COMP2121A: Discrete Mathematics
Problem Solving Session 26/30 Oct, 2018

$f : [n] \to [n]$ $|\Omega| = n^n$

1. **Balls and Bins** Suppose each of n balls are thrown independently into n bins uniformly at random. Let X be the number of empty bins. Compute $E[X]$.

Range of $X = \{1, 2, \ldots, n-1\} = \Omega$

For $k \in \Omega$, $\Pr[X=k] = ?$ $E[X] = \sum_{k=1}^{n-1} k \cdot \Pr[X=k]$

Define $Y_i = \begin{cases} 1, & \text{Bin } i \text{ is empty} \\ 0, & \text{Bin } i \text{ is non-empty.} \end{cases}$

$X = \sum_{i=1}^{n} Y_i$

$E[Y_i] = \Pr[Y_i = 1] = \left(1 - \frac{1}{n}\right)^n$

all balls miss bin i

even if Y_i's not indep.

$$\boxed{E[X] = \sum_{i=1}^{n} E[Y_i] = n\left(1 - \frac{1}{n}\right)^n}$$

$n = 6$

$\Pr[X=k] = \binom{n}{k} \cdot \frac{1}{n^n} \cdot S(n, n-k)$

$\boxed{\left(\frac{n-k}{n}\right)^n \cdot \left(\frac{1}{n-k}\right)^n}$

#. of surjective functions from $[n]$ to $[n-k]$.

k empty

n balls

$n-k$ non-empty

2. Suppose $n \geq 2$ and m are positive integers. There are m identical coins to be distributed among n persons.

We describe a procedure called UNFAIR to divide the coins. The first person comes along and an integer x is picked uniformly at random from $\{0, 1, 2, \ldots, m\}$. Then, the first person takes x coins and goes home. In general, when the ith person comes along (where $i < n$), and there are r coins left, an integer y is picked uniformly at random from $\{0, 1, 2, \ldots, r\}$ and the ith person goes home with y coins. The nth (last) person just takes whatever that is left.

Define X_i to be the number of coins the ith person takes.

(a) Compute $E[X_1]$, the expected number of coins the first person receives.

$$\frac{1}{m+1} \sum_{i=0}^{m} i = \frac{1}{m+1} \cdot \frac{1}{2} m(m+1) = \frac{m}{2}$$

(b) Suppose $n \geq 3$. Given that the first person receives x coins, what is the expected number of coins the second person receives? (Compute $E[X_2|X_1 = x]$.)

Remaining no. of coins $= m - x$

$$E[X_2 | X_1 = x] = \frac{m-x}{2}$$

(c) Assume $n \geq 3$. Compute $E[X_2]$.

$$E[X_2] = \sum_x \Pr[X_1 = x] \cdot E[X_2 | X_1 = x]$$

$$= \frac{m - E[X_1]}{2} = \frac{m}{4}$$

(d) For general $i \leq n$, compute $E[X_i]$.

If $i < n$, $E[X_i] = \frac{m}{2^i}$

$i = n$, $E[X_n] = m - \sum_{i=1}^{n-1} E[X_i] = \frac{m}{2^{n-1}}$

$n-1$

$0|0|_m \cdots | 0$

We next consider another procedure called FAIR. First, compute the set S of all integer solutions to the equation $x_1 + x_2 + x_3 + \ldots + x_n = m$, where each $x_i \geq 0$. A solution $(x_1, x_2, \ldots, x_n)$ is picked uniformly at random from S, and for each $1 \leq i \leq n$, the ith person receives x_i coins. non-negative

(a) What is the size of S?

$$\frac{(m+n-1)!}{m!\,(n-1)!} = \binom{m+n-1}{m} = \binom{m+n-1}{n-1}$$

(b) Suppose X_1 is the number of coins received by the first person. What is the probability that $X_1 = k$, where $0 \leq k \leq m$? Express your answer in terms of n, m and k. $S_k = \{\vec{x} \in S : x_1 = k\}$.

