Lecture 16: Minimum Spanning Trees

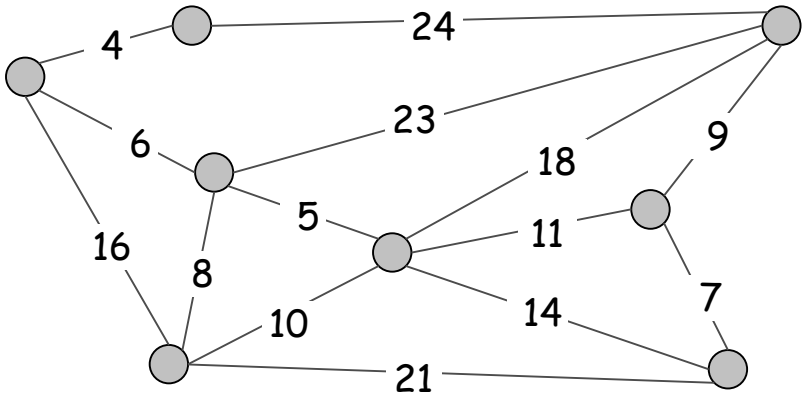
Version of March 27, 2018

Minimum Spanning Trees

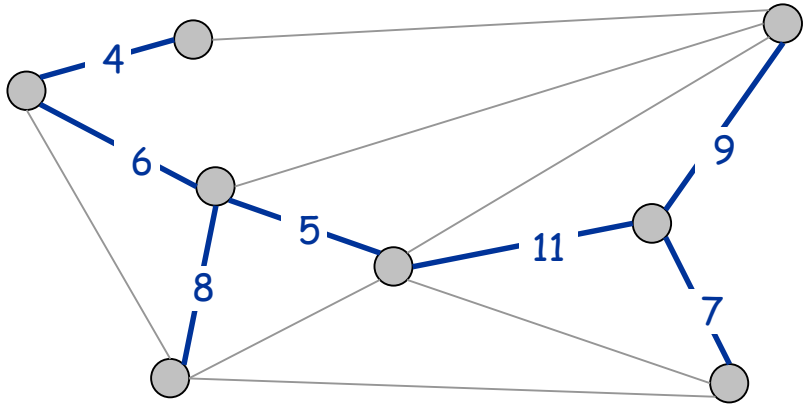
- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Minimum Spanning Tree

Minimum spanning tree. Given a connected undirected graph $G = (V, E)$ with real-valued edge weights $w(e)$, an MST is a subset of the edges $T \subseteq E$ such that T is a tree that connects all nodes whose sum of edge weights is minimized.



$G = (V, E)$



$T, \Sigma_{e \in T}\, w(e) = 50$

Applications: telephone networks, electrical and hydraulic systems , TV cables, computer networks, road systems

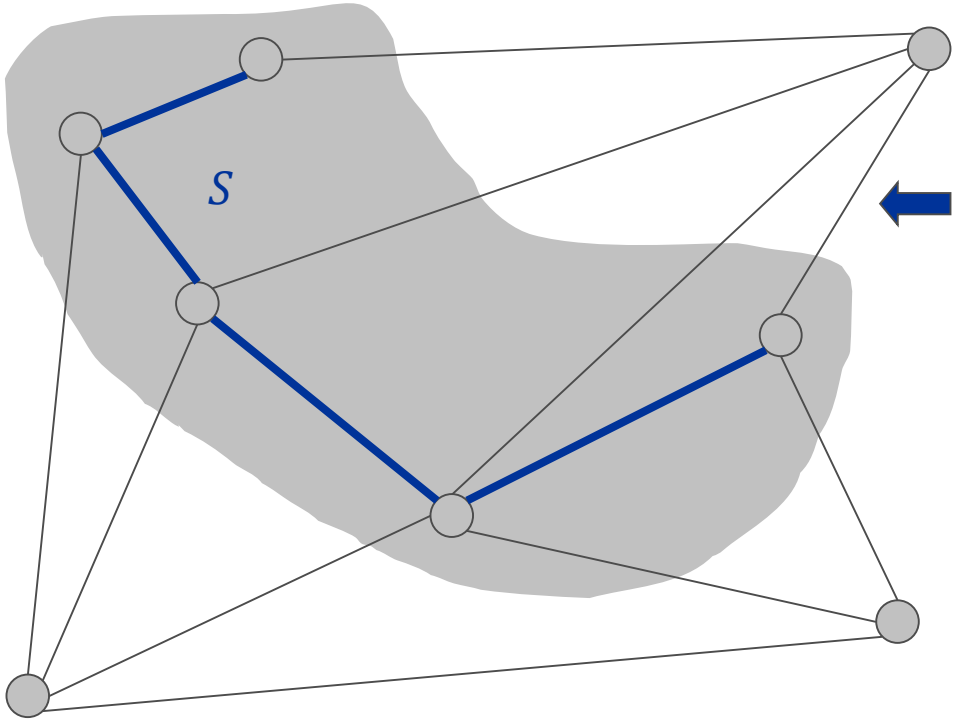
Minimum Spanning Trees

- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Prim's Algorithm: Idea

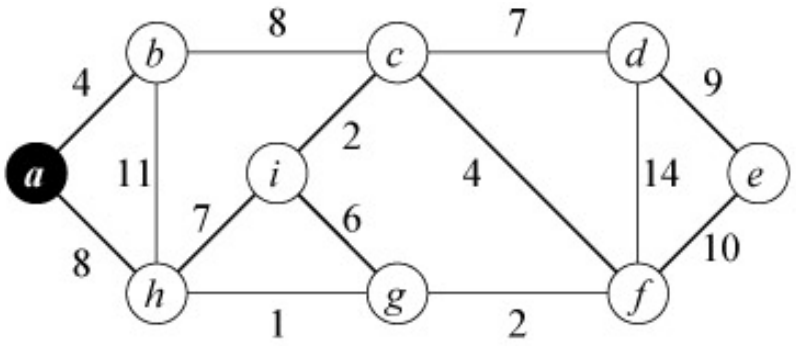
Prim's algorithm

- Initialize S = {any one node}.
- Add min cost edge $e = (u, v)$ with $u \in S$ and $v \in V - S$ to T.
- Add v to S.
- Repeat until $S = V$

