THE UNIVERSITY OF HONG KONG

FACULTY OF ENGINEERING

DEPARTMENT OF COMPUTER SCIENCE

COMP2121A/B Discrete Mathematics

Date: 14 Dec, 2017 Time: 2:30pm - 5:30pm

Write Your **University Number here** and also on **every page**: ____________________

Please read the following before you begin.

1. Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is your responsibility to ensure that your calculator operates satisfactorily, and you must record the **name and type of the calculator** used here:

   ____

2. You are advised to spend around 5 minutes to go through all the questions before you begin writing. Allocate the time for each part of the question carefully.
3. Please write your answer in the designated space. If you run out of space, you can use the blank side of the previous page. Please indicate clearly when you are doing so.
4. Attempt ALL questions. The full mark for the exam is 100 points.

For marking use:

Question 1	Question 5
Question 2	Question 6
Question 3	Question 7
Question 4	Question 8

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1 Sets, Relations & Functions (15 points)

Let A and B be two sets, where $|A| = m$ and $|B| = n$.

(a) (3 pt) Let $P(S)$ denote the power set of a set S. Prove or disprove that $P(A \cup B) = P(A) \cup P(B)$.

(b) (3 pt) How many injective functions are there from A to B?

(c) (4 pt) How many symmetric relations on A can there be?

(d) (2 pt) Given $f(n) = n^2 + 200n + 4$, show that $f(n) = O(n^2)$.
(*Hint*: Start with the definition of big-O notation, find the constants that can show $f(n) = O(n^2)$.)

(e) (3 pt) List the functions below in increasing order of big-O complexity:

$$n, \quad 2^n, \quad n \log n, \quad \log n, \quad 2n^5 - 10n^3 + n, \quad \sqrt{n}, \quad n^2 + \log n, \quad (\log n)^2, \quad n!$$

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2 Logic (10 points)

At least two students from a class lined up from left to right, where the leftmost is considered as the first. Write each of the following statements as a logical expression using predicates, quantifiers and logical connectives. The only predicate that you may use is

$F(x, y)$, meaning that "x is somewhere to the left of y in the line."

For example, in the line "CDA", both $F(C, A)$ and $F(C, D)$ are true.

However, you may define a logical expression for a predicate P and then use it to compose further expressions.

(a) (2 pt) Student x is in the line.

(b) (2 pt) Student x is first in line.

(c) (3 pt) Student x is immediately to the right of student y.

(d) (3 pt) Student x is second in line.

3 Proofs (10 points)

(a) (5 pt) Let $s_1, s_2, \ldots, s_{56}$ be 56 bit strings of length at most 9 (where we allow an empty string of length 0). Prove that there exist two strings, s_i and s_j, where $i \neq j$, which contain the same number of 0's and the same number of 1's. (For example, strings 010001 and 100100 contain the same number of 0's and 1's.)

(b) Among the many rooms in an old mansion with only one entrance door, there is a ghost in each room that has an odd number of doors. Moreover, starting from the entrance, each room is reachable by going through doorways.

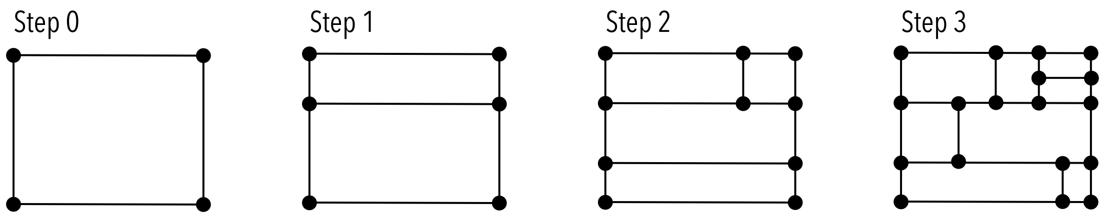
(i) (2 pt) Prove that there is a room in which there is a ghost.

(ii) (3 pt) Prove that, if there is exactly one room with a ghost, a man entering from outside can always reach that room by going through each doorway exactly once.

4 Rectangle Partitioning (15 points)

Suppose you are given a rectangular graph with 4 vertices and 4 edges. At each step, you may draw a horizontal or a vertical line within each rectangle of the graph, in order to divide each rectangle within the graph into two regions. Moreover, we assume the lines are drawn to maximize the number of vertices (i.e., the intersection points).

