



COMP 2012H Honors Object-Oriented Programming and Data Structures

Topic 20: AVL Trees

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Motivation

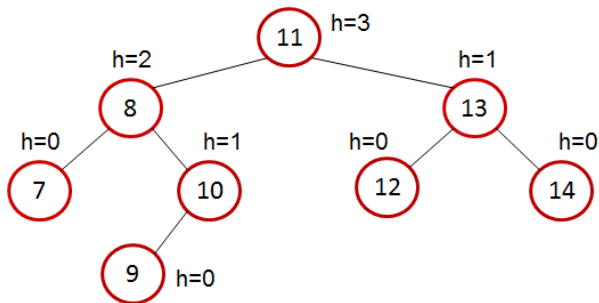
- A **binary search trees** (BST) supports **efficient** searching if it is well **balanced** — its nodes are fairly evenly distributed on both its left and right sub-trees.
- However, this is not always the case as **insertions** and **deletions** of tree nodes will generally make the resulting BST **unbalanced**.
- In the **worst case**, the tree is **de-generated** to a **sorted linked list** and the searching time is **linear time**.

Target: A balanced binary search tree

A BST with N nodes and a height of the order of $\log N$.

AVL (Adelson-Velsky and Landis) Trees

- An **AVL tree** is a **BST** where the **height of the two sub-trees** of **ANY** of its nodes may differ by **at most one**.
- Each node stores a **height** value, which is used to check if the tree is **balanced** or not.



AVL Trees

AVL Tree Properties

Every sub-tree of an AVL tree is itself an AVL tree. (An empty tree is an AVL tree too.)

- With this property, an **AVL tree** is **balanced** and it is guaranteed that its height is **logarithmic** in the number of nodes, N . i.e., order of **$\log(N)$** .
- Efficiency of its following tree operations can always be guaranteed.
 - ▶ **Searching**: order of **$\log(N)$ in the worst case**
 - ▶ **Insertion**: order of **$\log(N)$ in the worst case**
 - ▶ **Deletion**: order of **$\log(N)$ in the worst case**

AVL Tree Implementation I

```
template <typename T>          /* File: avl.h */
class AVL
{
private:
    struct AVLnode
    {
        T value;
        int height;
        AVL left;           // Left subtree is also an AVL object
        AVL right;          // Right subtree is also an AVL object
        AVLnode(const T& x) : value(x), height(0), left(), right() { }
    };

    AVLnode* root = nullptr;

    AVL& right_subtree() { return root->right; }
    AVL& left_subtree() { return root->left; }
    const AVL& right_subtree() const { return root->right; }
    const AVL& left_subtree() const { return root->left; }
```

AVL Tree Implementation II

```
int height() const;           // Find the height of tree
int bfactor() const;          // Find the balance factor of tree
void fix_height() const;      // Rectify the height of each node in tree
void rotate_left();           // Single left or anti-clockwise rotation
void rotate_right();          // Single right or clockwise rotation
void balance();               // AVL tree balancing

public:
    AVL() = default;           // Build an empty AVL tree by default
    ~AVL() { delete root; }    // Will delete the whole tree recursively!

    bool is_empty() const { return root == nullptr; }
    const T& find_min() const;   // Find the minimum value in an AVL
    bool contains(const T& x) const; // Search an item
    void print(int depth = 0) const; // Print by rotating -90 degrees

    void insert(const T& x); // Insert an item in sorted order
    void remove(const T& x); // Remove an item
};
```

AVL Tree Searching

- Searching in AVL trees is the **same** as in BST.

```
// Goal: To search for an item x in an AVL tree
// Return: (bool) true if found, otherwise false
template <typename T>
bool AVL<T>::contains(const T& x) const
{
    if (is_empty())                // Base case #1
        return false;

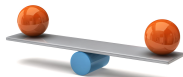
    else if (x == root->value) // Base case #2
        return true;

    else if (x < root->value) // Recursion on the left subtree
        return left_subtree().contains(x);

    else // Recursion on the right subtree
        return right_subtree().contains(x);
}
```

