Ejercicios Integrales

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February 6, 2022

Contents

1 Integrales con trucos algebraicos

1.1 $\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}dx$

$$\begin{aligned}
\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}dx &= \int \frac{x^{\frac{3}{5}}+x^{\frac{1}{6}}}{x^{\frac{1}{2}}}dx \\
&= \int x^{\frac{3}{5}-\frac{1}{2}}dx + \int x^{\frac{1}{6}-\frac{1}{2}}dx \\
&= \int x^{\frac{1}{10}}dx + \int x^{-\frac{2}{6}}dx \\
&= \int x^{\frac{1}{10}}dx + \int x^{-\frac{1}{3}}dx \\
&= \frac{10}{11}x^{\frac{11}{10}} + \frac{3}{2}x^{\frac{2}{3}}
\end{aligned}$$

1.2 $\int \frac{1}{\sqrt{x-1}+\sqrt{x+1}}dx$

$$\begin{aligned}
\int \frac{1}{\sqrt{x-1}+\sqrt{x+1}}dx &= \int \frac{1}{\sqrt{x-1}+\sqrt{x+1}} \cdot \frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}dx \\
&= \int \frac{\sqrt{x-1}-\sqrt{x+1}}{x-1-x-1}dx \\
&= -\frac{1}{2}\int \sqrt{x-1}-\sqrt{x+1}dx \\
&= -\frac{1}{2}\int (x-1)^{\frac{1}{2}}dx + \frac{1}{2}\int (x+1)^{\frac{1}{2}}dx \\
&= -\frac{1}{2}(x-1)^{\frac{3}{2}} \cdot \frac{2}{3} + \frac{1}{2}(x+1)^{\frac{3}{2}} \cdot \frac{2}{3} \\
&= \frac{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}}{3}
\end{aligned}$$

1.3 $\int \frac{e^x+e^{2x}+e^{3x}}{e^{4x}}dx$

$$\begin{aligned}
\int \frac{e^x+e^{2x}+e^{3x}}{e^{4x}}dx &= \int e^{x-4x}dx + \int e^{2x-4x}dx + \int e^{3x-4x}dx \\
&= \int e^{-3x}dx + \int e^{-2x}dx + \int e^{-x}dx \\
&= -\frac{1}{3}e^{-3x} - \frac{1}{2}e^{-2x} - e^{-x}
\end{aligned}$$

1.4 $\int \frac{a^x}{b^x}dx$

$$\begin{aligned}
\int \frac{a^x}{b^x}dx &= \int \frac{e^{x\ln(a)}}{e^{x\ln(b)}} \\
&= \int e^{x\ln(a)-x\ln(b)}dx \\
&= \int e^{x(\ln(a)-\ln(b))}dx \\
&= \int e^{x\ln(\frac{a}{b})}dx \\
&= \int e^{x\ln(\frac{a}{b})} \cdot \frac{\ln(\frac{a}{b})}{\ln(\frac{a}{b})}dx \\
&= \frac{1}{\ln(\frac{a}{b})} \int e^{x\ln(\frac{a}{b})} \ln(\frac{a}{b})dx \\
&= \frac{1}{\ln(\frac{a}{b})} e^{x\ln(\frac{a}{b})} \\
&= \frac{1}{\ln(\frac{a}{b})} \cdot \frac{a^x}{b^x}
\end{aligned}$$

1.5 $\int \tan^2 x dx$

$$\begin{aligned}
\int \tan^2 x dx &= \int \frac{\sin^2 x}{\cos^2 x}dx \\
&= \int \frac{1-\cos^2 x}{cos^2 x}dx \\
&= \int \frac{1}{\cos^2 x} - 1dx \\
&= \int \sec^2 x dx - x \\
&= \tan x - x
\end{aligned}$$

