L09: Permutations and Combinations

- Objectives
  - **Permutations**
  - Combinations
  - Generalized Permutations and Combinations
- Reading: Rosen 6.3, 6.5

Permutations

- **Example 1**
  In how many ways can we select three students from a group of five students to stand in line for a photo?

- **Solution:**
  Here the order in which the students are selected matters. There are 5 ways to select the first student to stand at the start of the line. Once this student has been selected, there are 4 ways to select the second student to stand in the line from the remaining 4 students. After the first and the second students have been selected, there are 3 ways to select the third student. By the product rule, there are a total of $5 \times 4 \times 3 = 60$ ways

Permutations

- **Example 2**
  In how many ways can we arrange all five of the students above in a line for a photo?

- **Solution:**
  This example is similar to example 1, except that we are now selecting all the 5 students instead of 3. Using the same rationale as the previous question, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

Permutation

- **Definition**
  A **permutation** of a set of distinct elements is an ordered arrangement of the elements of the set.

- **Example**
  Let $S = \{1, 2, 3\}$. The ordered arrangement $(3, 1, 2)$ is a permutation of S.

k-Permutation

- **Definition**
  A ***k*-permutation** (or ***k*-element permutation**) of a set of n distinct elements is an ordered arrangement of $k \leq n$ elements of the set.

- **Example 5**
  Let $S = \{1, 2, 3\}$. The ordered arrangement $(3, 2)$ is a 2-permutation of *S*.

- **Definition**
  A ***n*-permutation** of a set of n distinct elements is simply called a permutation of the set.

Number of *k*-Permutations

- **Theorem**
  Let n and k be integers with $0 \leq k \leq n$. The number of *k*-permutations of a set with n distinct elements, denoted by $P(n, k)$, is equal to

$$P(n, k) = n(n-1) \ldots (n-k+1)$$
$$= \prod_{i=0}^{k-1} (n-i)$$
$$= \frac{n!}{(n-k)!}$$

Example

- **Example 6**
  How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

- **Solution:**
  The order matters as it matters which person wins which prize. Therefore the number of ways to pick three prize winners is the number of *ordered* selections of three elements from a set of 100 elements, i.e., the number of 3-permutations of a set of 100 elements.

$$P(100,3) = 100 \times 99 \times 98 \; = \; 970200 \text{ ways.}$$

Example

- **Example 7**
  Suppose a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wants. How many possible orders can the saleswoman use when visiting these cities?

- **Solution:**
  Each travel order gives a different trip schedule, so order matters here. Since the first city has been fixed, so she only needs to choose the order for the remaining 7 cities, which is $P(7,7) = 7! = 5040$ possible orders.

Example

- **Example 8**
  How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

- **Solution:**
  Since the string ABC must occur together, we can consider it as a single letter (i.e. denote it by S). So now we only need to find the number of permutations of six SDEFGH. The answer is $P(6,6) = 6! = 720$.

Example

- **Example**
  Twelve people sit down at a round table. We consider two seating arrangements equivalent if each person has the same person to the right in both seating charts. How many different seating charts are there?

- **Solution**
  In a circular arrangement, we first have to fix the position of the first person, which can be performed only in *one way* (since every position is considered the same when no one is already sitting in any of the seats). Once we have fixed the position of the first person, we can arrange the remaining persons in $(12 - 1)!$ ways.

Outline

- Permutations
- **Combinations**
- Generalized Permutations and Combinations

k-Combinations

- **Example 9**
  How many different committees of three students can be formed from a group of four students?

- **Solution**
  We need only to find the number of subsets containing 3 elements from the set of 4 students. There are 4 such subsets. Therefore there are 4 ways to choose the 3 students for the committee, where the order in which these students are chosen does not matter.

k-Combinations

- **Definition**
  A ***k*-combination** of a set of *n* distinct elements is an unordered selection of $k \leq n$ elements from the set.

- **Example 10**
  Let $S = \{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from *S*.

