Lecture 18: Maximum Flow

Version of April 8, 2019

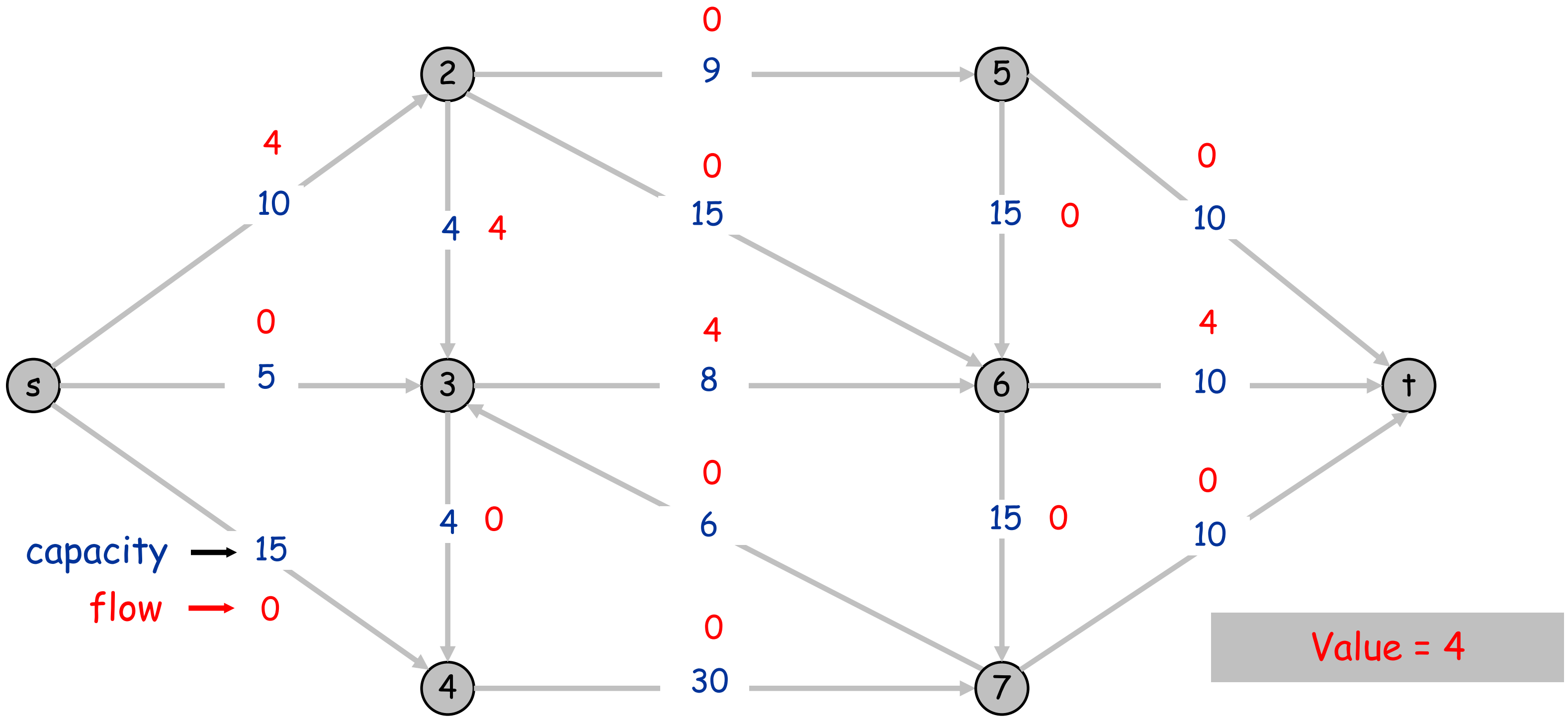
Flow

Input: A directed connected graph $G = (V, E)$, where

- every edge $e \in E$ has a capacity $c(e)$;
- a source vertex s and a target vertex t.

Output: A flow $f : E \to \mathbf{R}$ from s to t, such that

- For each $e \in E$, $0 \le f(e) \le c(e)$ (capacity)
- **For each** $\boldsymbol{v \in V - \{s, t\}}$, $\sum_{e \text{ out of } v} \boldsymbol{f(e)} = \sum_{e \text{ into } v} \boldsymbol{f(e)}$ (conservation)



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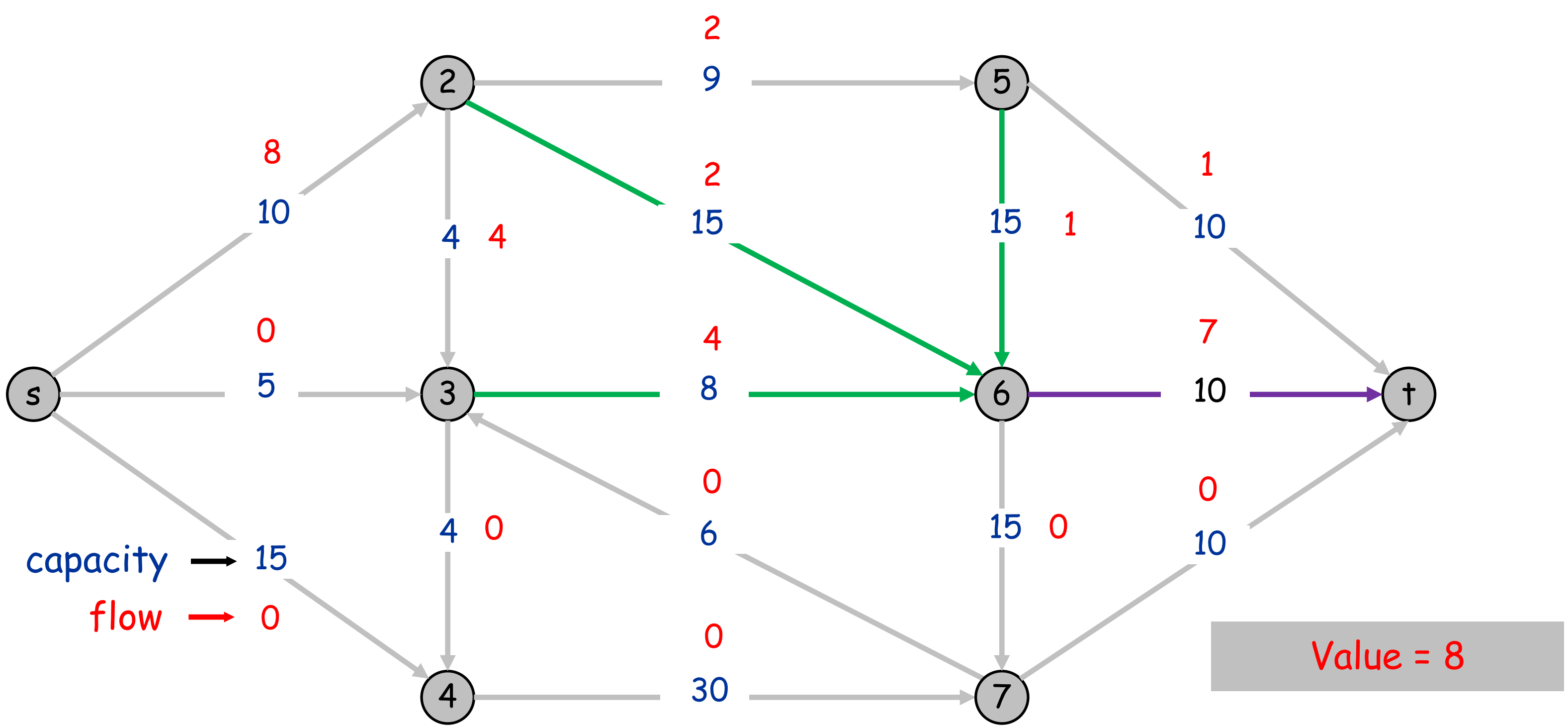


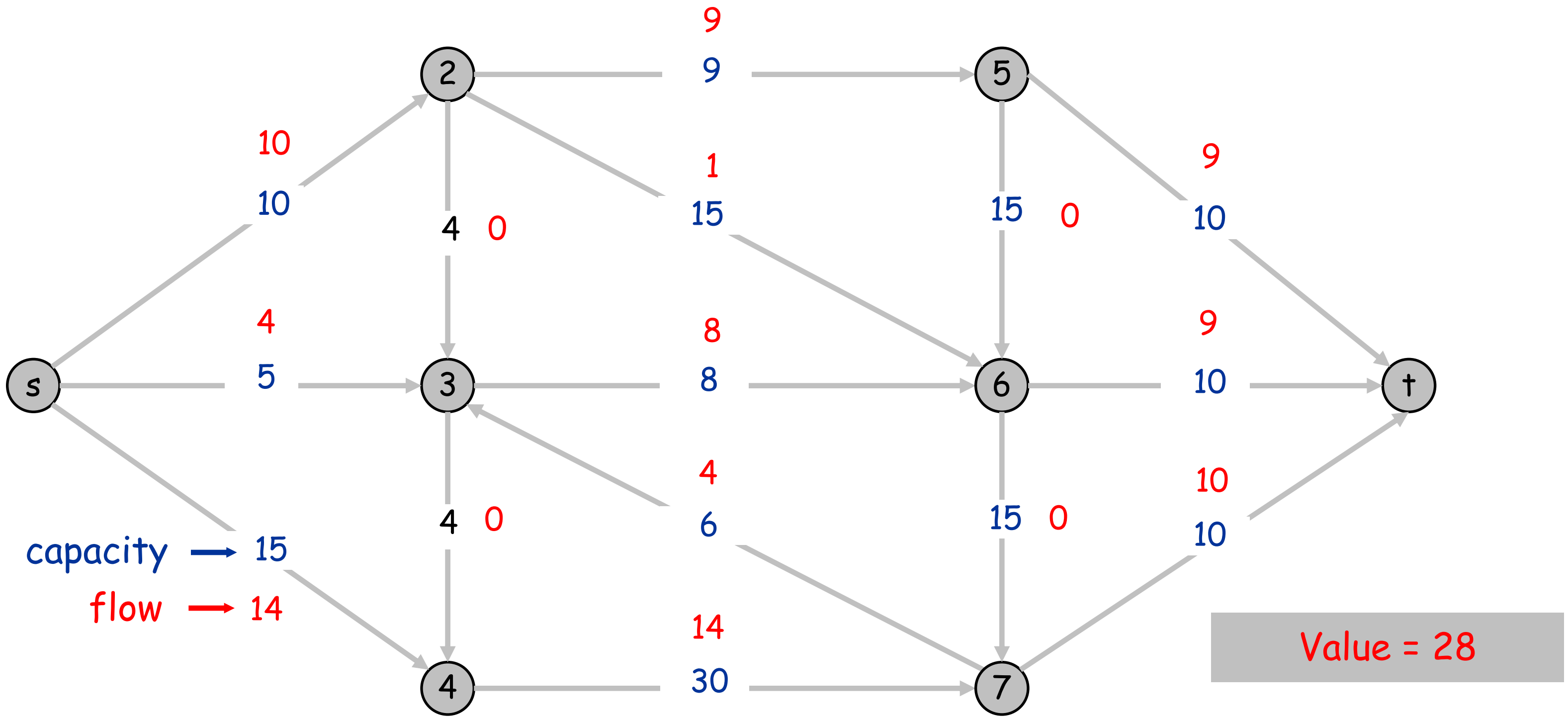
Maximum Flow

Def: The value of a flow f is $|f| = \sum_v f(s,v) = \sum_v f(v,t)$

The maximum flow problem is to find the flow with maximum value.

Example: The flow below is a maximum flow.

Q: How can we be sure this flow achieves the maximum value possible?



Flow Applications

Direct applications

- Water flowing in pipes
- Electricity flows
- Vehicle traffic flows
- Communication network traffic flows

Indirect applications

- Bipartite matching
- Circulation-demand problem
- Baseball elimination
- Airline scheduling
- Fairness in car sharing (carpool)
- …

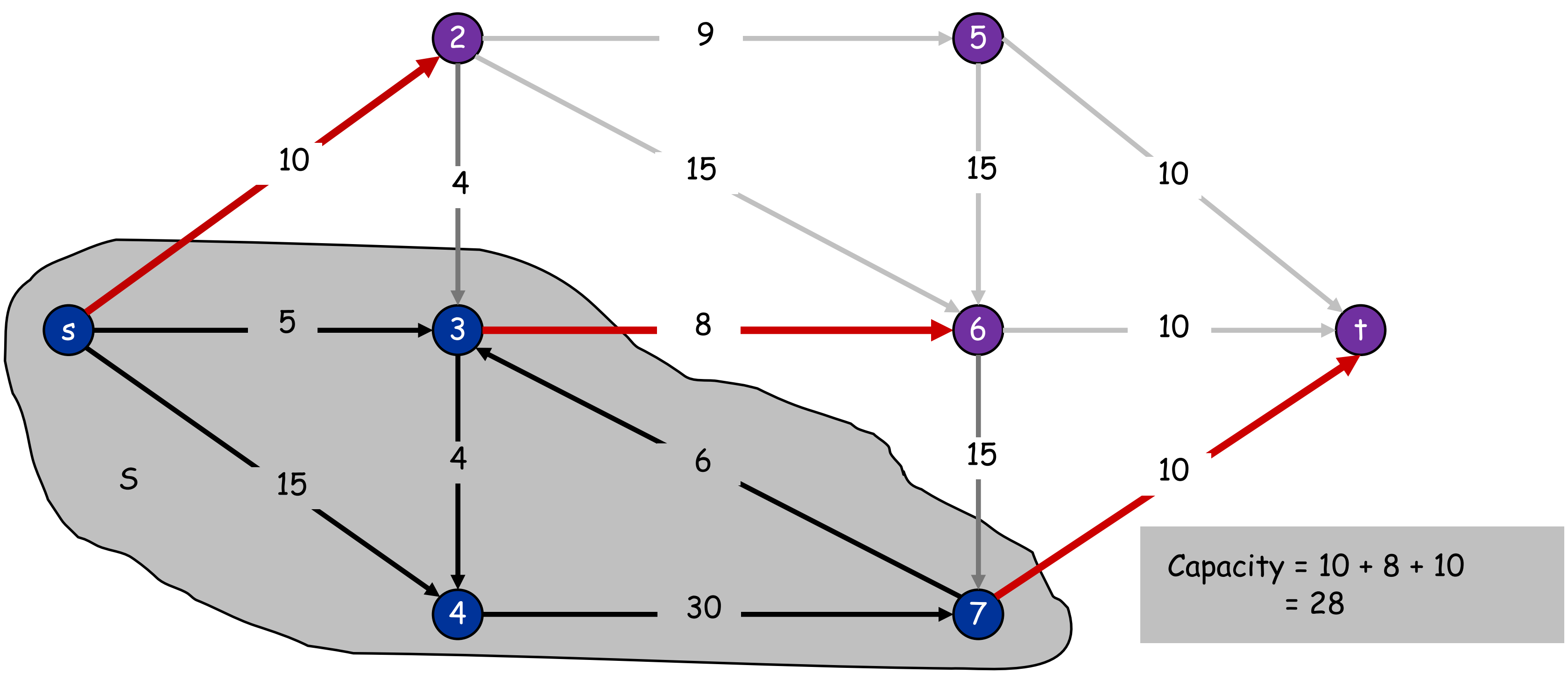
s-t Cut

Def: An s-t cut is a partition (S,T) of V with $s \in S$ and $t \in T$.