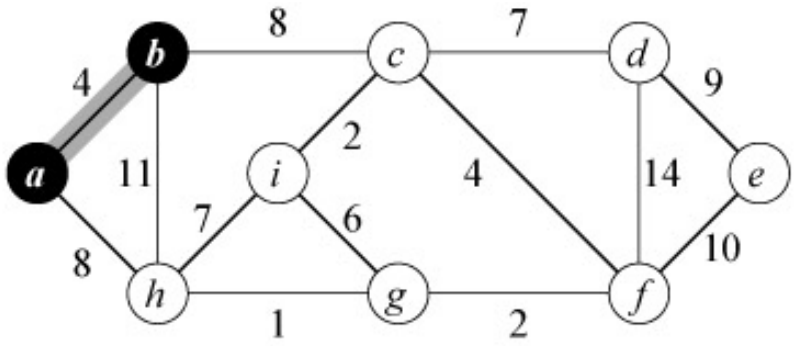


Prim's Algorithm: Example

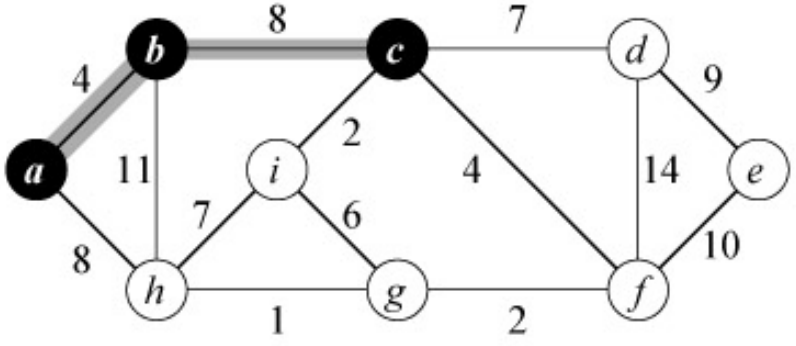
(a)



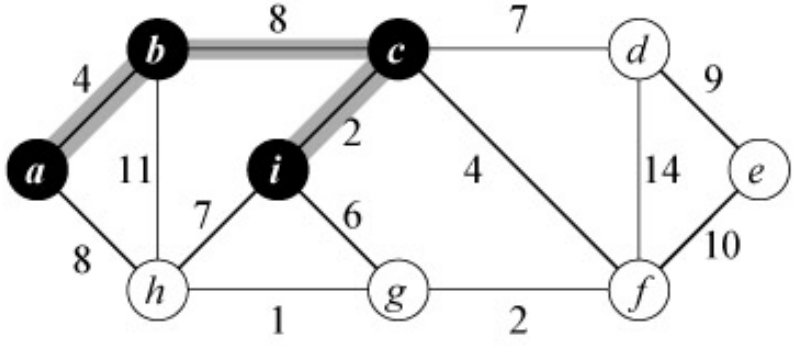
(b)



(c)

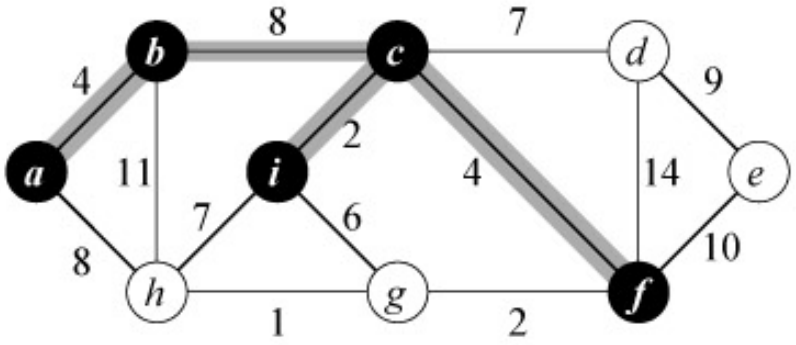


(d)

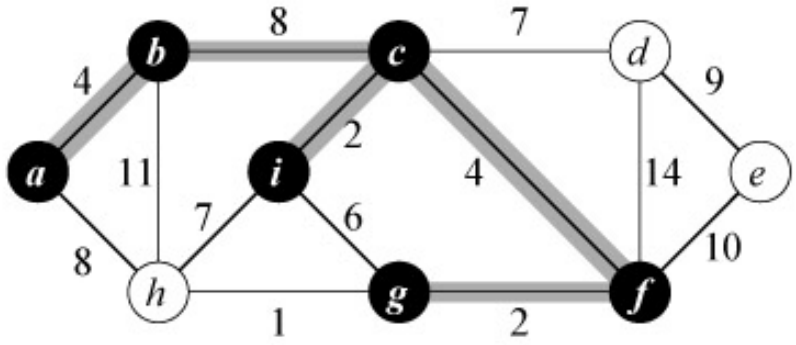


Prim's Algorithm: Example (continued)

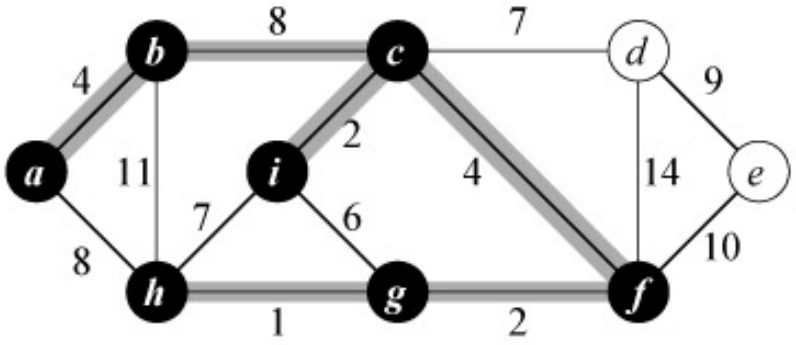
(e)



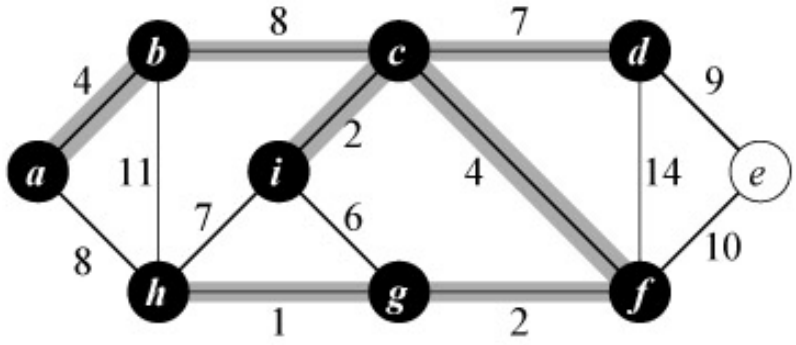
(f)



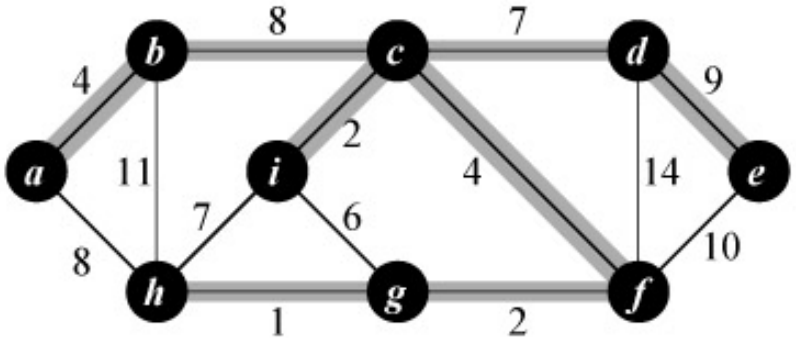
(g)



(h)



(i)



Prim's Algorithm: Implementation

Implementation.

