L15: Expectation and Variance

- Reading: Rosen 7.4

Expected Value

- **Definition**

  The expected value (or expectation or mean) of a random variable X is

$$E(X) = \sum_{x \in S} p(s) X(s)$$

- **Example**

  Let X be the number that comes up when a fair dice is rolled. What is the expected value of X?

- **Solution**

  The random variable X takes the values 1, 2, 3, 4, 5, or 6. Each has probability 1/6. It follows that

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}.$$

Expected Value

- Recall $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$
- **Theorem**

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$

- **Example**
  Suppose that a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. We know that
  $p(X = 3) = 1/8$
  $p(X = 2) = 3/8$
  $p(X = 1) = 3/8$
  $p(X = 0) = 1/8$
  So $E(X) = \frac{1}{8} \cdot 3 + \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot 1 + \frac{1}{8} \cdot 0 = 1.5$

Example: Binomial Distribution

- Recall that a random variable X with the binomial distribution with parameter n and p, q has
$$p(X = k) = C(n, k)p^k q^{n-k}$$
- We can derive its expectation through a rather complicated sequence of derivations:
$$E(x) = np$$

Linearity of Expectations

- **Theorem**
  If $X_i, i = 1, 2, \dots, n$ are random variables on S, and if a and b are real numbers, then
  - $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$
  - $E(aX + b) = aE(X) + b$
- **Proof**

$$
\begin{aligned}
&E(X_1 + X_2 + \cdots + X_n) \\
&= \sum_{s \in S} p(s)\left[X_1(s) + X_2(s) + \cdots + X_n(s)\right] \\
&= \sum_{s \in S} p(s)X_1(s) + \sum_{s \in S} p(s)X_2(s) + \cdots + \sum_{s \in S} p(s)X_n(s) \\
&= E(X_1) + E(X_2) + \cdots + E(X_n).
\end{aligned}
$$

Proof (cnt'd)

For the second result, note that

$$
\begin{aligned}
E(aX + b) &= \sum_{s \in S} p(s) \left[aX(s) + b \right] \\
&= a \sum_{s \in S} p(s) X(s) + b \sum_{s \in S} p(s) \\
&= a\, E(X) + b,
\end{aligned}
$$

because $\sum_{s \in S} p(s) = 1$.

Example: Binomial Distribution

- X is the number of successes in n independent Bernoulli trials, with success probability p on each trial
- $X = X_1 + X_2 + \cdots + X_n$,
  where $X_i = 1$ with prob. p and 0 otherwise
  These X_i's are often called indicator random variables
- $E(X_i) = p \cdot 1 + (1 - p) \cdot 0 = p$
- $E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = np$

Example: the Hatcheck Problem

- **Question**
  A new employee started a job checking hats, but forgot to put the claim check numbers on the hats. So, the *n* customers just receive a random hat from those remaining. What is the expected number of hat returned correctly?
- **Solution**
  Let *X* be the random variable that equals the number of people who receive the correct hat. We can write
  $X = X_1 + X_2 + \cdots + X_n$,
  where $X_i = 1$ if the *i*th person receives the correct hat and 0 otherwise. (Note that they are not independent.)
  $E(X_i) = 1/n$.
  $E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = 1$

Example: Hiring Problem

```
Hire-Assistant(n):
randomly permute all n candiates
best ← 0
for i ← 1 to n
    interview candidate i
    if candidate i is better than best then
        fire best
        hire candidate i
        best ← i
```

- Questions: What's the expected number of hires?

Hiring Problem: Solution

- Let $X_i = 1$ if you hire candidate i and 0 otherwise.
- Let X = number of hires $= X_1 + \cdots + X_n$.
- $E(X_i) = p(X_i = 1) = 1/i$.
  - Among the first i candidates, the best has probability $1/i$ to be placed at the last position.
- $E(X) = E(X_1) + \cdots + E(X_n) \; = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} = \Theta(\log n)$.

