L02: Predicate Logic (First-order logic)

- Outline
  - Predicates
  - Quantifiers
  - Quantifiers with Restricted domains
  - Logical Equivalences involving Quantifiers
  - Negating Quantified Expressions
  - Nested Quantifiers

- Reading
  - Kenneth Rosen, Section 1.4-1.5

Predicate Logic

- Suppose we know that "every COMP student is required to take either COMP 2711 or COMP 2711H".
- No rules of propositional logic allow us to conclude the truth of the statement "Chan Tai Man, a COMP student, is required to take either COMP 2711 or COMP 2711H".
- We now study predicate logic which is more powerful than propositional logic.

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Predicate

- **Definition:** Informally, a **predicate** is a statement that may be true or false depending on the choice of values of its variables. Each choice of values produces a proposition.
- More formally, a statement involving n variables x_1, x_2, … , x_n, denoted by $P(x_1, x_2, \ldots, x_n)$, is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \ldots, x_n)$ and P is called an **n-ary predicate**.
- **Example:** $P(x)$ denotes the statement "x is greater than 3". x is the variable, and P is the predicate "is greater than 3"

Examples

- Let $P(x)$ denote the statement "$x>3$". What are the truth values of $P(4)$ and $P(2)$?
- Let $A(x)$ denote the statement "student x is required to take either COMP 2711 or COMP 2711H". Suppose Alice is a COMP student and Bob is a CHEM student. What are the truth values of A(Alice) and A(Bob)?
- Let $Q(x, y)$ denote the statement "$x=y+3$". What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

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Universal Quantification

- We saw that when the variables in a propositional function are assigned values, the resulting proposition has a certain truth value.
- Sometimes we may want to say that a predicate is true over a set of values.
- **Definition:** The **universal quantification** of $P(x)$ is the statement "for all elements x in the domain such that $P(x)$".
- Denote as $\forall x\, P(x)$. We read it as "for all x $P(x)$" or "for every x $P(x)$".
- Here $\forall$ is called the **universal quantifier**.
- An element for which $P(x)$ is false is called a **counterexample** of $\forall x\, P(x)$.

Domain

- The **domain** or **universe** is the set of all possible values of a variable.
- The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not well defined.
- Generally, an implicit assumption is made that the domain is nonempty. Otherwise $\forall x\, P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.

Examples

- Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?
- Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?
- Let $P(x)$ be the statement "$x^2 > 0$". What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?
- What is the truth value of "$\forall x\, (x^2 \geq x)$" if the domain consists of all real numbers?
  What is its truth value if the domain consists of all integers?

Existential Quantification

- **Definition:** The **existential quantification** of $P(x)$ is the statement "there exists an element x in the domain such that $P(x)$".
- The notation $\boldsymbol{\exists x\, P(x)}$ denotes the existential quantification of $P(x)$.
- Here $\exists$ is called the **existential quantifier**.
- Note that $\exists x$ means "there exists **at least one** x in the domain" but not "there exists one and only one x in the domain" or "there exists a unique x in the domain".

Universal and Existential Quantifiers

- When the domain has n elements $x_1, x_2, \ldots, x_n$

  $\forall x\, P(x)$ is the same as $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$,
  $\exists x\, P(x)$ is the same as $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Universal and Existential Quantifiers

Universal and Existential Quantifiers		
Statement	When is it true?	When is it false?
$\forall x\, P(x)$	$P(x)$ is true for every x.	There exists an x for which $P(x)$ is false.
$\exists x\, P(x)$	There exists an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- Give a counter example when $\forall x\ P(x)$ is false
- Give an example when $\exists x\ P(x)$ is true

Examples

- Let $P(x)$ be the statement "$x > 3$". What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?
- Let $Q(x)$ be the statement "$x = x + 1$". What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Example

- $P(x)$: student x has learned C++
- $Q(x)$: student x has learned Python
- Domain is all students in this class
- Express the following statements using predicate and quantifiers:
  1. “every student in this class has learned C++”
  2. “some student in this class has learned C++”
  3. “every student in this class has learned either C++ or Python”

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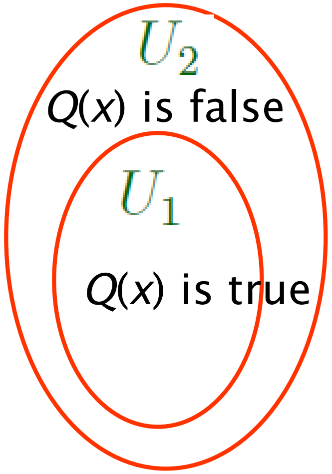
Restricted Domains

