

Ejercicios Integrales

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1 Integrales con trucos algebraicos

1.1 $\int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}} dx$

$$\begin{aligned}
 \int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}} dx &= \int \frac{x^{\frac{3}{5}} + x^{\frac{1}{6}}}{x^{\frac{1}{2}}} dx \\
 &= \int x^{\frac{3}{5} - \frac{1}{2}} dx + \int x^{\frac{1}{6} - \frac{1}{2}} dx \\
 &= \int x^{\frac{1}{10}} dx + \int x^{-\frac{2}{6}} dx \\
 &= \int x^{\frac{1}{10}} dx + \int x^{-\frac{1}{3}} dx \\
 &= \frac{10}{11} x^{\frac{11}{10}} + \frac{3}{2} x^{\frac{2}{3}}
 \end{aligned}$$

1.2 $\int \frac{1}{\sqrt{x-1} + \sqrt{x+1}} dx$

$$\begin{aligned}
 \int \frac{1}{\sqrt{x-1} + \sqrt{x+1}} dx &= \int \frac{1}{\sqrt{x-1} + \sqrt{x+1}} \cdot \frac{\sqrt{x-1} - \sqrt{x+1}}{\sqrt{x-1} - \sqrt{x+1}} dx \\
 &= \int \frac{\sqrt{x-1} - \sqrt{x+1}}{x-1-x-1} dx \\
 &= -\frac{1}{2} \int \sqrt{x-1} - \sqrt{x+1} dx \\
 &= -\frac{1}{2} \int (x-1)^{\frac{1}{2}} dx + \frac{1}{2} \int (x+1)^{\frac{1}{2}} dx \\
 &= -\frac{1}{2} (x-1)^{\frac{3}{2}} \cdot \frac{2}{3} + \frac{1}{2} (x+1)^{\frac{3}{2}} \cdot \frac{2}{3} \\
 &= \frac{\sqrt{(x+1)^3} - \sqrt{(x-1)^3}}{3}
 \end{aligned}$$

$$\mathbf{1.3} \quad \int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}} dx$$

$$\begin{aligned} \int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}} dx &= \int e^{x-4x} dx + \int e^{2x-4x} dx + \int e^{3x-4x} dx \\ &= \int e^{-3x} dx + \int e^{-2x} dx + \int e^{-x} dx \\ &= -\frac{1}{3}e^{-3x} - \frac{1}{2}e^{-2x} - e^{-x} \end{aligned}$$

$$\mathbf{1.4} \quad \int \frac{a^x}{b^x} dx$$

$$\begin{aligned} \int \frac{a^x}{b^x} dx &= \int \frac{e^{x \ln(a)}}{e^{x \ln(b)}} \\ &= \int e^{x \ln(a) - x \ln(b)} dx \\ &= \int e^{x(\ln(a) - \ln(b))} dx \\ &= \int e^{x \ln(\frac{a}{b})} dx \\ &= \int e^{x \ln(\frac{a}{b})} \cdot \frac{\ln(\frac{a}{b})}{\ln(\frac{a}{b})} dx \\ &= \frac{1}{\ln(\frac{a}{b})} \int e^{x \ln(\frac{a}{b})} \ln(\frac{a}{b}) dx \\ &= \frac{1}{\ln(\frac{a}{b})} e^{x \ln(\frac{a}{b})} \\ &= \frac{1}{\ln(\frac{a}{b})} \cdot \frac{a^x}{b^x} \end{aligned}$$

$$\mathbf{1.5} \quad \int \tan^2 x dx$$

$$\begin{aligned} \int \tan^2 x dx &= \int \frac{\sin^2 x}{\cos^2 x} dx \\ &= \int \frac{1 - \cos^2 x}{\cos^2 x} dx \\ &= \int \frac{1}{\cos^2 x} - 1 dx \\ &= \int \sec^2 x dx - x \\ &= \tan x - x \end{aligned}$$

1.6 $\int \frac{1}{a^2+x^2} dx$

$$\begin{aligned}\int \frac{1}{a^2+x^2} dx &= \int \frac{1}{a^2(1+\frac{x^2}{a^2})} dx \\ &= \frac{1}{a^2} \int \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{a}{a} dx \\ &= \frac{1}{a} \int \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a} dx \\ &= \frac{1}{a} \int \frac{1}{1+u^2} du \\ &= \frac{1}{a} \cdot \tan^{-1}\left(\frac{x}{a}\right)\end{aligned}$$

