Divide and Conquer for Integer Multiplication
A Review

Outline

- Quick D&C Integer Multiplication Review
- High level Example for 4X4 D&C simple integer multiplication
- High level Example for 4X4 D&C Karatsuba integer multiplication
- Full worked example of 4X4 simple integer multiplication
- Full worked example of 4X4 Karatsuba integer multiplication

Background

- Straightforward "long multiplication"of 2 n-bit integer words requires $O(n^2)$ time (scalar multiplications and additions).
- In class, we saw how to use divide and conquer ideas to develop two different multiplication algorithms.
- These slides assume you know the algorithms

- The first algorithm satisfied the recurrence

$$\text{For } n > 1, \quad T(n) = 4T(n/2) + n. \qquad T(1) = 1$$

$$\Longrightarrow T(n) = O(n^2)$$

- The 2nd , Karatsuba Multiplication, satisfied the recurrence

$$\text{For } n > 1, \quad T(n) = 3T(n/2) + n. \qquad T(1) = 1$$

$$\Longrightarrow T(n) = O\left(n^{\log_2 3}\right) = O(n^{1.585\ldots})$$

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The first divide-and-conquer algorithm for integer multiplication

4 bit by 4 bit example

$$7 \qquad 13 \qquad = \qquad 91$$

$$0100 \qquad 1101 \qquad = \qquad 01011011$$

$$\mathbf{1} \cdot \mathbf{2^2} + \mathbf{3} \qquad \mathbf{3} \cdot \mathbf{2^2} + 1 \qquad = \qquad \mathbf{1} \cdot \mathbf{3} \times \mathbf{2^4} + [\mathbf{1} \cdot \mathbf{1} + 3 \cdot \mathbf{3}] \times \mathbf{2^2} + \mathbf{3} \cdot 1$$

$$\boxed{a\mathbf{2^2} + b} \times \boxed{c\mathbf{2^2} + d} = \boxed{ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd}$$

The first divide-and-conquer algorithm for integer multiplication

4 bit by 4 bit example

$$a\mathbf{2^2} + b \quad \times \quad c\mathbf{2^2} + d \quad = \quad ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd$$

ac $\quad$ ad $\quad$ bc $\quad$ bd

The first divide-and-conquer algorithm for integer multiplication

4 bit by 4 bit example

$$a\mathbf{2^2} + b \quad \times \quad c\mathbf{2^2} + d \quad = \quad ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd$$

ac

$a = a_1\mathbf{2^1} + a_0 \times \mathrm{c} = c_1\mathbf{2^1} + c_0$

ad

$a_1\mathbf{2^1} + a_0 \times d_1\mathbf{2^1} + d_0$

bc

$b_1\mathbf{2^1} + b_0 \times c_1\mathbf{2^1} + c_0$

bd

$b_1\mathbf{2^1} + b_0 \times d_1\mathbf{2^1} + d_0$

The first divide-and-conquer algorithm for integer multiplication

4 bit by 4 bit example

$$a\mathbf{2^2} + b \times c\mathbf{2^2} + d = ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd$$

ac

$a = a_1\mathbf{2^1} + a_0 \times \mathrm{c} = c_1\mathbf{2^1} + c_0$

$a_1c_1\mathbf{2^2} + [a_1c_0 + a_0c_1]\mathbf{2^1} + a_0c_0$

ad

$a_1\mathbf{2^1} + a_0 \times d_1\mathbf{2^1} + d_0$

$a_1d_1\mathbf{2^2} + [a_1d_0 + a_0d_1]\mathbf{2^1} + a_0d_0$

bc

$b_1\mathbf{2^1} + b_0 \times c_1\mathbf{2^1} + c_0$

$b_1c_1\mathbf{2^2} + [b_1c_0 + b_0c_1]\mathbf{2^1} + b_0c_0$

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$b_1\mathbf{2^1} + b_0 \times d_1\mathbf{2^1} + d_0$

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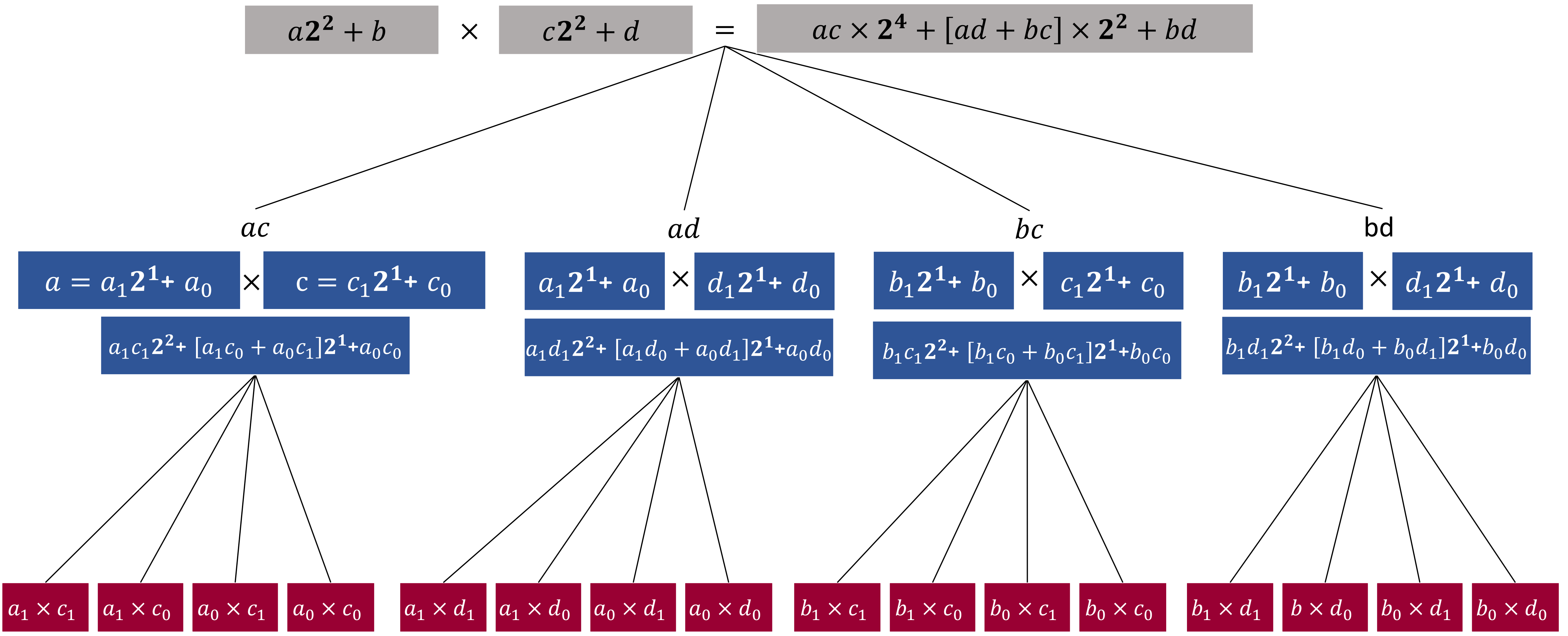
The first divide-and-conquer algorithm for integer multiplication

4 bit by 4 bit example



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Karatsuba Multiplication

4 bit by 4 bit example

$$7 \qquad 13 \qquad = \qquad 91$$

$$0100 \qquad 1101 \qquad = \qquad 01011011$$

$$1 \cdot \mathbf{2}^{\mathbf{2}} + 3 \qquad 3 \cdot \mathbf{2}^{\mathbf{2}} + 1 \qquad = \quad 1 \cdot 3 \times \mathbf{2}^{\mathbf{4}} + [1 \cdot 1 + 3 \cdot 3] \times \mathbf{2}^{\mathbf{2}} + 3 \cdot 1$$

Old Method

$$a\mathbf{2}^{\mathbf{2}} + b \quad \times \quad c\mathbf{2}^{\mathbf{2}} + d \quad = \quad ac \times \mathbf{2}^{\mathbf{4}} + [ad + bc] \times \mathbf{2}^{\mathbf{2}} + bd$$

$$1 \cdot \mathbf{2}^{\mathbf{2}} + 3 \qquad 3 \cdot \mathbf{2}^{\mathbf{2}} + 1 \qquad = \quad 1 \cdot 3 \times \mathbf{2}^{\mathbf{4}} + [(1 + 3)(3 + 1) - 1 \cdot 3 - 3 \cdot 1] \times \mathbf{2}^{\mathbf{2}} + 3 \cdot 1$$

Rewrite of same expression

$$a\mathbf{2}^{\mathbf{2}} + b \quad \times \quad c\mathbf{2}^{\mathbf{2}} + d \quad = \quad ac \times \mathbf{2}^{\mathbf{4}} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2}^{\mathbf{2}} + bd$$

Karatsuba Multiplication

4 bit by 4 bit example

$$a\mathbf{2^2} + b \quad \times \quad c\mathbf{2^2} + d \quad = \quad (i) \times \mathbf{2^4} + [(ii) - (i) - (iii)] \times \mathbf{2^2} + (iii)$$

$(i) = ab$

$(ii) = (a + b) \times (c + d)$

$(iii) = bd$

Karatsuba Multiplication

4 bit by 4 bit example

$a\mathbf{2^2} + b$ $\times$ $c\mathbf{2^2} + d$ $=$ $(i) \times \mathbf{2^4} + [(ii) - (i) - (iii)] \times \mathbf{2^2} + (iii)$

$(i) = ab$

$a = a_1\mathbf{2^1} + a_0 \times c = c_1\mathbf{2^1} + c_0$

$(ii) = (a+b) \times (c+d)$

$a + b = (a_1 + b_1)\mathbf{2^1} + (a_0 + b_0) \times c + d = (c_1 + d_1)\mathbf{2^1} + (c_0 + d_0)$

$(iii) = bd$

$\boldsymbol{b} = b_1\mathbf{2^1} + b_0 \times d = d_1\mathbf{2^1} + d_0$

Karatsuba Multiplication

4 bit by 4 bit example

$$a\mathbf{2}^{\mathbf{2}} + b \quad \times \quad c\mathbf{2}^{\mathbf{2}} + d \quad = \quad (i) \times \mathbf{2}^{\mathbf{4}} + [(ii) - (i) - (iii)] \times \mathbf{2}^{\mathbf{2}} + (iii)$$

$(i) = ab$

$$a = a_1\mathbf{2}^{\mathbf{1}} + a_0 \quad \times \quad c = c_1\mathbf{2}^{\mathbf{1}} + c_0$$

$(ii) = (a + b) \times (c + d)$

$$a + b = (a_1 + b_1)\mathbf{2}^{\mathbf{1}} + (a_0 + b_0) \quad \times \quad c + d = (c_1 + d_1)\mathbf{2}^{\mathbf{1}} + (c_0 + d_0)$$

$(iii) = bd$

$$\boldsymbol{b} = b_1\mathbf{2}^{\mathbf{1}} + b_0 \quad \times \quad d = d_1\mathbf{2}^{\mathbf{1}} + d_0$$

$a_1 \times c_1$ | $(a_1 + a_0)(c_1 + c_0)$ | $a_0 \times c_0$

$(a_1 + b_1)(c_1 + d_1)$ | $[(a_1 + b_1) + (a_0 + b_0)][(c_1 + d_1) + (c_0 + d_0)]$ | $(a_0 + b_0)(c_0 + d_0)$

$b_1 \times d_1$ | $(b_1 + b_0)(d_1 + d_0)$ | $b_0 \times d_0$

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First divide and conquer algorithm for multiplication : Example

0	1	1	0	$\times$	1	1	0	1

$$\boxed{\left(a\mathbf{2^2}+b\right)\left(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}\right)=ac\times\mathbf{2^4}+[ad+bc]\times\mathbf{2^2}+bd}$$

First divide and conquer algorithm for multiplication : Example

0 1 1 0 × 1 1 0 1

0 1 × 1 1

$$\boxed{\left(a\mathbf{2^2}+b\right)\left(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}\right)=ac\times\mathbf{2^4}+[ad+bc]\times\mathbf{2^2}+bd}$$

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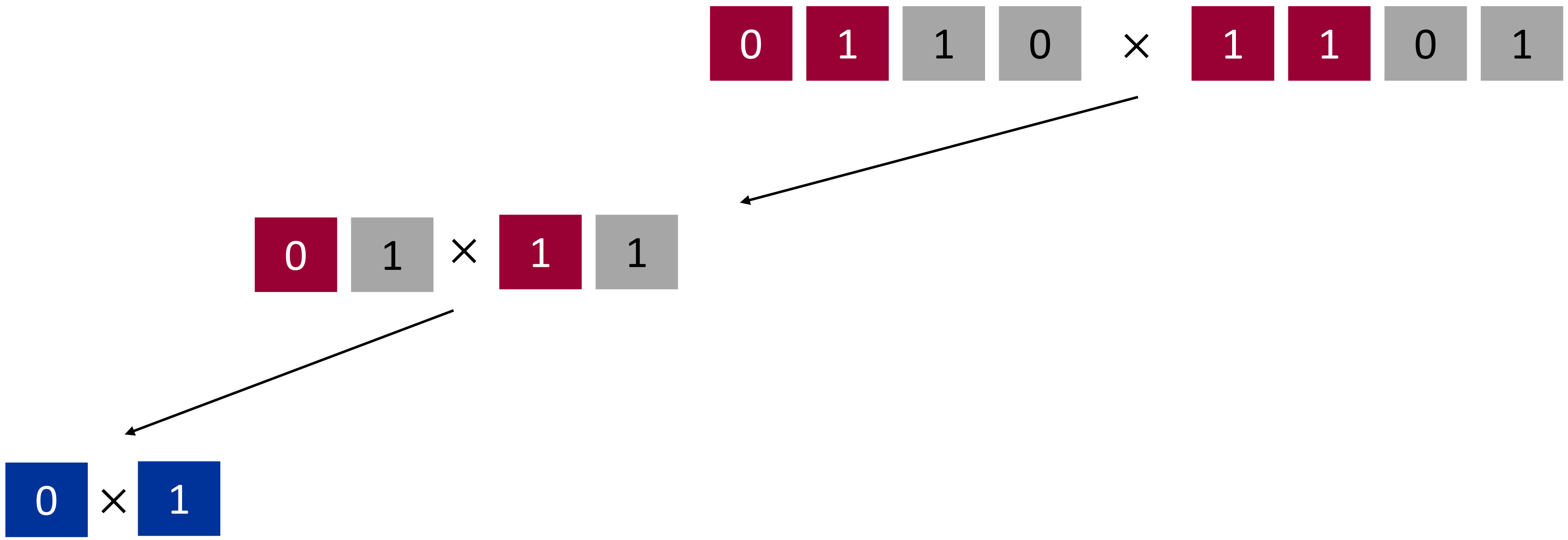

$$\boxed{(a\mathbf{2}^{2} + b)(\boldsymbol{c}\mathbf{2}^{2} + \mathrm{d}) = ac \times \mathbf{2}^{4} + [ad + bc] \times \mathbf{2}^{2} + bd}$$

First divide and conquer algorithm for multiplication : Example

0 1 1 0 × 1 1 0 1

0 1 × 1 1

0

$$\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[ad+bc]\times\mathbf{2}^{\mathbf{2}}+bd$$

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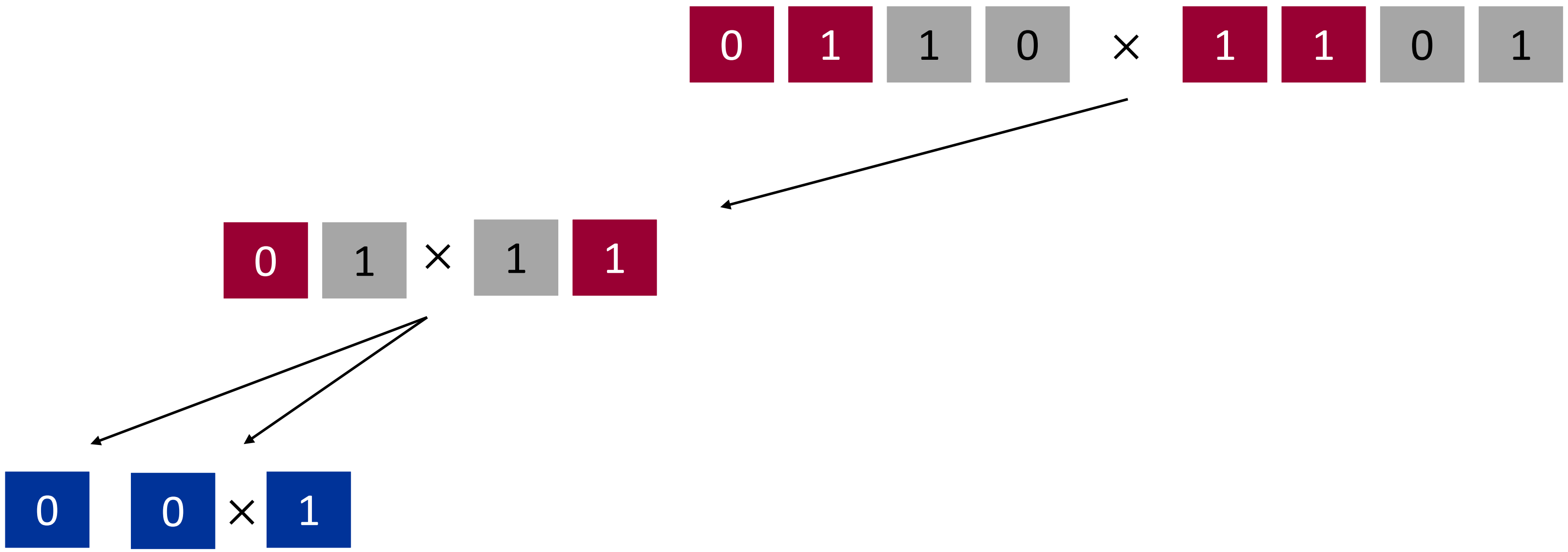

$$\boxed{(a\mathbf{2}^{\mathbf{2}} + b)(\boldsymbol{c}\mathbf{2}^{\mathbf{2}} + \mathrm{d}) = ac \times \mathbf{2}^{\mathbf{4}} + [ad + bc] \times \mathbf{2}^{\mathbf{2}} + bd}$$

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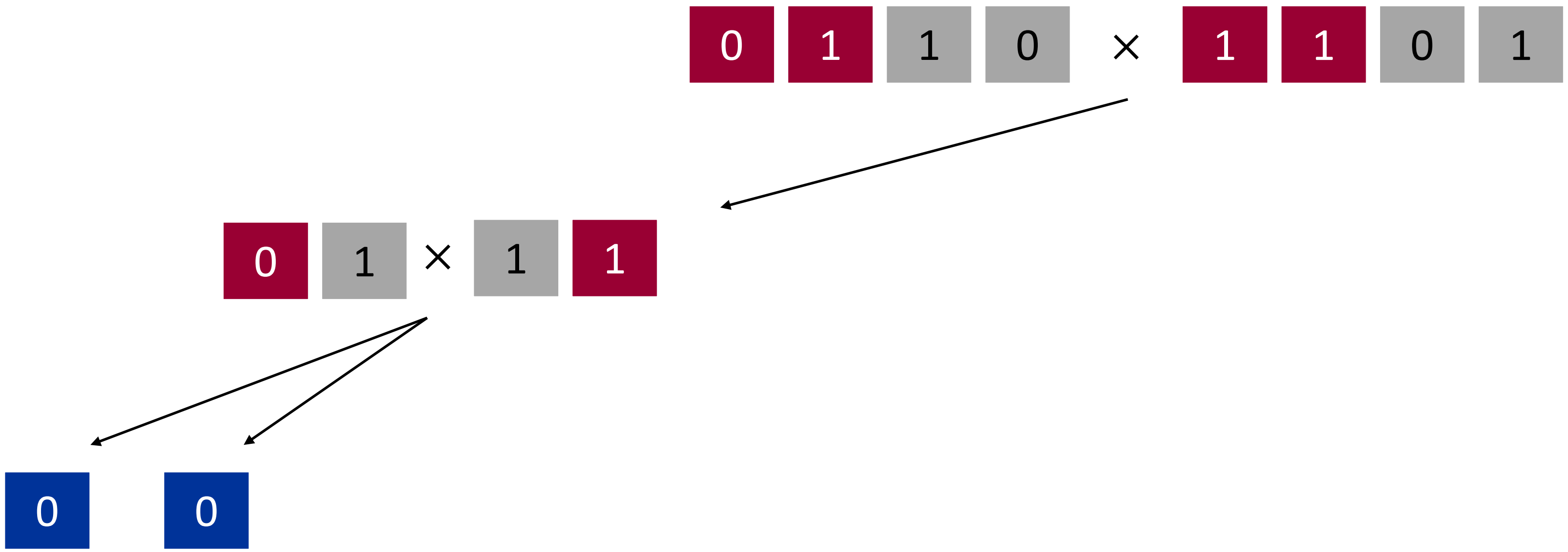

$$\boxed{\left(a\mathbf{2^2} + b\right)\left(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}\right) = ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd}$$

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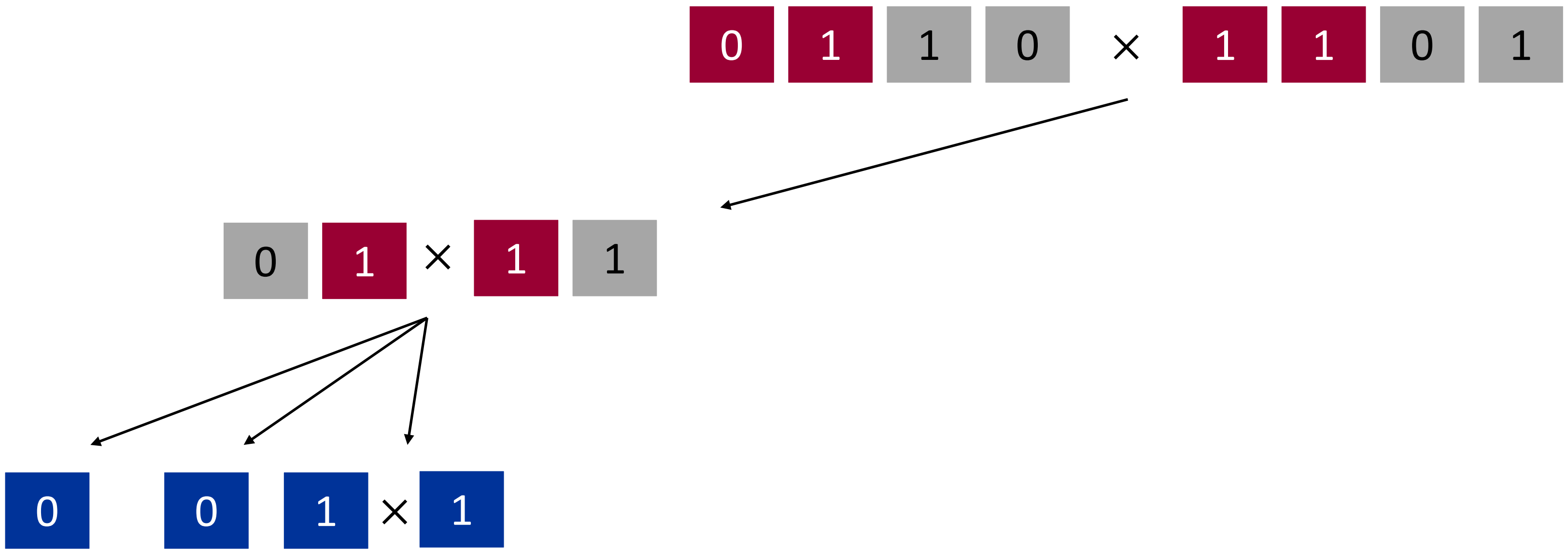