Def: The capacity of the cut (S,T) is $c(S,T) = \sum_{e \text{ from } S \text{ to } T} c(e)$

Claim: The value of any s-t flow cannot exceed the capacity of any s-t cut.

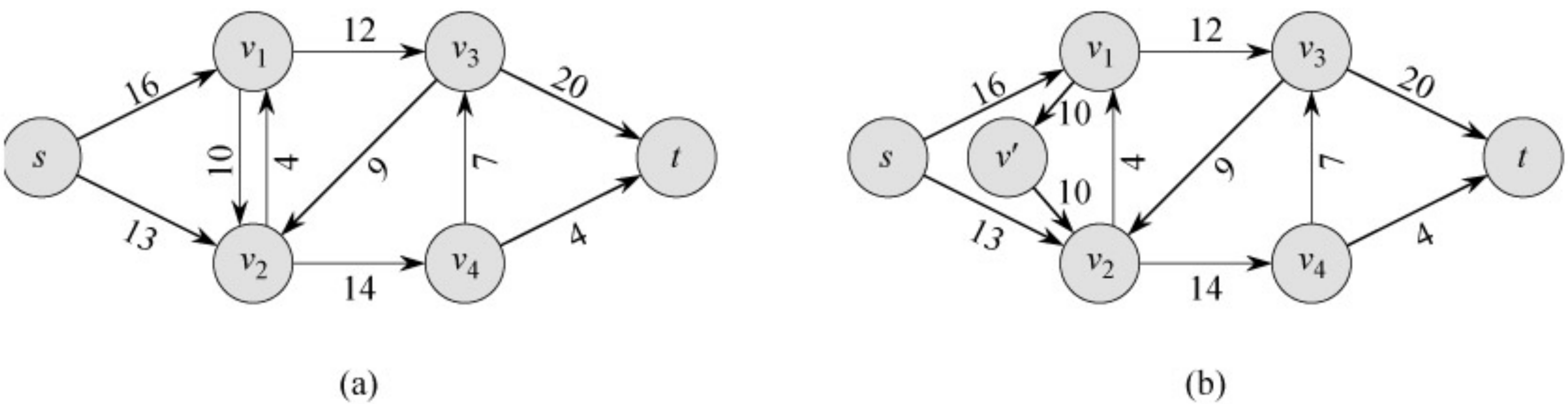
Observation (proved later): An s-t cut with capacity matching the value of a flow is a "proof" that the flow is a max flow.



Assumptions

Antiparallel edges

- $(u, v), (v, u) \in E$
- Models two-way traffic
- Causes problems in algorithms
- But can be removed by adding an auxiliary vertex
- Will assume no antiparallel edges



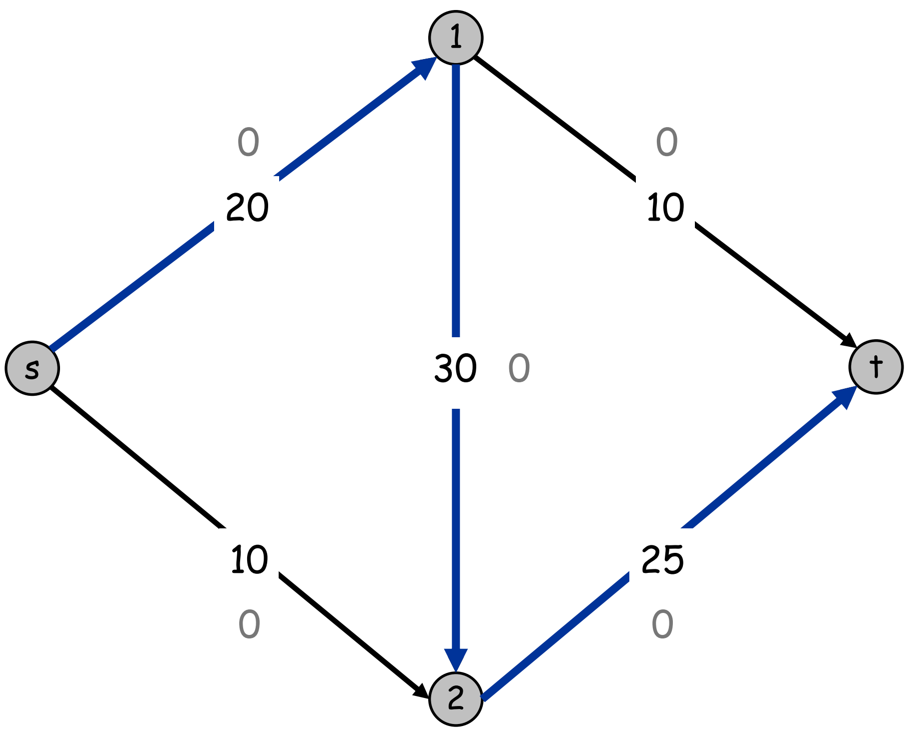
Also assume

- No edges going into s
- No edges going out of t

Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s-t path P where each edge has $f(e) \leq c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.



Flow value = 0

Towards a Max Flow Algorithm

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- Find an s-t path P where each edge has $f(e) \leq c(e)$.
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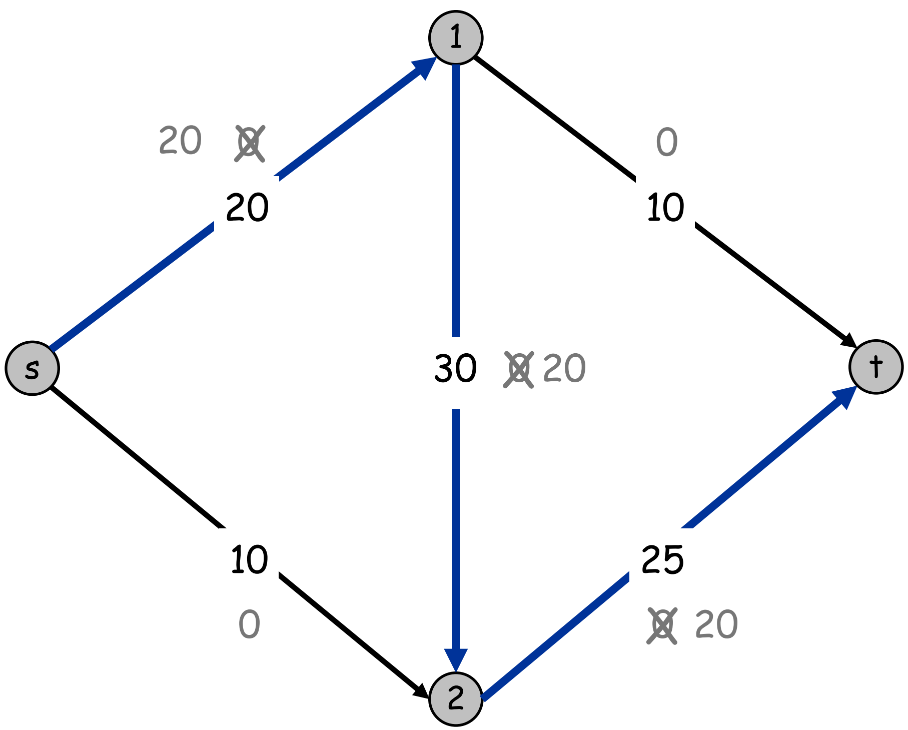

Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

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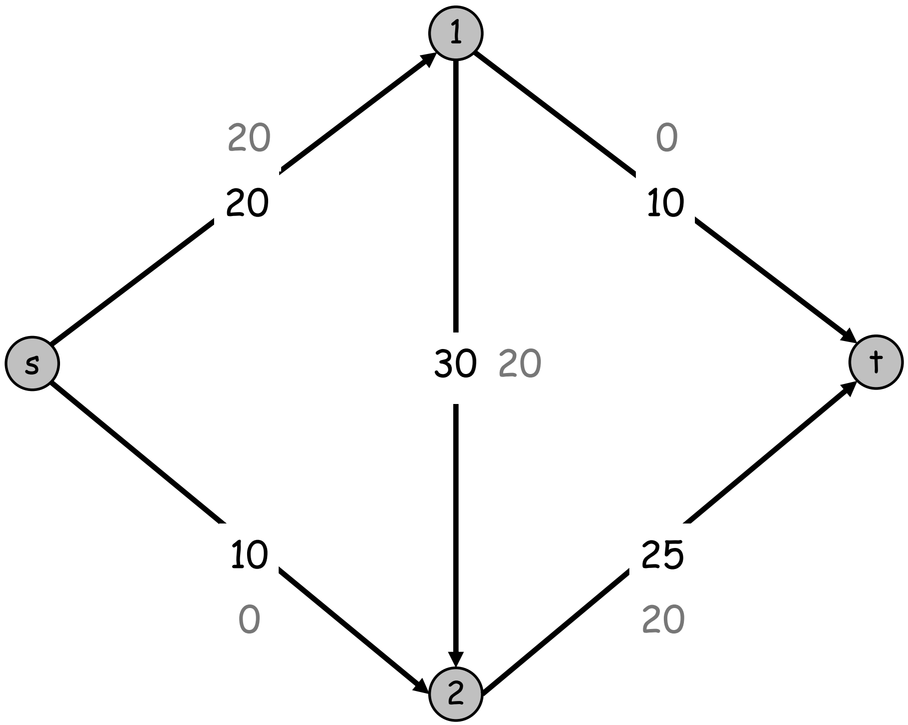

Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.
- Start with $f(e) = 0$ for all edge $e \in E$.
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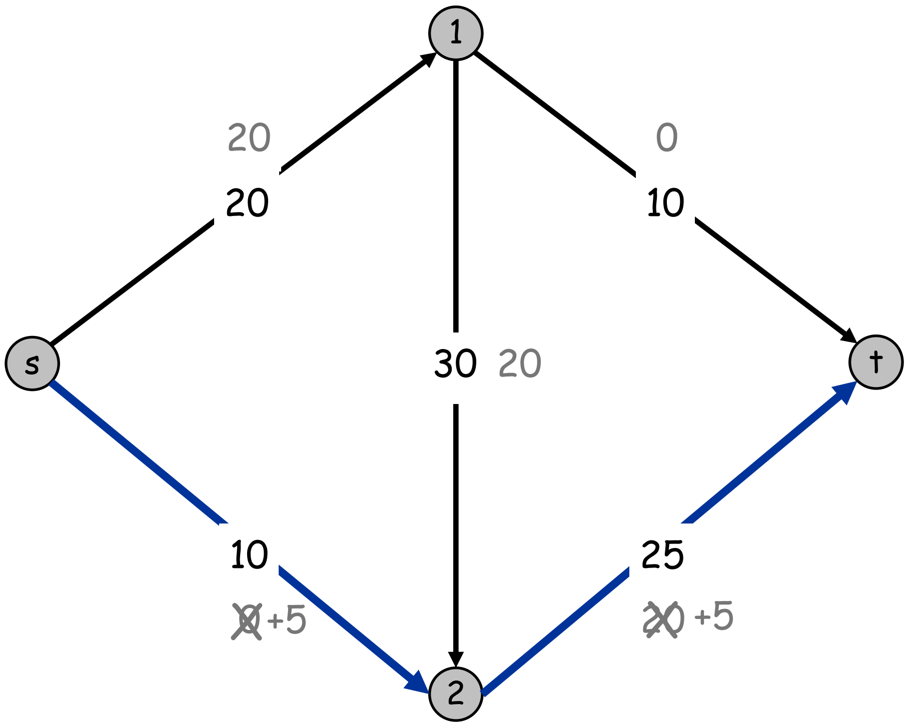

Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

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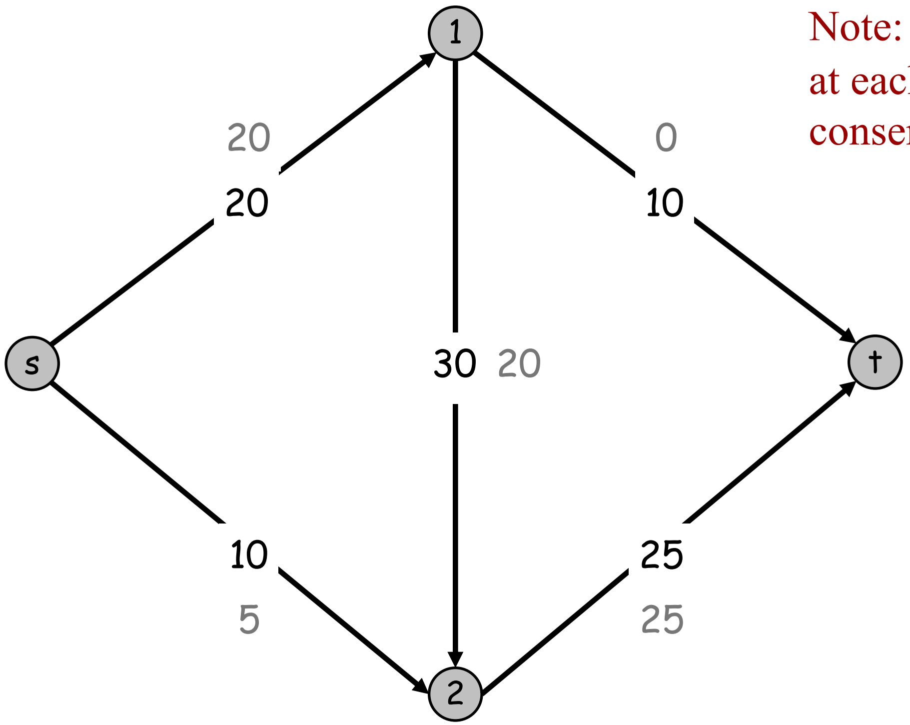