Example: Balls and Bins

- **Question**
  Throw n balls into n bins randomly. How many boxes will be empty in expectation?
- **Solution**
  - Let $X_i = 1$ box i is empty, and 0 otherwise.
  - Let $X =$ number of empty boxes $= X_1 + \cdots + X_n$.
  - $E(X_i) = p(X_i = 1) = \left(1 - \frac{1}{n}\right)^n$.
  - $E(X) = E(X_1) + \cdots + E(X_n) = n\left(1 - \frac{1}{n}\right)^n \approx n/e$.

Average-case Analysis of Algorithms

- **Definition**
  Let X be the random variable representing the running time (i.e., # of instructions) of an algorithm, when the input is drawn from a certain distribution (usually, uniform). Then the average-case running time of the algorithm is $E(X)$.

Example: Linear Search

```
procedure linear search(x:integer,
                a_1, a_2, …, a_n: distinct integers)
i := 1
while (i ≤ n and x ≠ a_i)
      i := i + 1
if i ≤ n then location := i
else location := 0
return location
```

- Assume that x is one of the a_i's with equal probability
- For asymptotic analysis, it's enough to count the number of iterations, denoted as X
- $E(X) = \frac{1}{n} \cdot 1 + \frac{1}{n} \cdot 2 + \cdots + \frac{1}{n} \cdot n = \frac{n+1}{2} = \Theta(n)$

Example: Insertion Sort

<u>Insertion-Sort(A):</u>

for $j \leftarrow 2$ **to** n **do**

  $key \leftarrow A[j]$

  $i \leftarrow j - 1$

  while $i \geq 1$ **and** $A[i] > key$ **do**

    $A[i+1] \leftarrow A[i]$

    $i \leftarrow i - 1$

  endwhile

  $A[i+1] \leftarrow key$

endfor

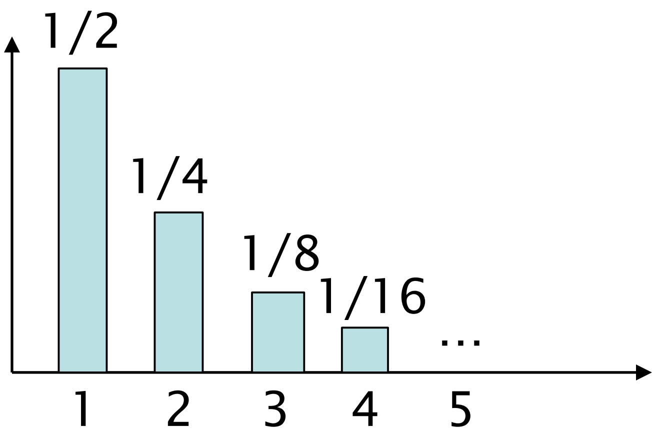
sorted	key	unsorted

Example: Insertion Sort

- Assumption: The input is a random permutation.
- For asymptotic analysis, it's sufficient to count the total number of iterations of the inner loop, denoted X
- X_i: the # of iterations of the inner loop for inserting a_i
  - $E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$
- a_i can be inserted into one of the first i locations with equal probability
  - The first i elements of a random permutation is a random permutation of i elements
  - The rank of the last element in a random permutation is one of $1, \ldots, i$ with equal probability
- $E(X_i) = \frac{1}{i} \cdot 1 + \frac{1}{i} \cdot 2 + \cdots + \frac{1}{i} \cdot i = \frac{i+1}{2}$
- $E(X) = \sum_{i=2}^{n} \frac{i+1}{2} = \frac{n^2+3n-4}{4} = \Theta(n^2)$

Geometric Distribution

- Experiment: Flip a fair coin until it turns up heads. Let random variable $X =$ the number of flips.