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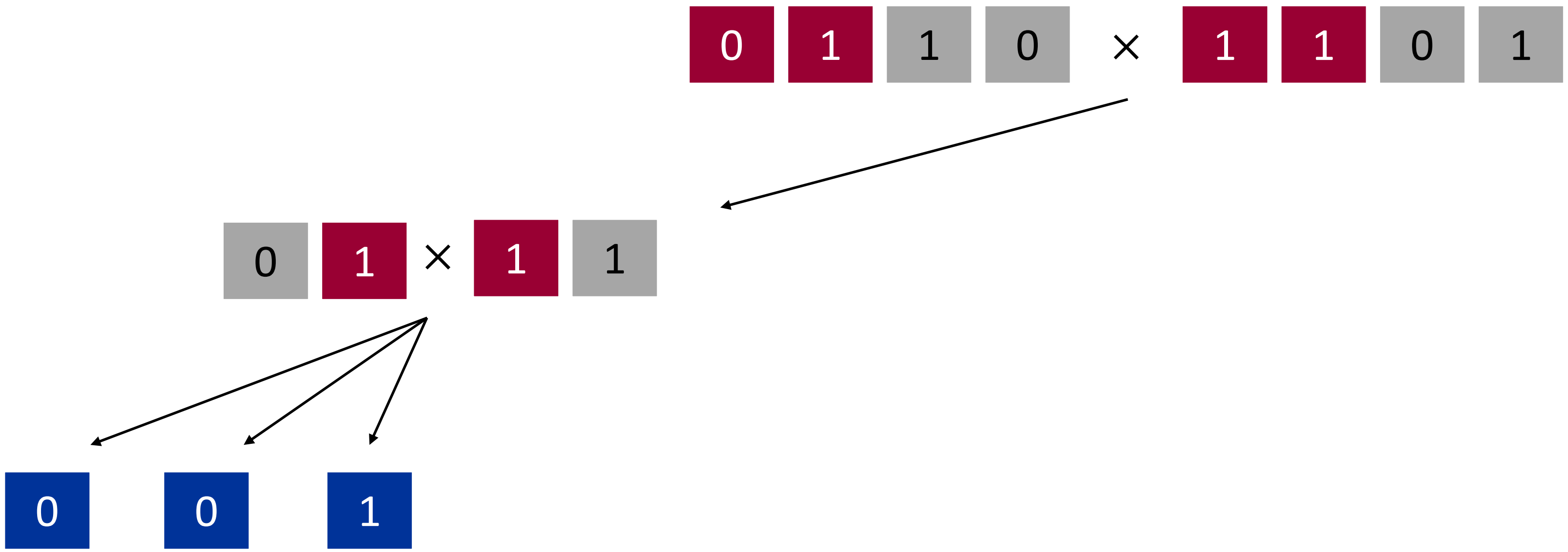

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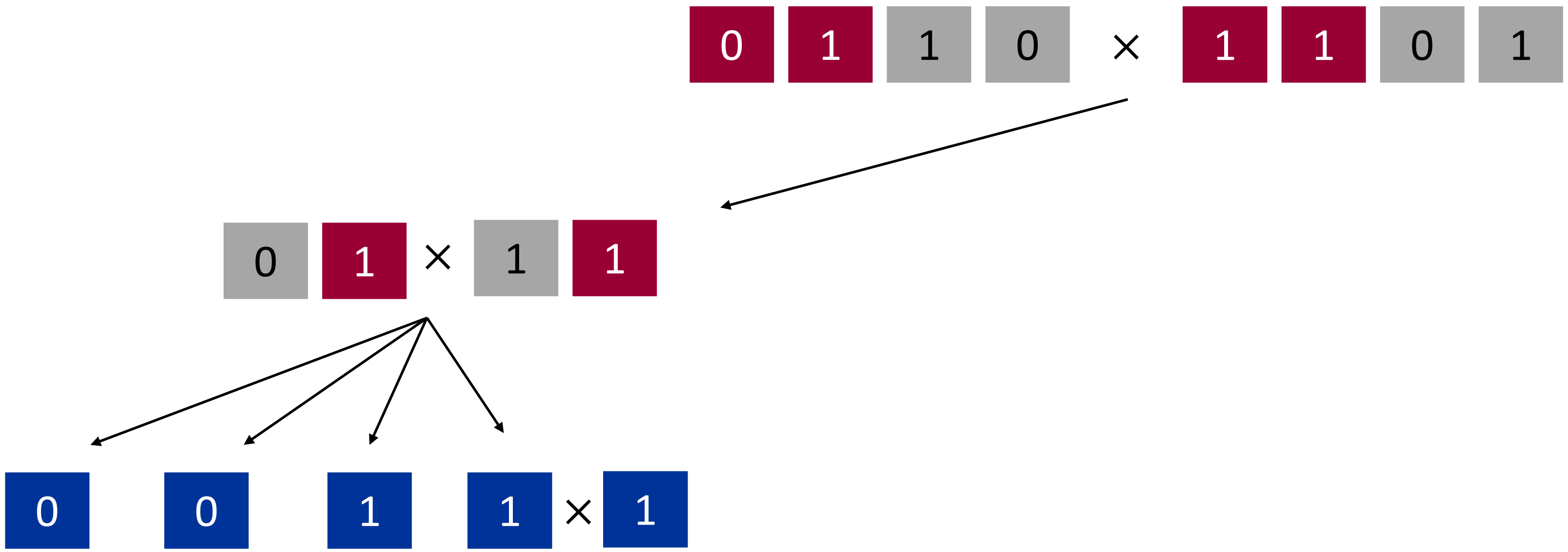

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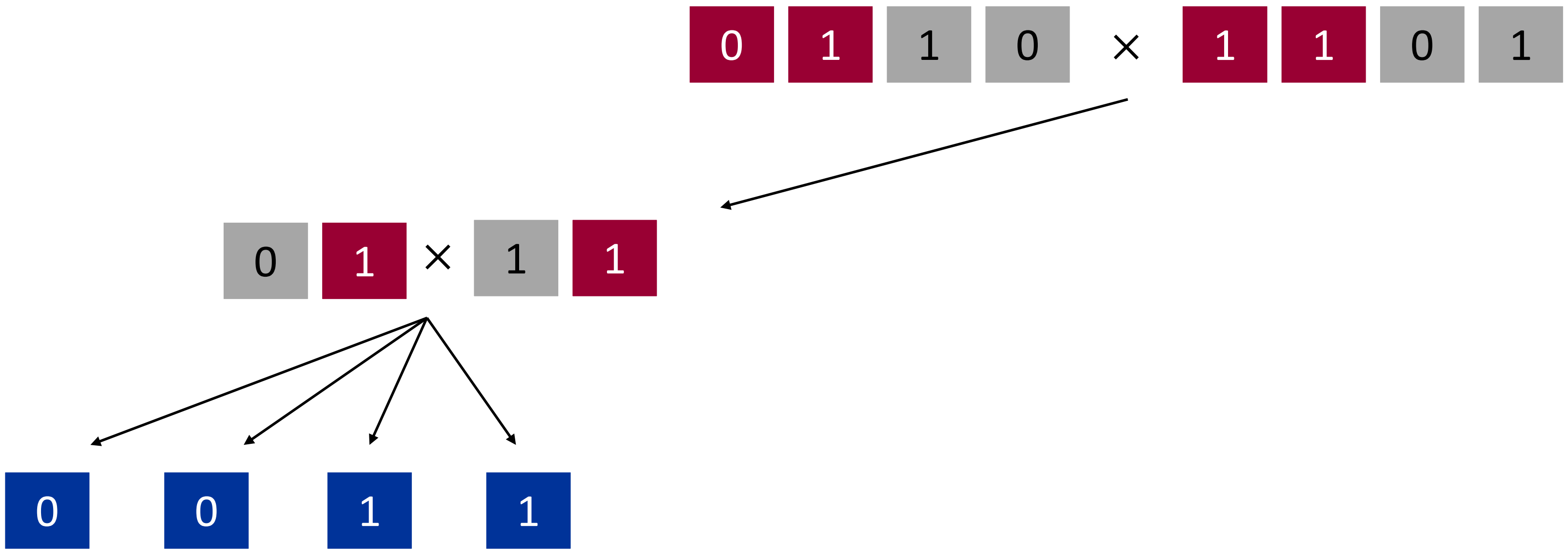

$$\boxed{(a\mathbf{2^2} + b)(\mathbf{c2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd}$$

First divide and conquer algorithm for multiplication : Example

$$0\,1\,1\,0 \times 1\,1\,0\,1$$

$$0 \times 2^2 + (0 + 1) \times 2^1 + 1 = 1\,1$$

0 0 1 1

$$\boxed{(a\mathbf{2}^{\mathbf{2}} + b)(\boldsymbol{c}\mathbf{2}^{\mathbf{2}} + \mathrm{d}) = ac \times \mathbf{2}^{\mathbf{4}} + [ad + bc] \times \mathbf{2}^{\mathbf{2}} + bd}$$

First divide and conquer algorithm for multiplication : Example

0 1 1 0 × 1 1 0 1

1 1

$$\boxed{\left(a\mathbf{2^2}+b\right)\left(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}\right)=ac\times\mathbf{2^4}+[ad+bc]\times\mathbf{2^2}+bd}$$

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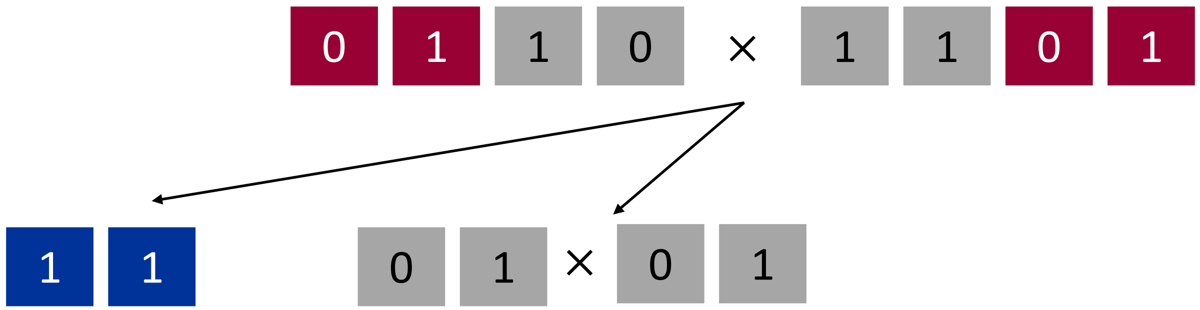

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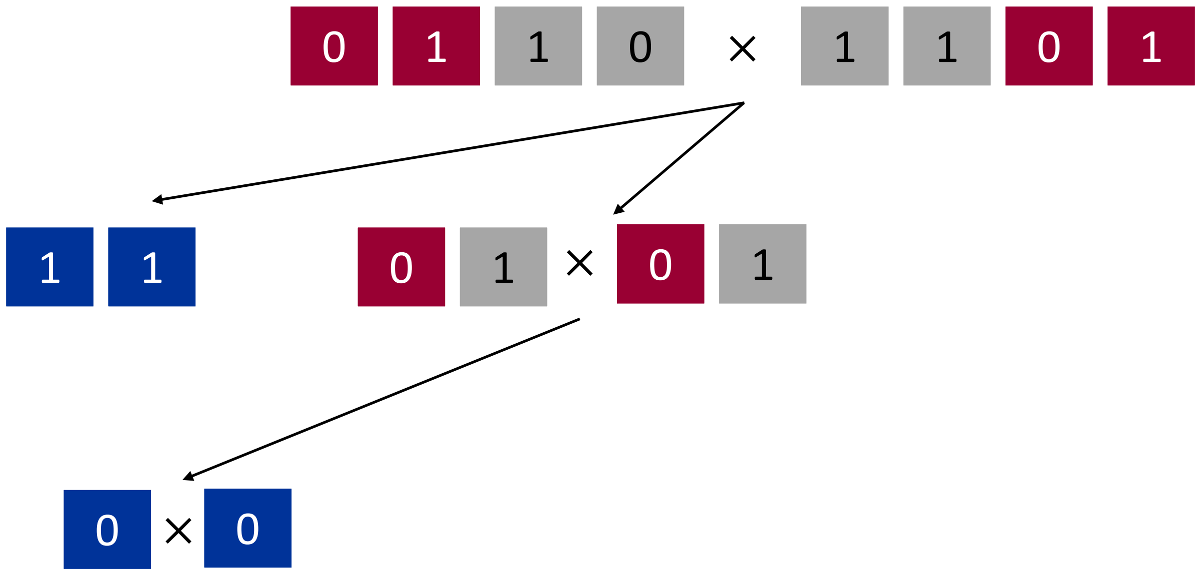

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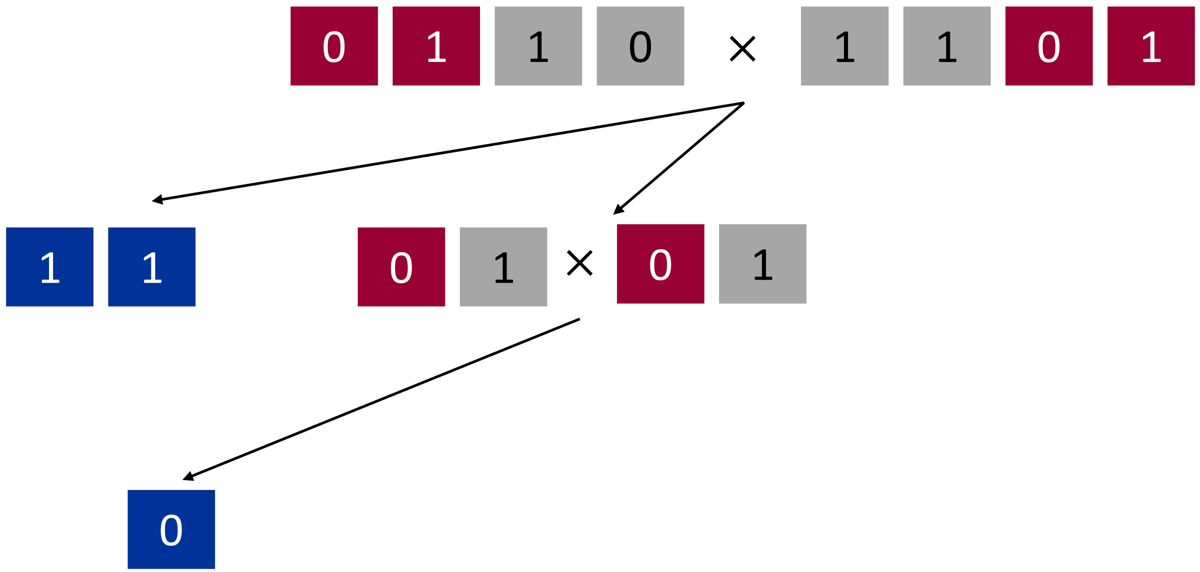

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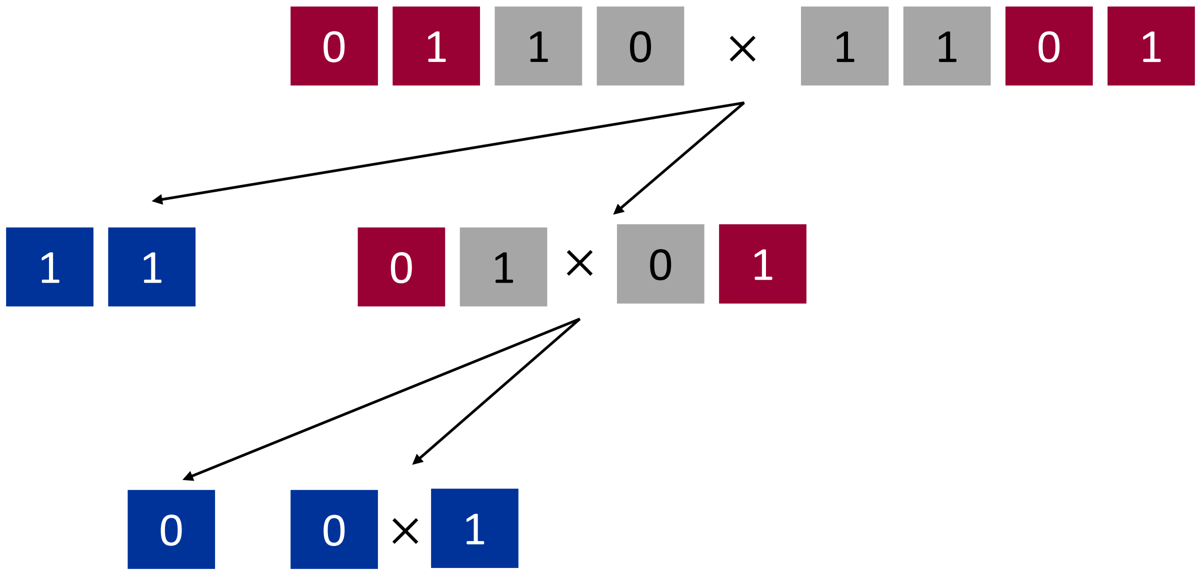

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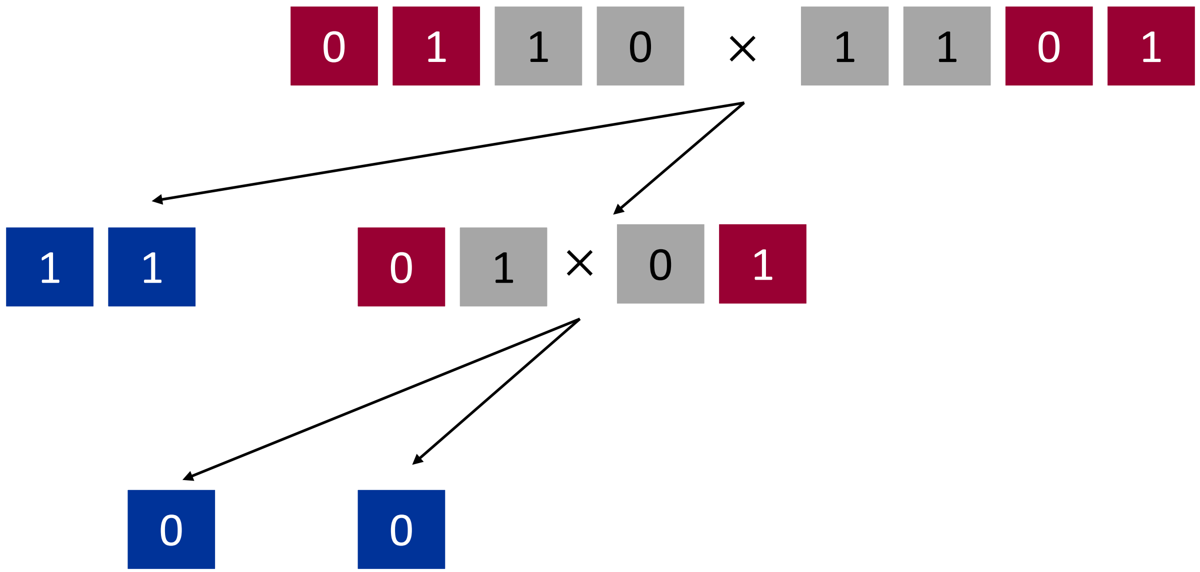

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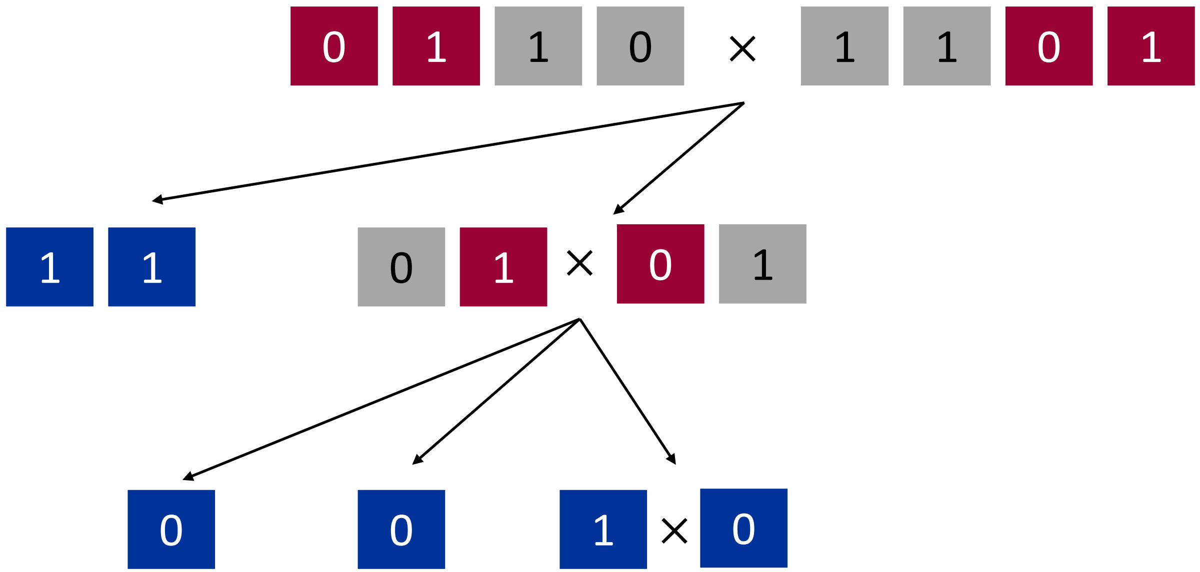

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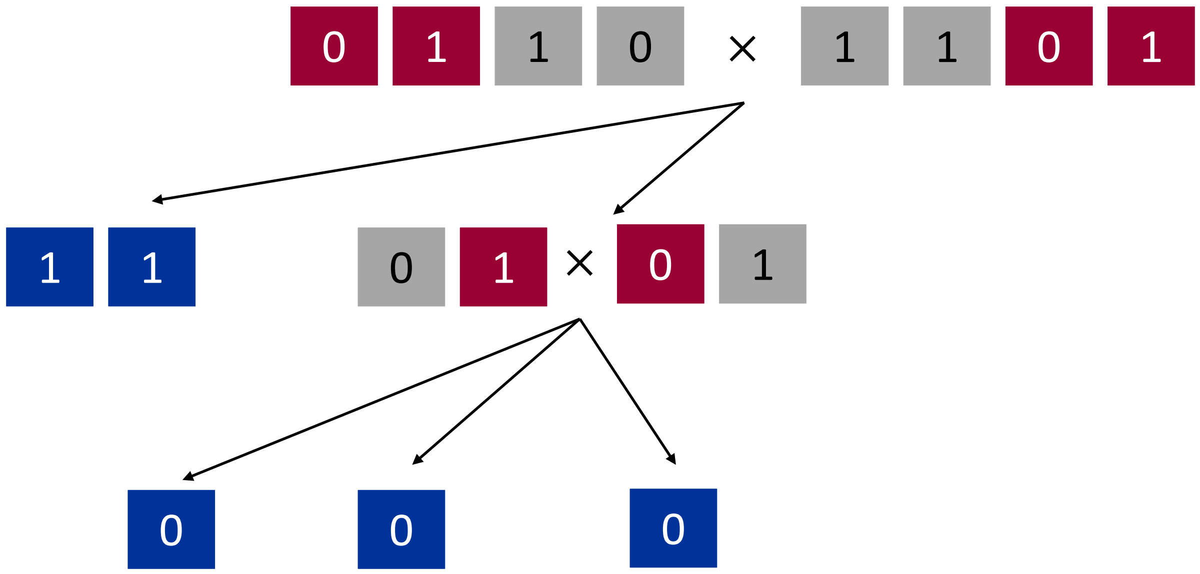

