THE UNIVERSITY OF HONG KONG

FACULTY OF ENGINEERING

DEPARTMENT OF COMPUTER SCIENCE

COMP2121 Discrete Mathematics: Quiz 2

Date: 6 Nov, 2018 Time: 9:30am - 10:20am

Write your **university number here** and also on **every page**: ______________________

Please read the following before you begin.

1. Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is your responsibility to ensure that your calculator operates satisfactorily, and you must record the **name and type of the calculator** here:

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2. This is a **closed book** quiz.
3. You are advised to spend around 5 minutes to go through all the questions before you begin writing. Allocate the time for each part of the question carefully.
4. Please write your answer in the designated space.
5. Give your answers correct to **3 decimal places if necessary**.
6. Attempt ALL questions. The full mark for the exam is 100 points.

For marking use:

Question 1

Question 2

Question 3

Question 4

Question 5

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Question 1 (24 points)

(a) (8 pt) How many positive integers not exceeding 200 are divisible by 5 or 7?

Solution: Let A denote the set of integers not exceeding 200 that are divisible by 5, then $|A| = \lfloor 200/5 \rfloor = 40$. Assume B denotes the set of integers not exceeding 200 that are divisible by 7, then $|B| = \lfloor 200/7 \rfloor = 28$. Then we know that $A \cap B$ denotes the set of integers not exceeding 200 that are both divisible by 5 or 7, or in other words, divisible by 35. We can get $|A \cap B| = \lfloor 200/35 \rfloor = 5$. We know that $A \cup B$ denotes the set of positive integers not exceeding 200 that are divisible by either 5 or 7, and we can calculate $|A \cup B| = |A| + |B| - |A \cap B| = 40 + 28 - 5 = 63$.

(b) (8 pt) How many ways are there to arrange the letters a, b, c, d, e in a row such that the pattern *ace* does not exist within the arrangement?

Solution: Since there are in total $P(5,5) = 5! = 120$ ways to arrange the letters a, b, c, d, e without any constraint. If pattern *ace* exists, *ace* can be treated as a whole, and there are 3 arrangeable elements in total. Hence there are $P(3,3) = 3! = 6$ ways for the arrangement. Therefore there are $120 - 6 = 114$ ways to arrange the letters such that pattern *ace* do not exists.

(c) (8 pt) What is the minimum number of students, each of whom comes from one of the 13 districts, that must be enrolled in a university to guarantee that there are at least 50 students who come from the same district?

Solution: If a university enrolled x students, there will be at least $\lceil \frac{x}{13} \rceil$ students coming from the same district. To garantee that there are at least 50, $x = 13 \times 49 + 1 = 638$ students must be enrolled.

Question 2 (16 points)

A deck of cards contains 52 cards. There are 13 different kinds of cards, with four suits of cards of each kind. These kinds are twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings and aces. There are also four suits: spades, clubs, hearts and diamonds, each containing 13 cards. A *pair* means two cards in the same kind. A five-card poker hand contains a *two pairs* if it contains two pairs in different kinds and another card in a third kind. A *flush* means all cards are of the same suit. A *straight* means all cards have consecutive kinds. (Note that an ace can only be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight. However, those with an ace in between, like Q-K-A-2-3, are NOT straights.)

(a) (8 pt) What is the probability that a five-card poker hand does not contains a flush or a two pairs?

Solution: We use S to denote the set of all possible combinations of five cards, F to denote the set of five cards of different kinds containing a flush, P to denote the set of five cards containing a two pairs, E to denote the set of five-card poker hand does not contains a flush or a two pairs. Obviously we know that $E = S - (F \cup P)$, and we want to calculate $P(E) = |E|/|S|$.

First, we can obtain that $|F| = 4 \times \binom{13}{5} = 5148$, and also $|P| = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1} = 123552$. As it is impossible to have five cards containing a flush and a two pairs at the same time, $|F \cap P| = 0$. As a result, $|F \cup P| = |F| + |P| - |F \cap P| = 5148 + 123552 - 0 = 128700$. We can calculate that $|S| = \binom{52}{5} = 2598960$. So $P(E) = |E|/|S| = (2598960 - 128700)/2598960 = 0.950$.

(b) (8 pt) What is the probability that a five-card poker hand contains either a flush or a straight but not both?

Solution: We use S to denote the set of all possible combinations of five cards, F to denote the set of five cards of different kinds containing a flush, T to denote the set of five cards containing a straight, E' to denote the set of five cards of different kinds and contain either a flush or a straight, but not both. Then $E' = (F \cup T) - (F \cap T)$, and we want to calculate $P(E') = |E'|/|S|$.

