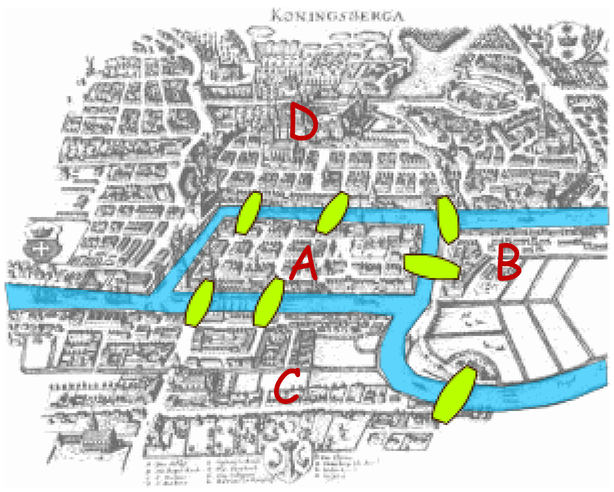
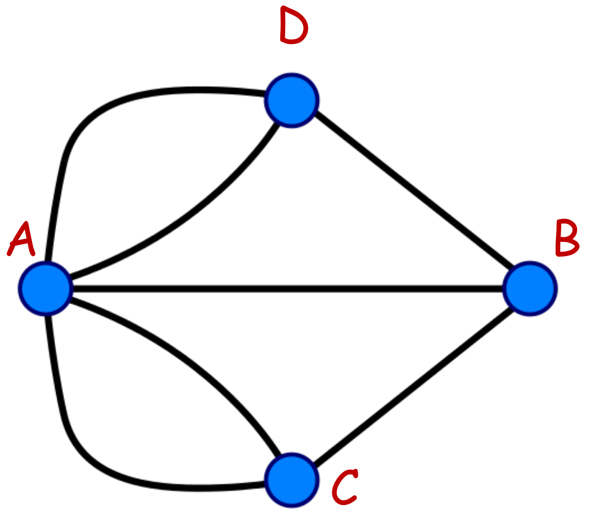
Lecture 14: Introduction to Graphs

Version of March 10, 2019

The Seven Bridges of Königsberg

Q: Can you find a path to cross all seven bridges, each exactly once?





Q: (Reformulated as a graph problem) Can you find a path in the graph that includes every edge exactly once?

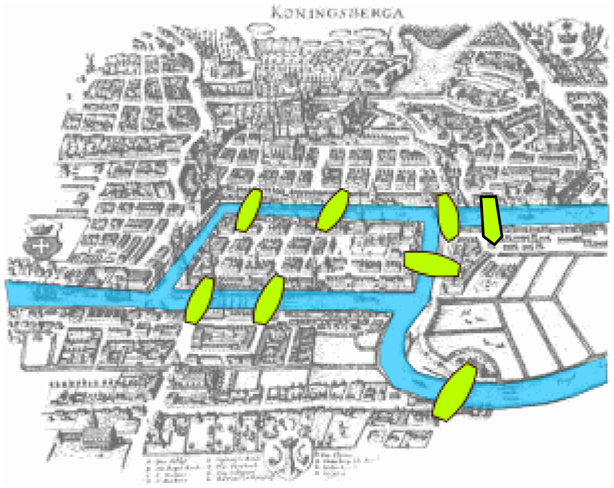
A: Not possible.

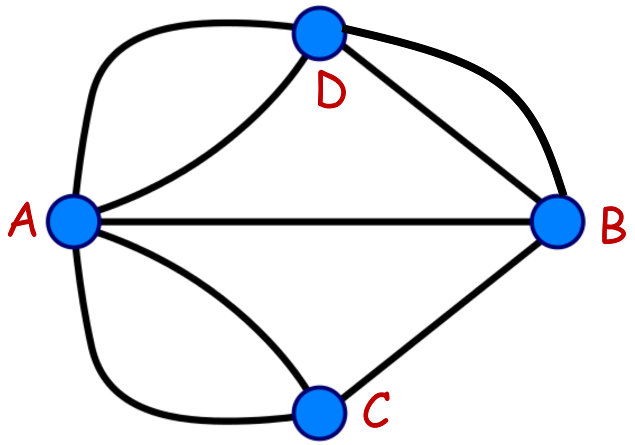
Theorem: A graph has such a path (known as an Euler path) iff there are 0 or 2 vertices with an odd degree.

Q: Can a graph have exactly one vertex with an odd degree?