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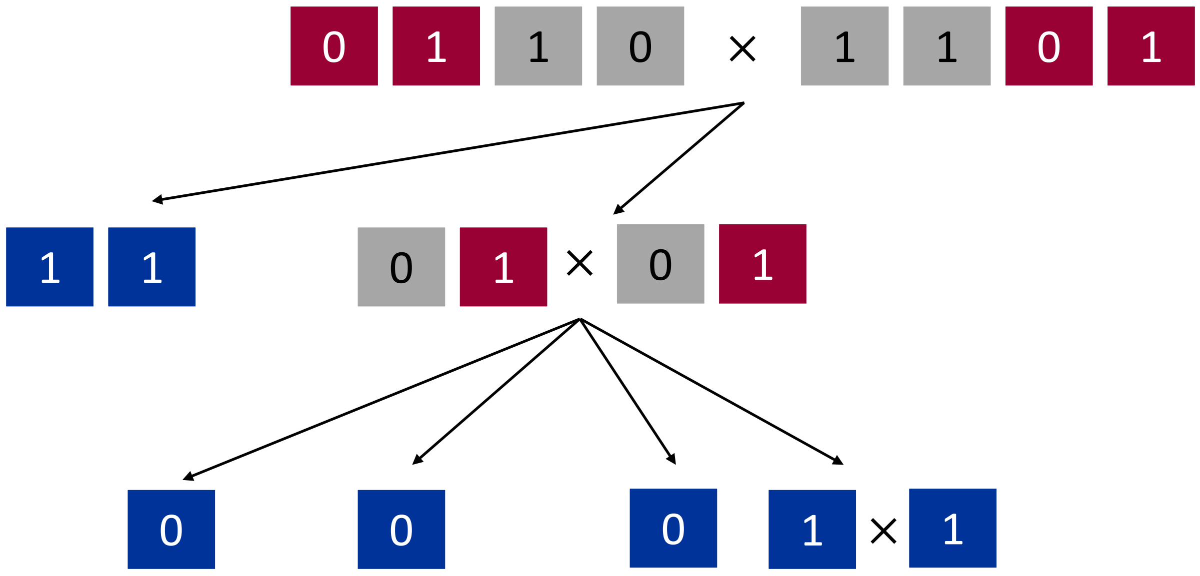

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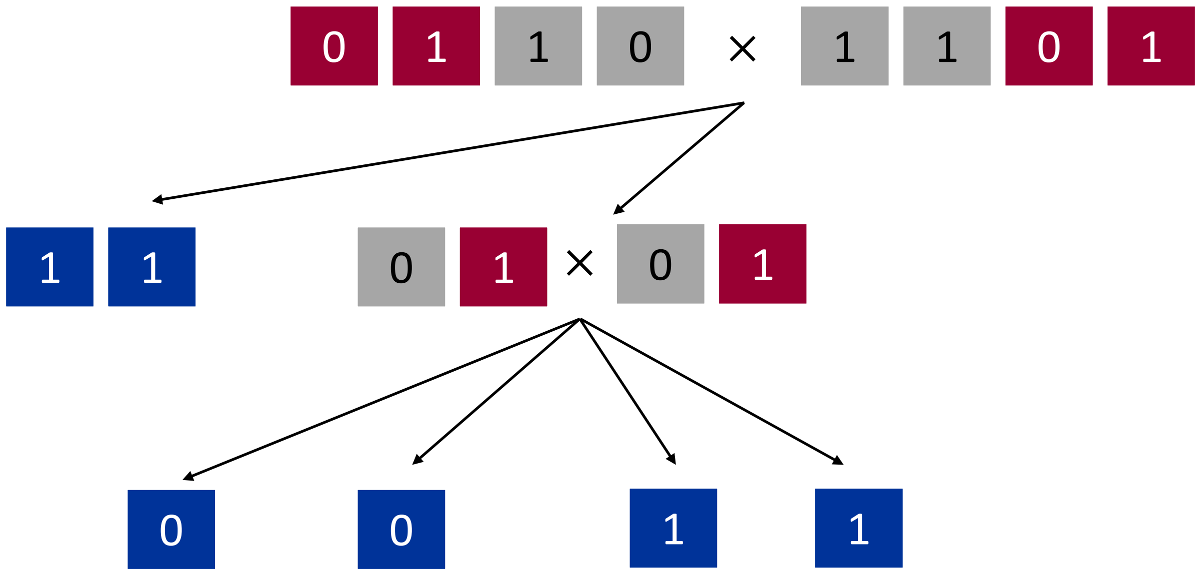

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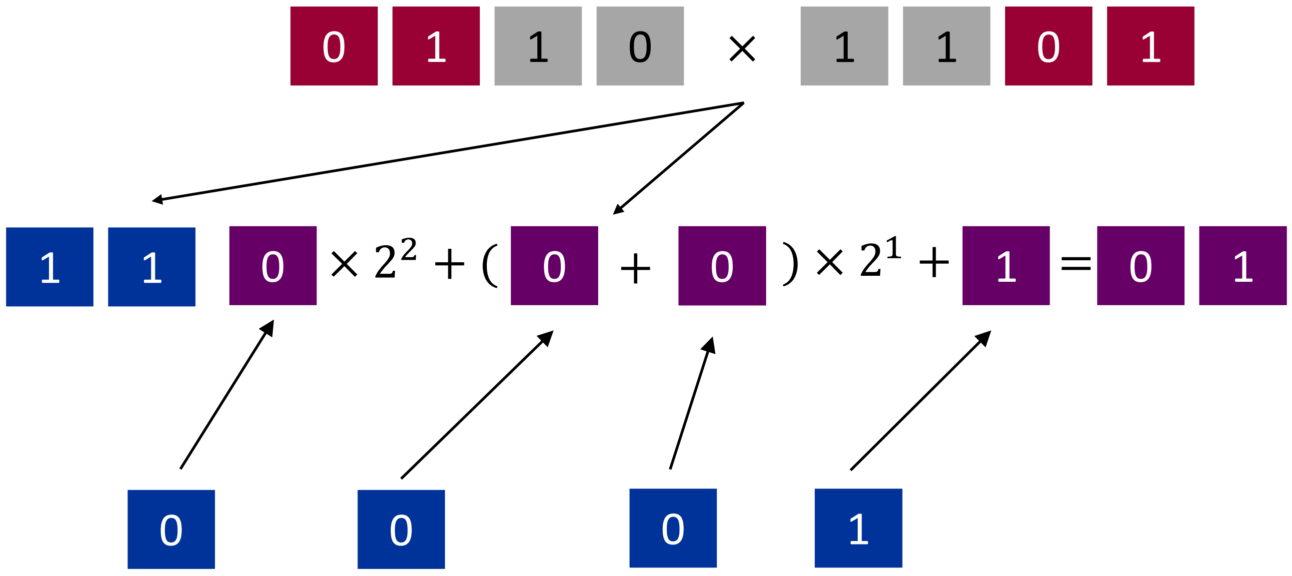

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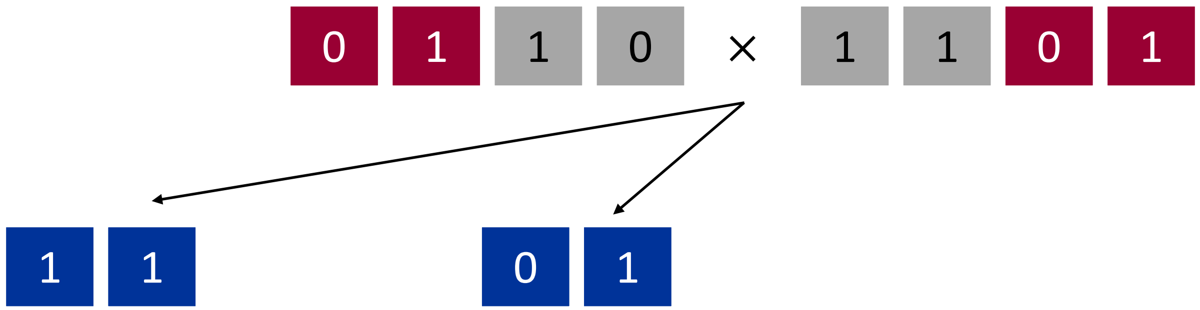

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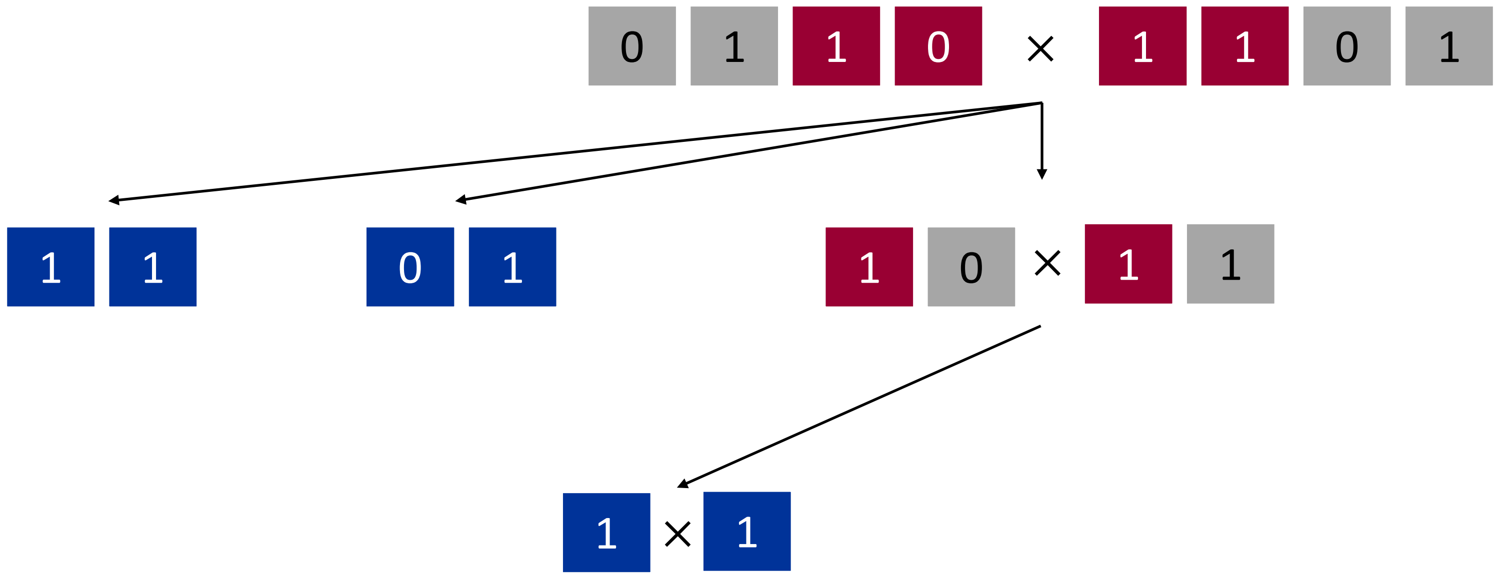

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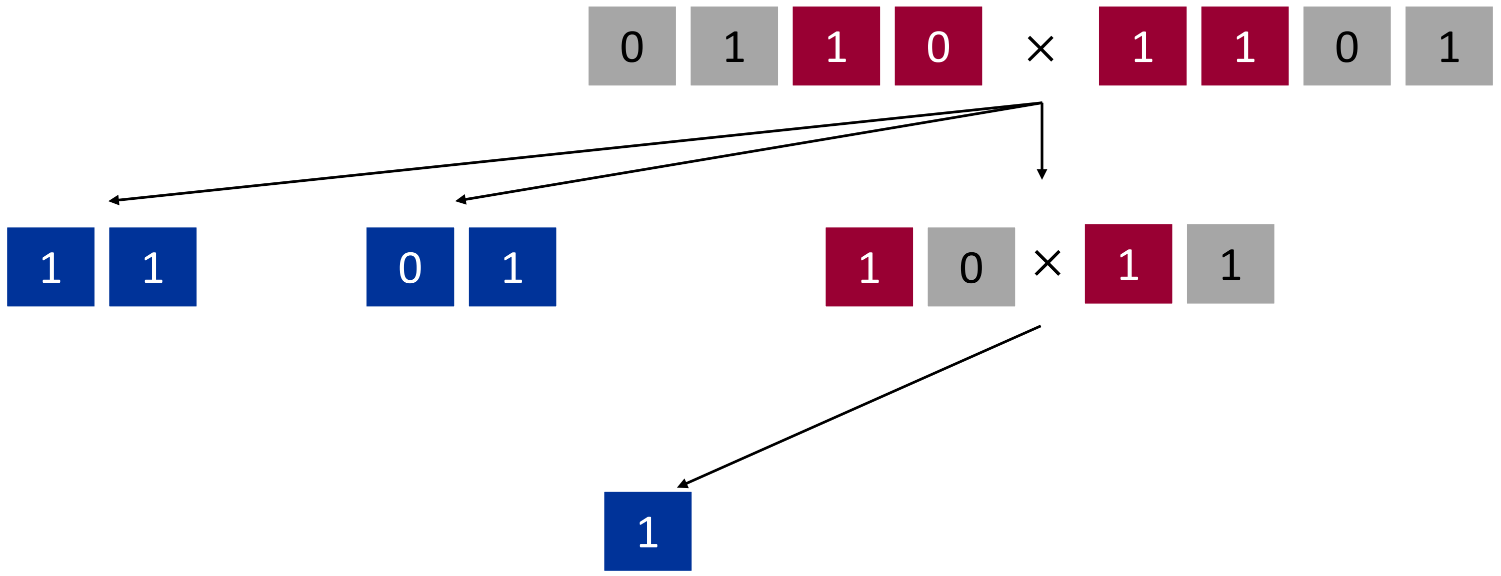

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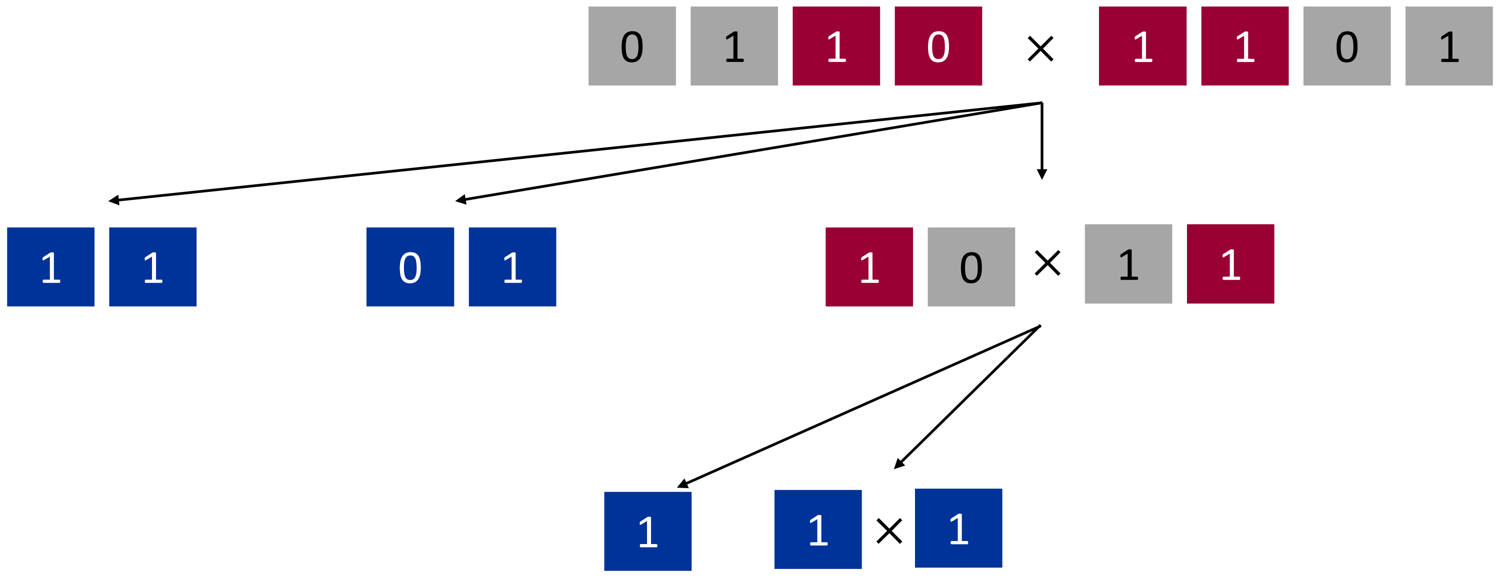

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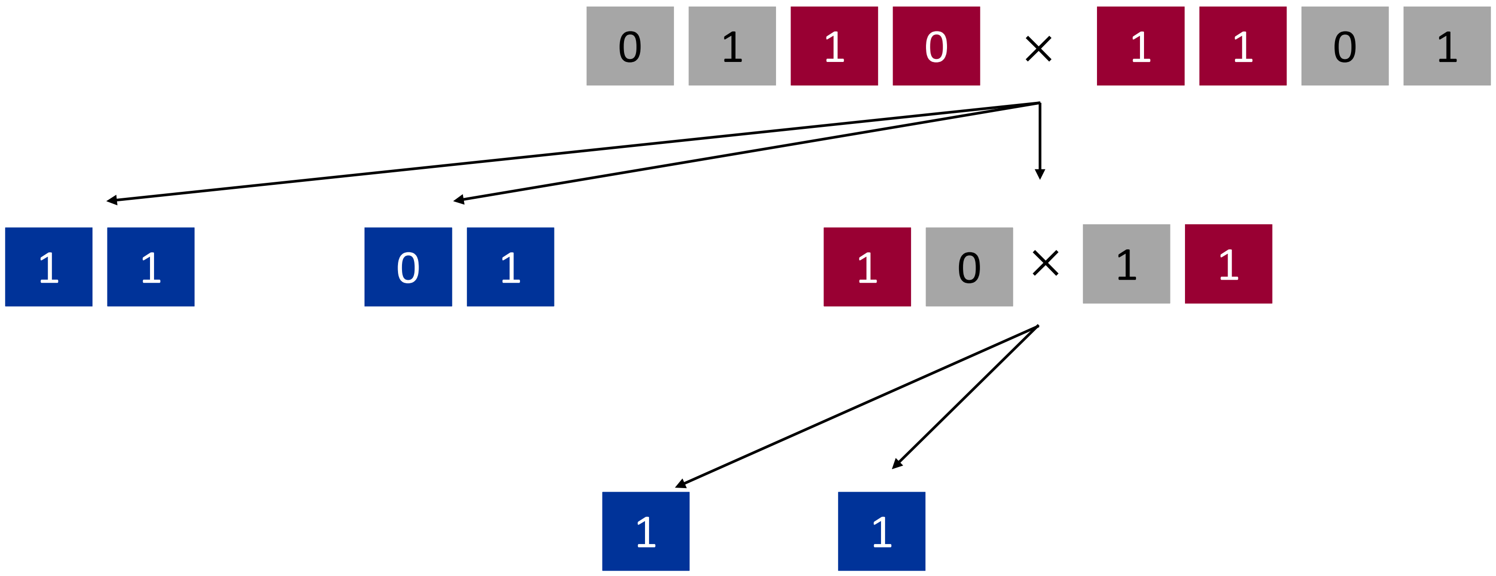

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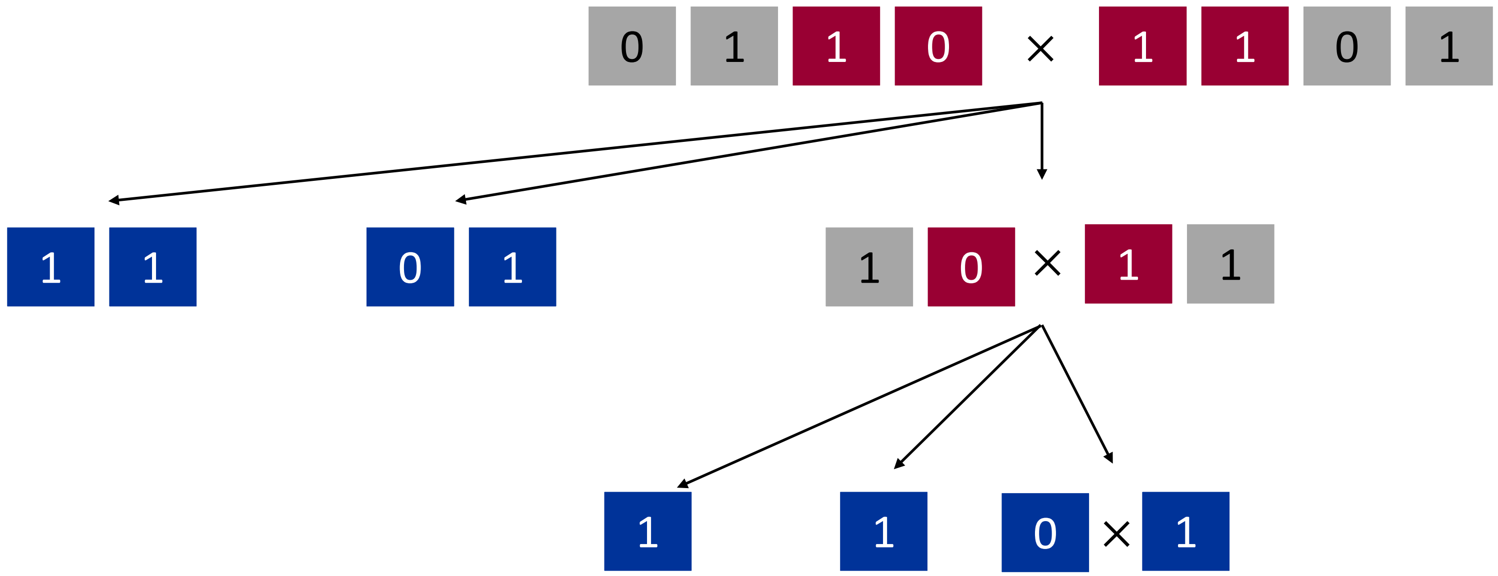