- Maintain set of explored nodes S.
- For each unexplored node v, maintain ***cheapest*** edge from v to node in S.
- Maintain all nodes in a priority queue with this cheapest edge as key

Prim(G, r):

for each $v \in V$ **do**
    $v.key \leftarrow \infty, v.p \leftarrow nil, v.color \leftarrow white$
$r.key \leftarrow 0$
create a min priority queue Q **on** V
while $Q \neq \emptyset$
    $u \leftarrow$ **Extract-Min(**Q**)**
    $u.color \leftarrow black$
    for each $v \in Adj[u]$ **do**
        if $v.color = white$ **and** $w(u,v) < v.key$ **then**
            $v.p \leftarrow u$
            $v.key \leftarrow w(u,v)$
            Decrease-Key($Q, v, w(u,v)$**)**

Note: In the end, the parent pointers form the MST.

Running time:
$O(E \log V)$

Q: Decrease-key needs the location of the key in the heap. How to get that?

Minimum Spanning Trees

- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Cut Lemma

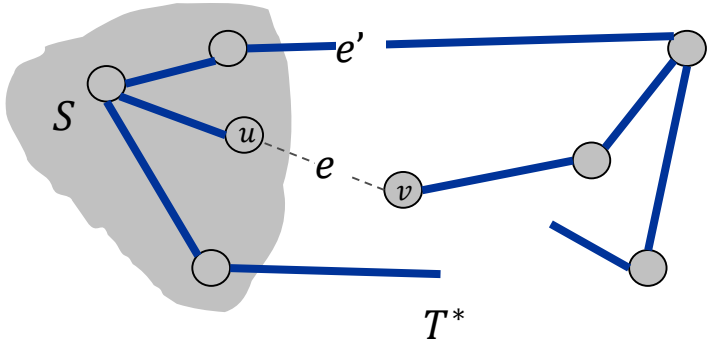
Simplifying assumption. All edge weights are distinct.

Cut lemma. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then every MST must contain e.

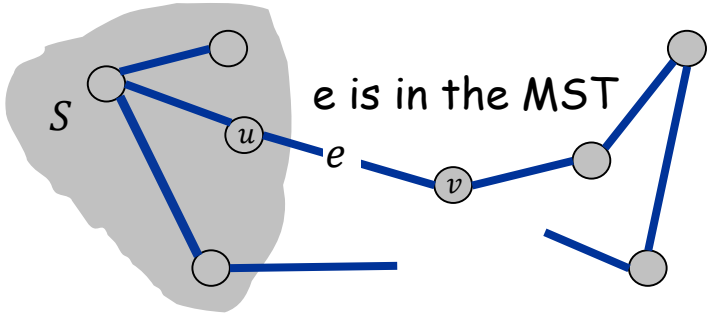
Correctness of Prim's Algorithm: Apply lemma between the black and white vertices.

Pf of Cut Lemma. (exchange argument)

- Let T^* be any MST.
- Let $e = (u, v)$ and suppose $e \notin T^*$.
- There is a path in T^* that connects u to v, which must cross cut separating S from $V - S$ using some other edge $e' \in T^*$ with $w(e') > w(e)$.



- If we replace e' in T^* with e, then T^* is still a spanning tree, but the total cost will be lowered, **contradicting fact that T^* is an MST.**

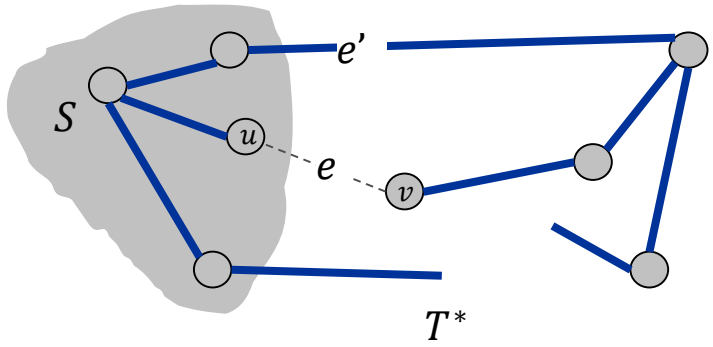


=> e is in every MST!

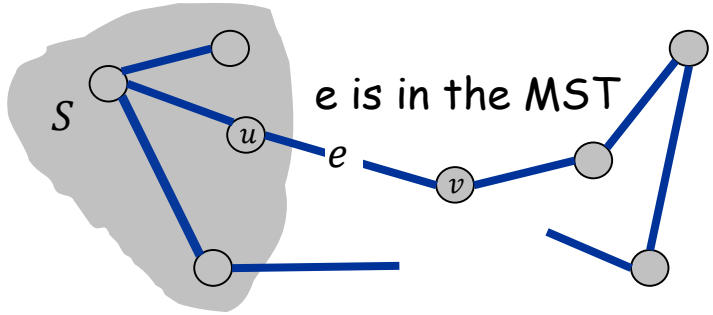
Cut Lemma: Deeper

Pf of Cut Lemma. (exchange argument)

- Let T^* be any MST.
- Let $e = (u, v)$ and suppose $e \notin T^*$.
- There is a path in T^* that connects u to v, which must cross cut separating S from $V - S$ using other edge $e' \in T^*$ with $w(e') > w(e)$.



- If we replace e' in T^* with e, then T^* is still a spanning tree, but the total cost will be lowered, **contradicting fact that T^* is an MST.**



Why is boxed statement correct?

Adding e to T^* creates a cycle that contains both e & the unique path in T^* connecting u to v.

Removing ANY edge from the cycle will keep the graph connected (and keep it a tree).

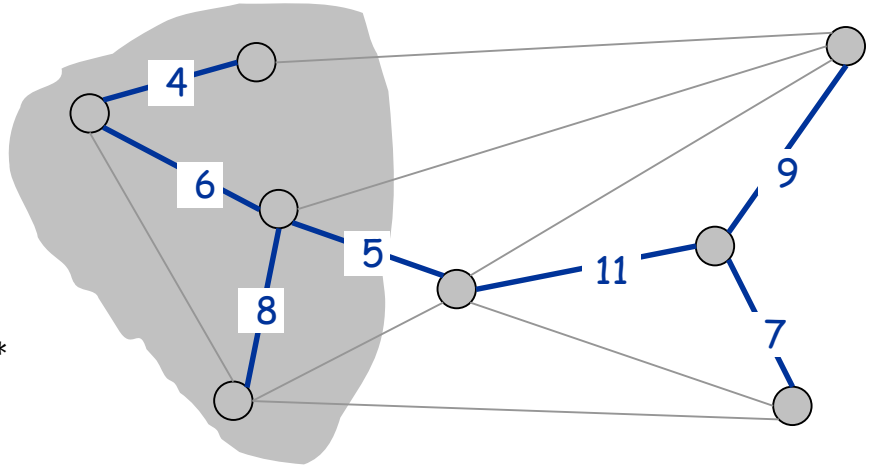
In particular, $T^* \cup \{e\} - \{e'\}$, is a tree!

