) = B\lambda(\lambda-1)\rho^{\lambda-2}$$
$$\lambda = 0 \qquad R(\rho) = B$$

A 3.2.11 Dielektrischer Hohlzylinder im Transversalfeld



$\vec{E}_0 = E_0 \vec{e}_y$

$$\Delta \varphi = 0$$

$$\varrho \partial_\varrho (\varrho \partial_\varrho \varphi) + \partial_\alpha^2 \varphi = 0$$

in allen offenen Bereichen.

Randbedingungen:

$\varrho = 0$: φ beschr. $\quad \varphi = 0$ (o.B.d.A.)

$\varrho = a$: $[\![\vec{E}_t]\!] = \vec{0} \Rightarrow [\![\varphi]\!] = 0$ (o.B.d.A.)

$[\![D_n]\!] = 0 \Rightarrow (\varepsilon_r \partial_\varrho \varphi)\big|_{\varrho = a+} = (\partial_\varrho \varphi)\big|_{\varrho = a^-}$; $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$

Bsp.

$$f(r,\theta) = r^2 \cos(\theta) \qquad \vec{r} = r\,\vec{e_r}$$

$$\vec{\nabla} \times [\vec{\nabla} \times (\vec{r} f)] \qquad \vec{r} f = r^3 \cos(\theta)\,\vec{e_r} + 0\,\vec{e_\alpha} + 0\,\vec{e_\theta}$$

$$\vec{\nabla} \times (\vec{r} f) = \vec{e_\theta} \left[\cancel{\frac{\partial_\alpha (r^3 \cos(\theta))}{r \sin(\theta)}} - \cancel{\frac{\partial_r (r \cdot 0)}{r}} \right] +$$

$$+ \vec{e_\alpha} \frac{\cancel{\partial_r (r \cdot 0)} - \partial_\theta (r^{\cancel{3}\,2} \cos(\theta))}{\cancel{r}}$$

$$\vec{\nabla} \times (\vec{r} f) = r^2 \sin(\theta)\,\vec{e_\alpha}$$

$$\vec{\nabla} \times \left[\vec{\nabla} \times (\vec{r} f)\right] = \vec{e_r} \frac{\partial_\theta (\sin(\theta)\, r^2 \sin(\theta)] +}{r \sin(\theta)} + \vec{e_\theta} \left[- \frac{\partial_r (r^3 \sin(\theta)}{r} \right]$$

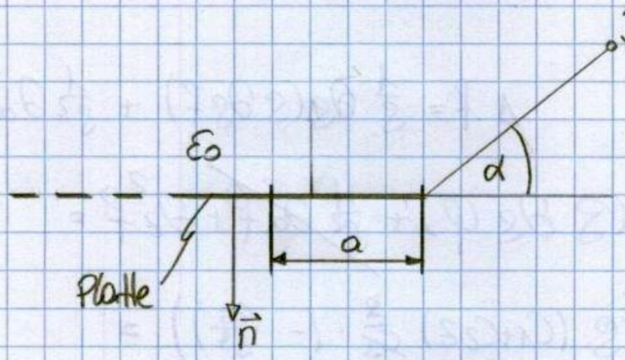
$$= \vec{e_r} \frac{\partial_\theta (r^2 \sin^2(\theta))}{r \sin(\theta)} - \vec{e_\theta} \frac{3 r^2 \sin(\theta)}{r} =$$

$$= \vec{e_r} \frac{r^{\cancel{2}}\, 2 \cancel{\sin(\theta)} \cos(\theta)}{\cancel{r} \cancel{\sin(\theta)}} - \vec{e_\theta} \frac{3 r^{\cancel{2}} \sin(\theta)}{\cancel{r}} =$$

$$= 2 r \cos(\theta)\,\vec{e_r} - 3 r \sin(\theta)\,\vec{e_\theta}$$

A 3.2.12 Kantenfeld

$$\varphi(\rho,\alpha) = -U\sqrt{\rho/a}\,\cos(\alpha/2)$$



$$\vec{n}\cdot[\![\vec{D}]\!] = \sigma$$

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{E} = -\vec{\nabla}\varphi \qquad \vec{\nabla} = \vec{e_\rho}\,\partial_\rho + \frac{1}{\rho}\,\vec{e_\alpha}\,\partial_\alpha$$

$$\vec{E} = -\left(-U\cos(\alpha/2)\,(+\tfrac{1}{2})\sqrt{\tfrac{a}{\rho}}\cdot\left(\tfrac{1}{a}\right)\vec{e_\rho} + \frac{1}{\rho}(+U)\sqrt{\tfrac{\rho}{a}}\,\tfrac{1}{2}\sin(\alpha/2)\,\vec{e_\alpha}\right) =$$

$$= \frac{U}{2\cancel{a}}\sqrt{\frac{\cancel{a}}{\rho a}}\cos\left(\frac{\alpha}{2}\right)\vec{e_\rho} - \frac{U}{2\cancel{\rho}}\sqrt{\frac{\cancel{\rho}}{a\rho}}\sin\left(\frac{\alpha}{2}\right)\vec{e_\alpha} =$$

$$= \frac{U}{2}\,\frac{1}{\sqrt{\rho a}}\left(\cos(\alpha/2)\,\vec{e_\rho} - \sin(\alpha/2)\,\vec{e_\alpha}\right)$$

$$[\![\vec{D}]\!] = \vec{D}^{(2)} - \vec{D}^{(1)} = \varepsilon_0\vec{E}(\rho,-\pi) - \varepsilon_0\vec{E}(\rho,\pi) =$$

$$= \varepsilon_0\,\frac{+U}{2\sqrt{\rho a}}\,\vec{e_\alpha} + \varepsilon_0\,\frac{U}{2\sqrt{\rho a}}\,\vec{e_\alpha} = \varepsilon_0\,\frac{U}{\sqrt{\rho a}}\,\vec{e_\alpha}$$

$$\vec{n}\cdot[\![\vec{D}]\!] = \varepsilon_0\,\frac{U}{\sqrt{\rho a}} = \sigma$$

$$Q' = \int_0^a \sigma\,d\rho = \varepsilon_0\,\frac{U}{\sqrt{a}}\int \rho^{-\frac{1}{2}}\,d\rho = \varepsilon_0\,\frac{U}{\sqrt{a}}\left(2\rho^{\frac{1}{2}}\right)\Big|_0^a =$$

$$= \varepsilon_0\,\frac{U}{\sqrt{a}}\,2\sqrt{a} = 2\varepsilon_0 U$$

A. 3.2.13. Widerstand

(i) $\varphi(\rho,z) = (C_1 + C_2 z)\ \ln(c_3/\rho)$

$\Delta\varphi = 0 \qquad \Delta f = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho f) + \frac{1}{\rho^2}\partial_\alpha^2 f + \partial_z^2 f$

$\Delta\varphi = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho\varphi) + \cancel{\frac{1}{\rho^2}\partial_\alpha^2 f} + \partial_z^2 f =$

$= \frac{1}{\rho}\partial_\rho\left(\rho\,(C_1+C_2 z)\frac{\rho}{c_3}\cdot\left(-\frac{c_3}{\rho^2}\right)\right) =$

$= \frac{1}{\rho}\partial_\rho\left(-\cancel{\rho}\,(C_1+C_2 z)\frac{1}{\cancel{\rho}}\right) = 0$

$\vec{E} = -\vec{\nabla}\varphi = -\left(\vec{e_\rho}\,\partial_\rho\varphi + \vec{e_\alpha}\frac{1}{\rho}\partial_\alpha\varphi + \vec{e_z}\,\partial_z\varphi\right)$

$\vec{E} = -\left((C_1+C_2 z)\frac{\rho}{c_3}\left(-\frac{c_3}{\rho^2}\right)\vec{e_\rho} + \frac{1}{\rho}\,0\,\vec{e_\alpha} + C_2\ln(c_3/\rho)\,\vec{e_z}\right) =$

$= (C_1+C_2 z)\frac{1}{\rho}\vec{e_\rho} - C_2\ln(c_3/\rho)\vec{e_z}$

$z=0:\ E_\rho = 0 \Rightarrow C_1\cdot\frac{1}{\rho} = 0 \Rightarrow C_1 = 0$

$\rho=b:\ E_z = 0 \Rightarrow -C_2\ln\left(\frac{c_3}{b}\right) = 0 \Rightarrow \ln\left(\frac{c_3}{b}\right) = 0 \Rightarrow \frac{c_3}{b} = 1$

$c_3 = b$

$\rho=a:\ E_z = \frac{K_z}{\gamma_s} \Rightarrow -C_2\ln\left(\frac{c_3}{a}\right) = \frac{K_z}{\gamma_s} \Rightarrow C_2 = \frac{-K_z}{\gamma_s\ln(\frac{b}{a})}$

$\{\rho,\alpha,z\}$

(ii) Durchflutungssatz: $\vec{H} = \frac{I}{2\pi\rho}\vec{e_\alpha}$

(iii) $\vec{S} = \vec{E}\times\vec{H} = \left[(C_1+C_2 z)\frac{1}{\rho}\vec{e_\rho} - C_2\ln(c_3/\rho)\vec{e_z}\right]\times\left[\frac{I}{2\pi\rho}\vec{e_\alpha}\right] =$

$= (C_1+C_2 z)\frac{1}{\rho}\frac{I}{2\pi\rho}\vec{e_z} + C_2\ln\left(\frac{c_3}{\rho}\right)\frac{I}{2\pi\rho}\vec{e_\rho} =$

$= (\cancel{C_1}^{\,0}+C_2 z)\frac{I}{2\pi\rho^2}\vec{e_z} + \frac{C_2 I}{2\pi\rho}\ln(c_3/\rho)\vec{e_\rho} =$