Number of *k*-Combinations

- **Theorem**
  Let n and k be integers with $0 \leq k \leq n$. The number of *k*-combinations of a set with *n* distinct elements, denoted by $C(n, k)$ or $\binom{n}{k}$, is equal to

$$C(n, k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

Proof

The k-permutations of the set can be obtained by forming the $C(n,k)$ k-combinations of the set, and then ordering the elements in each k-combination, which can be done in $P(k,k)$ (i.e., $k!$) ways. Consequently,

$$P(n,k) = C(n,k) \cdot P(k,k)$$

This implies that

$$C(n,k) = \frac{P(n,\ k)}{P(k,k)} = \frac{n!/(n-k)!}{k!\,(k-k)!} = \frac{n!}{k!\,(n-k)!}$$

Example

- **Example 11**
  How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

- **Solution**
  The order does not matter as we are only interested in the set of cards selected, not their order of being selected.
  Therefore there are $C(52,5) = \frac{52!}{5!47!}$ ways.
  When 5 cards are selected, there are 47 remaining cards. That is, there is a 1:1 correspondence between sets of 5 cards and sets of 47 cards. Therefore there are $C(52,47) = C(52,5) = \frac{52!}{5!47!}$ ways.

Corollary

- **Corollary**
  Let *n* and *k* be integers with $0 \le k \le n$. Then
  $$C(n,k) = C(n, n-k)$$
- **Proof**
  From the previous theorem, it follows that
  $$C(n,k) = \frac{n!}{k!\,(n-k)!}$$
  $$C(n,n-k) = \frac{n!}{(n-k)!\,[n-(n-k)]!} = \frac{n!}{(n-k)!\,k!}$$
  Hence, $C(n,k) = C(n,n-k)$.

Combinatorial Proof and Bijection Principle

- **Definition**
  A **combinatorial proof** of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.

- **Bijection principle**
  Two sets U and V have the same size, i.e., $|U| = |V|$, if and only if there is a bijection function f from U to V (i.e., one-to-one and onto)

Combinatorial Proof and Bijection Principle

- **Alternative (combinatorial) proof of corollary**
  Suppose S is a set with n distinct elements.
  Every subset A of S with $k \leq n$ elements corresponds to the subset $S - A$ of S with $n - k$ elements.
  Consequently, $C(n, k) = C(n, n - k)$.

Examples

- **Example 12**
  How many bit strings of length n contain exactly k 1's?
- **Hint**
  e.g., # bit strings of length 5 containing 3 ones = # ways of choosing 3 positions out of 5 positions
  e.g., 01011 corresponds to $\{2,4,5\} \subset \{1,2,3,4,5\}$

- Revisit the playoff example

Examples

- **Example 13**
  Suppose there are 40 faculty members in the computer science department and 30 in the mathematics department. How many ways are there to select a committee to develop a discrete mathematics course at a university if the committee is to consist of four faculty members from the computer science department and three from the mathematics department?
- **Solution**
  There are $C(40, 4)$ ways to select the 4 computer science faculty members. For each fixed set of selected CSE faculty members, there are $C(30, 3)$ ways to select the 3 Math faculty members, so there are a total of $C(40,4) \times C(30,3) = \frac{40!}{4!36!} \frac{30!}{3!27!}$ possible ways.

Outline

- Permutations
- Combinations
- **Generalized Permutations and Combinations**

Permutations with Indistinguishable Objects

- **Example 15**
  How many different strings can be made by reordering the letters of the word *SUCCESS*?
- **Solution:**
  Since some of the letters repeat in SUCCESS, so the number of permutations for 7 distinct letters is not the correct answer here. SUCCESS contains 3 S's, 2 C's, 1 E and 1 U. The three indistinguishable S's can be placed among the 7 positions in $C(7,3)$ different ways, leaving 4 positions free. Then the two C's can be placed in $C(4,2)$ ways, leaving 2 free positions. … By the product rule, the number of different ways is

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!\,2!\,1!\,1!}$$

Permutations with Indistinguishable Objects

- **Theorem**

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …, and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\ldots n_k\,!}$$

Distributing Objects into Boxes

Many counting problems can be solved by enumerating the different ways objects can be placed into boxes where the order these objects are placed into the boxes does not matter.

The objects can be either *distinguishable* (labeled) or *indistinguishable* (unlabeled). Similarly, the boxes can also be either *distinguishable* or *indistinguishable*.

Here we first consider the case where both the boxes and the objects are distinguishable.