$x_2 + \ldots + x_n = m - k$ m-k coins n-2

$$|S_k| = \frac{(m-k+n-2)!}{(m-k)!\,(n-2)!} = \binom{m+n-k-2}{n-2} \quad \text{Ans} = \frac{|S_k|}{|S|}$$

(c) Prove that for all positive integers $n \geq 2$ and $m \geq 1$,

$$\sum_{k=0}^{m} k \cdot \binom{m+n-k-2}{n-2} = \frac{m}{n} \cdot \binom{m+n-1}{m}.$$

$$\sum_{k=0}^{m} k \cdot \Pr[X_1 = k] = \frac{m}{n}$$

$$E[X_1] \overset{?}{=} \frac{m}{n} \checkmark$$

Let X_i be the r.v. denoting no of coins received by person i.

$\sum_i X_i = m$

$\sum_{i=1}^{n} E[X_i] = m \Rightarrow E[X_i] = \frac{m}{n}$

$\forall i, j.\ E[X_i] = E[X_j]$

$B = \sum_{i=1}^n X_i \qquad E[B] = \sum_{i=1}^n E[X_i] = np$

$E[X_i] = p \qquad X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

$\text{var}(B) = E[B^2] - E[B]^2$

3. Let $B = \mathsf{Bin}(n,p)$, i.e., flipping n biased coins, each having heads with probability p. Compute $E[B^2]$.

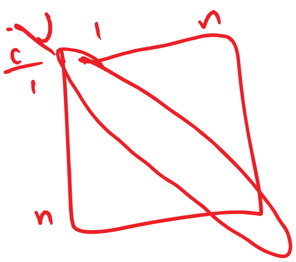
(For general $k \geq 2$, how to compute $E[B^k]$?)

$E[B^2] = \sum_{k=0}^n k^2 \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k}$

$\hookrightarrow = E\left[\sum_{i=1}^n X_i \cdot \sum_{j=1}^n X_j\right]$

$= \sum_{i=1}^n \sum_{j=1}^n E[X_i X_j]$

$= n \cdot p + (n^2 - n) \cdot p^2$



Case (1) $i = j$

$E[X_i X_j] = E[X_i^2] = E[X_i] = p$

Case (2) $i \neq j$

$E[X_i X_j] \underset{\text{indep}}{=} E[X_i] E[X_j] = p^2$

$\text{var}(B) = E[B^2] - E[B]^2$

$= np + n^2p^2 - np^2 - (np)^2$

$= np(1-p) = npq, \quad q = 1-p$

$E[B^3] = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n E[X_i X_j X_k]$

$\boxed{\begin{array}{l} \text{① all different} \\ \text{② two the same} \\ \text{③ all the same} \end{array}}$

Moment generating function.

Given r.v. B, $\varphi(t) = E[e^{tB}] \qquad \varphi(0) = 1$

$\varphi'(t) = E\left[\frac{d}{dt} e^{tB}\right] = E[B \cdot e^{tB}], \quad \varphi'(0) = E[B]$

$\varphi''(t) = E[B^2 e^{tB}], \quad \varphi''(0) = E[B^2]$

$\varphi^{(k)}(0) = E[B^k]$.

$\varphi_B(t) = E[e^{t(X_1 + \dots + X_n)}] = E[e^{tX_1} \cdot e^{tX_2} \cdot \dots \cdot e^{tX_n}]$.

$= \prod_{i=1}^n E[e^{tX_i}]$

$E[e^{tX_i}] = (1-p)\cdot e^0 + pe^t = pe^t + 1 - p.$

$\varphi_B(t) = (pe^t + (1-p))^n$

$\varphi'(t) = n(pe^t + 1-p)^{n-1} \cdot pe^t = np\, e^t \cdot (pe^t + 1-p)^{n-1}$

$\varphi''(t) = np\{ e^t \cdot (pe^t + 1-p)^{n-1}$

$+ e^t \cdot (n-1)(pe^t + 1-p)^{n-2} \cdot pe^t \}.$

$= np\{ e^t \cdot (pe^t + 1-p)^{n-1} + pe^{2t} \cdot (n-1) \cdot (pe^t + 1-p)^{n-2} \}$

$\varphi''(0) = np\{ 1 + p\cdot(n-1) \} = np + n^2p^2 - np^2$

$\varphi'''(t) = \ldots$