The Seven Bridges of Königsberg

Solution: Build one more bridge to remove 2 odd-degree vertices.





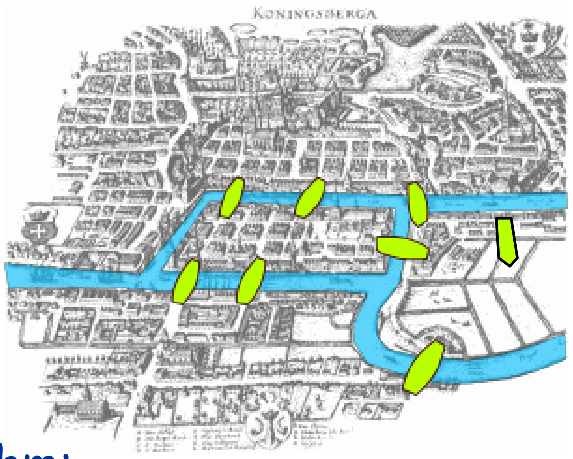
Algorithm:

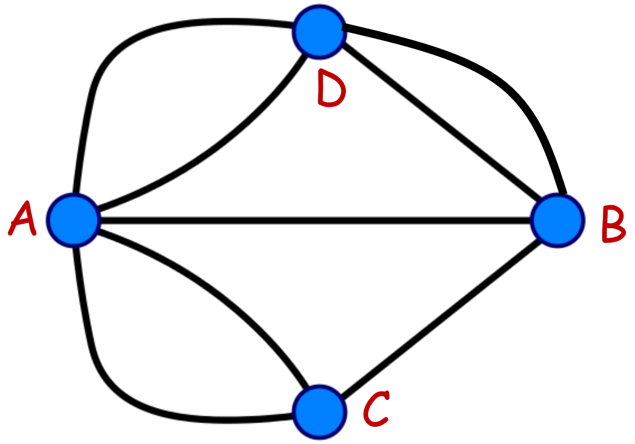
```
$u \leftarrow$ any odd-degree vertex
if no such vertex exists
    $u \leftarrow$ any vertex
while $u$ has an edge not taken yet
    take that edge $(u, v)$
    $u \leftarrow v$
```

But, this algorithm may get stuck...

The Seven Bridges of Königsberg

Solution: Build one more bridge to remove 2 odd-degree vertices.





Algorithm:

```
$u \leftarrow$ any odd-degree vertex
Find-Path($u$)

while there are still edges not yet taken
    $u \leftarrow$ any previously seen vertex that
        is endpoint of untaken edge
    Find-Path($u$)
    insert $p$ into existing path at $u$

Find-Path($u$):
while $u$ has an edge not taken yet
    take that edge $(u,v)$
    $u \leftarrow v$
```

Hierholzer's algorithm (1873)

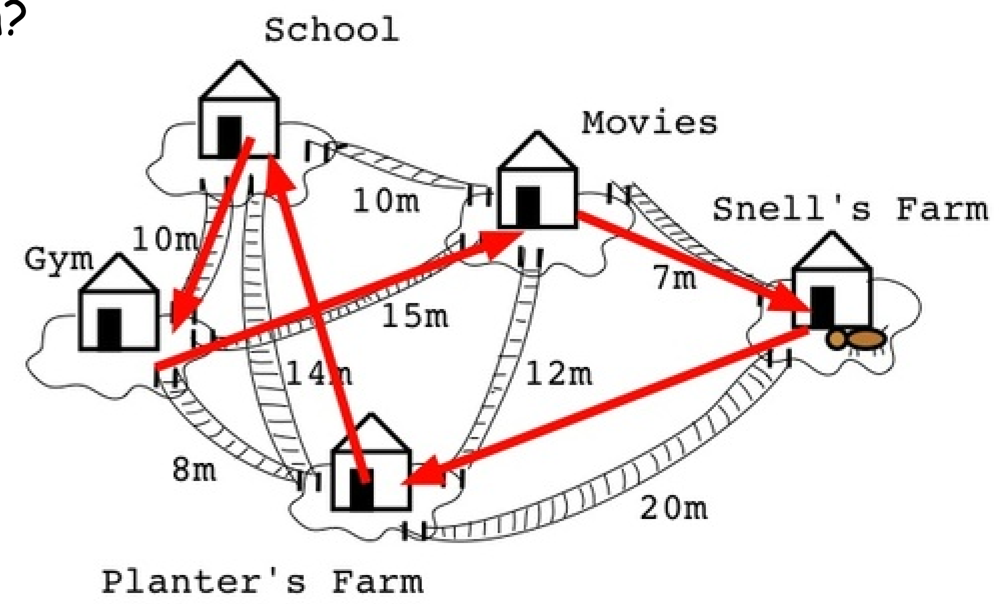
Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

Because they model ``relationships'' graphs are ubiquitous.
Instead of solving problems in one application, we concentrate on designing algorithms to solve problems in abstract graphs.
These can then be used in many different application areas!