$\Rightarrow$ $w(T^* \cup \{e\} - \{e'\}) = w(T^*) + w(e) - w(e') < w(T^*)$

Contradicting fact that T^* was supposed to be a MINIMUM Spanning Tree.

Uniqueness of MST

Theorem: The MST is unique.



Pf:

- Let T^* be *some* MST.
- Consider any edge $e \in T^*$
- Removing e from T^* breaks T^* into two parts S and $V - S$
- e must be the min cost edge crossing the cut $(S, V - S)$.

  (If not, replace e with the min cost edge and ***improve*** *the MST.)*
- Applying the cut lemma on S, we know that every MST must contain e.
- Applying the above argument to every edge in T^*, we have
  - **Every edge e in T^* must be contained in every MST**
  - Any spanning tree contains exactly $V - 1$ edges.
  - So, every MST contains exactly the same $V - 1$ edges as T^*, i.e., is T^*

Note: If there are edges with equal weights, then the MST may not be unique.

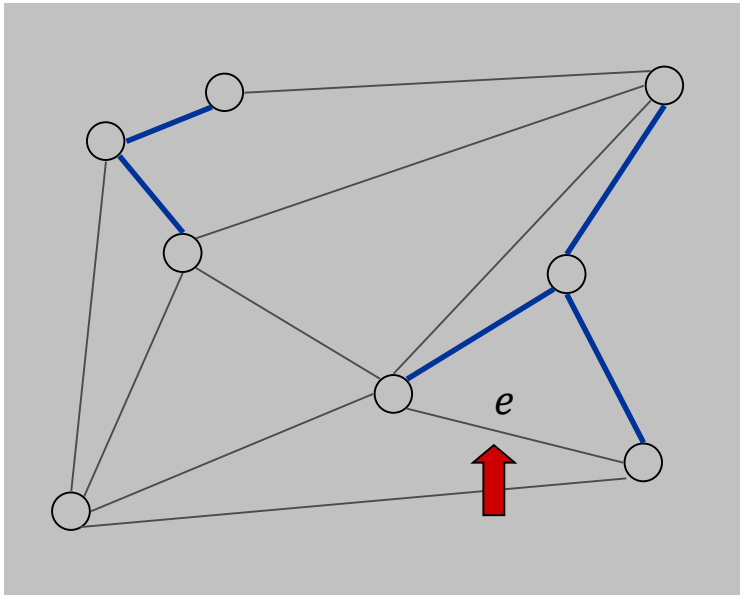
Minimum Spanning Trees

- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Kruskal's Algorithm: Idea

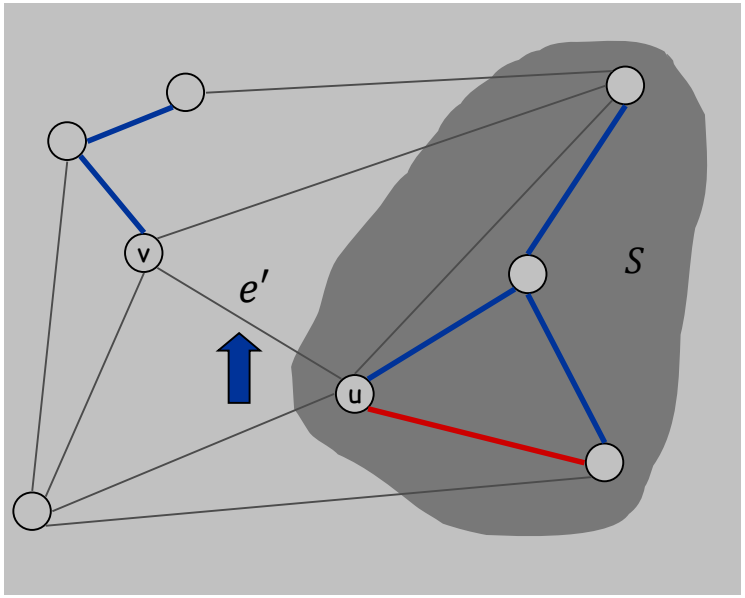
Kruskal's algorithm.

- Starts with an empty tree T
- Consider edges in ascending order of weight.
- Case 1: If adding e to T creates a cycle, discard e.
- Case 2: Otherwise, insert $e = (u, v)$ into T according to cut lemma



