HKUST – Department of Computer Science and Engineering

COMP 2711: Discrete Math Tools for Computer Science

Spring 2017 Final Examination

Date: 17 May 2017 Time: 08:30–11:30 Venue: LG1 Table Tennis Room

Name: ______________________ Student ID: ________________

Email: ______________________

Instructions

- This is a closed book exam. It consists of 19 pages and 14 questions.
- Please write your name, student ID, email on this page.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Solutions can be written in terms of binomial coefficients, factorials, and the $C(n, k)$, $P(n, k)$ notation. For example, you can write $\binom{5}{3} + \binom{4}{2}$ instead of 16. Avoid use nonstandard notation such as ${}_nP_k$ or ${}_nC_k$. Calculators may be used for the exam (but are not necessary).
- You do not have to show the steps unless required otherwise.

Questions	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Points	4	6	8	8	5	6	4	5	6	8	10	8	10	12	100
Score															

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

I declare that the answers submitted for this examination are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University regulations governing academic integrity.

Student's Name: ______________________________

Student's Signature: ______________________________

Problem 1: [4 pts] Suppose that p and q are distinct primes. Use the principle of inclusion-exclusion to find $\phi(pq)$, the number of integers not exceeding pq that are relatively prime to pq.

Answer: $pq - p - q + 1$.

Problem 2: [6 pts] For each of the following parts, find the best big-O function for the function. Choose your answer from among the following:

$$1,\ \log_2 n,\ n,\ n\log_2 n,\ n^2,\ n^3, \ldots,\ 2^n,\ n!$$

(a) $1 + 4 + 7 + \cdots + (3n+1)$

(b) $1 + 3 + 5 + 7 + \cdots + (2n-1)$

(c) $\dfrac{3 + 2n^4 + 4n}{3n^3 + 3n}$

(d) $1 + 2 + 3 + \cdots + (n^2 - 1) + n^2$

(e) $\lceil n+2 \rceil \cdot \lceil n/3 \rceil$

(f) $3n^4 + \log_2 n^8$

Answer: (a) n^2

(b) n^2

(c) n

(d) n^4

(e) n^2

(f) n^4

Problem 3: [8 pts] For each of the following parts, find the "best" big-O notation to describe the complexity of the algorithm. Choose your answers from the following:

$$1,\ \log_2 n,\ n,\ n\log_2 n,\ n^2,\ n^3, \ldots,\ 2^n,\ n!$$

(a) A linear search to find the smallest number in a list of n numbers

(b) An algorithm that prints all bit strings of length n.

(c) The number of print statements in the following:

$i := 1, j := 1$
while $i \le n$
  while $j \le i$
    print "hello";
    $j := j + 1$
  $i := i + 1$

(d) The number of print statements in the following:

while $n > 1$
  print "hello";
  $n := \lfloor n/2 \rfloor$

Answer: (a) n

(b) 2^n

(c) n

(d) $\log_2 n$

Problem 4: [8 pts] An experiment consists of picking at random a bit string of length five. Consider the following events:

E_1: the bit string chosen begins with 1;

E_2: the bit string chosen ends with 1;

E_3: the bit string chosen has exactly three 1s.

Answer each of the following parts.

(a) Find $p(E_3|E_2)$.

(b) Find $p(E_3|E_1 \cap E_2)$.

(c) Determine whether E_1 and E_2 are independent.

(d) Determine whether E_2 and E_3 are independent.

Answer: (a) $\binom{4}{2}/2^4$.

(b) $\binom{3}{1}/2^3$.

(c) Yes, because $p(E_1|E_2) = p(E_1)$.

(d) No, because $p(E_3|E_2) \neq p(E_3)$.

Problem 5: [5 pts] Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You roll a die to determine which urn to choose: if you roll a 1 or 2 you choose urn 1; if you roll a $3, 4, 5$, or 6 you choose urn 2. Once the urn is chosen, you draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

Answer: $\frac{\frac{2}{10}\cdot\frac{1}{3}}{\frac{2}{10}\cdot\frac{1}{3}+\frac{12}{15}\cdot\frac{2}{3}} = 1/9$

Problem 6: [6 pts] Suppose that we flip n fair coins. Let X_n be the random variable that is obtained by subtracting the number of tails obtained from the number of heads obtained.

(a) What is the expected value of X_4?

(b) What is the variance of X_4?

