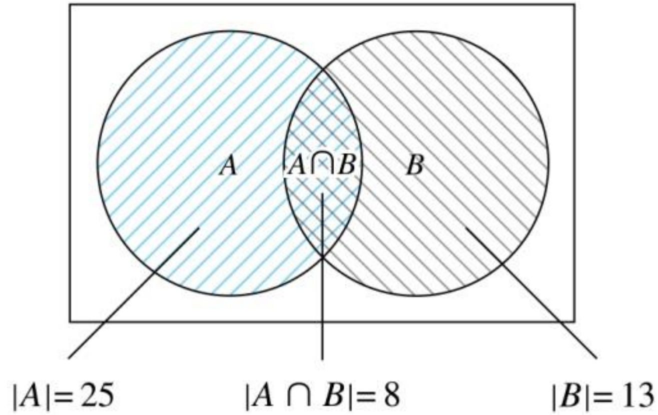
L11: The Inclusion-Exclusion Principle

- Reading
  - Rosen, 8.5, 8.6

Two Finite Sets

- Example:
  In a discrete mathematics class every student is a major in computer science or mathematics, or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$

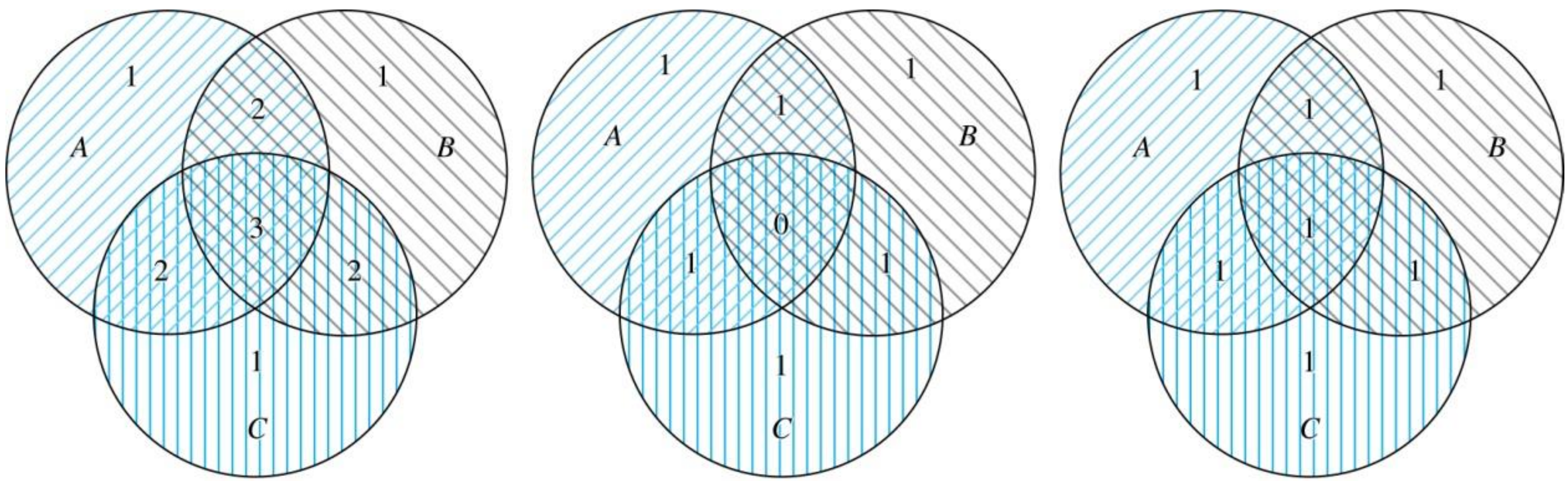


Two Finite Sets

- Example: How many positive integers not exceeding 1000 are divisible by 7 or 11?
- Solution:
  - A: The set of integers divisible by 7, $|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$
  - B: The set of integers divisible by 11, $|B| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$
  - $|A \cap B|$: Set of integers divisible by both 7 and 11: $|A \cap B| = \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor = 12$
  - $|A \cup B|$: Set of integers divisible by either 7 or 11: $|A \cup B| = |A| + |B| - |A \cap B| = 220$.