Note: Adding flow along a *path* at each step maintains flow conservation at every vertex

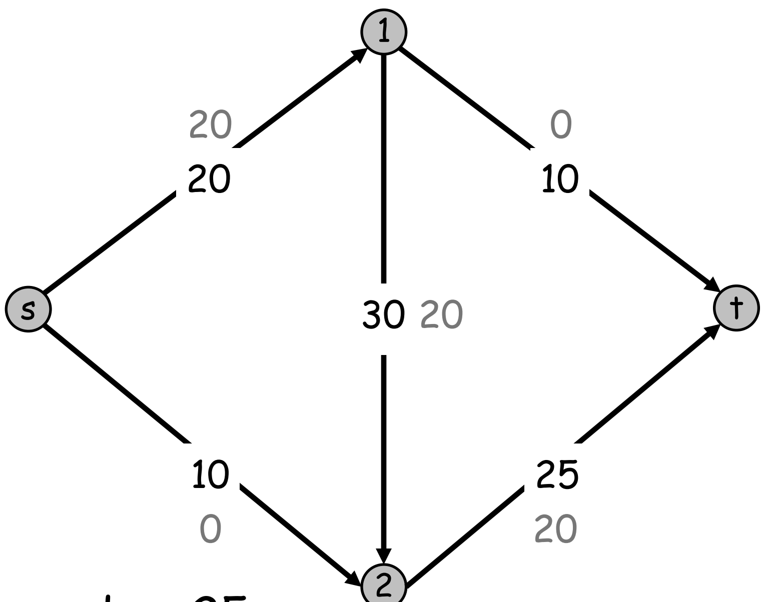
Flow value = 25

Towards a Max Flow Algorithm

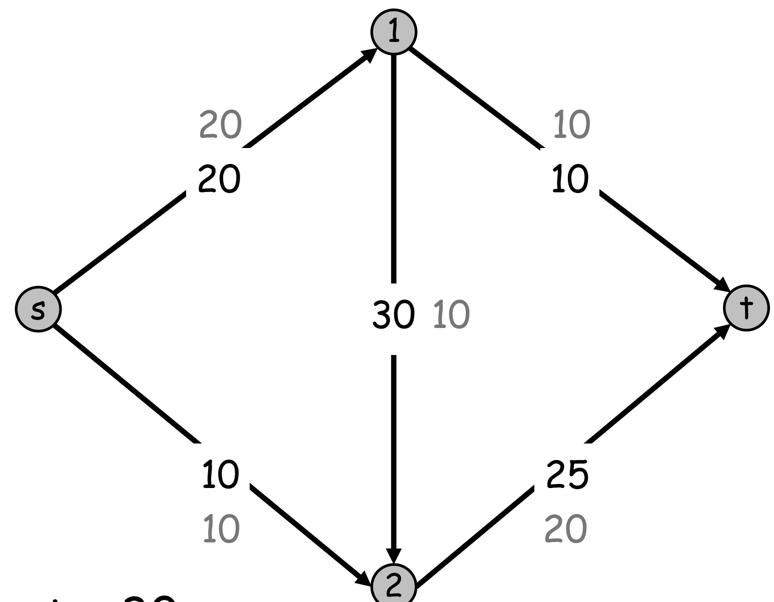
Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s-t path P where each edge has $f(e) \leq c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.

Doesn't Work:
local optimality $\neq$ global optimality

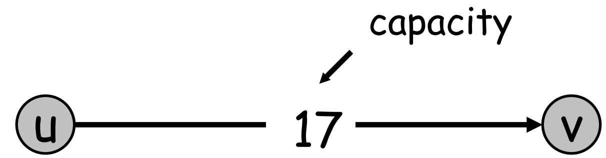


greedy = 25



opt = 30

Residual Graph

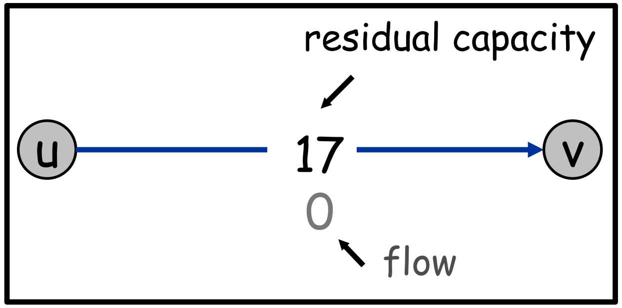


Original edge: $e = (u, v) \in E$.

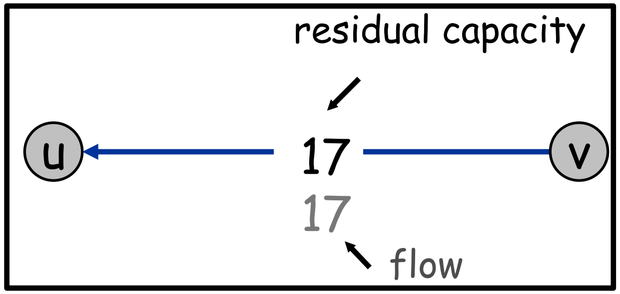
- Flow $f(e)$, capacity $c(e)$.

Create (New) Residual edges:

a. If $f(u,v) = 0$, it has one residual edge (u,v) with residual capacity $c_f(u,v) = c(u,v)$



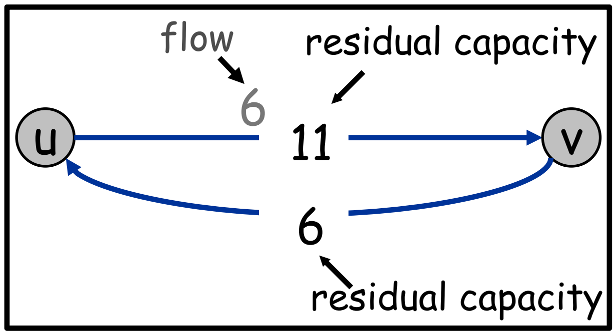
b. If $f(u,v) = c(u,v)$, it has one residual edge (v,u) with residual capacity $c_f(v,u) = f(u,v)$



c. If $0 < f(u,v) < c(u,v)$, it has two residual edges:

i. (u,v) with $c_f(u,v) = c(u,v) - f(u,v)$

ii. (v,u) with $c_f(v,u) = f(u,v)$

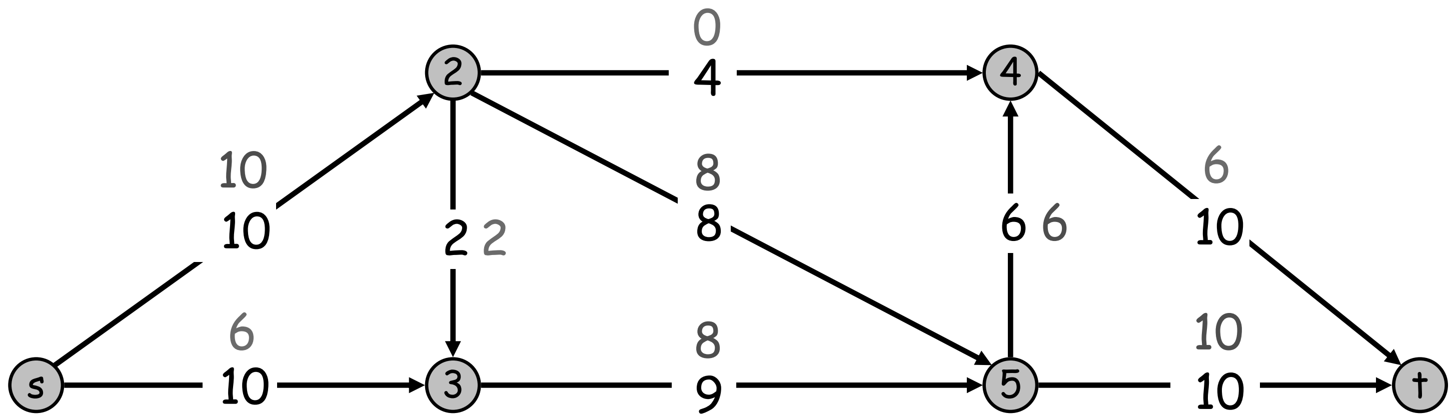


Residual graph: $G_f = (V, E_f)$.

- Vertices are the same vertices
- Edges are all the residual edges
- Residual capacity is "available remaining capacity"

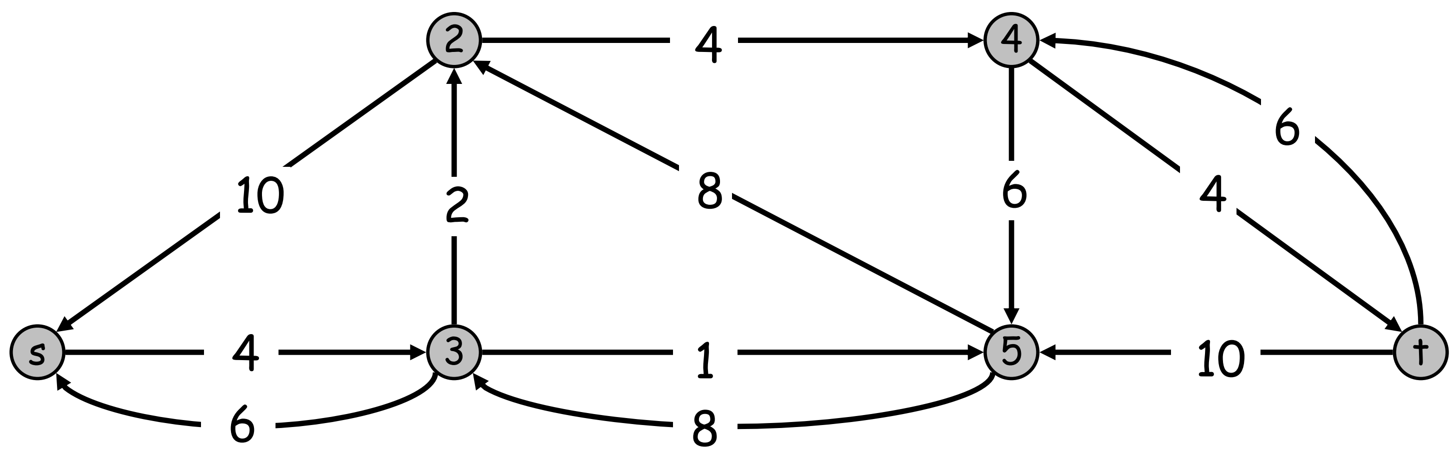
A Graph G, flow f and associated residual Graph G_f

G:



Flow value = 16

G_f:



Ford-Fulkerson Algorithm

Greedy algorithm.

1. Start with $f(e) = 0$ for all edges $e \in E$.
2. Construct Residual Graph G_f for current flow f(e)=0

3. While there exists some s-t path P IN G_f

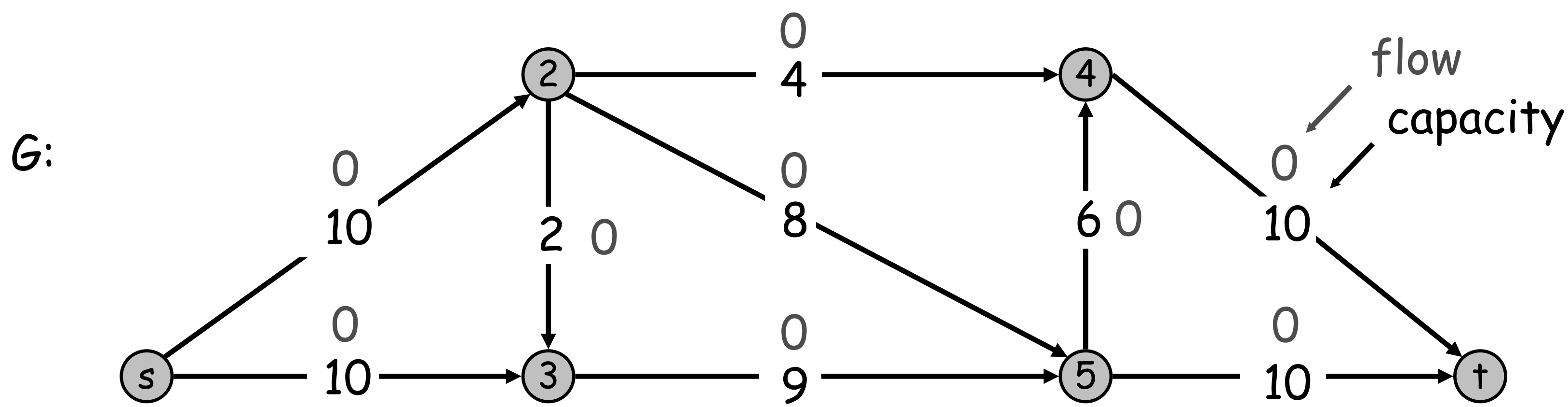
4. Let $c_f(p) \leftarrow \min\{c_f(e): e \in P\}$
   This is the maximum amount of flow that can be pushed through residual capacity of P's edges

