COMP2121: Discrete Mathematics
Homework 1 **Due Date:** Oct 5, 2018

Rules: Discussion of the problems is permitted, but writing the assignment together is not (i.e. you are not allowed to see the actual pages of another student).

Course Outcomes

- **[O1. Abstract Concepts]**
- **[O2. Proof Techniques]**
- **[O3. Basic Analysis Techniques]**

1. (10 pt) [O1] Express the following statements using the propositions p "He has the ID card", q "He has the password", and r "He opens the door" together with logical connectives.

   (a) "He has the ID card and opens the door."

   (b) "He does not have the password and opens the door."

   (c) "If he has the password and does not have the ID card, he does not open the door."

   (d) "He opens the door if and only if he has the ID card and has the password."

   (e) "If he does not open the door, then he either does not have the ID card or does not have the password."

   Solution:

   (a) $p \wedge r$ 2 pt

   (b) $\neg q \wedge r$ 2 pt

   (c) $(q \wedge \neg p) \rightarrow \neg r$ 2 pt

   (d) $r \leftrightarrow (p \wedge q)$ 2 pt

   (e) $\neg r \rightarrow (\neg p \vee \neg q)$ 2 pt

2. (10 pt) [O1, O2] Show that $(p \rightarrow q) \rightarrow r$ and $\neg r \rightarrow (p \wedge \neg q)$ are logically equivalent.

   (a) using truth tables.

   (b) using logical equivalence.

   Solution:

   (a) using truth tables. 5 pt

p	q	r	$p \to q$	$(p \to q) \to r$	$\neg r$	$p \wedge \neg q$	$\neg r \to (p \wedge \neg q)$
T	T	T	T	T	F	F	T
T	T	F	T	F	T	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	F	F	T
F	T	F	T	F	T	F	F
F	F	T	T	T	F	F	T
F	F	F	T	F	T	F	F

(b) using logical equivalence. 5 pt

$$
\begin{aligned}
&(p \to q) \to r \\
\Leftrightarrow\, & r \vee \neg(p \to q) \\
\Leftrightarrow\, & \neg r \to \neg(p \to q) \\
\Leftrightarrow\, & \neg r \to \neg(q \vee \neg p) \\
\Leftrightarrow\, & \neg r \to (p \wedge \neg q)
\end{aligned}
$$

3. (15 pt) **[O1]** Consider the following conjecture in number theory:

Every even number is the difference of two primes.

(a) Express the statement in terms of quantifiers, variable(s), equality and inequality symbols ($<, >, =$), logical operators ($\wedge, \vee, \to$) and predicates $P(n)$: n is a prime number and $E(n)$: n is an even number.

(b) Express the negation of (a) **without** using the logical operator $\neg$.

[Be careful to define the domain(s) of your variable(s)]

Solution:

(a) $\forall n\,[E(n) \to \exists x, y\,(P(x) \wedge P(y) \wedge (x - y = n))]$ 7 pt

(b) $\exists n\,[E(n) \wedge \forall x, y\,(P(x) \wedge P(y) \to (x - y > n) \vee (x - y < n))]$ 8 pt
if $\neq$ is used, -1 pt

4. (10 pt) **[O2]** Assuming the truth of the theorem which states that "$\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square," prove that $\sqrt{3} + \sqrt{5}$ is irrational. (Hint: Suppose a is irrational and b is an integer, you can prove that $a + b$ is irrational by contradiction.)

Solution:

Obviously, 15 is not a perfect square, hence $\sqrt{15}$ is irrational. 2pt for proving $\sqrt{15}$ is irrational

We will prove $8+2\sqrt{15}$ is irrational by contradiction. Assume that $8+2\sqrt{15}$ is rational and $8 + 2\sqrt{15} = a/b$, then we have $\sqrt{15} = (a/b - 8)/2 = (a - 8b)/2b$ is rational. But we show that $\sqrt{15}$ is irrational at the beginning, which is a contradiction. Thus, the assumption is incorrect and $8 + 2\sqrt{15}$ must be irrational. 4pt for proving $8+2\sqrt{15}$ is irrational

Since $8+2\sqrt{15}$ is irrational, it is not a perfect square. So, $\sqrt{8+2\sqrt{15}} = \sqrt{3}+\sqrt{5}$ is irrational, which completes the proof. 4pt for proving $\sqrt{8+2\sqrt{15}}$ is irrational

5. (18 pt) **[O2, O3]** Use mathematical induction to prove the following.

   (a) Prove that $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$ where $H_n = 1 + 1/2 + \cdots + 1/n$ denotes the n-th harmonic number.

   (b) Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

   Solution:

   (a) Let $P(n)$ denote that $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$ holds for n.
   Base Case: When $n = 1$, $(n+1)H_n - n = 2 \times 1 - 1 = 1 = H_1$. So we know that $P(1)$ is true. 2pt for base case
   Inductive Step: We have to show that $P(n+1)$ is true given the assumption that $P(n)$ is true. By induction hypothesis, we have 1 pt for induction hypothesis 6 pt for inductive step

$$\begin{aligned} H_1 + H_2 + \cdots + H_n + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\ &= (n+1)(H_{n+1} - 1/(n+1)) - n + H_{n+1} \\ &= (n+1)H_{n+1} - 1 - n + H_{n+1} \\ &= (n+2)H_{n+1} - (n+1) \end{aligned}$$

   So we conclude that $P(n+1)$ is true.
   Now by mathematical induction, we know that $P(n)$ is true for all $n \geq 1$.

