# L06: GCDs and Congruences

- **Greatest Common Divisor (GCD)**
- Multiplicative Inverses
- Solving Linear Congruences
- The Chinese Remainder Theorem

- Reading: Rosen 4.3, 4.4, 4.5

# Review of Primary School Knowledge

- **Definition**
  A positive integer $p$ greater than $1$ is called prime if the only positive factors of $p$ are $1$ and $p$. A positive integer that is greater than $1$ and is not prime is called *composite*.
- **Theorem** (The Fundamental Theorem of Arithmetic)
  Every positive integer greater than $1$ can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.
  (Will prove later.)

# Greatest Common Divisor

- **Definition**
  Let $a$ and $b$ be integers, not both zero. The largest integer $d$ such that $d \mid a$ and also $d \mid b$ is called the greatest common divisor of $a$ and $b$, denoted by $\gcd(a, b)$.
- One can find the gcd by prime factorizations
- **Example**
  $120 = 2^3 \cdot 3 \cdot 5 \qquad 500 = 2^2 \cdot 5^3$
  $\gcd(120,500) = 2^{\min(3,2)} \cdot 3^{\min(1,0)} \cdot 5^{\min(1,3)} = 2^2 \cdot 3^0 \cdot 5^1 = 20$
- **Definition**
  The integers $a$ and $b$ are relatively prime if $\gcd(a, b) = 1$.
  - Example: 17 and 22

# Euclidean Algorithm

- However, factoring large numbers is hard!
  - No efficient algorithms exist
- **Lemma**
  Let $a = bq + r$, where $a$, $b$, $q$, and $r$ are integers. Then $\gcd(a, b) = \gcd(b, r)$.
- **Proof**
  - Suppose that $d$ divides both $a$ and $b$. Then $d$ also divides $a - bq = r$. Hence, any common divisor of $a$ and $b$ must also be any common divisor of $b$ and $r$.
  - Suppose that $d$ divides both $b$ and $r$. Then $d$ also divides $bq + r = a$. Hence, any common divisor of $a$ and $b$ must also be a common divisor of $b$ and $r$.
  - Therefore, $\gcd(a, b) = \gcd(b, r)$.

# Euclidean Algorithm

- Idea: To obtain maximum efficiency, choose the smallest $r$, i.e., $r = a \bmod b$ (suppose $a > b$), and iterate.

<u>**gcd(**$a,b$**):**</u>
$x \leftarrow a$
$y \leftarrow b$
**while** $y \neq 0$
    $r \leftarrow x \bmod y$
    $x \leftarrow y$
    $y \leftarrow r$
**return** $x$

Example:

$$
\begin{aligned}
&\gcd(287, 91) \\
&= \gcd(91, 14) \\
&= \gcd(14, 7) \\
&= 7
\end{aligned}
$$

- Correctness of algorithm follows from previous lemma
- Termination is obvious
- Running time will be analyzed later

# gcds as Linear Combinations

- **Theorem** (Bézout's Theorem)
  If $a$ and $b$ are positive integers, then there exist integers $s$ and $t$ such that $\gcd(a, b) = sa + tb$.
- Example

