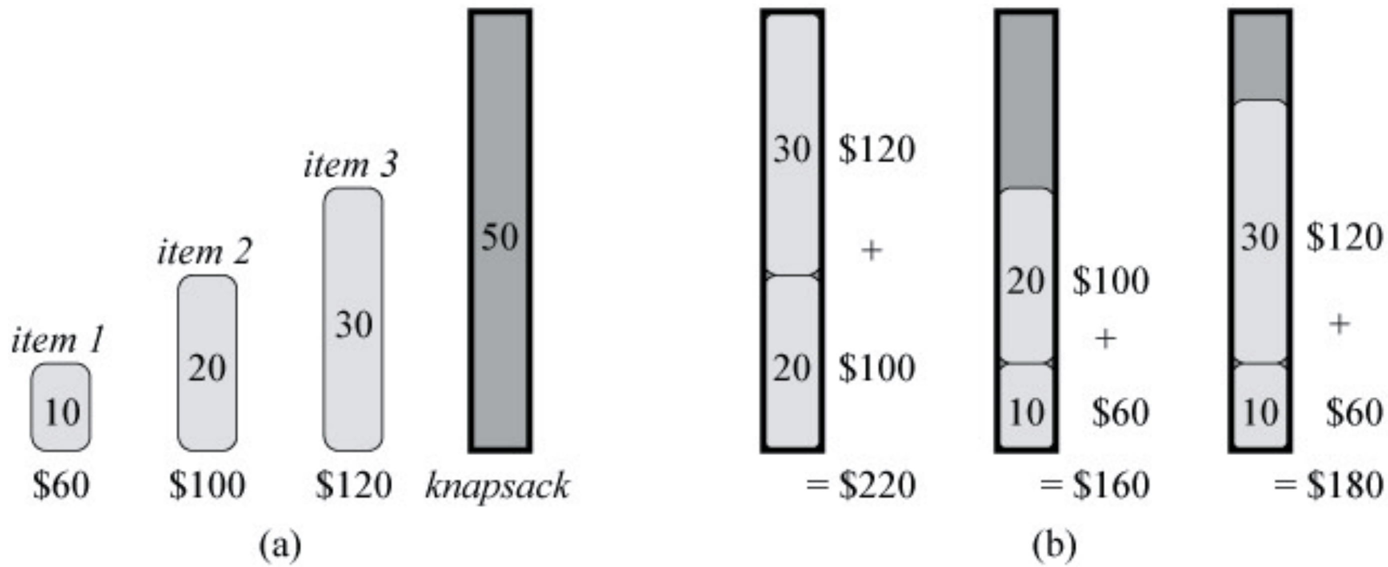
Lecture 12: 2D Dynamic Programming

Version of February 26, 2018

The 0/1 Knapsack Problem



Input: A set of n items,
where item i has weight w_i and value v_i,
and a knapsack with capacity W.

Goal: Find $x_1, \dots, x_n \in \{0,1\}$ satisfying $\sum_{i=1}^{n} x_i w_i \leq W$
that maximizes $\sum_{i=1}^{n} x_i v_i$.

Recall: Greedy doesn't work

The Recurrence

Definition: Let $V[j]$ be the largest obtainable value for a knapsack with capacity j.

Recurrence: First Attempt

If Optimal Solution for knapsack of size j chooses item i, remainder of optimal solution is optimal solution for subproblem of filling knapsack of size $j - w_i$

$$V[j] = \max(0,\ v_1 + V[j - w_1],\ v_2 + V[j - w_2], \ldots,\ v_n + V[j - w_n])$$
$$V[j] = 0, j \leq 0$$

WRONG: This may pick the same item more than once! Non-legal Solution!

New definition: Let $V[i, j]$ be the largest obtained value for a knapsack with capacity j, choosing ONLY from the first i items.

Recurrence:

Doesn't choose i

$$V[i, j] = \max(V[i-1, j],\ \ v_i + V[i-1, j - w_i])$$
$$V[i, j] = 0, i = 0\ or\ j = 0$$

Chooses i

So Far

Input: A set of n items; item i has weight w_i and value v_i; a knapsack with capacity W.
Goal: Find $x_1, \ldots, x_n \in \{0,1\}$ such that $\sum_{i=1}^{n} x_i w_i \leq W$ and $\sum_{i=1}^{n} x_i v_i$ is maximized.