   (b) Let $P(n)$ denote that $21 \mid (4^{n+1} + 5^{2n-1})$.
   Base Case: When $n = 1$, $21 \mid (4^{n+1} + 5^{2n-1} = 4^2 + 5 = 21)$. So we know that $P(1)$ is true. 2pt for base case
   Inductive Step: We have to show that $P(n+1)$ is true given the assumption that $P(n)$ is true. By induction hypothesis, we have $21 \mid (4^{n+1} + 5^{2n-1})$ which means $4^{n+1} + 5^{2n-1} = 21k$, where $k \in \mathbb{Z}$. Then 1 pt for induction hypothesis 6 pt for inductive step

$$\begin{aligned} 4^{(n+1)+1} + 5^{2(n+1)-1} &= 4 \times 4^{n+1} + 25 \times 5^{2n-1} \\ &= 4 \times (4^{n+1} + 5^{2n-1}) + 21 \times 5^{2n-1} \\ &= 4 \times 21k + 21 \times 5^{2n-1} \\ &= 21 \times (4k + 5^{2n-1}) \end{aligned}$$

   Since $4k + 5^{2n-1} \in \mathbb{Z}$, so we have $21 \mid (4^{(n+1)+1} + 5^{2(n+1)-1})$, which makes us conclude that $P(n+1)$ is true.
   Now by mathematical induction, we know that $P(n)$ is true for all $n \geq 1$.

6. (10 pt) **[O1, O2]** Decide whether the following statements about big-O notation are true or not.

   (a) Let $f(n) = \sqrt{n} + 5$, then $f(n) = \Omega(\log n)$.

(b) Let $f(n) = n \log n - 4$, then $f(n) = O(n^2)$.

(c) Let $f(n) = n + \log n$, then $f(n) = O(\log^2 n)$.

(d) Let $f(n) = 2n^3 + 5n^2 \log n + 4$, then $f(n) = O(n^3)$.

(e) Let $f(n) = 5 \log n + \sqrt{n} + 2$, then $f(n) = \Theta(\sqrt{n})$.

Solution:

(a) True 2 pt

(b) True 2 pt

(c) False 2 pt

(d) True 2 pt

(e) True 2 pt

7. (15 pt) **[O2, O3]** A relation R is defined on the set Z of integers by xRy if $3x - 7y$ is even. Prove that R is an equivalence relation.

Solution: We will show that R is reflexive, symmetric and transitive.

(a) To show that R is reflexive, i.e. $\forall x \in Z, xRx$. Let $x \in Z$. Then $3x - 7x = -4x = 2(-2x)$, which is even. Thus, xRx for each $x \in Z$. 5 pt

(b) To show that R is symmetric, i.e. $\forall x, y \in Z$, if xRy then yRx. Assume xRy that is $3x - 7y$ is even, and we have $3x - 7y = 2k$ for some $k \in Z$. Now $3y - 7x = 3x - 7y - 10x + 10y = 2k - 10x + 10y$ by assumption. Then $3y - 7x = 2(k - 5x + 5y)$, where $k - 5x + 5y \in Z$. Thus $3y - 7x$ is even and we have yRx. 5 pt

(c) To show that R is transitive, i.e. $\forall x, y, z \in Z$, if xRy and yRz, then xRz. Let $x, y, z \in Z$. 5 pt

$$
\begin{aligned}
xRy \text{ and } yRz \Rightarrow & 3x - 7y \text{ is even and } 3y - 7z \text{ is even} \\
\Rightarrow & 3x - 7y = 2k \text{ for some } k \in Z \text{ and } 3y - 7z = 2t \text{ for some } t \in Z \\
\Rightarrow & 3x - 7y + 3y - 7z = 2k + 2t \text{ for some } k, t \in Z \\
\Rightarrow & 3x - 7z = 2k + 2t + 4y \text{ for some } k, t \in Z \\
\Rightarrow & 3x - 7z = 2(k + t + 2y), \text{ where } k + t + 2y \in Z \\
\Rightarrow & 3x - 7z \text{ is even } \Rightarrow xRz.
\end{aligned}
$$

8. (12 pt) **[O2, O3]** Let $f : X \to Y$ be a function. Show that the following statements are equivalent.

(a) There exists a function $g : Y \to X$ such that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.

(b) f is a bijection.

(Hint: Prove (a)$\Rightarrow$(b) and (a)$\Leftarrow$(b). When proving (a)$\Leftarrow$(b), if you use the inverse of function f, you should prove that it exists first.)

Solution:

(a)$\Rightarrow$(b):

(1) To show that f is injective by contradiction. Suppose there exist two different $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, which is a contradiction. Thus, f is injective. 4 pt

(2) To show that f is surjective. For any $y \in Y$, we have $f(g(y)) = y$. So there exists an $x = g(y) \in X$ such that $f(x) = y$. Thus, f is surjective. 4 pt

Since f is both injective and surjective, it is bijective.

(a)$\Leftarrow$(b): Since f is a bijection, for each $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. Thus, the inverse of function f, denoted by f^{-1}, exists. Let $g(y) = f^{-1}(y)$ for each $y \in Y$. Then $g(f(x)) = x$ and $f(g(y)) = y$ follows for all $x \in X$ and $y \in Y$. 4 pt