The Traveling Salesman Problem

Q: How to visit all places with the shortest total distance, and come back to origin?



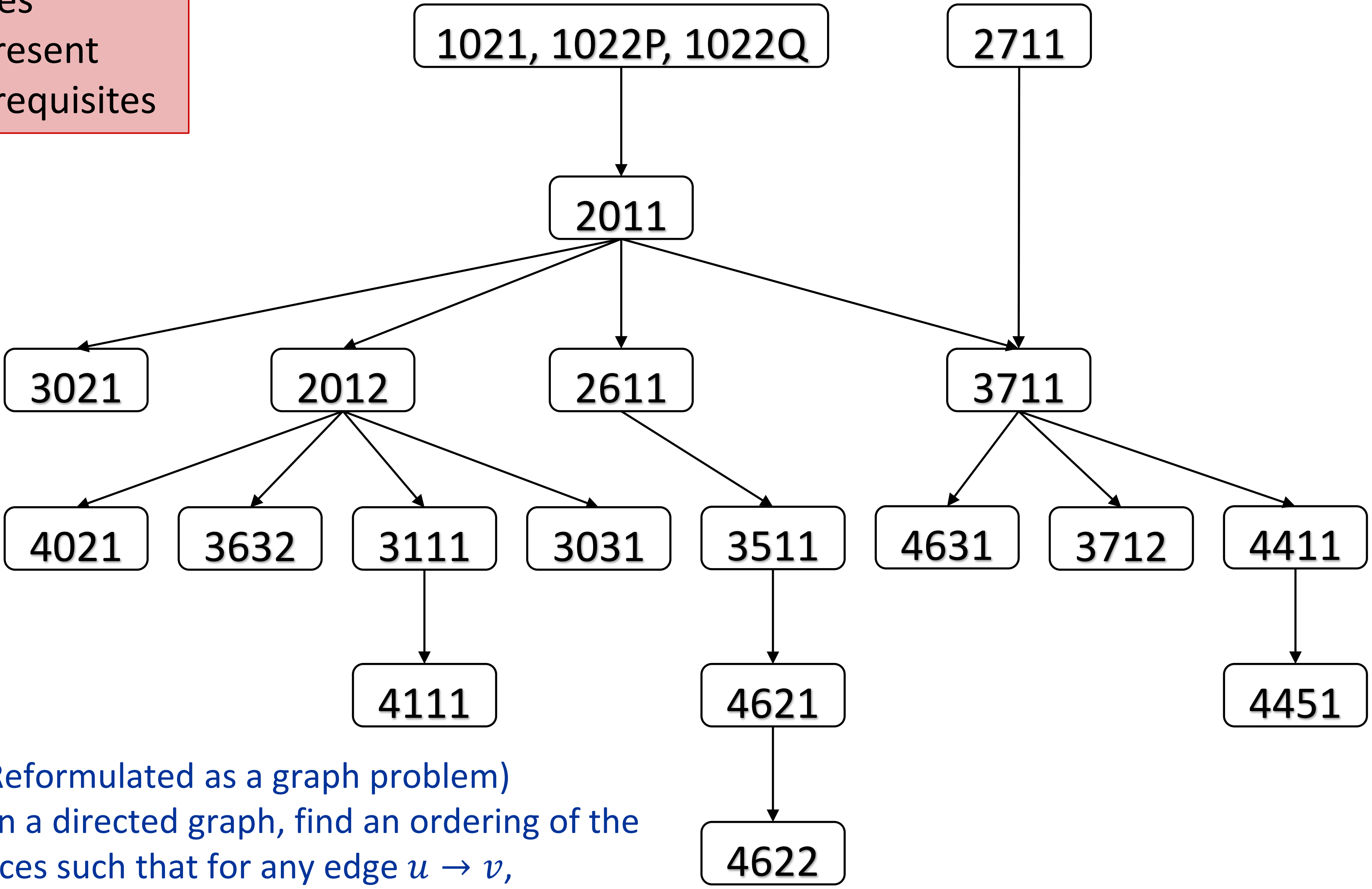
Q: (Reformulated as a graph problem) Given a graph where edges have weights (lengths), how to find a cycle with minimum total weight that includes all vertices?

