COMP2121A: Discrete Mathematics
Problem Solving Session 4 23 Oct, 2018

1. In the forest, there are two animals *giraffe* and *hippo*. We know that the weather of the forest is sunny with probability 0.6 and rainy with probability 0.4. On sunny days, the giraffe and the hippo appear independently with probability 0.7 and 0.8. On rainy days, the giraffe and the hippo appear independently with probability 0.2 and 0.5.

   (a) What is the probability that the hippo appears?
   Solution: We use G/H to denote the events that the giraffe/hippo appears respectively. We use S/R to denote the event that weather is sunny/rainy respectively. $\mathbf{Pr}[H] = \mathbf{Pr}[H, S] + \mathbf{Pr}[H, R] = 0.6 \cdot 0.8 + 0.4 \cdot 0.5 = 0.68$.

   (b) Conditioning on the appearance of the giraffe, what is the probability that hippo also appears?
   Solution: $\mathbf{Pr}[G] = \mathbf{Pr}[G, S] + \mathbf{Pr}[G, R] = 0.6 \cdot 0.7 + 0.4 \cdot 0.2 = 0.5$.
   $\mathbf{Pr}[G, H] = \mathbf{Pr}[G, H, S] + \mathbf{Pr}[G, H, R] = 0.6 \cdot 0.7 \cdot 0.8 + 0.4 \cdot 0.2 \cdot 0.5 = 0.376$.
   $\mathbf{Pr}[H|G] = \frac{\mathbf{Pr}[G, H]}{\mathbf{Pr}[G]} = \frac{0.376}{0.5} = 0.752$.

   (c) Are the events that the giraffe appears and the hippo appears independent?
   Solution: Note that $\mathbf{Pr}[H] \neq \mathbf{Pr}[H|G]$. The two events are correlated.

2. Suppose A is a set of size n. Recall that a relation R on A is a subset of $A \times A$. If an element x is related to an element y by R, we write $(x, y) \in R$.

   We now form a random relation R in the following way. Suppose p is a real number in $[0, 1]$. *Independently* for each $(x, y) \in A \times A$, the event "$(x, y) \in R$" occurs with probability p.

   For each of the following statements, do the following.

   (1) Translate into an equivalent logical statement with quantifiers.

   (2) Calculate the probability that the statement holds.

   Example Statement. The relation R is empty.

   (1) $\forall x \in A, \forall y \in A, (x, y) \notin R$.

   (2) Since each pair (x, y) is not included in R with probability $1 - p$ independently, the probability that the statement holds is $(1 - p)^{n^2}$.

   (a) The relation R is reflexive.
   Solution: (1) $\forall x \in A, (x, x) \in R$. (2) p^n.

   (b) The relation R is symmetric.
   Solution: (1) $\forall x \in A, \forall y \in A, (x, y) \in R \Rightarrow (y, x) \in R$. (2) $(p^2 + (1 - p)^2)^{\binom{n}{2}}$.

   (c) The relation R is anti-symmetric, i.e. for all $a, b \in A$, if both (a, b) and (b, a) are contained in R, then $a = b$.
   Solution: (1) $\forall x \in A, \forall y \in A, (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$. (2) $(1 - p^2)^{\binom{n}{2}}$.

(d) The relation R is symmetric and anti-symmetric.

Solution: (1) $\forall x \in A, \forall y \in A, (x, y) \in R \Rightarrow x = y$. (2) $(1-p)^{n(n-1)}$.

(e) The relation R corresponds to a function from A to itself. For instance, if $(x, y) \in R$, this means that when the input is x, the output is y.

Solution: (1) Let $E := \forall x \in A, \exists y \in A, (x, y) \in R \land (\forall z \in A, (x, z) \in R \Rightarrow y = z)$. (2) $n^n p^n (1-p)^{n(n-1)}$.

(f) The relation R corresponds to an injective function from A to itself.

Solution: (1) $E \land (\forall x_1, \forall x_2, \forall y, (x_1, y) \in R \land (x_2, y) \in R \Rightarrow x_1 = x_2)$. (2) $n! p^n (1-p)^{n(n-1)}$.

3. A robbery took place. A witness saw a couple driving away in a yellow car. The couple consisted of a black man with a beard and a moustache and a blond woman with hair in a ponytail. Mr and Mrs. Collins matched this description and were brought to court. Since the witness could not recognize them, the jury resorted to probability theory.

   (a) Let n be the number of couples in town. Assume for simplicity that the n couples have been generated independently and let p be the probability that a generated couple matches the description.
   What is the probability that exactly k couples match the description?
   Solution:
   $$\binom{n}{k} p^k (1-p)^{n-k}.$$

   (b) The witness saw a couple matching the description, namely the robbers. And since it was dark, the witness did not get other information about the robbers apart from the description. What is the probability that k more couples match the description, *in addition* to the robbers?
   Solution:
   $$\frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{1-(1-p)^n}.$$

   (c) We assume that among couples that match the description, every couple is equally likely to be the robbers. The Collins match the description. What is the probability that they are innocent? Give an estimate for $p = 1/1,000,000$ and $n = 1,000,000$.

Solution:

$$\begin{aligned}
\mathbf{Pr}[\text{Guilty}] &= \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \cdot \frac{1}{k+1} \\
&= \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)! \cdot (k+1)} p^k (1-p)^{n-1-k} \\
&= \frac{1}{np} \sum_{k=0}^{n-1} \binom{n}{k+1} p^{k+1} (1-p)^{n-1-k} \\
&= 1 - \binom{n}{1} p (1-p)^{n-1} \approx 1 - \frac{1}{e}
\end{aligned}$$

Therefore, the probability that they are innocent equals $\frac{1}{e}$.