$= -\frac{K_z I z}{\gamma_s\ln(\frac{b}{a})\,2\pi\rho^2}\vec{e_z} - \frac{K_z I}{\gamma_s\ln(\frac{b}{a})\,2\pi\rho}\ln\left(\frac{b}{\rho}\right)\vec{e_\rho}$

A 3.2.14 Elektrostatische Potentiale

$$\vec{E} = \frac{E_0}{a^2}\left[(x^2-y^2)\vec{e_x} - 2xy\,\vec{e_y}\right]$$

(i) $\vec{E} = -\nabla\varphi = -(\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z)\,\varphi(x,y,z)$

$$\varphi_x = -\frac{E_0}{a^2}(x^2-y^2) \Rightarrow \varphi = \int -\frac{E_0}{a^2}(x^2-y^2)\,dx + c(y)$$

$$\varphi_y = +\frac{E_0}{a^2}2xy \Rightarrow \varphi = \int \frac{E_0}{a^2}2xy\,dy + c(x)$$

$$\varphi_z = 0$$

$$\varphi = -\frac{E_0}{a^2}\left(\frac{x^3}{3} - y^2x\right) + c(y) \quad |\,\partial_y$$

$$+\frac{E_0}{a^2}2xy + c'(y) \overset{!}{=} +\frac{E_0}{a^2}2xy$$

$$\Rightarrow c'(y) = 0 \Rightarrow c(y) = \int c'(y)\,dy = c \in \mathbb{R}$$

$$\varphi(x,y,z) = -\frac{E_0}{a^2}\left(\frac{x^3}{3} - y^2x\right) + c$$

(ii.) $\vec{D} = \vec{\nabla}\times\vec{V} = \varepsilon_0\vec{E} = \frac{\varepsilon_0 E_0}{a^2}\left[(x^2-y^2)\vec{e_x} - 2xy\,\vec{e_y}\right]$

$\vec{V} = V_x\vec{e_x} + V_y\vec{e_y} + V_z\vec{e_z}$

$\vec{\nabla}\times\vec{V} = (\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z)\times(V_x\vec{e_x} + V_y\vec{e_y} + V_z\vec{e_z}) =$

$= \partial_x V_y\,\vec{e_z} - \partial_x V_z\,\vec{e_y} - \partial_y V_x\,\vec{e_z} + \partial_y V_z\,\vec{e_x} + \partial_z V_x\,\vec{e_y} - \partial_z V_y\,\vec{e_x}$

$= (\partial_y V_z - \partial_z V_y)\vec{e_x} + (\partial_z V_x - \partial_x V_z)\vec{e_y} + (\partial_x V_y - \partial_y V_x)\vec{e_z}$

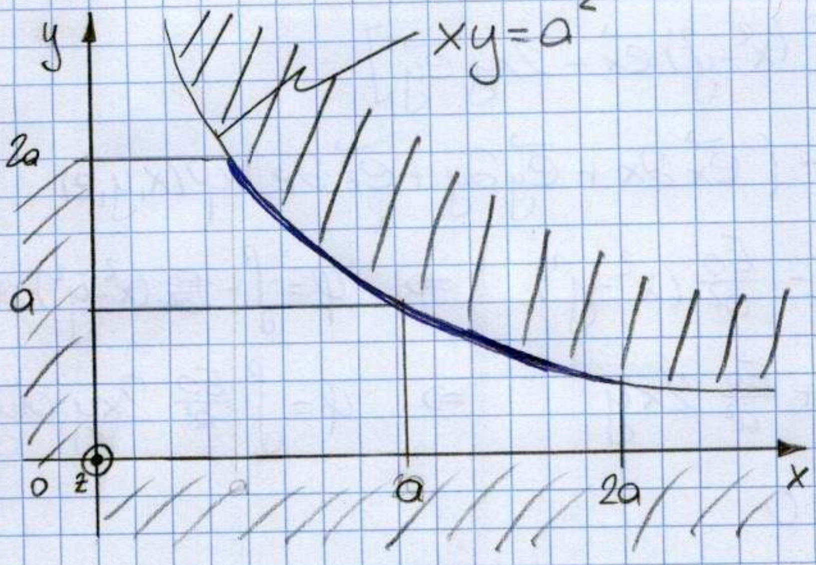
$\Rightarrow \partial_y V_z - \partial_z V_y = \frac{\varepsilon_0 E_0}{a^2}(x^2-y^2)$

$\partial_z V_x - \partial_x V_z = -\frac{\varepsilon_0 E_0}{a^2}2xy$

$\partial_x V_y - \partial_y V_x = 0$

$\vec{\nabla}\cdot\vec{E} = \frac{E_0}{a^2}(2x - 2x) = 0 \Rightarrow V = V\vec{e_z}$

A 3.2.15 Elektrischer Fluß



$$\Psi' = \int_{C} \vec{n} \cdot \vec{D} \, ds \qquad \vec{D} = \varepsilon_0 \vec{E}$$

Ansatz: $\varphi = Cxy \qquad \varphi(x, \frac{a^2}{x}) = U$

$$Cx \frac{a^2}{x} = U \quad \Rightarrow \quad C = \frac{U}{a^2}$$

$$\vec{E} = -\vec{\nabla}\varphi \qquad \vec{E} = -\frac{U}{a^2}(y\vec{e_x} + x\vec{e_y})$$

$$\vec{D} = \varepsilon_0 \vec{E} \qquad \vec{D} = -\frac{\varepsilon_0 U}{a^2}(y\vec{e_x} + x\vec{e_y})$$

$$\vec{V} = V\vec{e_z} \qquad \vec{D} = \vec{\nabla} \times \vec{V}$$

$$\left(\cancel{\Psi' = \int_{C} \vec{s} \cdot \vec{D} \, ds = \int_{C} \vec{s} \cdot (\vec{\nabla} \times \vec{V}) \, ds \overset{\text{Stokes}}{=} \int_{C} \vec{s} \cdot \vec{V} \, ds} \right)$$

$$\Psi' = V(2a, \tfrac{a}{2}) - V(a/2, 2a) = \frac{15}{4} \varepsilon_0 U$$

Gl (3.66)

A. 3.2.16 Randfeld eines Plattenkondensators

$$\varphi = \frac{U}{2\pi} u$$

$$\vec{V} = V \vec{e}_z \quad \text{mit} \quad V = \frac{\varepsilon_0 U}{2\pi} v$$

$$x = \frac{a}{2\pi} \left[1 + v + e^{v} \cos(u) \right]$$

$$y = \frac{a}{2\pi} \left[u + e^{v} \sin(u) \right]$$

z-Achse: $x=0, y=0$

$$\left.\begin{array}{l} \mathrm{I}: \ 0 = 1 + v + e^{v} \cos(u) \\ \mathrm{II}: \ 0 = u + e^{v} \sin(u) \end{array}\right\} \text{z-Achse.}$$

$\mathrm{II}: \Rightarrow \ u = 0$

$0 = 1 + v + e^{v} \quad \Rightarrow \quad v = -1{,}2785$

A.3.2.17 Asymptotisches Randfeld

$\varphi = \frac{U}{2}$ bei $\alpha = \pi$ $\qquad \varphi(\rho, \pi) = \frac{U}{2}$

$\varphi = -\frac{U}{2}$ bei $\alpha = -\pi$ $\qquad \varphi(\rho, -\pi) = -\frac{U}{2}$

$\varphi = 0$ $\quad \rho \to \infty$ $\qquad \lim\limits_{\rho \to \infty} \varphi(\rho, \alpha) = 0$

$$\cancel{\varphi = A_2 \rho^{-k} B_2 \sin(k\alpha)}$$

$$\cancel{k = \frac{1}{2},\ A_2 = 1,\ B_2 = \frac{U}{2}}$$

$$\cancel{\varphi(\rho, \alpha) =}$$

$$\varphi(\rho, \alpha) = \left[A_1 + A_2 \ln\left(\frac{a}{\rho}\right)\right](B_1 + B_2\alpha)$$

$$\frac{U}{2} = \left[A_1 + A_2 \ln\left(\frac{a}{\rho}\right)\right](B_1 + B_2\pi)$$

$$-\frac{U}{2} = \left[A_1 + A_2 \ln\left(\frac{a}{\rho}\right)\right](B_1 - B_2\pi)$$

$$\lim_{\rho \to \infty} \left[A_1 + A_2 \ln\left(\frac{a}{\rho}\right)\right](B_1 + B_2\pi) = 0$$

$\Rightarrow$ $A_2 = 0$ $\qquad A_1 = 1.$

$$\left.\begin{array}{l} B_1 + B_2\pi = \frac{U}{2} \\ B_1 - B_2\pi = -\frac{U}{2} \end{array}\right\} +$$

$$B_1 = 0 \quad \Rightarrow \quad B_2 = \frac{U}{2\pi}$$

$$\varphi(\rho, \alpha) = \frac{U}{2\pi}\alpha$$

$$\vec{E} = -\frac{U}{2\pi\rho}\vec{e_\alpha} \qquad \vec{D} = \vec{\nabla} \times \vec{V}$$

$$\vec{V} = V\vec{e_z}$$

DGL lösen.