From previous part we know that $|F| = 5148$ and $|S| = 2598960$. We can calculate that $|T| = 10 \times 4^5 = 10240$, and $|F \cap T| = 4 \times 10 = 40$. As a result, $|F \cup T| = |F| + |T| - |F \cap T| = 5148 + 10240 - 40 = 15348$. $|E| = |F \cup T| - |F \cap T| = 15348 - 40 = 15308$. So $P(E') = |E'|/|S| = 15308/2598960 = 0.00589$.

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Question 3 (16 points)

A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a '1' 70% of the time and a '0' 30% of the time. When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received incorrectly (as a 1) is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8, and the probability that it is received incorrectly (as a 0) is 0.2.

(a) (8 pt) Find the probability that a '0' is received.

Solution: For $X \in \{0, 1\}$, let $\{X\}$ denote the event that X is received, and $[X]$ denote the event that X is sent. Hence, $P(\{X\} \mid [Y])$ is the probability that "X" is received given "Y" sent, where $X, Y \in \{0, 1\}$. Then we know that $P([1]) = 0.7$, $P([0]) = 0.3$, $P(\{0\} \mid [0]) = 0.9$, $P(\{1\} \mid [0]) = 0.1$, $P(\{1\} \mid [1]) = 0.8$ and $P(\{0\} \mid [1]) = 0.2$.

$$\begin{aligned} P(\{0\}) =& P(\{0\} \mid [0])P([0]) + P(\{0\} \mid [1])P([1]) \\ =& 0.9 \times 0.3 + 0.2 \times 0.7 \\ =& 0.41 \end{aligned}$$

(b) (8 pt) Use Bayes' Theorem to find the probability that a '1' was sent, given that a '0' was received.

Solution:

$$\begin{aligned} P([1] \mid \{0\}) =& \frac{P(\{0\} \mid [1])P([1])}{P(\{0\})} \\ =& \frac{P(\{0\} \mid [1])P([1])}{P(\{0\} \mid [0])P([0]) + P(\{0\} \mid [1])P([1])} \\ =& \frac{0.2 \times 0.7}{0.9 \times 0.3 + 0.2 \times 0.7} \\ =& \frac{14}{41} \approx 0.341 \end{aligned}$$

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Question 4 (24 points)

Consider the equation $x_1 + x_2 + x_3 + x_4 = 13$, where each x_i is a non-negative integer for $i = 1$ to 4.

(a) (8 pt) Let S denote the collection of solutions to the above equation. What is $|S|$?

(b) (8 pt) For each solution $s \in S$, let $k(s)$ be the number of x_i's such that $x_i = 0$. How many solutions s are there in S such that $k(s) = 1$?

(c) (8 pt) Suppose we pick a solution s uniformly at random from S, and denote the random variable $K = k(s)$. What is $E[K]$?

Solution:

(a)

$$\binom{13+4-1}{3} = \binom{16}{3} = 560$$

(b) It is equivalent to choose one of the four variables to be 0 and the remaining three variables to be larger than or equal to 1. Thus,

$$\binom{4}{1} \times \binom{13-3+3-1}{2} = 264$$

(c)

$$E[K] = \frac{264}{560} \times 1 + \frac{\binom{4}{2} \times \binom{13-2+2-1}{1}}{560} \times 2 + \frac{\binom{4}{3} \times \binom{13-1+1-1}{0}}{560} \times 3 = 0.75$$

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Question 5 (20 points)

There is a box containing the same amount of two objects, balls and cubes. Each object is colored red, green or blue uniformly at random in advance. Consider the experiment in which two objects are drawn randomly from the box with **replacement**, i.e., the first drawn object is returned to the box before the second object is drawn.

(a) (10 pt) Given that at least one of the two drawn objects is *red*, what is the probability that the other one is also *red*?

(b) (10 pt) Given that at least one of the two drawn objects is a *red ball*, what is the probability that the other one is also *red*?

Solution:

(a)

$$\frac{\Pr[\text{both objects are red}]}{\Pr[\text{at least one object is red}]} = \frac{\frac{1}{3}\cdot\frac{1}{3}}{1-\frac{2}{3}\cdot\frac{2}{3}} = \frac{1}{5} \text{ or } 0.2$$

(b)

$$\frac{\Pr[\text{both objects are red and at least one is a ball}]}{\Pr[\text{at least one object is a red ball}]} = \frac{\frac{1}{6}^2 + 2\cdot\frac{1}{6}\cdot\frac{1}{6}}{1-\frac{5}{6}\cdot\frac{5}{6}} = \frac{3}{11} \text{ or } \approx 0.273$$

$$\text{or } = \frac{\frac{1}{3}^2\cdot(1-\frac{1}{2}\cdot\frac{1}{2})}{1-\frac{5}{6}\cdot\frac{5}{6}}$$

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