Lecture 20: Stable Matching

The Stable Marriage Problem (input)

Goal. Given n men, n women, and their preference lists, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference lists

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference lists

The Stable Marriage Problem (input)

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- Participants rate members of opposite sex.
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- Each woman lists men in order of preference from best to worst.

- Problem historically stated in terms of men/women but actually has broad applications
  - Matching Medical students and Hospital residency places in US
  - Auction mechanisms for sponsored internet search
  - JUPAS
  - ……

The Stable Marriage Problem (output)

A set of n m-w pairs that constitute a perfect and stable matching

Perfect matching: everyone is matched *monogamously*.

- Each man is matched to exactly one woman.
- Each woman is matched to exactly one man.

Stability: no incentive for any unmatched pair to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to their current partners.
- Unstable pair m-w could each improve by divorcing their current partners and getting together.

Stable matching: perfect matching with no unstable pairs.

Stable marriage/matching problem.
Given the preference lists of n men and n women, find a stable matching (if one exists).

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Lists

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Lists

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

No. X-B is an unstable pair.

X and B prefer each other to their current matches (C and Y, respectively).

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Lists

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Lists

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	1st (favorite)	2nd	3rd (least favorite)
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Lists

	1st (favorite)	2nd	3rd (least favorite)
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Lists

Stable Matching Problem

Q. Do stable matchings always exist?
A. Yes. (This is not obvious; we will prove later)

Q. Is the stable matching unique?
A. No. It is possible that there are several stable matchings

favorite ↓ (1st) … least favorite ↓ (3rd)

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Men's Preference Lists

favorite ↓ (1st) … least favorite ↓ (3rd)

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Women's Preference Lists

Shaded boxes are stable matching X-B, Y-A, Z-C
Previous page had stable matching X-A, Y-B, Z-C for same lists!

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962]
Intuitive algorithm that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Shapley won Nobel Prize in Economics (partially) for this in 2012.
Gale was not eligible because he had died in 2008.

Proof of Correctness: Termination

Claim. Algorithm terminates after at most n^2 iterations of while loop.

Pf. A man starts from the first woman in his list and then continues in decreasing order of preference, without proposing to the same woman again. There are only n^2 possible proposals. ▪

Proof of Correctness: Perfection

Perfection means that in the end of the algorithm each man and woman gets matched to exactly one partner.

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that man Z is not matched upon termination of algorithm.
- Then some woman, say A, is not matched upon termination.
- By Observation 2, A was never proposed to.
- But, Z must have proposed to everyone, since he ends up single.
- Contradiction! ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to their partner in Gale-Shapley matching S*.

- Case 1: Z never proposed to A.
  $\Rightarrow$ Z prefers his GS partner to A. ← men propose in decreasing order of preference
  $\Rightarrow$ A-Z is stable.

- Case 2: Z proposed to A.
  $\Rightarrow$ A rejected Z (right away or later)
  $\Rightarrow$ A prefers her GS partner to Z. ← women only trade up
  $\Rightarrow$ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

S*
A-Y
B-Z
. . .

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue or stack.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - if m matched to w then `wife[m]=w` and `husband[w]=m`
  - Otherwise, set entry to `0` if unmatched

Men proposing.

- For each man, maintain a list (linked list or array) of women, ordered by preference.

Efficient Implementation

Women rejecting/accepting.

- Does woman `w` prefer man `m` to man `m'`?
- Naïve implementation requires O(n) time for this comparison.
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to man 6
since `inverse[3]` < `inverse[6]`

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Understanding the Solution

Q. For a given problem instance, several stable matchings might exist. Recall that Gale-Shapely gives us freedom to decide *which* man proposes. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

