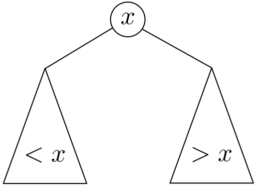
AVL Trees

Version of April 4, 2019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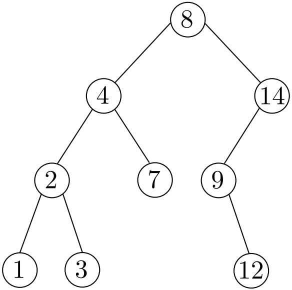
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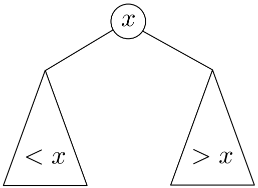

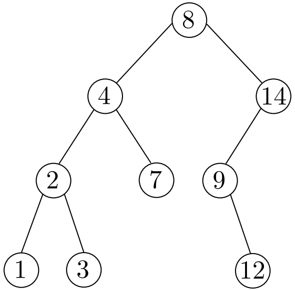

Binary-search-tree property

For every node x

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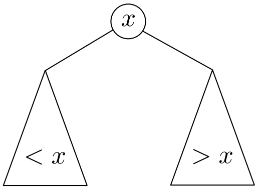
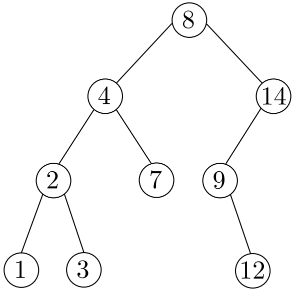
Binary Search Trees





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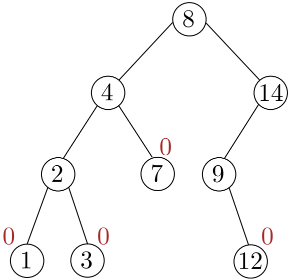

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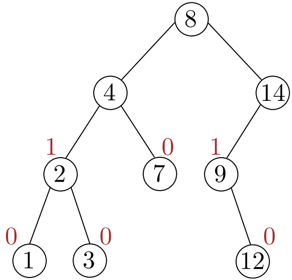

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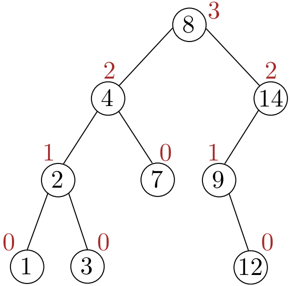

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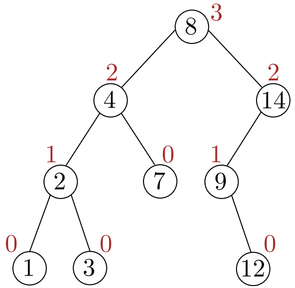

Question

Let n be the size of a binary search tree. How can we keep its height $O(\log n)$ under insertion and deletion?

Balanced Binary Search Tree: AVL Tree

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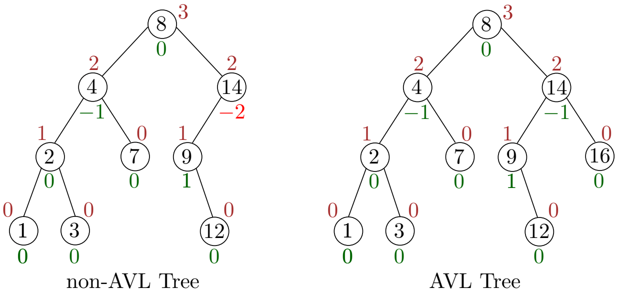

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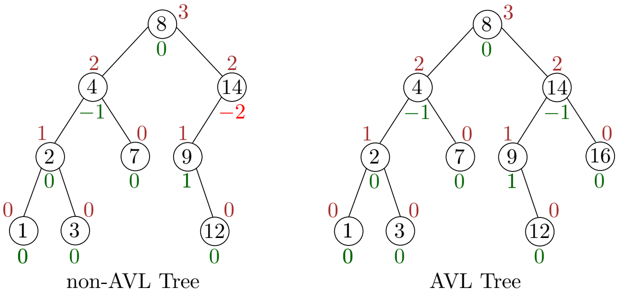

- The *balance factor* of a node is the height of its right subtree minus the height of its left subtree.
- A node with balance factor 1, 0 or -1 is considered *balanced*.

