THE UNIVERSITY OF HONG KONG

FACULTY OF ENGINEERING

DEPARTMENT OF COMPUTER SCIENCE

COMP2121 Discrete Mathematics: Quiz 3

Date: 27 Nov, 2018 Time: 9:30am - 10:20am

Write your **university number here** and also on **every page**: ____________________

Please read the following before you begin.

1. Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is your responsibility to ensure that your calculator operates satisfactorily, and you must record the **name and type of the calculator** here:

   ____

2. This is a **closed book** quiz.
3. You are advised to spend around 5 minutes to go through all the questions before you begin writing. Allocate the time for each part of the question carefully.
4. Please write your answer in the designated space.
5. Attempt ALL questions. The full mark for the exam is 100 points.

For marking use:

Question 1

Question 2

Question 3

Question 4

Question 5

Question 6

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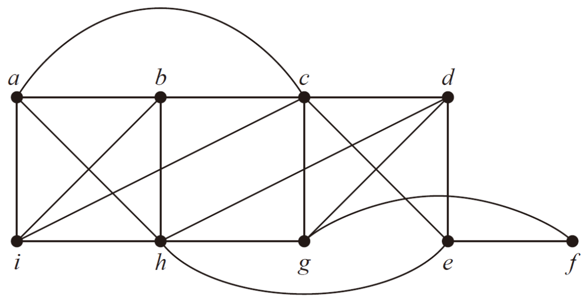
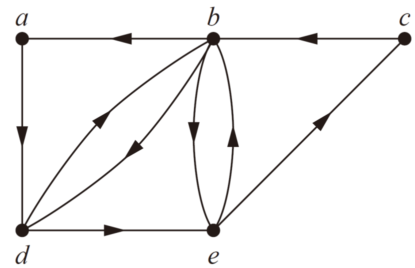
Question 1 (20 points)

1. Does each of the following graphs have an Euler circuit? If so, give one. (5 points each)

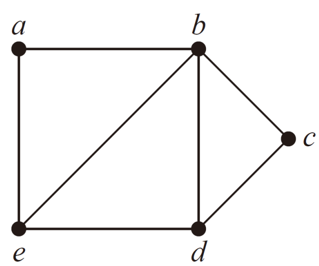


(1)

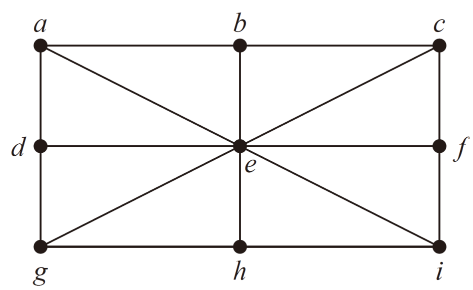


(2)

2. Does each of the following graphs have a Hamiltonian cycle? If so, give one. (5 points each)



(1)



(2)

Solution:

1. (1) Yes. One Euler circuit is $a-i-h-g-d-e-f-g-c-e-h-d-c-a-b-i-c-b-h-a$

   (2) Yes. One Euler circuit is $a-d-b-d-e-b-e-c-b-a$.

2. (1) Yes. One Hamiltonian circuit is $a-b-c-d-e-a$.

   (2) Yes. One Hamiltonian circuit is $a-b-c-f-i-h-g-d-e-a$.

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Question 2 (15 points)

Prove that a simple graph and its complement cannot both be disconnected.

(Hint: You may start with a disconnected graph G, and consider the cases in which two arbitrary vertices v and w are (i) in the same connected components of G; and (ii) not in the same connected components of G. Deduce in both cases that $\bar{G}$, the complement of G, is connected.)

Solution: If G is disconnected, and let v and w be vertices of G. If v and w lie in different components of G, then they are adjacent in $\bar{G}$. If v and w lie in the same components of G, then let z lies in another component and we have $v \to z \to w$ being a path in $\bar{G}$. In any case, any two vertices can be connected by a path in $\bar{G}$ and hence $\bar{G}$ is connected.