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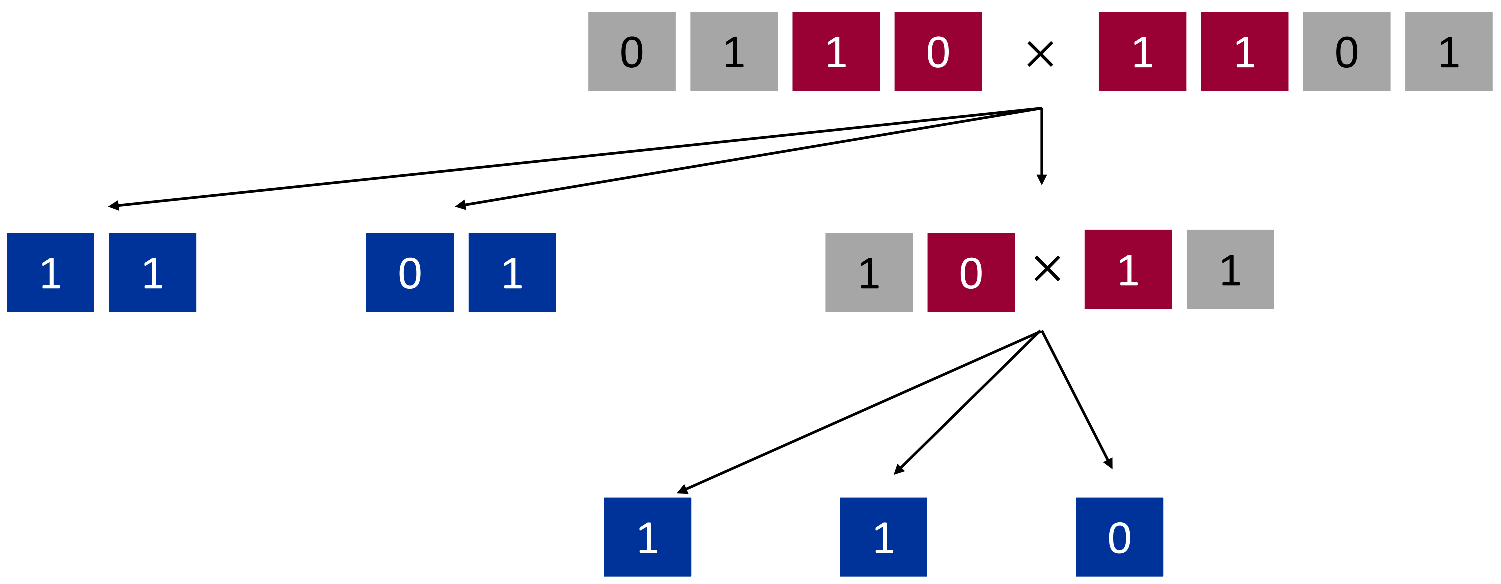
First divide and conquer algorithm for multiplication : Example



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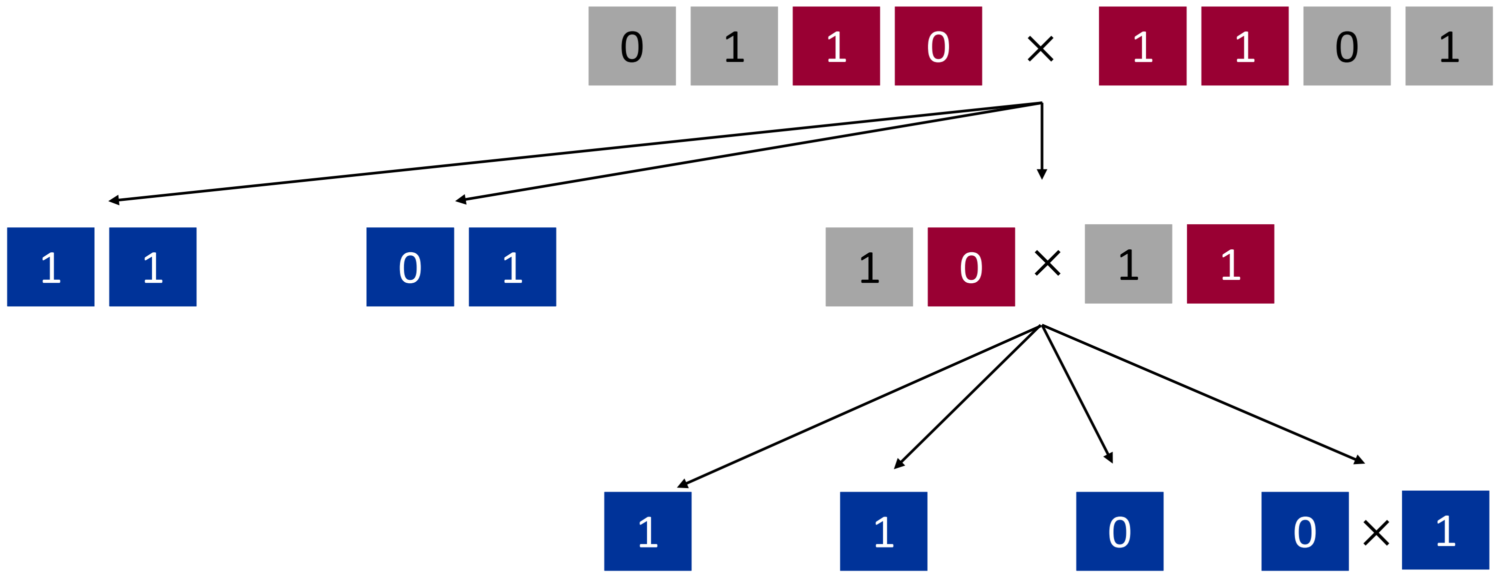

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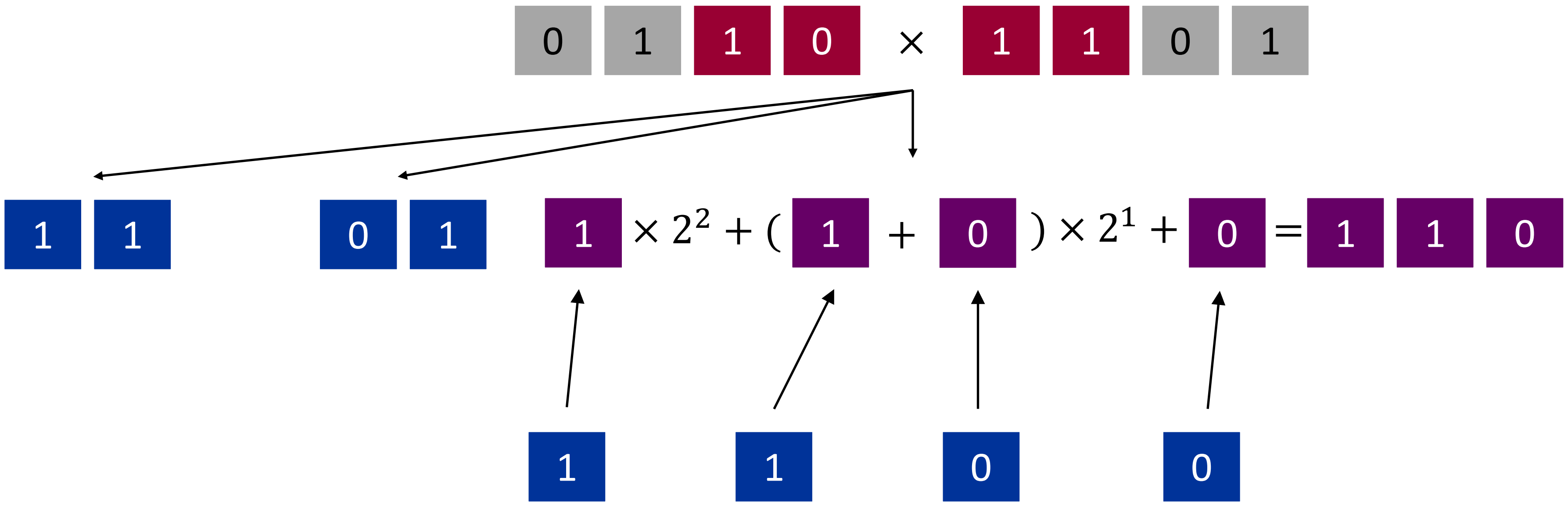

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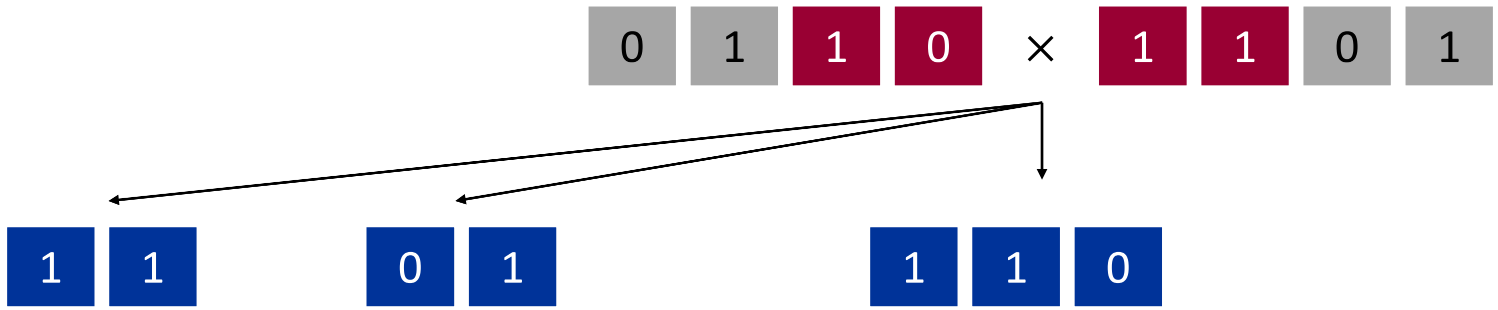
First divide and conquer algorithm for multiplication : Example



$$\left(a\mathbf{2}^{2}+b\right)\left(\boldsymbol{c}\mathbf{2}^{2}+\mathrm{d}\right)=ac\times\mathbf{2}^{4}+[ad+bc]\times\mathbf{2}^{2}+bd$$

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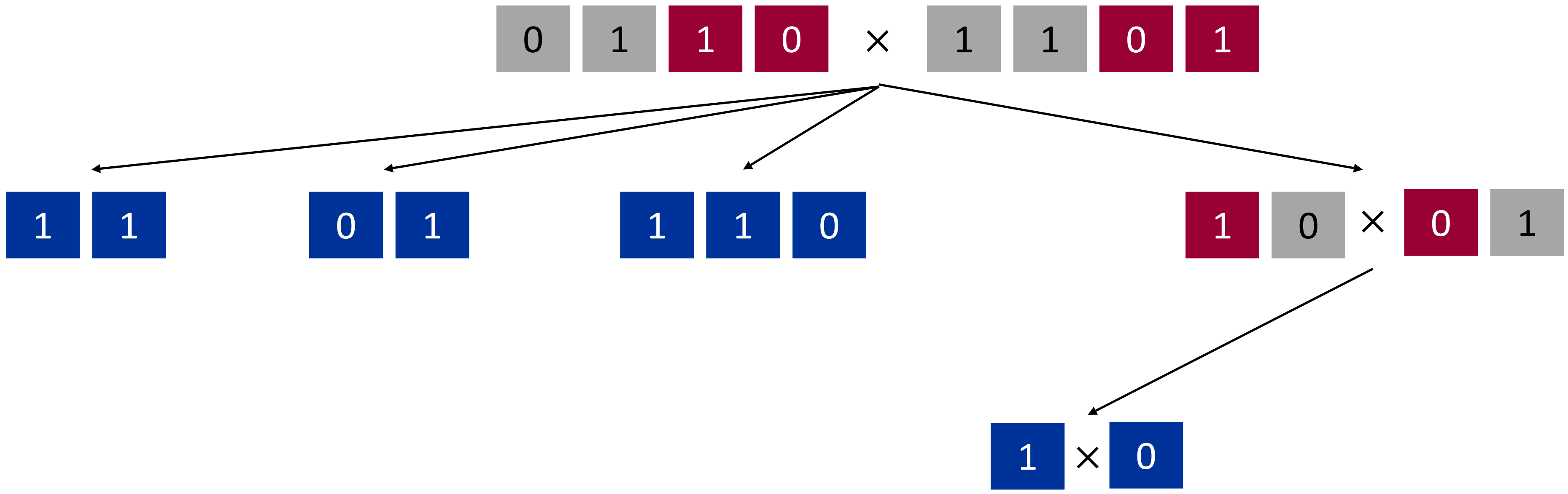

$$\boxed{\left(a\mathbf{2}^{\mathbf{2}} + b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}} + \mathrm{d}\right) = ac \times \mathbf{2}^{\mathbf{4}} + [ad + bc] \times \mathbf{2}^{\mathbf{2}} + bd}$$

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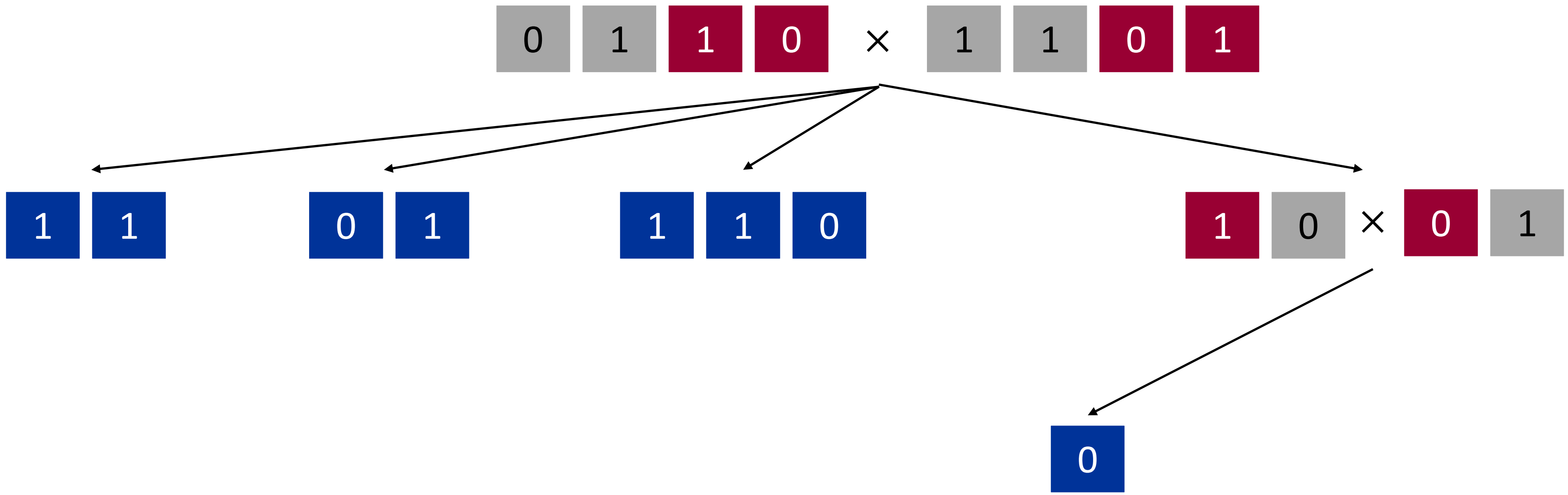

$$\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[ad+bc]\times\mathbf{2}^{\mathbf{2}}+bd$$

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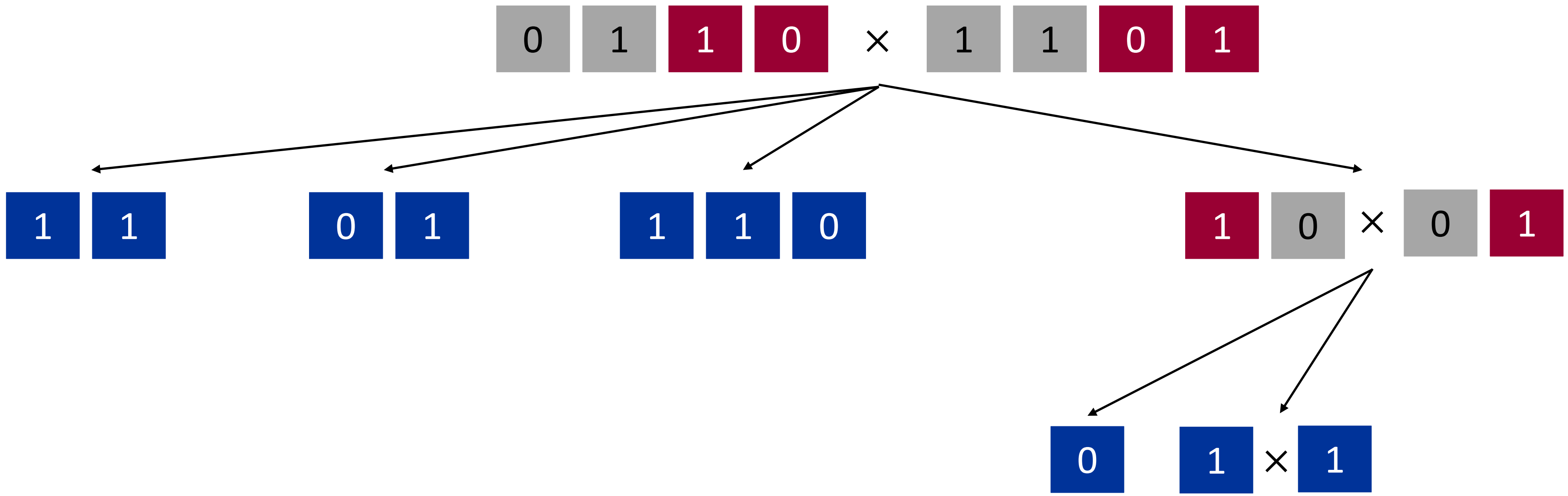

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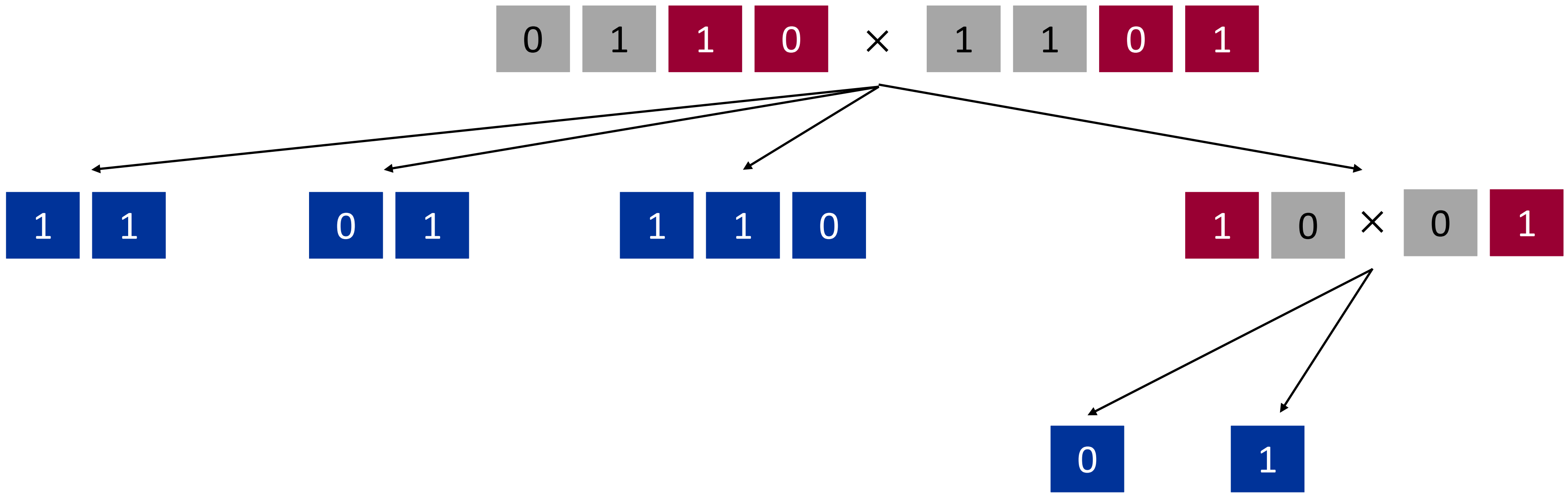

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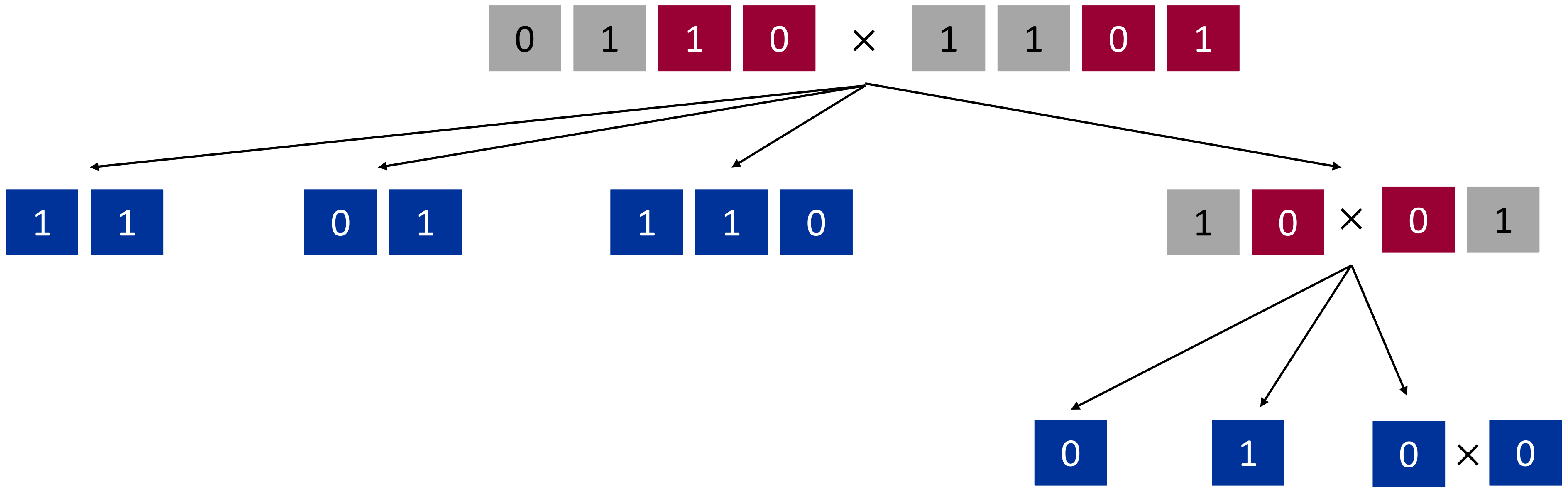