A.3.2.18 Kugelkondensator

$$\varepsilon = \varepsilon(\theta, \alpha) \qquad Q = C\,U$$

$$\Delta\varphi = 0 \qquad U = \int_{\mathcal{C}} \vec{s}\cdot\vec{E}\,ds$$

$$\vec{E} = -\vec{\nabla}\varphi \qquad Q = \int_{A} \vec{n}\cdot\vec{D}\,dA$$

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{E} = E_r\,\vec{e_r} \qquad E_r = \frac{K}{r^2}$$

$$U = \int_{r_1}^{r_2} E_r\,dr = K\cdot\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \Rightarrow K = \frac{U}{\frac{1}{r_1} - \frac{1}{r_2}}$$

$$Q = \int_{\mathcal{K}} \underbrace{D_r\vec{e_r}}_{\vec{D}} \cdot \underbrace{\vec{e_r}}_{\vec{n}}\,dA$$

($\mathcal{K}$... Kugeloberfläche)

$$\vec{D} = \varepsilon\vec{E} = \varepsilon(\theta,\alpha)\,E_r\,\vec{e_r} = \varepsilon(\theta,\alpha)\frac{K}{r^2}\vec{e_r} = D_r\,\vec{e_r}$$

$$Q = \int_{\mathcal{K}} D_r\,dA = \int_{\mathcal{K}} \varepsilon(\theta,\alpha)\frac{K}{r^2}\,dA = \int_0^{\pi}\int_0^{2\pi} \varepsilon(\theta,\alpha)\frac{K}{r^2}\,r^2\sin(\theta)\,d\theta\,d\alpha$$

$$Q = \int_0^{\pi}\int_0^{2\pi} \varepsilon(\theta,\alpha)\,K\sin(\theta)\,d\theta\,d\alpha = C\cdot U$$

$$Q = \int_0^{\pi}\int_0^{2\pi} \varepsilon(\theta,\alpha)\frac{U}{\frac{1}{r_1}-\frac{1}{r_2}}\cdot\sin(\theta)\,d\theta\,d\alpha$$

$$\Rightarrow C = \frac{1}{\frac{1}{r_1}-\frac{1}{r_2}}\int_0^{\pi}\int_0^{2\pi} \varepsilon(\theta,\alpha)\sin(\theta)\,d\theta\,d\alpha$$

A 3.2.19 Homogen polarisierte Kugel

homogen elektrisch starr polarisiert : $\vec{P} = P\vec{e_z}$, $P = const.$

$S = 0$, $\cancel{\sigma = 0}$ $\Rightarrow$ $\vec{n} \cdot [\![\vec{D}]\!] = 0$

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ $\qquad \vec{\nabla} \cdot \vec{D} = 0$ $\qquad \vec{\nabla} \times \vec{E} = \vec{0}$

$\vec{E} = E\vec{e_z}$ $\qquad \vec{D} = D\vec{e_z}$

$E = const.$

$$r = a: \left. \begin{array}{l} \vec{e_r} \times [\![\vec{E}]\!] \overset{!}{=} \vec{0} \\ [\![\vec{E}]\!] = \vec{E}^{(2)} - \vec{E}^{(1)} \end{array} \right\} \Rightarrow \vec{E}^{(1)} = 0 = E\vec{e_z}$$

$\vec{E}^{(2)}$: $\llcorner$ 0 lt. Vs. $\qquad \vec{D} = \vec{P}$

$$\vec{e_r} \cdot [\![\vec{D}]\!] = \sigma = -D\underbrace{\vec{e_r} \cdot \vec{e_z}}_{\cos(\theta)} = -D\cos(\theta)$$

$$[\![\vec{D}]\!] = 0 - D\vec{e_z}$$

$$\sigma = -D \cdot \cos(\theta) = -P \cdot \cos(\theta)$$

A 3.2.20 Grundtypen

$$\varphi(r,\theta) = \begin{cases} U \frac{r}{a} \cos(\theta) & r \leq a \\ U \left(\frac{a}{r}\right)^2 \cos(\theta) & r \geq a \end{cases} \qquad U = \text{const.}$$

(i)

$$\vec{E} = -\vec{\nabla}\varphi = -\vec{\nabla}\varphi = -\left(\vec{e}_r \partial_r \varphi + \vec{e}_\theta \frac{\partial_\theta \varphi}{r} + \cancel{\vec{e}_\alpha \frac{\partial_\alpha \varphi}{r\sin(\theta)}}\right)$$

von α unabh.

$$\vec{E} = \begin{cases} -\frac{U}{a}\cos(\theta)\vec{e}_r + \frac{U}{a}\sin(\theta)\vec{e}_\theta & r \leq a \\ 2Ua^2 \frac{1}{r^3}\cos(\theta)\vec{e}_r + \frac{Ua^2}{r^3}\sin(\theta)\vec{e}_\theta & r \geq a \end{cases}$$

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{D} = \varepsilon\vec{E}$$

$$\vec{\nabla}\cdot\vec{f} = \frac{\partial_r(r^2 f_r)}{r^2} + \frac{\partial_\theta[\sin(\theta) f_\theta]}{r\sin(\theta)} + \cancel{\frac{\partial_\alpha f_\alpha}{r\sin(\theta)}}$$

$r \leq a$:

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial_r\left(-\frac{Ur^2}{a}\cos(\theta)\right)}{r^2} + \frac{\partial_\theta\left[\sin^2(\theta)\cdot\frac{U}{a}\right]}{r\sin(\theta)} =$$

$$= \frac{-\frac{2Ur}{a}\cos(\theta)}{\frac{r^2}{1}} + \frac{\frac{2U}{a}\sin(\theta)\cos(\theta)}{\frac{r\sin(\theta)}{1}} =$$

$$= -\frac{2Ur\cos(\theta)}{a r^2} + \frac{2U\cos(\theta)}{ar} = -\frac{2U}{ar}\cos(\theta) + \frac{2U}{ar}\cos(\theta)$$

$$= 0$$

$r \geq a$:

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial_r\left(\frac{2Ua^2}{r}\cos(\theta)\right)}{r^2} + \frac{\partial_\theta\left(\frac{Ua^2}{r^3}\sin^2(\theta)\right)}{r\sin(\theta)} =$$

$$= \frac{-\frac{2Ua^2}{r^2}\cos(\theta)}{\frac{r^2}{1}} + \frac{\frac{Ua^2}{r^3}2\sin(\theta)\cos(\theta)}{\frac{r\sin(\theta)}{1}} =$$

$$= -\frac{2Ua^2\cos(\theta)}{r^4} + \frac{2Ua^2\cos(\theta)}{r^4} = 0$$

$$\Rightarrow \rho = 0 \qquad \vec{n}\cdot[\![\vec{D}]\!] = \sigma$$

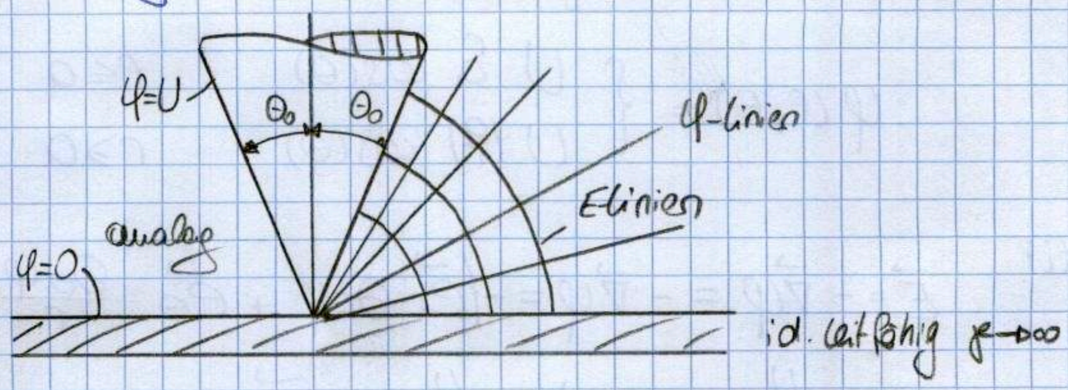
~~$\vec{E}$ stetig $\Rightarrow$ $\vec{D}$ stetig $\Rightarrow$ $[\![\vec{D}]\!] = 0 \Rightarrow \sigma = 0$~~

$$\vec{E}(r=a,\theta) = \begin{cases} -\frac{U}{a}\cos(\theta)\vec{e}_r + \frac{U}{a}\sin(\theta)\vec{e}_\theta \\ \frac{2U}{a}\cos(\theta)\vec{e}_r + \frac{U}{a}\sin(\theta)\vec{e}_\theta \end{cases} \ominus$$

$$[\![\vec{E}]\!] = \frac{3U}{a}\cos(\theta)\vec{e}_r \qquad \vec{n} = \vec{e}_r \Rightarrow \sigma = \frac{3U}{a}\cos(\theta)$$

A. 3.2.21 Kegelspitze auf Platte

(i)



Kugelkoordinaten.

$$\Delta\varphi=0$$
$$\varphi(r,\Theta=\Theta_0,\alpha)=U$$
$$\varphi(r,\Theta=\tfrac{\pi}{2},\alpha)=0$$

$$\cancel{\varphi=U\cos(\Theta-\Theta_0)}$$

$$\varphi=\varphi(\Theta)$$

$$\Delta\varphi=\frac{\partial_\Theta\left[\sin(\Theta)\,\partial_\Theta\varphi\right]}{r^2\sin(\Theta)}=0$$

$$\partial_\Theta\left[\sin(\Theta)\,\partial_\Theta\varphi\right]=0$$

$$\partial_\Theta\varphi=\frac{C(r,\alpha)}{\sin(\Theta)}$$

$$\varphi=\int\frac{C(r,\alpha)}{\sin(\Theta)}\,d\Theta=C_1\ln\left(\tan\left(\tfrac{\Theta}{2}\right)\right)+C_2$$

$$\varphi\left(\tfrac{\pi}{2}\right)=C_1\ln(1)+C_2=0\ \Rightarrow\ C_2=0$$
$$\varphi(\Theta_0)=C_1\ln\left(\tan\left(\tfrac{\Theta_0}{2}\right)\right)=U\ \Rightarrow\ C_1=\frac{U}{\ln\left(\tan\left(\frac{\Theta_0}{2}\right)\right)}$$

Rest: (siehe Lsg.)