- **Example**
  What does each of the following statements mean, assuming that the domain in each case consists of all real numbers?
  (a) $\forall x < 0\ (x^2 > 0)$
  (b) $\forall y \neq 0\ (y^3 \neq 0)$
  (c) $\exists z > 0\ (z^2 = 2)$
- **Solution**
  - (a) Equivalent to $\forall x\,(x < 0 \rightarrow x^2 > 0)$. True
  - (b) Equivalent to $\forall y\,(y \neq 0 \rightarrow y^3 \neq 0)$. True
  - (b) Equivalent to $\exists z\,(z > 0 \land z^2 = 2)$. True

Quantifier with Restricted Domain

- Let U_1 and U_2 be two domains with $U_1 \subseteq U_2$. Suppose $U_1 = \{ x \in U_2 \mid Q(x) \text{ is true} \}$.
  Then, a statement $P(x)$ about U_2 may also be interpreted as a statement about U_1
  (a) $\forall x \in U_1\ (P(x))$ is equivalent to
  $$\forall x \in U_2\ (Q(x) \rightarrow P(x))$$
  (b) $\exists x \in U_1\ (P(x))$ is equivalent to
  $$\exists x \in U_2\ (Q(x) \wedge P(x))$$



Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators.
- Example
  - $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$, which is not a valid proposition

Binding Variables

- A variable can be
  - bound by a quantifier
    - the part of a logical expression bound by a quantifier is called its scope
  - set to a particular value
  - otherwise, free
- Example: $\exists x\big(P(x) \wedge Q(x)\big) \vee \forall x\, R(x) \vee S(x)$
- All variables in a propositional function must be bound or set to a particular value to turn it into a proposition.

Example

- Consider the following argument:

Premise 1: “All lions are fierce”

Premise 2: “Some lions do not drink coffee”

Conclusion: “Some fierce creatures do not drink coffee”

Let $P(x)$, $Q(x)$, and $R(x)$ be the statements “x is a lion”, “x is fierce”, and “x drinks coffee”, respectively.

Assume that the domain consists of all creatures.

Express the statements using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.

- **Solution**

$$\begin{array}{l} \forall x\,(P(x) \rightarrow Q(x)) \\ \exists x\,(P(x) \land \neg R(x)) \\ \hline \exists x\,(Q(x) \land \neg R(x)) \end{array}$$

- **Remark**

  In the next section, we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.

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Logical Equivalence

- **Definition**

  Statements involving predicates and quantifiers are **logically equivalent** if and only if they have the same truth value no matter which predicates are substituted into these statements and which (common) domain is used for the variables in these propositional functions.

  We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Logical Equivalence

- **Example**
  Show that $\forall x\,(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent
- **Solution**
  - Suppose $\forall x\,(P(x) \wedge Q(x))$ is true. That is, if a is in the domain, then $P(a)$ and $Q(a)$ are both true. Thus, $\forall x\, P(x)$ is true and so is $\forall x\, Q(x)$. Thus $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true.
  - Suppose $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true. Then, $\forall x\, P(x)$ is true and $\forall x\, Q(x)$ is true. Thus, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true. Thus, for all a, $P(a) \wedge Q(a)$ is true. Thus $\forall x\,(P(x) \wedge Q(x))$ is true.

Logical Equivalence (cont'd)

- The previous logical equivalence shows that we can distribute a universal quantifier over a conjunction.

$$\forall x\, (Q(x) \wedge P(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

- We can also distribute an existential quantifier over a disjunction:

$$\exists x\, (P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$$

However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction (why?):

$$\forall x\, (P(x) \vee Q(x)) \not\equiv \forall x\, P(x) \vee \forall x\, Q(x)$$

$$\exists x\, (P(x) \wedge Q(x)) \not\equiv \exists x\, P(x) \wedge \exists x\, Q(x)$$

Logical Equivalence (cont'd)

$$\exists x\,(P(x) \wedge Q(x)) \not\equiv \exists x\,P(x) \wedge \exists x\,Q(x)$$

Example:

$P(x)$: $2x + 1 = 5$

$Q(x)$: $x^2 = 9$

$\exists x\,P(x) \wedge \exists x\,Q(x)$ is true because $P(2)$ and $Q(3)$ are true, but $\exists x\,(P(x) \wedge Q(x))$ is false because there is no one integer a such that $P(a)$ and $Q(a)$ are both true.

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Motivation: negation

- **Example**
  Let $P(x)$ denote the statement "student x has iPhone".
  The domain is all students in this class

  Express the following statement as a universal quantification: "every student in the class has an iPhone".

  Then express the negation of the statement using an existential Quantifier.

Motivation: negation

- **Example**

  Express the following statement as an existential quantification: “there is a student in the class who has an iPhone”.

  Then express the negation of the statement using a universal quantifier.

De Morgan's Laws for Quantifiers

De Morgan's Laws for Quantifiers			
Negation	Equivalent statement	When is it true?	When is it false?
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	There exists an x for which $P(x)$ is false.	$P(x)$ is true for every x.
$\neg\exists x\, Q(x)$	$\forall x\, \neg Q(x)$	$Q(x)$ is false for every x.	There exists an x for which $Q(x)$ is true.

