COMP2121: Discrete Mathematics
Homework 3 **Due Date:** 7pm, 30 Nov, 2018

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]**
- **[O2. Proof Techniques]**
- **[O3. Basic Analysis Techniques]**

1. (6 points) **[O1]** How many vertices and how many edges does each of these graphs have?

   (a) Complete graph K_n.

   (b) Cycle graph C_n.

   (c) Complete bipartite graph $K_{n,m}$.

   Solution: (2 points each)

   (a) n vertices, $\frac{n(n-1)}{2}$ edges.

   (b) n vertices, n edges.

   (c) $n+m$ vertices, nm edges.

2. (9 points) **[O1]** For which values of $n \geq 3$ do these graphs have an Euler circuit?

   (a) Complete graph K_n.

   (b) Cycle graph C_n.

   (c) Complete bipartite graph $K_{n,n}$.

   Solution: (3 points each)

   As we know, a connected graph has an Euler circuit iff all vertices have even degrees.

   (a) Whenever n is odd, every vertex in K_n has even degree $n-1$ and hence K_n has an Euler circuit.

   (b) For any $n \geq 3$, every vertex in C_n has even degree 2 and hence C_n has an Euler circuit.

   (c) Whenever n is even, every vertex in $K_{n,n}$ has even degree n and hence $K_{n,n}$ has an Euler circuit.

3. (10 points) **[O2]** Show that a simple graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

   Solution:

It is trivial when $n \le 2$. Then we consider the cases when $n \ge 3$. For $n \ge 3$ and any positive integer $k \le n-1$, we have $k(n-k) = (n-1) + (k-1)(n-(k+1)) \ge n-1$.

For any graph with k vertices, we know that it has at most $\frac{k(k-1)}{2}$ edges. Suppose $G(V,E)$ is disconnected, then we can find some $S \subsetneq V$ such that subgraphs $G_s(S, E_s)$ and $G_{v \setminus s}(V \setminus S, E \setminus E_s)$ are disconnected. Suppose $|S| = k \le n-1$, then $|V \setminus S| = n-k$ and $|E_s| \le \frac{k(k-1)}{2}$, $|E \setminus E_s| \le \frac{(n-k)(n-k-1)}{2}$. Hence for $1 \le k \le n-1$

$$\begin{aligned}
|E| &\le \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\
&= \frac{n^2-n}{2} - k(n-k) \\
&\le \frac{n^2-n}{2} - (n-1) \\
&= \frac{(n-1)(n-2)}{2}
\end{aligned}$$

Thus, for a simple graph with more than $\frac{(n-1)(n-2)}{2}$ edges, it must be connected.

4. (14 points) **[O2]** Prove the following statements:

   (a) Complete graph K_n is not a planar graph when $n \ge 5$.

   (b) Complete bipartite graph $K_{n,n}$ is not a planar graph when $n \ge 3$.

   Solution: (7 points each)

   (a) When $n \ge 5$, we have $\frac{n(n-1)}{2} = \frac{(n-3)(n-4)}{2} + 3n - 6 > 3n - 6$. Since any graph G with $e > 3v - 6$ is not a planar graph and K_n has $v = n$ and $e = \frac{n(n-1)}{2}$, we conclude that K_n is not a planar graph.

   (b) When $n \ge 3$, we have $n^2 = (n-2)^2 + 4n - 4 > 4n - 4$. Since any simple triangle-free graph G with $e > 2v - 4$ is not a planar graph and $K_{n,n}$ is a simple triangle-free graph with $v = 2n$ and $e = n^2$, we conclude that $K_{n,n}$ is not a planar graph.

5. (36 points) **[O3]** Suppose K_n is a complete graph whose vertices are indexed by $[n] = \{1, 2, 3, ..., n\}$, where $n \ge 4$. In this question, a cycle is identified solely by the collection of edges it contains; there is no particular orientation or starting point associated with a cycle. (Give your answers in terms of n for the following questions.)

   (a) How many Hamiltonian cycles are there in K_n?

   (b) How many Hamiltonian cycles in K_n contain the edge $\{1, 2\}$?

   (c) How many Hamiltonian cycles in K_n contain both the edges $\{1, 2\}$ and $\{2, 3\}$?