Three Finite Sets

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Example

- 1232 students have taken a course in Spanish;
- 879 have taken a course in French;
- 114 have taken a course in Russian;
- 103 have taken courses in both Spanish and French;
- 23 have taken courses in both Spanish and Russian;
- 14 have taken courses in both French and Russian;
- 2092 students have taken a course in at least one of Spanish French and Russian.
- How many students have taken all 3 languages?
- Solutions:

  $|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$,

  we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

  Solving for $|S \cap F \cap R|$ yields 7.

Inclusion-Exclusion Principle

- **Theorem**
  Let $A_1, A_2, \ldots, A_n$ be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| +$$

$$\sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof

- We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation.
- Suppose a is a member of exactly r of the sets.
- This element is counted:
  - $C(r, 1)$ times by $\sum |A_i|$
  - $C(r, 2)$ times by $\sum \left|A_i \cap A_j\right|$
  - $C(r, m)$ times by the $\sum$ involving m sets.
- Thus, it is counted exactly
$$C(r, 1) - C(r, 2) + C(r, 3) - \cdots + (-1)^{r+1} C(r, r)$$
times.

Proof (cnt'd)

- Recall that a corollary of the binomial theorem states
  $C(r,0) - C(r,1) + C(r,2) - C(r,3) + \cdots + (-1)^r C(r,r) = 0$.
- Hence,

$$C(r,1) - C(r,2) + C(r,3) - \cdots + (-1)^{r+1} C(r,r) = C(r,0) = 1$$

Number of Onto Functions

- Example

  How many surjective (or onto) functions are there from a set with six elements to a set with three elements?

- Solution

  Suppose the function is from A to B. We know that there are 3^6 different functions.

  Let $B = \{b_1, b_2, b_3\}$. Let P_i be the set of functions that do not map to b_i. We want to find $3^6 - |P_1 \cup P_2 \cup P_3|$.

  $|P_1| = |P_2| = |P_3| = 2^6$

  $|P_1 \cap P_2| = |P_1 \cap P_2| = |P_1 \cap P_2| = 1$

  $|P_1 \cap P_2 \cap P_3| = 0$

  Then, we can use the inclusion-exclusion principle to find $|P_1 \cup P_2 \cup P_3| = C(3,1)2^6 - C(3,2)1^6 + 0$.

Example

- How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job?
- Solution:
  Each assignment is an onto function from the jobs to the employees.

Derangement

- Example (Hatcheck Problem):
  A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats randomly. What is the probability that no one receives the correct hat?
- Remark
  The answer is the number of ways that the hats can be arranged so that there is no hat in its original position divided by $n!$, which is the number of permutations of n hats.

Derangement

- Definition
  A **derangement** is a permutation of objects that leaves no object in its original position.
- Example
  The permutation 21453 is a derangement of 12345 because no number is left in its original position. However, 21543 is not a derangement of 12345 because this permutation leaves 4 unchanged.

Number of Derangements

- Theorem

  The number of derangements of a set with n elements is

  $$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!}\right]$$

- Proof

  Let P_i be the number of permutations where i remains unchanged. We have

  $D_n = n! - |P_1 \cup P_2 \cup \cdots \cup P_n|$

Proof (cont'd)

- We note that

$$
\begin{aligned}
&|P_i| = (n-1)! \\
&\left|P_i \cap P_j\right| = (n-2)! \\
&\cdots \\
&\left|P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_m}\right| = (n-m)! \\
&\cdots
\end{aligned}
$$

- For each m, there are $C(n,m)$ ways to pick $P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_m}$, it follows that

$$
\begin{aligned}
&|P_1 \cup P_2 \cup \cdots \cup P_n| \\
&= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n+1}\binom{n}{n}(n-n)! \\
&= \frac{n!}{1!\,(n-1)!}(n-1)! - \frac{n!}{2!\,(n-2)!}(n-2)! + \cdots + (-1)^{n+1}\frac{n!}{n!\,0!}0! \\
&= n!\left[\frac{1}{1!} - \frac{1}{2!} + \cdots + (-1)^{n+1}\frac{1}{n!}\right]
\end{aligned}
$$

Proof (cont'd)

- The probability of a random permutation being a derangement is thus

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

which approaches $1/e$ as $n \to \infty$.

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786