De Morgan's Laws for Quantifiers (cont'd)

- When the domain has n elements $x_1, x_2, \ldots, x_n$, it follows that

  $\neg\forall x\, P(x)$ is the same as $\neg\,(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n))$,

  which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)$ by De Morgan's laws,

  and this is the same as $\exists x\, \neg P(x)$.

- Similarly, $\neg\exists x\, P(x)$ is the same as $\neg\,(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n))$, which by De Morgan's laws is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n)$, and this is the same as $\forall x\, \neg P(x)$.

Examples

- **Example**

  What is the negation of the statement

  $\forall x\ (x^2 > x)$?

- **Example**

  What is the negation of the statement

  $\exists x\ (x^2 = 2)$?

Examples

- **Example**

  Show that $\neg\forall x\,(P(x)\rightarrow Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

- **Solution**

$$\begin{aligned}\neg\forall x\,(P(x)\rightarrow Q(x)) &\equiv \exists x\,\neg\,(P(x)\rightarrow Q(x))\\ &\equiv \exists x\,\neg\,(\neg\,P(x) \vee Q(x))\\ &\equiv \exists x\,(P(x) \wedge \neg Q(x))\end{aligned}$$

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Nested Quantifiers

- **Example**

  Assume that the domain for the variables x and y consists of all real numbers.

  The statement

  $$\forall x \, \forall y \, (x + y = y + x)$$

  says that $x + y = y + x$ for all real numbers x and y.

  This is the commutative law for the addition of real numbers.

Nested Quantifiers

- **Examples**

The statement

$$\forall x \, \exists y \, (x + y = 0)$$

says that for every real number x, there is a real number y such that $x + y = 0$.

This states that every real number has an additive inverse.

What is the truth value of this quantification?

$$\exists y \, \forall x \, (x + y = 0)$$

It is false since there is no value of y that satisfies the equation $x + y = 0$ for all values of x.

- **Remark:** This example illustrates that the order in which quantifiers appear makes a difference.

Quantifications of Two Variables

- The following table summarizes the meanings of the different possible quantifications involving two variables.

Quantifications of Two Variables		
Statement	When is it true?	When is it false?
$\forall x \forall y\, P(x, y)$ $\forall y \forall x\, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y\, P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y\, P(x, y)$	There is an x such that $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y\, P(x, y)$ $\exists y \exists x\, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Example

- Let $Q(x, y, z)$ be the statement "$x + y = z$". What are the truth values of the statements

$$\forall x \, \forall y \, \exists z \; Q(x, y, z)$$

$$\exists z \, \forall x \, \forall y \; Q(x, y, z),$$

where the domain of the variables is all real numbers?

- **Solution:**
- Suppose x and y are assigned values. Then there exists a real number z such that $x + y = z$. Thus the first statement is true.
- There is no value of z that satisfies the equation $x + y = z$ for all values of x and y. Thus the second statement is false

Examples

- **Example**

  Translate the statement “the sum of two positive integers is always positive” into a logical expression.

- **Example**

  Translate the statement “every nonzero real number has a multiplicative inverse”.

Examples

- **Example**

Translate the statement

$$\forall x\,(C(x) \lor \exists y\,(C(y) \land F(x, y)))$$

into English, where $C(x)$ is "x has a computer", $F(x, y)$ is "x and y are friends", and the domain for both x and y consists of all students in the school.

- **Solution**

For every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.

Every student either has a computer or has a friend who has a computer.

Examples

- **Example**

Translate the statement

$$\exists x \forall y \forall z \, ((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y, z))$$

into English, where $F(a, b)$ means a and b are friends and the domain for x, y, and z consists of all students in the school.

- **Solution**

The statement says that there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends.

In other words, there is a student none of whose friends are also friends with each other.

Examples

- Express the statement "if a person is female and is a parent, then this person is someone's mother" as a logical expression using the following predicates, with a domain consisting of all people.
  - $F(x)$: x is female
  - $P(x)$: x is a parent
  - $M(x, y)$: x is y's mother
- Express the statement "everyone has exactly one best friend" as a logical expression using the following predicates, with a domain consisting of all people.
  - $B(x, y)$: y is x's best friend

Negating Nested Quantifiers

- **Example**

  Express the negation of the statement $\forall x \exists y\,(xy = 1)$ so that no negation precedes a quantifier.

Examples

- **Example**

  Use quantifiers to express the statement "there is a woman who has taken a flight on every airline in the world".

- **Solution**

  Let $P(w,f)$ be "woman w has taken flight f "

  Let $Q(f,a)$ be "f is a flight on airline a "

  $\exists w \, \forall a \, \exists f \, (\, (P(w, f) \wedge Q(f, a) \,)$

- **Example**

  Use quantifiers to express the statement "there does not exist a woman who has taken a flight on every airline in the world" so that no negation precedes a quantifier.