(c) What is the expected value of X_n?

(d) What is the variance of X_n?

Answer: Let $Y_i = 1$ is the i-th flip is a heads and $Y_i = -1$ if it is a tails. So we have $X_n = Y_1 + \cdots + Y_n$, $E[Y_i] = 0, V[Y_i] = 1$.

(a) 0.

(b) 4.

(c) 0.

(d) n.

Problem 7: [4 pts] Each pixel in a 32×8 vertical display is turned on or off with equal probability. The display shows a horizontal line if all 8 pixels in a given row are turned on. Let X denote the number of horizontal lines that the display shows

(a) What is the expected value of X?

(b) What is the variance of X?

Answer: (a) $32/2^8 = 1/8$.

(b) $32(\frac{1}{2^8})(\frac{2^8-1}{2^8}) = \frac{255}{2048}$.

Problem 8: [5 pts] Some fundamental principles for reasoning about nonnegative integers are:

1. The Induction Principle,
2. The Strong Induction Principle,
3. The Well Ordering Principle.

Identify which, if any, of the above principles (or their variants) is captured by each of the following inference rules.

(a) $$\frac{P(0),\ \forall m(\forall k \le m,\ P(k)) \to P(m+1)}{\therefore \forall n,\ P(n)}$$

(b) $$\frac{P(b),\ \forall k \ge b,\ P(k) \to P(k+1)}{\therefore \forall k \ge b,\ P(k)}$$

(c) $$\frac{\exists n,\ P(n)}{\therefore \exists m,\ [P(m) \wedge (\forall k,\ P(k) \to k \ge m)]}$$

(d) $$\frac{P(0),\ \forall k \ge 1,\ P(k) \to P(k+1)}{\therefore \forall n,\ P(n)}$$

(e) $$\frac{\forall m,\ (\forall k < m,\ P(k)) \to P(m)}{\therefore \forall n,\ P(n)}$$

Answer: (a) The Strong Induction Principle.

(b) The Induction Principle.

(c) The Well Ordering Principle.

(d) None.

(e) None.

Problem 9: [6 pts] Give a recursive definition for each of the following sets. Remember to include both the base case and the recursive step.

(a) $\{3, 7, 11, 15, 19, 23, \dots\}$

(b) $\{\dots, -5, -3, -1, 1, 3, 5, \dots\}$

(c) The set of strings $1, 111, 11111, 1111111, \dots$

Answer: (a) $3 \in S$; $x \in S \to x + 4 \in S$

(b) $1 \in S$; $x \in S \to x \pm 2 \in S$

(c) $1 \in S$; $x \in S \to x11 \in S$

Problem 10: [8 pts] For each of the following parts, either give an example or prove that such a graph does not exist.

(a) A simple graph with 6 vertices, whose degrees are $2, 2, 2, 3, 4, 4$

(b) A simple graph with 8 vertices, whose degrees are $0, 1, 2, 3, 4, 5, 6, 7$

(c) A simple graph with 6 vertices and 16 edges

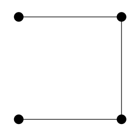
(d) A graph with 4 vertices that has an Euler path but no Euler circuit.

Answer: (a) None. It is not possible to have one vertex of odd degree.

(b) None. It is not possible to have a vertex of degree 7 and a vertex of degree 0 in this graph.

(c) None. The largest number of edges in a simple graph with 6 vertices is 15.

(d) An example below has an Euler path but no Euler circuit.



Problem 11: [10 pts] Consider the following sequence of predicates:

$$
\begin{array}{lcl}
Q_1(x_1) & = & x_1 \\
Q_2(x_1, x_2) & = & x_1 \to x_2 \\
Q_3(x_1, x_2, x_3) & = & (x_1 \to x_2) \to x_3 \\
Q_4(x_1, x_2, x_3, x_4) & = & ((x_1 \to x_2) \to x_3) \to x_4 \\
Q_5(x_1, x_2, x_3, x_4, x_5) & = & (((x_1 \to x_2) \to x_3) \to x_4) \to x_5 \\
 & \vdots &
\end{array}
$$

Let T_n be the number of different true/false assignments to the variables $x_1, x_2, \ldots, x_n$ for which $Q_n(x_1, x_2, \ldots, x_n)$ is true. For example, $T_2 = 3$ since $Q_2(x_1, x_2)$ is true for 3 different assignments to the variables x_1 and x_2:

x_1	x_2	$Q_2(x_1, x_2)$
T	T	T
T	F	F
F	T	T
F	F	T

(a) Express T_{n+1} in terms of T_n, assuming $n \geq 1$.