AVL Tree Insertion and Rotation

- To **insert** an item in an AVL tree
 - ▶ **Search** the tree and **locate** the place where the new item should be inserted to.
 - ▶ **Create a new node** with the item and **attach** it to the tree.
- The **insertion** may cause the AVL tree **unbalanced**
⇒ tree balancing by **rotation(s)**
- Types of rotation
 - ▶ **single rotation**
 - ▶ **double rotation** (i.e., two single rotations)



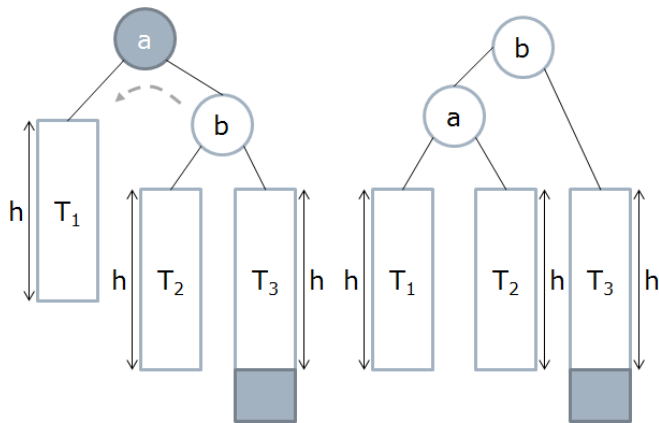
AVL Tree Insertion and Rotation ..

Insertion may violate the AVL tree property in 4 cases:

1. **Left (anti-clockwise) rotation** [single rotation]:
Insertion into the **right sub-tree of the right child** of a node
2. **Right (clockwise) rotation** [single rotation]:
Insertion into the **left sub-tree of the left child** of a node
3. **Left-right rotation** [double rotation]:
Insertion into the **right sub-tree of the left child** of a node
4. **Right-left rotation** [double rotation]:
Insertion into the **left sub-tree of the right child** of a node

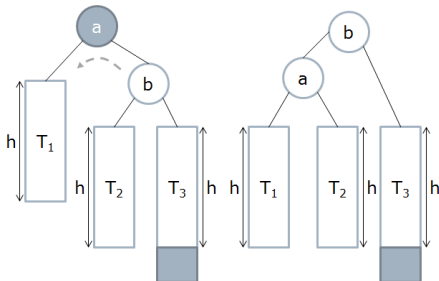
AVL Left (Anti-clockwise) Rotation

Left rotation at node **a**.



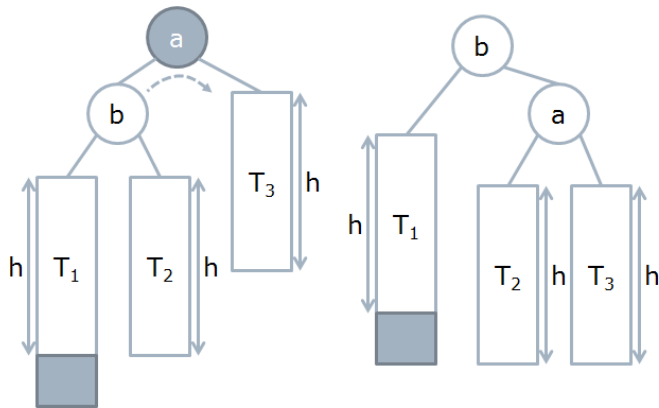
AVL Code: Left Rotation

```
/* Goal: To perform a single left (anti-clockwise) rotation */
template <typename T>
void AVL<T>::rotate_left() // The calling AVL node is node a
{
    AVLnode* b = right_subtree().root; // Points to node b
    right_subtree() = b->left;
    b->left = *this; // Note: *this is node a
    fix_height(); // Fix the height of node a
    this->root = b; // Node b becomes the new root
    fix_height(); // Fix the height of node b, now the new root
}
```



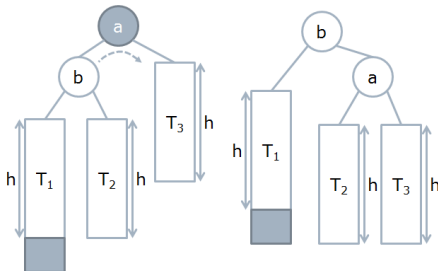
AVL Right (Clockwise) Rotation

Right rotation at node **a**.