5. Push $c_f(p)$ units of flow along the edges $e \in P$
   by adding $c_f(p)$ to f(e) for every $e \in P$

6. Construct Residual Graph G_f for new current flow f(e)

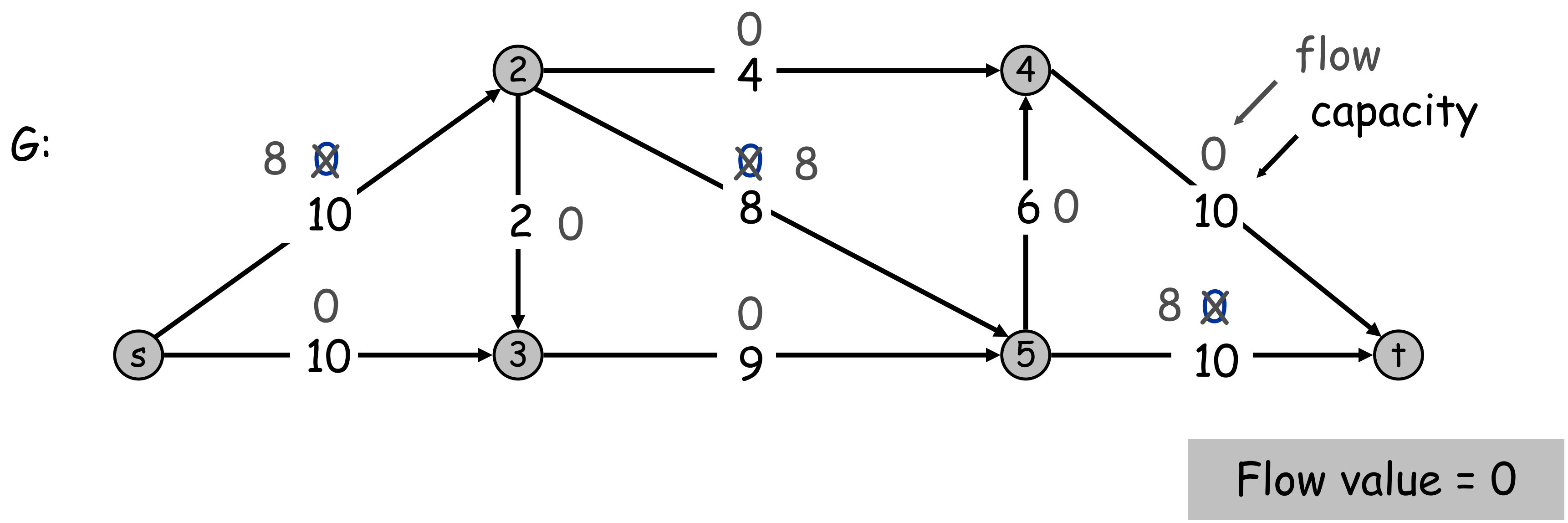
Claim: When algorithm gets stuck, current flow is maximal!

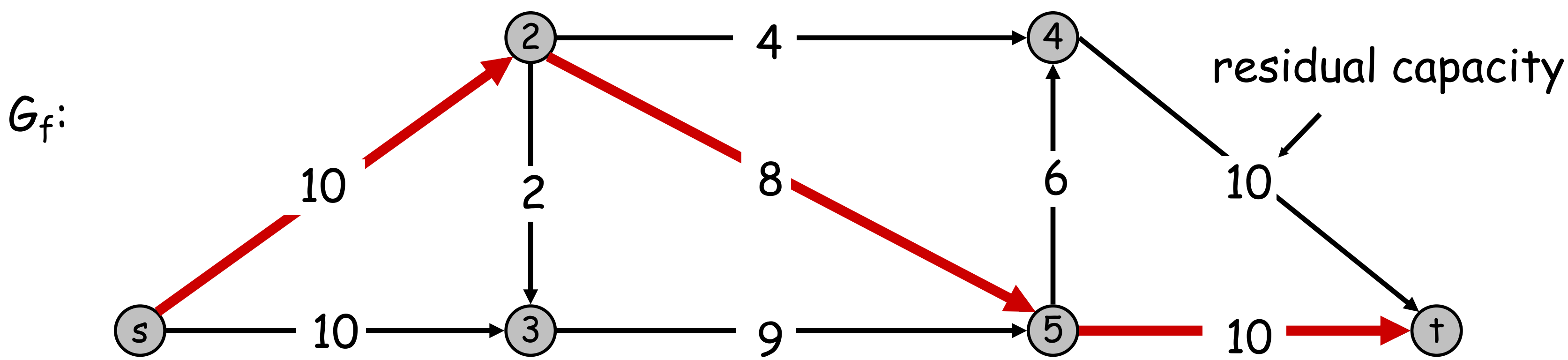
Ford-Fulkerson Algorithm



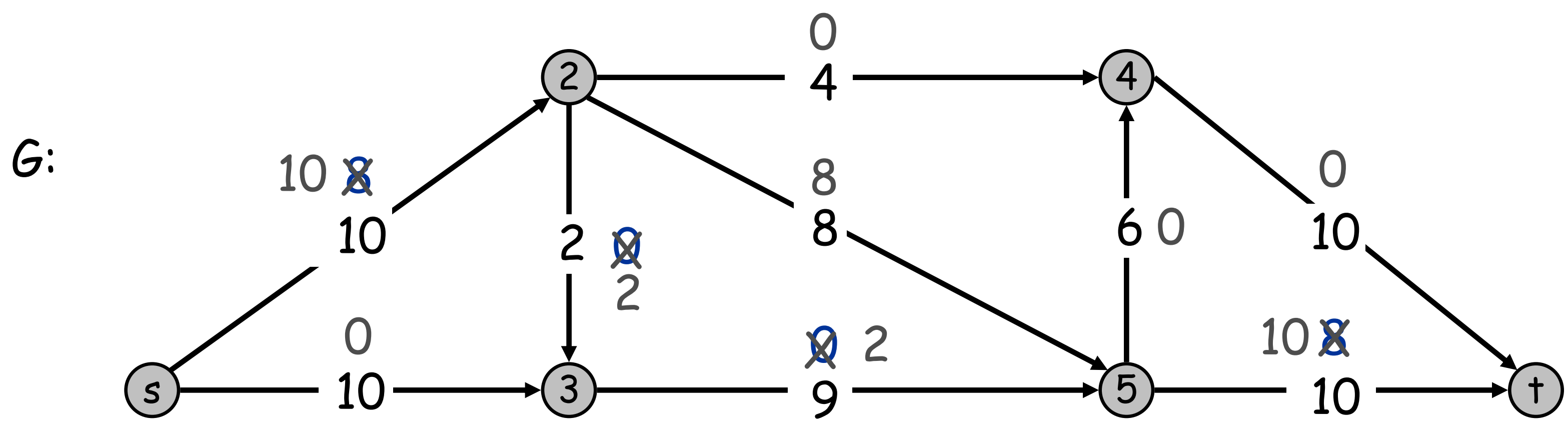
Flow value = 0

Ford-Fulkerson Algorithm

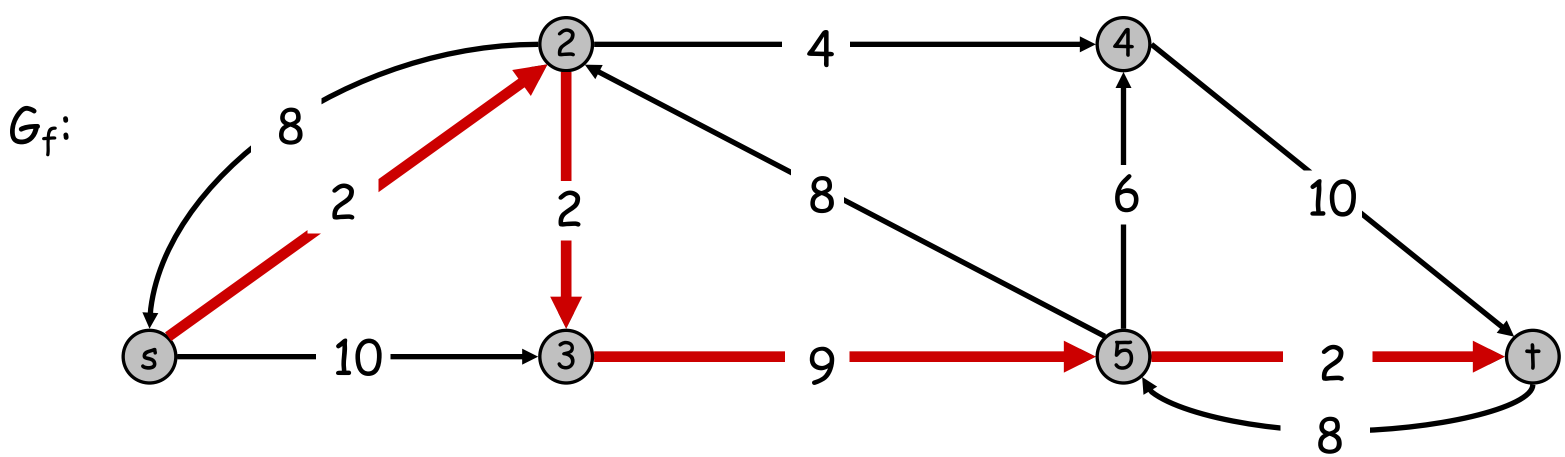




Ford-Fulkerson Algorithm

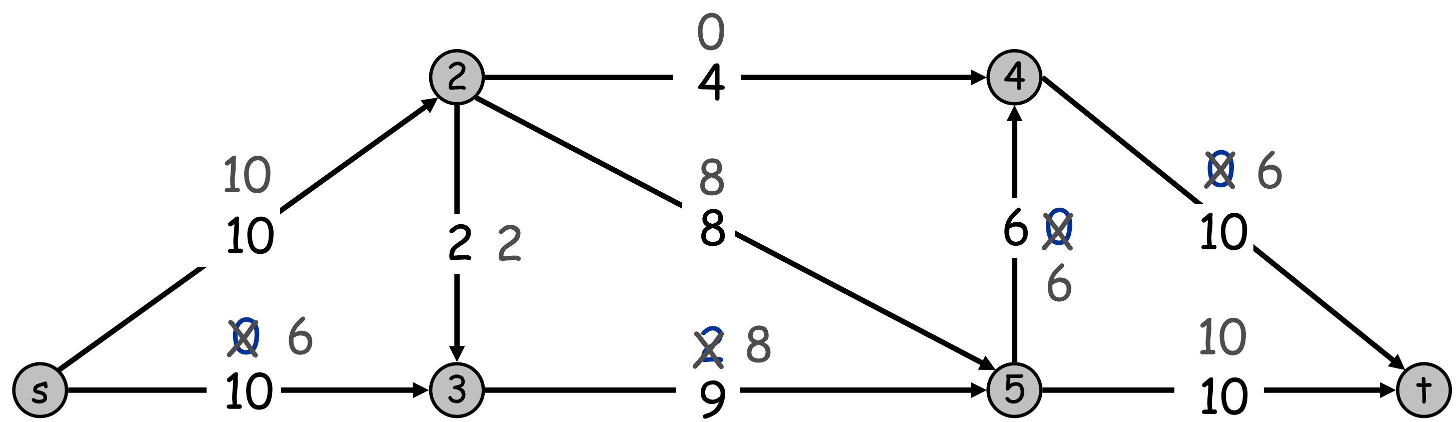


Flow value = 8



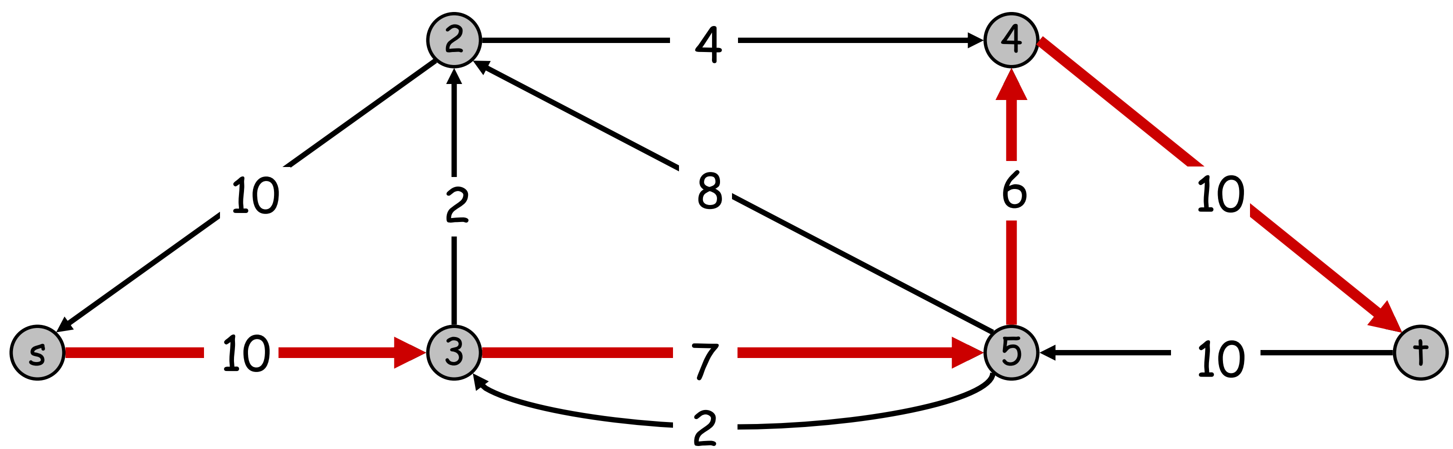
Ford-Fulkerson Algorithm

G:

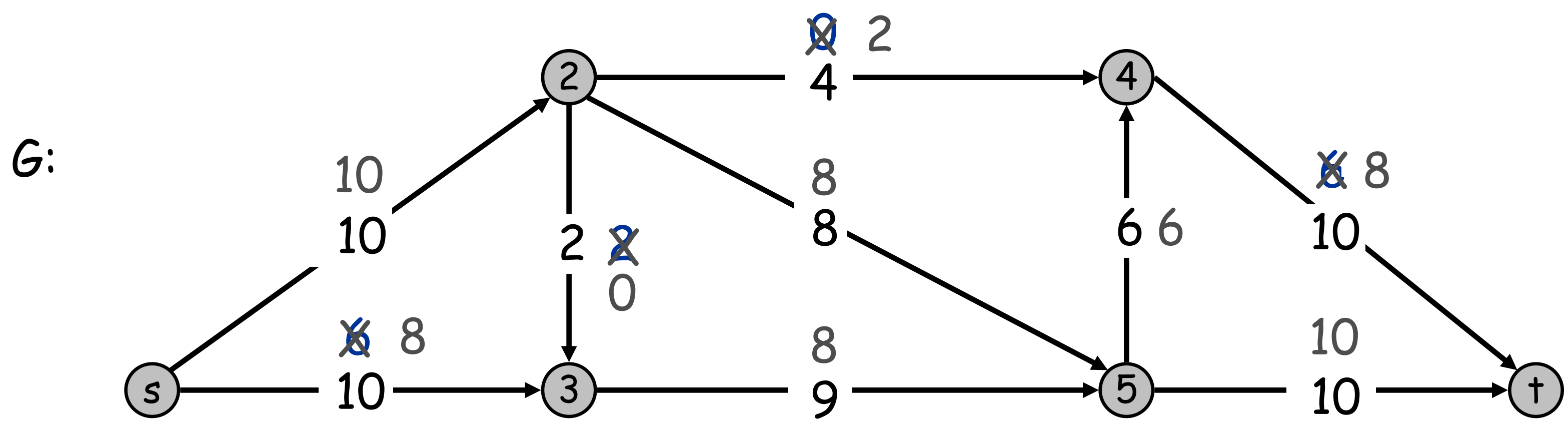


Flow value = 10

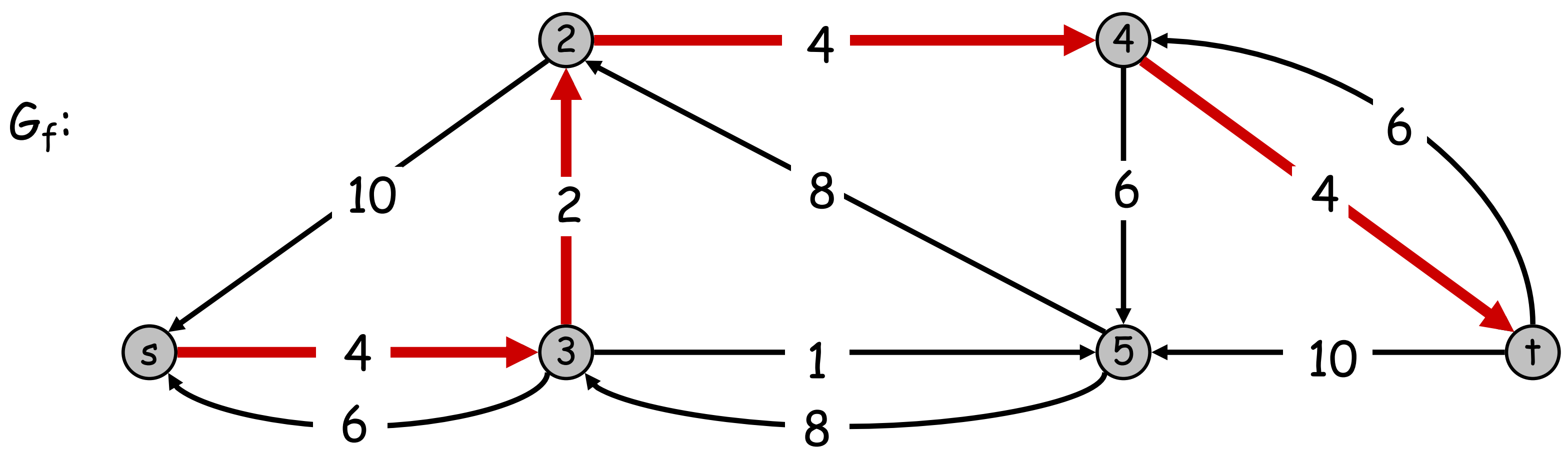
G_f:



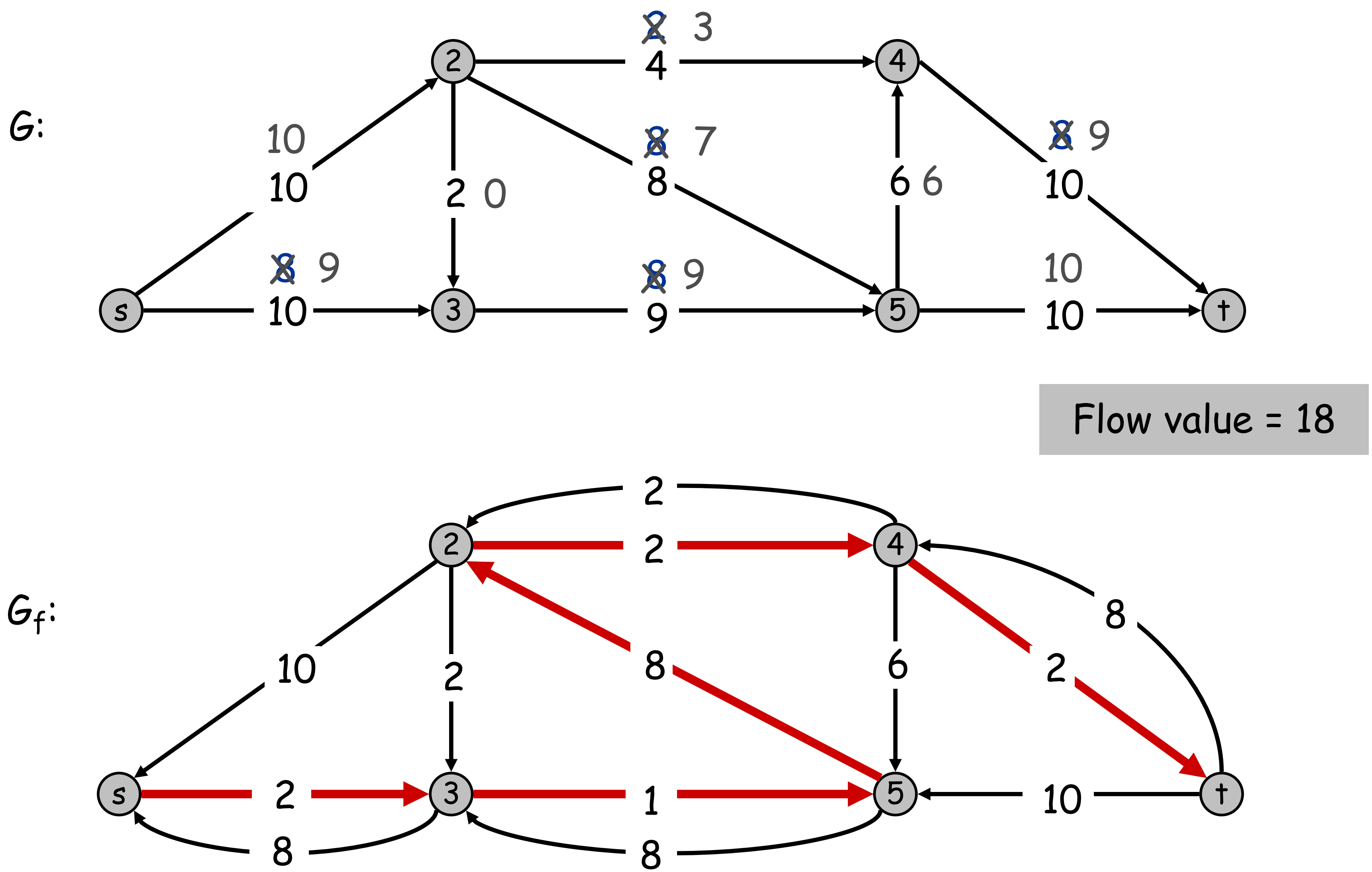
Ford-Fulkerson Algorithm



Flow value = 16

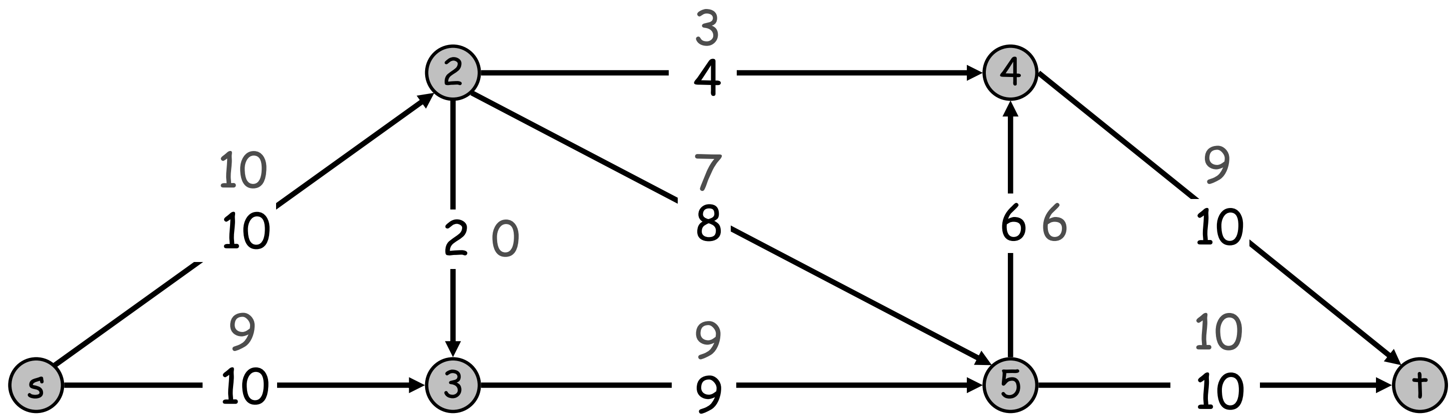


Ford-Fulkerson Algorithm



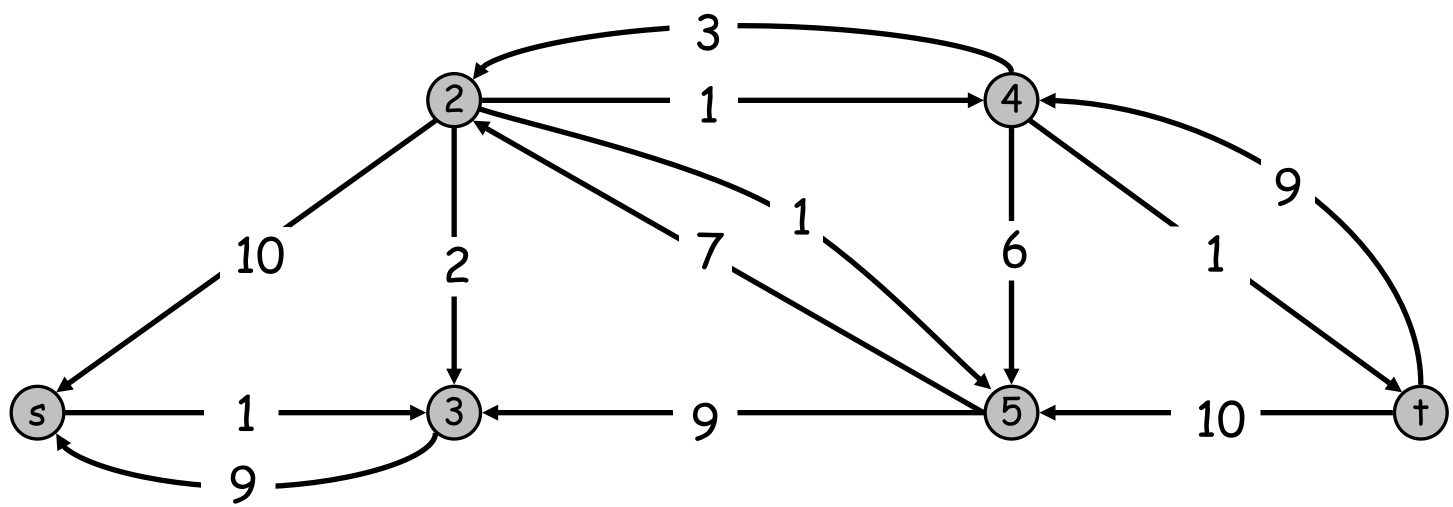
Ford-Fulkerson Algorithm

G:



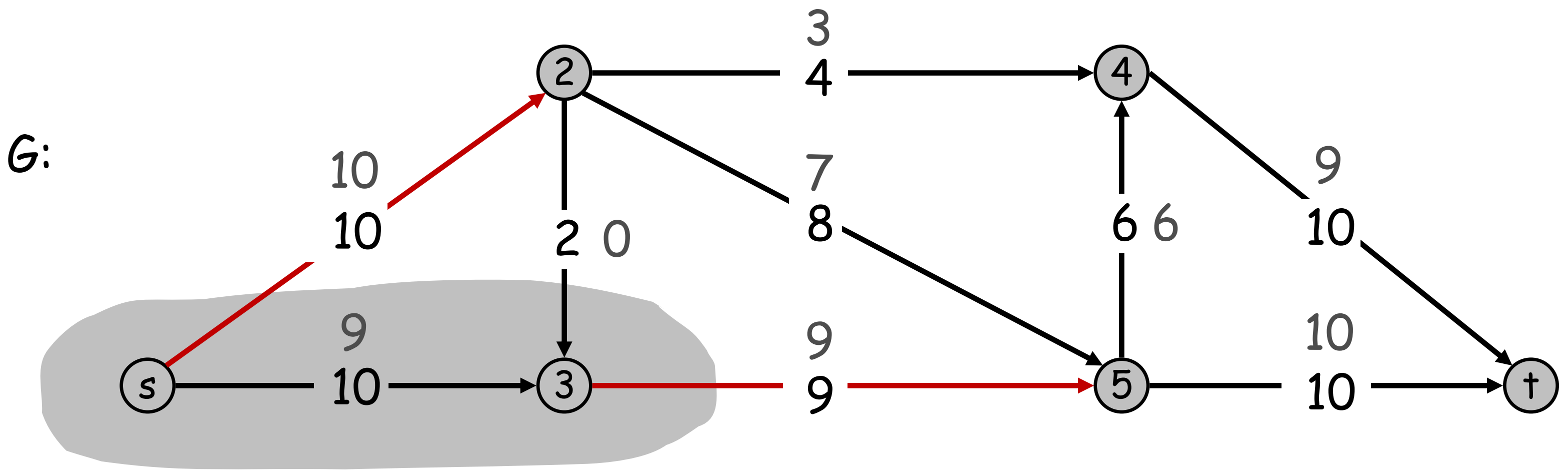
Flow value = 19

G_f:



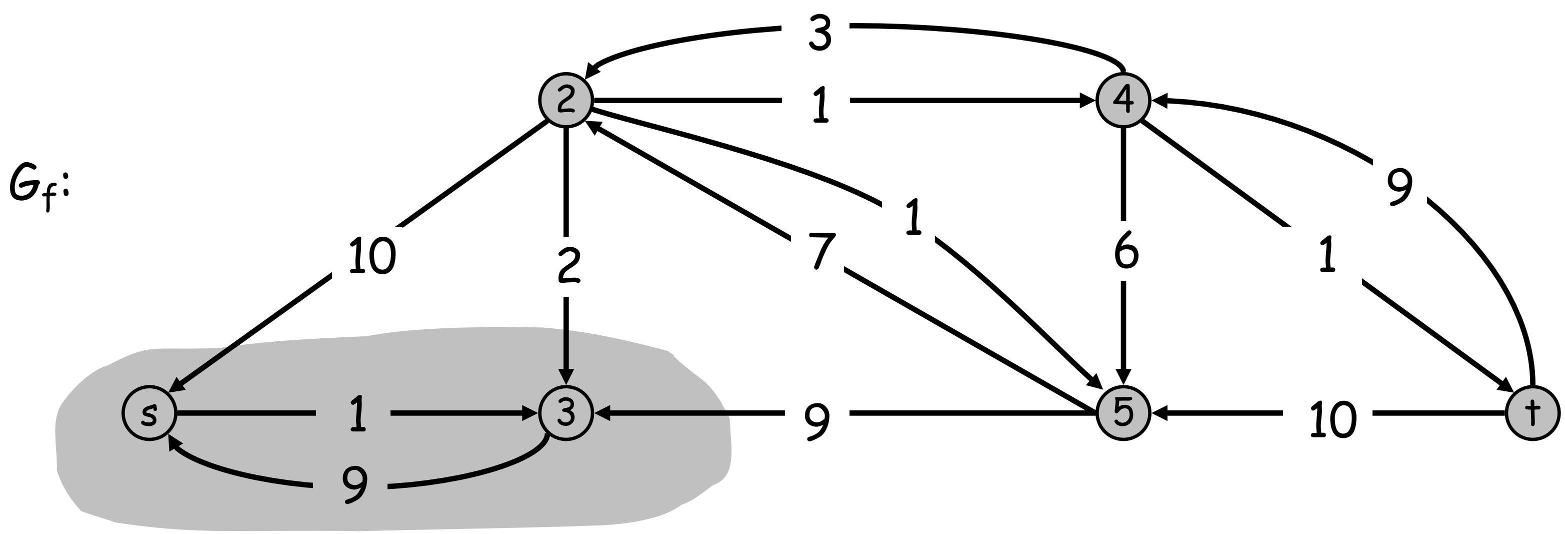
No s-t path exists in G_f. Algorithm stops! Current flow is optimally maximal.

Ford-Fulkerson Algorithm



Cut capacity = 19

Flow value = 19



Ford-Fulkerson Algorithm

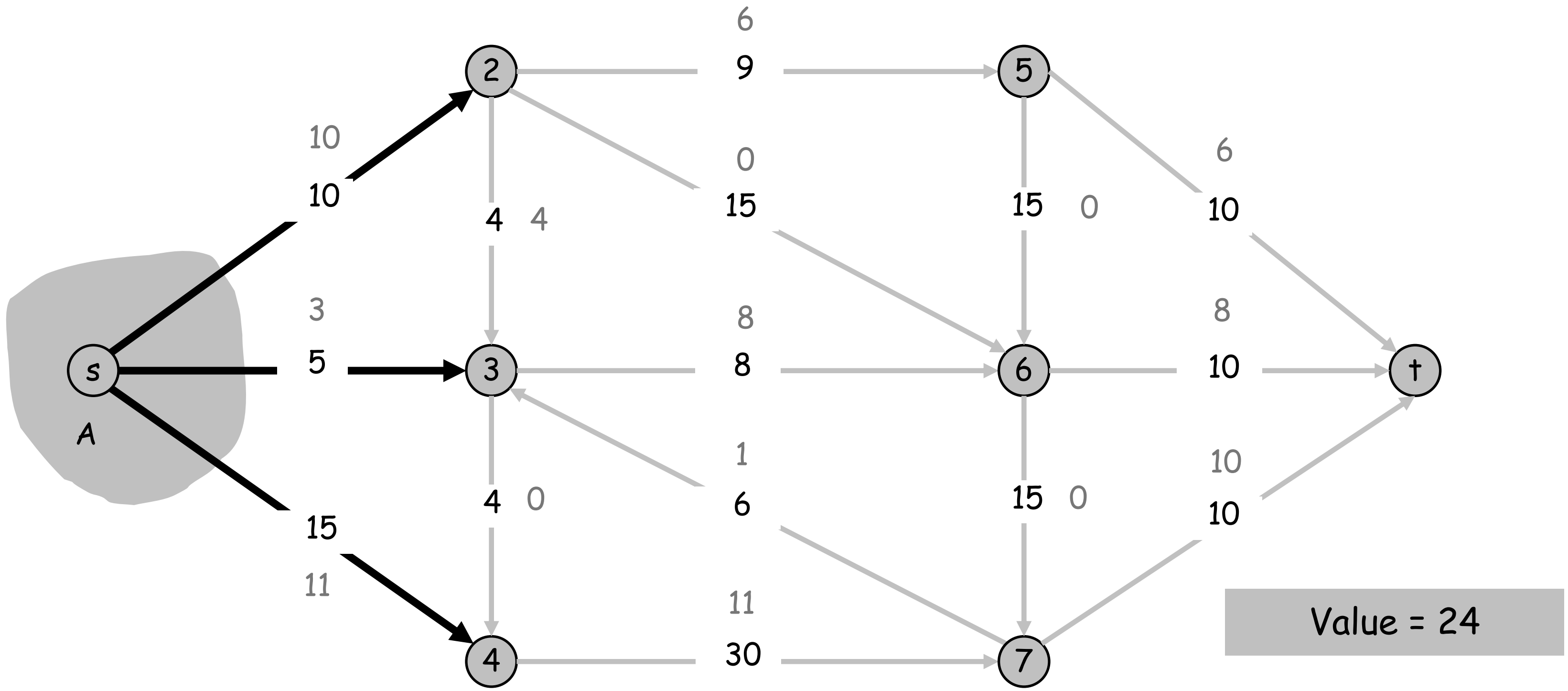
```
Ford-Fulkerson(G,s,t) {
for each (u,v) ∈ E do
     f(u,v) ← 0
     c_f(u,v) ← c(e)
     c_f(v,u) ← 0
while there exists path P in residual graph G_f do
     c_f(p) ← min{c_f(e): e ∈ P}
     for each edge (u,v) ∈ P do
          if (u,v) ∈ E then
               f(u,v) ← f(u,v) + c_f(p)
               c_f(u,v) ← c_f(u,v) − c_f(p)
               c_f(v,u) ← c_f(v,u) + c_f(p)
          else
               f(v,u) ← f(v,u) − c_f(p)
               c_f(v,u) ← c_f(v,u) + c_f(p)
               c_f(u,v) ← c_f(u,v) − c_f(p)
```

Net flow and cuts

Def: Let f be any flow, and let (S, T) be any s-t cut. Then, the net flow across the cut is

$$f(S,T) = \sum_{e \text{ from } S \text{ to } T} f(e) - \sum_{e \text{ from } T \text{ to } S} f(e)$$

Net flow lemma: For any s-t cut (S, T), $f(S,T) = |f|$.



Net flow and cuts

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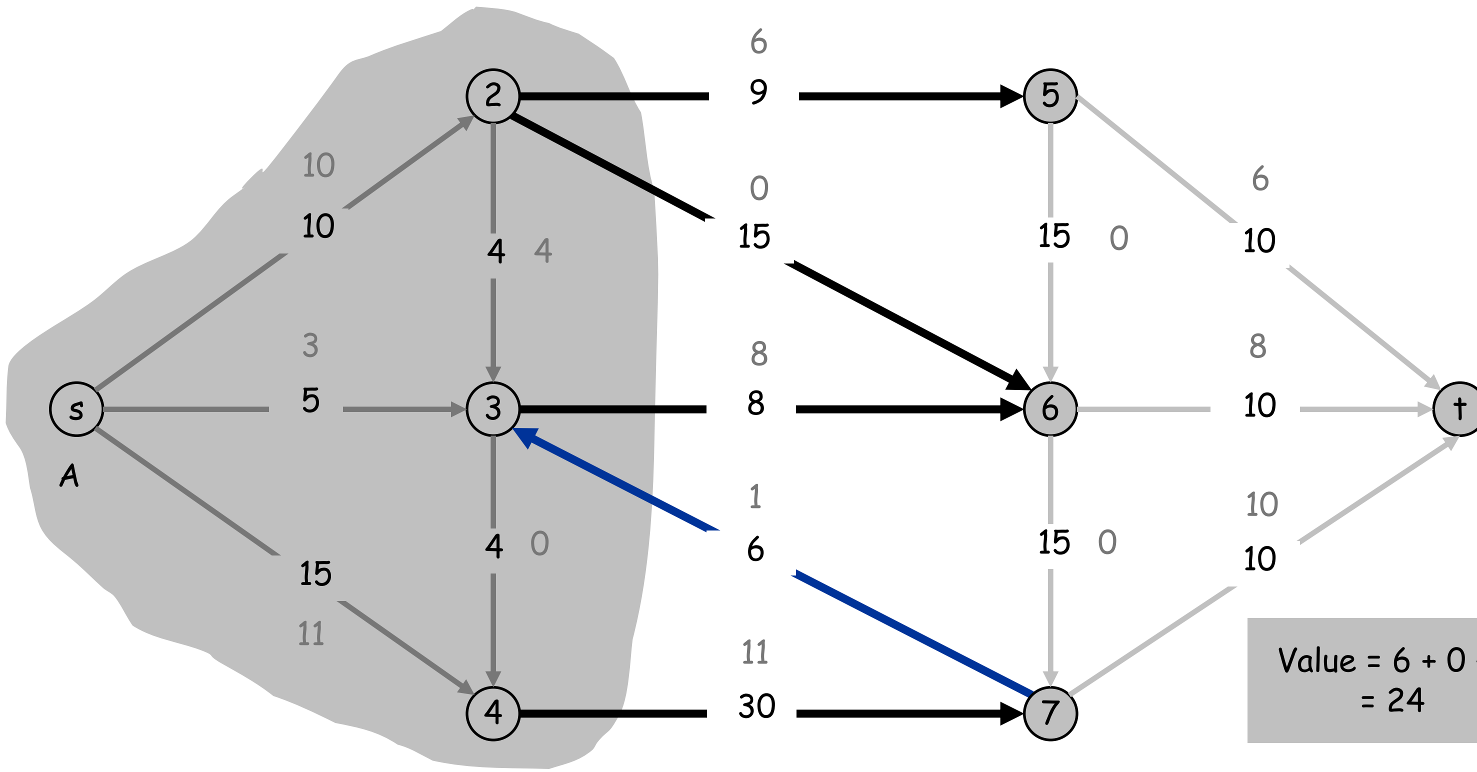

Value = 6 + 0 + 8 - 1 + 11
= 24

Net flow and cuts

Def: Let f be any flow, and let (S, T) be any s-t cut. Then, the **net flow** across the cut is

$$f(S,T) = \sum_{e \text{ from } S \text{ to } T} f(e) - \sum_{e \text{ from } T \text{ to } S} f(e)$$

