Problem 1

(a) $A = \Omega(B)$
(b) $A = \mathrm{O}(B), A = \Theta(B), A = \Omega(B)$
(c) $A = \mathrm{O}(B)$
(d) $A = \Omega(B)$
(e) $A = \Omega(B)$
(f) $A = \mathrm{O}(B), A = \Omega(B), A = \Theta(B)$
(g) $A = \Omega(B)$

Problem 2

(a)

$$
\begin{aligned}
h &= \log_{\frac{5}{3}} n \\
T(n) &= T\left(\frac{3n}{5}\right) + 1 \\
&= T\left(\frac{3}{5} * \frac{3n}{5}\right) + 1 + 1 = T\left(\left(\frac{3}{5}\right)^2 n\right) + 2 \\
&= T\left(\frac{3}{5} * \left(\frac{3}{5}\right)^2 n\right) + 1 + 2 = T\left(\left(\frac{3}{5}\right)^3 n\right) + 3 \\
&\dots \\
&= T\left(\left(\frac{3}{5}\right)^h n\right) + h \\
&= T(1) + \log_{\frac{5}{3}} n \\
&= O(\log n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
h &= \log_{\frac{5}{3}} n \\
T(n) &= T\left(\frac{3n}{5}\right) + n \\
&= \left[T\left(\frac{3}{5} * \frac{3n}{5}\right) + \frac{3n}{5}\right] + n = T\left(\left(\frac{3}{5}\right)^2 n\right) + \frac{3n}{5} + n \\
&= \left[T\left(\frac{3}{5} * \left(\frac{3}{5}\right)^2 n\right) + \left(\frac{3}{5}\right)^2 n\right] + \left(\frac{3}{5}\right) n + n \\
&= T\left(\left(\frac{3}{5}\right)^3 n\right) + \left(\frac{3}{5}\right)^2 n + \left(\frac{3}{5}\right) n + n \\
&\dots \\
&= T\left(\left(\frac{3}{5}\right)^h n\right) + \left(\frac{3}{5}\right)^{h-1} n + \cdots + \left(\frac{3}{5}\right)^2 n + n \\
&= T(1) + \left(\frac{3}{5}\right)^{h-1} n + \cdots + \left(\frac{3}{5}\right)^2 n + n \\
&= T(1) + \left[\left(\frac{3}{5}\right)^{h-1} + \cdots + \left(\frac{3}{5}\right)^2 + 1\right] n \\
&= T(1) + \left(\frac{1 - \left(\frac{3}{5}\right)^h}{1 - \frac{3}{5}}\right) n \\
&= \frac{5}{2} n - \frac{3}{2} \\
&= O(n)
\end{aligned}
$$

(c)

$$
\begin{aligned}
h &= \log_3 n \\
T(n) &= 9T\left(\frac{n}{3}\right) + n^2 \\
&= 9\left[9T\left(\frac{\frac{n}{3}}{3}\right) + \left(\frac{n}{3}\right)^2\right] + n^2 = 9^2 T\left(\frac{n}{3^2}\right) + 9\left(\frac{n}{3}\right)^2 + n^2 \\
&= 9^2\left[9T\left(\frac{\frac{n}{3^2}}{3}\right) + \left(\frac{n}{3^2}\right)^2\right] + 9n^2 + n^2 = 9^3 T\left(\frac{n}{3^3}\right) + 9^2\left(\frac{n}{3^2}\right)^2 + 9\left(\frac{n}{3}\right)^2 + n^2 \\
\ldots \\
&= 9^h T\left(\frac{n}{3^h}\right) + 9^{h-1}\left(\frac{n}{3^{h-1}}\right)^2 + \cdots + 9\left(\frac{n}{3}\right)^2 + n^2 \\
&= 9^h T(1) + \left[9^{h-1}\left(\frac{1}{3^{h-1}}\right)^2 + \cdots + 9\left(\frac{1}{3}\right)^2 + 1\right] n^2 \\
&= 9^{\log_3 n} + hn^2 \\
&= n^2 + n^2 \log_3 n \\
&= O(n^2 \log_3 n)
\end{aligned}
$$