Case 1

e creates cycle.
Don't add e to T

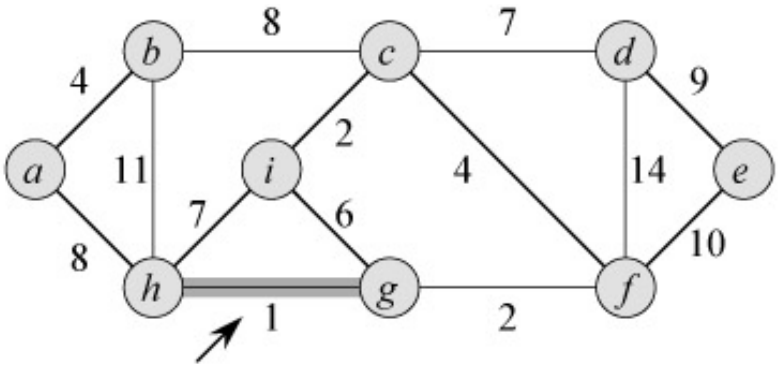


Case 2

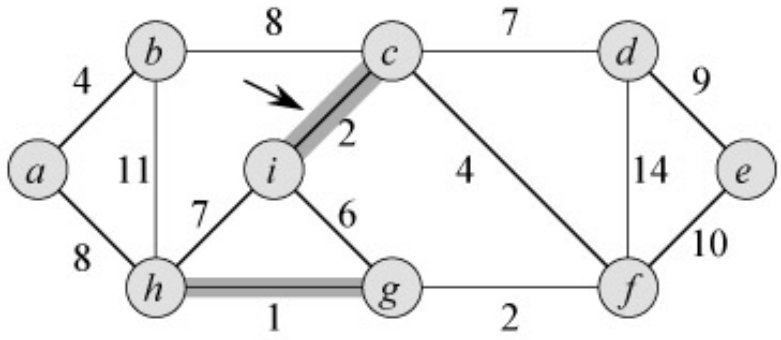
e' doesn't create cycle.
Add e' to T

Kruskal's Algorithm: Example

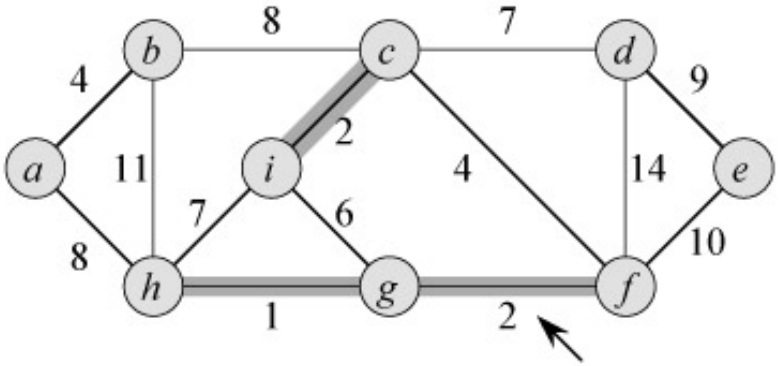
(a)



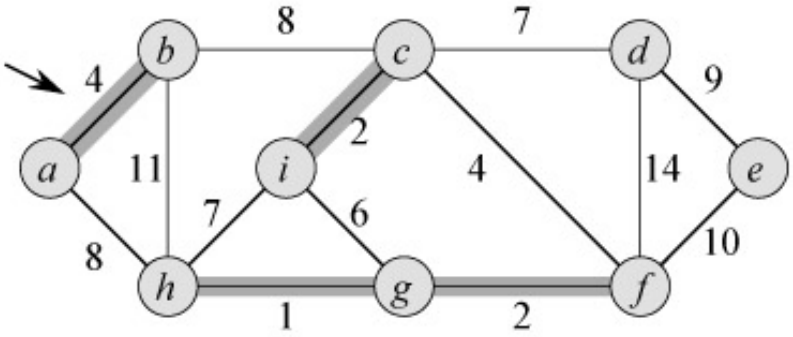
(b)



(c)

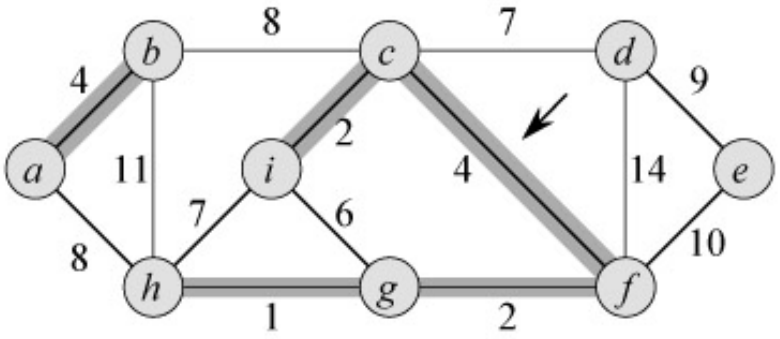


(d)

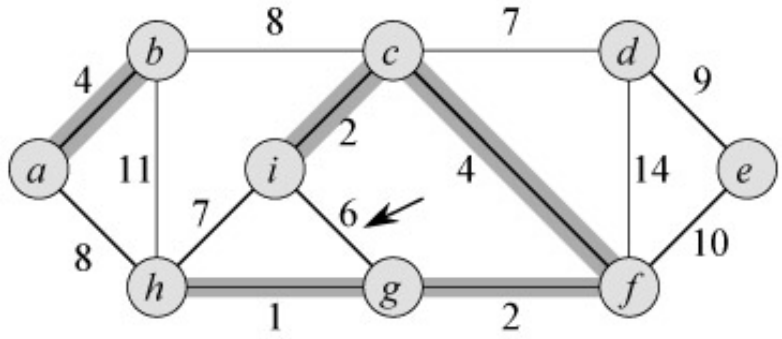


Kruskal's Algorithm: Example (continued)

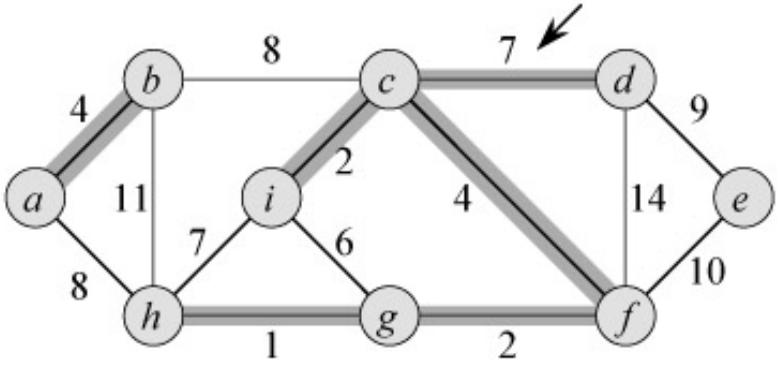
(e)



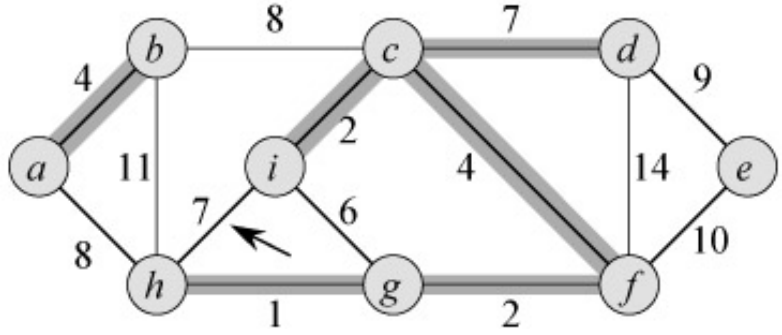
(f)



(g)

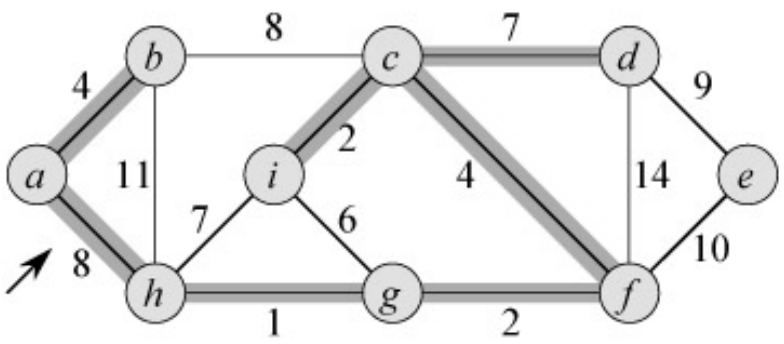


(h)