Subproblem:

$V[i, j]$ is the largest obtained value for knapsack with capacity j, choosing ONLY from items 1 … i.

Recurrence: $V[i,j] = \max(V[i-1,j],\ v_i + V[i-1, j-w_i])$

With initial condition, $\forall i, \quad V[i,0] = 0$

Find Order for filling in table: For i = 1 to n
For j = 1 to W

Required Solution: $V[n, W]$

Note that the DP is 2-Dimensional (2 variables) and not 1-D.

The Algorithm

let $V[0..n, 0..W]$ **be a new 2D array of all** 0
for $i \leftarrow 1$ **to** n **do**
  for $j \leftarrow 1$ **to** W **do**
    if $w[i] \leq j$ **and** $v[i] + V[i-1, j-w[i]] > V[i-1, j]$ **then**
      $V[i,j] \leftarrow v[i] + V[i-1, j-w[i]]$
    else
      $V[i,j] \leftarrow V[i-1, j]$
return $V[n, W]$

Running time: $\Theta(nW)$

Space: $\Theta(nW)$, but can be improved to $\Theta(n+W)$

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

Reconstructing the Solution

Idea: Remember the optimal decision for each subproblem in keep[i,j]

```
let V[0..n, 0..W] and keep[0..n, 0..W] be a new array of all 0
for i ← 1 to n do
    for j ← 1 to W do
        if w[i] ≤ j and v[i] + V[i − 1, j − w[i]] > V[i − 1, j] then
            V[i, j] ← v[i] + V[i − 1, j − w[i]]
            keep[i, j] ← 1
        else
            V[i, j] ← V[i − 1, j]
            keep[i, j] ← 0
K ← W
for i ← n downto 1 do
    if keep[i, K] = 1 then
        print i
        K ← K − w[i]
```

Running time: $\Theta(nW)$

Space: $\Theta(nW)$, cannot be improved to $\Theta(n+W)$ due to the $keep$ array.

Longest Common Subsequence

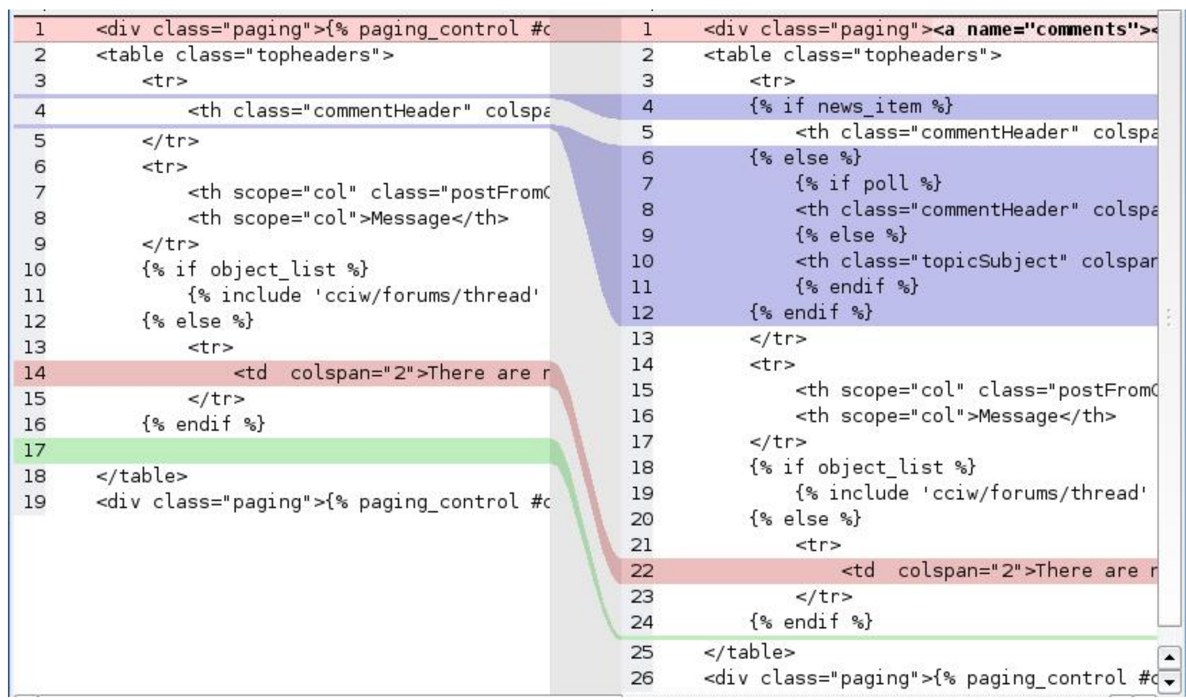
Problem: Given two sequences $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_n)$, we say that Z is a common subsequence of X and Y if Z has a strictly increasing sequence of indices i and j of both X and Y such that we have $x_{i_p} = y_{j_p} = z_p$ for all $p = 1, 2, \ldots, k$. The goal is to find the longest common subsequence of X and Y.