A: Don't know.

- Don't have an algorithm that runs in polynomial time. (Conjecture is that such an algorithm doesn't exist.)
- This is actually equivalent to the P = NP problem (still open).

Partial COMP course dependency chart

Edges represent prerequisites

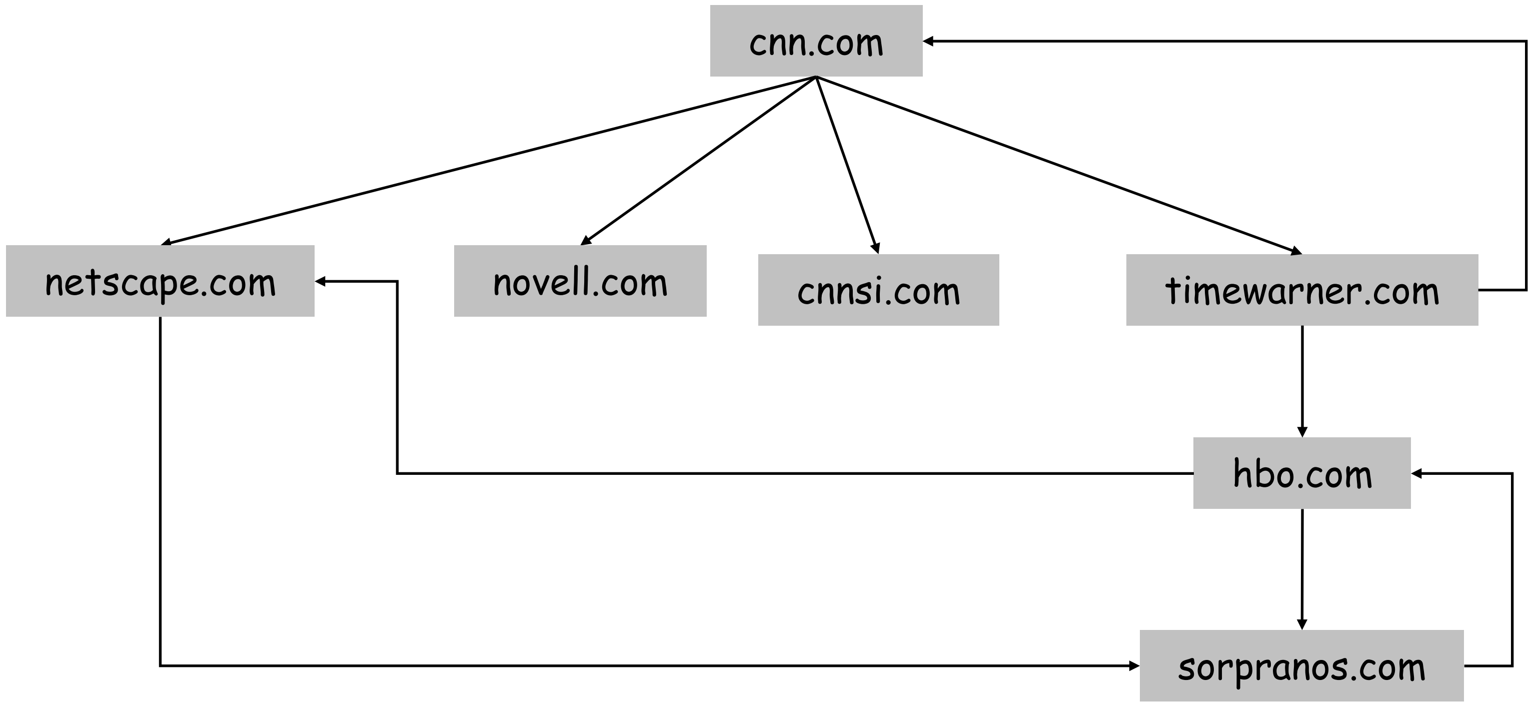


Q: (Reformulated as a graph problem)
Given a directed graph, find an ordering of the vertices such that for any edge $u \rightarrow v$, u is ordered before v, or declare that there is a cycle in the graph.

World Wide Web

Web graph.