1.6 $\int \frac{1}{a^2+x^2} dx$

$$\begin{aligned}
\int \frac{1}{a^2+x^2} dx &= \int \frac{1}{a^2(1+\frac{x^2}{a^2})} dx \\
&= \frac{1}{a^2} \int \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{a}{a} dx \\
&= \frac{1}{a} \int \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a} dx \\
&= \frac{1}{a} \int \frac{1}{1+u^2} du \\
&= \frac{1}{a} \cdot \tan^{-1}(\frac{x}{a})
\end{aligned}$$

1.7 $\int \frac{1}{\sqrt{a^2-x^2}}$

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2-x^2}} &= \int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}} dx \\
&= \int \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{a} dx \\
&= \int \frac{1}{\sqrt{1-u^2}} du \\
&= \sin^{-1}(\frac{x}{a})
\end{aligned}$$

1.8 $\int \frac{1}{1+\sin x} dx$

$$\begin{aligned}
\int \frac{1}{1+\sin x} dx &= \int \frac{1-\sin x}{\cos^2 x} dx \\
&= \int \sec^2(x) dx - \int \sec(x) \cdot \tan(x) dx \\
&= \tan x - \sec x
\end{aligned}$$

1.9 $\int \frac{8x^2+6x+4}{x+1} dx$

$$\begin{aligned}
\int \frac{8x^2+6x+4}{x+1} &= \int 8x - 2 + \frac{6}{x+1} dx \\
&= \int 8x dx - 2 \int 1 dx + 6 \int \frac{1}{x+1} dx \\
&= 4x^2 - 2x + 6\ln(x+1)
\end{aligned}$$

1.10 $\int \frac{1}{\sqrt{2x-x^2}}dx$

$$\begin{aligned}\int \frac{1}{\sqrt{2x-x^2}}dx &= \int \frac{1}{\sqrt{1-(x-1)^2}}dx \\ &= \sin^{-1}(x-1)\end{aligned}$$

2 Reducciones

2.1 $\int \ln^n(x)dx$

$$\int \ln(x)dx = x\ln(x) - x$$
$$\int \ln^n(x)dx = x\ln^n(x) - n\int \ln^{n-1}(x)dx$$
$$\int \ln^n(x)dx = \sum_{i=0}^{n}(-1)^i\frac{n!}{(n-i)!}x\ln^{n-i}(x)$$

2.1.1 Demostracion Primera Igualdad

$$\begin{aligned}\int \ln(x)dx &= \int 1\cdot\ln(x)dx \\ &= x\cdot\ln(x) - \int x\cdot\frac{1}{x}dx \\ &= x\cdot\ln(x) - \int 1dx \\ &= x\cdot\ln(x) - x\end{aligned}$$

2.1.2 Demostracion Segunda Igualdad

$$\begin{aligned}\int \ln^n(x)dx &= \int 1\cdot\ln^n(x)dx \\ &= x\cdot\ln^n(x) - \int x\cdot n\ln^{n-1}(x)\cdot\frac{1}{x}dx \\ &= x\cdot\ln^n(x) - \int n\ln^{n-1}(x)dx \\ &= x\cdot\ln^n(x) - n\int \ln^{n-1}(x)dx\end{aligned}$$

2.1.3 Demostracion Tercera Igualdad

Caso Base

$$\begin{aligned}\sum_{i=0}^{1}(-1)^i\frac{1!}{(1-i)!}x\ln^{n-i}(x) &= x\ln(x) - x \\ &= \int \ln(x)dx\end{aligned}$$