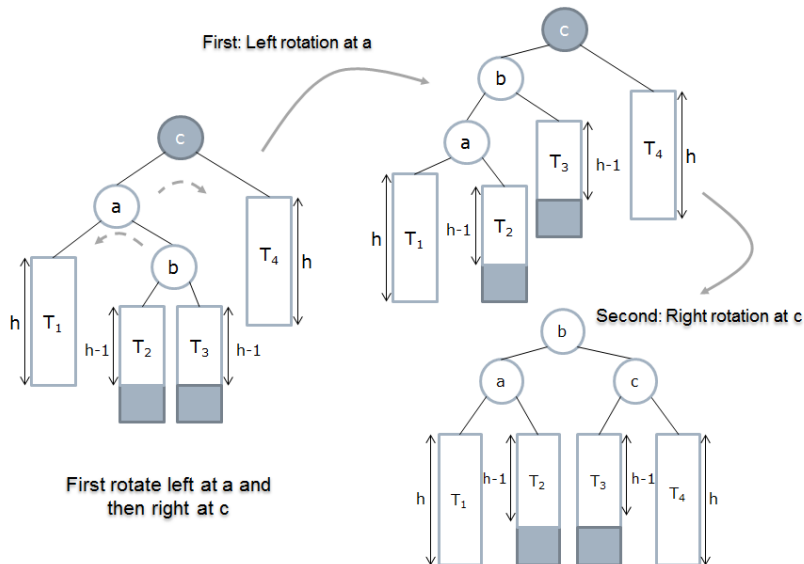


AVL Code: Right Rotation

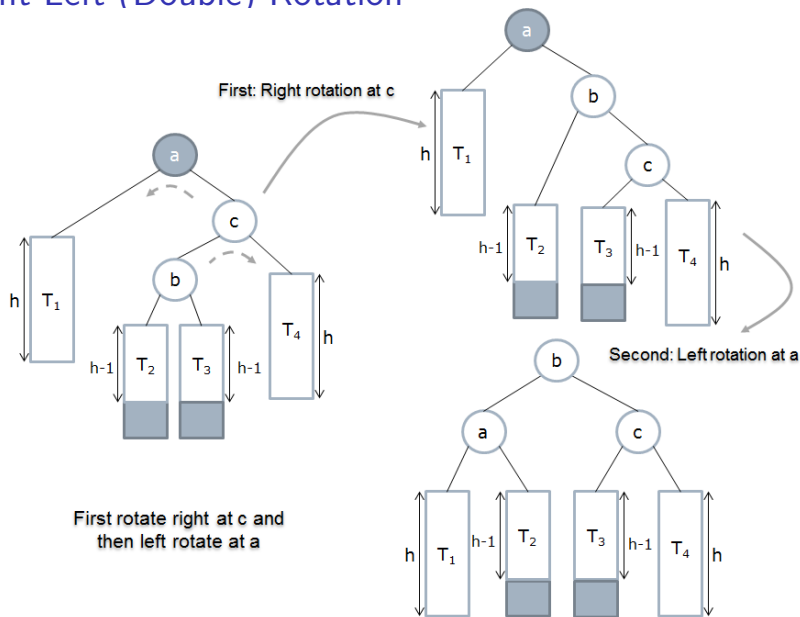
```
/* Goal: To perform right (clockwise) rotation */
template <typename T>
void AVL<T>::rotate_right() // The calling AVL node is node a
{
    AVLnode* b = left_subtree().root; // Points to node b
    left_subtree() = b->right;
    b->right = *this; // Note: *this is node a
    fix_height(); // Fix the height of node a
    this->root = b; // Node b becomes the new root
    fix_height(); // Fix the height of node b, now the new root
}
```



Left-Right (Double) Rotation



Right-Left (Double) Rotation



AVL Code: Insertion

```
/* To insert an item x to AVL tree and keep the tree balanced */

template <typename T>
void AVL<T>::insert(const T& x)
{
    if (is_empty())                // Base case
        root = new AVLnode(x);

    else if (x < root->value)
        left_subtree().insert(x); // Recursion on the left sub-tree

    else if (x > root->value)
        right_subtree().insert(x); // Recursion on the left sub-tree

    balance(); // Re-balance the tree at every visited node
}
```