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Question 3 (15 points)

Suppose that a king has gathered his $2n$ knights of the Round Table for an important council. Every two knights are either friends or enemies, and each knight has no more than $n-1$ enemies among the other $2n-1$ knights. This question asks whether the king can seat all his knights (but not himself) around the Round Table so that each knight has two friends for his neighbors.

1. Show that this question can be modeled as a graph problem determining whether there is a Hamiltonian circuit. (7 points)
2. Show that there is a way for the king to seat his knights as required. (8 points)

Solution:

1. We construct a graph in which each vertex represents a knight and two knights are connected in the graph if they are friends. Then a Hamiltonian circuit in the graph exactly corresponds to a seating of the knights at the Round Table such that adjacent knights are friends.
2. It suffices to show that there exists a Hamiltonian circuit in the graph of part (a). The degree of each vertex in the graph is at least $2n-1-(n-1) = n \geq (2n/2)$. Hence, by Dirac's Theorem, the graph has a Hamiltonian circuit.

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Question 4 (15 points)

Suppose that a connected planar simple graph with e edges and v vertices only contains cycles of length 5 or more. Show that $3e \leq 5v - 10$ if $v \geq 5$.

Solution: Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph. Because each region has degree greater than or equal to 5, it follows that

$$2e = \sum_{\text{all regions } R} \deg(R) \geq 5r.$$

By Euler's Formula, we have $r = e - v + 2$ Thus, $2e \geq 5(e - v + 2)$, which implies $3e \leq 5v - 10$.

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Question 5 (15 points)

Suppose G_1, G_2 and G_3 are connected simple graphs on the same vertex set V (where $|V| = 17$) such that the union of G_1, G_2 and G_3 is the complete graph on V. Prove that at least one of G_1, G_2 and G_3 is non-planar.

Solution: Suppose G_1, G_2 and G_3 are all planar. Then since $e \leq 3v - 6$, G_1 has at most 45 edges. Similarly, G_2 and G_3 also have at most 45 edges. But the union of G_1, G_2 and G_3 is a complete graph K_{17} having exactly 136 edges $>$ (45+45+45) edges, which is a contradiction.

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Question 6 (20 points)

Consider setting up a timetable for seven lectures, with the constraints that those pairs of lectures with asterists in the following table should not coincide in the timetable, to avoid student time clashes for the lectures. What is the smallest number of lecture time slots that are needed to schedule all seven lectures?

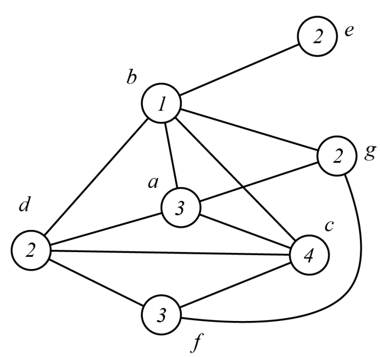
Solve this problem by reducing it into a graph problem. State clearly your reduction and how you arrive at the answer.

	a	b	c	d	e	f	g
a	-	*	*	*	-	-	*
b	*	-	*	*	*	-	*
c	*	*	-	*	-	*	-
d	*	*	*	-	-	*	-
e	-	*	-	-	-	-	-
f	-	-	*	*	-	-	*
g	*	*	-	-	-	*	-

Solution:

Problem reduction: We reduce the problem to a graph coloring problem. The vertices of the graph are the lectures, and an edge corresponds to an asterisk in the table. The problem then reduces to finding a proper graph coloring, where the different colors represent different lecture time slots, so that no adjacent vertices (lectures) are colored the same (being assigned to same time slots). The chromatic number of the graph will give us the answer.

Construct graph G corresponding to the above table. Note that the vertices $\{a, b, c, d\}$ form a complete subgraph in G, and hence at least 4 colors are need for a proper coloring of G. A 4-coloring of G is given by:



Hence, at least 4 different time slots are needed for the timetable.

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