A. Yes, it always returns the (unique) matching that is optimal for the men (proof omitted).

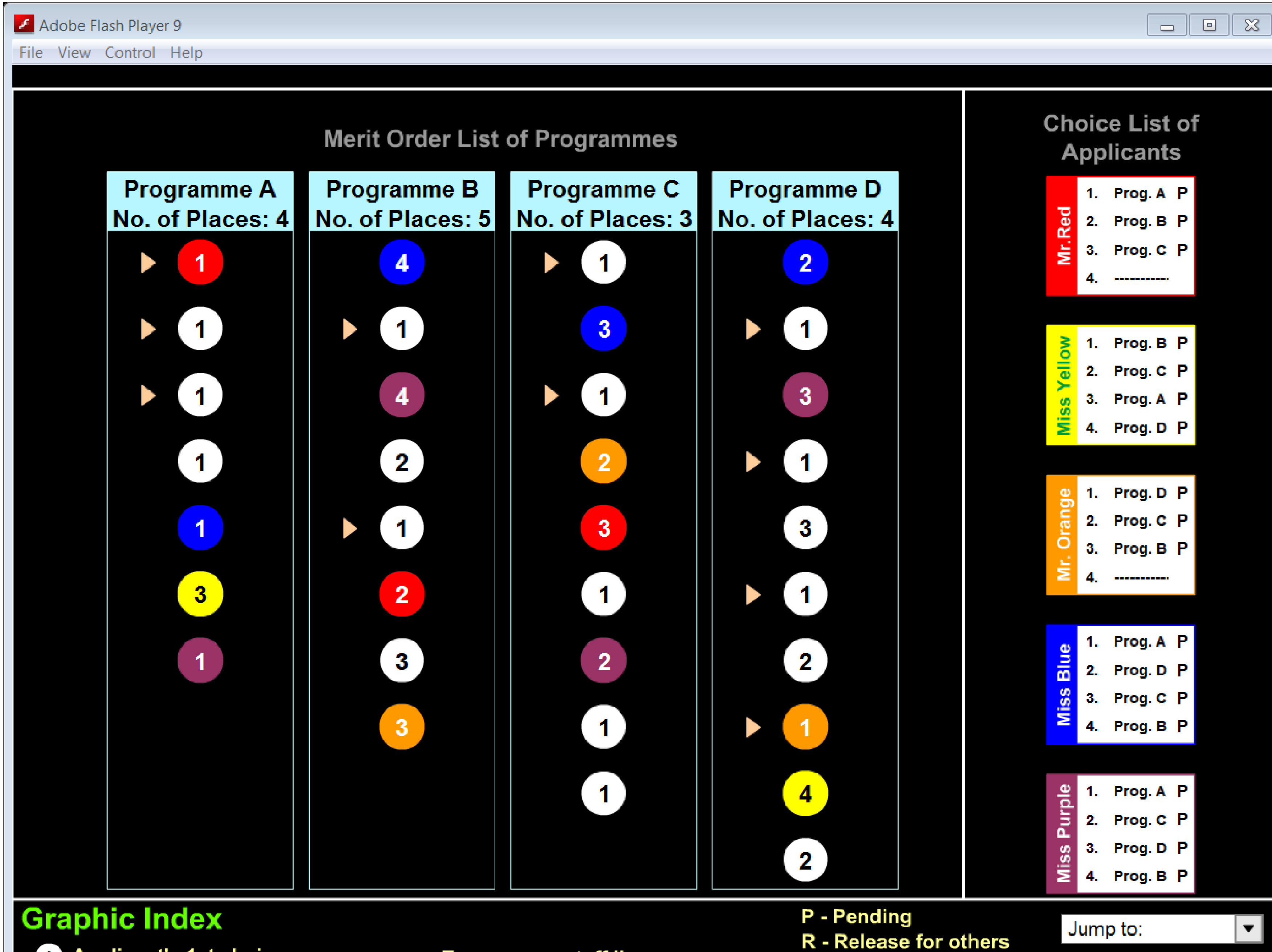
Optimal for the men means: each man gets his best possible partner in any possible stable matching.

Observation. Man and Women are not equivalent in the algorithm. Men propose, women accept/reject.

To get a woman optimal algorithm, have women propose and men accept/reject.

Several variants of the problem exist with multiple applications.

JUPAS Admission Scheme



JUPAS Admission Scheme

Applicants correspond to men that order their programme choices (which correspond to women).

Differences w.r.t. to the basic version of the problem:

1. Many more applicants than programmes (places)
2. A programme is matched to many (instead of one) applicants. # it's matched to is its *intake*

http://www.jupas.edu.hk/

The JUPAS computer system will match (the "iteration process") the order of preference you have assigned to your programme choice list with the position you have been placed in each merit order list of these programmes.

You will then be made an offer of the highest priority on your programme choice list for which you have the required rating.

That is, the matching is optimal (i.e., fair) for the applicants!