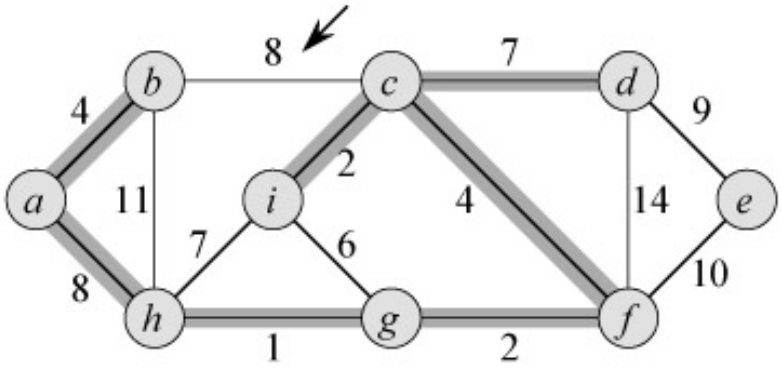


Kruskal's Algorithm: Example (continued)

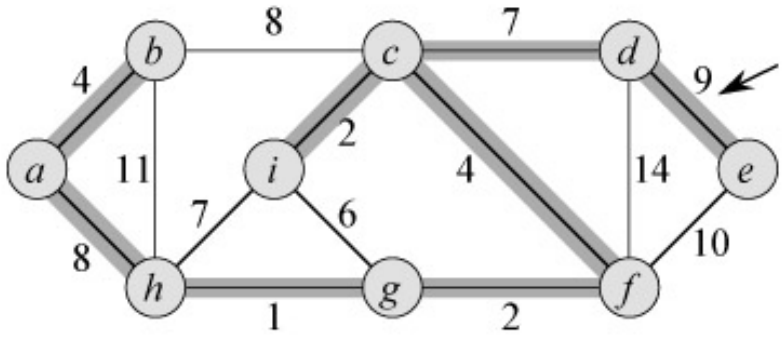
(i)



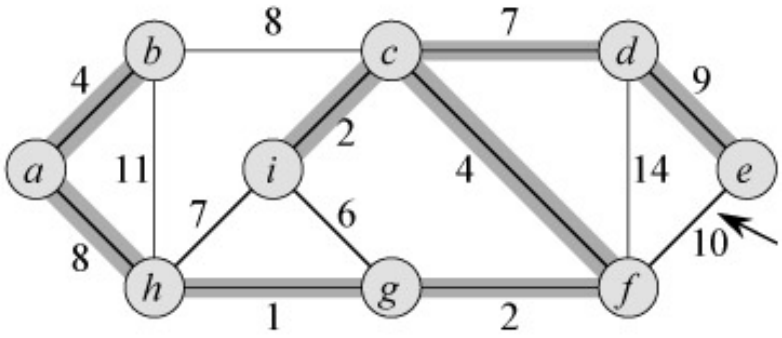
(j)



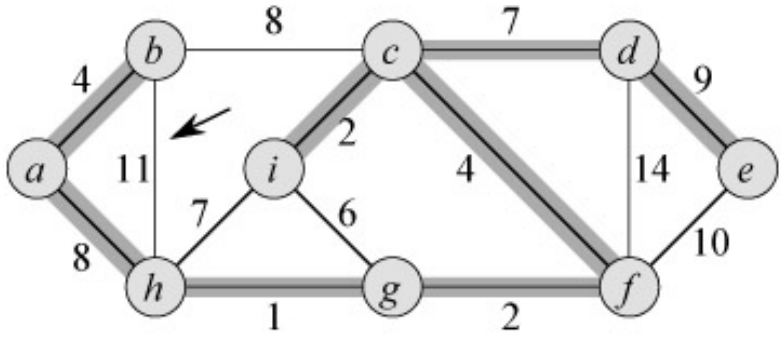
(k)



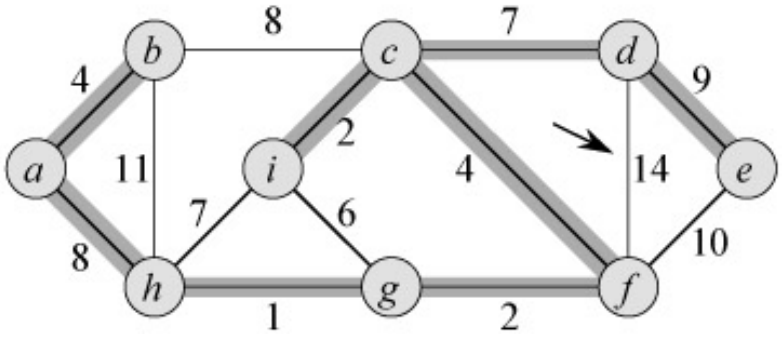
(l)



(m)



(n)



Minimum Spanning Trees

- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Kruskal's Algorithm: Implementation

Key question: How to check whether adding e to T will create a cycle?

- Use DFS?
  - Would result in $O(E \cdot V)$ total time.
- Can we do the checking in $O(\log V)$ time?

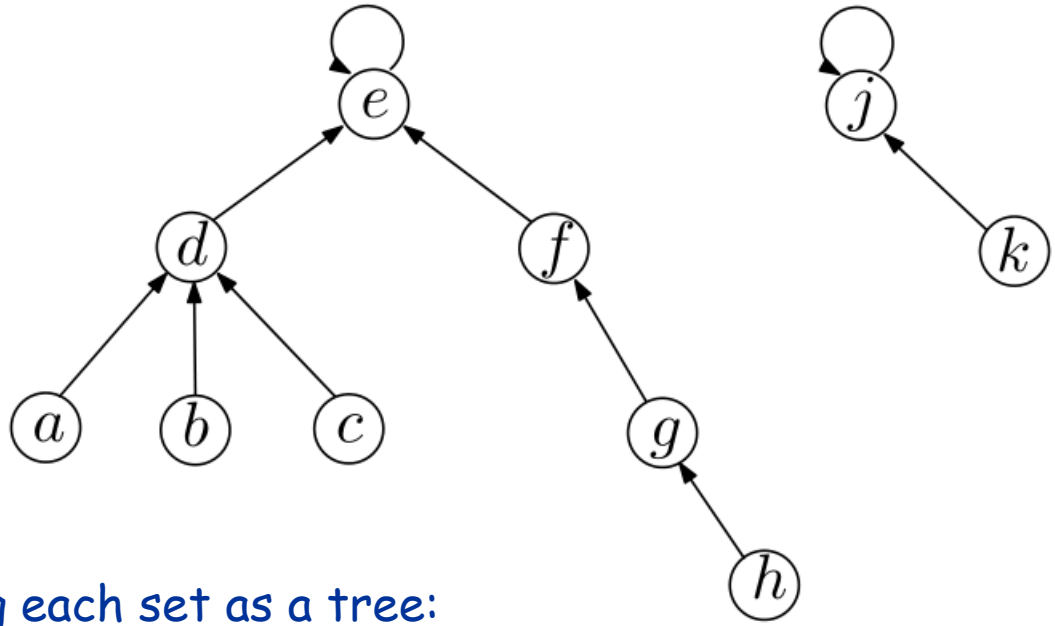
Observations:

- The actual structure of each component of T does not matter.
  - Each component can be considered as a set of nodes.
- After an edge is added, two sets "union" together.