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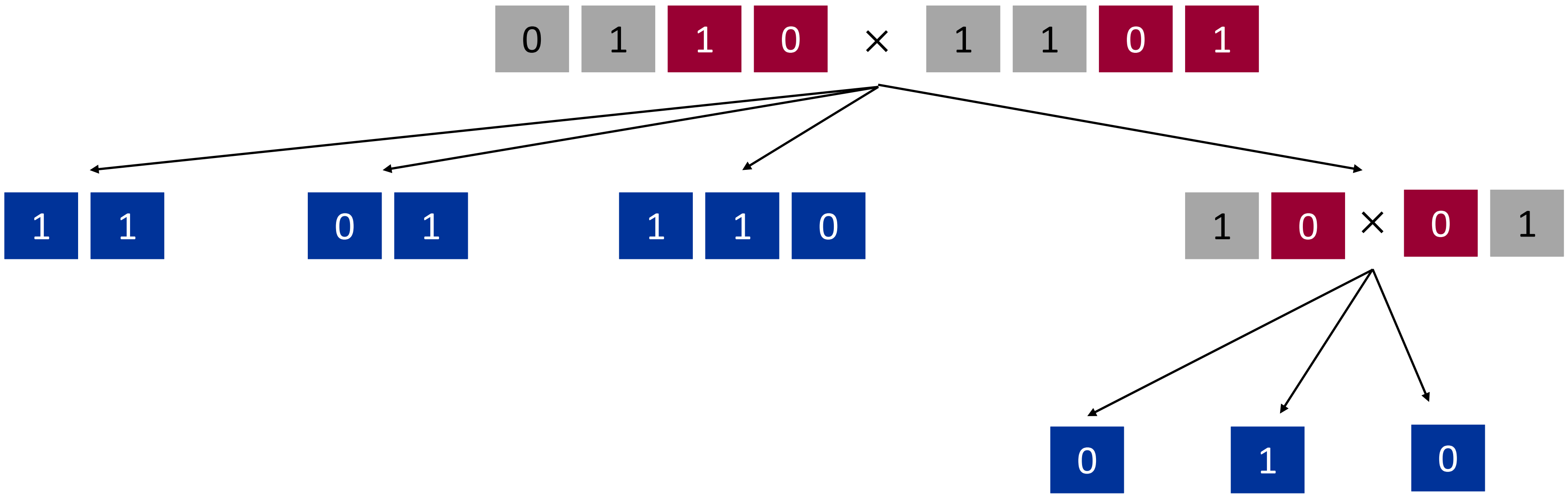

$$\boxed{\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[ad+bc]\times\mathbf{2}^{\mathbf{2}}+bd}$$

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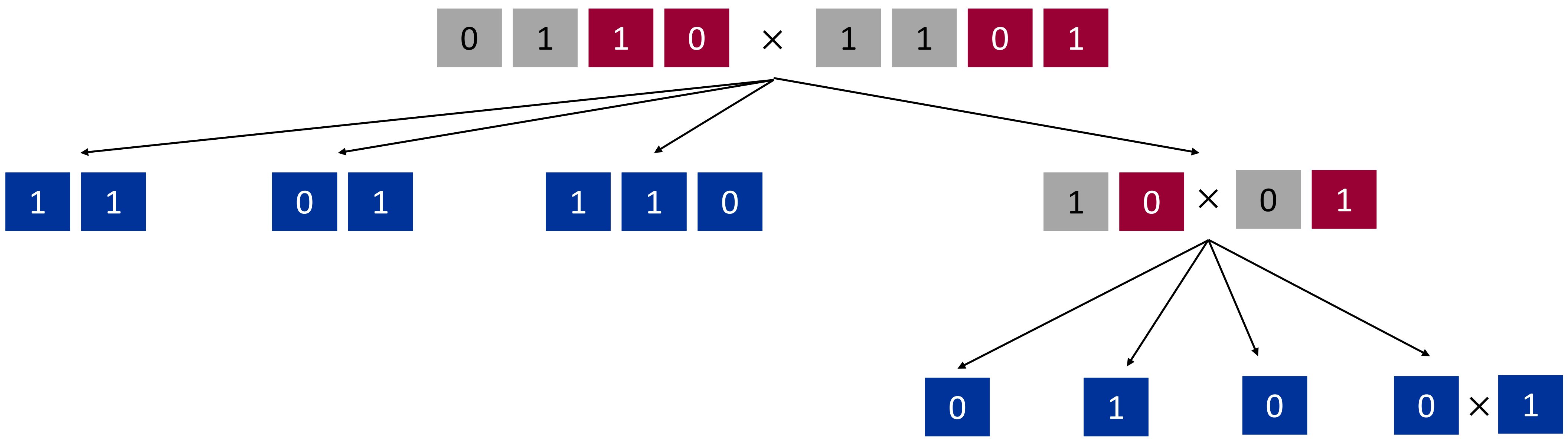

$$\left(a\mathbf{2^2} + b\right)\left(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}\right) = ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd$$

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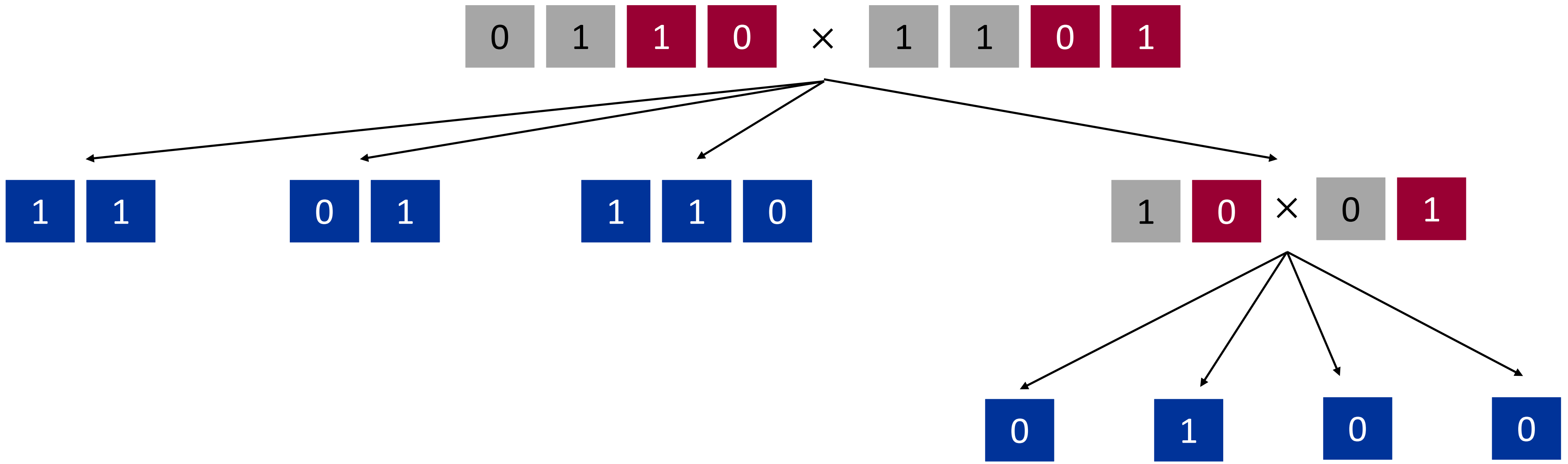

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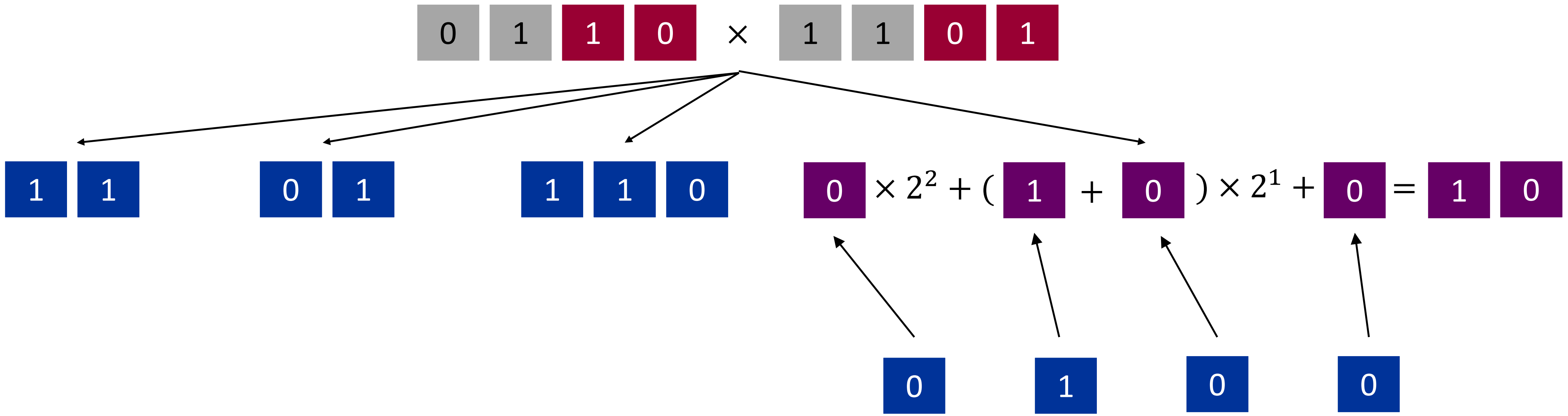

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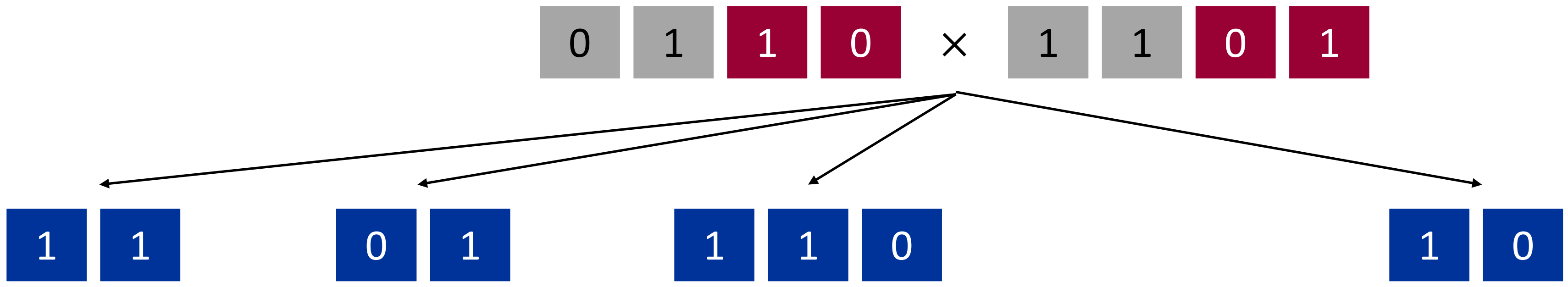

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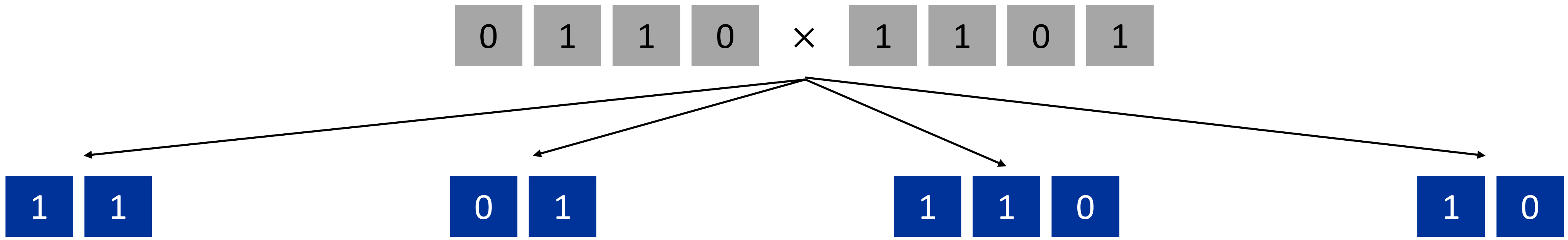

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First divide and conquer algorithm for multiplication : Example

$$11 \times 2^4 + (01 + 110) \times 2^2 + 10 = 1001110$$

11	01	110	10

$$\boxed{(a\mathbf{2^2} + b)(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd}$$

First divide and conquer algorithm for multiplication : Example

0	1	1	0	×	1	1	0	1	=	1	0	0	1	1	1	0

$$\boxed{\left(a\mathbf{2^2} + b\right)\left(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}\right) = ac \times \mathbf{2^4} + [ad + bc] \times \mathbf{2^2} + bd}$$

Outline

- Quick D&C Integer Multiplication Review
- High level Example for 4X4 D&C simple integer multiplication
- High level Example for 4X4 D&C Karatsuba integer multiplication
- Full worked example of 4X4 simple integer multiplication
- Full worked example of 4X4 Karatsuba integer multiplication

Karatsuba Multiplication: Example

0	1	1	0	×	1	1	0	1

$$\boxed{(a\mathbf{2^2} + b)(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [(a+b) \times (c+d) - ac - bd] \times \mathbf{2^2} + bd}$$

Karatsuba Multiplication: Example

0 1 1 0 × 1 1 0 1

0 1 × 1 1

$$\left(a\mathbf{2}^{\mathbf{2}} + b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}} + \mathrm{d}\right) = ac \times \mathbf{2}^{\mathbf{4}} + [(a+b)\times(c+d) - ac - bd] \times \mathbf{2}^{\mathbf{2}} + bd$$

Karatsuba Multiplication: Example

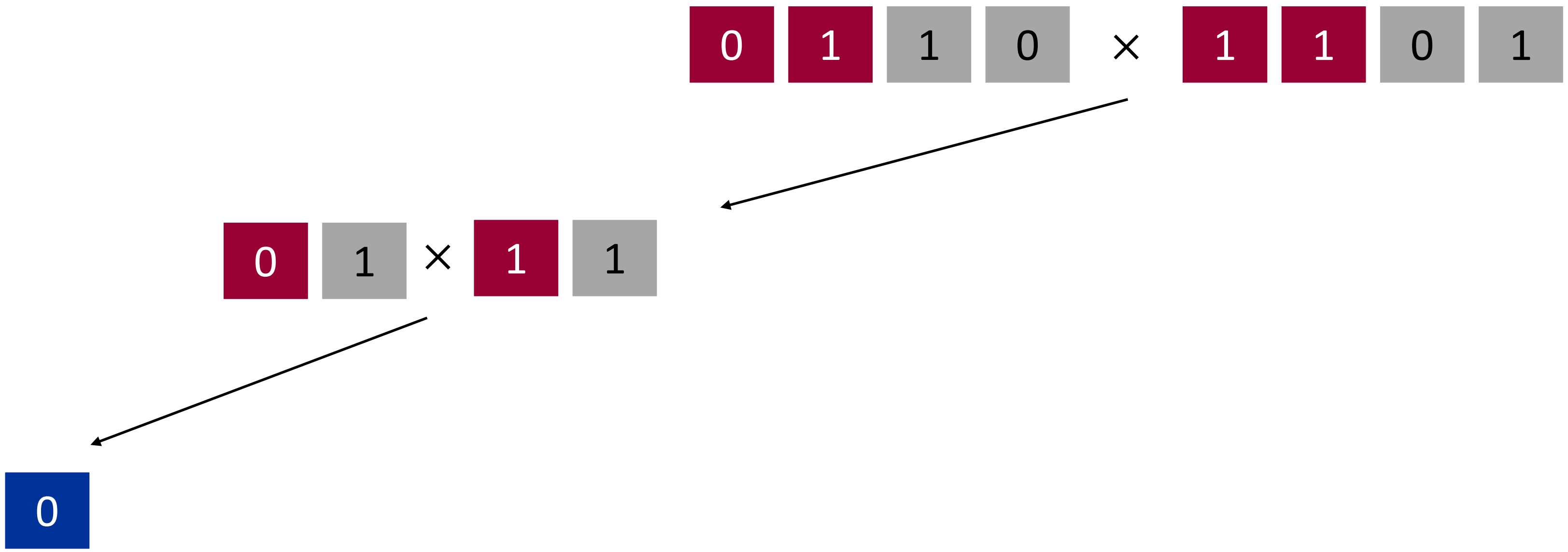
0 1 1 0 × 1 1 0 1

0 1 × 1 1

0 × 1

$$\left(a\mathbf{2^2} + b\right)\left(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}\right) = ac \times \mathbf{2^4} + [(a+b) \times (c+d) - ac - bd] \times \mathbf{2^2} + bd$$

Karatsuba Multiplication: Example



$$\boxed{\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2}^{\mathbf{2}}+bd}$$

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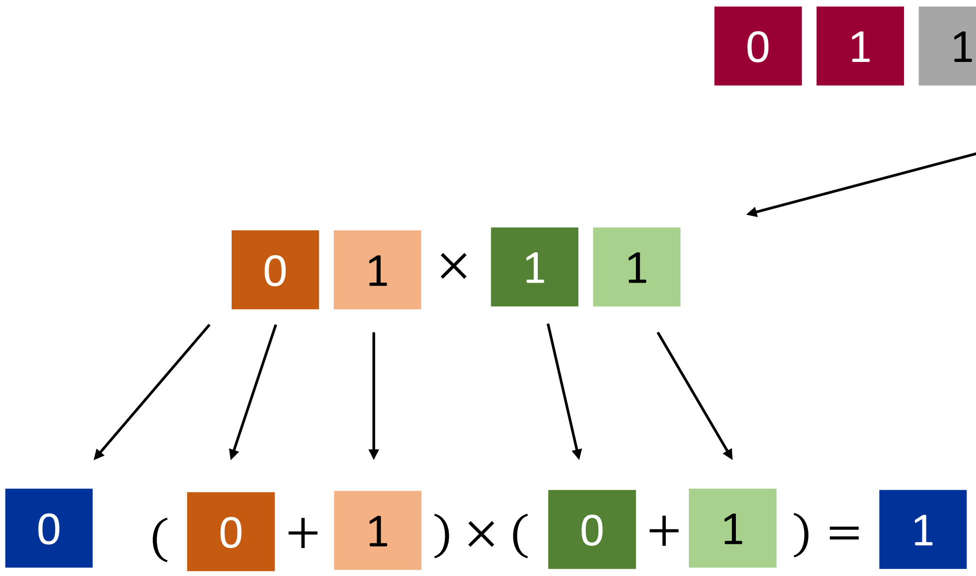

$$(a\mathbf{2}^{\mathbf{2}} + b)(\boldsymbol{c}\mathbf{2}^{\mathbf{2}} + \mathrm{d}) = ac \times \mathbf{2}^{\mathbf{4}} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2}^{\mathbf{2}} + bd$$

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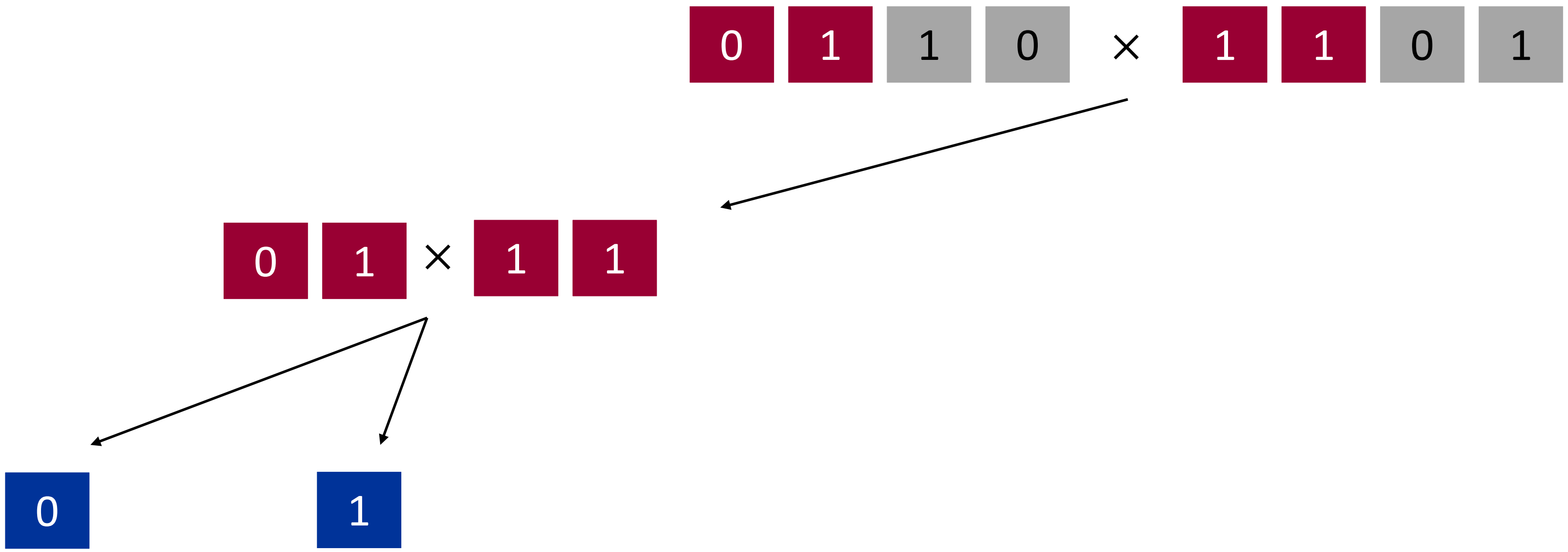

$$\boxed{(a\mathbf{2}^{\mathbf{2}} + b)(\boldsymbol{c}\mathbf{2}^{\mathbf{2}} + \mathrm{d}) = ac \times \mathbf{2}^{\mathbf{4}} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2}^{\mathbf{2}} + bd}$$

