COMP2121A: Discrete Mathematics
Problem Solving Session 2 Sep 18, 2017

Review. Proof methods, $p \to q \Leftrightarrow \neg p \vee q$

1. direct proof
2. proof by contradiction, assume $p \wedge \neg q$, try to find a contradiction
3. proof by contrapositive, prove $\neg q \to \neg p$
4. mathematical induction, show that $P(x)$ is true for all $x \in \bigcup_{i \geq 0} S_i$
   (a) base case, prove $P(x)$ is true for all $x \in S_0$
   (b) inductive step, given $P(x)$ is true for all $x \in \bigcup_{0 \leq i \leq k-1} S_i$, prove $P(x)$ is true for all $x \in S_k$

Questions.

1. Given a real number x and an positive integer n, show an efficient method to evaluate x^n with only multiplications and additions.

   Solution: Please refer to page 28 of the lecture slides 02-proof or the written notes for the solution.

2. Define $f(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3$. Use mathematical induction to prove that $f(n) = \left[\frac{n\cdot(n+1)}{2}\right]^2$ for all positive integers.

Solution:

- Base Case: When $n = 1$, $f(1) = 1 = \left[\frac{1\cdot(1+1)}{2}\right]^2$. The statement is true.
- Inductive Step: Next we show the statement is true when $n = k + 1$, given the statement is true when $n = k$. By induction hypothesis, we have $f(k) = \left[\frac{k\cdot(k+1)}{2}\right]^2$. Then

$$\begin{aligned} f(k+1) &= f(k) + (k+1)^3 \\ &= \left[\frac{k\cdot(k+1)}{2}\right]^2 + (k+1)^3 \\ &= \frac{(k^2+4k+4)\cdot(k+1)^2}{4} \\ &= \frac{(k+2)^2\cdot(k+1)^2}{4} \\ &= \left[\frac{(k+1)\cdot(k+2)}{2}\right]^2 \end{aligned}$$

  Therefore, the statement is true when $n = k + 1$.

Then by mathematical induction, we conclude that the statement is true for all positive integers.

3. Prove the following statement. There exist irrational numbers x and y such that x^y is rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$. Is it rational or not?)

   Solution: Let $x = y = \sqrt{2}$. Then if x^y is rational we are done.
   Otherwise, let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We have $x^y = 2$. In either case, we can find such x and y.

4. Given a finite set A of n points on the plane (2-dimensional space) such that for any two points x, y in A, the line containing x and y must contain another point z in A. Prove that all points in A are on the same line.[1]

   (a) Is the following proof (induction on the number of points) correct? If not, where is the bug?

   - Base case: for point set of size 3 the statement is true.
   - Inductive step: assume this statement is true for point set of size $k \geq 3$. Consider the case when we have a point set A of size $k+1$. We argue as follows.

     i. Pick A' of k points from the given point set A. Let x be the other point in A but not in A'.

     ii. By *induction hypothesis*, points in A' are on the same line.

     iii. Pick any y in A', the line going through x, y contains another point z in A.

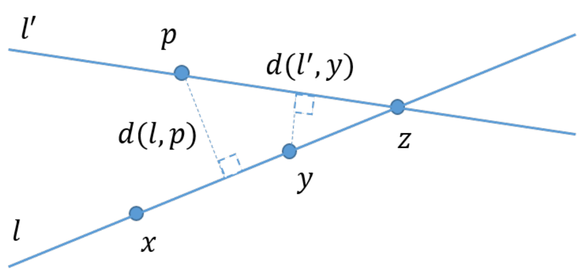
     iv. Thus, x, y and z are on the same line.

     v. So x and all points in A' are on the same line.

   Solution: In Step (ii), the induction hypothesis does not apply to the set A'. We are given A with the property that for any two points x, y in A, there is another point z in A on the line (x, y). However, this property might not be true for A'. It is possible that the point z on line (x, y) is not in the set A'.

   (b) Can you give a proof by contradiction?

   Solution: We assume there is a finite point set A as described by the statement, and they are not all on the same line.



   Consider a line l passing through three points x, y, z. Let p be a point not on l. Draw the perpendicular to the line l through the point p. At least 2 points, say y and z, are on the same side. Consider the line l' passing through p and z. We have the distance from y to line l' is smaller than the distance from p to l, i.e. $\text{dist}(y, l') < \text{dist}(p, l)$.

   By the property of set A, there is another point q on line l'. Given points p, q, z and y, We can apply the same process to find a distance from a new point to a new line that is smaller than $\text{dist}(y, l')$. We can keep doing this to find infinite

[1] This is equivalent to Sylvester-Gallai theorem, which is named after James Joseph Sylvester, who posed it as a problem in 1893, and Tibor Gallai, who published one of the first proofs of this theorem in 1944.

amount of distances, which contradicts with the fact that the number of points in A is finite.