Need such a "union-find" data structure:

- Maintain a collection of sets to support the following two operations:
- **`Find-Set(`**u**`)`**: For a given node u, find which set this node belongs to.
- **`Union`**(u, v): For two given nodes u and v, merge the two sets containing u and v together.

The union-find data structure



Representing each set as a tree:

- The trees in the union-find data structure are NOT the same as the partial MST trees!
- The root of the tree is the representative node of all nodes in that tree (i.e., use the root's ID as the unique ID of the set).
- Every node (except the root), has a pointer pointing to its parent.
  - The root has a parent pointer to itself.
  - No child pointers (unlike BST), so a node can have many children.

Make-Set(x) and Find-Set(x)

Create-Set(x):

```
Make-Set(x):
x.parent ← x
x.height ← 0
```

Find-Set(x):

```
Find-Set(x):
while x != x.parent do
      x ← x.parent
return x
```

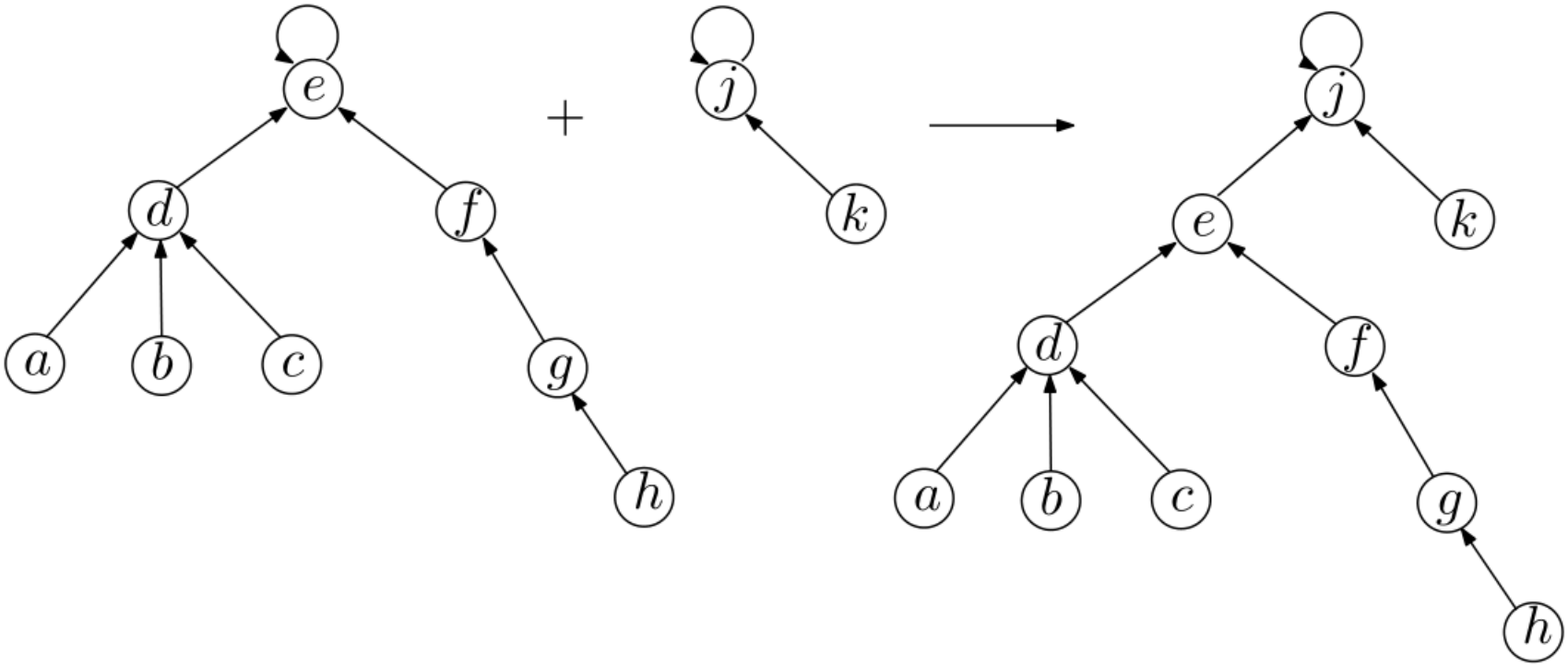
Running time proportional to the height of the tree.

Union(x, y)

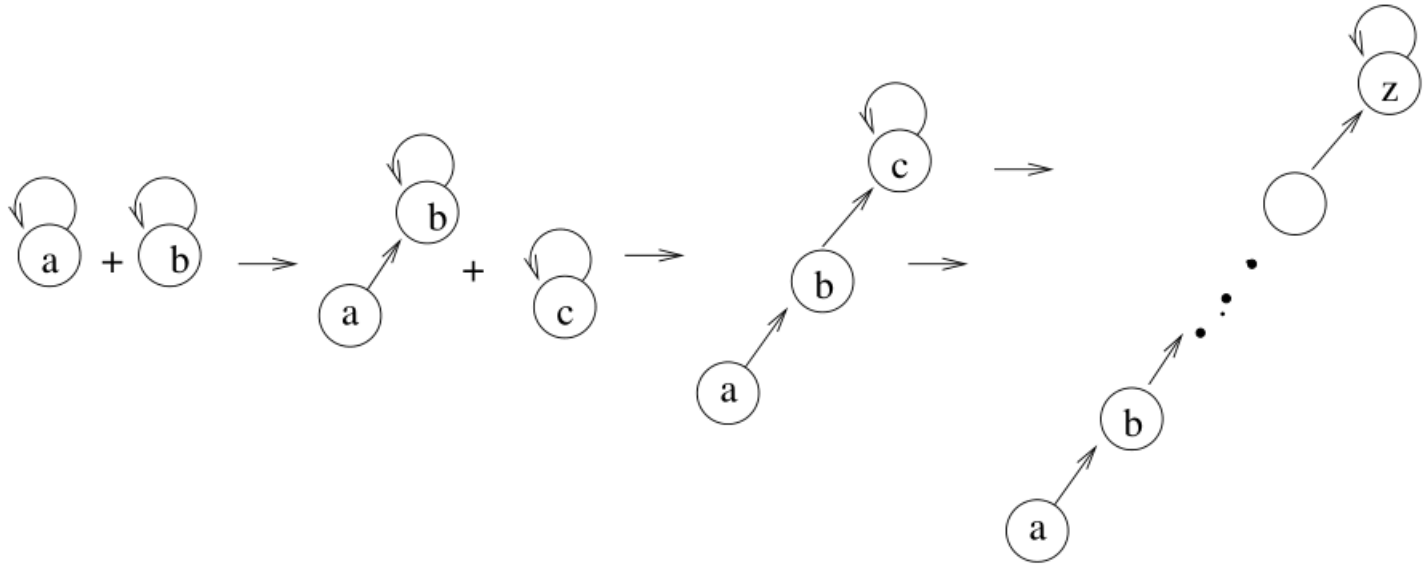
Assumption: x and y are the roots of their trees.