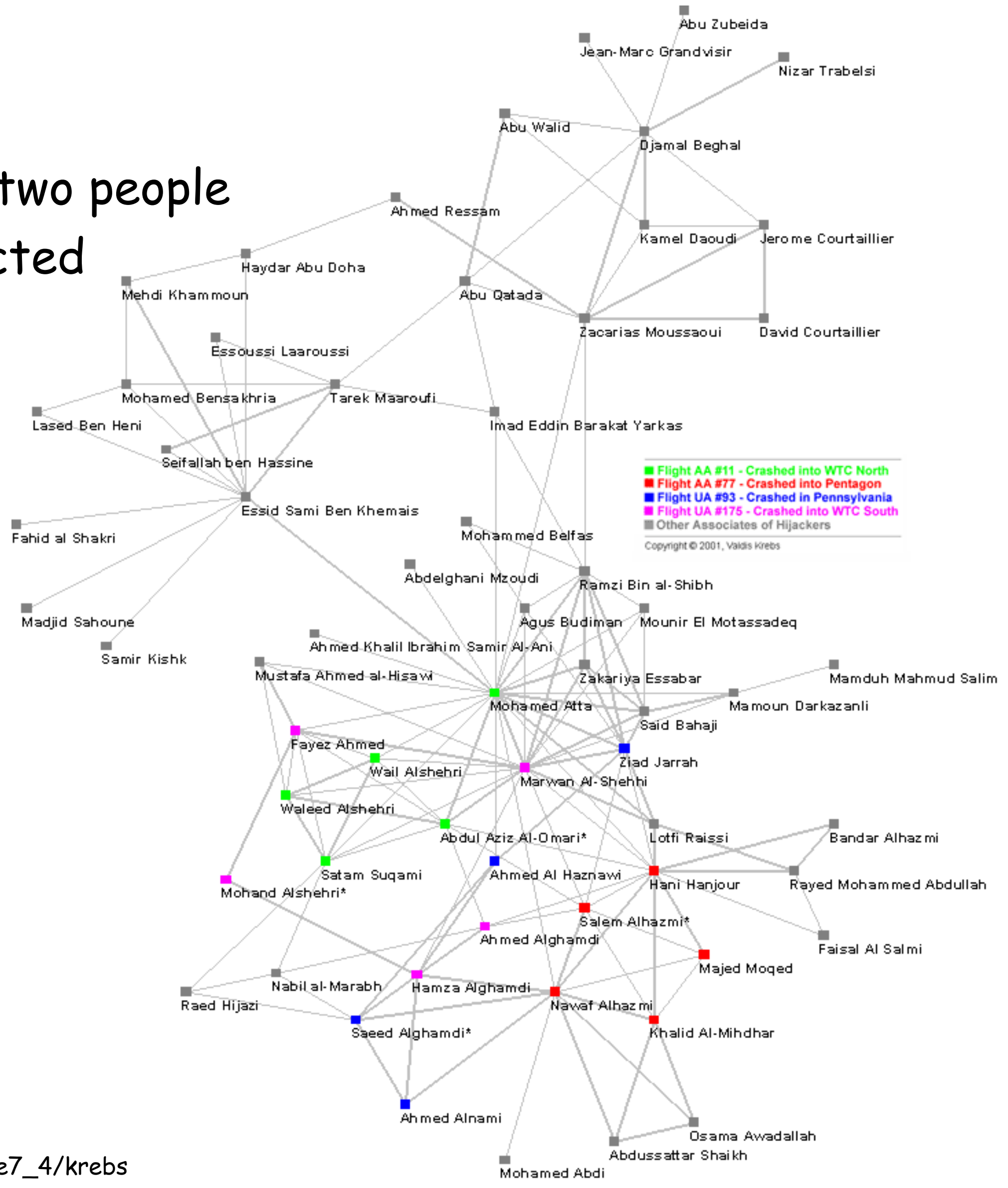
- Node: web page.
- Edge: hyperlink from one page to another (directed).



Social Networks

Social network graph.

- Nodes: people.
- Edges: relationship between two people
  - Can be directed or undirected



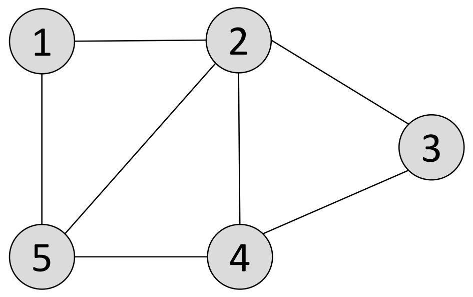
Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs

Undirected and Directed Graphs

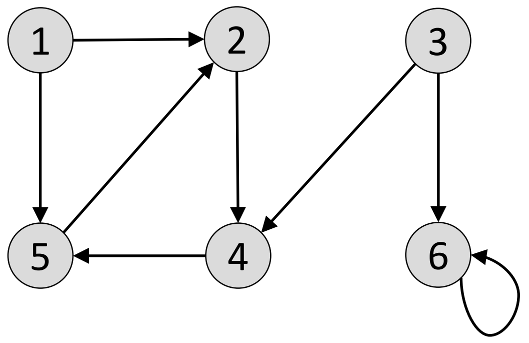
Graph. $G = (V, E)$

- V: set of nodes (vertices).
- E: set of edges between pairs of nodes.
- Abusing notation, we also use V and E to denote the number of nodes and edges. We sometimes also use $n = |V|, m = |E|$.
- There are two different types of graphs: Undirected and Directed

Undirected graph



Directed graph



Undirected Graphs

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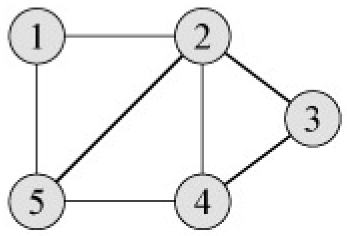
Undirected graphs

- Edges have no direction (or both directions)
- $\deg(v) = \#$ edges at v
- $\sum_{v \in V} \deg(v) = 2E$



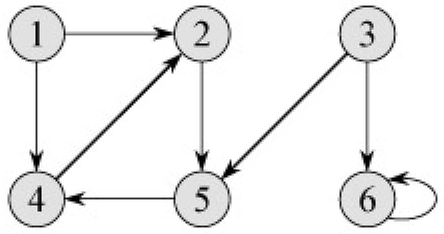
Directed Graphs

Graph. $G = (V, E)$

- V: set of nodes (vertices).
- E: set of edges between pairs of nodes.
- Abusing notation, we also use V and E to denote the number of nodes and edges. We sometimes also use $n = |V|$, $m = |E|$.

Directed graphs.

- Edges have directions
- If an edge has both directions, we will use two edges with opposite directions
- $\deg^{out}(v) = \#$ edge leaving v; $\deg^{in}(v) = \#$ edge entering v.
- $\sum_{v\in V} \deg^{out}(v) = \sum_{v\in V} \deg^{in}(v) = E$



Exercises

Q: Can an undirected graph have exactly one vertex with an odd degree?

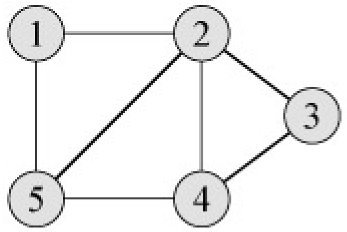
A: No, since $\sum_{v \in V} \deg(v) = 2E$, which is an even number.

Q: [Handshaking lemma] Suppose that the guests in a party shake hands with each other arbitrarily. Show that, no matter how they shake hands, the number of guests who shake hands an odd number of times must be even.

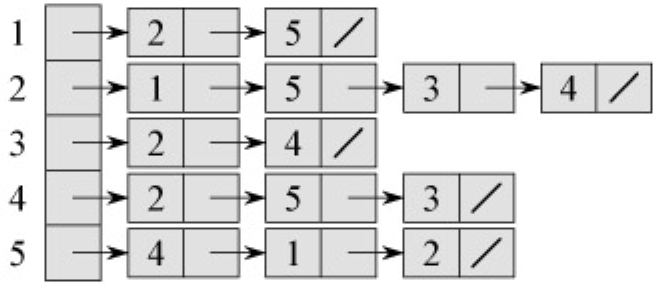
A: Model each guest as a node, a handshake as an edge. This is an undirected graph, so $\sum_{v \in V} \deg(v)$ is even. If odd number of people shake an odd number of times, then the total degree would be

$$\underbrace{\text{odd} + \text{odd} + \ldots + \text{odd}}_{\text{odd}} + \underbrace{\text{even} + \text{even} + \ldots + \text{even}}_{\text{no matter how many}} = \text{odd}$$

Graph Representation: Adjacency List and Adjacency Matrix



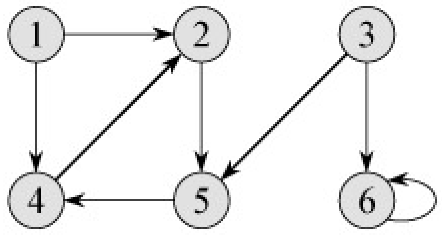
(a)



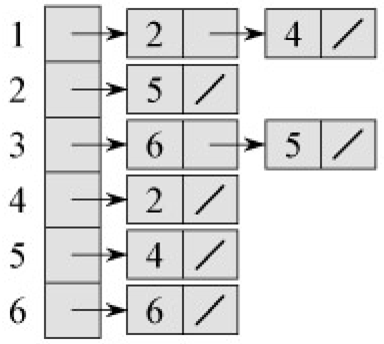
(b)

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}$$

(c)



(a)



(b)

$$\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

(c)

Graph Representation: Adjacency List and Adjacency Matrix

Adjacency list.

