1 Problem 1

The random variable ξ has Poisson distribution with the parameter λ. If $\xi = k$ we perform k Bernoulli trials with the probability of success p. Let us define the random variable η as the number of successful outcomes of Bernoulli trials. Prove that η has Poisson distribution with the parameter $p\lambda$.

1.1 Solution

Proof. Let $P(\eta = m)$ - probability that a random variable $\eta =$ m, then

$P(\eta = m) = \sum\limits_{k=m}^{\infty} C_k^m \cdot p^m \cdot (1-p)^{k-m} \cdot P(\xi = k) = \sum\limits_{k=m}^{\infty} C_k^m \cdot p^m \cdot (1-p)^{k-m} \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \sum\limits_{k=m}^{\infty} \frac{k!}{m!(k-m)!} \cdot p^m \cdot (1-p)^{k-m} \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \frac{p^m \cdot e^{-\lambda}}{m!} \sum\limits_{k=m}^{\infty} \frac{(1-p)^{k-m}}{(k-m)!} \lambda^k$

$k - m = k^{'}$

$=> P(\eta = m) = \frac{p^m \cdot e^{-\lambda}}{m!} \sum\limits_{k^{'}=0}^{\infty} \frac{(1-p)^{k^{'}}}{(k^{'})!} \lambda^{k^{'}+m} = \frac{(p\lambda)^m \cdot e^{-\lambda}}{m!} \sum\limits_{k^{'}=0}^{\infty} \frac{(1-p)^{k^{'}}}{(k^{'})!} \lambda^{k^{'}}$

Rather well-known fact that, $e^{(1-p)\lambda} = \sum\limits_{k^{'}=0}^{\infty} \frac{((1-p)\lambda)^{k^{'}}}{(k^{'})!}$

$=> P(\eta = m) = \frac{(p\lambda)^m}{m!} \cdot e^{-\lambda + (1-p)\lambda} = \frac{(p\lambda)^m}{m!} \cdot e^{-p\lambda}$, $p\lambda > 0$

$=> \eta$ has Poisson distribution with the parameter $p\lambda$.
Q.E.D. $\square$

2 Problem 2

A strict reviewer needs t_1 minutes to check assigned application to Deep|Bayes summer school, where t_1 has normal distribution with parameters $\mu_1 = 30$, $\sigma_1 = 10$. While a kind reviewer needs t_2 minutes to check an application, where t_2 has normal distribution with parameters $\mu_2 = 20$, $\sigma_2 = 5$. For each application the reviewer is randomly selected with 0.5 probability. Given that the time of review $t = 10$, calculate the conditional probability that the application was checked by a kind reviewer.

2.1 Solution

Let A_1 and A_2 be events, when the application was checked by a kind reviewer and strict reviewer respectively. B is event, when $t = 10$.

Using the Bayes theorem, we obtain:

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)} \ ;$$

It's obvious that: $P(A_1) = P(A_2) = 0.5$

Because t_1 has normal distribution with parameters $\mu_1 = 30$, $\sigma_1 = 10$

1) $P(B|A_1) = \int\limits_{-\infty}^{10} \frac{1}{10 \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-30)^2}{2 \cdot 10^2}} \, dx$

let $p = \frac{x-30}{10} => P(B|A_1) = \int\limits_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \, dp$

2) Similarly,

$P(B|A_2) = \int\limits_{-\infty}^{10} \frac{1}{5 \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-20)^2}{2 \cdot 5^2}} \, dx$

$l = \frac{x-20}{5} => P(B|A_2) = \int\limits_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{l^2}{2}} \, dl$

3) $P(A_1|B) = \dfrac{0.5 \cdot \int\limits_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \, dp}{0.5 \cdot \int\limits_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{l^2}{2}} \, dl + 0.5 \cdot \int\limits_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \, dp} = 0.5$

Answer: 0.5