   (d) How many Hamiltonian cycles in K_n contain both the edges $\{1, 2\}$ and $\{3, 4\}$?

(e) Suppose that M is a set of $k \leq \frac{n}{2}$ edges in K_n with the property that no two edges in M share a vertex. How many Hamiltonian cycles in K_n contain all the edges in M? Give your answer in terms of n and k.

(f) How many Hamiltonian cycles in K_n **do not** contain any edge from $\{1,2\}$, $\{2,3\}$ and $\{3,4\}$?

Solution: (6 points each)

(a) First, we count the permutation of n vertices, which is $n!$. Since there is no particular orientation (thus divided by 2) or starting point (thus divided by n) associated with a cycle, we have $\frac{n!}{2n} = \frac{(n-1)!}{2}$.

(b) We consider the edge $\{1,2\}$ as a special vertex. Then we count the permutation of the remaining $n-1$ vertices (including the special vertex) and get $\frac{(n-2)!}{2}$ from the result of part (a). Observe that for the special vertex (the edge $\{1,2\}$) in the permutation, putting 1 in front of 2 and putting 2 in front of 1 are two different cycles. Thus, we have $2 \times \frac{(n-2)!}{2} = (n-2)!$.

(c) Notice that the edges $\{1,2\}$ and $\{2,3\}$ must be connected by vertex 2. Thus, we consider two edge as a special vertex. Then we count the permutation of the remaining $n-2$ vertices (including the special vertex) and get $\frac{(n-3)!}{2}$ from the result of part (a). Observe that for the special vertex (the edges $\{1,2\}$ and $\{2,3\}$) in the permutation, putting 1 in front of 3 and putting 3 in front of 1 are two different cycles. Thus, we have $2 \times \frac{(n-3)!}{2} = (n-3)!$.

(d) Similar to part (b), we consider the edges $\{1,2\}$ and $\{3,4\}$ as two special vertices. Then we count the permutation of the remaining $n-2$ vertices (including the special vertices) and get $\frac{(n-3)!}{2}$ from the result of part (5a). Observe that for each special vertex in the permutation, putting one endpoint first and putting another endpoint first are two different cycles. Thus, we have $2^2 \times \frac{(n-3)!}{2} = 2 \times (n-3)!$.

(e) Similar to part (d), we consider each edge in M as a special vertex. Then we count the permutation of the remaining $n-k$ vertices (including the special vertices) and get $\frac{(n-k-1)!}{2}$. Observe that for each special vertex in the permutation, putting one endpoint first and putting another endpoint first are two different cycles. Thus, we have $2^k \times \frac{(n-k-1)!}{2} = 2^{k-1} \times (n-k-1)!$.

(f) Let $e_1 = \{1,2\}$, $e_2 = \{2,3\}$ and $e_3 = \{3,4\}$. Let A_i denote the set of Hamiltonian cycles that contain edge e_i. Then it is equivalent to compute

$$\left| \bigcap_{i=1}^{3} \bar{A}_i \right| = |A_\emptyset| - |A_1| - |A_2| - |A_3| + |A_{1,2}| + |A_{1,3}| + |A_{2,3}| - |A_{1,2,3}|.$$

From part (a), we have $|A_\emptyset| = \frac{(n-1)!}{2}$.
From part (b), we have $|A_1| = |A_2| = |A_3| = (n-2)!$.
From part (c), we have $|A_{1,2}| = |A_{2,3}| = (n-3)!$.
From part (d), we have $|A_{1,3}| = 2 \times (n-3)!$.

Similar to part (c), we can get $|A_{1,2,3}| = (n-4)!$.

Thus, we obtain $\left|\bigcap_{i=1}^{3} \bar{A}_i\right| = \frac{(n-1)!}{2} - 3 \times (n-2)! + 4 \times (n-3)! - (n-4)!$.

6. (10 points) **[O1, O2]** Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuit of length 4 or less. Prove that if $v \geq 4$ then $e \leq \frac{5}{3}v - \frac{10}{3}$.