Induccion

$$\begin{aligned}
\sum_{i=0}^{n+1}(-1)^i\frac{(n+1)!}{(n+1-i)!}x\ln^{n+1-i}(x) &= x\ln^{n+1}(x)+\sum_{i=1}^{n+1}(-1)^i\frac{(n+1)!}{(n+1-i)!}x\ln^{n+1-i}(x)\\
&= x\ln^{n+1}(x)+(n+1)\sum_{i=1}^{n+1}(-1)^i\frac{n!}{(n+1-i)!}x\ln^{n+1-i}(x)\\
&= x\ln^{n+1}(x)-(n+1)\sum_{i=0}^{n}(-1)^i\frac{n!}{(n+1-(i+1))!}x\ln^{n+1-(i+1)}(x)\\
&= x\ln^{n+1}(x)-(n+1)\sum_{i=0}^{n}(-1)^i\frac{n!}{(n-i)!}x\ln^{n-i}(x)\\
&= x\ln^{n+1}(x)-(n+1)\int\ln^n(x)dx\\
&= x\ln^{n+1}(x)-\int(n+1)\ln^n(x)x\frac{1}{x}dx\\
&= \int 1\cdot\ln^{n+1}(x)dx\\
&= \int\ln^{n+1}(x)dx
\end{aligned}$$

2.2 $\int x^n e^x dx$

$$\begin{aligned}
\int xe^x dx &= xe^x - e^x\\
\int x^n e^x dx &= x^n e^x - n\int x^{n-1}e^x dx\\
\int x^n e^x dx &= \sum_{i=0}^{n}(-1)^i\frac{n!}{(n-i)!}x^{n-i}e^x
\end{aligned}$$

2.2.1 Demostracion Primera Igualdad

$$\begin{aligned}
\int xe^x dx &= xe^x - \int 1\cdot e^x dx\\
&= xe^x - e^x
\end{aligned}$$

2.2.2 Desmotracion Segunda Igualdad

$$\begin{aligned}
\int x^n e^x dx &= x^n e^x - \int nx^{n-1}e^x dx\\
&= x^n e^x - n\int x^{n-1}e^x dx
\end{aligned}$$

2.2.3 Demostracion Tercera Igualdad

Caso Base

$$\begin{aligned}\sum_{i=0}^{1}(-1)^i\frac{1!}{(1-i)!}x^{1-i}e^x &= xe^x - e^x\\ &= \int xe^x\end{aligned}$$

Induccion

$$\begin{aligned}\sum_{i=0}^{n+1}(-1)^i\frac{(n+1)!}{(n+1-i)!}x^{n+1-i}e^x &= x^{n+1}e^x + (n+1)\sum_{i=1}^{n+1}(-1)^i\frac{n!}{(n+1-i)!}x^{n+1-i}e^x\\ &= x^{n+1}e^x + (n+1)\sum_{j=0}^{n}(-1)^{j+1}\frac{n!}{(n-j)!}x^{n-j}e^x\\ &= x^{n+1}e^x - (n+1)\sum_{j=0}^{n}(-1)^{j}\frac{n!}{(n-j)!}x^{n-j}e^x\\ &= x^{n+1}e^x - (n+1)\int x^n e^x dx\\ &= \int x^{n+1}e^x dx\end{aligned}$$

3 Sustituciones

3.1 $\int e^x \sin(e^x) dx$

$$\int e^x \sin(e^x) dx = -\cos(e^x)$$

3.2 $\int xe^{-x^2} dx$

$$\begin{aligned}\int xe^{-x^2}dx &= -\frac{1}{2}\int -2xe^{-x^2}dx\\ &= -\frac{1}{2}\int e^u dx\\ &= -\frac{1}{2}e^{-x^2}\end{aligned}$$

3.3 $\int \frac{\ln x}{x} dx$

$$\begin{aligned}\int \frac{\ln x}{x}dx &= \int \ln(x)\cdot\frac{1}{x}dx\\ &= \int udu\\ &= \frac{1}{2}\ln^2(x)\end{aligned}$$

3.4 $\int \frac{e^x}{e^{2x}+2e^x+1} dx$

$$\begin{aligned}
\int \frac{e^x}{e^{2x}+2e^x+1} dx &= \int \frac{1}{u^2+2u+1} du \\
&= \int \frac{1}{(u+1)^2} du \\
&= \int (u+1)^{-2} du \\
&= -(e^x+1)^{-1}
\end{aligned}$$