- A node-indexed array of lists.
- Given node u, retrieving all neighbors in $\Theta(\deg(u))$ time
- Given u, v, checking if (u, v) is an edge takes $\Theta(\deg(u))$ time.
- Space: $\Theta(V + E)$.

Adjacency matrix.

- A $V \times V$ matrix.
- Given node u, retrieving all neighbors in $\Theta(V)$ time
- Given u, v, checking if (u, v) is an edge takes $O(1)$ time.
- Space: $\Theta(V^2)$.

Note:

- Adjacency list are more commonly used, since most graphs are sparse.
- Usually, assume no self-loops and duplicated edges.
  - Thus, for undirected graphs, $0 \leq E \leq V(V-1)/2$
  - For directed graphs, $0 \leq E \leq V(V-1)$
- Can convert from one to the other in $\Theta(V^2)$ time.

Q: How to represent weights?

Paths and Connectivity

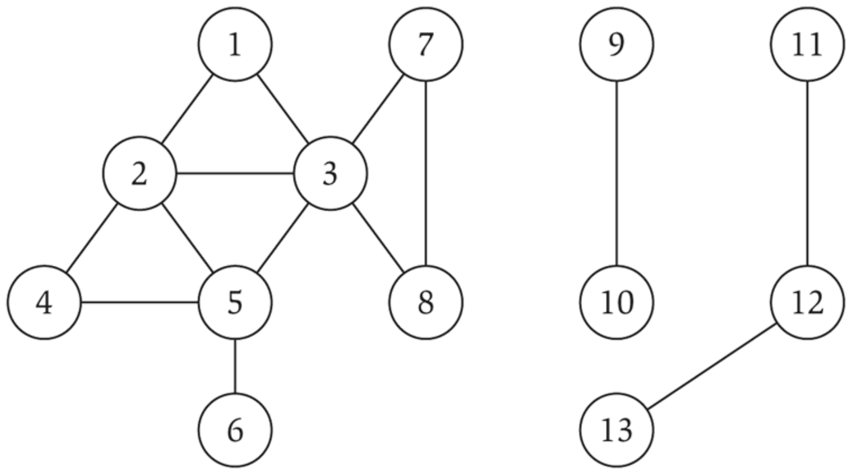
Def. A path in a (directed or undirected) graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \dots, v_{k-1}, v_k$ such that (v_i, v_{i+1}) is an edge. The length of the path is $k - 1$ (i.e., # edges in the path).

Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.

Theorem: For a connected graph, $E \geq V - 1$.

Def. A cycle is a path $v_1, v_2, \dots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$,
Def. A cycle is simple if the first $k - 1$ nodes are all distinct.



Exercise

Q: Suppose in a wireless network of n mobile devices, each device is within communication range with at least $n/2$ other devices (assuming n is an even number). Show that all devices are connected.

Reformulated as a graph problem: Let G be an undirected graph where each node has degree $\geq n/2$. Show that G is connected.

Pf: Consider any two nodes u and v in G. There are two cases:

- If there is an edge (u, v), then u and v are connected.
- If there is no direct edge between u and v, then they must have a common neighbor, say w, because
  - There are $n - 2$ nodes other than u and v.
  - u and v each have $\geq n/2$ neighbors among these nodes.
- Thus there is a path between u and v.
- The above argument holds for any two nodes u, v, so the graph G is connected.

Q: If the threshold $n/2$ is changed to $n/2 - 1$, does the claim still hold?

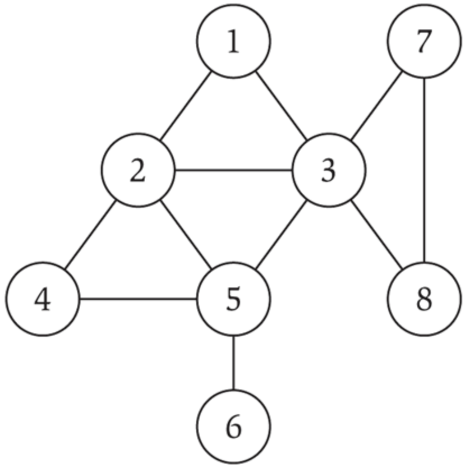
Connectivity and Shortest Path

s-t connectivity problem. Given two nodes s and t, is there a path from s to t?

s-t shortest path problem. Given two node s and t, what is the shortest path from s to t?