(b) Use induction to prove that $T_n = \frac{1}{3}(2^{n+1} + (-1)^n)$ for $n \geq 1$.

Answer: (a) $T_{n+1} = T_n + 2(2^n - T_n) = 2^{n+1} - T_n$.

(b) Base case ($n = 1$): $T_1 = 1 = \frac{1}{3}(2^2 - 1)$.

Inductive hypothesis: Assume $T_n = \frac{1}{3}(2^{n+1} + (-1)^n)$ for $n \geq 1$.

Inductive step:

$$
\begin{aligned}
T_{n+1} &= 2^{n+1} - T_n \\
&= 2^{n+1} - \frac{1}{3}(2^{n+1} + (-1)^n) \\
&= \frac{2}{3}2^{n+1} - \frac{1}{3}(-1)^n \\
&= \frac{1}{3}2^{n+2} + \frac{1}{3}(-1)^{n+1} \\
&= \frac{1}{3}(2^{n+2} + (-1)^{n+1})
\end{aligned}
$$

Problem 12: [8 pts] Consider the following game:

You begin with a stack of n boxes. Then you make a sequence of moves. In each move, you divide one stack of boxes into two nonempty stacks. The game ends when you have n stacks, each containing a single box. You earn points for each move; in particular, if you divide one stack of height $a+b$ into two stacks with heights a and b, then you score ab points for that move. Your overall score is the sum of the points that you earn for each move. As an example, suppose that we begin with a stack of $n = 10$ boxes. Then the game might proceed as shown in the following figure. On each line of the figure, the underlined stack is divided in the next step.

Stack Heights										Score
$\underline{10}$										
5	$\underline{5}$									25 points
$\underline{5}$	3	2								6
$\underline{4}$	3	2	1							4
2	$\underline{3}$	2	1	2						4
$\underline{2}$	2	2	1	2	1					2
1	$\underline{2}$	2	1	2	1	1				1
1	1	$\underline{2}$	1	2	1	1	1			1
1	1	1	1	$\underline{2}$	1	1	1	1		1
1	1	1	1	1	1	1	1	1	1	1
						Total Score			=	45 points

Use induction to prove that your score is always $n(n-1)/2$ at the end of the game, no matter how you divide the stacks.

Answer: The proof is by strong induction. Let $P(n)$ be the proposition that every way of unstacking n blocks gives a score of $n(n-1)/2$.

Base case: If $n = 1$, then there is only one block. No moves are possible, and so the total score for the game is $1(1-1)/2 = 0$. Therefore, $P(1)$ is true.

Inductive step: Now we must show that $P(1), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 1$. So assume that $P(1), \ldots, P(n)$ are all true and that we have a stack of $n+1$ blocks. The first move must split this stack into substacks with positive sizes a and b where $a + b = n + 1$ and $0 < a, b \leq n$. Now the total score for the game is the sum of points for this first move plus points obtained by unstacking the two resulting substacks:

$$
\begin{aligned}
\text{total score} &= (\text{score for 1st move}) \\
&\quad +(\text{score for unstacking } a \text{ blocks}) \\
&\quad +(\text{score for unstacking } b \text{ blocks}) \\
&= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \qquad \text{by } P(a) \text{ and } P(b) \\
&= \frac{(a+b)^2 - (a+b)}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(a+b)((a+b)-1)}{2} \\
&= \frac{(n+1)n}{2}
\end{aligned}
$$

This shows that $P(1), P(2), \ldots, P(n)$ imply $P(n+1)$.

Problem 13: [10 pts] Recall that the sequence of Fibonacci numbers are defined as follows: $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that any two successive Fibonacci numbers are relatively prime.

Answer: Let $P(n)$ be the proposition that F_n and F_{n+1} are relatively prime.

Base case: $P(0)$ and $P(1)$ are obviously true.

Inductive step: Suppose $P(k)$ is true for $k \geq 1$, i.e., F_k and F_{k+1} are relatively prime. We need to show that $P(k+1)$ is true, i.e., F_{k+1} and F_{k+2} are relatively prime.