AVL Trees with Minimum Number of Nodes

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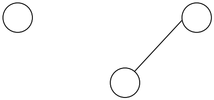

$n_0 = 1$ $n_1 = 2$

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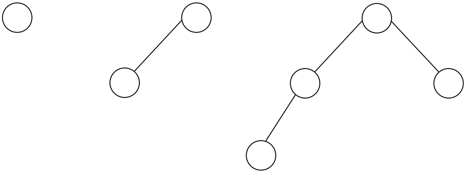

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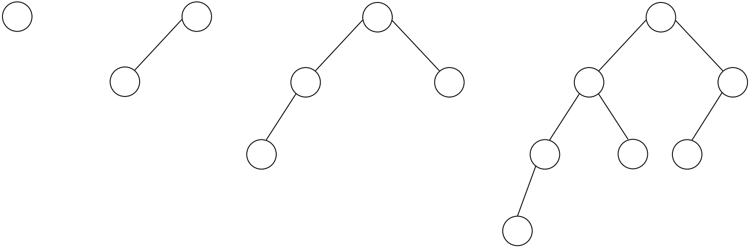

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Height of AVL Tree

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- If h is odd, let $i = (h-1)/2$. $\Rightarrow n > 2^{(h-1)/2} n_1 = 2^{(h-1)/2} \cdot 2 \Rightarrow h = O(\log n)$
- Thus, many operations (e.g., insertion, deletion, and search) on an AVL tree will take $O(logn)$ time

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Lemma:

$n_h = f_{h+2} - 1$

Proof: by induction

$$n_{h+1} \quad = \quad 1+n_h+n_{h-1} \quad = \quad 1+f_{h+2}-1+f_{h+1}-1 \quad = \quad f_{h+3}-1$$

Since $f_h \sim c\phi^h$ for golden ratio $\phi = \frac{1+\sqrt{5}}{2}$, this also immediately provides alternative derivation that $h = O(\log n)$.

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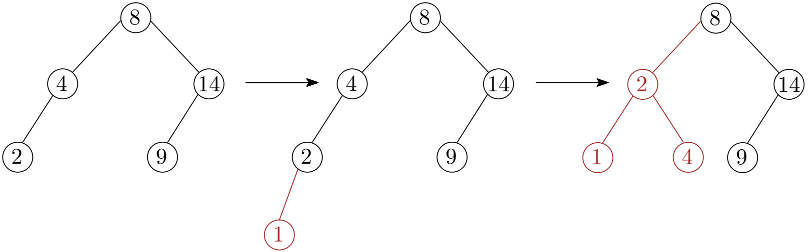

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Restore height balance

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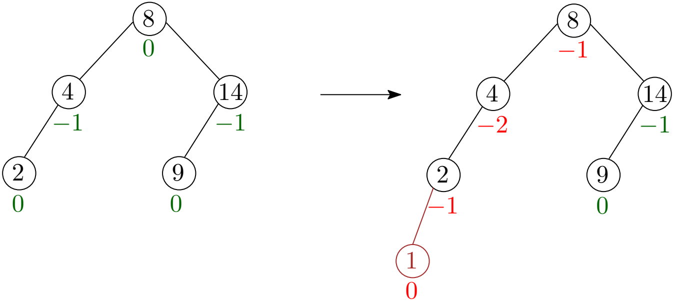