AVL Code: Balancing

```
/* Goal: To balance an AVL tree */
template <typename T>
void AVL<T>::balance()
{
    if (is_empty())
        return;

    fix_height();
    int balance_factor = bfactor();

    if (balance_factor == 2)           // Right subtree is taller by 2
    {
        if (right_subtree().bfactor() < 0) // Case 4: insertion to the L of RT
            right_subtree().rotate_right();
        return rotate_left();           // Cases 1 or 4: Insertion to the R/L of RT
    }
    else if (balance_factor == -2) // Left subtree is taller by 2
    {
        if (left_subtree().bfactor() > 0) // Case 3: insertion to the R of LT
            left_subtree().rotate_left();
        return rotate_right();           // Cases 2 or 3: insertion to the L/R of LT
    }
    // Balancing is not required for the remaining cases
}
```

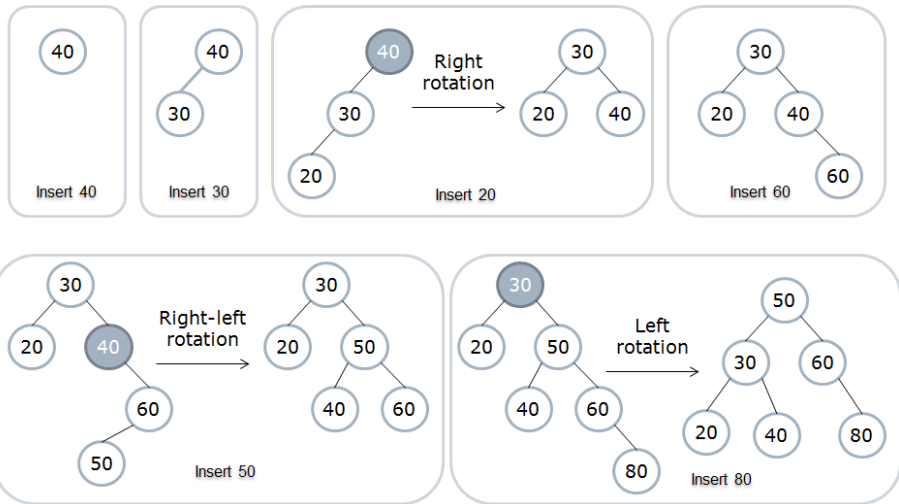
AVL Code: Balancing ..

```
/* To find the height of an AVL tree */
template <typename T>
int AVL<T>::height() const { return is_empty() ? -1 : root->height; }

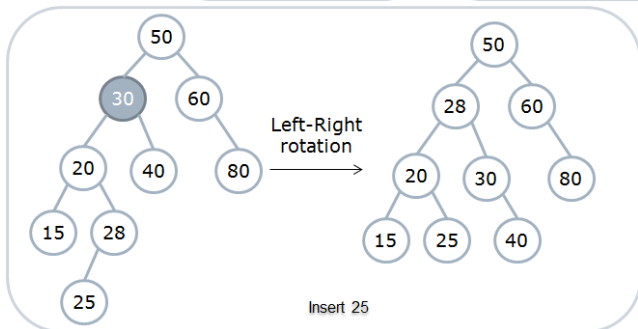
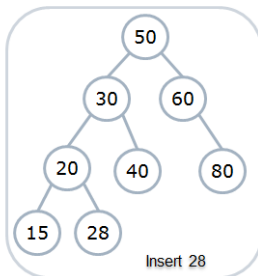
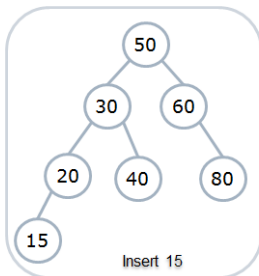
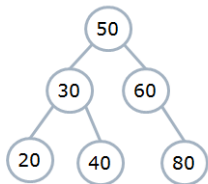
/* Goal: To rectify the height values of each AVL node */
template <typename T>
void AVL<T>::fix_height() const
{
    if (!is_empty())
    {
        int left_avl_height = left_subtree().height();
        int right_avl_height = right_subtree().height();
        root->height = 1 + max(left_avl_height, right_avl_height);
    }
}

/* balance factor = height of right sub-tree - height of left sub-tree */
template <typename T>
int AVL<T>::bfactor() const
{
    return is_empty() ? 0
        : right_subtree().height() - left_subtree().height();
}
```

Example: AVL Tree Insertion



Example: AVL Tree Insertion ..



