THE UNIVERSITY OF HONG KONG

FACULTY OF ENGINEERING

DEPARTMENT OF COMPUTER SCIENCE

COMP2121 Discrete Mathematics: Quiz 1

Date: 2 Oct, 2018 Time: 9:30am - 10:20am

Write your **university number here** and also on **every page**: ______________________

Please read the following before you begin.

1. Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is your responsibility to ensure that your calculator operates satisfactorily, and you must record the **name and type of the calculator** here:

   ____

2. This is a **closed book** quiz.
3. You are advised to spend around 5 minutes to go through all the questions before you begin writing. Allocate the time for each part of the question carefully.
4. Please write your answer in the designated space.
5. Attempt ALL questions. The full mark for the quiz is 100 points.

For marking use:

Question 1

Question 2

Question 3

Question 4

Question 5

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Question 1 (15 points)

Let p, q, and r be the propositions

p**:** Grizzly bears have been seen in the area.

q**:** Hiking is safe on the trail.

r**:** Berries are ripe along the trail.

Write these propositions, (a) to (c), using p, q, and r and logical connectives.

(a) (5 pts) Hiking is not safe on the trail if grizzly bears have been seen in the area.

Solution: $p \to \neg q$

(b) (5 pts) If grizzly bears have not been seen in the area or berries are not ripe along the trail, hiking is safe on the trail.

Solution: $(\neg p \vee \neg r) \to q$

(c) (5 pts) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

Solution: $r \to (q \leftrightarrow \neg p)$

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Question 2 (20 points)

Show that $(\neg p \to q) \to r$ and $(p \to r) \land (q \to r)$ are logically equivalent by logical equivalences.

Solution:

$$
\begin{array}{cr}
 & (\neg p \to q) \to r \\
\Leftrightarrow & (p \lor q) \to r \\
\Leftrightarrow & \neg(p \lor q) \lor r \\
\Leftrightarrow & (\neg p \land \neg q) \lor r \\
\Leftrightarrow & (\neg p \lor r) \land (\neg q \lor r) \\
\Leftrightarrow & (p \to r) \land (q \to r)
\end{array}
$$

Consequently $(\neg p \to q) \to r$ and $(p \to r) \land (q \to r)$ are logically equivalent.

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Question 3 (25 points)

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a white cat","x is a black cat","x can catch mice", and "x is a bad cat"respectively. Suppose the domain for x consists of all cats. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

(a) (5 pts) All white cats can catch mice.

Solution: $\forall x(P(x) \rightarrow R(x))$.

(b) (5 pts) No black cats are bad.

Solution: $\neg\exists x(Q(x) \wedge S(x))$ (or $\forall x(Q(x) \rightarrow \neg S(x))$ or $\forall x(\neg Q(x) \vee \neg S(x))$).

(c) (5 pts) All cats are bad or can catch mice.

Solution: $\forall x(S(x) \vee R(x))$ (or $\forall x(\neg S(x) \rightarrow R(x))$).

(d) (5 pts) If a cat is black or white, it can catch mice.

Solution: $\forall x((P(x) \vee Q(x)) \rightarrow R(x))$.

(e) (5 pts) Does (d) follow from (a), (b) and (c)? Why?

Solution: (d) is the conclusion from follow (a), (b) and (c).

Because from (a), we get $\forall x(P(x) \rightarrow R(x))$.

From (b), we get $\neg\exists x(Q(x) \wedge S(x)) \Leftrightarrow \forall x(\neg Q(x) \vee \neg S(x)) \Leftrightarrow \forall x(Q(x) \rightarrow \neg S(x))$

From (c), we get $\forall x(S(x) \vee R(x)) \Leftrightarrow \forall x(\neg S(x) \rightarrow R(x))$.