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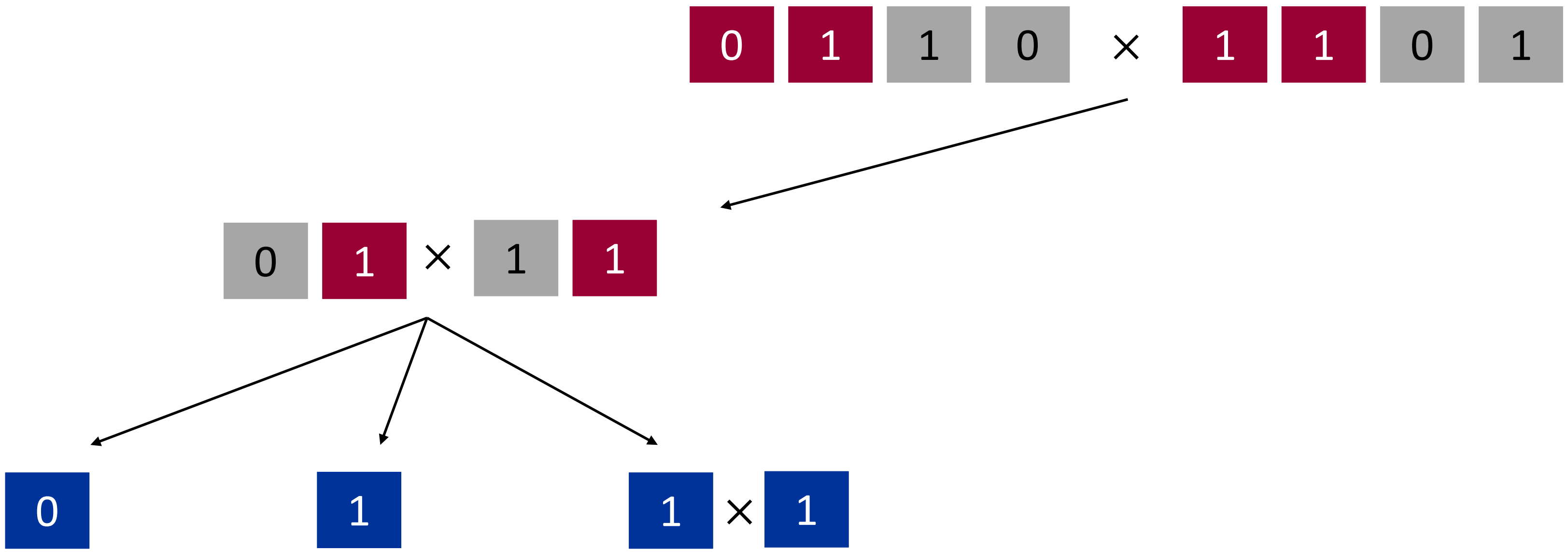
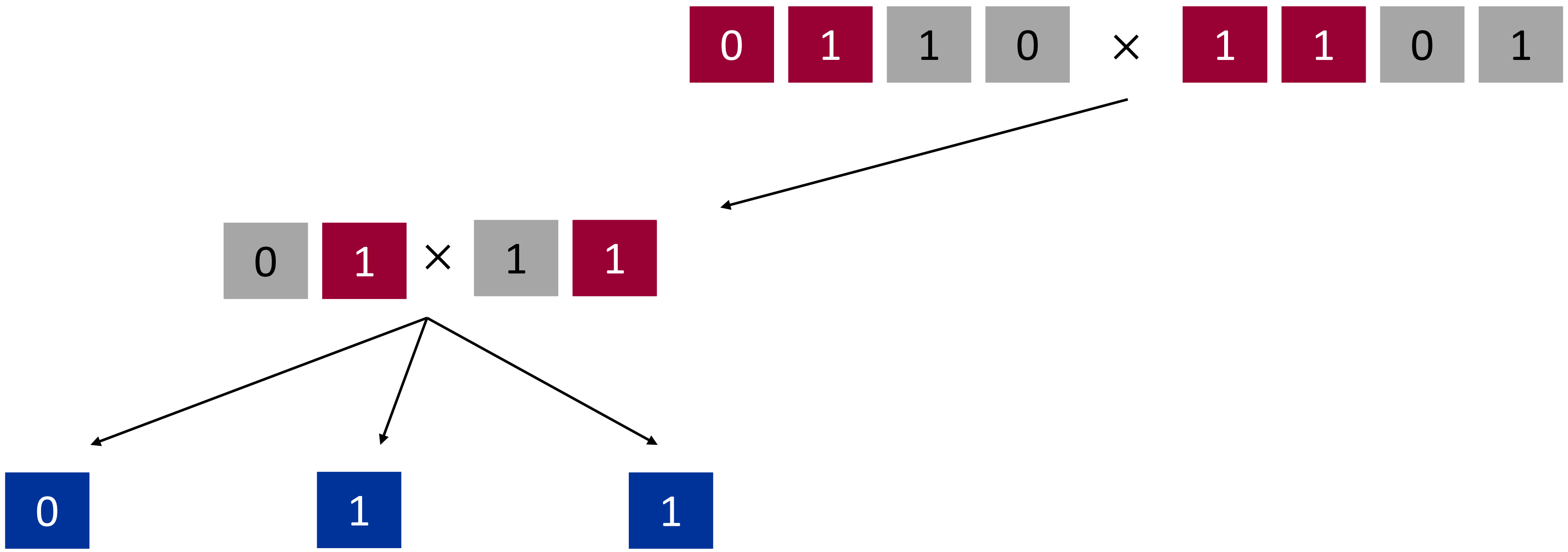


$$\left(a\mathbf{2^2}+b\right)\left(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}\right)=ac\times\mathbf{2^4}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2^2}+bd$$

Karatsuba Multiplication: Example



$$(a\mathbf{2^2}+b)(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}) = ac\times\mathbf{2^4}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2^2}+bd$$

Karatsuba Multiplication: Example

0 1 1 0 × 1 1 0 1

$0 \times 2^2 + (1 - 0 - 1) \times 2^1 + 1 = 1\ 1$

0 1 1

$$\left(a\mathbf{2^2} + b\right)\left(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}\right) = ac \times \mathbf{2^4} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2^2} + bd$$

Karatsuba Multiplication: Example

0 1 1 0 × 1 1 0 1

1 1

$$\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2}^{\mathbf{2}}+bd$$

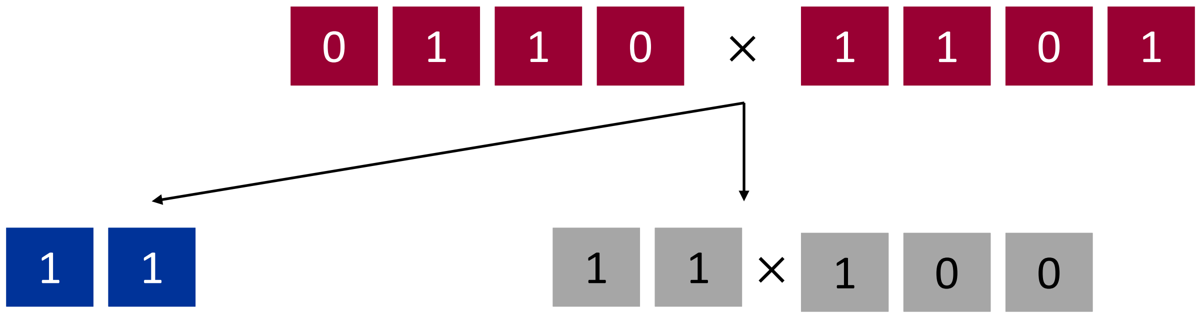
Karatsuba Multiplication: Example

0 1 1 0 × 1 1 0 1

1 1 (0 1 + 1 0) × (1 1 + 0 1)

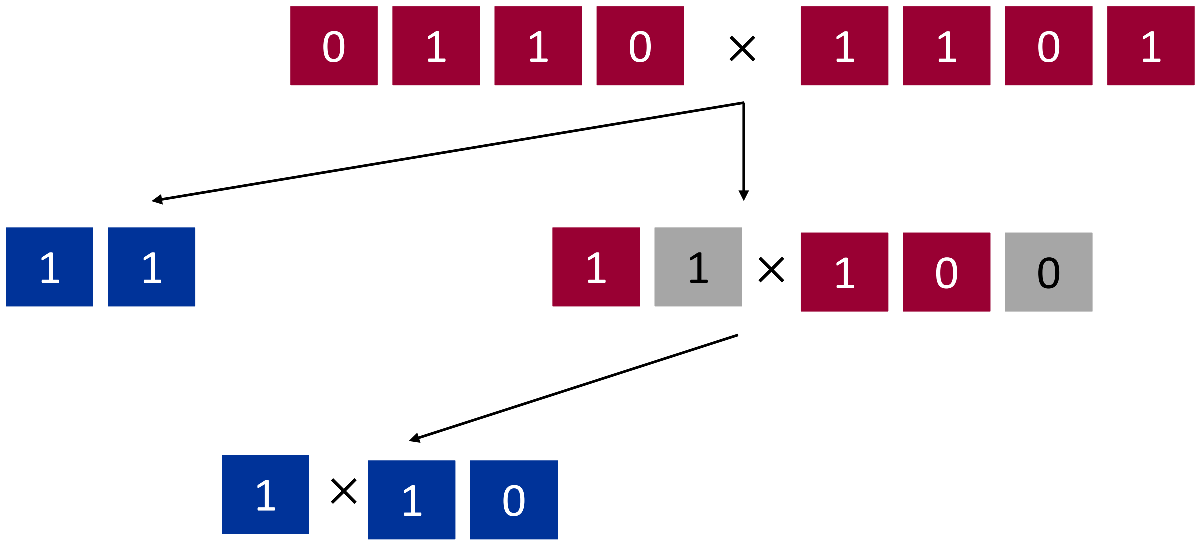
$$\boxed{\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2}^{\mathbf{2}}+bd}$$

Karatsuba Multiplication: Example



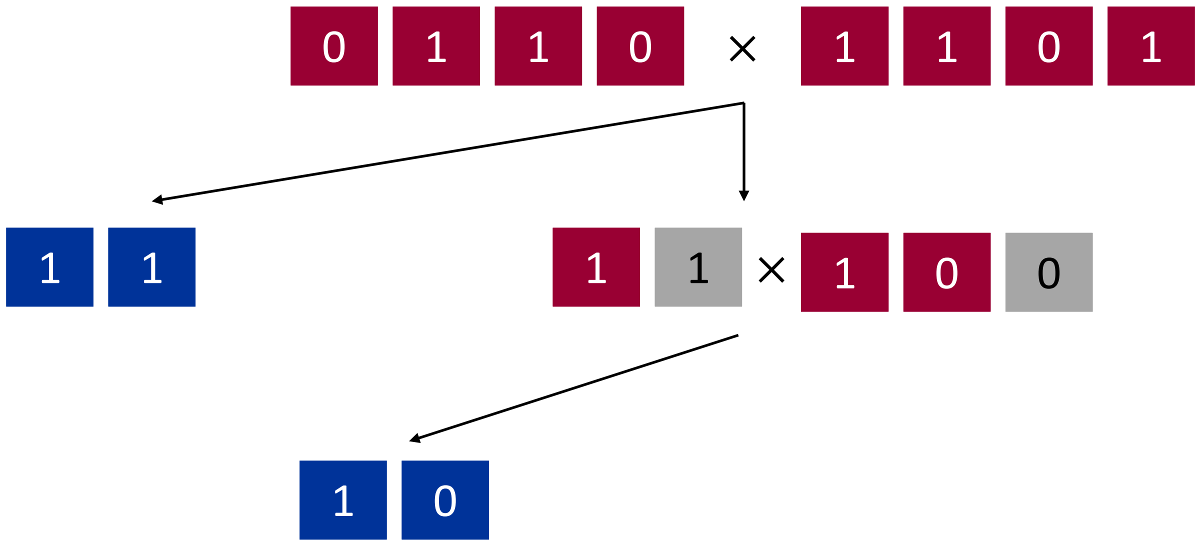
$$\boxed{\left(a\mathbf{2}^{\mathbf{2}}+b\right)\left(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}\right)=ac\times\mathbf{2}^{\mathbf{4}}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2}^{\mathbf{2}}+bd}$$

Karatsuba Multiplication: Example



$$\boxed{(a\mathbf{2}^{\mathbf{2}}+b)(\boldsymbol{c}\mathbf{2}^{\mathbf{2}}+\mathrm{d}) = ac\times \mathbf{2}^{\mathbf{4}} + [(a+b)\times(c+d)-ac-bd]\times\mathbf{2}^{\mathbf{2}}+bd}$$

Karatsuba Multiplication: Example



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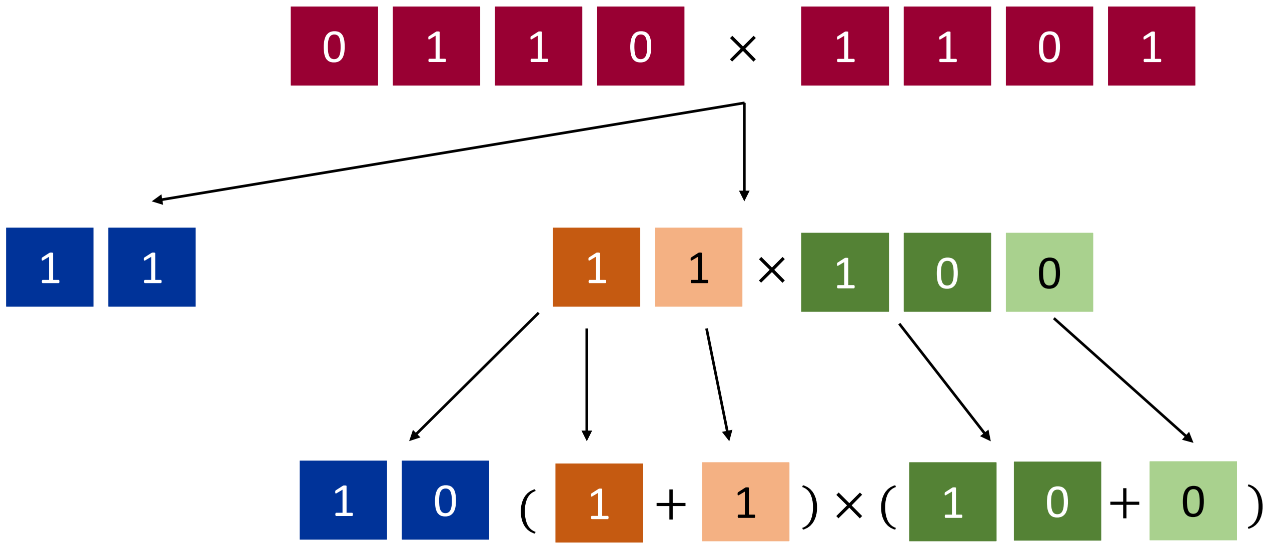

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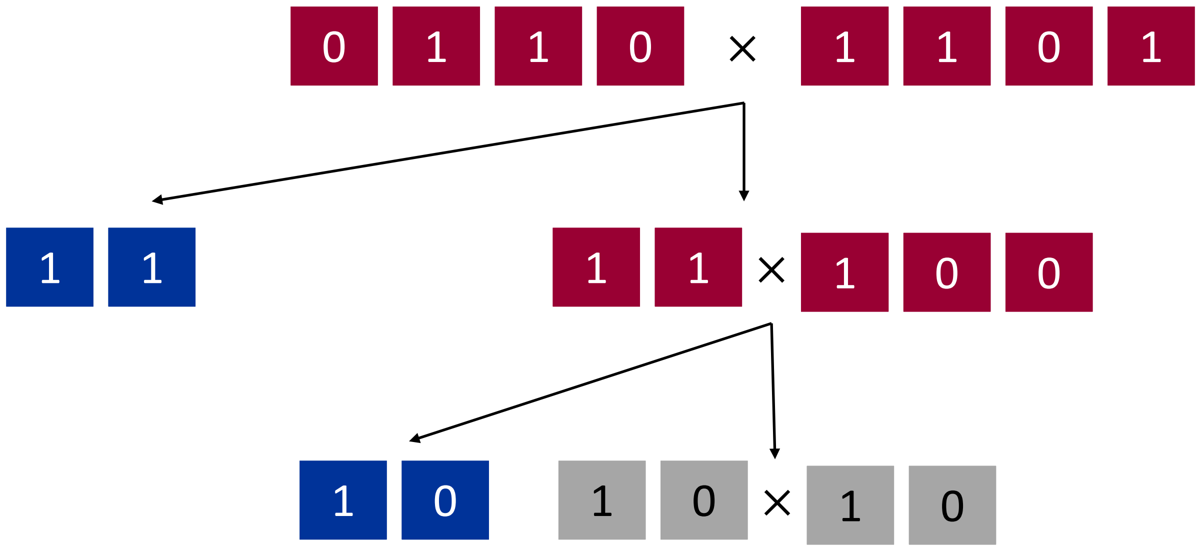
Karatsuba Multiplication: Example



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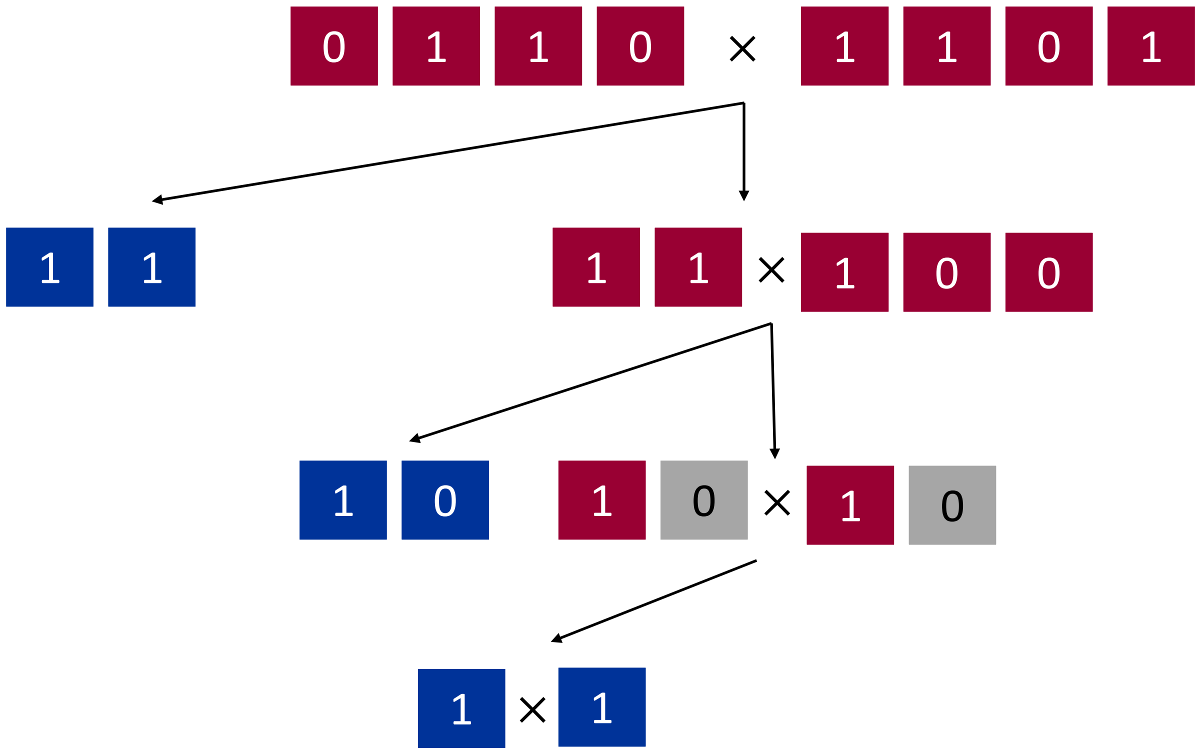

$$\boxed{(a\mathbf{2^2}+b)(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}) = ac\times\mathbf{2^4}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2^2}+bd}$$

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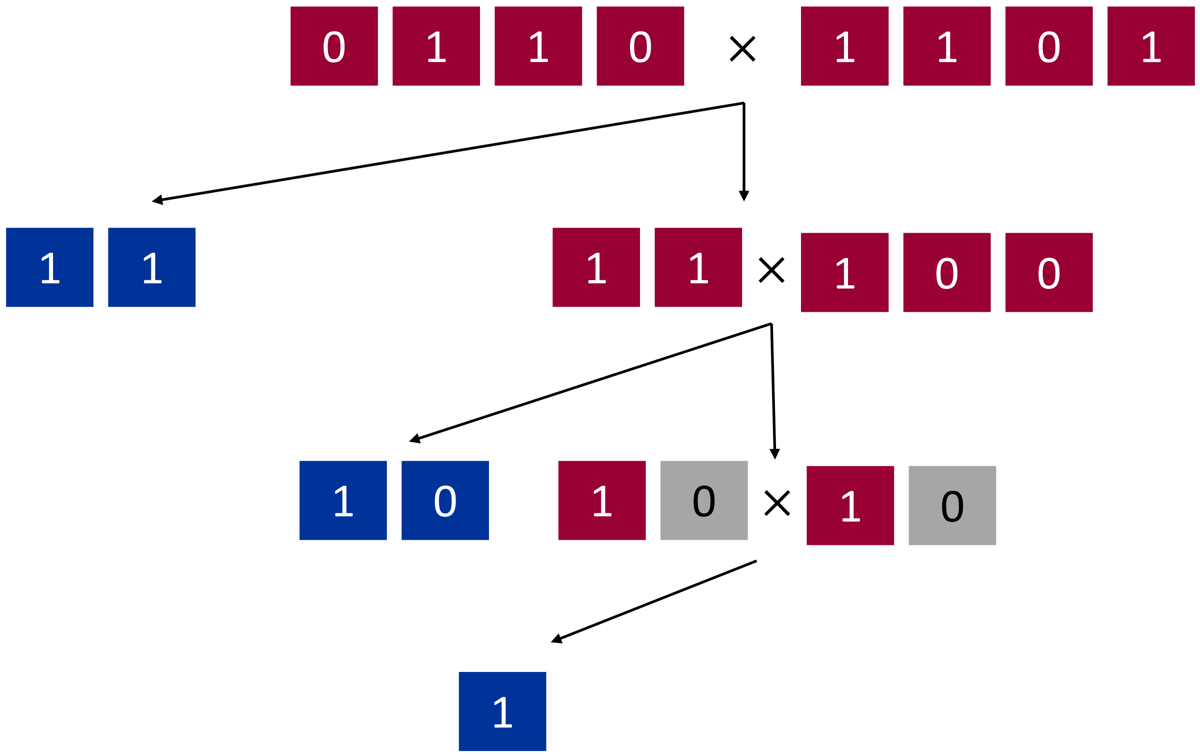