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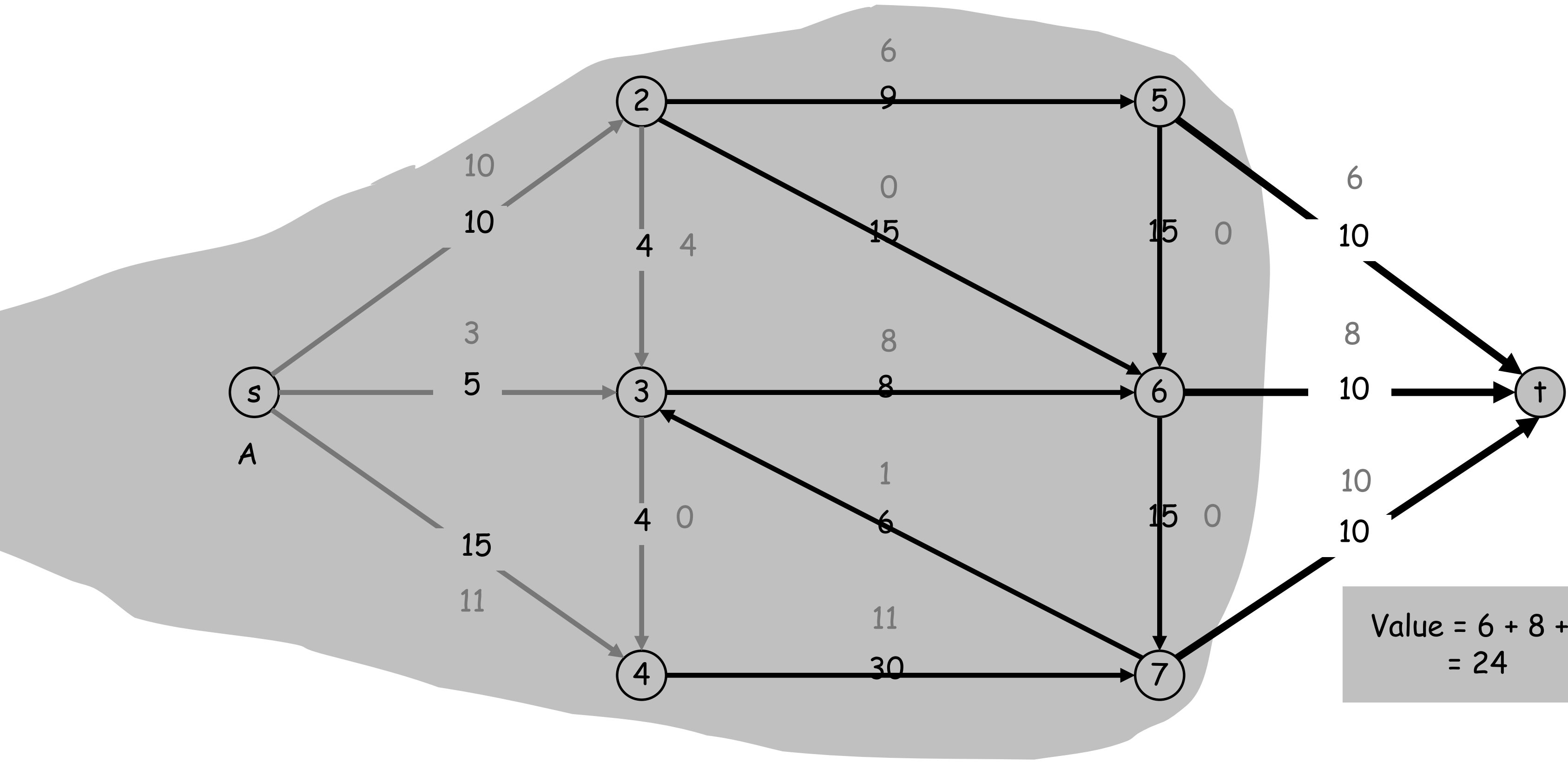


Value = 6 + 8 + 10
= 24

Net flow and cuts

Net flow lemma: Let f be any flow, and let (S, T) be any s-t cut. Then,

$$f(S,T) = \sum_{e \text{ from } S \text{ to } T} f(e) - \sum_{e \text{ from } T \text{ to } S} f(e) = |f|$$

Proof:

$$\sum_{e \text{ out of } s} f(e) = |f| \qquad (1)$$

By flow conservation, for any vertex $v \in V - \{s, t\}$,

$$\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) = 0 \qquad (2)$$

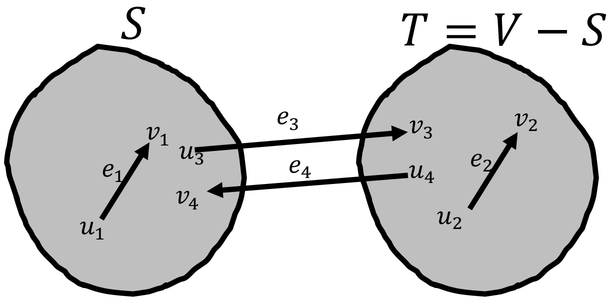
Sum (2) over all $v \in S - \{s\}$, together with (1). We see that

- For every edge e inside S, both $f(e)$ and $-f(e)$ appear
- For every edge e from S to T, only $f(e)$ appear
- For every edge e from T to S, only $-f(e)$ appear

Lemma is thus proved.

A Deeper Dive into the Proof

$$\text{Balance}(u,w) = \begin{cases} 0 = 1-1 & u \in S, w \in S \\ 0 = 0-0 & u \notin S, w \notin S \\ 1 = 1-0 & u \in S, w \notin S \\ -1 = 0-1 & u \notin S, w \in S \end{cases}$$



$\text{Balance}(u_1, v_1) = 0$
$\text{Balance}(u_2, v_2) = 0$
$\text{Balance}(u_3, v_3) = 1$
$\text{Balance}(u_4, v_4) = -1$

$$|f| = \sum_{e \text{ out of } s} f(e) - \sum_{v \in S - \{s\}} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right)$$

Definition of $f(e)$
Each parenthesis = 0

$$= \sum_{v \in S} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right)$$

Each $(u, w) = e \in E$ can appear in 0, 1 or 2 of these terms.

$$= \sum_{(u,w) = e \in E} \text{Balance}(u,w) f(e)$$

$$= \sum_{e \text{ from } S \text{ to } T} f(e) - \sum_{e \text{ from } T \text{ to } S} f(e) = f(S, T)$$

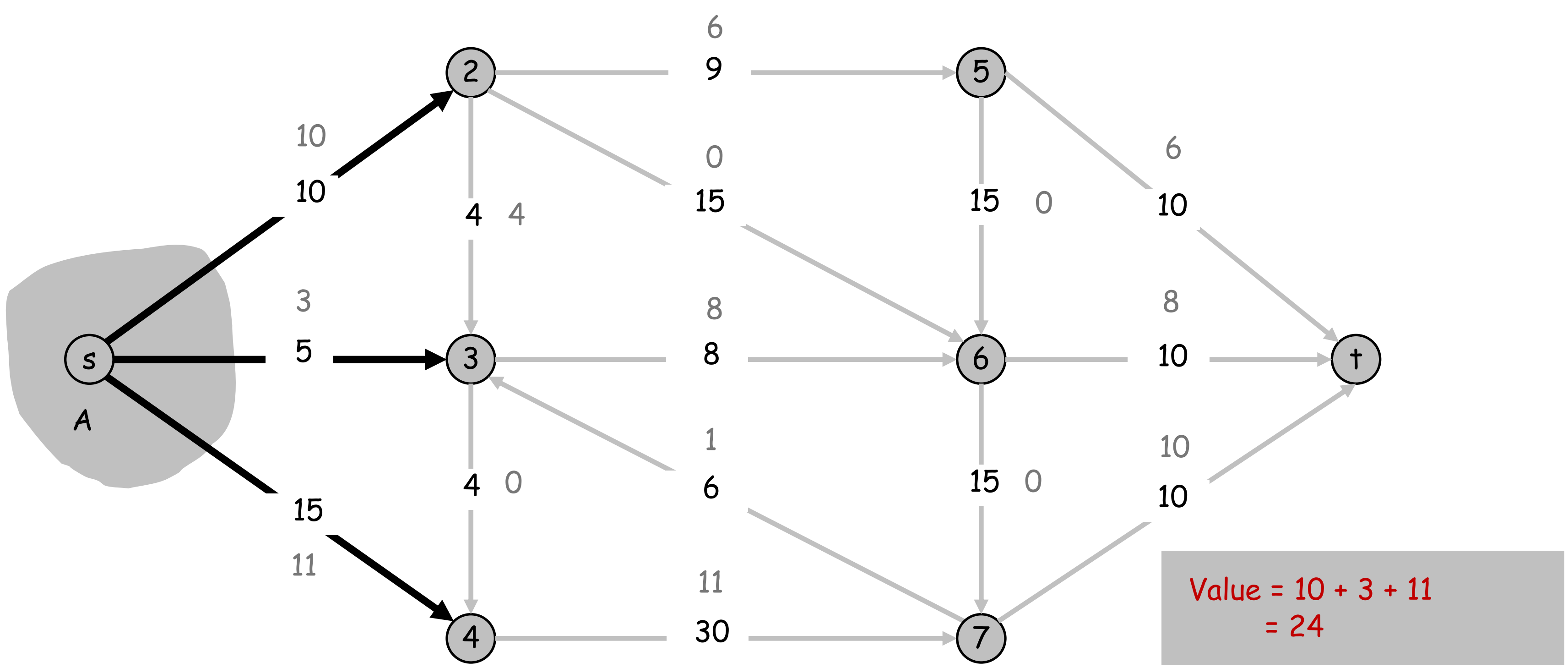
How Can we prove flows optimal (maximal)?

We just saw tools for calculating the value of a flow.

Given flow, how can we prove that the flow is optimal, i.e., can it be improved?

For example, the flow below has value=24.

This can be improved to have value=28



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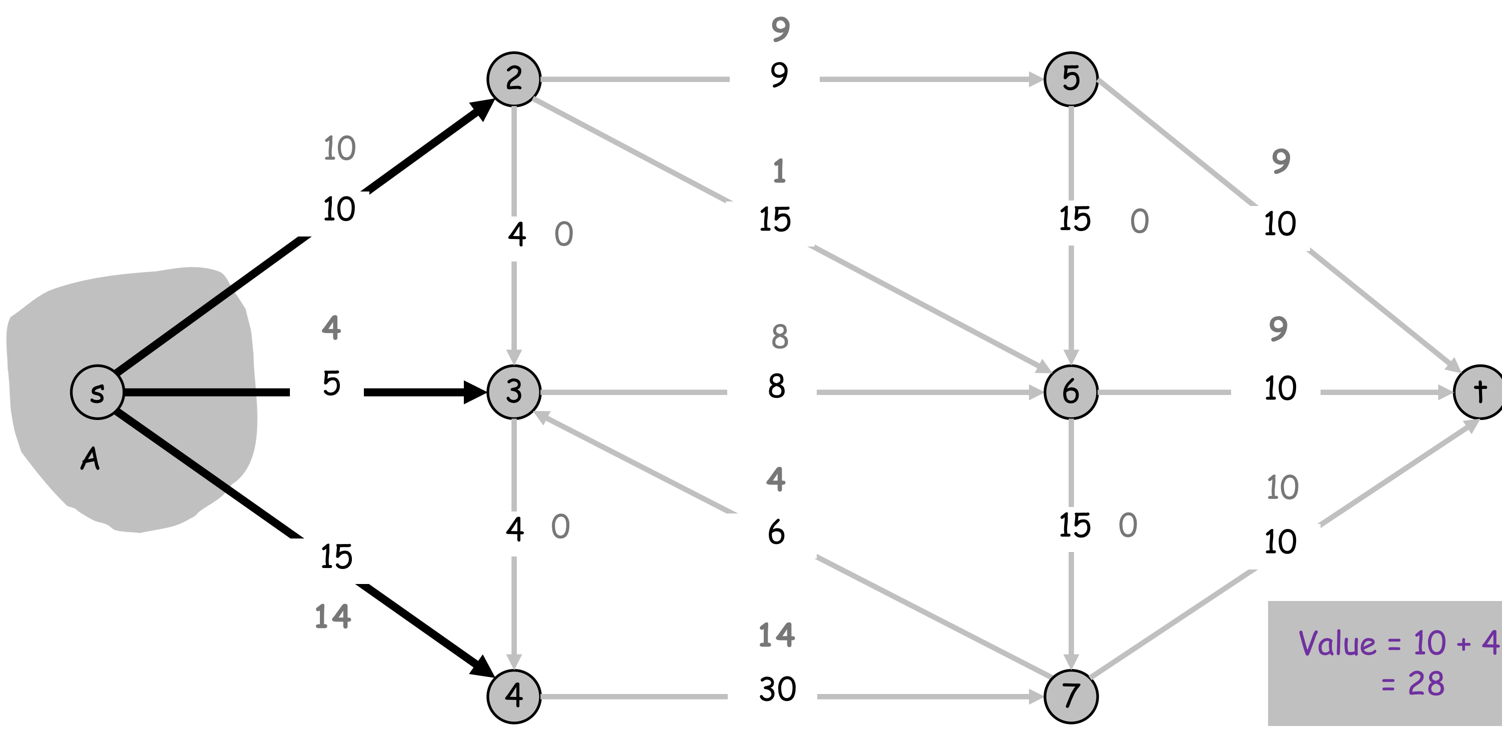
We just saw tools for calculating the value of a flow.

Given flow, how can we prove that the flow is optimal, i.e., can it be improved?

For example, the flow below has value=24.

This can be improved to have value=28

Is this the best?
How can we be sure?



Value = 10 + 4 + 14
= 28

Flow and Cuts

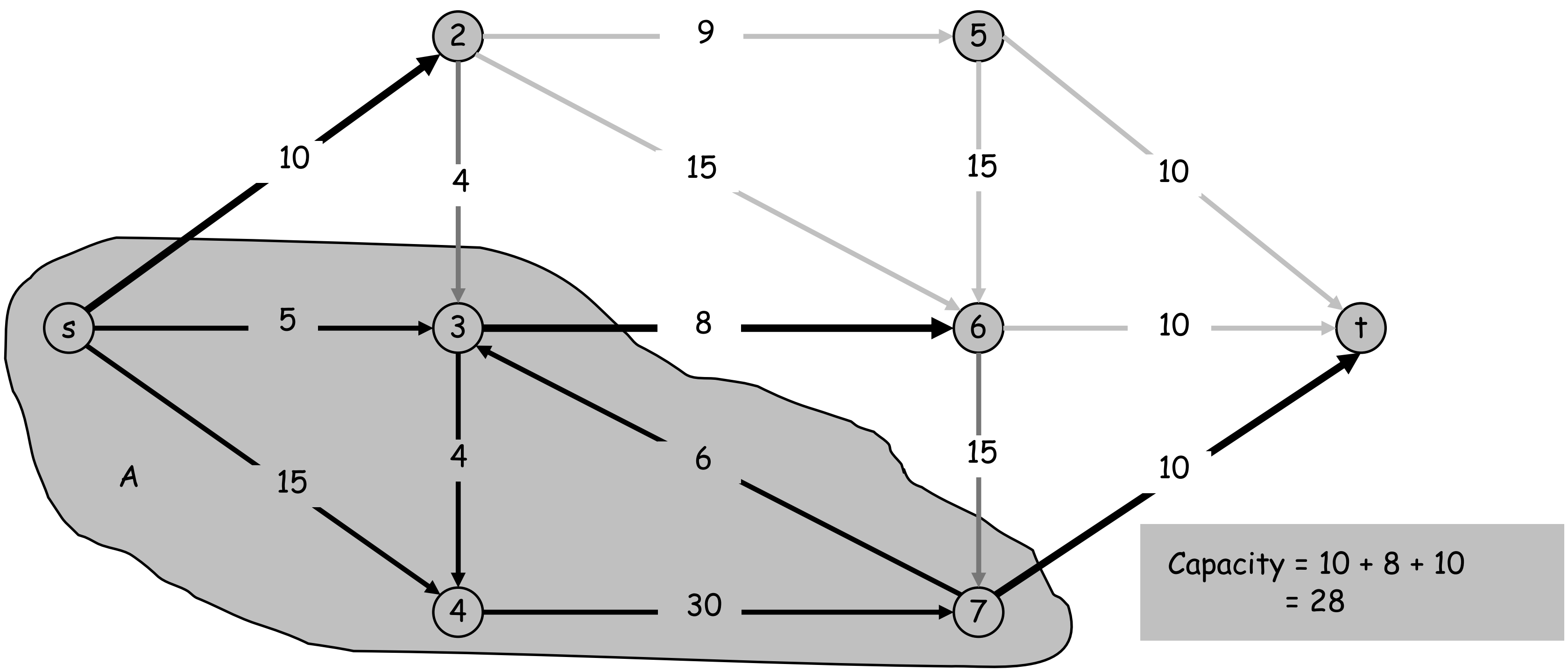
Def: The **capacity** of the cut (S,T) is $c(S,T) = \sum_{e \text{ from } S \text{ to } T} c(e)$

Claim: For any flow f and any s-t cut (S,T), $|f| \le c(S,T)$.