1.7 $\int \frac{1}{\sqrt{a^2-x^2}}$

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}} &= \int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}} dx \\ &= \int \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{a} dx \\ &= \int \frac{1}{\sqrt{1-u^2}} du \\ &= \sin^{-1}\left(\frac{x}{a}\right)\end{aligned}$$

1.8 $\int \frac{1}{1+\sin x} dx$

$$\begin{aligned}\int \frac{1}{1+\sin x} dx &= \int \frac{1-\sin x}{\cos^2 x} dx \\ &= \int \sec^2(x) dx - \int \sec(x) \cdot \tan(x) dx \\ &= \tan x - \sec x\end{aligned}$$

1.9 $\int \frac{8x^2+6x+4}{x+1} dx$

$$\begin{aligned}\int \frac{8x^2+6x+4}{x+1} &= \int 8x - 2 + \frac{6}{x+1} dx \\ &= \int 8x dx - 2 \int 1 dx + 6 \int \frac{1}{x+1} dx \\ &= 4x^2 - 2x + 6 \ln(x+1)\end{aligned}$$

1.10 $\int \frac{1}{\sqrt{2x-x^2}} dx$

$$\begin{aligned}\int \frac{1}{\sqrt{2x-x^2}} dx &= \int \frac{1}{\sqrt{1-(x-1)^2}} dx \\ &= \sin^{-1}(x-1)\end{aligned}$$

2 Reducciones

2.1 $\int \ln^n(x) dx$

$$\begin{aligned}\int \ln(x) dx &= x \ln(x) - x \\ \int \ln^n(x) dx &= x \ln^n(x) - n \int \ln^{n-1}(x) dx \\ \int \ln^n(x) dx &= \sum_{i=0}^n (-1)^i \frac{n!}{(n-i)!} x \ln^{n-i}(x)\end{aligned}$$

2.1.1 Demostracion Primera Igualdad

$$\begin{aligned}\int \ln(x) dx &= \int 1 \cdot \ln(x) dx \\ &= x \cdot \ln(x) - \int x \cdot \frac{1}{x} dx \\ &= x \cdot \ln(x) - \int 1 dx \\ &= x \cdot \ln(x) - x\end{aligned}$$

2.1.2 Demostracion Segunda Igualdad

$$\begin{aligned}\int \ln^n(x) dx &= \int 1 \cdot \ln^n(x) dx \\ &= x \cdot \ln^n(x) - \int x \cdot n \ln^{n-1}(x) \cdot \frac{1}{x} dx \\ &= x \cdot \ln^n(x) - \int n \ln^{n-1}(x) dx \\ &= x \cdot \ln^n(x) - n \int \ln^{n-1}(x) dx\end{aligned}$$

2.1.3 Demostracion Tercera Igualdad

Caso Base

$$\begin{aligned}\sum_{i=0}^1 (-1)^i \frac{1!}{(1-i)!} x \ln^{1-i}(x) &= x \ln(x) - x \\ &= \int \ln(x) dx\end{aligned}$$

Induccion

$$\begin{aligned}
\sum_{i=0}^{n+1} (-1)^i \frac{(n+1)!}{(n+1-i)!} x \ln^{n+1-i}(x) &= x \ln^{n+1}(x) + \sum_{i=1}^{n+1} (-1)^i \frac{(n+1)!}{(n+1-i)!} x \ln^{n+1-i}(x) \\
&= x \ln^{n+1}(x) + (n+1) \sum_{i=1}^{n+1} (-1)^i \frac{n!}{(n+1-i)!} x \ln^{n+1-i}(x) \\
&= x \ln^{n+1}(x) - (n+1) \sum_{i=0}^n (-1)^i \frac{n!}{(n+1-(i+1))!} x \ln^{n+1-(i+1)}(x) \\
&= x \ln^{n+1}(x) - (n+1) \sum_{i=0}^n (-1)^i \frac{n!}{(n-i)!} x \ln^{n-i}(x) \\
&= x \ln^{n+1}(x) - (n+1) \int \ln^n(x) dx \\
&= x \ln^{n+1}(x) - \int (n+1) \ln^n(x) x \frac{1}{x} dx \\
&= \int 1 \cdot \ln^{n+1}(x) dx \\
&= \int \ln^{n+1}(x) dx
\end{aligned}$$

2.2 $\int x^n e^x dx$

$$\begin{aligned}
\int x e^x dx &= x e^x - e^x \\
\int x^n e^x dx &= x^n e^x - n \int x^{n-1} e^x dx \\
\int x^n e^x dx &= \sum_{i=0}^n (-1)^i \frac{n!}{(n-i)!} x^{n-i} e^x
\end{aligned}$$

