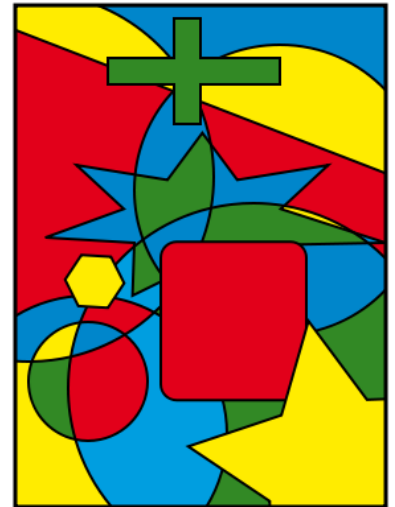


Part I Logic

- L01 Propositional Logic
- L02 Predicate Logic
- L03 Inference Rules and Proof Techniques

What is Logic?

- Puzzles
- Prerequisite of law schools (LSAT)
 - Riders A, B, C, D, E, F, and G take busses 1, 2, 3 to work.
 - Neither E nor G takes bus 1 on a day when B does.
 - G does not take bus 2 on a day when D does.
 - When A and F take the same bus, it is always bus 3.
 - C always takes bus 3.
 - Who takes which bus?
- Rules of mathematical reasoning
- Automated reasoning and artificial intelligence
 - Four color theorem
 - Still in progress



L01: Propositional Logic

- Outline
 - Propositions
 - Compound Propositions
 - Propositional Equivalences
- Reading
 - Kenneth Rosen, section 1.1-1.3

Proposition

- **Definition**

A **proposition** is a declarative statement (i.e., a sentence that declares a fact) that is either true or false, but not both

- **Remark**

Propositions are the basic building blocks of logic. The area of logic that deals with propositions is called **propositional logic**.

- **Definition**

The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and false, denoted by F, if it is a false proposition.

Examples

- Each of the following declarative statements is a proposition:
 - (a) Beijing is the capital of China.
 - (b) COMP 271 1 is an elective course for the COMP program.
 - (c) $2 + 2 = 2^2$
 - (d) $1 + 1 = 3$

Propositions (a) and (c) are true but (b) and (d) are false.

Examples

- These statements are not propositions:
 - (a) No parking
 - (b) Who has an iPad?
 - (c) $y = \log(x+1)$
 - (d) $x^2 - 3x + 1 = 0$

- (c) and (d) can become propositions if x and y are assigned values
- Logical paradoxes are not propositions
 - (a) This sentence is false. (Liar paradox)
 - (b) All I know is that I know nothing. (Socratic paradox)

Outline

- Introduction to Propositions
- **Compound Propositions**
- Propositional Equivalence

Logical Operator and Truth Table

- **Logical operators** or **logical connectives** can be used to turn existing propositions into new propositions.
- The definition of a logical operator can be given in the form of a **truth table** by enumerating **all possible truth values of the proposition(s)** involved.

Negation

- **Definition**

Let p be a proposition. The **negation** of p , denoted by $\neg p$ or \bar{p} and read as “not p ”, is the statement “**it is not the case that p** ”. The truth value of $\neg p$ is the opposite of the truth value of p .

p	$\neg p$
T	F
F	T

- **Example:**

- p = “Tom’s iPhone has at least 32GB of memory”
- $\neg p$ = “It is not the case that Tom’s iPhone has at least 32GB of memory”
- $\neg p$ = “Tom’s iPhone has less than 32GB of memory”₉

Conjunction

- **Definition**

Let p and q be propositions. The **conjunction** of p and q , denoted by $p \wedge q$, is the proposition “ p and q ”.

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- **Definition**

The **disjunction** of p and q , denoted by $p \vee q$, is the proposition “ p or q ”.

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

- **Definition**

The **exclusive or** of p and q , denoted by $p \oplus q$, is the proposition that is true when **exactly one of p and q** is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statement

- **Definition**

The **conditional statement** $p \rightarrow q$ is the proposition “if p , then q ”.

p is called the **hypothesis** (or **premise**) and q is called the **conclusion** (or **consequence**).

Also called an **implication**.

- Note: There does not need to be any connection between the hypothesis and the conclusion. The true value of $p \rightarrow q$ depends only on the truth values of p and q .

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples

- “If Beijing is the capital of China, then Beijing is a city of China”.
- True.
- “If Beijing is a city of China, then Beijing is the capital of China.”
- True.
- “If the moon is made of green cheese, then I have more money than Bill Gates. ”
- True.
- “If the moon is not made of green cheese, then I have more money than Bill Gates. ”
- False.

Necessary and sufficient conditions

- If $p \rightarrow q$ is true, then p is a **sufficient condition** of q , and q is a **necessary condition** of p
- Example:
 - “If I am elected, then taxes will be lowered.”
 - “I am elected” is a sufficient condition for “lower taxes”.
 - Other candidate may also lower taxes.
 - “Lower taxes” is a necessary condition for “I am elected”
 - If taxes are not lowered, then I am definitely not elected.

