COMP 3711

Tutorial 3a

Question 1

Using the *Master Theorem*, give asymptotic tight bounds for $T(n)$

(a) $T(1) = 1$

$T(n) = 3T(n/4) + n \quad \text{if } n > 1$

(b) $T(1) = 1$

$T(n) = 3T(n/4) + 1 \quad \text{if } n > 1$

(c) $T(1) = 1$

$T(n) = 4T(n/2) + n^2 \quad \text{if } n > 1$

(d) $T(1) = 1$

$T(n) = 4T(n/3) + n^2 \quad \text{if } n > 1$

Question 1

Using the *Master Theorem*, give asymptotic tight bounds for $T(n)$

(e) $T(1) = 1$

$T(n) = 9T(n/3) + n^2$ if $n > 1$

(f) $T(1) = 1$

$T(n) = 10T(n/3) + n^3$ if $n > 1$

(g) $T(1) = 1$

$T(n) = 99T(n/10) + n^2$ if $n > 1$

(h) $T(1) = 1$

$T(n) = 101T(n/10) + n^2$ if $n > 1$

Recall the version of the Master Theorem for equalities we saw
Let $a \geq 1, b > 1$ and $c \geq 0$ be constants.

$$T(n) = aT(n/b) + f(n), \qquad c = \log_b a$$

1. If $f(n) = \theta(n^{c-\epsilon})$ for some $\epsilon > 0$ => $\mathrm{T}(n) = \theta(n^c)$
2. If $f(n) = \theta(n^c)$ => $\mathrm{T}(n) = \theta(n^c \log n)$
3. If $f(n) = \theta(n^{c+\epsilon})$ for some $\epsilon > 0$ & $af(n/b) \leq df(n)$ for some $d < 1$ and large enough n => $\mathrm{T}(n) = \theta(f(n))$

Let $f(n) = \theta(n^k)$ for some k. This simplifies to

What about 2[nd] part of (3)?

1. If $k < c$ => $\mathrm{T}(n) = \theta(n^c)$
2. If $k = c$ => $\mathrm{T}(n) = \theta(n^c \log n)$
3. If $k > c$ => $\mathrm{T}(n) = \theta(f(n)) = \theta(n^k)$

Note: $af(n/b) \leq df(n)$ becomes $a\left(\frac{n}{b}\right)^k \leq dn^k$, i.e $\frac{a}{b^k} \leq d$.
This is true iff $\beta = c - k =$ $\boldsymbol{\log_b a}$**-** $\boldsymbol{k\log_b b \leq \log_b d}$.
Recall $k > c$. Then $\beta < 0$. Set $d = b^\beta < 1$. Then $\log_b d = \beta$ and 2[nd] part of condition 3 is satisfied.

We use the specialized version of Master Theorem from previous page:

Let $a, b \geq 1$ and $c \geq 0$ be constants.

$T(n) = aT(n/b) + f(n),\quad c = \log_b a$. Let $f(n) = \theta(n^k)$ for some k.

1. If $k < c$ => $T(n) = \theta(n^c)$
2. If $k = c$ => $T(n) = \theta(n^c \log n)$
3. If $k > c$ => $T(n) = \theta(f(n)) = \theta(n^k)$

(a) $T(n) = 3T(n/4) + n$: This is case 3 because $1 > \log_4 3$.

$$\Rightarrow \quad T(n) = \Theta(n).$$

(b) $T(n) = 3T(n/4) + 1$: This is case 1 because $0 < \log_4 3$.

$$\Rightarrow \quad T(n) = \Theta\left(n^{\log_4 3}\right) = \Theta(n^{0.7924\ldots}).$$

(c) $T(n) = 4T(n/2) + n^2$: This is case 2 because $2 = \log_2 4$.

$$\Rightarrow \quad T(n) = \Theta(n^2 \log n).$$

(d) $T(n) = 4T(n/3) + n^2$: This is case 3 because $2 > \log_3 4$.

$$\Rightarrow \quad T(n) = \Theta(n^2).$$

We use specialized version of Master Theorem from the previous page:

Let $a, b \geq 1$ and $c \geq 0$ be constants.

$T(n) = aT(n/b) + f(n), \qquad c = \log_b a.$ Let $f(n) = \theta(n^k)$ for some k.

1. If $k < c$ => $T(n) = \theta(n^c)$
2. If $k = c$ => $T(n) = \theta(n^c \log n)$
3. If $k > c$ => $T(n) = \theta(f(n)) = \theta(n^k)$

(e) $T(n) = 9T(n/3) + n^2$: This is case 2 because $2 = \log_3 9$.

$$\Rightarrow \quad T(n) = \Theta(n^2 \log n).$$

(f) $T(n) = 10T(n/3) + n^3$: This is case 3 because $3 > \log_3 10$.

$$\Rightarrow \quad T(n) = \Theta(n^3).$$

(g) $T(n) = 99T(n/10) + n^2$: This is case 3 because $2 > \log_{10} 99$.

$$\Rightarrow \quad T(n) = \Theta(n^2).$$

(h) $T(n) = 101T(n/10) + n^2$: This is case 1 because $2 < \log_{10} 101$.

$$\Rightarrow \quad T(n) = \Theta\left(n^{\log_{10} 101}\right).$$