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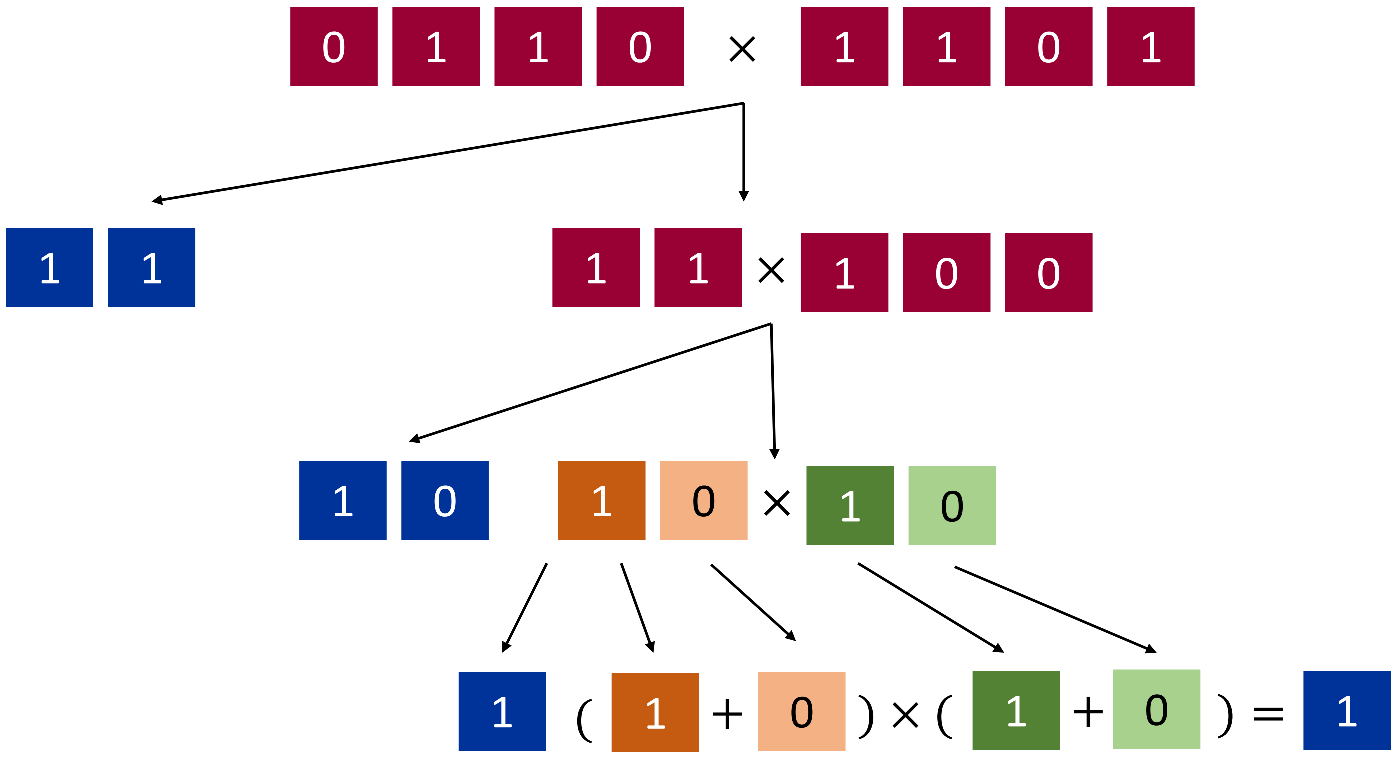
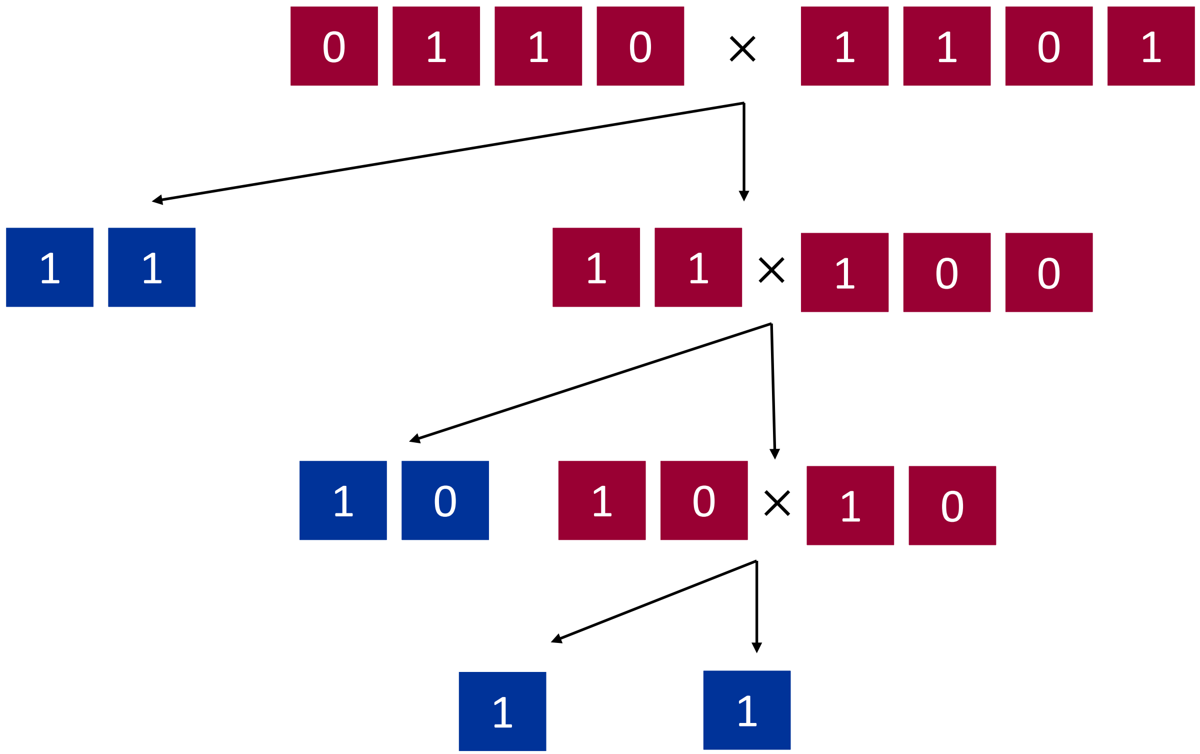


$$\left(a\mathbf{2^2} + b\right)\left(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}\right) = ac \times \mathbf{2^4} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2^2} + bd$$

Karatsuba Multiplication: Example



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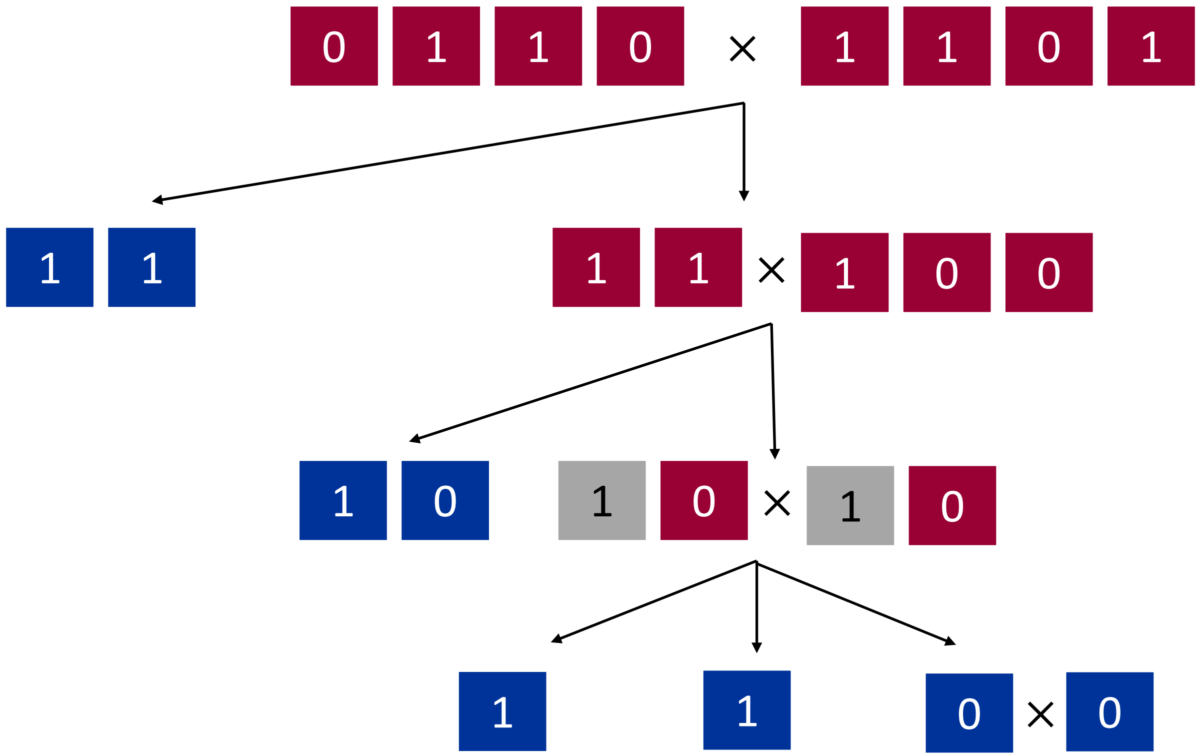

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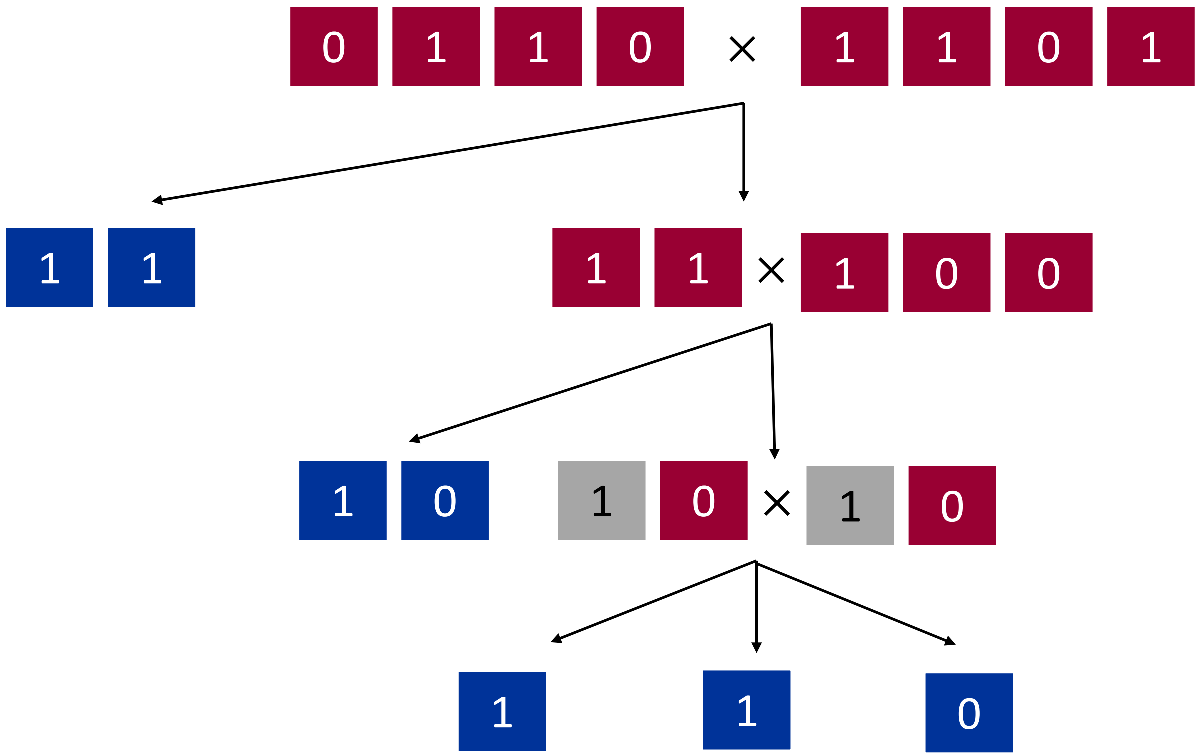

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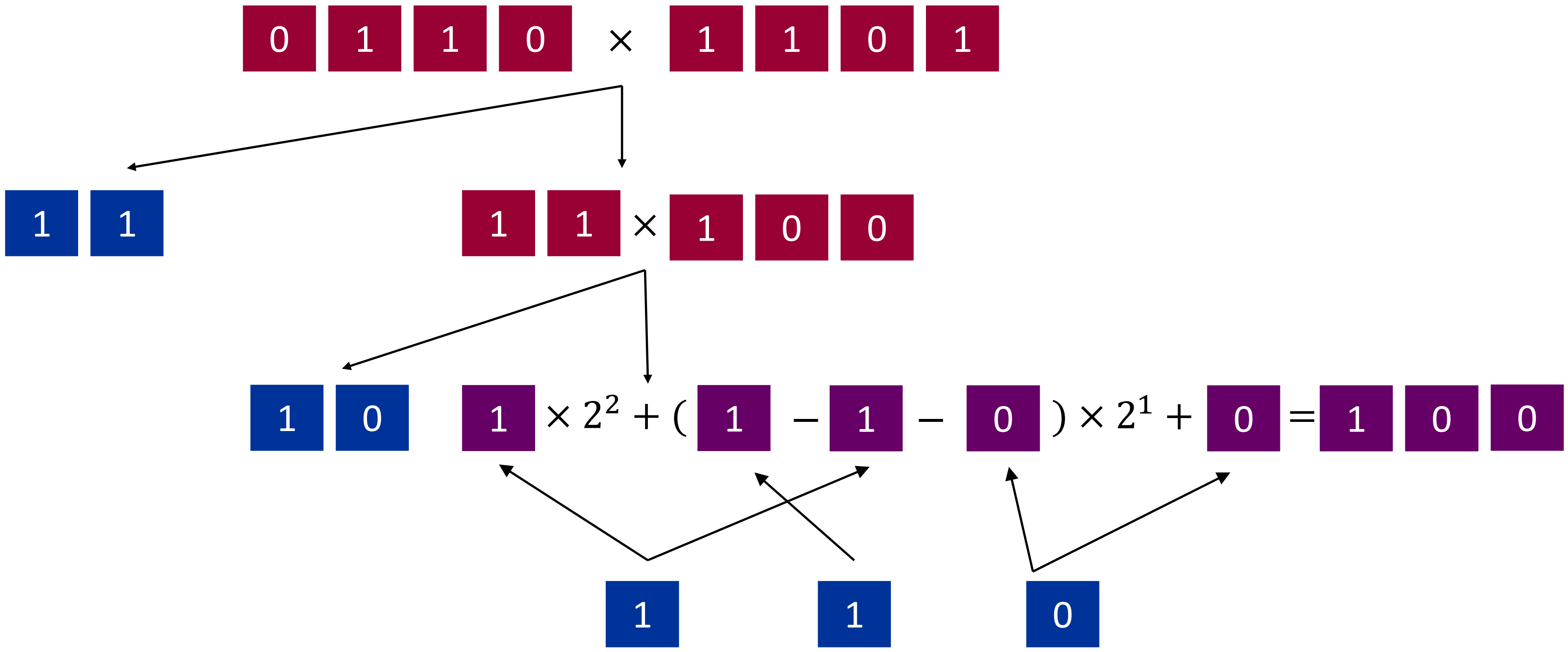
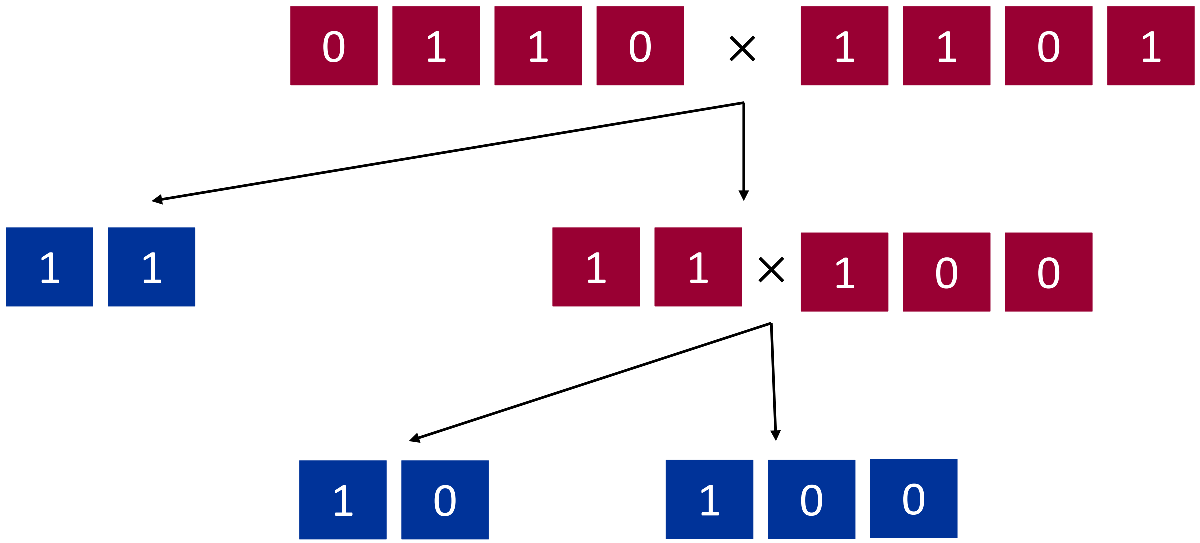


$$\boxed{(a2^2 + b)(c2^2 + \mathrm{d}) = ac \times 2^4 + [(a+b) \times (c+d) - ac - bd] \times 2^2 + bd}$$

Karatsuba Multiplication: Example



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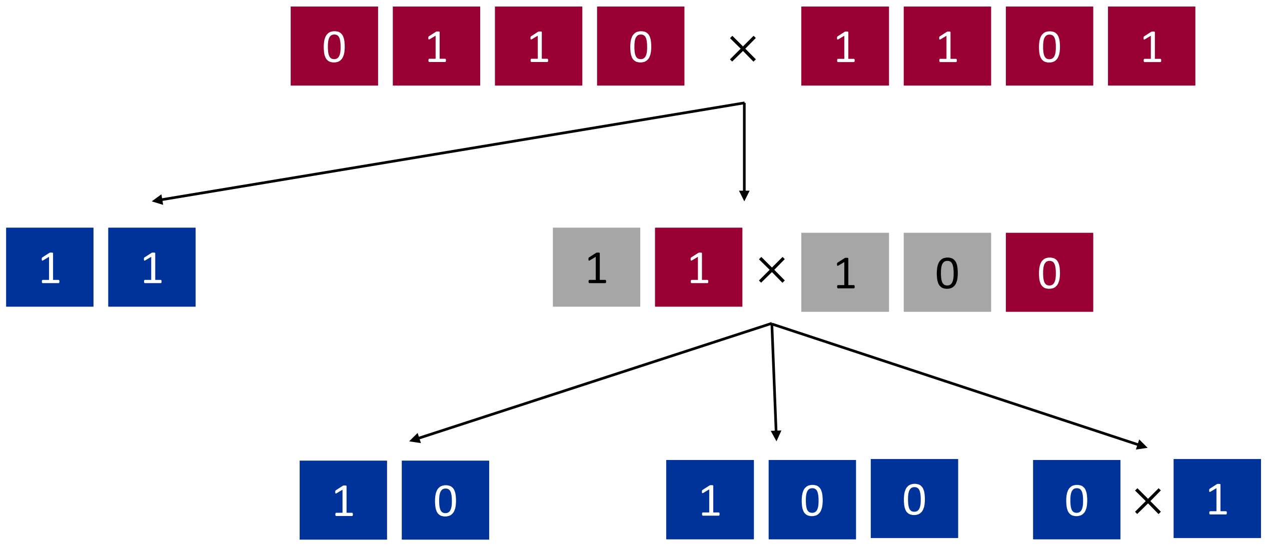

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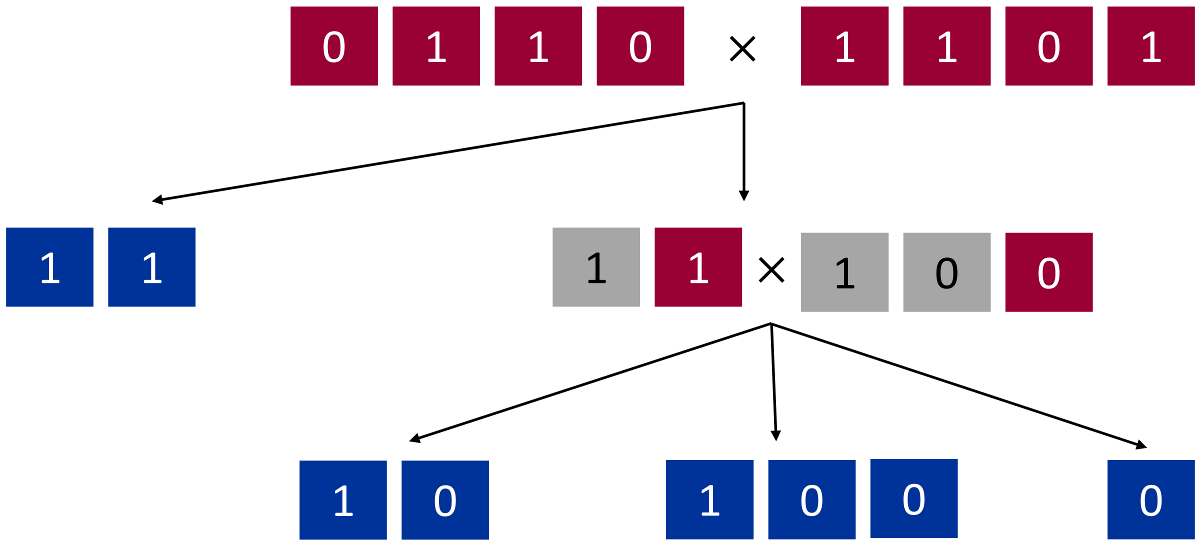
Karatsuba Multiplication: Example



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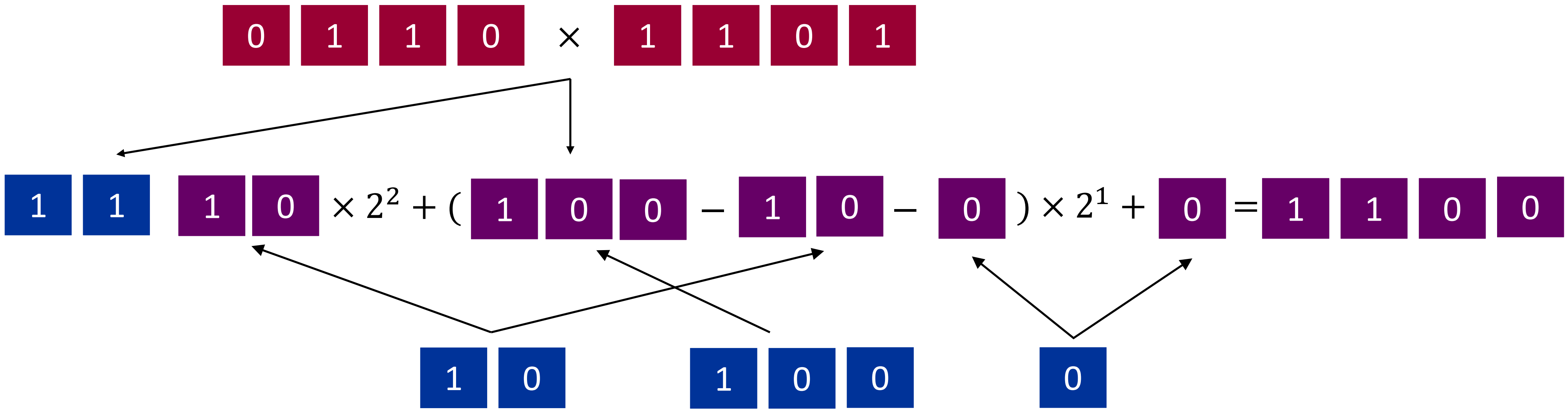

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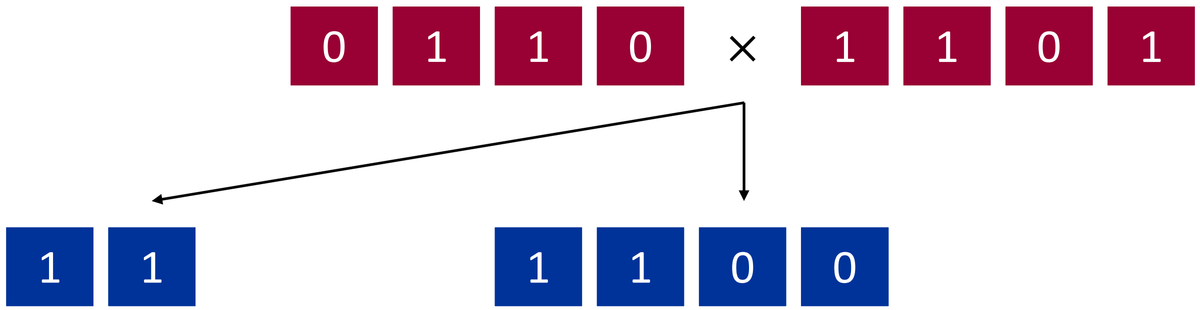