A. 3.2.23 Influenzierteles elektrisches Moment

$\vec{E_0} = E_0 \vec{e_z}$ $\qquad \varphi = -E_0\left[1-\left(\frac{a}{r}\right)^3\right]z \quad , \quad r \geq a.$

$\vec{E} = -\vec{\nabla}\varphi$ $\qquad z = r\cdot\cos(\theta)$

(i) $\varphi = -E_0\left[1-\left(\frac{a}{r}\right)^3\right] r\cdot\cos(\theta)$

$$\vec{\nabla} f = \vec{e_r}\,\partial_r f + \vec{e_\theta}\,\frac{\partial_\theta f}{r} + \cancel{\vec{e_\alpha}\,\frac{\partial_\alpha f}{r\sin(\theta)}}$$ nur von r,θ abh.

$$-\vec{\nabla}\varphi = E_0\left[\left\{+3\left(\frac{a}{r}\right)^2\cdot\frac{a}{r^2}\cdot r\cdot\cos(\theta) + \left[1-\left(\frac{a}{r}\right)^3\right]\cdot\cos(\theta)\right\}\vec{e_r} + \right.$$
$$\left. + \left\{-\left[1-\left(\frac{a}{r}\right)^3\right]\sin(\theta)\right\}\vec{e_\theta}\right] =$$
$$= E_0\left[\left\{3\left(\frac{a}{r}\right)^3\cos(\theta) + \cos(\theta) - \left(\frac{a}{r}\right)^3\cos(\theta)\right\}\vec{e_r} + \right.$$
$$\left. + \left[\left(\frac{a}{r}\right)^3 - 1\right]\sin(\theta)\,\vec{e_\theta}\right] =$$
$$= E_0\left[\left[2\left(\frac{a}{r}\right)^3+1\right]\cos(\theta)\,\vec{e_r} + \left[\left(\frac{a}{r}\right)^3-1\right]\sin(\theta)\,\vec{e_\theta}\right]$$

$$\vec{E}(r=a,\theta) = E_0\;3\cos(\theta)\,\vec{e_r}$$

(ii) $\sigma = \varepsilon_0 E_r = 3\varepsilon_0 E_0 \cos(\theta)$

$$\vec{p} = \int_{\partial V} \sigma\, r\, dA$$

A.3.2.25 Dipol im Hohlraum eines Dielektrikums

$$\varphi_0 = \frac{p_0}{4\pi\varepsilon_0} \frac{\cos(\theta)}{r^2} \qquad \vec{p_0} = p_0 \vec{e_z}$$

$$\varphi = \left(A r + \frac{B}{r^2}\right) \cos(\theta)$$

$r<a$: $\varphi = \left(A_1 r + \frac{B_1}{r^2}\right) \cos(\theta)$

$r>a$: $\varphi = \left(A_2 r + \frac{B_2}{r^2}\right) \cos(\theta)$

Randbedingungen zu erfüllen:

$r \to 0$: $\varphi \to \varphi_0 \Rightarrow B_1 = \frac{p_0}{4\pi\varepsilon_0}$

$r \to \infty$: $\varphi \to 0 \Rightarrow A_2 = 0$

$r = a$: $[\![E_\theta]\!] = 0 \Rightarrow [\![\varphi]\!] = 0 \Rightarrow \left(A_1 r + \frac{p_0}{4\pi\varepsilon_0} \frac{1}{r^2}\right) \cos(\theta) = \frac{B_2}{r^2} \cos\theta$

$[\![D_r]\!] = 0 \Rightarrow$

A. 3.2.26 Ein drehsymmetrisches Qudrupolfeld

(i)

$$\vec{E} = -\vec{\nabla}\varphi$$

$$\vec{\nabla}\varphi = \vec{e_r}\,\partial_r\varphi + \vec{e_\theta}\,\frac{\partial_\theta\varphi}{r} + \cancel{\vec{e_\alpha}\,\frac{\partial_\alpha\varphi}{r\sin(\theta)}}$$ von α unabh.

$$\varphi = \frac{Qa^2}{4\pi\varepsilon_0}\,\frac{3\cos^2(\theta)-1}{r^3}$$

$$\partial_r\varphi = \frac{Qa^2}{4\pi\varepsilon_0}\left(3\cos^2(\theta)-1\right) - 3r^{-4}$$

$$\partial_\theta\varphi = \frac{-Qa^2}{4\pi\varepsilon_0 r^3}\,6\cos(\theta)\sin(\theta)$$

$$\vec{E} = \frac{Qa^2}{4\pi\varepsilon_0}\left(\frac{3(3\cos^2(\theta)-1)}{r^4}\,\vec{e_r} + \frac{6\cos(\theta)\sin(\theta)}{r^4}\,\vec{e_\theta}\right)$$

(ii) $\vec{D} = \vec{\nabla}\times\vec{V}$ $\qquad \vec{V} = V\,\vec{e_\alpha}$

$$\frac{Qa^2}{4\pi}\,\frac{3}{r^4}\left(\left(3\cos^2(\theta)-1\right)\vec{e_r} + 2\sin(\theta)\cos(\theta)\,\vec{e_\theta}\right) =$$

$$= \vec{\nabla}\times\vec{V} = \frac{\partial_\theta[\sin(\theta)V]}{r\sin(\theta)}\,\vec{e_r} - \frac{\partial_r(rV)}{r}\,\vec{e_\theta}$$

$$\Rightarrow \begin{cases} \frac{Qa^2}{4\pi}\,\frac{3}{r^{\cancel{4}3}}\left(3\cos^2(\theta)-1\right) = \frac{\partial_\theta[\sin(\theta)V]}{\cancel{r}\sin(\theta)} \\ -\frac{Qa^2}{{}_2\cancel{4}\pi}\,\frac{3}{r^{\cancel{4}3}}\,\cancel{2}\sin(\theta)\cos(\theta) = +\frac{\partial_r(rV)}{\cancel{r}} \end{cases}$$

$$rV = -\frac{3Qa^2}{2\pi}\sin(\theta)\cos(\theta)\int \overset{r^{-3}}{\frac{1}{r^3}}\,dr + c(\theta)$$

$$= -\frac{3Qa^2}{2\pi}\sin(\theta)\cos(\theta)\,\frac{r^{-2}}{-2} + c(\theta) =$$

$$= \frac{3Qa^2}{4\pi}\sin(\theta)\cos(\theta)\,\frac{1}{r^2} + c(\theta)$$

$$V = \frac{3Qa^2}{4\pi}\,\frac{\sin(\theta)\cos(\theta)}{r^3} + \frac{c(\theta)}{r}$$

$$\sin(\theta)V = \frac{3Qa^2}{4\pi}\,\frac{\sin^2(\theta)\cos(\theta)}{r^3} + \frac{c(\theta)}{r}$$

$$\partial_\theta(\sin(\theta)V) = \frac{3Qa^2}{4\pi}\,\frac{2\sin(\theta)\cos^2(\theta) - \sin^3(\theta)}{r^3} + \frac{c'(\theta)}{r}$$

$$\frac{\partial_\theta(\sin(\theta)V)}{\sin(\theta)} = \frac{3Qa^2}{4\pi}\,\frac{2\cos^2(\theta) - \overset{1-\cos^2(\theta)}{\sin^2(\theta)}}{r^3} + \frac{c'(\theta)}{r\sin(\theta)}$$ $\Rightarrow c'(\theta)=0 \Rightarrow c'(\theta)=c$

A 3.3.1 Punktuelle Ladungsinjektion

$$\partial_t \rho + \underbrace{\cancel{\vec{\nabla}\cdot(\rho\vec{v})}}_{\vec{v}=\vec{0}} + \frac{\gamma}{\varepsilon}\rho = 0$$

$\int_V \rho\, dV = Q_0$

$\int_{V/0} \rho\, dV = 0$

$\int \rho^* d\bar{V} = 0$

Merkhinweis:

$$T_R = \frac{\cancel{l}}{\gamma \cancel{A}} \cdot \frac{\varepsilon \cancel{A}}{\cancel{l}} = \frac{\varepsilon}{\gamma}$$

$t=0$: Q_0 injiziert

$$\partial_t \rho = -\frac{\gamma}{\varepsilon}\rho$$

$$\frac{\partial_t \rho}{\rho} = -\frac{\gamma}{\varepsilon}$$

$$\ln(\rho) = -\frac{\gamma}{\varepsilon}t + \bar{c}$$

$$\rho(t) = c \cdot e^{-\frac{\gamma}{\varepsilon}t}$$

$$\rho(0) = c = Q_0$$

$$\rho(t) = Q_0 \cdot e^{-t/T_R} \qquad T_R = \frac{\varepsilon}{\gamma}$$

für $\vec{r} = \vec{0}$

$\rho(\vec{r}, t) = 0$ für $r > 0$

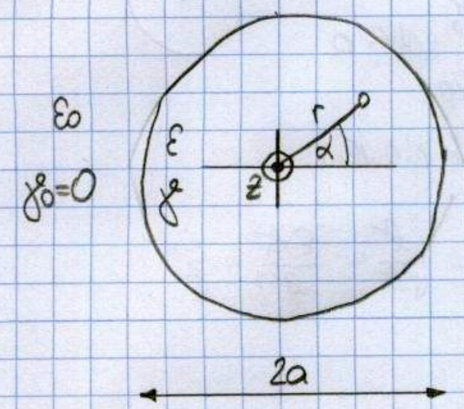
$\dot{Q}(V) + I(\partial V) = 0$ ... Ladungserhaltung

$$\dot{Q} = \left(Q_0\, e^{-t/T_R}\right)^{\cdot} = -\frac{Q_0}{T_R}\, e^{t/T_R}$$

$$I(\partial V) = -\dot{Q}(V)$$

$$\vec{J}(r,t) = \frac{-\dot{Q}\,\vec{e_r}}{A} = +\frac{Q_0}{T_R} \cdot e^{t/T_R} \cdot \frac{1}{4\pi r^2}\,\vec{e_r}$$