(d)

$$
\begin{aligned}
h &= \log_3 n \\
T(n) &= 7T\left(\frac{n}{3}\right) + n^2 \\
&= 7\left[7T\left(\frac{\frac{n}{3}}{3}\right) + \left(\frac{n}{3}\right)^2\right] + n^2 = 7^2 T\left(\frac{n}{3^2}\right) + 7\left(\frac{n}{3}\right)^2 + n^2 \\
&= 7^2\left[7T\left(\frac{\frac{n}{3^2}}{3}\right) + \left(\frac{n}{3^2}\right)^2\right] + 7\left(\frac{n}{3}\right)^2 + n^2 \\
&= 7^3 T\left(\frac{n}{3^3}\right) + 7^2\left(\frac{n}{3^2}\right)^2 + 7\left(\frac{n}{3}\right)^2 + n^2 \\
\ldots \\
&= 7^h T\left(\frac{n}{3^h}\right) + 7^{h-1}\left(\frac{n}{3^{h-1}}\right)^2 + \cdots + 7\left(\frac{n}{3}\right)^2 + n^2 \\
&= 7^h T\left(\frac{n}{3^h}\right) + \left[7^{h-1}\left(\frac{1}{3^{h-1}}\right)^2 + \cdots + 7\left(\frac{1}{3}\right)^2 + 1\right] n^2 \\
&= 7^h T(1) + \left(\frac{1-\left(\frac{7}{9}\right)^h}{1-\frac{7}{9}}\right) n^2 \\
&= 7^{\log_3 n} + \frac{9}{2}\left(n^2 - 7\log_3 n\right) \\
&= \frac{9}{2}n^2 + n^{\log_3 7} - \frac{63}{2}\log_3 n \\
&= O(n^2)
\end{aligned}
$$

(e)

$$
\begin{aligned}
h &= \log_3 n \\
T(n) &= 5T\left(\frac{n}{3}\right) + n \\
&= 5\left[5T\left(\frac{1}{3} * \frac{n}{3}\right) + \frac{n}{3}\right] + n = 5^2 T\left(\frac{n}{3^2}\right) + \left(\frac{5}{3}\right)n + n \\
&= 5^2\left[5T\left(\frac{1}{3} * \frac{n}{3^2}\right) + \frac{n}{3^2}\right] + 2n = 5^3 T\left(\frac{n}{3^3}\right) + \left(\frac{5}{3}\right)^2 n + \frac{5}{3}n + n \\
\cdots \\
&= 5^h T\left(\frac{n}{3^h}\right) + \left(\frac{5}{3}\right)^{h-1} n + \cdots + \left(\frac{5}{3}\right)n + n \\
&= 5^h T(1) + \left[\left(\frac{5}{3}\right)^{h-1} + \left(\frac{5}{3}\right)^{h-2} + \cdots + \left(\frac{5}{3}\right) + 1\right]n \\
&= 5^{\log_3 n} + \frac{3}{2}\left(n^{\log_3 5} - n\right) \\
&= n^{\log_3 5} + \frac{3}{2}n^{\log_3 5} - \frac{3}{2}n \\
&= O\left(n^{\log_3 5}\right)
\end{aligned}
$$

Problem 3

(a)

$$
\begin{aligned}
T(n) &= 4T\left(\frac{n}{2}\right) + n^2 \\
&= 4\left[4T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 = 4^2 T\left(\frac{n}{2^2}\right) + 4\left(\frac{n}{2}\right)^2 + n^2 \\
&= 4^2\left[4T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + 4\left(\frac{n}{2^2}\right)^2 + n^2 \\
&= 4^3 T\left(\frac{n}{2^3}\right) + 4^2\left(\frac{n}{2^2}\right)^2 + 4\left(\frac{n}{2^2}\right)^2 + n^2
\end{aligned}
$$