$$\gcd(6,14) = (-2) \cdot 6 + 1 \cdot 14$$

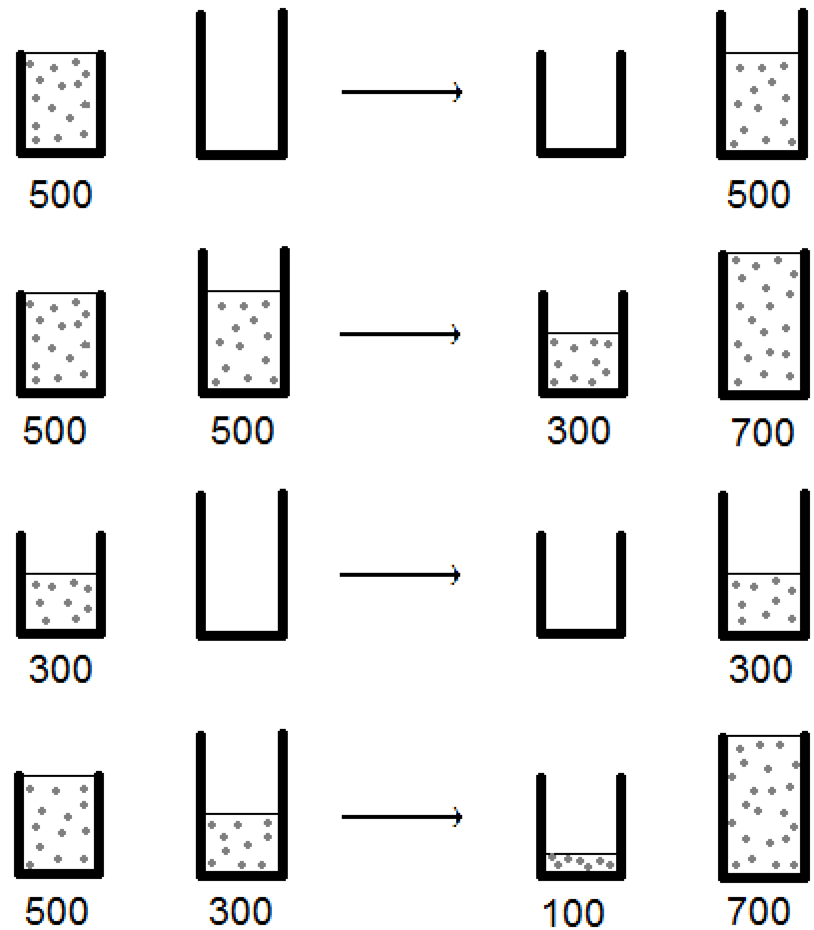
- Instead of proving this theorem directly, we give an algorithm to find such $s$ and $t$.

# The Extended Euclidean Algorithm

- **Example**
  Express $\gcd(252,198)$ as a linear combination of $252$ and $198$.
- **Solution**
  - First find $\gcd(252,198)$
    1) $252 = 1\cdot198 + 54$
    2) $198 = 3\cdot54 + 36$
    3) $54 = 1\cdot36 + 18$
    4) $36 = 2\cdot18$
    5) $\gcd(252,198) = 18$
  - Rewriting:
    - $54 = 252 - 1\cdot198$
    - $36 = 198 - 3\cdot54$
    - $18 = 54 - 1\cdot36$
  - Substituting:
    $18 = 54 - 1\cdot(198 - 3\cdot54)$
    $= 4 \cdot 54 - 1 \cdot 198$
    $= 4 \cdot (252 - 1\cdot198) - 1 \cdot 198$
    $= 4 \cdot 252 - 5 \cdot 198$

# Puzzle: Water Measuring

- Given
  - Two bottles: one has volume of 500 ml and the other one 700 ml.
  - Infinite water supply
- Goal: Get exactly 100 ml of water
- This follows exactly from
  $100 = \gcd(500,700)$
  $= 3 \times 500 - 2 \times 700$
- Corollary: Any multiple of the gcd can be obtained.



# Outline

- Greatest Common Divisor (GCD)
- **Multiplicative Inverses**
- Solving Linear Congruences
- The Chinese Remainder Theorem

# Multiplicative Inverses

- **Definition**
  The (multiplicative) inverse of $a$ modulo $m$ is some $b$ such that $ab \equiv 1 \pmod{m}$.
- By default "inverse" means "multiplicative inverse".
- **Examples**

$\mathbf{Z}_5$:

| $a$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a^{-1}$ | 1 | 3 | 2 | 4 |

$\mathbf{Z}_7$:

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a^{-1}$ | 1 | 4 | 5 | 2 | 3 | 6 |

$\mathbf{Z}_6$:

| $a$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a^{-1}$ | 1 | X | X | X | 5 |

$\mathbf{Z}_8$:

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $a^{-1}$ | 1 | X | 3 | X | 5 | X | 5 |

# Multiplicative Inverses

- **Theorem**
  For any $a \in \mathbf{Z}_m, m > 1$, if $\gcd(a, m) = 1$ then $a$ has a unique inverse in $\mathbf{Z}_m$.
- **Corollary**
  For any prime $p$, every nonzero $a \in \mathbf{Z}_p$ has a multiplicative inverse.
- **Proof of Theorem**
  Since $\gcd(a, m) = 1$, by Bézout's Theorem, there are integers $s$ and $t$ such that $sa + tm = 1$.
  - Hence, $sa + tm \equiv 1 \pmod{m}$.
  - Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$, or
  $$(s \bmod m) \cdot_m a = 1$$
  - Consequently, $s \bmod m$ is the inverse of $a$ in $\mathbf{Z}_m$.