AVL Tree Deletion

To **delete an item** from an AVL tree.

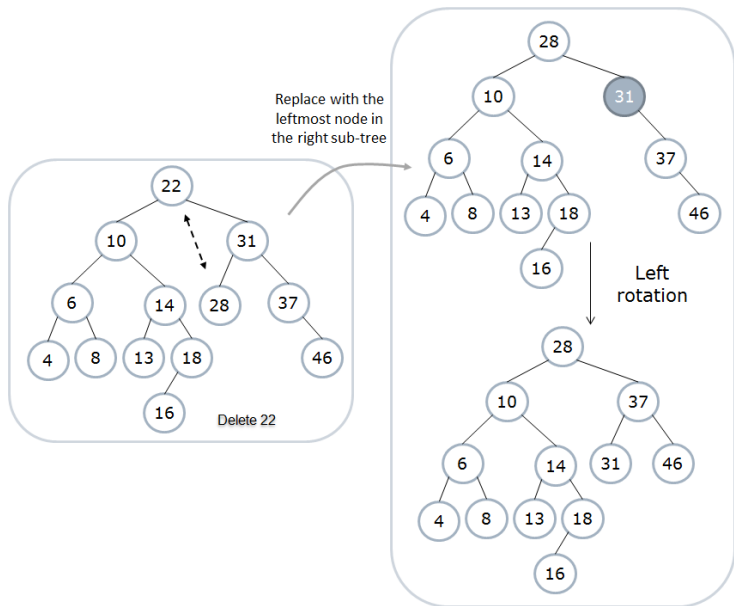


1. **Search** and **locate** the node with the required key.
2. **Delete** the node like deleting a node in BST.
3. A node deletion may result in a **unbalanced** tree
⇒ **Re-balance** the tree by **rotation(s)**.
 - ▶ **single rotation**
 - ▶ **double rotation** (i.e. two single but different rotations)

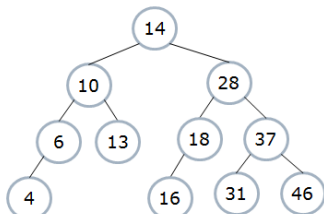
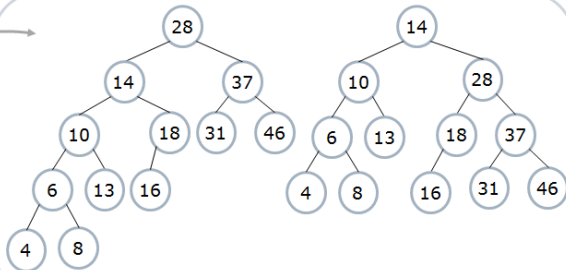
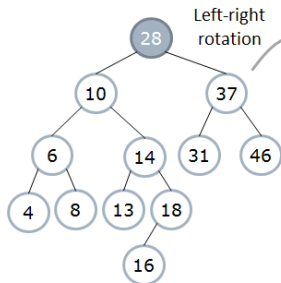
AVL Tree Deletion ..

- Similar to node deletion in BST, 3 cases need to be considered
 1. The node to be removed is a leaf node
 - ⇒ Delete the leaf node immediately
 2. The node to be removed has 1 child
 - ⇒ Adjust a pointer to bypass the deleted node
 3. The node to be removed has 2 children
 - ⇒ Replace the node to be removed with either the
 - maximum node in its left sub-tree, or
 - minimum node in its right sub-treeThen remove the max/min node depending on the choice above.
- Removing a node can render multiple ancestors unbalanced
 - ⇒ every sub-tree affected by the deletion has to be re-balanced.

