COMP2121: Discrete Mathematics
Homework 2 **Due Date:** Nov 2, 2018

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]** Understand abstract mathematical concepts which are fundamental to computer science, e.g., logic, sets, functions, basic probability, graph theory.
- **[O2. Proof Techniques]** Be able to perform abstract thinking and present logical argument using techniques such as mathematical induction, proof by contradiction.
- **[O3. Basic Analysis Techniques]** Be able to apply formal reasoning to analyze and enumerate the possible outcomes of a computational problem e.g. model and compute the number of operations using recursion, counting and combinatorics.

1. (18 points) **[O1]** How many positive integers between 100 and 999 inclusive

   (a) are divisible by 4?

   (b) are divisible by both 4 and 7?

   (c) are divisible by neither 3, 4 nor 7?

   (d) contain the digit 5 at least once?

   (e) contain the digit 5 exactly once?

   (f) have distinct digits? (That is, no digit appears more than once.)

   Solution: (3 points each)

   Let the number of positive integers between 100 and 999 inclusive that are divisible by a positive integer x be $f(x)$.

   (a) $f(4) = \lfloor 999/4 \rfloor - \lfloor 99/4 \rfloor = 249 - 24 = 225$.

   (b) $f(4 \times 7) = \lfloor 999/(4 \times 7) \rfloor - \lfloor 99/(4 \times 7) \rfloor = 35 - 3 = 32$.

   (c) $f(3) = 300$. $f(7) = 128$. $f(3 \times 7) = 43$. $f(3 \times 4) = 75$. $f(3 \times 4 \times 7) = 10$. By the principle of inclusion and exclusion the required number is $(999 - 99) - 300 - 225 - 128 + 75 + 43 + 32 - 10 = 387$.

   (d) If a number does not contain the digit 6, then there are 8 choices for the first digit, 9 choices for the second digit, and 9 choices for the third digit. Thus the required number is $(999 - 99) - 8 \times 9 \times 9 = 252$.

   (e) $9 \times 9 + 8 \times 9 + 8 \times 9 = 225$.

   (f) $9 \times 9 \times 8 = 648$.

2. (18 points) **[O1]** How many permutations of the letters ABCDEFGHI are there

(a) that end with a letter OTHER THAN C?

(b) that contain the string HI?

(c) that contain the string ACD?

(d) that contain the strings AB, DE and GH?

(e) if the letter A is somewhere to the left of the letter E?

(f) if the letter A is somewhere to the left of the letter E and there is exactly one letter between A and E?

Solution: (3 points each)

(a) $8 \times 8! = 322560$. (Or $9! - 8! = 322560$.)

(b) $8! = 40320$.

(c) $7! = 5040$.

(d) $6! = 720$.

(e) From every permutation where A is somewhere to the left of E, we can get a "symmetric" one by switching the positions of A and E. Thus the required number is $9!/2 = 181440$.

(f) $7 \times 7! = 35280$.

3. (6 points) **[O2]** A group of 15 students are to select 5 courses. Each student selects exactly 1 course, and no course is selected by more than 4 students. Show that at least 3 courses are selected by 3 or more students.

Solution: Suppose on the contrary at most 2 courses are selected by 3 or more students. Let n be the number of courses that are selected by 3 or more student. Then the assumption simply says $n \leq 2$. Since every course is selected by at most 4 students, the total number of students is at most $4n + 2(5 - n) = 2n + 10 \leq 2 \times 2 + 10 = 14$, a contradiction.

4. (12 points) **[O1]** How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 31$$

where x_i, $i = 1, 2, 3, 4, 5$, is a non-negative integer such that

(a) $x_i > 3$ for $i = 1, 2, 3, 4, 5$?

(b) $x_1 \geq 2, x_2 \geq 4, x_3 \geq 5, x_4 \geq 7, x_5 \geq 12$?

(c) $x_1 \leq 5$?

(d) $x_1 < 4$ and $x_2 > 8$?

Solution:

(3 pt each)

(a) $\binom{31-4\times5+5-1}{4} = 1365$.

(b) $\binom{31-2-4-5-7-12+5-1}{4} = 5$.
(c) $\binom{31+5-1}{4} - \binom{31-6+5-1}{4} = 28609$.
(d) $\binom{31-9+5-1}{4} - \binom{31-4-9+5-1}{4} = 7635$

5. (12 points) **[O1,O2]** Suppose a card is chosen at random from a standard 52-card deck. Let A be the event that the card is a face card (jack, queen or king). Let B be the event that the card is from one of the red suits (hearts or diamonds).

   (a) What is $\Pr(A)$? What is $\Pr(B)$?
   (b) What is $\Pr(A \cap B)$? Are A and B independent?
   (c) What is $\Pr(A \cup \overline{B})$?

   Solution: (4 points each)

   (a) $\Pr(A) = \frac{3\times 4}{52} = \frac{3}{13}$, $\Pr(B) = \frac{13+13}{52} = \frac{1}{2}$.
   (b) $\Pr(A \cap B) = \Pr(\text{the card is a face card from one of the red suits}) = \frac{3+3}{52} = \frac{3}{26}$. Since $\Pr(A \cap B) = \Pr(A)\Pr(B)$, the events A and B are independent.
   (c)
   $$\begin{aligned}\Pr(A \cup \overline{B}) &= 1 - \Pr(\overline{A \cup \overline{B}}) = 1 - \Pr(\overline{A} \cap B) = 1 - [\Pr(B) - \Pr(A \cap B)] \\ &= 1 - [\tfrac{1}{2} - \tfrac{3}{26}] = \tfrac{8}{13}.\end{aligned}$$

6. (10 points) **[O1,O3]** A company analysed that the chance of a male customer trying their new product is 30%, while that of a female customer trying their new product is 65%. They also know that 70% of their customers are female.