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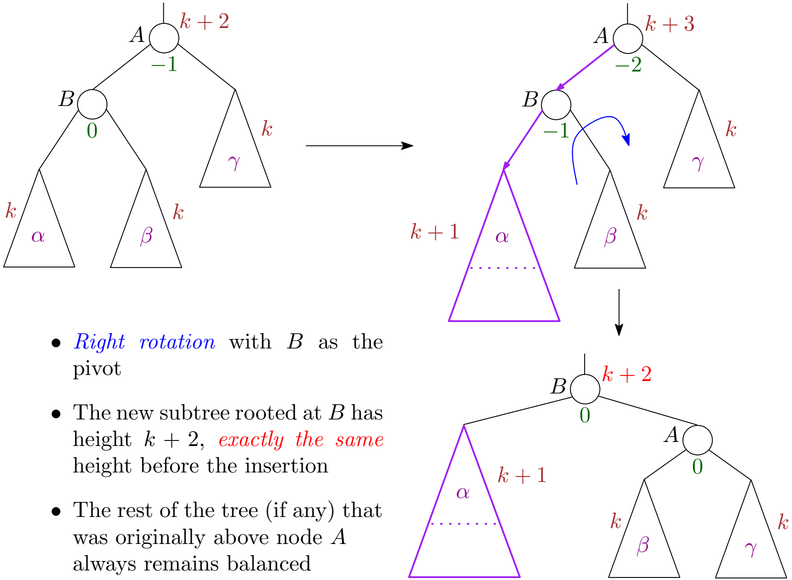
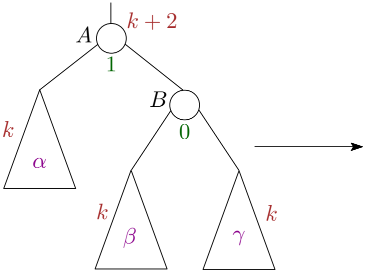
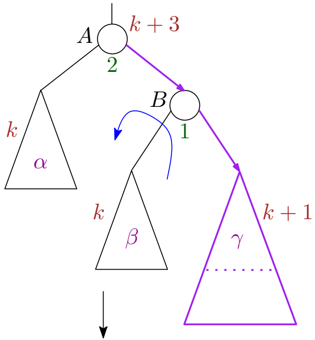
- Cases 1 and 4 are mirror image symmetries with respect to A, as are cases 2 and 3



- *Right rotation* with B as the pivot
- The new subtree rooted at B has height $k+2$, *exactly the same* height before the insertion
- The rest of the tree (if any) that was originally above node A always remains balanced

Insertion: Right-Right Case





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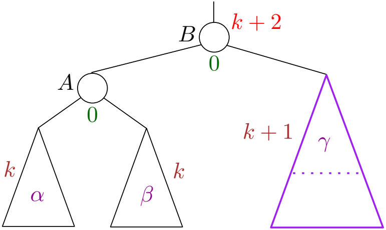

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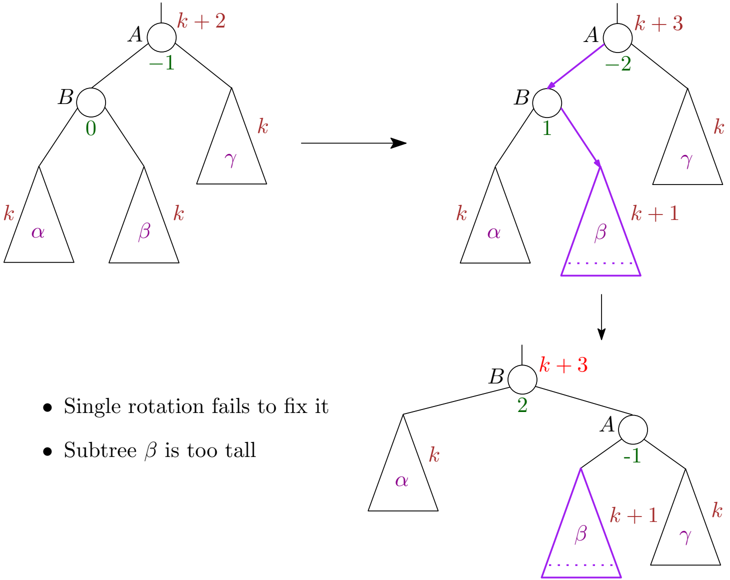
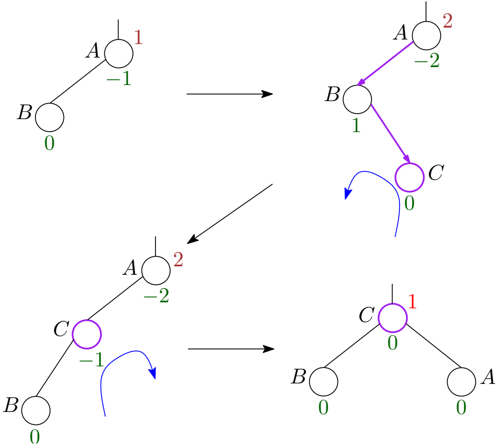


- Single rotation fails to fix it
- Subtree β is too tall

When subtree α, β and γ are empty, $k = -1$. Insert C:



Left-Right Case: Special Case

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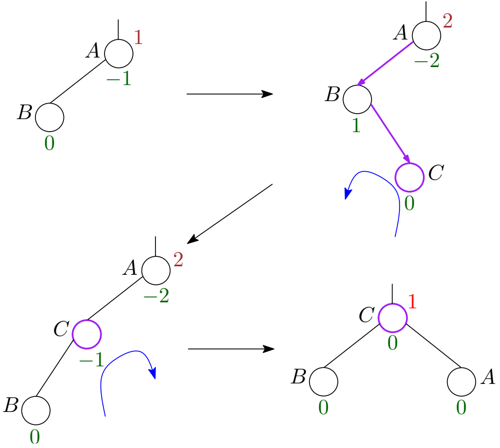
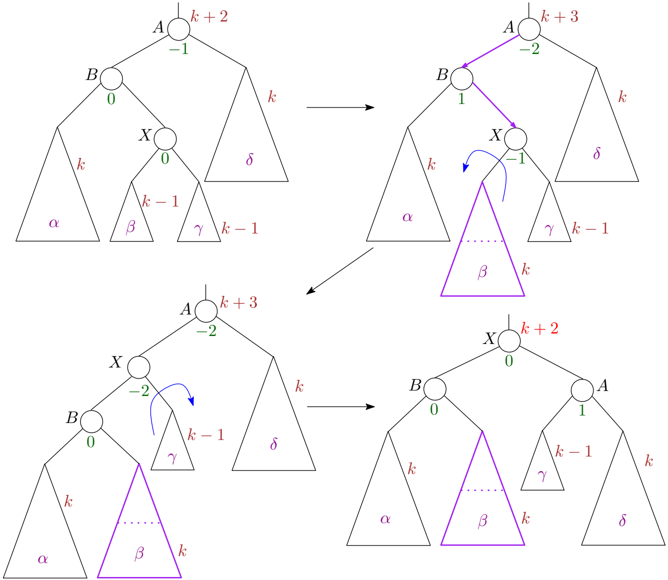
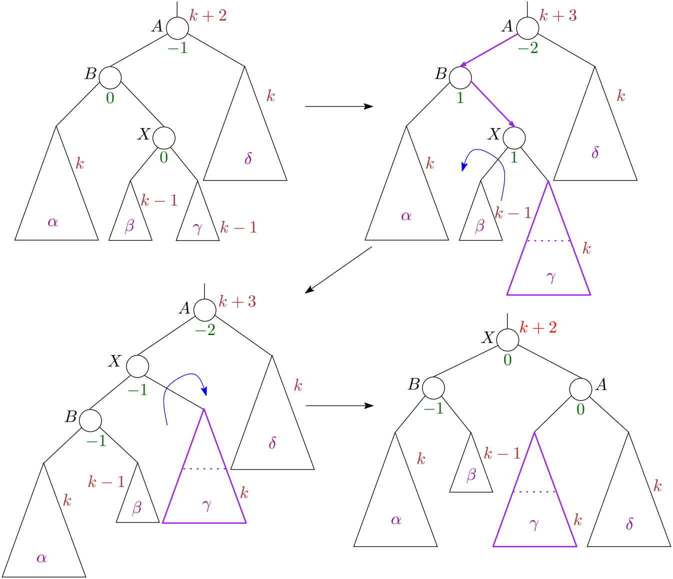


- Left rotation and then right rotation with C as the pivot. Done!

Left-Right Case: General Case 1



Left-Right Case: General Case 2



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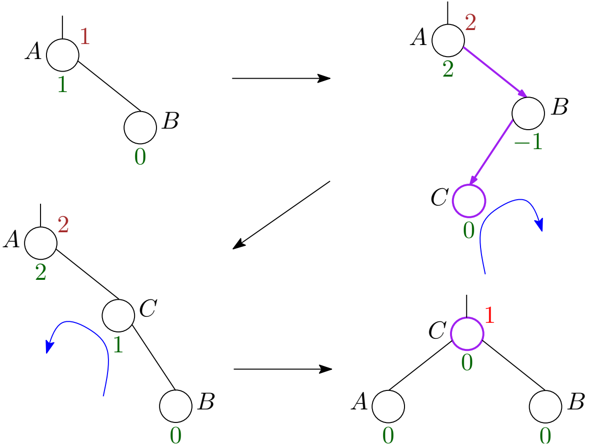