- General case: each coin turns up heads with prob. p
- **Definition**
  A random variable X has geometric distribution with parameter $p \leq 1$ if $p(X = k) = (1-p)^{k-1}p,\ k = 1, 2, 3, \ldots$.
- **Theorem:** $E(X) = 1/p$

Example: Coupon Collector

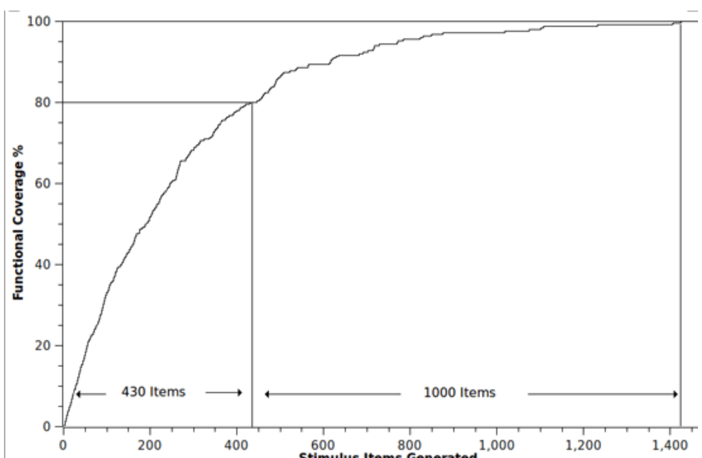
- **Question**
  Each box of cereal contains a coupon. There are n different types of coupons. Assuming a box contains each type of coupon equally likely, how many boxes do you need to open to have at least one coupon of each type?

Coupon Collector: Solution

- Stage i = time between i and $i+1$ distinct coupons.
- Let X_i = number of steps you spend in stage i.
- Let X = number of steps in total $= X_0 + X_1 + \cdots + X_{n-1}$.

$$E[X] = \sum_{i=0}^{n-1} E[X_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=1}^{n} \frac{1}{i} = \Theta(n \log n)$$

geometric distribution with $p = (n-i)/n$



Independent Random Variables

- **Definition**
  The random variables X and Y on a sample space S are independent if
  $$p(X = r_1 \; and \; Y = r_2) = p(X = r_1) \cdot \; p(Y = r_2)$$
  for all r_1, r_2
- **Theorem**
  If X and Y are independent variables on a sample space S, then $E(XY) = E(X)E(Y)$.
- **Note**
  Linearity of expectation holds no matter the random variables are independent or not.

Example

- Let X and Y be random variables that count the number of heads and the number of tails when a fair coin is flipped twice.
- Show that X and Y are not independent.
  - $p(X = 2\ and\ Y = 2) = 0$
  - $p(X = 2) = 1/4$, $p(Y = 2) = 1/4$
- Check
  - $E(XY) = 1 \cdot \frac{1}{2} = \frac{1}{2}$
  - $E(X) = E(Y) = 1$
  - $E(XY) \neq E(X)E(Y)$
- Check $E(X + Y) = 2 = E(X) + E(Y)$

Proof

- From the definition of expected value and because X and Y are independent random variables, it follows that

$$\begin{aligned}
E(XY) &= \sum_{s\in S} X(s)Y(s)\,p(s) \\
&= \sum_{r_1\in X(S),r_2\in Y(S)} r_1 r_2 \cdot p\big(X(s)=r_1 \text{ and } Y(s)=r_2\big) \\
&= \sum_{r_1\in X(S),r_2\in Y(S)} r_1 r_2 \cdot p\big(X(s)=r_1\big)\cdot p\big(Y(s)=r_2\big) \\
&= \left[\sum_{r_1\in X(S)} r_1\, p\big(X(s)=r_1\big)\right]\cdot\left[\sum_{r_2\in Y(S)} r_2\, p\big(Y(s)=r_2\big)\right] \\
&= E(X)E(Y).
\end{aligned}$$