Other Ways to Express conditional statements in English

- Equivalent ways of expressing $p \rightarrow q$:
 - if p , then q
 - q if p
 - q when p
 - p implies q
 - q follows from p
 - p is a sufficient condition for q
 - q is a necessary condition for p
 - p only if q (falsity of q implies falsity of p)
 - “The company can succeed only if it has sufficient funding.”

Converse, Contrapositive

- **Definition**

The **converse** of $p \rightarrow q$ is $q \rightarrow p$.

The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

The **inverse** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

$p \rightarrow q$ and its contrapositive are **equivalent**.

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Biconditional Statement

- **Definition**

The **biconditional statement** $p \leftrightarrow q$ is the proposition “ p if and only if q ”.

The statement is true when p and q have the same truth value and is false otherwise.

- Equivalent ways of expressing

$p \leftrightarrow q$:

- p if and only if q
- p iff q
- p is necessary and sufficient for q
- if p then q , and conversely

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ambiguity in natural language

- Sometimes, “if ... then ...” in natural language actually means “if and only if”.
- “If you finish your meal, then you can have dessert”
- What if you don’t finish your meal?
 - In natural language: then you can’t have dessert.
 - In logic: unspecified
- In this course, we should avoid such ambiguities.

Precedence of Logical Operators

- Multiple logical operators can be used to construct **compound propositions**.
- Parentheses can be used for clarity

$$p \vee \neg q \rightarrow p \wedge q,$$

may be written more clearly as
 $(p \vee \neg q) \rightarrow (p \wedge q).$

- Remark
 - Boolean operations in programming languages (e.g., &&, ||, !, == in C++) correspond to \vee , \wedge , \neg , \leftrightarrow
 - “if ... then ...” statements in programming languages do **not** correspond to \rightarrow

Operator	Precedence
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5

Outline

- Introduction to Propositions
- Compound Propositions
- **Propositional Equivalence**

Tautology and Contradiction

- **Definition**

A **tautology** is a compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.

A **contradiction** is a compound proposition that is always false.

A **contingency** is a compound proposition that is neither a tautology nor a contradiction.

Tautology and Contradiction

- **Example**

$p \vee \neg p$ is a tautology

$p \wedge \neg p$ is a contradiction

$p \rightarrow \neg p$ is a contingency

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \rightarrow \neg p$
T	F	T	F	F
F	T	T	F	T

Logical Equivalence

- Compound propositions that always have the same truth values are called **logically equivalent**
- **Equivalent definition**
The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology.
- The notation is $p \equiv q$ (or $p \Leftrightarrow q$).
- **Remark**
The symbol \equiv is not a logical operator and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.

The use of \equiv , \leftrightarrow , $=$

- \equiv denotes logical equivalence regardless of truth values of the propositional variables
- \leftrightarrow is a binary logical operator
- $=$ is not used on propositions. It is only used between two mathematical quantities.
- Examples:
 - $(p \rightarrow q) \leftrightarrow (\neg p \vee q) \equiv T$
 - $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
 - $T \leftrightarrow F \equiv F$
 - $(2 = 3) \wedge (5 \neq 4) \equiv F$

Logical Equivalence

- **Example**

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q)$ \leftrightarrow $\neg p \wedge \neg q$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Logical Equivalence

- **Example**

Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Propositional Equivalences

- The following table summarizes the major propositional equivalences:

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

Propositional Equivalences (cont'd)

Equivalence	Name
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \top$ $p \wedge \neg p \equiv \text{F}$	Negation laws

Propositional Equivalences (cont'd)

Equivalence	Name
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$ $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$ $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	Involving conditional statements
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$	Involving biconditional statements

Examples

- **Example**

Use De Morgan's laws to express the negations of “Alice will send a secret message or Bob will send a secret message” and “today is Saturday and today is a holiday”.

- **Example**

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Revisit the Knight and Knave puzzle

- Let p be “A is knight” and q be “B is knight”
- A: “We are both knaves”: $\neg p \wedge \neg q$
- B says nothing
- Two possibilities
 - If A's claim is true, then $\neg p \wedge \neg q$ is true.
If A is telling the truth, then he is knight and p is true.
However, $\neg p \wedge \neg q$ and p cannot both be true.
 - If A's claim is false, then $\neg p \wedge \neg q$ is false, i.e., $\neg(\neg p \wedge \neg q)$ is true. Using de Morgan's law,
 $\neg(\neg p \wedge \neg q) \equiv p \vee q$.
Since A's claims is false, he is a khave, so p is false.
Therefore, q must be true.

Examples

- **Example**

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

- **Answer**

$$\begin{aligned}\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge (\neg(\neg p \wedge q)) \\ &\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) \\ &\equiv \neg p \wedge (p \vee \neg q) \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \\ &\equiv F \vee (\neg p \wedge \neg q) \\ &\equiv \neg p \wedge \neg q\end{aligned}$$

Examples

- **Example**

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology by developing a series of logical equivalences.