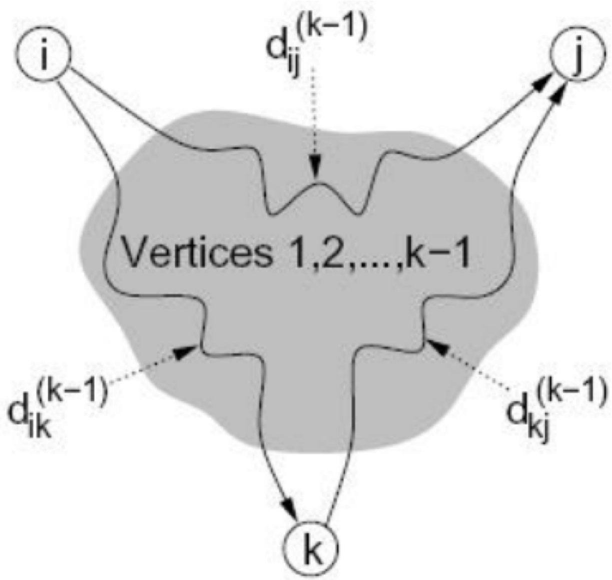
In the lecture note slides on the Floyd-Warshall Algorithm we said that it was possible to reduce the space requirement from $O(n^3)$ to $O(n^2)$ by

-not keeping each of the n nXn matrices $D^{(i)}$ but
-instead keeping only ONE matrix and reusing it.

We then wrote the code for doing that.

Why does this space-reduced code work and give the correct answer?

Recurrence



$$d_{ij}^{(k)} = min\left\{d_{ij}^{(k-1)}, \quad d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\}$$

When computing $d_{ij}^{(k)}$, there are two cases:

- Case 1: k is not a vertex on the shortest path from i to j
  => then the path uses only vertices in $\{1,2,\ldots,k-1\}$. $d_{ij}^{(k-1)}$

- Case 2: k is an intermediate node on the shortest path from i to j,
  => path can be split into shortest subpath from i to k and a subpath from k to j.
  Both subpaths use only vertices in $\{1,2,\ldots,k-1\}$ $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

The Floyd-Warshall Algorithm

Floyd-Warshall(G):

$d_{ij}^{(0)} = w(i,j)$ **for all** $1 \le i,j \le n$
for $k \leftarrow 1$ **to** n
  let $D^{(k)}$ **be a new** $n \times n$ **matrix**
  for $i \leftarrow 1$ **to** n
    for $j \leftarrow 1$ **to** n
      if $d_{ik}^{(k-1)} + d_{kj}^{(k-1)} < d_{ij}^{(k-1)}$ **then**
        $d_{ij}^{(k)} \leftarrow d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$
      else
        $d_{ij}^{(k)} \leftarrow d_{ij}^{(k-1)}$
return $D^{(n)}$

Floyd-Warshall II (G):

$d_{ij} = w(i,j)$ **for all** $1 \le i,j \le n$
for $k \leftarrow 1$ **to** n
  for $i \leftarrow 1$ **to** n
    for $j \leftarrow 1$ **to** n
      if $d_{ik} + d_{kj} < d_{ij}$ **then**
        $d_{ij} \leftarrow d_{ik} + d_{kj}$
return D

Surprising discovery: If we just drop all the superscripts,
i.e., the algorithm just uses one $n \times n$ array D => the algorithm still works! **WHY?**

We will show that

> if at the START of the k'th stage in F-W II $d_{ij} = d_{ij}^{(k-1)}$
>
> => at the END of the k'th stage in F-W II $d_{ij} = d_{ij}^{(k)}$

Note that if this statement is correct
=> at the very end of the algorithm, $d_{ij} = d_{ij}^{(n)}$ holds the correct answer

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Floyd-Warshall(G):
d_ij^(0) = w(i,j) for all 1 ≤ i,j ≤ n
for k ← 1 to n
    let D^(k) be a new n×n matrix
    for i ← 1 to n
        for j ← 1 to n
            if d_ik^(k-1) + d_kj^(k-1) < d_ij^(k-1) then
                d_ij^(k) ← d_ik^(k-1) + d_kj^(k-1)
            else
                d_ij^(k) ← d_ij^(k-1)
return D^(n)
```

```
Floyd-Warshall II (G):
d_ij = w(i,j) for all 1 ≤ i,j ≤ n
for k ← 1 to n
    for i ← 1 to n
        for j ← 1 to n
            if d_ik + d_kj < d_ij then
                d_ij ← d_ik + d_kj
return D
```

We will show that if at the START of the k'th stage in F-W II $\boldsymbol{d_{ij} = d_{ij}^{(k-1)}}$

=> at the END of the k'th stage in F-W II $\boldsymbol{d_{ij} = d_{ij}^{(k)}}$

Observation is that during kth stage of F-W II the items in the form d_{ik} are not changed. This is because $d_{kk} = 0$, so when processing

$$\textbf{if } d_{ik} + d_{kk} < d_{ik} \textbf{ then } \; d_{ik} \leftarrow d_{ik} + d_{kk},$$

the **if** statement is not activated and $d_{ik} = d_{ik}^{(k-1)}$ doesn't change during entire k'th phase.

Similarly, no items of the form d_{kj} change at all during the k'th stage so $d_{kj} = d_{kj}^{(k-1)}$ during the entire k'th phase

Floyd-Warshall(G):

$d_{ij}^{(0)} = w(i,j)$ **for all** $1 \le i,j \le n$
for $k \leftarrow 1$ **to** n
  let $D^{(k)}$ **be a new** $n \times n$ **matrix**
  for $i \leftarrow 1$ **to** n
    for $j \leftarrow 1$ **to** n
      if $d_{ik}^{(k-1)} + d_{kj}^{(k-1)} < d_{ij}^{(k-1)}$ **then**
        $d_{ij}^{(k)} \leftarrow d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$
      else
        $d_{ij}^{(k)} \leftarrow d_{ij}^{(k-1)}$
return $D^{(n)}$

Floyd-Warshall II (G):

$d_{ij} = w(i,j)$ **for all** $1 \le i,j \le n$
for $k \leftarrow 1$ **to** n
  for $i \leftarrow 1$ **to** n
    for $j \leftarrow 1$ **to** n
      if $d_{ik} + d_{kj} < d_{ij}$ **then**
        $d_{ij} \leftarrow d_{ik} + d_{kj}$
return D

We will show that if at the START of the k'th stage in F-W II $d_{ij} = d_{ij}^{(k-1)}$
=> at the END of the k'th stage in F-W II $d_{ij} = d_{ij}^{(k)}$

Just saw that during the entire k'th phase $d_{ik} = d_{ik}^{(k-1)}$ and $d_{kj} = d_{kj}^{(k-1)}$.

Thus the if-then statement

if $d_{ik} + d_{kj} < d_{ij}$ **then** $d_{ij} \leftarrow d_{ik} + d_{kj}$

will be activated if and only if the if-then statement

if $d_{ik}^{(k-1)} + d_{kj}^{(k-1)} < d_{ij}^{(k-1)}$ **then** $d_{ij}^{(k)} \leftarrow d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

is activated.

$$\boxed{\text{=> at the end of the phase } d_{ij} = d_{ij}^{(k)}}$$