   Solution: Since every simple circuit has a length of at least 5 and $n \geq 4$, each region R has a degree $deg(R) \geq 5$. Then we have $2e = \sum_R deg(R) \geq 5r$. From Euler's formula $r = e - v + 2$ we get $2e \geq 5(e - v + 2)$, which gives $e \leq \frac{5}{3}v - \frac{10}{3}$.

7. (15 points) **[O1, O3]** Suppose there is a connected planar simple graph G with v vertices such that all its regions are triangles (a cycle consisting of three edges).

   (a) (3 points) Into how many regions does a representation of the planar graph G split the plane?

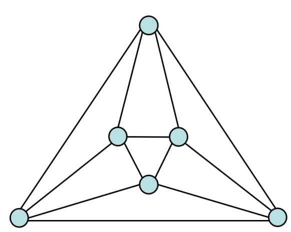
   (b) (12 points) Suppose the vertices of the planar graph G are colored in three colors. A region is called to be tricolored (or bicolored) if its vertices are colored in exactly three (or two) different colors. Similarly, a monocolored region is the one with all its vertices colored in exactly one color. Prove that the number of tricolored regions is always even no matter how the vertices are colored.

   (Hint: If you place a new vertex inside a region (triangle) of G and connect it with all vertices of that region, then all regions are still triangles and the parity of the total number of regions stays the same.)

   Solution:

   (a) Since all the regions are triangles, we have $\frac{3r}{2} = e$. Then, by Euler's Formula, we have $r = \frac{3r}{2} - v + 2$. After rearrangement, we have $r = 2v - 4$.

   (b) (Cannot use mathematical induction on v: Suppose the statement is true for $v = 5$. We still can not prove the statement is true for the following graph with $v = 6$, which cannot be constructed from any graph in question with $v = 5$.)



Possible solution 1: Let the colors be numbered 1, 2 and 3. Each region is characterized by the triple (i, j, k) of colors of its vertices. Let us transform the planar graph G into another planar graph G' as follows. We place a new vertex inside each monocolored region (i.e., a region of type (i, i, i), $i = 1, 2$ or

3), color this new vertex by a color $j \neq i$ and connect it with all vertices of this monocolored region. Notice that this transformation preserves the number of tricolored region. Then, it suffices to prove that the number of tricolored regions of G' is even. Moreover, we know that the total number of regions of G is even from part (a) and the transformation preserves its parity, hence the total number of regions of G' is even. Observe that G' only has bicolored and tricolored regions. Thus, it suffices to prove that the number of bicolored regions of G' is even.

Notice that every bicolored region of G' has a monocolored edge, i.e. its two endpoints share the same color. At the same time, every monocolored edge only belongs to two bicolored regions. This implies that the number of bicolored regions of G' is even, as required.

Possible solution 2: We first construct a *dual graph* G^* of the planar graph G. Notice that each edge e of G has a corresponding *dual edge* e^*, whose endpoints are the *dual vertices* corresponding to the regions on either side of e in G. Next, for each monocolored edge e of G, i.e. its two endpoints share the same color, we delete its corresponding dual edge e^* in G^*.

Let's denote the resulting graph as G^0. Then, observe that for each dual vertex v^* in G^0, the degree $deg(v^*)$ of v^* can only be 0, 2 or 3. If $deg(v^*)$ equals 0, 2 or 3, then v^*'s corresponding region in G is monocolored, bicolored or tricolored respectively. Since the total degree $\sum_{v^* \in G^0} deg(v^*)$ of all dual vertices in G^0 is equal to twice the number of dual edges remaining in G^0, $\sum_{v^* \in G^0} deg(v^*)$ is even. Moreover, we have

$$\sum_{v^* \in G^0} deg(v^*) = 2*\text{number of bicolored regions}+3*\text{number of tricolored regions}.$$

Thus, the number of tricolored regions in G is even.

Possible solution 3: Let m denote the total number of bicolored edges in G, i.e. its two endpoints have different colors. Then for each region in G, we count the total number of bicolored edges and get $2m$ since each bicolored edge belongs to two regions. Observe that each tricolored region has 3 bicolored edges, bicolored region has 2 and monocolored region has 0. Thus, we have

$$2m = 2 * \text{number of bicolored regions} + 3 * \text{number of tricolored regions},$$

and the number of tricolored regions in G is even.