3.5 $\int e^{e^x} e^x dx$

$$\begin{aligned}
\int e^{e^x} e^x dx &= \int e^u du \\
&= e^{e^x}
\end{aligned}$$

3.6 $\int \frac{x}{\sqrt{1-x^4}} dx$

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^4}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du \\
&= \frac{1}{2} \sin^{-1}(x^2)
\end{aligned}$$

3.7 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\begin{aligned}
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= 2 \int e^u du \\
&= 2e^{\sqrt{x}}
\end{aligned}$$

3.8 $\int x\sqrt{1-x^2} dx$

$$\begin{aligned}
\int x\sqrt{1-x^2} dx &= -\frac{1}{2} \int -2x\sqrt{1-x^2} dx \\
&= -\frac{1}{2} \int \sqrt{u+1} du \\
&= -\frac{1}{2} \cdot \frac{2}{3} \cdot (1-x^2)^{\frac{3}{2}} \\
&= -\frac{1}{3} \cdot (1-x^2)^{\frac{3}{2}}
\end{aligned}$$

3.9 $\int \ln(\cos x)\tan(x)dx$

$$\begin{aligned}
\int \ln(\cos x)\tan(x)dx &= -\int \ln(\cos x) \cdot -\sin(x) \cdot \frac{1}{\cos x} \\
&= -\int \ln(u) \cdot \frac{1}{u} du \\
&= -\frac{1}{2}\ln^2(\cos x)
\end{aligned}$$

3.10 $\int \frac{\ln(\ln x)}{x \ln x}$

$$\begin{aligned}
\int \frac{\ln(\ln x)}{x \ln x} &= \int \frac{\ln(\ln x)}{\ln x} \cdot \frac{1}{x} \\
&= \int \frac{\ln u}{u} du \\
&= \frac{1}{2}\ln^2(\ln(x))
\end{aligned}$$

3.11 $\int_2^{\sqrt{2}} \frac{1}{x^3\sqrt{x^2-1}}dx$

$$\begin{aligned}
\int_2^{\sqrt{2}} \frac{1}{x^3\sqrt{x^2-1}}dx &= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{1}{\sec^3(u)\sqrt{\sec^2(u)-1}}\sec(u)\cdot\tan(u)du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{\tan(u)}{\sec^2(u)\sqrt{\frac{1}{\cos^2(u)}-1}}du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{\tan(u)}{\sec^2(u)\sqrt{\frac{1-\cos^2(u)}{\cos^2(u)}}}du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{\tan(u)}{\sec^2(u)\sqrt{\frac{\sin^2(u)}{\cos^2(u)}}}du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{\tan(u)}{\sec^2(u)|\tan(u)|}du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{1}{\sec^2(u)}du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \cos^2(u)du \\
&= \int_{\sec^{-1}(2)}^{\sec^{-1}(\sqrt(2))} \frac{1+\cos(2u)}{2}du \\
&= \frac{1}{2}(\sec^{-1}(\sqrt{2})-\sec^{-1}(2)+\frac{1}{2}\int_{2\sec^{-1}(2)}^{2\sec^{-1}(\sqrt(2))}\cos(u)du) \\
&= \frac{1}{2}(\sec^{-1}(\sqrt{2})-\sec^{-1}(2)+\frac{1}{2}\sin(2\sec^{-1}(\sqrt{2}))-\frac{1}{2}\sin(2\sec^{-1}(2))) \\
&= \frac{1}{2}(\frac{\pi}{4}-\frac{\pi}{3}+\frac{1}{2}\sin(\frac{\pi}{2})-\frac{1}{2}\sin(\frac{2\pi}{3}) \\
&= \frac{1}{2}(-\frac{\pi}{12}+\frac{1}{2}-\frac{\sqrt{3}}{4}) \\
&= \frac{1}{2}\cdot\frac{6-(\pi+3\sqrt{3})}{12} \\
&= \frac{6-(\pi+3\sqrt{3})}{24} \\
&\approx -0.097
\end{aligned}$$