...

$$
\begin{aligned}
&= 4^h T\left(\frac{n}{2^h}\right) + 4^{h-1}\left(\frac{n}{2^{h-1}}\right)^2 + \cdots + 4\left(\frac{n}{2^2}\right)^2 + n^2 \\
&= 4^h T\left(\frac{n}{n}\right) + \left[4^{h-1}\left(\frac{1}{2^{h-1}}\right)^2 + \cdots + 4\left(\frac{n}{2^2}\right)^2 + 1\right] n^2 \\
&= 4^h T(1) + hn^2 \\
&= 4^{\log_2 n} + n^2 \log_2 n \\
&= n^2 + n^2 \log_2 n \\
&= O(n^2 \log n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
T(n) &= 5T\left(\frac{n}{2}\right) + n^2 \\
&= 5\left[5T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 = 5^2 T\left(\frac{n}{2^2}\right) + 5\left(\frac{n}{2}\right)^2 + n^2 \\
&= 5^2\left[5T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + 5\left(\frac{n}{2}\right)^2 + n^2 = 5^3 T\left(\frac{n}{2^3}\right) + 5^2\left(\frac{n}{2^2}\right)^2 + 5\left(\frac{n}{2}\right)^2 + n^2
\end{aligned}
$$

...

$$
\begin{aligned}
&= 5^h T\left(\frac{n}{2^h}\right) + 5^{h-1}\left(\frac{n}{2^{h-1}}\right)^2 + \cdots + 5\left(\frac{n}{2}\right)^2 + n^2 \\
&= 5^h T\left(\frac{n}{n}\right) + \left[\left(\frac{5}{2^2}\right)^{h-1} + \left(\frac{5}{2^2}\right)^{h-2} + \cdots + \frac{5}{2^2} + 1\right] n^2 \\
&= 5^h T(1) + \left(\frac{1 - \left(\frac{5}{4}\right)^h}{1 - \frac{5}{4}}\right) n^2 \\
&= 5^{\log_2 n} T(1) - 4\left(n^2 - 5^{\log_2 n}\right) \\
&= 2n^{\log_2 5} + 4n^{\log_2 5} - 4n^2 \\
&= 6n^{\log_2 5} - 4n^2 \\
&= O\left(n^{\log_2 5}\right)
\end{aligned}
$$

Problem 4

1. Merge(S^1, S^2): L($S^1 \cup S^2$)

```
create a new array to store the output A
i ← 1, j ← 1
for k ← 1 to (n1+n2)

    if j > n2 then
        terminate program
    if i > n1 then
        A[k]←S2[j]
        j←j+1

    if S1[i].y < S2[j].y then
        i←i+1
    else if S1[i].y > S2[j].y then
        A[k]←S1[j]
        i←i+1
    else
        A[k]←S1[i]
        k←k+1
        A[k]←S2[j]
        i←i+1
        j←j+1
```

2. Correctness:
   S_1 and S_2 are sorted by x-coordinate
   $\forall p_1 \in S_1, \forall p_2 \in S_2\ (p_1.x < p_2.x)$

   By property 1(a), the points are sorted by x in ascending order and by y in descending order. Because if $x_i < x_j$ *and* $y_i < y_j$ for i < j, then $p_i < p_j$ and contradict the property.

   if $p_1.y < p_2.y$ then ignore p_1,
   else if $p_1.y > p_2.y$ then store p_1,
   else store both.

   $O(n_1 + n_2)$:
   Go through from k = 1 to k = (n_1 + n_2)

3. FINDL(S):

```
n ← size of S
mid ← n / 2
if n = 1 then return S
L ← FINDL(S[0 to mid])
R ← FINDL(S[mid + 1 to n])
return Merge(L, R)
```

4. T(n) = 2T(n/2) + O(n) = O(nlogn)
   Correctness:
   Recursively find L(first half of S) and L(second half of S) and merge them into one.