A. 3.3.2 Ladungsrelaxation im Kreiszylinder



$$\rho(r,t) = \rho_0 \, e^{-t/T_R}$$

$$\vec{J} = J(r,t)\,\vec{e_r}$$

$\vec{e_r} \hat{=} \vec{e_\varsigma}$ (wegen Verwechslung)

$$\dot{Q}(V) = -I(\partial V)$$

$$Q' = r^2 \pi \, \rho(r,t)$$

$$\dot{Q}' = r^2 \pi \, \dot{\rho} = -r^2 \pi \, \frac{\rho_0}{T_R} \, e^{-t/T_R}$$

$$2 \cancel{\pi} \cancel{r} \, J = -\dot{Q}' = r^{\cancel{2}} \cancel{\pi} \, \frac{\rho_0}{T_R} \, e^{-t/T_R}$$

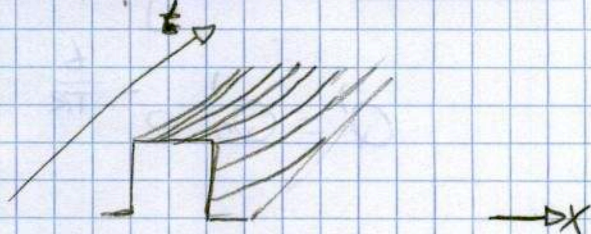
$$\vec{J} = \frac{\rho_0 r}{2 T_R} \, e^{-t/T_R} \, \vec{e_r}$$

$$\lim_{t \to \infty} \rho(r,t) = \lim_{t \to \infty} \rho_0 \, e^{-t/T_R} = 0$$

$$\sigma = \frac{\rho_0 \cdot \cancel{\pi} a^{\cancel{2}}}{2 \cancel{\pi} \cancel{a}} = \frac{\rho_0 a}{2}$$

gleichförmig an Mantelfläche verteilt.

A 3.3.3 Ladungsrelaxation in Schichtenstruktur



$$\varrho(t) = \pm \varrho_0 e^{-t/\tau_R}$$

$$\int_V \varrho \, dV + \int_{\partial V} \sigma \, dA = Q = \varrho_0 \cdot a \cdot A$$

A… Bezugsfläche

$$\varrho a + \sigma = \varrho_0 \cdot a$$

$$\pm a \varrho_0 e^{-t/\tau_R} + \sigma = \varrho_0 a$$

$$\sigma = \pm \varrho_0 a (1 - e^{-t/\tau_R})$$

$$\vec{E} = E \vec{e_x} \quad , \quad \vec{D} = \varepsilon_0 \vec{E}$$

$$Q(V) = \Psi(\partial V)$$

$$\varrho A a = \varrho \cdot V = D \cdot A = \varepsilon E \cdot A$$

$$\varrho \cancel{A} a = \varepsilon E \cancel{A}$$

$$E = \frac{a}{\varepsilon} \varrho \quad \dots \quad a < x < a+b$$

$$\varrho \cancel{A} \cdot x = \varepsilon E \cdot \cancel{A}$$

$$E = \frac{x}{\varepsilon} \varrho \quad \dots \quad 0 < x < a$$

$$E(-[x-(2a+b)]) \quad \dots \quad a+b < x < 2a+b$$

A. 3.3.4. Ladungsrelaxation in einem Stab

$$Q' = Q_0' e^{-\frac{t}{\tau_R}}$$ aus Relaxationsgleichung

$$\vec{J}(x,t) = J \vec{e_x}$$

$$\dot{Q}(V) = -I(\partial V)$$ ... Satz von der Erh. d. el. Ladung

$$\vec{J} = \frac{I}{A} \vec{e_x}$$

$$Q = Q' \cdot l = Q_0' \, l \, e^{-\frac{t}{\tau_R}}$$ $l \to x$

$$\dot{Q} = -l \, Q_0' \frac{1}{\tau_R} e^{-\frac{t}{\tau_R}}$$

$$\vec{J} = \frac{l}{A} Q_0' \frac{1}{\tau_R} e^{-\frac{t}{\tau_R}} \vec{e_x}$$ $\cancel{\vec{J}(x,t) = \frac{x}{A} \frac{Q_0'}{\tau_R} e^{-\frac{t}{\tau_R}} \vec{e_x}}$

A 3.3.5 Konvektiver Ladungstransport.

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) + \frac{\gamma}{\varepsilon}\rho = 0$$

$\vec{v} = v\vec{e}_x \qquad v = \text{const.}$

$$\cancel{\partial_t \rho}_{\text{zeitunabh.}} + \cancel{\partial_x(\rho\vec{v})}_{\text{so nicht}} + \frac{\gamma}{\varepsilon}\rho = 0$$

$$v\,\partial_x(\rho) + \frac{\gamma}{\varepsilon}\rho = 0$$

$$\ln(\rho) = -\frac{\gamma}{v\varepsilon}x + \bar{c}$$

$$\rho(x) = c\,e^{-\frac{\gamma}{v\varepsilon}x} \qquad c = \rho_0$$

$$\rho(x) = \rho_0\,e^{-\frac{\gamma}{v\varepsilon}x}$$

$$\vec{E} = E(x)\,\vec{e}_x \qquad \vec{D} = D\,\vec{e}_x$$

$$\cancel{\partial_t(\ldots)} + \vec{\nabla}\cdot\Big[\vec{v}\,\underbrace{\vec{\nabla}\cdot(\varepsilon\vec{E})}_{\partial_x(\varepsilon E(x))}\Big] + \vec{\nabla}\cdot(\gamma\vec{E}) = 0$$

$$\vec{\nabla}\cdot\big[v\,\partial_x(\varepsilon E(x))\vec{e}_x\big] + \partial_x(\gamma\vec{E}) = 0$$

$$\partial_x\big(v\,\partial_x(\varepsilon E(x))\big) + \partial_x(\gamma\vec{E}) = 0$$

$$v\varepsilon\,\partial_{xx}E(x) + \gamma\,\partial_x\vec{E}(x) = 0$$

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\partial_x D(x) = \rho_0\,e^{-\frac{\gamma}{v\varepsilon}x}$$

$$\partial_x E(x) = \frac{\rho_0}{\varepsilon}\,e^{-\frac{\gamma}{v\varepsilon}x}$$

$$E(x) = \frac{\rho_0}{\cancel{\varepsilon}}\left(-\frac{v\cancel{\varepsilon}}{\gamma}\,e^{-\frac{\gamma}{v\varepsilon}x}\right) + \bar{c}$$

$$E(0) = -\frac{\rho_0}{\cancel{\varepsilon}}\frac{v\cancel{\varepsilon}}{\gamma} + c = E_0$$

$$c = E_0 + \frac{\rho_0 v}{\gamma} \Rightarrow E(x) = \frac{\rho_0 v}{\gamma}\left(1 - e^{-\frac{\gamma}{v\varepsilon}x}\right) + E_0$$

$$\vec{J} = \gamma E + \rho \vec{v} = \text{... einsetzen ...} = (\gamma E_0 + \rho_0 v)\,\vec{e_x}$$

A.3.4.1. Leiter mit exzentrischer Bohrung

$\vec{J} = J\,\vec{e_z}$

$\vec{J_1} = \frac{I}{a^2\pi}\,\vec{e_z}$

$J = \frac{I}{\pi(a^2-b^2)} = \text{const.}$

$\vec{J_2} = -\frac{I}{b^2\pi}\,\vec{e_z}$

$I = H \cdot \ell$

$$\nabla \times \vec{H} = \vec{J} = J\,\vec{e_z}$$

$$\vec{e_z}\,\frac{1}{\varrho}\left[\partial_\varrho(\varrho H_\alpha) - \partial_\alpha H_\varrho\right] = J\,\vec{e_z}$$

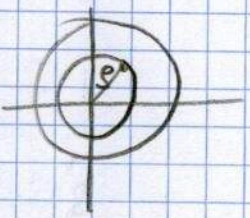
$$\partial_z H_\varrho - \partial_\varrho H_z = 0$$

$$\frac{1}{\varrho}\,\partial_\alpha H_z - \partial_z H_\alpha = 0$$

$H \cdot \cancel{2\pi}\cancel{\varrho} = J \cdot \varrho^{\cancel{2}}\cancel{\pi}$

$H = \frac{J \cdot \varrho}{2}$

$\vec{B} = \mu_0 \frac{J}{2}\varrho\,\vec{e_\alpha}$



$$\vec{B} = \frac{\mu_0 J}{2}\,\varrho\,\vec{e_\alpha} \qquad \vec{e_\alpha} = -\sin(\alpha)\,\vec{e_x} + \cos(\alpha)\,\vec{e_y}$$

