COMP 2711H: Final Exam
2016 Fall Semester – Sunil Arya
9 December, 12:30 pm – 3:30 pm

You have 3 hours to solve 8 problems for a total of 100 marks. Good luck!

Problem 1. (5 points)

(a) Define a free tree.

(b) State 3-4 interesting properties of free trees (obviously these should not be explicitly included as part of your definition).

Problem 2. (15 points)

The *rooted Fibonacci trees* T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

(a) Draw the first six rooted Fibonacci trees.

(b) Write three recurrence relations: (i) for the number of leaves, (ii) for the number of internal nodes, and (iii) for the total number of nodes, in the rooted Fibonacci tree T_n.

(c) Establish a lower and upper bound on the total number of nodes in the rooted Fibonacci tree T_n, that is exponential in n. You may use any method you like to prove this; however, you must present the complete details of your proof (that is, do not use any facts proved in class, without proving them).

Problem 3. (15 points)

Recall that a *derangement* is a permutation of n distinct objects that leaves no object in its original position.

(a) Establish a formula for D_n, the number of derangements of n objects. Justify your answer fully.

(b) Calculate D_4 using your formula. Explicitly show all the derangements of the numbers $1, 2, 3, 4$ and verify that your formula is correct in this special case.

Problem 4. (10 points)

Recall that the sum of the numbers in the i-th row of Pascal's triangle is 2^i. What is the sum of the *squares* of the numbers in the i-th row of Pascal's triangle? Justify your result using a combinatorial argument.

Hint: The answer to this problem can be expressed as a binomial coefficient.

Problem 5. (15 points)

Consider the following recurrence relation for the running time $T(n)$ of an algorithm:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 3\,T(n/2) + n \qquad \text{if } n > 1. \end{aligned}$$

Derive a good asymptotic upper bound on the running time $T(n)$ using three different methods: (a) iteration, (b) recursion trees, and (c) mathematical induction. You may assume that n is a power of 2.

Problem 6. (15 points)

(a) State and prove Fermat's little theorem. Your proof must be from first principles (that is, do not use any facts proved in class, without proving them).

(b) Prove that $n^7 - n$ is divisible by 42.

Hint: One proof of this is based on Fermat's little theorem.

Problem 7. (10 points)

(a) Let X be a random variable with mean μ and standard deviation σ. Prove *Chebyshev's inequality*, that is, show that for any positive real α,

$$p(|X - \mu| \geq \alpha\sigma) \leq 1/\alpha^2.$$

(b) Suppose a fair coin is tossed $10,000$ times. Use Chebyshev's inequality to find an interval around the mean such that the probability that the number of heads lies in this interval is at least 75%.

Problem 8. (15 points)

(a) Prove that n lines divide the plane into $(n^2 + n + 2)/2$ regions if any two of these lines have exactly one point in common and no three pass through a common point.

(b) Prove that n planes divide three-dimensional space into $(n^3 + 5n + 6)/6$ regions if any three of these planes have exactly one point in common and no four contain a common point.