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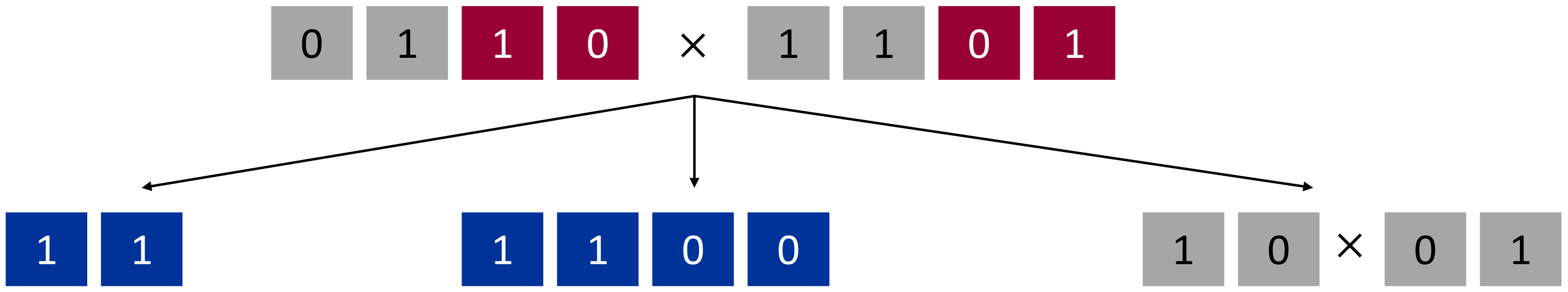

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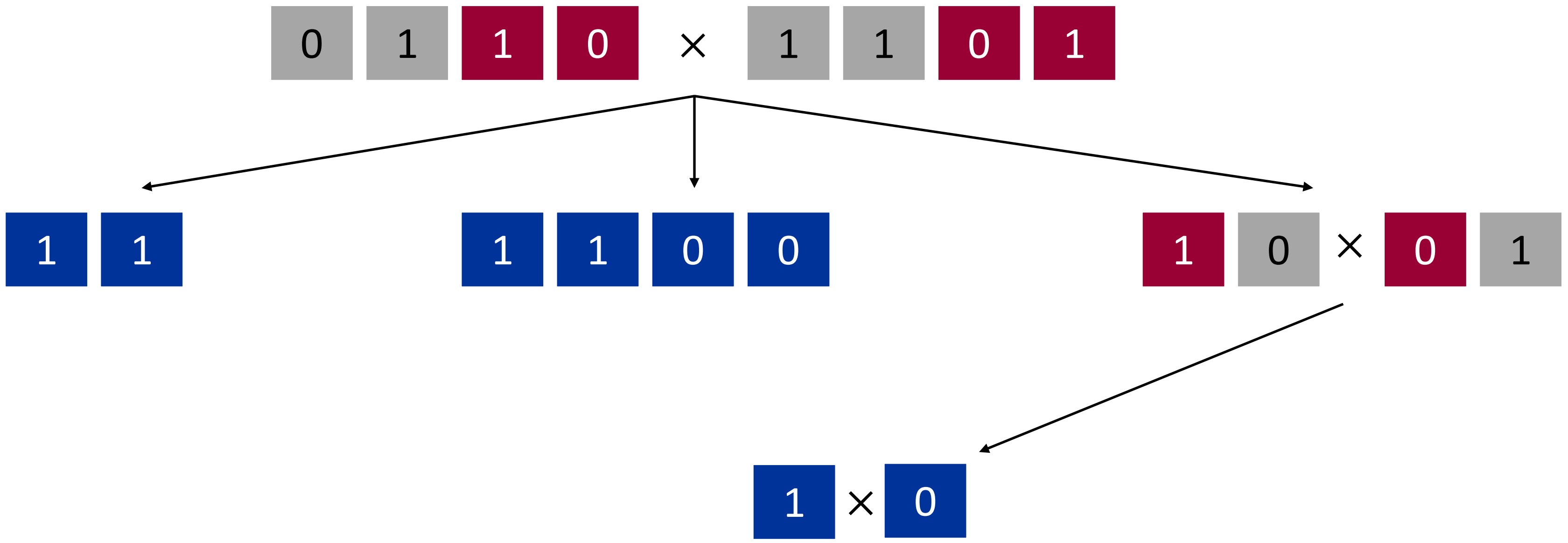

$$\boxed{(a\mathbf{2^2}+b)(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}) = ac\times\mathbf{2^4}+[(a+b)\times(c+d)-ac-bd]\times\mathbf{2^2}+bd}$$

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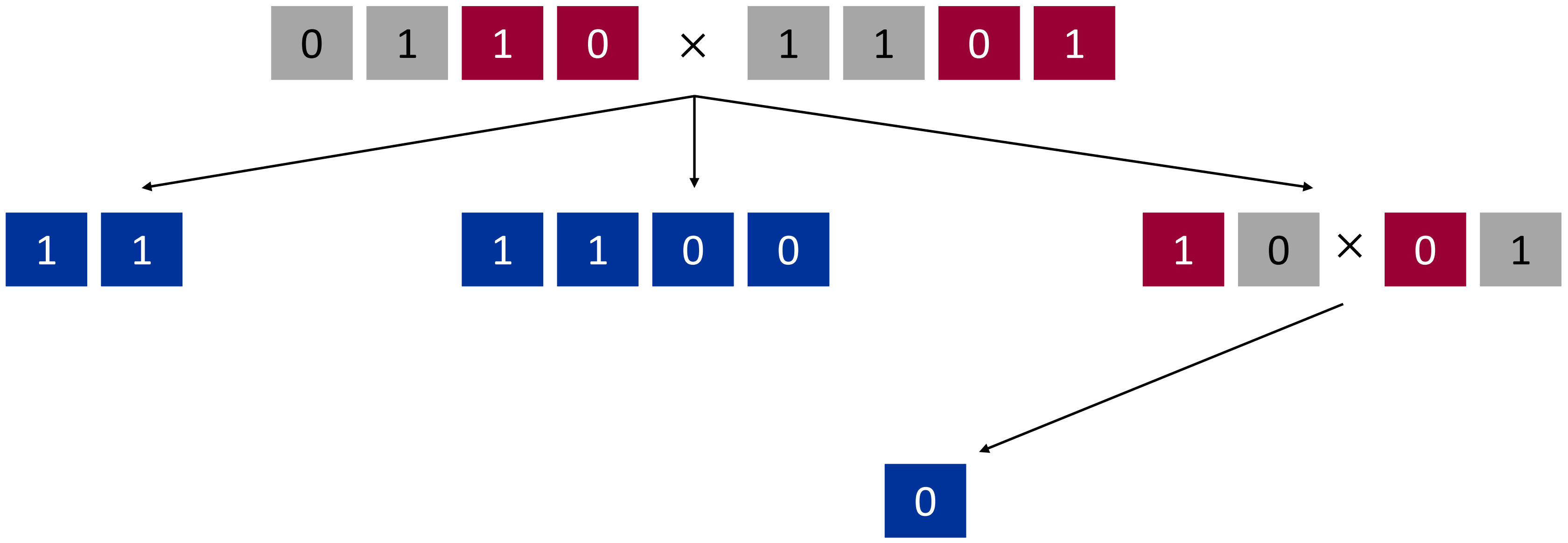

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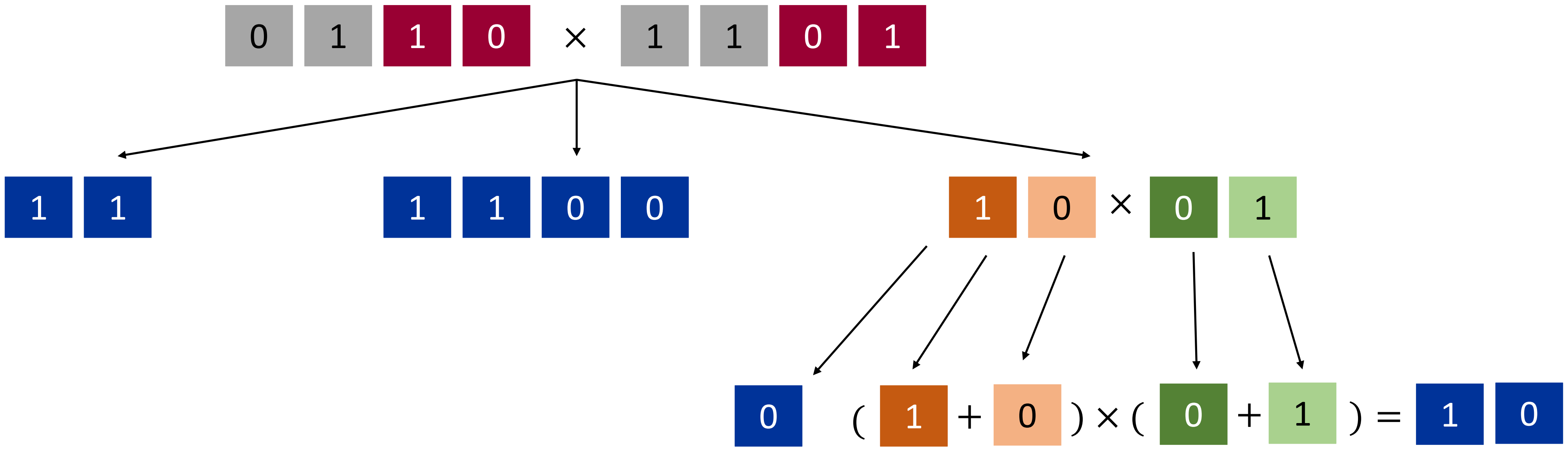
Karatsuba Multiplication: Example



$$\boxed{(a\mathbf{2}^{2} + b)(\boldsymbol{c}\mathbf{2}^{2} + \mathrm{d}) = ac \times \mathbf{2}^{4} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2}^{2} + bd}$$

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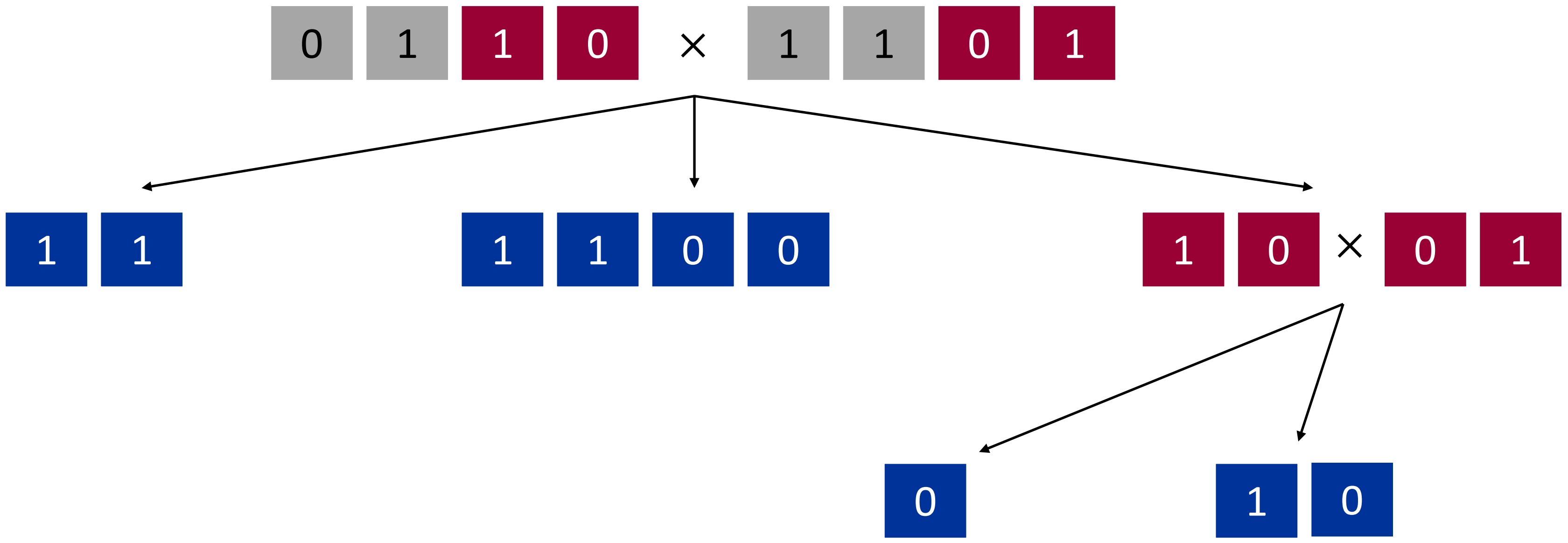

$$\boxed{(a2^2 + b)(c2^2 + \mathrm{d}) = ac \times 2^4 + [(a + b) \times (c + d) - ac - bd] \times 2^2 + bd}$$

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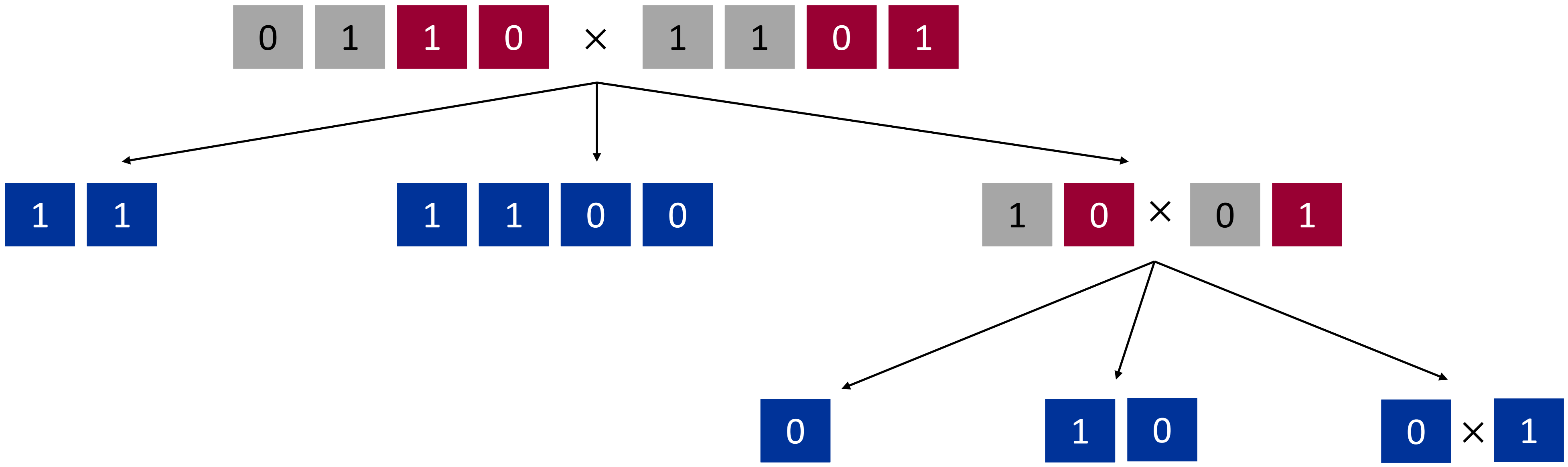

$$\boxed{\left(a\mathbf{2}^{2} + b\right)\left(\boldsymbol{c}\mathbf{2}^{2} + \mathrm{d}\right) = ac \times \mathbf{2}^{4} + [(a+b) \times (c+d) - ac - bd] \times \mathbf{2}^{2} + bd}$$

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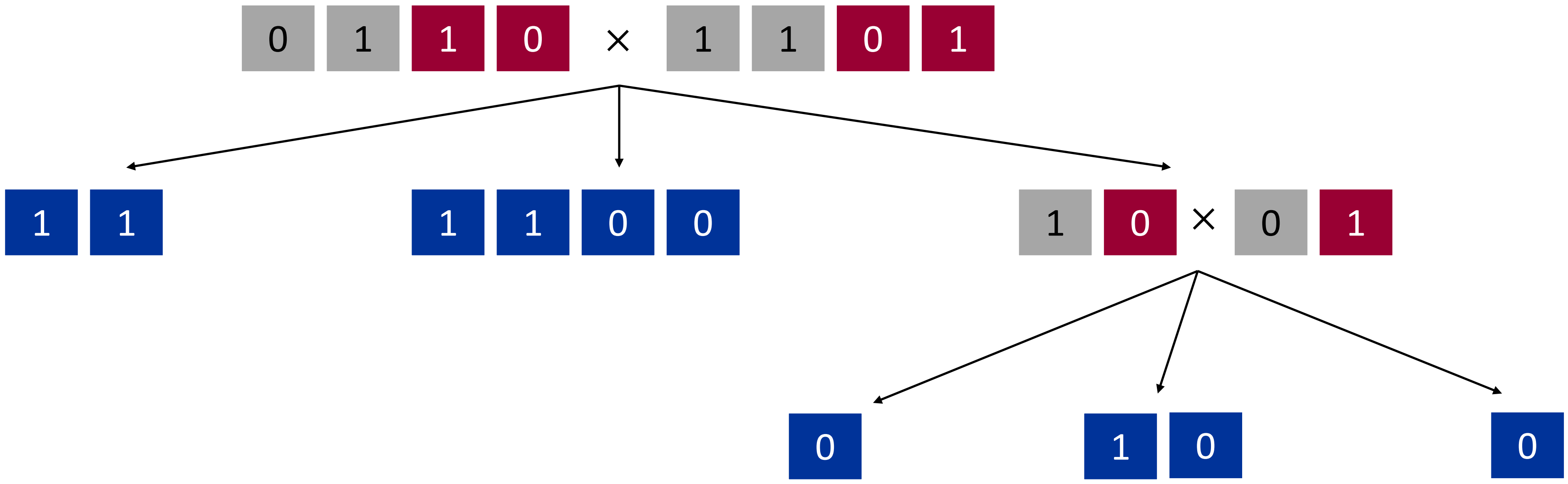

$$\boxed{(a\mathbf{2^2} + b)(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2^2} + bd}$$

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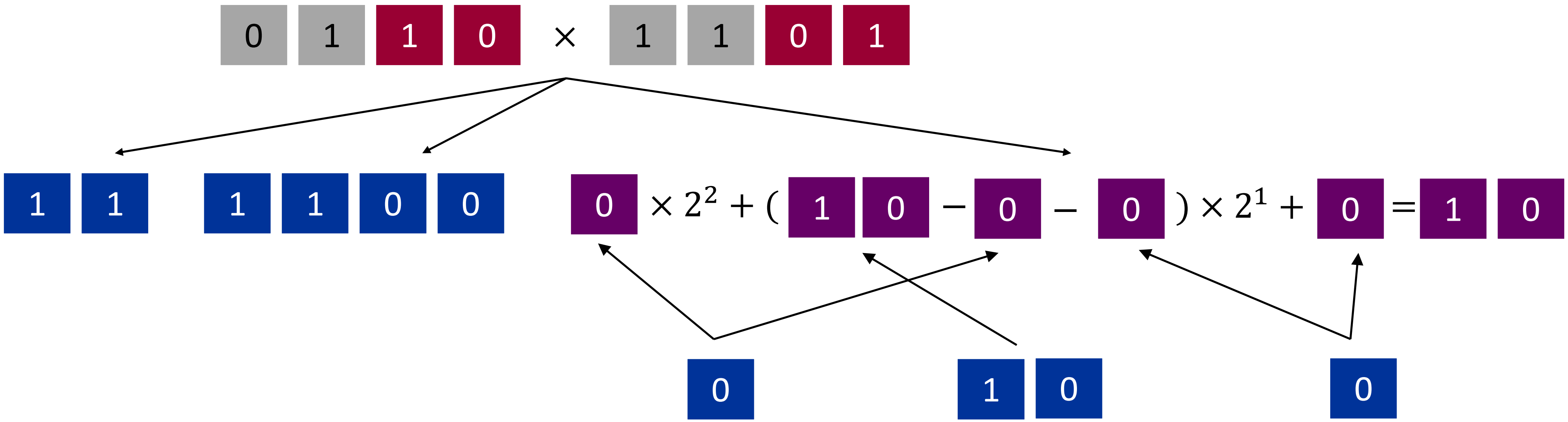

$$\left(a\mathbf{2^2}+b\right)\left(\boldsymbol{c}\mathbf{2^2}+\mathrm{d}\right)=ac\times\mathbf{2^4}+\left[(a+b)\times(c+d)-ac-bd\right]\times\mathbf{2^2}+bd$$

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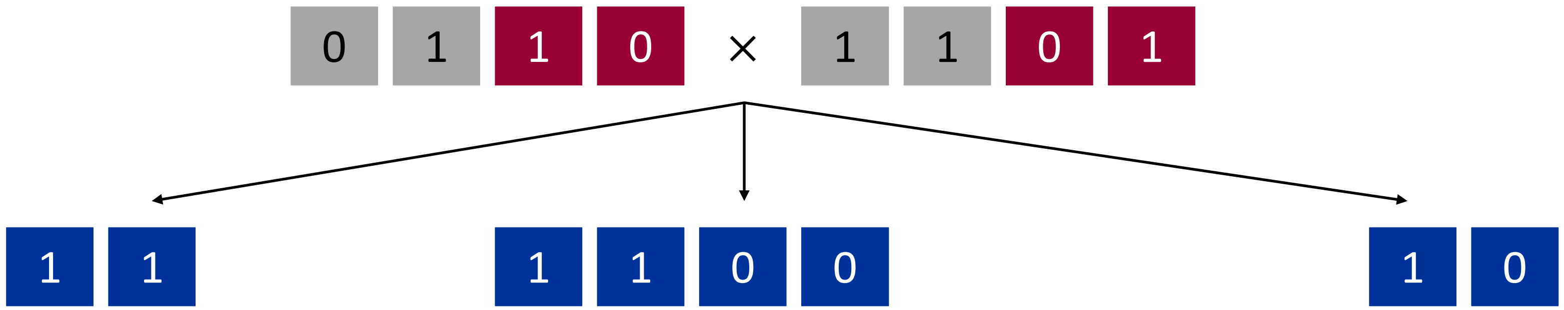

$$\boxed{(a\mathbf{2^2} + b)(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [(a + b) \times (c + d) - ac - bd] \times \mathbf{2^2} + bd}$$

Karatsuba Multiplication: Example

$$\boxed{1}\,\boxed{1} \times 2^4 + (\,\boxed{1}\,\boxed{1}\,\boxed{0}\,\boxed{0} - \boxed{1}\,\boxed{1} - \boxed{1}\,\boxed{0}\,) \times 2^2 + \boxed{1}\,\boxed{0} = \boxed{1}\,\boxed{0}\,\boxed{0}\,\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{0}$$

$$\boxed{1}\,\boxed{1} \qquad \boxed{1}\,\boxed{1}\,\boxed{0}\,\boxed{0} \qquad \boxed{1}\,\boxed{0}$$

$$\boxed{(a\mathbf{2^2} + b)(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [(a+b) \times (c+d) - ac - bd] \times \mathbf{2^2} + bd}$$

Karatsuba Multiplication: Example

0	1	1	0	$\times$	1	1	0	1	$=$	1	0	0	1	1	1	0

$$\boxed{(a\mathbf{2^2} + b)(\boldsymbol{c}\mathbf{2^2} + \mathrm{d}) = ac \times \mathbf{2^4} + [(a+b) \times (c+d) - ac - bd] \times \mathbf{2^2} + bd}$$