Distributing Objects into Boxes

- **Example 17**
  How many ways are there to distribute hands of five cards to each of four players from the standard deck of 52 cards?
- **Solution**
  Here, the cards are the distinguishable objects and the players are the distinguishable boxes.
  First player can be dealt 5 cards in $C(52,5)$ ways.
  The second player: $C(47,5)$ ways.
  The third player: $C(42,5)$ ways.
  The fourth player: $C(37,5)$ ways.
  By the product rule, the total number of ways is
  $C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!5!} \cdot \frac{47!}{42!5!} \cdot \frac{42!}{37!5!} \cdot \frac{37!}{32!5!} = \frac{52!}{5!5!5!5!32!}$

Distributing Objects into Boxes

- **Theorem**
  The number of ways to distribute *n* distinguishable objects into *k* distinguishable boxes so that n_i objects are placed into box *i*, *i* = 1, 2, …, *k*, is equal to

$$\frac{n!}{n_1!\,n_2!\ldots n_k!}$$

Distributing Objects into Boxes

- **Remark**

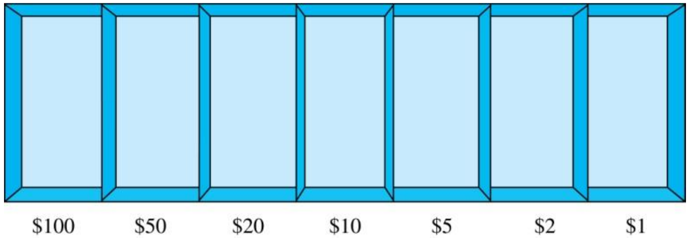
The solution of Example 17 is equal to the number of **permutations** of 52 objects, with five **indistinguishable** objects of each of four different types and 32 of a fifth type.

Consider 52 positions. Positions assigned to objects of first type in a permutation corresponds to cards dealt by first player, and so on.

That is, we can define a bijection between the kind of permutations and distributions of cards to players.

Combinations with Repetition

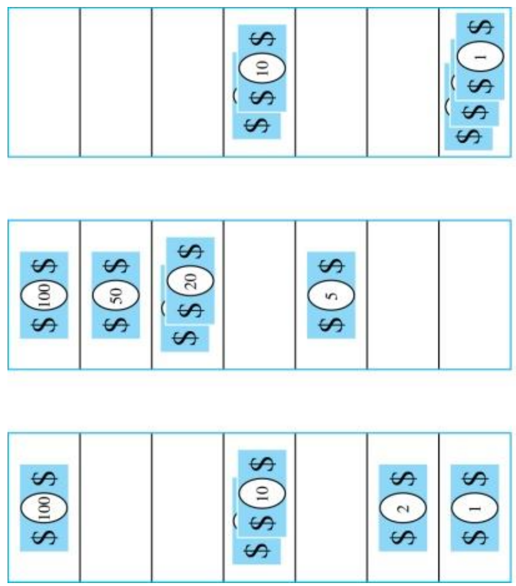
- **Example**
  How many ways are there to select five bills from the following denominations: $\$1$, $\$2$, $\$5$, $\$10$, $\$20$, $\$50$, and $\$100$? Assume there are enough bills of each denomination
- **Solution**
  Place the selected bills in the appropriate position of a cash box illustrated below:



- Note: This is the same as distributing 5 indistinguishable objects into 7 distinguishable boxes.

Solution (cnt’d)

- Some possible ways of placing the five bills:



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- \# of ways to select 5 bills from 7 denominations
  = # of ways to arrange 6 bars and 5 stars in a row:

$$C(11,5) = \frac{11!}{5!\,6!}$$

Combinations with Repetition

- **Theorem**
  The number of r-combinations from a set with n elements when repetition of elements is allowed is
  $$C(n+r-1, r)$$
- **Proof**: Each such r-combination can be represented by a list of $n-1$ bars and r stars. The bars mark the n cells, each corresponding to a distinct element. The stars between two bars correspond to the repetitions of an element.
  The list has $n+r-1$ positions, and we choose r of them to place the stars. Or equivalently, we choose $n-1$ positions to place the bars.

Combinations with Repetition

- **Example**
  How many solutions does the equation
  $$x_1 + x_2 + x_3 = 11$$
  have, where x_1, x_2, and x_3 are nonnegative integers?
- **Solution**
  Each solution corresponds to a way to select 11 items with repetition from a set with 3 elements; x_1 items of type 1, x_2 of type two, and x_3 of type three. By the previous theorem, it follows that there are
  $$C(3 + 11 - 1, 11) = C(13, 2) = 78$$
  solutions.

Combinations with Repetition

- **Example**
  Suppose that a cookie shop has 4 different kinds of cookies. How many different ways can 6 cookies be chosen?
- **Solution**
  The number of ways to choose 6 cookies is the number of 6-combinations with repetition from a set with 4 elements. By previous theorem, the number of such combinations is

$$C(4+6-1,6)=C(9,6)=C(9,3)=84$$