Proof:

$$|f| = \sum_{e \text{ from } S \text{ to } T} f(e) - \sum_{e \text{ from } T \text{ to } S} f(e)$$

$$\le \sum_{e \text{ from } S \text{ to } T} f(e) \le \sum_{e \text{ from } S \text{ to } T} c(e) = c(S,T)$$



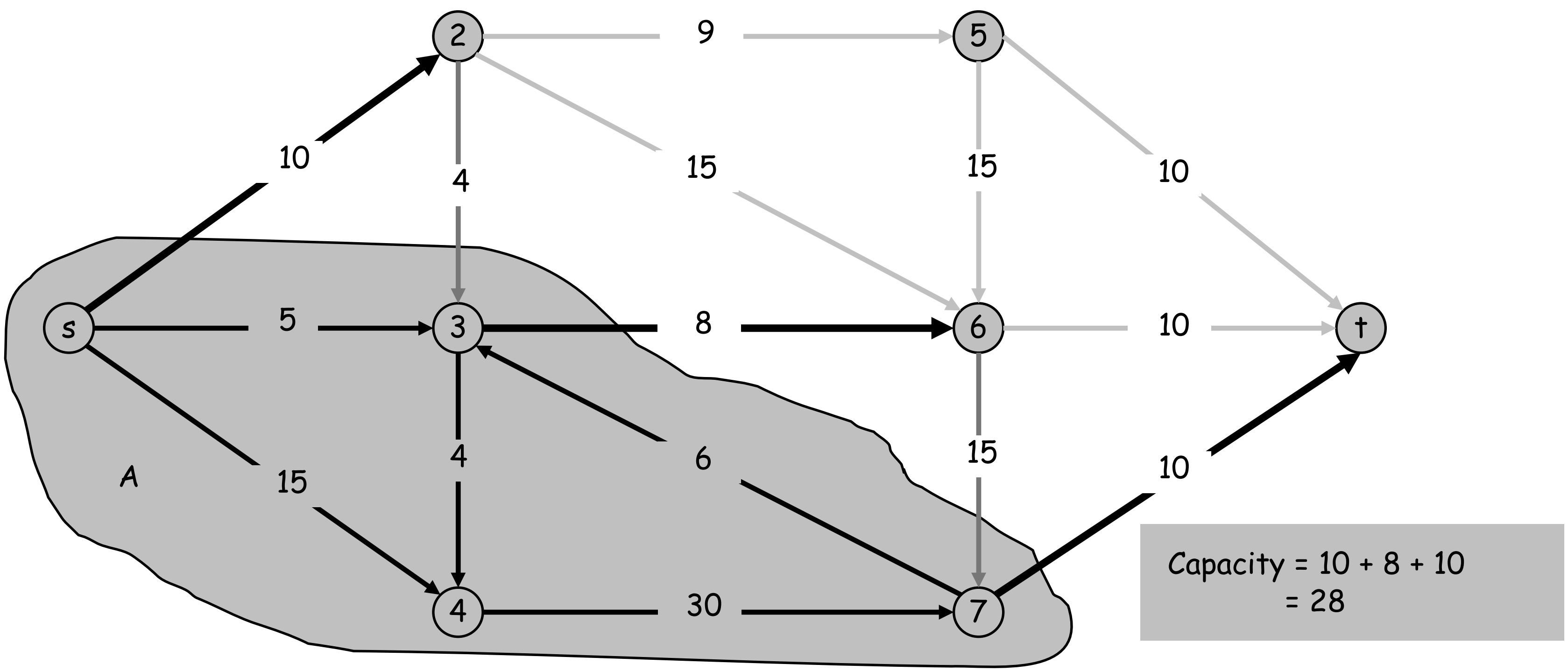
Flow and Cuts

Def: The capacity of the cut (S,T) is $c(S,T) = \sum_{e \text{ from } S \text{ to } T} c(e)$

Claim: For any flow f and any s-t cut (S,T), $|f| \le c(S,T)$.

Example Usage: A few pages ago, we found a flow with value $|f| = 28$ for the graph below. The cut (S,T), with $S = \{s,3,4,7\}$ has $c(S,T) = 28$ so that flow is maximum, since no flow can be better!

We now make this into a theorem!



Correctness of Ford-Fulkerson Algorithm

Max-Flow min-cut theorem: Let f be any flow.
Then the following three statements are equivalent:

(1) f is a maximum flow.

(2) The residual graph G_f has no path from s to t.

(3) $|f| = c(S, T)$ for some s-t cut (S, T).

Proof: (1) ⇒ (2), or ¬(2) ⇒ ¬(1): If there is a path in G_f, we can improve f.

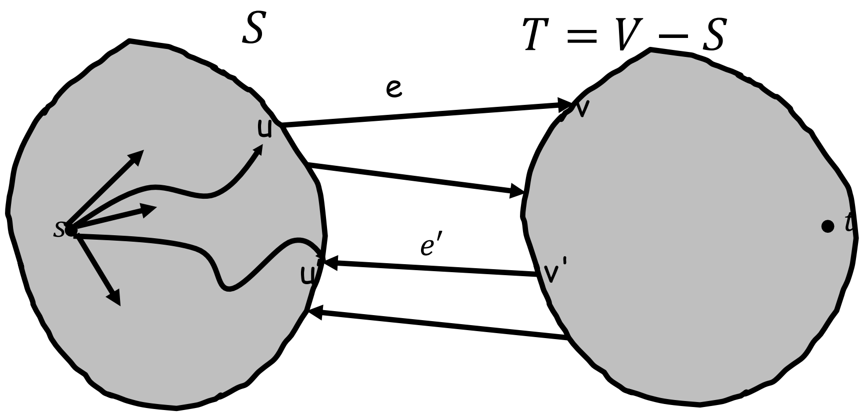
(2) ⇒ (3):

- Need to find an s-t cut (S, T) such that $|f| = c(S, T)$
- By net flow lemma, $|f| = f(S, T)$, so must find a cut such that
  a) all edges e from S to T are full, i.e., $f(e) = c(e)$
  b) all edges e from T to S are empty, i.e., $f(e) = 0$
- Consider $S =$ set of all nodes reachable from s in G_f.
- S cannot include t due to (2), so it is a valid s-t cut
- And this cut must meet the two conditions above!

(3) ⇒ (1): By the claim from last page.

Dive Deeper

The statement: (2: If G_f has no path from s to t) $\Rightarrow$ (3: $|f| = c(S,T)$ for some s-t cut (S,T)).



$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (u,v) \notin E \end{cases}$$

G_f is edges with $c_f(u,v) > 0$

S is vertices that can be reached from s with edges in G_f.

a) If $e = (u,v)$ is an edge from S to T, and $c_f(u,v) > 0$,
=> then $v \in S$ [path from s to u, followed by edge (u,v)],
contradicting $v \in T$.
=> For all edges $e = (u,v)$ from S to T, $c_f(u,v) = 0$, *i.e.*, $f(u,v) = c(u,v)$.

b) If $e' = (v', u')$ is an edge from T to S, and $c_f(u', v') > 0$,
=> then $v' \in S$ [path from s to u', followed by edge (u', v')],
contradicting $v' \in T$.
=> For all edges $e = (v', u')$ from T to S, $c_f(u', v') = 0$, *i.e.*, $f(v', u') = 0$.

Ford-Fulkerson: Running time analysis

Q: Which path to choose in the residual graph?

A: Ford-Fulkerson doesn't specify.

- The choice does not affect correctness
- But it does affect running time
- Note that one iteration of the loop can find one augmenting path in $O(E)$ time using BFS or DFS so full run-time of FF is $O(\#\ iterations\ X\ \ E)$

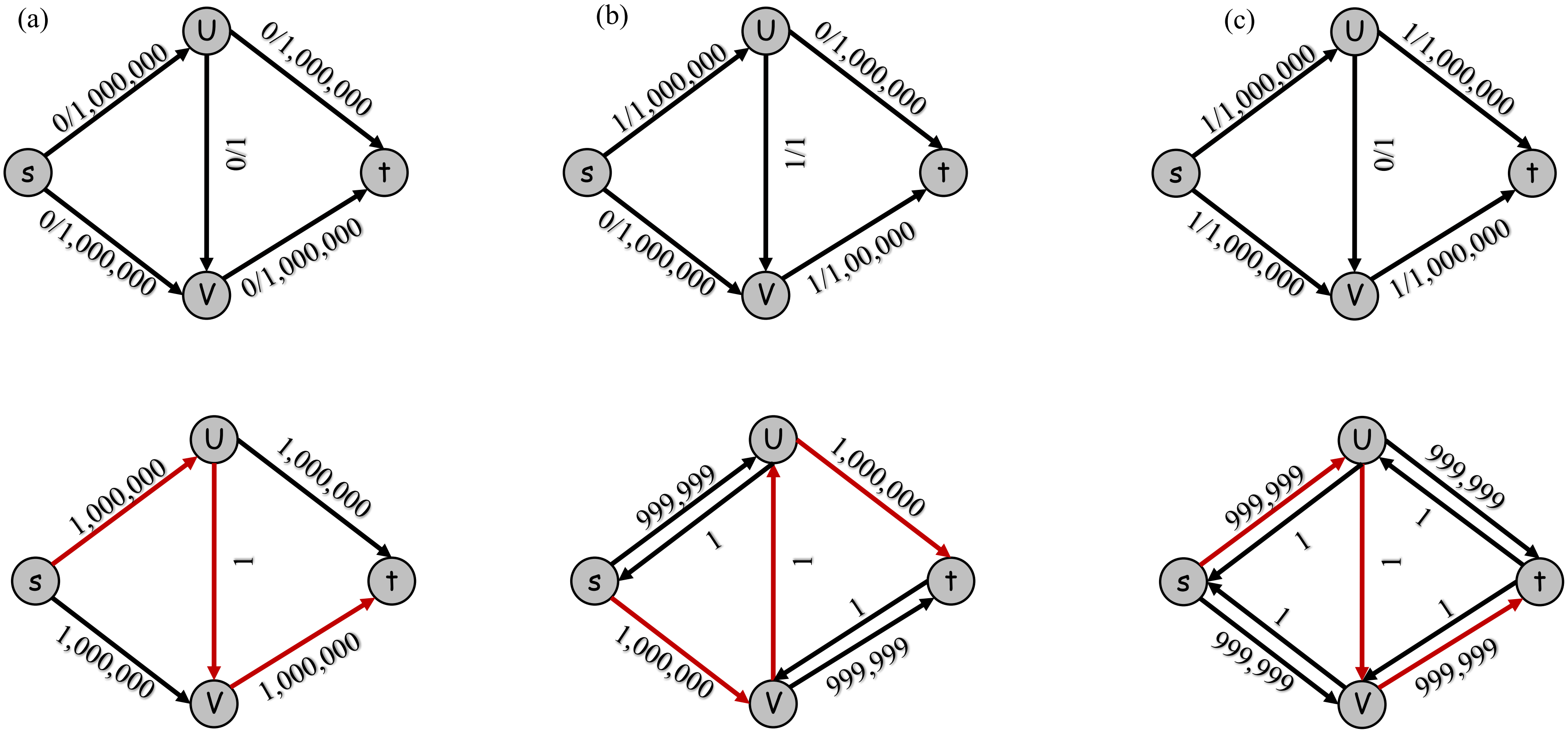
Claim: When all capacities are integers, Ford-Fulkerson takes at most $|f^*|$ iterations, where f^* is a maximum flow.

Proof: Each iteration increases $|f|$ by at least 1.

Integrality property: if all edge capacities are integers, then there exists a max flow for which every flow value is an integer and the F-F algorithm constructs such a flow.

Proof: The flow created by F-F is an integral flow since all (residual) capacities created are integral, so all changes to flows are additions/subtractions of integers.

Bad example



This up/down process will continue, adding only 1 unit of flow per augmenting path.
The final algorithm will require 1,000,000 augmenting steps!
If we had chosen s,u,t as first augmenting path, algorithm only uses 2 steps!

When capacities are irrational numbers, the algorithm might never terminate!

Edmonds-Karp: Choosing the shortest augmenting path

Idea: Choose the shortest (in terms of # edges) path in residual graph

- Can be done in $O(E)$ time using BFS.

Theorem: If we always choose the shortest path in the residual graph to augment the flow, then the Ford-Fulkerson algorithm terminates in $O(VE)$ iterations.

Proof: See textbook (not required).

Corollary: The Ford-Fulkerson algorithm can be implemented to run in $O(VE^2)$ time.

More advanced algorithms

- Push-relabel algorithms, $O(V^2E)$ time, and perform well in practice (see textbook for details)
- Theoretically best algorithm: $O(VE)$ time [King, Rao, Tarjan, 1994] [Orlin, 2013]