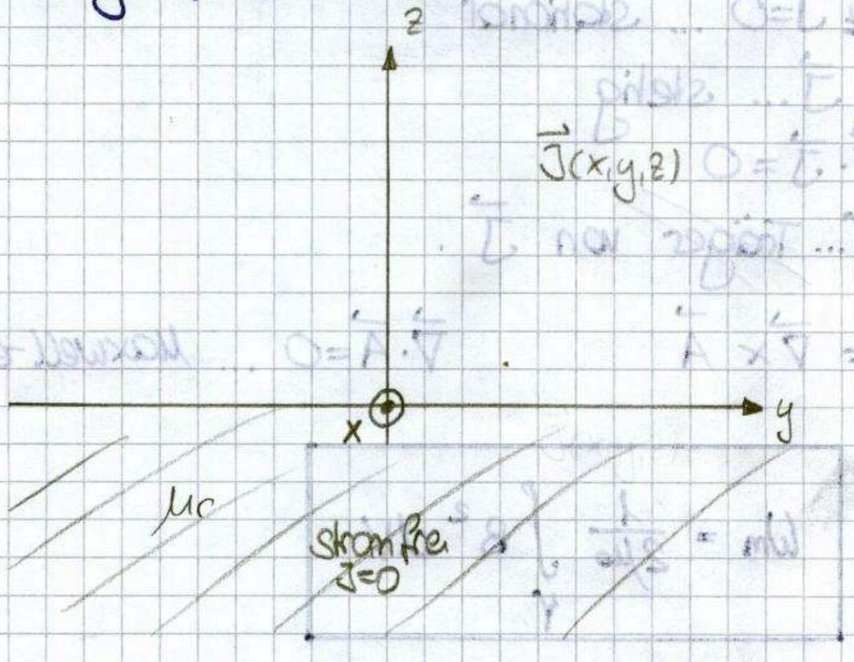
$$\vec{B_1} = \frac{\mu_0 J}{2}\,\varrho\left(-\sin(\alpha)\,\vec{e_x} + \cos(\alpha)\,\vec{e_y}\right) = \frac{\mu_0 J}{2}\left(-y\,\vec{e_x} + x\,\vec{e_y}\right)$$

$$\vec{B_2} = -\frac{\mu_0 J}{2}\left(-y\,\vec{e_x} + (x-c)\,\vec{e_y}\right)$$

$$\vec{B} = \vec{B_1} + \vec{B_2} = \frac{\mu_0 J}{2}\left[(\cancel{-y}\cancel{+y})\,\vec{e_x} + (\cancel{x}-\cancel{x}+c)\,\vec{e_y}\right] =$$

$$= \frac{\mu_0 J}{2}\,c\,\vec{e_y}$$

A. 3.4.3 Erweiterung der Spiegelungsmethode für Magnetfelder



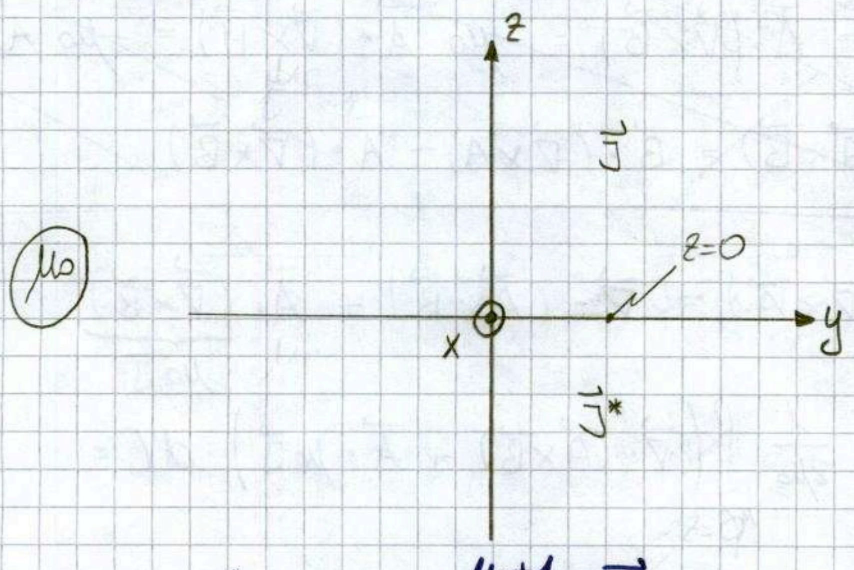
$z>0$:

$$\vec{J}(x,y,z) = J_x(x,y,z)\vec{e}_x + J_y(x,y,z)\vec{e}_y + J_z(x,y,z)\vec{e}_z$$

zusätzlich

$$\vec{J}^*(x,y,z) = \frac{\mu_r - 1}{\mu_r + 1}\left[J_x(x,y,-z)\vec{e}_x + J_y(x,y,-z)\vec{e}_y + J_z(x,y,-z)\vec{e}_z\right]$$

für $z<0$.



$z=0$: $\quad \vec{J}(x,y,z) = \frac{\mu_r + 1}{\mu_r - 1}\,\vec{J}^*(x,y,z)$

A. 3.4.4. Energieinhalt

$\partial_t \vec{J} = 0$ ... stationär

$\vec{J}$ ... stetig

$\vec{\nabla} \cdot \vec{J} = 0$

$\mathcal{V}$ ... ~~Träger~~ von $\vec{J}$.

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\qquad$ $\vec{\nabla} \cdot \vec{A} = 0$ ... Maxwell-Eichung

$$\boxed{W_m = \frac{1}{2\mu_0} \int_{\mathcal{V}} B^2 \, dV}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} \quad | \cdot \mu_0 \Rightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\cancel{B^2 = \vec{B} \cdot \vec{B} = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot \underbrace{(\vec{B} \times \vec{\nabla})}_{-\vec{\nabla} \times \vec{B}} =}$$

$$\cancel{= -\vec{A} \cdot (\vec{\nabla} \times \vec{B}) = -\mu_0 \vec{A} \cdot \underbrace{(\vec{\nabla} \times \vec{H})}_{\vec{J}} = -\mu_0 \vec{A} \cdot \vec{J}}$$

$$\cancel{\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times A) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})}$$

$$B^2 = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot (\vec{A} \times \vec{B}) + \vec{A} \cdot \underbrace{(\vec{\nabla} \times \vec{B})}_{\mu_0 \vec{J}}$$

$$W_m = \frac{1}{2\mu_0} \int_{\mathcal{V}} \left(\vec{\nabla} \cdot (\vec{A} \times \vec{B}) + \vec{A} \cdot \mu_0 \vec{J} \right) dV =$$

$$= \frac{1}{2} \int_{\mathcal{V}} \vec{A} \cdot \vec{J} \, dV + \underbrace{\frac{1}{2\mu_0} \underbrace{\int_{\mathcal{V}} \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \, dV}_{\int_{\partial \mathcal{V}} \vec{n} \cdot (\vec{A} \times \vec{B}) \, dA}}_{R \to \infty \Rightarrow 0}$$

A. 3.4.5. Ebenes Strömungsfeld

$$\partial_t \vec{J} \approx 0$$

$$\vec{J} \cdot \vec{e_z} = 0$$

$$\vec{H} = H(\vec{r})\,\vec{e_z} \qquad \vec{r} \text{... Ortsvektor in xy-Ebene}$$

$$\vec{r} = \vec{r}(x,y)$$

$$\vec{J} = J_x\,\vec{e_x} + J_y\,\vec{e_y}$$

z.z.: $I(\mathcal{A}) = \ell\,\underbrace{[H(\vec{r_2}) - H(\vec{r_1})]}$

$\Rightarrow$ wegunabhängiges Integral mit Stammfunktion H

$$I(\mathcal{A}) = \int_{\mathcal{A}} \vec{n}\cdot\vec{J}\,dA = \int_{\mathcal{A}} \vec{n}\cdot(\vec{\nabla}\times\vec{H})\,dA = \int_{\mathcal{C}} \vec{s}\cdot\vec{H}\,ds =$$

$$J = \vec{\nabla}\times\vec{H}$$

$$= \int_{\mathcal{C}} \vec{s}\cdot\vec{e_z}\,\overbrace{H(r)}^{H(x,y)}\,ds = \int_{r_1}^{\vec{r_2}} H(r)\,dr$$

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$$

A. 3.4.6. Besondere Darstellung der mag. Flußdichte

$\vec{e}$ ... bel. Richtung (konst. Einsvektor)

$$\vec{B} = \vec{\nabla} \times \left[\vec{\nabla} \times (\vec{e}\varphi) + \vec{e}\psi \right]$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{\nabla} \times [\ldots] \right) = 0. \quad \checkmark$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{\nabla} \cdot \vec{A} = 0$$

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \left(\vec{\nabla} \times (\vec{e}\varphi) + \vec{e}\psi \right) = \underbrace{\vec{\nabla} \cdot \left(\vec{\nabla} \times (\vec{e}\varphi) \right)}_{=0} + \vec{\nabla} (\vec{e}\psi) \stackrel{!}{=} 0$$

$$\vec{\nabla} (\vec{e}\psi) = 0$$

$$\vec{e} \cdot \vec{B} =$$

$$\vec{e} \cdot (\vec{\nabla} \times \vec{B}) =$$

A. 3.4.7 Kräftefreies Magnetfeld

$$\vec{B} = B_0 \left[\sin\left(\tfrac{z}{a}\right)\vec{e}_x + \cos\left(\tfrac{z}{a}\right)\vec{e}_y \right] \qquad B_0 = \text{const.},\ a = \text{const.}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{e}_x \left[\partial_y A_z - \cancel{\partial_z A_y}\right] + \vec{e}_y \left[\cancel{\partial_z A_x} - \partial_x A_z\right]$$

$$\partial_x \cancel{A_y} - \partial_y \cancel{A_x} = 0 \quad \checkmark$$

$$\vec{A} = A_z \vec{e}_z \qquad \vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \partial_z A_z = 0$$

$$\partial_y A_z = \sin\left(\tfrac{z}{a}\right) B_0 \qquad -\partial_x A_z = \cos\left(\tfrac{z}{a}\right) B_0$$

$$\Rightarrow A_z = \text{const.}$$

$$A_z = B_0 y \sin\left(\tfrac{z}{a}\right) + C$$

$$\partial_x A_z = \partial_x C \overset{!}{=} -B_0 \cos\left(\tfrac{z}{a}\right)$$

$$\Rightarrow C = -B_0 x \cos\left(\tfrac{z}{a}\right)$$

$$A_z = B_0 y \sin\left(\tfrac{z}{a}\right) - B_0 x \cos\left(\tfrac{z}{a}\right)$$