Variance and Standard Deviation

- **Definition**
  Let X be a random variable on the sample space S. The variance of X, denoted by $V(X)$ is
  $$V(X) = \sum_{s \in S} \left(X(s) - E(X)\right)^2 p(s) = E\left(\left(X - E(X)\right)^2\right)$$
- **Definition**
  The standard deviation of X, denoted by $\sigma(X)$, is $\sqrt{V(X)}$
- **Example**
  Bernoulli trial: $X = 1$ with probability p, 0 otherwise
  $$E(X) = p$$
  $$V(X) = (1 - p)^2 p + (0 - p)^2 \cdot (1 - p) = (1 - p)p$$

Variance

- **Theorem**

$$V(X) = E(X^2) - E(X)^2$$

- **Proof**

$$\begin{aligned} V(X) &= \sum_{s \in S} \left(X(s) - E(X)\right)^2 p(s) \\ &= \sum_{s \in S} X^2(s)p(s) - 2E(X)\sum_{s \in S} X(s)p(s) + E(X)^2 \sum_{s \in S} p(s) \\ &= E(X^2) - 2E(X)^2 + E(X)^2 \\ &= E(X^2) - E(X)^2. \end{aligned}$$

- **Corollary**
  $V(aX) = a^2V(X)$, for any real number $a \geq 0$.

Example

- **Question**
  What is the variance of a random variable X, where X is the number that comes up when a fair dice is rolled?
- **Solution**
- $E(X) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$
- Using the first definition:
  $$V(X) = \frac{(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2}{6} = \frac{35}{12}$$
- Using the second definition:
  $$E(X^2) = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{91}{6}$$
  $$V(X) = E(X^2) - E(X)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Bienaymé's Formula

- **Theorem**
  If X and Y are two independent random variables on a sample space S, then
  $$\mathrm{V}(X+Y)=V(X)+V(Y)$$

  Furthermore, if $X_i, i=1,2,\ldots,n$, are pairwise independent random variables on S, then
  $$V(X_1+\cdots+X_n)=V(X_1)+\cdots+V(X_n)$$

Proof of Bienaymé's Formula

We note that

$$\begin{aligned} V(X+Y) &= E\big((X+Y)^2\big) - E(X+Y)^2 \\ &= E(X^2+2XY+Y^2) - \big(E(X)+E(Y)\big)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \end{aligned}$$

Because X and Y are independent, we have $E(XY) = E(X)E(Y)$. It follows that

$$V(X+Y) = \big(E(X^2) - E(X)^2\big) + \big(E(Y^2) - E(Y)^2\big) = V(X) + V(Y).$$

- The proof of the case with n pairwise independent random variables can be constructed by generalizing the proof of the case for two random variables.

Examples

- **Question**
  X = the sum of numbers of two dice
  What's $V(X)$?
- **Solution**
  Write $X = X_1 + X_2$, where X_1 is the number on the first dice, X_2 is the number on the second dice
  We already know $V(X_1) = V(X_2) = \frac{35}{12}$
  So $V(X) = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}$.
- **Question**
  X follows the binomial distribution. What's $V(X)$?
- **Solution**
  Write $X = X_1 + \cdots + X_n$, where $X_i = 1$ with prob. p
  We already know $V(X_i) = p(1-p)$
  So $V(X) = np(1-p)$.

Example

- Suppose we take a measurement by repeating the same experiment many times. Each trial returns a random measurement that is correct in expectation, but with some deviation. More precisely, let X_i be the measurement from the i-th experiments we have
$$E(X_i) = \mu, V(X_i) = \sigma^2$$
If we repeat the experiments n times independently, and report the average $Y = (X_1 + \cdots + X_n)/n$. What is $E(Y)$ and $V(Y)$?
- **Solution**
$$E(Y) = \frac{E(X_1) + \cdots + E(X_n)}{n} = \mu$$
$$V(Y) = \frac{V(X_1) + \cdots + V(X_n)}{n^2} = \frac{\sigma^2}{n}$$