Def: The length of the path (in terms of number of edges) is the distance from s to t.

The problem can be defined on either an undirected or directed graph.



Trees

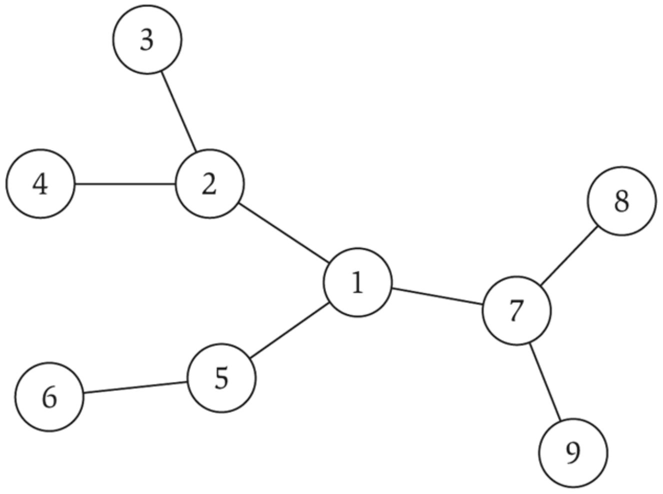
Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Def. An undirected graph is a forest if it does not contain a cycle (i.e., a collection of trees).

Theorem (simpler version of Theorem B.4 in textbook):
Let G be an undirected graph. Any two of the following statements imply the third (hence G is a tree).

(1) G is connected.
(2) G does not contain a cycle.
(3) $E = V - 1$.

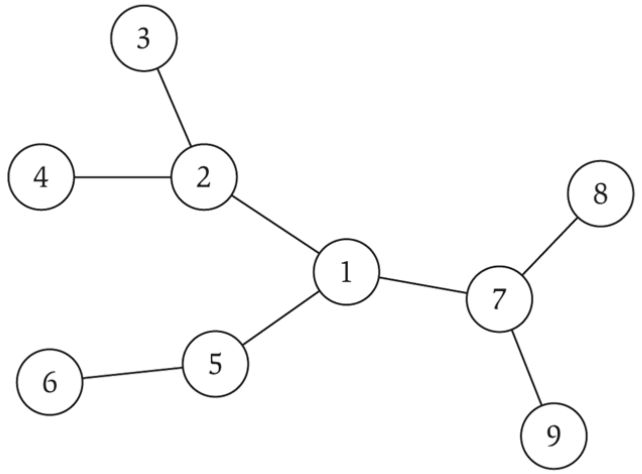
Proof: (Omitted)



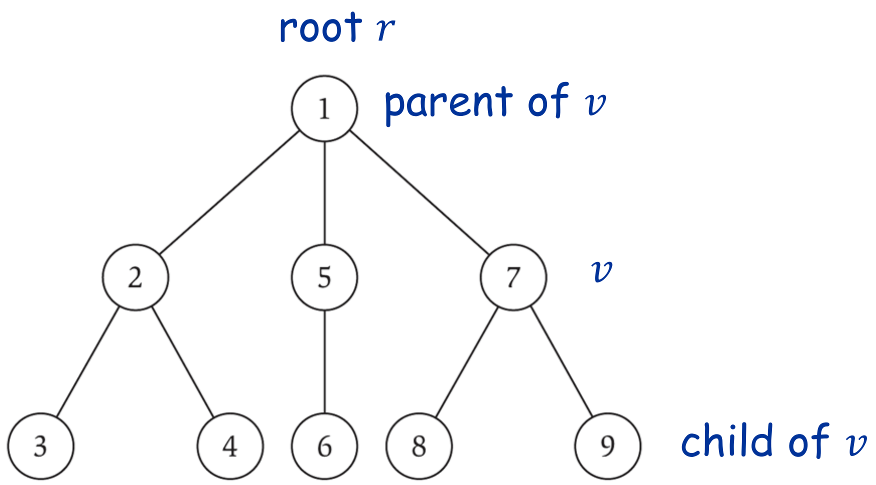
E=8, V=9

Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.



a tree



the same tree, rooted at 1