... nicht Maxwell-geeicht!!!

$$\vec{f} = \vec{J} \times \vec{B} \quad \ldots \text{ Lorentz-Kraftdichte.}$$

A. 3.5.1 Äußerer ebener Quadrupol

$$\vec{B} = \frac{B_0}{a}(x\vec{e}_x - y\vec{e}_y) \qquad \frac{B_0}{a} = \text{const.}$$

(i) $$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{\nabla}\cdot\vec{A} = 0$$

$$\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{e}_x(\partial_y A_z - \partial_z A_y) + \vec{e}_y(\partial_z A_x - \partial_x A_z) + \vec{e}_z(\partial_x A_y - \partial_y A_x) \overset{!}{=} \frac{B_0}{a}(x\vec{e}_x - y\vec{e}_y)$$

$$\vec{\nabla}\cdot\vec{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z = 0$$

$\Rightarrow$
$$\partial_y A_z - \cancel{\partial_z A_y} = \frac{B_0}{a}x$$
$$\cancel{\partial_z A_x} - \partial_x A_z = -\frac{B_0}{a}y$$
$$\partial_x A_y - \partial_y A_x = 0 \;\checkmark \quad \Leftrightarrow \quad \partial_x A_y = \partial_y A_x \qquad 0=0 \checkmark$$
$$(*)\ \partial_x A_x + \partial_y A_y + \partial_z A_z = 0$$

Annahme (f. ebene Magnetfelder): $\vec{A} = A_z\vec{e}_z$

$\Rightarrow$ aus $(*)$ $\quad \partial_z A_z = 0 \quad \Rightarrow \quad A_z = \text{const}(x,y)$

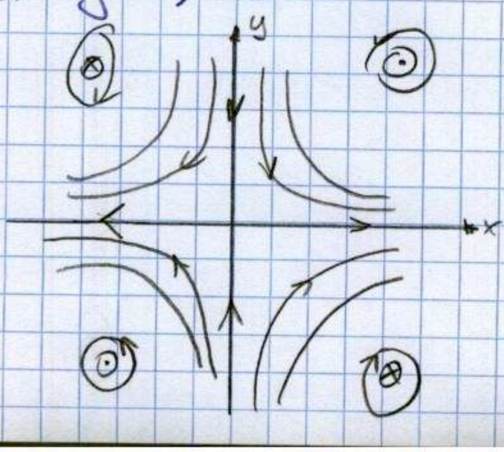
$$A_z = \int \frac{B_0}{a}x\,dy + c(x) = \frac{B_0}{a}xy + c(x)$$

$$\partial_x A_z = \frac{B_0}{a}y + c'(x) \overset{!}{=} \frac{B_0}{a}y$$

$$\Rightarrow c'(x) = 0 \quad \Rightarrow c(x) = c \in \mathbb{R}.$$

$$\vec{A} = \left(\frac{B_0}{a}xy + c\right)\vec{e}_z$$

(ii) & (iii)



A. 3.5.2. Schraubenfeder

$$\vec{A} = \underbrace{K_1 \varrho}_{A_\alpha} \vec{e}_\alpha + \underbrace{K_2 \ln\left(\frac{\varrho}{a}\right)}_{A_z} \vec{e}_z$$

(i)

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{e}_\varrho \left(\frac{1}{\varrho} \partial_\alpha A_z - \partial_z A_\alpha \right) + \vec{e}_\alpha \left(\cancel{\partial_z A_\varrho} - \partial_\varrho A_z \right) + \vec{e}_z \frac{1}{\varrho} \left(\partial_\varrho (\varrho A_\alpha) - \cancel{\partial_\alpha A_\varrho} \right)$$

$$\vec{B} = \vec{e}_\varrho \left(\frac{1}{\varrho} \cancel{\partial_\alpha \left[K_2 \ln\left(\frac{\varrho}{a}\right) \right]} - \cancel{\partial_z \left[K_1 \varrho \right]} \right) + \vec{e}_\alpha \left(- \partial_\varrho \left[K_2 \ln\left(\frac{\varrho}{a}\right) \right] \right) + \vec{e}_z \frac{1}{\varrho} \left(\partial_\varrho \left[\varrho^2 K_1 \right] \right) = - K_2 \frac{\cancel{a}}{\varrho} \cdot \frac{1}{\cancel{a}} \cdot \vec{e}_\alpha + \frac{1}{\cancel{\varrho}} 2 \cancel{\varrho} K_1 \vec{e}_z =$$

$$= - K_2 \frac{1}{\varrho} \vec{e}_\alpha + 2 K_1 \vec{e}_z$$

(ii) Schraublinien

(iii) Linienleiter mit Kreiszylinderspule

A. 3.5.3 Randwertproblem

$\mu_r \to \infty: \quad B = \mu \cdot H \to H = 0 \qquad \vec{J} = \vec{\nabla} \times \vec{H} = \vec{0}$

hochpermeabel

$$\vec{n} \times [\![\vec{H}]\!] = \vec{K}$$

$$[\![\vec{H}]\!] = \vec{H}^{(2)} - \underbrace{\vec{H}^{(1)}}_{=0} = \vec{H}^{(2)} = \vec{H}$$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{A} = A(x,y)\, \vec{e_z}$

$\vec{\nabla} \times \vec{H} = \vec{0} \qquad\qquad \vec{B} = \mu_0 \cdot \vec{H}$

$\Rightarrow \vec{\nabla} \times \vec{B} = \vec{0}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{0}$

$$\left(\begin{array}{l} \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = 0 \\ \vec{\nabla}(\partial_z A(x,y)) - \Delta \vec{A} = 0 \end{array} \right)$$

$(\vec{e_x}\partial_x + \vec{e_y}\partial_y + \cancel{\vec{e_z}\partial_z}) \times (A(x,y)\,\vec{e_z}) = -\partial_x A(x,y)\,\vec{e_y} + \partial_y A(x,y)\,\vec{e_x}$

$(\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z) \times (\partial_y A(x,y)\,\vec{e_x} - \partial_x A(x,y)\,\vec{e_y}) =$

$= -\partial_{xx} A\, \vec{e_z} - \partial_{yy} A\, \vec{e_z} = 0$

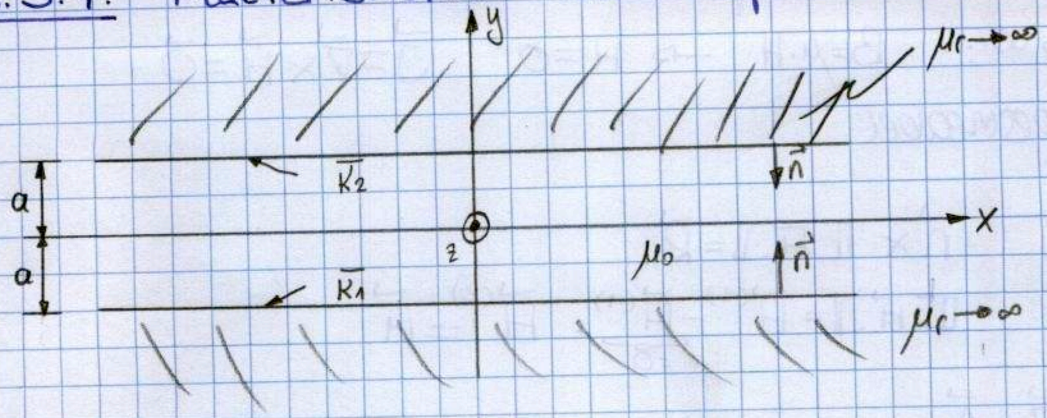
$\partial_{xx} A + \partial_{yy} A = 0$

$\vec{n} \times [\![\vec{B}]\!] = \mu \vec{K}$

$\vec{e_y} \times \vec{B} = \mu \vec{K}$

$-\partial_y A(x,y)\;\vec{e_z} = \mu \vec{K}$

A. 3.5.4. Flächenströme am Spalt



$y=0$: $\vec{B} = B_0 \cos(\omega t - kx)\,\vec{e_y}$

$B_0 = 1{,}0\,T$ $\quad a = 2mm \quad k = 100 m^{-1} \quad \omega = 314 s^{-1}$

$\vec{B} = \mu_0 \vec{H}$ $\qquad \vec{n} \times [\![\vec{H}]\!] = \vec{K}$

$\vec{n} \times [\![\vec{B}]\!] = \mu_0 \vec{K}$

$|y|>a$: $\vec{H} = \vec{0}$ wegen $\mu_r \to \infty$

$\Rightarrow [\![\vec{H}]\!] = \vec{H}$

$\vec{A} = A\,\vec{e_z}$ $\qquad \vec{B} = \vec{\nabla} \times \vec{A} = (\vec{e_x}\partial_x + \vec{e_y}\partial_y + \vec{e_z}\partial_z) \times (A_z \vec{e_z}) =$

$= -\partial_x A_z\,\vec{e_y} + \partial_y A_z\,\vec{e_x}$

$\vec{\nabla} \cdot \vec{A} = \partial_z A_z = 0$

$\partial_y A_z\,\vec{e_x} - \partial_x A_z\,\vec{e_y} = B_0 \cos(\omega t - kx)\,\vec{e_y}$

$\partial_y A_z\big|_{y=0} = 0$

$\partial_x A_z = -B_0 \cos(\omega t - kx)$

$A_z = -\int B_0 \cos(\omega t - kx)\,dx = \frac{B_0}{k} \sin(\omega t - kx)$

Sep.ansatz: $A = A_z \cdot f(y)$ $\qquad \partial_{xx} A + \partial_{yy} A = 0$

$A = \frac{B_0}{k} \cdot \sin(\omega t - kx)\,f(y)$

$\partial_x A = -\frac{B_0}{k} k \cos(\omega t - kx)\,f(y)$ $\qquad \partial_{xx} A = +B_0 \sin(\omega t - kx)(-k) f(y)$