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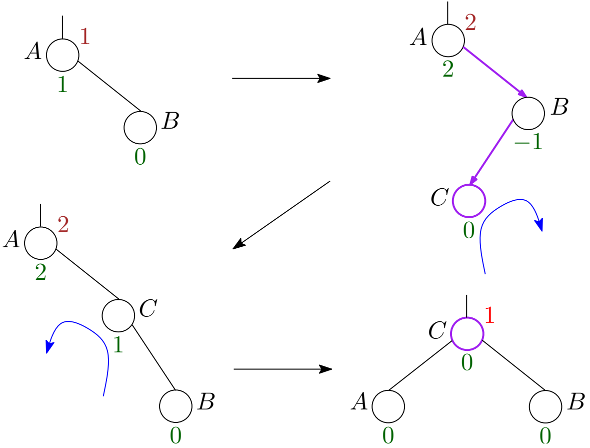

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Note that in all cases, height of rebalanced subtree is unchanged! This means no further tree modifications are needed.

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    $\Rightarrow$ Only *one* rotation needed.
  - In deletion, rotation restores *balance* but final *height* of rotated subtree might decrease by one. If this occurs, need to continue walking up path towards root, continuing to restoring balance by rotations when necessary.

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  - In insertion a (single/double) rotation restored balance *and* kept height of rebalanced subtree unchanged.
    $\Rightarrow$ Only *one* rotation needed.
  - In deletion, rotation restores *balance* but final *height* of rotated subtree might decrease by one. If this occurs, need to continue walking up path towards root, continuing to restoring balance by rotations when necessary.
  - Since path has length $O(h)$ this might require $O(h) = O(\log n)$ rotations.

Deletion (continued)

For each node along the path, restore the height balance if needed by doing single or double rotations

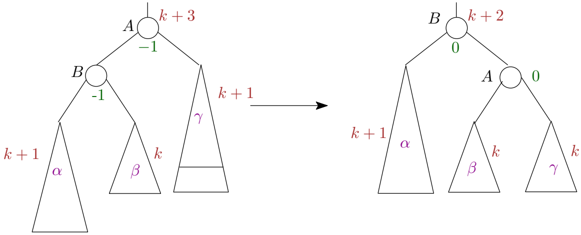
- Very similar to insertion with one major caveat
  - In insertion a (single/double) rotation restored balance *and* kept height of rebalanced subtree unchanged.
    $\Rightarrow$ Only *one* rotation needed.
  - In deletion, rotation restores *balance* but final *height* of rotated subtree might decrease by one. If this occurs, need to continue walking up path towards root, continuing to restoring balance by rotations when necessary.
  - Since path has length $O(h)$ this might require $O(h) = O(\log n)$ rotations.

$\Rightarrow$ Deletion can also be done in $O(\log n)$ time.

Deletion Example

Diagram below illustrates example in which subtree rooted at A has height $k+3$. An item is deleted from subtree γ, reducing its height from $k+1$ to k, leading to an imbalance.

After a single rotation, the subtree is now rooted at B with no imbalance. But, B has height $k+2$. This might cause an imbalance further up the tree, so the algorithm might need to continue walking upwards, correcting that imbalance.



Going Further I

AVL trees can be used to implement other operations in addition to `Insert` and `Delete`. Two particularly useful ones are

- `Split(T, x)` : Splits AVL tree T into two AVL trees, L and R. L contains all nodes in T with key values $\leq x$ and R contains all nodes with key values $> x$.
- `Join(L, R)` : Takes as input two AVL trees L, R such that all key values in L are less than all key values in R. Combines them into one output AVL tree containing all of the nodes.

Both of these operations can be implemented in $O(\log n)$ time where, in `Split(T, x)`, n is the number of nodes in T and, in `Join(L, R)`, n is the total number of nodes in L and R together.

Going Further II

AVL trees are one particular type of *Balanced* Search trees, yielding $O(\log n)$ behavior for dictionary operations.

There are many other types of Balanced Search Trees, e.g.

- red-black trees
- B-trees
- (a, b) trees
  $(2, 3)$ and $(2, 3, 4)$ trees are special cases
- treaps (randomized BSTs)
- splay trees (only $O(\log n)$ in amortized sense)
- skip lists (randomized; not trees, but behave similarly)