Example:



Let $r(n)$ be the number of non-overlapping rectangular regions partitioning the initial rectangle at step n, $e(n)$ be the number of edges in the graph at step n, and $v(n)$ be the number of vertices in the graph formed at step n.

(a) (3 pt) Give a recurrence relation and the initial condition for $r(n)$, and solve $r(n)$ for $n = 0, 1, 2, \ldots$

(b) (3 pt) Show that $e(n) = e(n-1) + 3 \cdot r(n-1)$. By substituting $r(n-1)$ by the answer in part (a), solve $e(n)$ for $n = 0, 1, 2, \ldots$

(c) (3 pt) Give a recurrence relation and the initial condition for $v(n)$, and solve $v(n)$ for $n = 0, 1, 2, \ldots$

(d) (3 pt) Show how $r(n)$, $e(n)$ and $v(n)$ can be related via Euler's formula for planar graphs. Pay attention to how the number of regions is defined in a planar graph.

(e) (3 pt) Using the sum of the degrees of the vertices in the graph, derive a relationship between $v(n)$ and $e(n)$. Verify your answer with the formulas in parts (a)-(c) above.

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5 Describing Situations with Probability (15 points)

Suppose Ω is a sample space, and $A, B \subseteq \Omega$ are events. Express each of the following situations in the form $\Pr[\mathsf{expr}_1] \geq \mathsf{expr}_2$ or $\Pr[\mathsf{expr}_1] > \mathsf{expr}_2$, where expr_1 and expr_2 must contain only mathematical symbols or numbers and may not contain English words.

Attention: The events A and B are interpreted as subsets of Ω. Hence, it will be incorrect to apply logical operators on them.

(a) (3 pt) There is a chance of at least 50% that event A happens.

Pr[ ] >

(b) (3 pt) There is a chance of at least 90% that at least one of the events A and B happens.

Pr[ ] >

(c) (3 pt) The chance that both events A and B happen is strictly less than 1%.

Pr[ ] >

(d) (3 pt) If event A happens, then event B happens with probability strictly less than 0.1.

Pr[ ] >

(e) (3 pt) The following statement is true with probability strictly less than 0.1: "If event A happens, then event B happens."

Pr[ ] >

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6 Random Hunting (10 points)

Suppose there are m hunters and n rabbits. Each hunter shoots one out of the n rabbits uniformly at random independently. Let X be the number of rabbits that are shot.

1. (5 pt) Suppose r is a non-negative integer. Compute $\Pr[X = r]$ in terms of m and n.

   For the special case $m = 8$, $n = 6$ and $r = 4$, give the numerical answer to 8 decimal places.

2. (5 pt) Compute $E[X]$ in terms of m and n.

   For the special case $m = 8$ and $n = 6$, give the numerical answer to 8 decimal places.

7 Degree Sequence (15 points)

The *degree sequence* of a graph is the list of the degrees of its vertices. Below are potential degree sequences of a graph containing 10 vertices.

(A) 2, 2, 2, 3, 4, 4, 4, 5, 6, 7

(B) 2, 2, 2, 3, 3, 3, 3, 4, 5, 5

(C) 2, 2, 3, 4, 5, 5, 5, 8, 8, 8

(D) 1, 1, 1, 1, 2, 2, 2, 2, 3, 3

(a) (3 pt) Which of the above can be the degree sequence of a simple undirected graph?

(b) (3 pt) Which of the above can be the degree sequence of a simple undirected graph that is connected?

(c) (3 pt) Which of the above can be the degree sequence of a simple undirected graph that is connected and planar?

(d) (3 pt) What is the minimum number of edges that must be added to a graph having the degree sequence in (c) in order to form a (multi-)graph that contains an Euler circuit?

(e) (3 pt) How many ways are there to add the edges to achieve the condition in (d)?

8 Cycle (10 points)

(a) (5 pt) Suppose $n \geq 3$. Prove that a connected simple undirected graph with n vertices such that every vertex has degree 2 must be a cycle.

(b) Given an undirected cycle on n vertices (where $n \geq 3$), suppose a uniformly random direction is assigned to each edge independently.

(i) (2 pt) What is the probability that a directed cycle is formed?

(ii) (3 pt) What is expectation of the length of the longest directed path?
You may use asymptotic notation. If you have time, you can explain your answer briefly.

END OF PAPER