3.12 Area circulo de radio r

$$
\begin{aligned}
2\int_{-r}^{r}\sqrt{r^2-x^2}dx &= 2\int_{-r}^{r}\sqrt{r^2(1-\frac{x^2}{r^2})}dx \\
&= 2\int_{-r}^{r}r\sqrt{1-\frac{x^2}{r^2}}dx \\
&= 2r\int_{-r}^{r}\sqrt{1-\frac{x^2}{r^2}}dx \\
&= 2r\int_{-r}^{r}\sqrt{1-\frac{x^2}{r^2}}\cdot\frac{r}{r}dx \\
&= 2r^2\int_{-r}^{r}\sqrt{1-\frac{x^2}{r^2}}\cdot\frac{1}{r}dx \\
&= 2r^2\int_{-1}^{1}\sqrt{1-x^2}dx \\
&= \pi r^2
\end{aligned}
$$

4 Por Partes

4.1 Expresar $\int \ln(\ln x)dx$ en terminos de $\int \frac{1}{\ln(x)}$ dx

$$
\begin{aligned}
\int \ln(\ln x)dx &= \int \ln(\ln x)\cdot\frac{x}{x}dx \\
&= \int \ln(u)e^u du \\
&= e^u\ln(u) - \int \frac{1}{u}e^u du \\
&= x\ln(\ln(x)) - \int \frac{1}{\ln x}\cdot x\cdot\frac{1}{x}dx \\
&= x\ln(\ln(x)) - \int \frac{1}{\ln x}dx
\end{aligned}
$$

5 Integrales por definicion

5.1 $\int_0^b x^3dx$

Sea $t_i = \frac{ib}{n}$ para cada i, notar que esto es una particion de $[0, b]$

5.1.1 Suma Inferior

$$m_i = \inf\{x^3 : t_{i-1} \leq x \leq t_i\}$$

Sabemos que x^3 es una funcion creciente en $[0, \infty)$ por lo tanto $m_i = \frac{(i-1)^3b^3}{n^3}$.

$$
\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \frac{(i-1)^3 b^3}{n^3} \cdot (\frac{ib}{n} - \frac{(i-1)b}{n}) \\
&= \sum_{i=1}^{n} \frac{(i-1)^3 b^3}{n^3} \cdot \frac{b}{n} \\
&= \frac{b^4}{n^4} \cdot \sum_{i=1}^{n} (i-1)^3 \\
&= \frac{b^4}{n^4} \cdot \sum_{i=0}^{n-1} i^3 \\
&= \frac{b^4}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \\
&= \frac{b^4}{4n^2} \cdot (n-1)^2 \\
&= \frac{b^4}{4n^2} \cdot (n^2 - 2n + 1) \\
&= \frac{b^4}{4} - \frac{b^4}{2n} + \frac{b^4}{4n^2}
\end{aligned}
$$

5.1.2 Suma Superior

$$M_i = \sup\{x^3 : t_{i-1} \leq x \leq t_i\}$$

Sabemos que x^3 es una funcion creciente en $[0, \infty)$ por lo tanto $M_i = \frac{i^3 b^3}{n^3}$.

$$
\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M_i \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \frac{i^3 b^3}{n^3} \cdot (\frac{ib}{n} - \frac{(i-1)b}{n}) \\
&= \sum_{i=1}^{n} \frac{i^3 b^3}{n^3} \cdot \frac{b}{n} \\
&= \frac{b^4}{n^4} \cdot \sum_{i=1}^{n} i^3 \\
&= \frac{b^4}{n^4} \cdot \sum_{i=1}^{n} i^3 \\
&= \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= \frac{b^4}{4n^2} \cdot (n+1)^2 \\
&= \frac{b^4}{4n^2} \cdot (n^2 + 2n + 1) \\
&= \frac{b^4}{4} + \frac{b^4}{2n} + \frac{b^4}{4n^2}
\end{aligned}
$$