Added up all known facts, we get

$$
\begin{aligned}
& \forall x((P(x) \rightarrow R(x)) \wedge (Q(x) \rightarrow \neg S(x)) \wedge (\neg S(x) \rightarrow R(x))) \\
\Rightarrow \quad & \forall x((P(x) \rightarrow R(x)) \wedge (Q(x) \rightarrow R(x))) \\
\Leftrightarrow \quad & \forall x((\neg P(x) \vee R(x)) \wedge (\neg Q(x) \vee R(x))) \\
\Leftrightarrow \quad & \forall x((\neg P(x) \wedge \neg Q(x)) \vee R(x)) \\
\Leftrightarrow \quad & \forall x(\neg(P(x) \vee Q(x)) \vee R(x)) \\
\Leftrightarrow \quad & \forall x((P(x) \vee Q(x)) \rightarrow R(x))
\end{aligned}
$$

which is exactly the same as (d).

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Question 4 (20 points)

The Fibonacci numbers are defined as follows. $f(1) = f(2) = 1$, $f(n) = f(n-1) + f(n-2)$ for all $n \geq 3$. Use mathematical induction to prove that for all $n \geq 2$,

$$f(n-1)f(n+1) = f(n)^2 + (-1)^n.$$

Solution:

Base Case: $f(3) = f(1) + f(2) = 2$. When $n = 2$, $f(1)f(3) = 2 = f(2)^2 + (-1)^2$. So we know the statement is true for $n = 2$.

Inductive Step: We are going to show that $f(k)f(k+2) = f(k+1)^2 + (-1)^{k+1}$ is true for $n = k+1$ given the assumption that $f(k-1)f(k+1) = f(k)^2 + (-1)^k$ is true for $n = k \geq 2$. By induction hypothesis, we have $f(k)^2 = f(k-1)f(k+1) - (-1)^k$. Then by definition,

$$\begin{aligned}
f(k)f(k+2) &= f(k)[f(k+1) + f(k)] \\
&= f(k)f(k+1) + f(k)^2 \\
&= f(k)f(k+1) + f(k-1)f(k+1) - (-1)^k \\
&= [f(k) + f(k-1)]f(k+1) + (-1)^{k+1} \\
&= f(k+1)f(k+1) + (-1)^{k+1} = f(k+1)^2 + (-1)^{k+1}.
\end{aligned}$$

By mathematical induction, we know that the statement is true for all $n \geq 2$.

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Question 5 (20 points)

Determine whether or not each of the following statements defines an equivalence relation on the set A. If it does, prove your answer. Otherwise, provide a counterexample to show that it does not.

1. (8 pts) A is the set of all circles in the same plane. A relation R is defined on A by aRb if and only if a and b have the same center.

   Solution: Yes, this is an equivalence relation.

   Reflexive: For each $a \in A$, since a has the same center as itself, we have aRa.

   Symmetric: For each $a, b \in A$ such that aRb, since a and b have the same center, b and a have the same center. Thus, we have bRa.

   Transitive: For each $a, b, c \in A$ such that aRb and bRc, since a and b have the same center and b and c have the same center, we have a and c also have the same center. Thus, we have aRc.

2. (6 pts) A is the set of all points in the same plane **without** the origin. A relation R is defined on A by aRb if and only if there is a line containing a and b that also contains the origin.

   Solution: Yes, this is an equivalence relation.

   Reflexive: For each $a \in A$, obviously there is a line containing a and the origin. Thus, we have aRa.

   Symmetric: For each $a, b \in A$ such that aRb, since there is a line containing a and b that also contains the origin, that line also contains b, a and the origin. Thus, we have bRa.

   Transitive: For each $a, b, c \in A$ such that aRb and bRc, we know there are one line l_1 containing a, b and the origin and one line l_2 containing b, c and the origin. Since b and the origin are two different points, the line containing b and the origin is unique. Hence, l_1 and l_2 are the same line. Then, l_1 (l_2) contains a, c and the origin and we have aRc.

3. (6 pts) A is the set of all points in the same plane. A relation R is defined on A by aRb if and only if there is a line containing a and b that also contains the origin.

   Solution: No, because the relation is not transitive.

   One of the many counterexamples: Consider the points $a = (1, 1)$, $b = (0, 0)$ and $c = (-1, 1)$. The line containing a and b also contains the origin (which is b). And the line containing b and c also contains the origin. Hence, we have aRb and bRc. But the line containing a and c does not pass through the origin. Thus, $(a, c) \notin R$ and the relation is not transitive.

END OF PAPER