Ex:

X:	A	B	A	C	B	D	A	B
Y:		B	D	C A	B		A	
Z:		B		C	B		A	

Application:

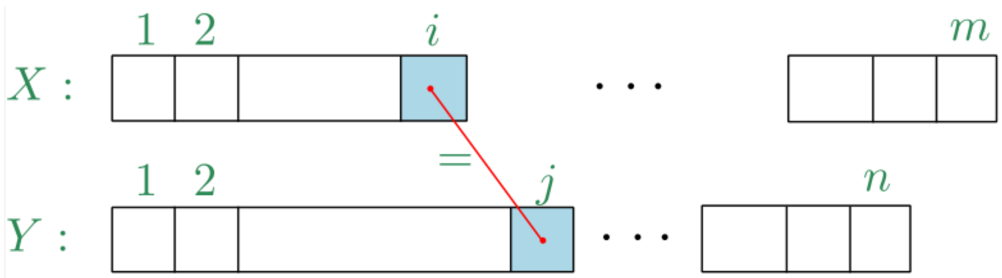
- diff
- genetics



The Recurrence

Def: Let $c[i, j]$ to be the length of the longest common subsequence of $X[1..i]$ and $Y[1..j]$.

Observations: The problem is equivalent to finding the maximum matching between X and Y such that matched pairs don't cross.



The recurrence:

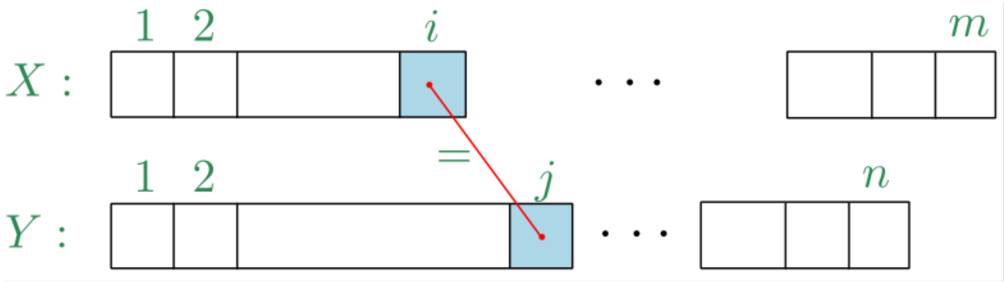
- Case 1: If $x_i = y_j$, then match x_i and y_j.
  Doing so will not miss the optimal solution.
  (If OPT doesn't match them to each other, we can change OPT so that they are matched.)

- Case 2: If $x_i \neq y_j$, then either x_i or y_j is not matched.
  So optimal solution reduces to either $c[i-1, j]$ or $c[i, j-1]$.

The Recurrence

Def: $c[i,j]$ is length of the longest common subsequence of $X[1..i]$ and $Y[1..j]$.

Observations: The problem is equivalent to finding the maximum matching between X and Y such that matched pairs don't cross.