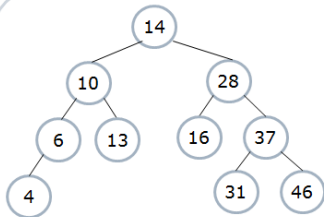
Example: AVL Tree Deletion



Example: AVL Tree Deletion ..

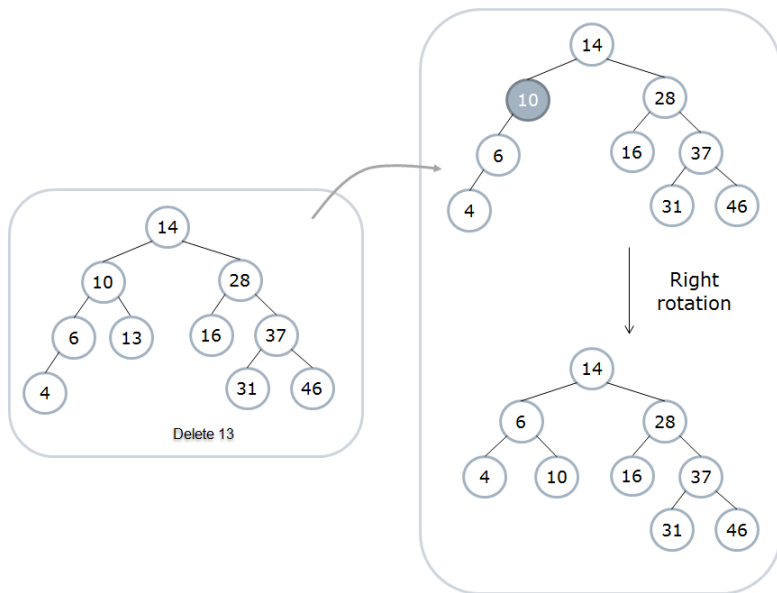


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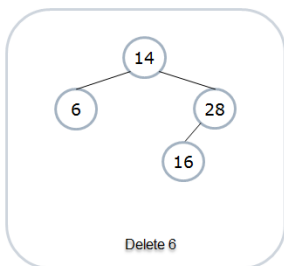
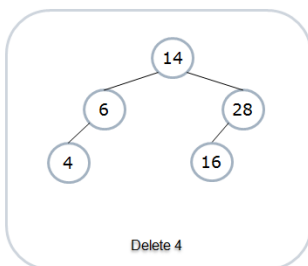
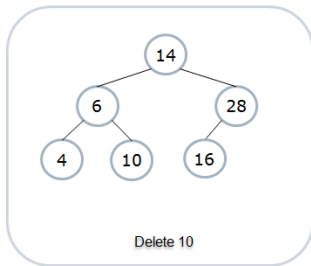
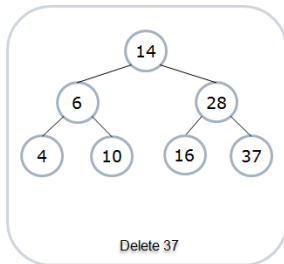
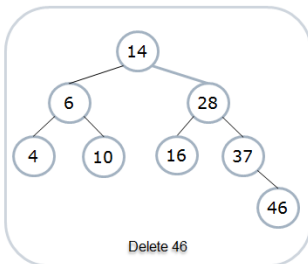
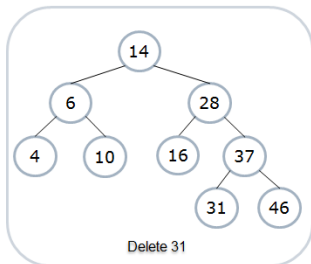


Delete 18

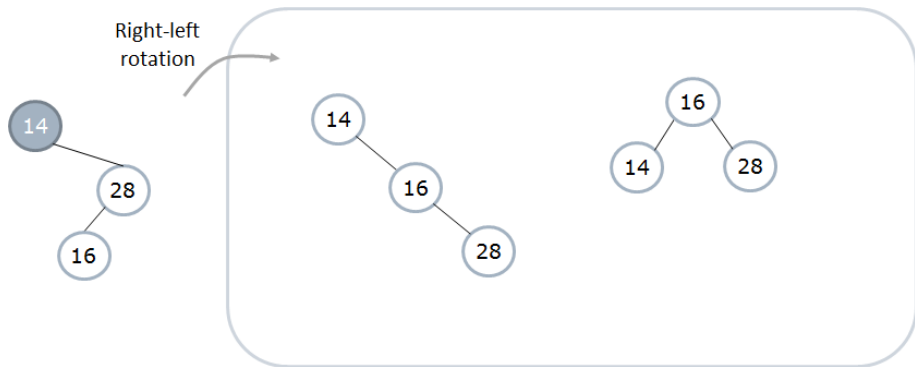
Example: AVL Tree Deletion ...