2.2.1 Demostracion Primera Igualdad

$$\begin{aligned}
\int x e^x dx &= x e^x - \int 1 \cdot e^x dx \\
&= x e^x - e^x
\end{aligned}$$

2.2.2 Desmotracion Segunda Igualdad

$$\begin{aligned}
\int x^n e^x dx &= x^n e^x - \int n x^{n-1} e^x dx \\
&= x^n e^x - n \int x^{n-1} e^x dx
\end{aligned}$$

2.2.3 Demostracion Tercera Igualdad

Caso Base

$$\begin{aligned}\sum_{i=0}^1 (-1)^i \frac{1!}{(1-i)!} x^{1-i} e^x &= x e^x - e^x \\ &= \int x e^x\end{aligned}$$

Induccion

$$\begin{aligned}\sum_{i=0}^{n+1} (-1)^i \frac{(n+1)!}{(n+1-i)!} x^{n+1-i} e^x &= x^{n+1} e^x + (n+1) \sum_{i=1}^{n+1} (-1)^i \frac{n!}{(n+1-i)!} x^{n+1-i} e^x \\ &= x^{n+1} e^x + (n+1) \sum_{j=0}^n (-1)^{j+1} \frac{n!}{(n-j)!} x^{n-j} e^x \\ &= x^{n+1} e^x - (n+1) \sum_{j=0}^n (-1)^j \frac{n!}{(n-j)!} x^{n-j} e^x \\ &= x^{n+1} e^x - (n+1) \int x^n e^x dx \\ &= \int x^{n+1} e^x dx\end{aligned}$$

3 Sustituciones

3.1 $\int e^x \sin(e^x) dx$

$$\int e^x \sin(e^x) dx = -\cos(e^x)$$

3.2 $\int x e^{-x^2} dx$

$$\begin{aligned}\int x e^{-x^2} dx &= -\frac{1}{2} \int -2x e^{-x^2} dx \\ &= -\frac{1}{2} \int e^u dx \\ &= -\frac{1}{2} e^{-x^2}\end{aligned}$$

3.3 $\int \frac{\ln x}{x} dx$

$$\begin{aligned}\int \frac{\ln x}{x} dx &= \int \ln(x) \cdot \frac{1}{x} dx \\ &= \int u du \\ &= \frac{1}{2} \ln^2(x)\end{aligned}$$

$$\mathbf{3.4} \quad \int \frac{e^x}{e^{2x} + 2e^x + 1} dx$$

$$\begin{aligned} \int \frac{e^x}{e^{2x} + 2e^x + 1} dx &= \int \frac{1}{u^2 + 2u + 1} du \\ &= \int \frac{1}{(u + 1)^2} du \\ &= \int (u + 1)^{-2} du \\ &= -(e^x + 1)^{-1} \end{aligned}$$

$$\mathbf{3.5} \quad \int e^{e^x} e^x dx$$

$$\begin{aligned} \int e^{e^x} e^x dx &= \int e^u du \\ &= e^{e^x} \end{aligned}$$

$$\mathbf{3.6} \quad \int \frac{x}{\sqrt{1-x^4}} dx$$

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^4}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{1}{2} \sin^{-1}(x^2) \end{aligned}$$

$$\mathbf{3.7} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\begin{aligned} \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= 2 \int e^u du \\ &= 2e^{\sqrt{x}} \end{aligned}$$

$$\mathbf{3.8} \quad \int x\sqrt{1-x^2} dx$$

$$\begin{aligned} \int x\sqrt{1-x^2} dx &= -\frac{1}{2} \int -2x\sqrt{1-x^2} dx \\ &= -\frac{1}{2} \int \sqrt{u+1} du \\ &= -\frac{1}{2} \cdot \frac{2}{3} \cdot (1-x^2)^{\frac{3}{2}} \\ &= -\frac{1}{3} \cdot (1-x^2)^{\frac{3}{2}} \end{aligned}$$

$$\mathbf{3.9} \quad \int \ln(\cos x) \tan(x) dx$$

$$\begin{aligned} \int \ln(\cos x) \tan(x) dx &= - \int \ln(\cos x) \cdot -\sin(x) \cdot \frac{1}{\cos x} \\ &= - \int \ln(u) \cdot \frac{1}{u} du \\ &= -\frac{1}{2} \ln^2(\cos x) \end{aligned}$$