The recurrence:

- Case 1: If $x_i = y_j$, then we match x_i and y_j.
- Case 2: If $x_i \neq y_j$, then either x_i or y_j is not matched.
  Optimal solution reduces to either $c[i-1,j]$ or $c[i,j-1]$.

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max\{c[i,j-1], c[i-1,j]\} & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

The Recurrence and Algorithm

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max\{c[i,j-1], c[i-1,j]\} & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

let $c[0..m, 0..n]$ **and** $b[0..m, 0..n]$ **be new arrays of all** 0
for $i \leftarrow 1$ **to** m
  for $j \leftarrow 1$ **to** n
    if $x_i = y_j$ **then**
      $c[i,j] \leftarrow c[i-1,j-1]+1$
      $b[i,j] \leftarrow$ " ↖ "   MATCH x_i, y_j
    else if $c[i-1,j] \geq c[i,j-1]$ **then**
      $c[i,j] \leftarrow c[i-1,j]$ x_i not matched
      $b[i,j] \leftarrow$ " ↑ "
    else
      $c[i,j] \leftarrow c[i,j-1]$
      $b[i,j] \leftarrow$ " ← " y_j not matched
Print-LCS(b, m, n)

Running time: $\Theta(mn)$

Space: $\Theta(mn)$, can be improved to $\Theta(m+n)$ if we only need to return the optimal length.

Reconstruct the Optimal Solution

Print-LCS (b, i, j) **:**

 if $i = 0$ **or** $j = 0$ **then return**
 if $b[i, j] =$ "↖" **then**
  Print-LCS $(b, i-1, j-1)$
  print x_i
 else if $b[i, j] =$ "↑"
  Print-LCS $(b, i-1, j)$
 else Print-LCS $(b, i, j-1)$

i	j	0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

Value of b[i,j] indicates whether

↖: x_i, y_j matched:
then write x_i and return LCS (i-1,j-1)

↑: x_i not matched
skip x_i and return LCS (i-1,j)

←: y_j not matched
skip y_j and return LCS (i,j-1)

Longest Common Substring

Problem: Given two strings $X = x_1 x_2 \dots x_m$ and $Y = y_1 y_2 \dots y_n$, we wish to find their longest common substring Z, that is, the largest k for which there are indices i and j with $x_i x_{i+1} \dots x_{i+k-1} = y_j y_{j+1} \dots y_{j+k-1}$.

Ex:

X : **DEADBEEF**

Y : **EATBEEF**

Z : **BEEF** //pick the longest contiguous substring

Note: Brute-force algorithm takes $O(n^4)$ time.

Different from LCS problem because, in this problem, letters have to be together.

The Recurrence

Def: $d[i,j] =$ the length of the longest common substring of $X[1..i]$ and $Y[1..j]$. (Does this work?)

Def: $d[i,j] =$ the length of the longest common substring of $X[1..i]$ and $Y[1..j]$ that ends at x_i and y_j.

Q: Wait, are we changing the problem?

A: Yes, but it's OK. Optimal solution to the original is just $\max_{i,j}\{d[i,j]\}$

Recurrence:

- If $x_i = y_j$, then the LCS of $X[1..i]$ and $Y[1..j]$
  is just the LCS of $X[1..i-1]$ and $Y[1..j-1]$, plus $x_i = y_j$
- If $x_i \neq y_j$, then there can't be a common substring ending at x_i and y_j!

$$\boxed{d[i,j] = \begin{cases} d[i-1,j-1]+1 & \text{if } x_i = y_j \\ 0 & \text{if } x_i \neq y_j \end{cases}}$$

The Algorithm

```
let d[0..m, 0..n] be a new array of all 0
l_m ← 0, p_m ← 0
for i ← 1 to m
    for j ← 1 to n
        if x_i = y_j then
            d[i, j] ← d[i − 1, j − 1] + 1
            if d[i, j] > l_m then
                l_m ← d[i, j]
                p_m ← i
for i ← p_m − l_m + 1 to p_m
    print x_i
```

Note: For this problem, reconstructing the optimal solution just needs the location of the LCS.

Running time: $\Theta(mn)$

Space: $\Theta(mn)$ but can be improved to $\Theta(m+n)$.