COMP2121A: Discrete Mathematics
Problem Solving Session 1 Sep 11, 2018

Review Propositional logic

1. proposition: a statement that either true or false, but not both
2. logical operators: $\neg$(negation), $\wedge$(and), $\vee$(or), $\oplus$(exclusive or), $\rightarrow$(implication), $\leftrightarrow$(biconditional)
3. truth table: used to determine the truth value of propositions

Predicates and Quantifiers

1. predicates: a statement that may be true or false depending on the values of its variables
2. domain: collection of values a variable would take
3. quantifiers: $\forall$(universal, for all), $\exists$(existential, there exists)

Questions

1. Express these system specifications using the propositions p "The message is scanned for viruses" and q "The message was sent from an unknown system" together with logical connectives (including negations).

   (a) "The message is scanned for viruses whenever the message was sent from an unknown system."

   (b) "The message was sent from an unknown system but it was not scanned for viruses."

   (c) "It is necessary to scan the message for viruses whenever it was sent from an unknown system."

   (d) "When a message is not sent from an unknown system, it is not scanned for viruses."

   Solution:

   (a) $q \rightarrow p$.

   (b) $\neg p \wedge q$.

   (c) $q \rightarrow p$.

   (d) $\neg q \rightarrow \neg p$.

2. Show that $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

   Solution:

$$
\begin{aligned}
& \neg p \to (q \to r) \\
\Leftrightarrow & p \vee (q \to r) \\
\Leftrightarrow & p \vee (\neg q \vee r) \\
\Leftrightarrow & \neg q \vee (p \vee r) \\
\Leftrightarrow & q \to (p \vee r)
\end{aligned}
$$

3. Let $P(x)$ be "x can speak English." and $Q(x)$ be "x knows C++.", where the universe of discourse is the set of all students in our class. Use quantifiers to express the following statements.

   (a) There is a student in our class who can speak English but does not know C++.

   (b) Every student in our class speaks English.

   (c) Every one that knows C++ in our class can speak English.

   Solution:

   (a) $\exists x(P(x) \wedge \neg Q(x))$.

   (b) $\forall x P(x)$.

   (c) $\forall x(Q(x) \rightarrow P(x))$.

4. The following four cards sit on a table:

Each card has a digit on one side and a letter on the other side. Which cards should you turn around to test the following statement: "whenever there is a vowel on one side of a card, there is an even digit on the other side"? (In English, vowels are letters A, E, I, O, U.)

Solution: [1] Let the four cards be the domain of variable x. And let the predicates $V(x)$ be statement that there is a vowel on card x, let $E(x)$ be statement that there is an even digit on card x. The statement "whenever there is a vowel on one side of a card, there is an even digit on the other side" can be interpreted as

$$\forall x[V(x) \to E(x)].$$

Card "E" should be turned to verify that there is an even digit on the other side. When there is an odd digit on the other side, the statement is not true.

Card "V" does not need to be turned; it is not a vowel and therefore it does not matter what kind of digit is on the other side.

Card "2" also does not need to be turned. Whether there is a vowel or a consonant on the other side, the card always satisfies the statement: after all, it is not stated that only cards with a vowel have an even digit on the other side!

Card "7" should be turned to verify that there is no vowel on the other side. When there is a vowel on the other side, the statement is not true.

[1]This puzzle is from `www.puzzle.dse.nl/logical/index_us.html`.

5. Given two statements p and q, we say that p is a stronger statement than q, if $p \to q$. Prove that if p_1 is stronger than p_2, then $(p_2 \to q)$ is stronger than $(p_1 \to q)$.

Solution: By definition, we need to show that $(p_1 \to p_2) \to ((p_2 \to q) \to (p_1 \to q))$.

$$
\begin{aligned}
& (p_2 \to q) \to (p_1 \to q) \\
\Leftrightarrow & \neg(p_2 \to q) \vee (p_1 \to q) \\
\Leftrightarrow & \neg(\neg p_2 \vee q) \vee (\neg p_1 \vee q) \\
\Leftrightarrow & (p_2 \wedge \neg q) \vee \neg p_1 \vee q \\
\Leftrightarrow & (p_2 \vee \neg p_1 \vee q) \wedge (\neg q \vee \neg p_1 \vee q) \\
\Leftrightarrow & p_2 \vee \neg p_1 \vee q \\
\Leftrightarrow & (p_1 \to p_2) \vee q.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& (p_1 \to p_2) \to ((p_2 \to q) \to (p_1 \to q)) \\
\Leftrightarrow & (p_1 \to p_2) \to ((p_1 \to p_2) \vee q) \\
\Leftrightarrow & \neg(p_1 \to p_2) \vee ((p_1 \to p_2) \vee q) \\
\Leftrightarrow & \text{True}.
\end{aligned}
$$