$$\mathbf{3.10} \quad \int \frac{\ln(\ln x)}{x \ln x}$$

$$\begin{aligned} \int \frac{\ln(\ln x)}{x \ln x} &= \int \frac{\ln(\ln x)}{\ln x} \cdot \frac{1}{x} \\ &= \int \frac{\ln u}{u} du \\ &= \frac{1}{2} \ln^2(\ln(x)) \end{aligned}$$

$$\mathbf{3.11} \quad \int_2^{\sqrt{2}} \frac{1}{x^3\sqrt{x^2-1}} dx$$

$$\begin{aligned}
\int_2^{\sqrt{2}} \frac{1}{x^3\sqrt{x^2-1}} dx &= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{1}{\sec^3(u)\sqrt{\sec^2(u)-1}} \sec(u) \cdot \tan(u) du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{\tan(u)}{\sec^2(u)\sqrt{\frac{1}{\cos^2(u)}-1}} du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{\tan(u)}{\sec^2(u)\sqrt{\frac{1-\cos^2(u)}{\cos^2(u)}}} du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{\tan(u)}{\sec^2(u)\sqrt{\frac{\sin^2(u)}{\cos^2(u)}}} du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{\tan(u)}{\sec^2(u)|\tan(u)|} du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{1}{\sec^2(u)} du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \cos^2(u) du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt{2})} \frac{1+\cos(2u)}{2} du \\
&= \frac{1}{2}(\sec^{-1}(\sqrt{2}) - \sec^{-1}(2)) + \frac{1}{2} \int_{2\sec^{-1}(2)}^{2\sec^{-1}(\sqrt{2})} \cos(u) du \\
&= \frac{1}{2}(\sec^{-1}(\sqrt{2}) - \sec^{-1}(2)) + \frac{1}{2} \sin(2\sec^{-1}(\sqrt{2})) - \frac{1}{2} \sin(2\sec^{-1}(2)) \\
&= \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{3} + \frac{1}{2} \sin\left(\frac{\pi}{2}\right) - \frac{1}{2} \sin\left(\frac{2\pi}{3}\right)\right) \\
&= \frac{1}{2}\left(-\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}\right) \\
&= \frac{1}{2} \cdot \frac{6 - (\pi + 3\sqrt{3})}{12} \\
&= \frac{6 - (\pi + 3\sqrt{3})}{24} \\
&\approx -0.097
\end{aligned}$$

3.12 Area circulo de radio r

$$\begin{aligned} 2 \int_{-r}^r \sqrt{r^2 - x^2} dx &= 2 \int_{-r}^r \sqrt{r^2 \left(1 - \frac{x^2}{r^2}\right)} dx \\ &= 2 \int_{-r}^r r \sqrt{1 - \frac{x^2}{r^2}} dx \\ &= 2r \int_{-r}^r \sqrt{1 - \frac{x^2}{r^2}} dx \\ &= 2r \int_{-r}^r \sqrt{1 - \frac{x^2}{r^2}} \cdot \frac{r}{r} dx \\ &= 2r^2 \int_{-r}^r \sqrt{1 - \frac{x^2}{r^2}} \cdot \frac{1}{r} dx \\ &= 2r^2 \int_{-1}^1 \sqrt{1 - x^2} dx \\ &= \pi r^2 \end{aligned}$$

4 Por Partes

4.1 Expresar $\int \ln(\ln x) dx$ en terminos de $\int \frac{1}{\ln(x)} dx$

$$\begin{aligned} \int \ln(\ln x) dx &= \int \ln(\ln x) \cdot \frac{x}{x} dx \\ &= \int \ln(u) e^u du \\ &= e^u \ln(u) - \int \frac{1}{u} e^u du \\ &= x \ln(\ln(x)) - \int \frac{1}{\ln x} \cdot x \cdot \frac{1}{x} dx \\ &= x \ln(\ln(x)) - \int \frac{1}{\ln x} dx \end{aligned}$$

5 Integrales por definicion

5.1 $\int_0^b x^3 dx$

Sea $t_i = \frac{ib}{n}$ para cada i, notar que esto es una particion de $[0, b]$

5.1.1 Suma Inferior

$$m_i = \inf\{x^3 : t_{i-1} \leq x \leq t_i\}$$

Sabemos que x^3 es una funcion creciente en $[0, \infty)$ por lo tanto $m_i = \frac{(i-1)^3 b^3}{n^3}$.