# Multiplicative Inverses are Unique

- **Proof of uniqueness**
  - Suppose $b, c$ are both inverses of $a$, i.e.,
    $$ab \equiv 1 \pmod{m} \qquad (1)$$
    $$ac \equiv 1 \pmod{m} \qquad (2)$$
  - Multiply both sides of $(1)$ by $c$:
    $$abc \equiv c \pmod{m}$$
  - Multiply both sides of $(2)$ by $b$:
    $$abc \equiv b \pmod{m}$$
  - So $b \equiv c \pmod{m}$, i.e., $a$ has a unique inverse in $\mathbf{Z}_m$.
- The inverse of $a$ is written as $a^{-1}$.
- Note: It's also true that if $\gcd(a, m) \neq 1$, $a^{-1}$ doesn't exist. (We will not prove this direction.)

# Finding Inverses

- Given $a, m$ such that $\gcd(a, m) = 1$, how to find the inverse of $a$ in $\mathbf{Z}_m$?
- Look at the proof of the previous theorem
  - Use the extended Euclidean algorithm to find $s$ and $t$ such that $sa + tm = 1$
  - $s$ mod $m$ is the multiplicative inverse of $a$ in $\mathbf{Z}_m$.
- **Example**
  Find an inverse of $3$ modulo $7$
- **Solution**
  Using the extended Euclidean algorithm: $7 = 2 \cdot 3 + 1$.
  we get $-2 \cdot 3 + 1 \cdot 7 = 1$, so $s = -2$.
  $-2 \bmod 7 = 5$ is the inverse of $3$ in $\mathbf{Z}_7$

# Finding Inverses

- **Example**
  Find the inverse of 101 modulo 4620

Working Backwards:

$4620 = 45 \cdot 101 + 75$
$101 = 1 \cdot 75 + 26$
$75 = 2 \cdot 26 + 23$
$26 = 1 \cdot 23 + 3$
$23 = 7 \cdot 3 + 2$
$3 = 1 \cdot 2 + 1$
$2 = 2 \cdot 1$

Since the last nonzero remainder is 1, gcd(101,4260) = 1

$1 = 3 - 1 \cdot 2$
$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$
$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$
$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$
$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$
$= 26 \cdot 101 - 35 \cdot 75$
$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$
$= -35 \cdot 4620 + 1601 \cdot 101$

1601 is an inverse of 101 modulo 42620

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# Solving Congruences

- Linear congruence
$$ax \equiv b \pmod{m}$$
- Given $a, b, m$, such that $\gcd(a, m) = 1$. How to find $x$?
- Solution:
  - Find $a^{-1}$
  - Multiply $a^{-1}$ on both sides
- Example
  - Solve $3x \equiv 4 \pmod{7}$
  - Find $3^{-1} = 5$
  - Multiply $5$ on both sides:
$$15x \equiv 20 \pmod{7}$$
$$x \equiv 6 \pmod{7}$$

# Solving Congruences

- **Corollary**
  If $\gcd(a, m) = 1$, the linear congruence
  $$ax \equiv b \pmod{m}$$
  has a unique solution in $\mathbf{Z}_m$
- **Proof**
  - Existence has already been proved by construction.
  - Uniqueness: Suppose it has two solutions $x_1, x_2$:
    $$ax_1 \equiv b \pmod{m}$$
    $$ax_2 \equiv b \pmod{m}$$
    Multiply both by $a^{-1}$:
    $$x_1 \equiv ba^{-1} \pmod{m}$$
    $$x_2 \equiv ba^{-1} \pmod{m}$$
    So, $x_1 \equiv x_2 \pmod{m}$.