5.1.3 Demostracion Integrabilidad

Uno de los criterios de integrabilidad para la integral de darboux es que $U(f,P) - L(f,P) \leq \varepsilon$, es decir que para todo ε existe una particion P tal que eso se cumpla.

$$
\begin{aligned}
U(f,P) - L(f,P) &\leq \varepsilon \\
\implies \frac{b^4}{4} + \frac{b^4}{2n} + \frac{b^4}{4n^2} - (\frac{b^4}{4} - \frac{b^4}{2n} + \frac{b^4}{4n^2}) &\leq \varepsilon \\
\implies 2 \cdot \frac{b^4}{2n} &\leq \varepsilon \\
\implies \frac{b^4}{n} &\leq \varepsilon
\end{aligned}
$$

Lo que significa que es integrable ya que si escogemos una particion lo suficientemente grande n es suficientemente grande para que lo ultimo tienda a 0.

5.1.4 Resultado

Primero demostraremos que $L(f,P) \leq \frac{b^4}{4}$

$$
\begin{aligned}
L(f,P) &\leq \frac{b^4}{4}\\
\implies \frac{b^4}{4} - \frac{b^4}{2n} + \frac{b^4}{4n^2} &\leq \frac{b^4}{4}\\
\implies \frac{b^4}{4n^2} - \frac{b^4}{2n} &\leq 0\\
\implies \frac{b^4 - b^4 2n}{4n^2} &\leq 0\\
\implies b^4(1-2n) &\leq 0\\
\implies 1-2n &\leq 0
\end{aligned}
$$

Lo cual es cierto ya que n es mayor que 1 para cualquier particion y siempre es positivo.

Por ultimo desmotraremos que $U(f,P) \geq \frac{b^4}{4}$

$$
\begin{aligned}
U(f,P) &\geq \frac{b^4}{4}\\
\implies \frac{b^4}{4} + \frac{b^4}{2n} + \frac{b^4}{4n^2} &\geq \frac{b^4}{4}\\
\implies \frac{b^4}{4n^2} + \frac{b^4}{2n} &\geq 0\\
\implies \frac{b^4 + b^4 2n}{4n^2} &\geq 0\\
\implies b^4(1+2n) &\geq 0\\
\implies 1+2n &\geq 0
\end{aligned}
$$

Lo cual de nuevo es siempre cierto dado que n es un natural mayor que 0.

Dado que $L(f,P) \leq \frac{b^4}{4} \leq U(f,P)$ y que x^3 es integrable, por definicion, $\int_0^b x^3 dx = \frac{b^4}{4}$

5.2 $f(x) \geq 0, x \in [a,b] \implies \int_a^b f(x)dx \geq 0$

Sabemos que f es integrable, mas aun sabemos que para toda suma inferior, $L(f,P) \leq \int_a^b f(x)dx$, si consideramos la particion $t_0 = a, t_1 = b$ tenemos que $L(f,P) = \sum_{i=1}^{1} m_i(b-a) = m_i(b-a)$ dado que $m_i = \inf\{f(x) : a \leq x \leq b\}$ y que todo f(x) en ese intervalo es postivo implica que $m_i \geq 0$. por lo tanto $m_i(b-a) \geq 0$.

5.2.1 $f(x) \geq g(x), x \in [a,b] \implies \int_a^b f(x)dx \geq \int_a^b g(x)dx$

consideramos la funcion $h(x) = f(x) - g(x)$, esta funcion es mayor que 0 en $[a,b]$. por el teorema anterior sabemos que $\int_a^b h(x) \geq 0$ por lo tanto $\int_a^b f(x) - g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx \geq 0$. lo que implica que $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.