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^n m_i \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^n \frac{(i-1)^3 b^3}{n^3} \cdot \left(\frac{ib}{n} - \frac{(i-1)b}{n} \right) \\
&= \sum_{i=1}^n \frac{(i-1)^3 b^3}{n^3} \cdot \frac{b}{n} \\
&= \frac{b^4}{n^4} \cdot \sum_{i=1}^n (i-1)^3 \\
&= \frac{b^4}{n^4} \cdot \sum_{i=0}^{n-1} i^3 \\
&= \frac{b^4}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \\
&= \frac{b^4}{4n^2} \cdot (n-1)^2 \\
&= \frac{b^4}{4n^2} \cdot (n^2 - 2n + 1) \\
&= \frac{b^4}{4} - \frac{b^4}{2n} + \frac{b^4}{4n^2}
\end{aligned}$$

5.1.2 Suma Superior

$$M_i = \sup\{x^3 : t_{i-1} \leq x \leq t_i\}$$

Sabemos que x^3 es una función creciente en $[0, \infty)$ por lo tanto $M_i = \frac{i^3 b^3}{n^3}$.

$$\begin{aligned}
U(f, P) &= \sum_{i=1}^n M_i \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^n \frac{i^3 b^3}{n^3} \cdot \left(\frac{ib}{n} - \frac{(i-1)b}{n} \right) \\
&= \sum_{i=1}^n \frac{i^3 b^3}{n^3} \cdot \frac{b}{n} \\
&= \frac{b^4}{n^4} \cdot \sum_{i=1}^n i^3 \\
&= \frac{b^4}{n^4} \cdot \sum_{i=1}^n i^3 \\
&= \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= \frac{b^4}{4n^2} \cdot (n+1)^2 \\
&= \frac{b^4}{4n^2} \cdot (n^2 + 2n + 1) \\
&= \frac{b^4}{4} + \frac{b^4}{2n} + \frac{b^4}{4n^2}
\end{aligned}$$

5.1.3 Demostracion Integrabilidad

Uno de los criterios de integrabilidad para la integral de darboux es que $U(f, P) - L(f, P) \leq \varepsilon$, es decir que para todo ε existe una particion P tal que eso se cumpla.

$$\begin{aligned}
U(f, P) - L(f, P) &\leq \varepsilon \\
\Rightarrow \frac{b^4}{4} + \frac{b^4}{2n} + \frac{b^4}{4n^2} - \left(\frac{b^4}{4} - \frac{b^4}{2n} + \frac{b^4}{4n^2} \right) &\leq \varepsilon \\
\Rightarrow 2 \cdot \frac{b^4}{2n} &\leq \varepsilon \\
\Rightarrow \frac{b^4}{n} &\leq \varepsilon
\end{aligned}$$

Lo que significa que es integrable ya que si escogemos una particion lo suficientemente grande n es suficientemente grande para que lo ultimo tienda a 0.

5.1.4 Resultado

Primero demostraremos que $L(f, P) \leq \frac{b^4}{4}$

$$\begin{aligned}
L(f, P) &\leq \frac{b^4}{4} \\
\Rightarrow \frac{b^4}{4} - \frac{b^4}{2n} + \frac{b^4}{4n^2} &\leq \frac{b^4}{4} \\
\Rightarrow \frac{b^4}{4n^2} - \frac{b^4}{2n} &\leq 0 \\
\Rightarrow \frac{b^4 - b^4 2n}{4n^2} &\leq 0 \\
\Rightarrow b^4(1 - 2n) &\leq 0 \\
\Rightarrow 1 - 2n &\leq 0
\end{aligned}$$

Lo cual es cierto ya que n es mayor que 1 para cualquier particion y siempre es positivo.

Por ultimo desmotraremos que $U(f, P) \geq \frac{b^4}{4}$

$$\begin{aligned}
U(f, P) &\geq \frac{b^4}{4} \\
\Rightarrow \frac{b^4}{4} + \frac{b^4}{2n} + \frac{b^4}{4n^2} &\geq \frac{b^4}{4} \\
\Rightarrow \frac{b^4}{4n^2} + \frac{b^4}{2n} &\geq 0 \\
\Rightarrow \frac{b^4 + b^4 2n}{4n^2} &\geq 0 \\
\Rightarrow b^4(1 + 2n) &\geq 0 \\
\Rightarrow 1 + 2n &\geq 0
\end{aligned}$$

Lo cual de nuevo es siempre cierto dado que n es un natural mayor que 0.

Dado que $L(f, P) \leq \frac{b^4}{4} \leq U(f, P)$ y que x^3 es integrable, por definicion, $\int_0^b x^3 dx = \frac{b^4}{4}$