# Revisiting the String Hash Function

- Consider the simpler case:
  $$h(s) = \left(\left((s[0] \cdot 31 + s[1]) \cdot 31 + s[2]\right) \bmod 2^{32}\right) \bmod n$$
- Note $\gcd(31, 2^{32}) = 1$
- Given any $s[2]$ and $b$, the congruence
  $$31x + s[2] \equiv b \pmod{2^{32}}$$
  has a solution. This means $h(s)$ depends on $x = s[0] \cdot 31 + s[1]$ and every $b$ is possible.
- Similarly, given any $s[1]$, $x$ can possibly take any value depending on $s[0]$.
- Other reasons:
  - Performance: x*31 = x << 5 – 1
  - Using 31 produces more balanced hashes over English text

# Checksums

- **Example**
  HKID numbers are of the format X123456(Y), where
  - X is one or two letters
  - Y is check digit, 0 to 9 or A.
- **How is it computed**
  - Replace the first two letters as follows:
    A = 10 B = 11 C = 12 D = 13 E = 14 F = 15 G = 16 H = 17 I = 18 J = 19
    K = 20 L = 21 M = 22 N = 23 O = 24 P = 25 Q = 26 R = 27 S = 28
    T = 29 U = 30 V = 31 W = 32 X = 33 Y = 34 Z = 35 empty = 36
  - Denote the digits as $x_1, \ldots, x_8$
  - $c = (9x_1 + 8x_2 + 7x_2 + 6x_3 + \cdots + 2x_8) \bmod 11$
  - Check digit $x_9 = 11 - c$
    If $x_9 = 11$, check digit = 0
    If $x_9 = 10$, check digit = A

# HKID Checksum: Single Error

- Note that for a valid HKID, we have
$$9x_1 + 8x_2 + 7x_3 + \cdots + 2x_8 + x_9 \equiv 0 \pmod{11}$$
- Suppose $x_2$ is mistyped as $x_2' \neq x_2$
- Suppose the checksum is still correct, i.e.,
$$9x_1 + 8x_2' + 7x_3 + \cdots + 2x_8 + x_9 \equiv 0 \pmod{11}$$
- Subtracting one congruence from the other:
$$8(x_2 - x_2') \equiv 0 \pmod{11}$$
- Since $\gcd(8, 11) = 1$, 8 has an inverse. Multiply both sides by $8$:
$$x_2 - x_2' \equiv 0 \pmod{11}$$
- This contradicts with the assumption $x_2' \neq x_2$ and they are both in $\{0, \dots, 9\}$
- Note: If one letter is wrong, it may not be detected!

# HKID Checksum: Transposition Error

- Note that for a valid HKID, we have
$$9x_1 + 8x_2 + 7x_3 + \cdots + 2x_8 + x_9 \equiv 0 \pmod{11}$$
- Suppose $x_2$ and $x_4$ are swapped, and $x_2 \neq x_4$
- Suppose the checksum is still correct, i.e.,
$$9x_1 + 8x_4 + 7x_3 + 6x_2 + \cdots + 2x_8 + x_9 \equiv 0 \pmod{11}$$
- Subtracting one congruence from the other:
$$2(x_4 - x_2) \equiv 0 \pmod{11}$$
- Since $\gcd(2, 11) = 1$, 2 has an inverse. Multiply both sides by $2^{-1}$:
$$x_4 - x_2 \equiv 0 \pmod{11}$$
- This contradicts with the assumption $x_2 \neq x_4$ and they are both in $\{0, .., 9\}$