We will prove so by contradiction. If F_{k+1} and F_{k+2} are not relatively prime, i.e., there is an integer $d \geq 2$ such that $F_{k+1} = da, F_{k+2} = db$. Then we have $F_k = F_{k+2} - F_{k+1} = db - da = d(b-a)$. We must have $b - a > 0$ since $F_{k+2} > F_{k+1}$ for $k \geq 1$. This means that $d | F_k$, which contradicts with the induction hypothesis that F_k and F_{k+1} are relatively prime.

Problem 14: [12 pts] The *2-3-averaged numbers* are a subset, $\mathbf{N}23$, of the real interval $[0,1]$ defined recursively as follows:

Base cases: $0, 1 \in \mathbf{N}23$.

Inductive step: If $a, b \in \mathbf{N}23$, then $\dfrac{2a+3b}{5} \in \mathbf{N}23$.

(a) Prove that $\left(\dfrac{3}{5}\right)^n \in \mathbf{N}23$ for all nonnegative integers n.

(b) It's obvious that all numbers in $\mathbf{N}23$ are rational numbers. How about the other direction, i.e., is every rational number in $[0,1]$ a member of $\mathbf{N}23$? If so, prove it; if not, find a counterexample, namely, find a rational number $c \in [0,1]$ and show that $c \notin \mathbf{N}23$.

(c) [**Bonus question**]
Prove that the product of any two 2-3-averaged numbers is also a 2-3-averaged number.

Answer: (a) Let $P(n)$ be the proposition that $(3/5)^n \in \mathbf{N}23$.
Base case: When $n = 0$, $(3/5)^0 = 1 \in \mathbf{N}23$. So, $P(0)$ is true.
Inductive hypothesis: Assume $P(n)$ is true for $n \geq 0$.
Inductive step:

$$\left(\frac{3}{5}\right)^{n+1} = \frac{3 \cdot 3^n}{5 \cdot 5^n} = \frac{2 \cdot 0 + 3 \cdot (\frac{3}{5})^n}{5}.$$

Since both 0 and $(\dfrac{3}{5})^n$ are in $\mathbf{N}23$, $\dfrac{2 \cdot 0 + 3 \cdot (\frac{3}{5})^n}{5}$ is also in $\mathbf{N}23$. This shows that $P(n) \to P(n+1)$.

(b) The rational number $1/4$ is not in $\mathbf{N}23$. Below we will show, by structural induction, that any number in $\mathbf{N}23$ is in the form of $\dfrac{x}{5^y}$ where x and y are both integers. Then if $\dfrac{1}{4} = \dfrac{x}{5^y}$, we would have $4x = 5^y$, which can't be true since the LHS is even while the RHS is odd.
The base cases $0, 1$ are both in this form of $\dfrac{x}{5^y}$. For the inductive step, if $a = \dfrac{x_1}{5^{y_1}}$, $b = \dfrac{x_2}{5^{y_2}}$, then

$$\frac{2a+3b}{5} = \frac{\frac{x_1}{5^{y_1}} + \frac{x_2}{5^{y_2}}}{5} = \frac{x_1 \cdot 5^{y_2} + x_2 \cdot 5^{y_1}}{5^{y_1+y_2+1}},$$

which is also in this form.

(c) [**Bonus question**]

Let $P(x)$ be the statement that $xy \in \mathbf{N}23$, $\forall y \in \mathbf{N}23$. We will show that $P(x)$ holds for all $x \in \mathbf{N}23$ by structural induction on x.

Base case: $P(0), P(1)$ are true, because $0 \cdot y = 0$ and $1 \cdot y = y$.

Inductive step: Assume $P(x_1), P(x_2)$ are true for $x_1, x_2 \in \mathbf{N}23$, i.e., $x_1 y \in \mathbf{N}23$ and $x_2 y \in \mathbf{N}23$ for any $y \in \mathbf{N}23$. We need to show that for any $y \in \mathbf{N}23$, $\dfrac{2x_1 + 3x_2}{5} \cdot y \in \mathbf{N}23$. Indeed,

$$\frac{2x_1 + 3x_2}{5} \cdot y \quad = \quad \frac{2x_1 y + 3x_2 y}{5}.$$

By the induction hypothesis, both $x_1 y$ and $x_2 y$ are in $\mathbf{N}23$, so by the recursive definition, $\dfrac{2x_1 y + 3x_2 y}{5}$ is also in $\mathbf{N}23$.