$-\cancel{B_0} k\,\cancel{\sin(\omega t - kx)}\,f(y) + \frac{\cancel{B_0}}{k} \cdot \cancel{\sin(\omega t - kx)}\,f''(y) = 0$

$-k\,f(y) + \frac{1}{k} f''(y) = 0$ $\qquad f''(y) - k^2 f(y) = 0$

$$f''(y) - k^2 f(y) = 0 \qquad f(y) = e^{\lambda y}$$

$$\lambda^2 \cancel{e^{\lambda y}} - k^2 \cancel{e^{\lambda y}} = 0$$

$$\lambda = \pm k$$

$$f(y) = \cancel{c_1}^{\frac{1}{2}} \cdot e^{ky} + \cancel{c_2}^{\frac{1}{2}} e^{-ky} = \frac{e^{ky} + e^{-ky}}{2} = \cosh(ky)$$

$$A = \frac{B_0}{k} \sin(\omega t - kx) \cosh(ky)$$

$$\partial_y A \Big|_{y=0} = \cancel{k} \frac{B_0}{\cancel{k}} \sin(\omega t - kx) \sinh(ky) \Big|_{y=0} = 0$$

$$\partial_x A \Big|_{y=0} = -\frac{B_0}{k} k \cos(\omega t - kx) \cosh(ky) \Big|_{y=0} = -B_0 \cos(\omega t - kx)$$

$$\vec{A} = A \vec{e_z}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\partial_x A \ \vec{e_y} + \partial_y A \ \vec{e_x} =$$

$$= + \frac{B_0}{\cancel{k}} \sin(\omega t - kx) \sinh(ky) \vec{e_x} + \frac{B_0}{\cancel{k}} \cos(\omega t - kx) \cosh(ky) \vec{e_y}$$

$$\vec{n} \times [\![\vec{B}]\!] = \mu_0 \vec{K} = \vec{n} \times \vec{B} = \mu_0 \vec{K}$$

$\sinh(ak) = -\sinh(-ak)$

$y=-a$: $\quad (\vec{e_y} \times \vec{e_x}) B_0 \sin(\omega t - kx) \sinh(ky) = -B_0 \sin(\omega t - kx) \sinh(ky) \Big|_{y=-a} \vec{e_z}$

$y=a$: $\quad (-\vec{e_y} \times \vec{e_x})$ ——— " ——— $= B_0 \sin(\omega t - kx) \sinh(ky) \Big|_{y=a} \vec{e_z}$

$$\underline{\underline{\vec{K} = \frac{B_0}{\mu_0} \sin(\omega t - kx) \sinh(ak) \vec{e_z}}}$$

A.3.5.6. Drehsymmetrisches Strömungsfeld

$$\vec{J} = \frac{2I_0}{\pi} \frac{\varrho a}{(a^2+z^2)^3} \left[\varrho z \vec{e}_\varrho + \frac{3}{4}(a^2+z^2)\vec{e}_z \right] \qquad I_0, a = \text{const.}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}$$

$$\vec{\nabla} \times \vec{B} = \vec{e}_\varrho \left(\cancel{\frac{1}{\varrho}\partial_\alpha B_z} - \partial_z B_\alpha \right) + \vec{e}_\alpha \left(\cancel{\partial_z B_\varrho - \partial_\varrho B_z} \right) + \vec{e}_z \frac{1}{\varrho}\left[\partial_\varrho(\varrho B_\alpha) - \cancel{\partial_\alpha B_\varrho} \right]$$

$$\cancel{\frac{1}{\varrho}\partial_\alpha B_z = \frac{2I_0}{\pi} \frac{\varrho a}{(a^2+z^2)^3} \varrho z}$$

$$\cancel{-\partial_\alpha B_\varrho = \frac{2I_0}{\pi} \frac{\varrho a}{(a^2+z^2)^3} \frac{3}{4}(a^2+z^2)}$$

$$\vec{B} = B_\alpha \vec{e}_\alpha$$

$$\Rightarrow \quad +\partial_z B_\alpha = -\frac{2I_0}{\pi} \frac{\varrho a}{(a^2+z^2)^3} \varrho z$$

$$\frac{1}{\varrho}\partial_\varrho(\varrho B_\alpha) = \frac{\cancel{2}I_0}{\pi} \frac{\varrho a}{(a^2+z^2)^{\cancel{3}2}} \frac{3}{\cancel{4}2} \cancel{(a^2+z^2)}$$

$$B_\alpha = -\frac{2I_0}{\pi} \varrho^2 a \underbrace{\int \frac{z}{(a^2+z^2)^3}\, dz}_{-\frac{1}{4(z^2+a^2)^2}} = +\frac{1}{2\cancel{4}(z^2+a^2)^2} \frac{\cancel{2}I_0}{\pi} \varrho^2 a$$

$$B_\alpha = \frac{I_0}{\pi} \frac{\varrho^2 a}{2(z^2+a^2)^2} + c(\varrho)$$

$$\partial_\varrho \left[\frac{I_0}{\pi} \frac{\varrho^3 a}{2(z^2+a^2)^2} + \varrho c(\varrho) \right] = \frac{I_0}{\pi} \frac{3\varrho^2 a}{2(z^2+a^2)^2} + (\varrho c(\varrho))'$$

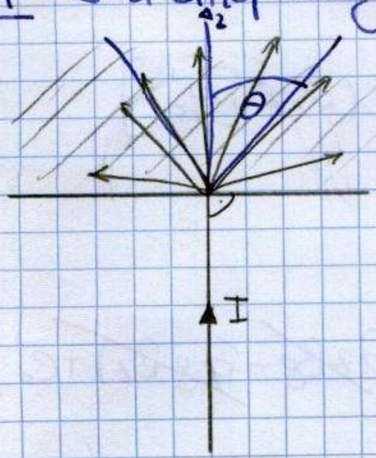
$$\frac{1}{\varrho}\left(\text{———} \;\text{''}\; \text{———} \right) = \frac{I_0}{\pi} \frac{3\varrho a}{2(z^2+a^2)^2} + \frac{(\varrho c(\varrho))'}{\varrho}$$

$$\Rightarrow \quad \frac{(\varrho c(\varrho))'}{\varrho} = 0 \quad \Leftrightarrow \quad (\varrho c(\varrho))' = 0 \quad \Leftrightarrow \quad \varrho c(\varrho) = c$$

$$c(\varrho) = \frac{c}{\varrho}$$

$$\underline{\underline{\frac{1}{\mu} B_\alpha = \frac{I_0}{2\pi} \frac{\varrho^2 a}{(z^2+a^2)^2} + \frac{c}{\varrho}}} \qquad (c \in \mathbb{R})$$

A 3.5.7 Stromeinspeisung



$$\vec{J} = \frac{I}{2r^2\pi}\,\vec{e_r}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

$$\vec{\nabla}\times\vec{H} = \vec{e_r}\,\frac{\partial_\theta[\sin[\theta]\,H_\alpha] - \cancel{\partial_\alpha H_\theta}}{r\sin(\theta)} \qquad \vec{H} = H_\alpha\,\vec{e_\alpha}$$

$$\partial_\theta[\sin(\theta)H_\alpha] = \frac{I}{2r\pi}\sin(\theta)$$

$$\sin(\theta)H_\alpha = -\frac{I}{2r\pi}\cos(\theta)$$

$$H_\alpha = -\frac{I}{2r\pi}\cot(\theta)$$

A.3.5.8 Homogen magnetisierte Kugel

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$\vec{H} = -\vec{\nabla}\varphi$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{B} = \mu_0(\vec{H} + \vec{M})$ $\qquad$ $\vec{M} = \text{const.}$

$$\Delta\varphi = 0$$

$$\vec{M} = M\vec{e_z}$$

A.5.1.1. Poynting-Satz für Sinusfelder

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \qquad \vec{B} = \mu \vec{H}$$
$$\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{J} \qquad \vec{D} = \varepsilon \vec{E}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \mathrm{Re}\left[\vec{\underline{E}}(\vec{r})\, e^{j\omega t}\right] + \mu\, \partial_t \mathrm{Re}\left[\vec{\underline{H}}(\vec{r})\, e^{j\omega t}\right] = 0$$

A 5.1.2. Einschalten eines Punktdipoles

$$\vec{\Pi} = \frac{p(t-\frac{r}{c_0})}{4\pi\epsilon_0 r}\,\vec{e_z}$$

$$\vec{A} - \frac{1}{c_0^2}\partial_t \vec{\Pi} \qquad \varphi = -\vec{\nabla}\cdot\vec{\Pi}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{A} = \frac{1}{c_0^2}\,\frac{\partial_t p(t-\frac{r}{c_0})}{4\pi\epsilon_0 r}\,\vec{e_z}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \qquad \text{lt. Lsg.}$$

A 5.1.3. Strahlungsleistung einer Dipolantenne

$$\vec{E} = c_0 \vec{B} \times \vec{e_r} = \frac{U}{r} \sin(\theta) \cos(\omega t - kr) \vec{e_\theta} \qquad kr \gg 1$$

$$\vec{E} = c_0 \mu \vec{H} \times \vec{e_r} = \text{———} \; " \; \text{———}$$

$$\vec{B} = \frac{U}{c_0 r} \sin(\theta) \cos(\omega t - kr) \vec{e_\alpha}$$

$$\omega t - kr = 0 + 2\pi n,\ n \in \mathbb{Z} \quad \cos(\omega t - kr) = 1$$

$$\omega t = kr + 2\pi n$$

$$t = \frac{kr + 2\pi n \omega}{\omega}$$

siehe Lösungen