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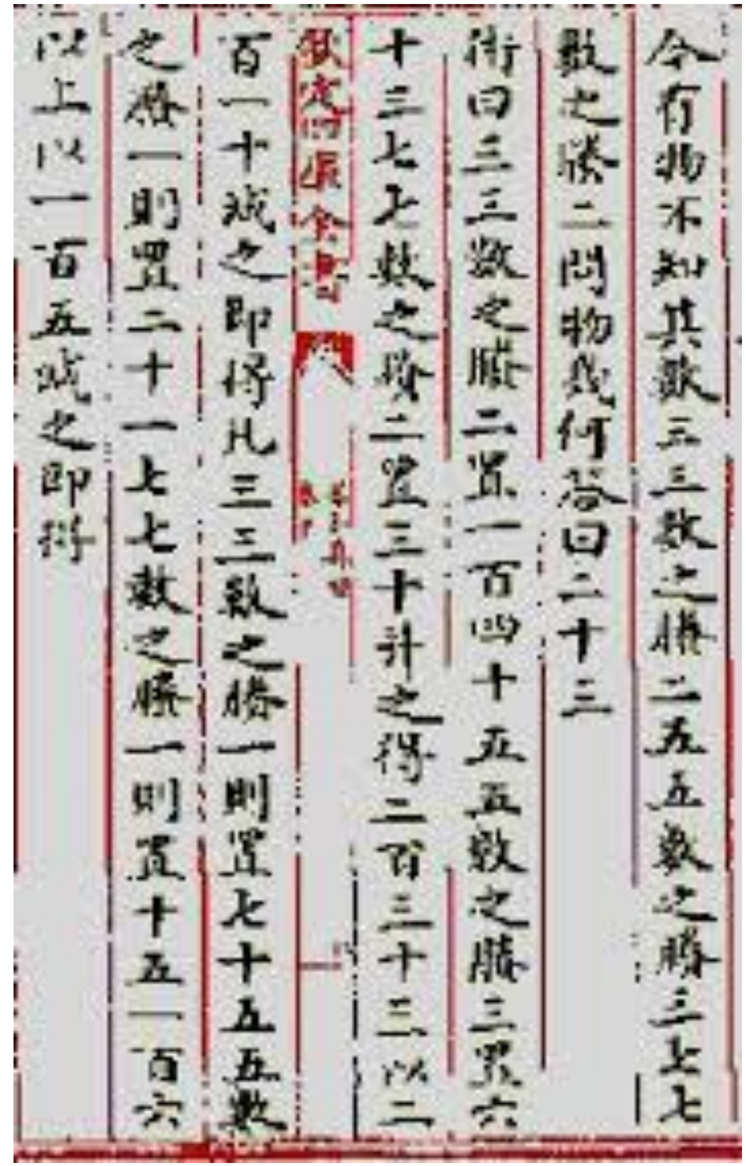
# Sun-Tsu's Problem

There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?

System of linear congruences:

$$x \equiv 2 \pmod{3},$$
$$x \equiv 3 \pmod{5},$$
$$x \equiv 2 \pmod{7}.$$


今有物不知其數三三數之賸二五五數之賸三七七
數之賸二問物幾何答曰二十三
術曰三三數之賸二置一百四十五五數之賸三置六
十三七七數之賸二置三十并之得二百三十三以二
百一十減之即得凡三三數之賸一則置七十五五數
之賸一則置二十一七七數之賸一則置十五一百六
以上以一百五減之即得

# The Chinese Remainder Theorem

- **Theorem**
  Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than one and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$
\begin{gathered}
x \equiv a_1 \pmod{m_1} \\
x \equiv a_2 \pmod{m_2} \\
\ldots \\
x \equiv a_n \pmod{m_n}
\end{gathered}
$$

  has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

- **Proof**
  We'll show that a solution exists by describing a way to construct the solution. (Uniqueness proof is left as exercise.)

# The Chinese Remainder Theorem

- **Proof**

Let $M_k = \frac{m}{m_k}, k = 1,2, \dots, n$

Since $\gcd(m_k, M_k) = 1, M_k$ has an inverse $y_k$ modulo $m_k$:

$$M_k y_k \equiv 1 \pmod{m_k}$$

We claim that this is a solution:

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n$$

Check:

$$x \equiv a_k \pmod{m_k}?$$

$a_j M_j y_j \equiv 0 \pmod{m_k}$ for any $j \neq k$;

$a_k M_k y_k \equiv a_k \pmod{m_k}$

# The Chinese Remainder Theorem

- Consider the 3 congruences from Sun-Tsu's problem:

  $x \equiv 2 \pmod{3}, \; x \equiv 3 \pmod{5}, \quad x \equiv 2 \pmod{7}.$
- Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.
- We see that
  - 2 is an inverse of $M_1 \pmod{3}$
  - 1 is an inverse of $M_2 \pmod{5}$
  - 1 is an inverse of $M_3 \pmod{7}$
- Hence,

$$\begin{aligned} x &= a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3 \\ &= 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 = 233 \equiv 23 \pmod{105} \end{aligned}$$