Tuesday, September 18, 2018 9:03 AM

Problem Solving Session 2

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COMP2121A: Discrete Mathematics
Problem Solving Session 2 Sep 18, 2017

Review. Proof methods, $p \to q \Leftrightarrow \neg p \vee q$

1. direct proof
2. proof by contradiction, assume $p \wedge \neg q$, try to find a contradiction
3. proof by contrapositive, prove $\neg q \to \neg p$
4. mathematical induction, show that $P(x)$ is true for all $x \in \bigcup_{i \geq 0} S_i$
   (a) base case, prove $P(x)$ is true for all $x \in S_0$
   (b) inductive step, given $P(x)$ is true for all $x \in \bigcup_{0 \leq i \leq k-1} S_i$, prove $P(x)$ is true for all $x \in S_k$

Questions.

1. Given a real number x and an positive integer n, show an efficient method to evaluate x^n with only multiplications and additions. Multiplication takes same time, indep of size of arguments.

$n=4$. $((x \times x) \times x) \times x$ 3 mult.
$n=8$ $(((x \times x) \times x) \times x) \times$ ~~[illegible]~~ 7 mult.
Straightforward method: $n-1$ multiplications.

$x \cdot x = x^2$
$x^2 \cdot x^2 = x^4$

$n \geq 2$. case $n = 2r$. First compute $y = x^r$, compute $= y \times y$ $(= x^{2r})$
case $n = 2r+1$ " " $y = x^r$ compute $= y \times y \times x$ $(= x^{2r+1})$

Base case $n=1$. return x
Correctness: by mathematical induction
$T(n) = $ #. of multiplications
$T(1) = 0$
$T(2r) = T(r) + 1$
$T(2r+1) = T(r) + 2$
$\lfloor 1.5 \rfloor = 1$

$T(n) \leq T(\lfloor \frac{n}{2} \rfloor) + 2$.
$\leq 2 \log_2 n$

$n=7$, 111
$n=2r+1$
$n=6$ 110
$n=2r$

$n=11$
1011
1+2+2

Given n, # of multiplications $= \lfloor \log_2 n \rfloor + $ (#. of 1's in binary -1)

Extra $\begin{cases} F(0)=F(1)=1 \\ F(n)=F(n-1)+F(n-2). \end{cases}$

2. Define $f(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3$. Use mathematical induction to prove that $f(n) = \left[\frac{n\cdot(n+1)}{2}\right]^2$ for all positive integers.

3. Prove the following statement. There exist irrational numbers x and y such that x^y is rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$. Is it rational or not?)

In class, we showed $\sqrt{2}$ is irrational.

Case Analysis.

case 1. $\sqrt{2}^{\sqrt{2}}$ is rational. $x = y = \sqrt{2}$ and x^y rational

easy case.

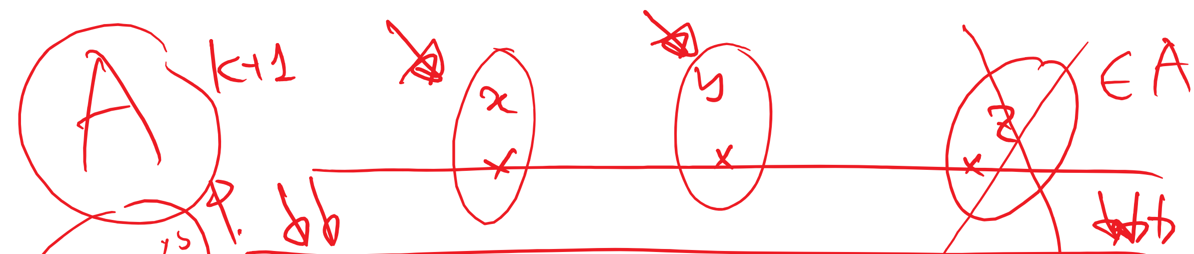
case 2 $x = \sqrt{2}^{\sqrt{2}}$ irrational.

can we find irrational y such that x^y is rational?

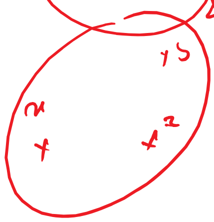
$y = \sqrt{2}$ irrational

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = 2 \text{ rational}$$

3



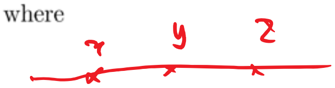
$p \Rightarrow q$



4. Given a finite set A of n points on the plane (2-dimensional space) such that for any two points x, y in A, the line containing x and y must contain another point z in A. Prove that all points in A are on the same line.[1]

$q \to p$ Trivial

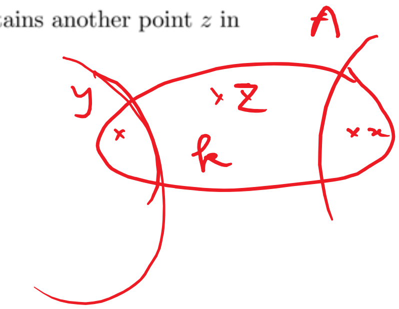
   (a) Is the following proof (induction on the number of points) correct? If not, where is the bug?



   - Base case: for point set of size 3 the statement is true. ✓
   - Inductive step: assume this statement is true for point set of size $k \geq 3$. Consider the case when we have a point set A of size $k + 1$. We argue as follows.

$k+1 \geq 4$

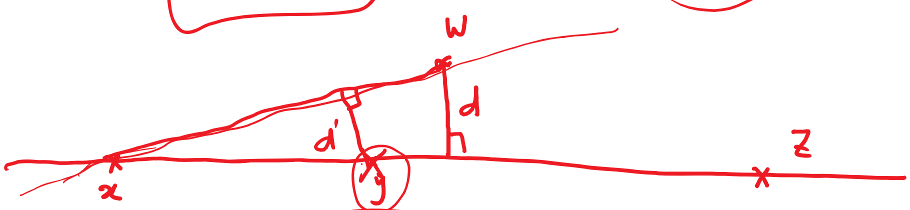
     i. Pick A' of k points from the given point set A. Let x be the other point in A but not in A'.
     ii. By *induction hypothesis*, points in A' are on the same line.
     iii. Pick any y in A', the line going through x, y contains another point z in A.
     iv. Thus, x, y and z are on the same line.
     v. So x and all points in A' are on the same line.



   (b) Can you give a proof by contradiction?

a

$d' < d$



Pick the "closest" witness

[1] This is equivalent to Sylvester-Gallai theorem, which is named after James Joseph Sylvester, who posed it as a problem in 1893, and Tibor Gallai, who published one of the first proofs of this theorem in 1944.