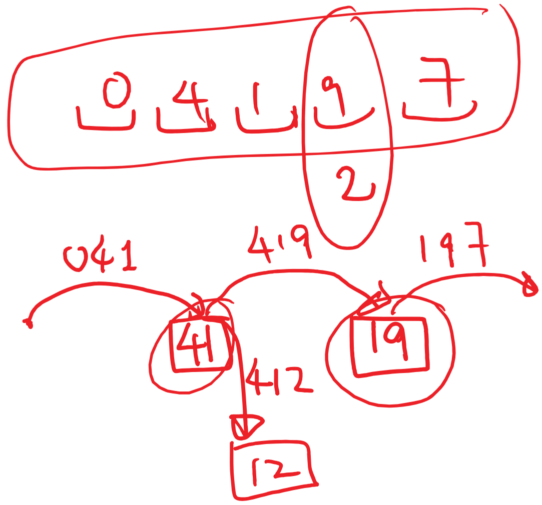
**COMP2121: Discrete Mathematics**
**Problem Solving Session 6** 16 Nov, 2018

1. Assume that a digital lock opens anytime a correct sequence of 3 digits (0, ..., 9) is entered. The brute force approach is to try out all possibilities, i.e., 000, 001, 002, ..., 999 and $10^3 \times 3$ digits have to be entered. However, note that if 135286 is entered, effectively the following 3-digit sequences have been tried: 135, 352, 528 and 286. Does there exist a method to construct a sequence of $10^3 + 4$ digits so that the sequence contains all possible 3-digit sequences as its subsequences. Prove or give arguments to support your claim. (Hint: Transform this problem into a graph problem, and then define the vertices and edges of this graph.)

Euler circuit

444



$V = \{ ij : 0 \leq i,j \leq 9 \}$

directed graph

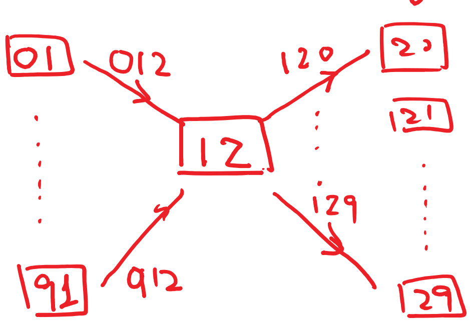
$E = \{ (ij, kl) : j = k \}.$

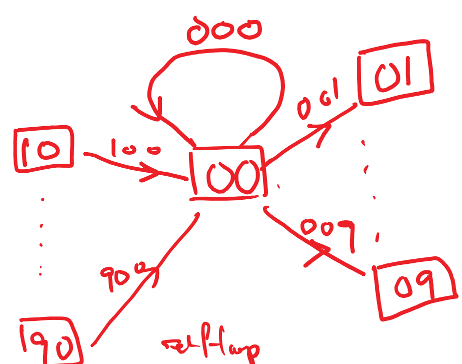
strongly connected

For a vertex in V, is in-degree the same as out-degree?

$i \neq j$.

$i = j$





out-deg = in-deg = 10.

self-loop

out-deg = in-deg = 9.

⇒ (Euler circuit)

out-deg = in-deg = 9.

⇒ $\boxed{\text{Euler circuit}}$

⇒ $\boxed{\text{Sequence of digits}}$ +1

starting node: $\boxed{ij} \to \boxed{\phantom{ij}}$

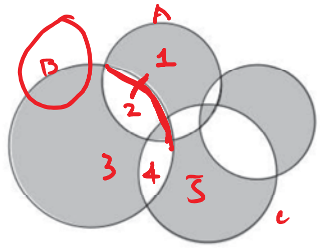
$2 + 10^3$

2. We draw some circles on the plane (say, $n$ in number). These divide the plane into a number of regions. Figure 0.1 shows such a set of circles, and also an "alternating" coloring of the regions with two colors. Now our question is: can we always color these regions this way?

You may follow the following steps to prove that the graph $G$ representing regions is bipartite (hence can be two-colored).

(a) Encode each region by 0, 1 strings such that the code of adjacent regions differ by exactly one bit.

(b) Argue that any cycle on the graph $G$ is of even length.




| | A | B | C | D |
|---|---|---|---|---|
| 1: | 1 | 0 | 0 | 0 |
| → 2: | 1 | 1 | 0 | 0 |
| 3: | 0 | 1 | 0 | 0 |

Figure 0.1: Two-coloring the regions formed by a set of circles.

(a) Suppose regions 1 and 2 share a non-zero length boundary corresponding to some circle B.

⇒ exactly 1 of these regions is inside circle B

Removing circle B, regions 1 and 2 will be merged into a single region ⇒ for all other circles $C \neq B$, either (i) both regions are contained in C or (ii) both regions are outside C.

⇒ They differ in exactly 1 bit.

1: 0 1 0 0

0 1 1 0

Consider a cycle in G.

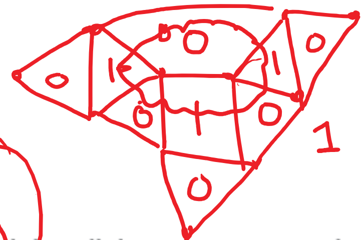
Consider a cycle in G.
$x$ is the starting point.

Observation : for every bit $i$, it is flipped even no. of times.

The total no. of flips is even.

$\Rightarrow$ The length of the cycle is even.

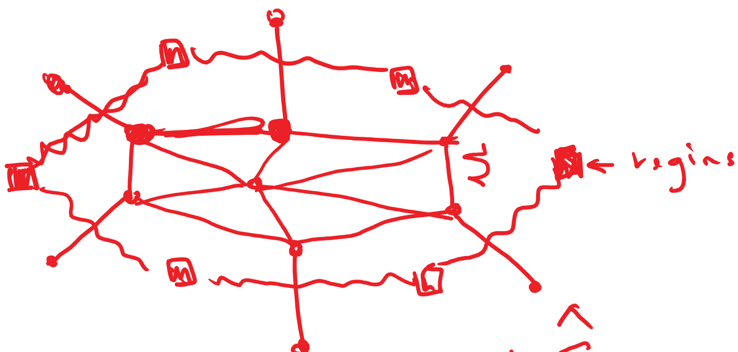
$\Rightarrow$ $G$ is bipartite.




regions

3. If a planar graph G has all degrees even, prove that the faces can be colored with two colors in such a way that any two faces with a common edge on their boundary get different colors.

[Hint: prove that any cycle (in some graph) is even.]




Consider a cycle in the dual graph $\hat{G}$

( • regions become vertices
• adjacent regions have a connecting edge. )

Each edge in cycle represents a crossing edge in G.

$\hat{G}$

Want to show the no. of crossing edges is even

Consider set S of vertices one side of G.

$\sum_{v \in S} \deg(v)$ is even

$= 2W + U$

$\Rightarrow U$ is even

$\Rightarrow$ length of cycle is even.

$U =$ no. of crossing edges

$W =$ no. of edges on the same side as S

edge in G

4