- If not, do Find-Set first

Idea: Set $x.parent \leftarrow y$



But, what if…



Solution (union by height):

- When we union two trees together, we always make the root of the taller tree the parent of shorter tree.
- Need to maintain the height of each tree

Union(x, y):
$a \leftarrow$ **Find-Set**(x)
$b \leftarrow$ **Find-Set**(y)
if $a.height \leq b.height$ **then**
    if $a.height = b.height$ **then**
        $b.height \leftarrow b.height + 1$
    $a.parent \leftarrow b$
else
    $b.parent \leftarrow a$

The union-find data structure: Analysis

Theorem: The running time of Find-Set and Union is $O(\log n)$

Pf: We will show (by induction) that for any tree with height h, its size is at least 2^h.

- At beginning, $h(x) = 0$, and $size(x) = 1$. We have $1 \geq 2^0$.
- Suppose the assumption is true for any x and y before Union(x, y). Let the size and height of the resulting tree be $size(x')$, and $h(x')$.

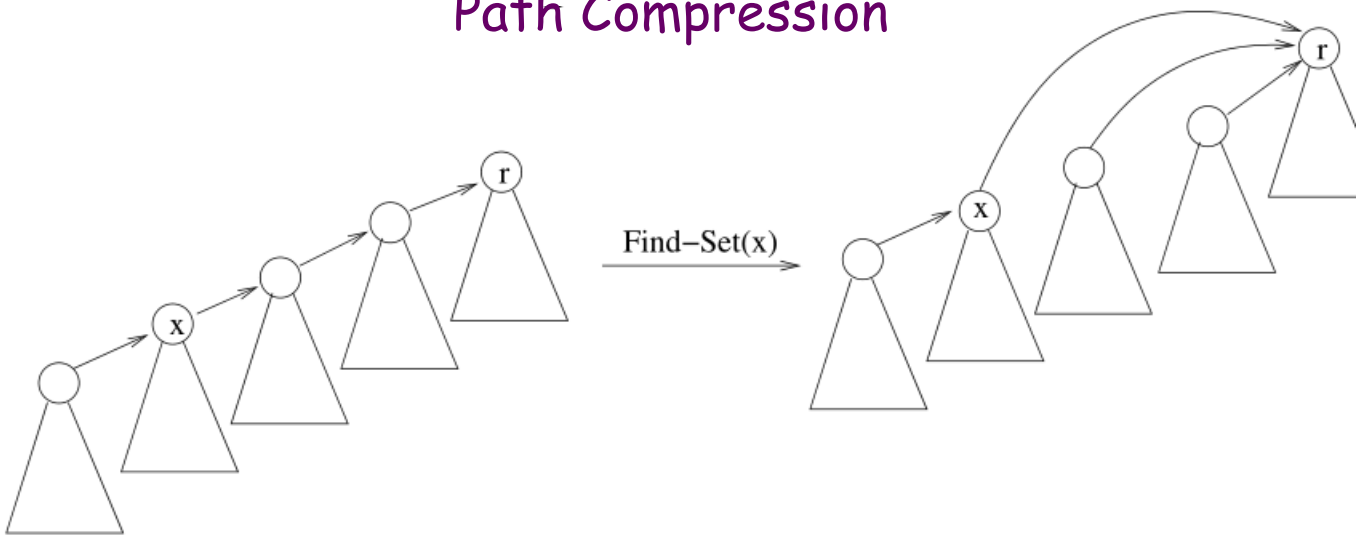
  - Case 1: $h(x) < h(y)$, we have
    $size(x') = size(x) + size(y) \geq 2^{h(x)} + 2^{h(y)} \geq 2^{h(y)} = 2^{h(x')}$.

  - Case 2: $h(x) = h(y)$, we have
    $size(x') = size(x) + size(y) \geq 2^{h(x)} + 2^{h(y)} = 2^{h(y)+1} = 2^{h(x')}$.

  - Case 3: $h(x) > h(y)$, similar to case 1.

- Consider any tree during the running of the algorithm. It must have $\leq n$ elements. Since a tree with height $\log n$ has at least n elements the tree has height $\leq \log n$. Thus all operations take $O(\log n)$ time.

Path Compression



Idea:

- We have visited a number of nodes after Find-Set(x), and have reached the root r.
- We already know that these nodes belong to the set represented by r.
- Why not just set the parent pointers of these nodes to r directly?
  - Future operations will be faster!

Analysis:

- This results in a running time that is practically a constant (but theoretically not).
- See textbook for details (not required).

Minimum Spanning Trees

- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Kruskal's Algorithm

```
MST-Kruskal(G):
for each vertex v ∈ V
    Make-Set(v)
sort the edges of G into increasing order by weight
for each edge (u,v) ∈ E taken in the above order
    if Find-Set(u) ≠ Find-Set(v) then
        output edge (u,v)
        Union(u,v)
```

Running time:

- $O(E \log E + E \log V) = O(E \log V)$

Note: If edges are already sorted and we use path compression, then the running time is close to $O(E)$.

Current best MST algorithm:

- An algorithm by Seth and Ramachandran (2002) has been shown to be optimal, but its running time is still unknown…

Minimum Spanning Trees

- Definition
- Prim's Algorithm
- The Cut Lemma
  - Correctness of Prim's Algorithm
  - Uniqueness of MSTs (under distinct weight assumption)
- Kruskal's Algorithm
  - Basic Idea
  - Union-Find Data Structure
  - Kruskal's algorithm
- Removing distinct weight assumption

Removing the distinct weight assumption

Our proofs of correctness for both Prim's and Kruskal's algorithm assumed that all the edges had different weights.

In fact, even if we remove this assumption (and allow some or even all edges to have the same weight) the algorithms still work. The only thing that needs to be changed is that, instead of choosing **the** smallest cost edge, we choose **a** smallest cost edge (breaking ties arbitrarily).

The proof of correctness for non-distinct edge weights is very similar to what we showed but a bit more complicated (see textbook for details).