Example: AVL Tree Deletion



Example: AVL Tree Deletion



AVL Code: Deletion I

```
/* To remove an item x in AVL tree and keep the tree balanced */

template <typename T>
void AVL<T>::remove(const T& x)
{
    if (is_empty())                // Item is not found; do nothing
        return;

    if (x < root->value)
        left_subtree().remove(x); // Recursion on the left sub-tree

    else if (x > root->value)
        right_subtree().remove(x); // Recursion on the right sub-tree

    else
    {
        AVL& left_avl = left_subtree();
        AVL& right_avl = right_subtree();
    }
}
```

AVL Code: Deletion II

```
// Found node has 2 children
if (!left_avl.is_empty() && !right_avl.is_empty())
{
    root->value = right_avl.find_min(); // Copy the min value
    right_avl.remove(root->value); // Remove node with min value
}

else // Found node has 0 or 1 child
{
    AVLnode* node_to_remove = root; // Save the node first
    *this = left_avl.is_empty() ? right_avl : left_avl;

    // Reset the node to be removed with empty children
    right_avl.root = left_avl.root = nullptr;
    delete node_to_remove;
}
}

balance(); // Re-balance the tree at every visited node
}
```

AVL Code: Find the Minimum Value

```
/* To find the minimum value stored in an AVL tree. */

template <typename T>
const T& AVL<T>::find_min() const
{
    // It is assumed that the calling tree is not empty
    const AVL& left_avl = left_subtree();

    if (left_avl.is_empty())    // Base case: Found!
        return root->value;

    return left_avl.find_min(); // Recursion on the left subtree
}
```

AVL Testing Code

```
/* File: avl.tpp
 *
 * It contains template header and all the template functions
 */

#include "avl.h"
#include "avl-balance.cpp"
#include "avl-bfactor.cpp"
#include "avl-contains.cpp"
#include "avl-find-min.cpp"
#include "avl-fix-height.cpp"
#include "avl-height.cpp"
#include "avl-insert.cpp"
#include "avl-print.cpp"
#include "avl-remove.cpp"
#include "avl-rotate-left.cpp"
#include "avl-rotate-right.cpp"
```

AVL Testing Code .. I

```
#include <iostream>      /* File: test-avl.cpp */
using namespace std;
#include "avl.tpp"

int main()
{
    AVL<int> avl_tree;
    while(true)
    {
        char choice; int value;
        cout << "Action: f/i/m/p/q/r (end/find/insert/min/print/remove): ";
        cin >> choice;

        switch(choice)
        {
            case 'f':
                cout << "Value to find: "; cin >> value;
                cout << boolalpha << avl_tree.contains(value) << endl;
                break;

            case 'i':
                cout << "Value to insert: "; cin >> value;
                avl_tree.insert(value);
                break;
```


AVL Testing Code .. II

```
    case 'm':
        if (avl_tree.is_empty())
            cerr << "Can't search an empty tree!" << endl;
        else
            cout << avl_tree.find_min() << endl;
        break;

    case 'p':
        avl_tree.print();
        break;

    case 'q': default:
        return 0;

    case 'r':
        cout << "Value to remove: "; cin >> value;
        avl_tree.remove(value);
        break;
    }
}
```

AVL Trees: Pros and Cons

Pros:

- Time complexity for searching is in the order of $\log(N)$ since AVL trees are always balanced.
- Insertion and deletions are also in the order of $\log(N)$ since the operation is dominated by the searching step.
- The tree re-balancing step adds no more than a constant factor to the time complexity of insertion and deletion.

Cons:

- A bit more space for storing the height of an AVL node.

That's all!

Any questions?