   (a) What is the probability that a customer who does not try their new product is a male?
   (b) What is the probability that a customer who tries their new product is a female?

   Solution: (5 points each)

   Suppose event M represents that a customer is a male, and $\overline{M}$ represents that a customer is not a male (i.e., a female). Also assume that E represents that a customer tries the new product and $\overline{E}$ represents that a customer does not try the product. Then we have $P(\overline{M}) = 0.7$, $P(M) = 1 - 0.7 = 0.3$, $P(E \mid M) = 0.3$, $P(\overline{E} \mid M) = 1 - 0.3 = 0.7$, $P(E \mid \overline{M}) = 0.65$, $P(\overline{E} \mid \overline{M}) = 1 - 0.65 = 0.35$.

   (a)
   $$\begin{aligned}P(M \mid \overline{E}) &= \frac{P(\overline{E} \mid M)P(M)}{P(\overline{E} \mid M)P(M) + P(\overline{E} \mid \overline{M})P(\overline{M})} \\ &= \frac{0.7 \times 0.3}{0.7 \times 0.3 + 0.35 \times 0.7} \\ &\sim 0.4615\end{aligned}$$

(b)

$$\begin{aligned} P(\overline{M} \mid E) &= \frac{P(E \mid \overline{M})P(\overline{M})}{P(E \mid M)P(M) + P(E \mid \overline{M})P(\overline{M})} \\ &= \frac{0.65 \times 0.7}{0.3 \times 0.3 + 0.65 \times 0.7} \\ &\sim 0.8349 \end{aligned}$$

7. (16 points) **[O3]** In a game there are two boxes, where inside each box there is a red ball and a blue ball. A player will draw a ball from box 1 and place it inside box 2, and then a ball is drawn from box 2. If balls of different colors are drawn, the player wins the game. Let event E be the event that a red ball is drawn from box 1, event F be that a blue ball is drawn from box 2 and event W be the event that the player wins the game.

   (a) Calculate P(F).

   (b) Find out if events E and F are independent or not.

   (c) Find out if events E and W are independent or not.

   (d) If the player needs to pay \$42 for the game when he/she loses, while a \$120 prize is given if the game is won by drawing a blue ball from box 2, and a \$60 prize is given if the game is won by drawing a red ball from box 2 instead. What is the expected net gain of one game?

   Solution: (4 points each)

   (a) $P(F) = P(F \cap E) + P(F \cap \overline{E}) = P(F \mid E)P(E) + P(F \mid \overline{E})P(\overline{E}) = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}$

   (b) $P(E \mid F) = P(E \cap F)/P(F) = P(F \mid E)P(E)/P(F) = \frac{1}{3} \cdot \frac{1}{2}/\frac{1}{2} = \frac{1}{3} \neq P(E)$, so event E and F are not independent.

   (c) To win after drawing a red ball from box 1, the player must draw a blue ball from box 2, so $P(W \mid E) = P(F \mid E) = \frac{1}{3}$, $P(W) = P(E \cap F) + P(\overline{E} \cap \overline{F}) = P(F \mid E)P(E) + P(\overline{F} \mid \overline{E})P(\overline{E}) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3} = P(W \mid E)$, so event E and W are independent.

   (d) To win by drawing a blue ball from box 2, the player must draw a red ball from box 1, so $P(W \cap F) = P(W \cap E) = P(W)P(E) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. Similarly $P(W \cap \overline{F}) = P(W \cap \overline{E}) = P(W)P(\overline{E}) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. Expected prize is $100P(W \cap F) + 50P(W \cap \overline{F}) - 30P(\overline{W}) = \frac{120}{6} + \frac{60}{6} - 42(1 - \frac{1}{3}) = \$2$.

8. (8 points) **[O3]** Suppose n contestants participate in a game consisting of two stages. In the first stage, the n contestants, **one by one**, attempt the game. Each contestant has a probability p of passing the first stage, independent of other players.

   If no contestant passes the first stage, then no one wins any money. Otherwise, the contestants passing the first stage compete in the second stage to share a cash prize of $M = 1,000,000$ dollars. In the second stage, each contestant draws a number independently uniformly at random from $\{1, 2, 3, 4, 5\}$, and the amount of money he receives is proportional to the number drawn. For example, if there are 3 contestants in the second stage and the 3 numbers drawn are 1, 2 and 5, then the prize M is shared between the 3 contestants in the ratio of 1 : 2 : 5. For instance, the contestant drawing the number 5 will win $\frac{5}{1+2+5} \times M$.

Suppose you are one of $n = 5$ contestants at the beginning of stage one of the game with $p = 0.2$. What is the expected amount you will win at the end of the whole game? Round your answer to the nearest dollar.

Solution: Let X_i be the amount a contestant i wins at the end of the whole game. Then $E[\sum_{i=1}^n X_i]$ is the expected total amount that all contestants win. We know

$$\sum_{i=1}^n X_i = \begin{cases} 0 & \text{, if no contestant passes the first stage;} \\ M & \text{, otherwise.} \end{cases}$$

Thus, we have

$$E[\sum_{i=1}^n X_i] = (1 - (1-p)^n)M.$$

By the linearity of the expected value operator, we have $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i]$. Since $E[X_i]$ is the same for each contestant i, we have

$$\sum_{i=1}^n E[X_i] = nE[X_i] = (1 - (1-p)^n)M.$$

Hence, $E[X_i] = (1 - (1-p)^n)M/n = (1 - (1-0.2)^5)1000000/5 = 134464$.