Economics of Financial Markets
July Exam Project

Colaiacomo Luca[1] **, Xhelaj Ken**[2]
A.A. 2022-2023 Quantitative Finance

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[1]luca.colaiacomo@studio.unibo.it
[2]ken.xhelaj@studio.unibo.it

Contents

Tools used for the analysis

We choose to conduct the entire analysis using Python. The software is open source and compared to many other programming languages, it has an intuitive use that fits well our needs and our previous coding experiences. Moreover it has extensive peer-to-peer support and a wide assortment of libraries with open access to their source. We rely on the following libraries to make the computations as intuitive and readable as possible:

- Numpy: introduces the *array* object, allowing the user to perform numerical computing and matrix algebra more efficiently.
- Pandas: introduces *DataFrame* and *Series* object, it interacts with Excel to import and export data, has several methods that allow preprocessing, manipulation and statistics computations.
- MatplotLib: graphing library, strongly inspired by MATLAB plotting style.
- PyPortfolioOpt: portfolio optimization and asset allocation libray. It interacts with Pandas DataFrames and Matplotlib graphs to compute returns, covariances, efficient frontiers, run optimization routines with a wide range of criteria and even provides a Black Litterman toolbox. We rely extensively on this library throughout the entire analysis.

We recommend that the interested reader installs and updates all these libraries to the latest version, to make sure the code runs on their terminal as well. We provide the code both in one single Jupyter Notebook (which we suggest to use, given the possibility to run and visualize the outputs one chunk of code at a time) and in separate source codes for each question. As coding style, we have favoured readability over elegance.

Finally we provide also Excel spreadsheets for each table computed.

1 Asset Allocation in the Italian Stock Market

1.1 Daily and Monthly Stock Statistics (Q 1-2)

1.1.1 Return analysis

Given the spreadsheet of prices per ticker, the first thing we have to do is clean the dataset from unnecessary columns.

Our end result is a Pandas DataFrame having the stock tickers as column labels and timestamps as row labels. We preliminarly inspect the prices timeseries and observe that we are provided data for 88 stocks. The data spans from January 1st 2015 to July 4th 2023, comprising then 2219 daily entries (103 monthly). Of the 88 stocks, not all have a full time series: Pirelli (I:PIRL); Illimity Bank (I:ILLB); Aquafil (I:SPAC); Equita Group (I:EQUI); Italgas (I:IG); Enav (I:ENAV); Poste Italiane (I:PST); Gambero Rosso (I:GAMB) all present some missing data.

The following securities have been delisted during the time sample provided: Astaldi (I:AST); Dea Capital (I:DEA), Banca Intermobiliare (I:BIM) Borgosesia RSP (I:BOR), Cattolica Assicurazioni (I:CASS).

We care both about missing data and delistings, since our end goal is building a portfolio and to have a better estimate we want to have as many observations as possible and active stocks during the entire period.

We construct a DataFrame for the daily and monthtly returns of each stock and one with their main descriptive statistics: mean, standard deviation, variance, skewness and kurtosis.

We present here only a sample of the results, while the complete tables can be found in the appendix.

Sample of daily returns statistics

	Mean	SD	Variance	Skewness	Kurtosis
I:LDO	0.000406336	0.023371919	0.000546247	-0.334182707	11.76002227
I:ECK	0.000651427	0.031898572	0.001017519	2.543398019	18.73465056
I:LRZ	0.000223826	0.030731711	0.000944438	1.037096265	11.64504118
I:PIRL	-9.94865E-06	0.022295283	0.00049708	-0.041668302	6.072628837
I:STL	0.000833429	0.024366738	0.000593738	-0.451300597	5.447452701
I:PINF	0.000270955	0.03538062	0.001251788	-0.550216643	95.226757

Sample of monthly returns statistics

	Mean	SD	Variance	Skewness	Kurtosis
I:LDO	0.009081459	0.110900047	0.01229882	0.199375341	2.783640575
I:ECK	0.025371239	0.298662767	0.089199448	7.56437303	68.29474565
I:LRZ	0.006059619	0.169457622	0.028715886	3.573282226	22.04243009
I:PIRL	-0.001640596	0.094643305	0.008957355	-0.43077611	0.917258478
I:STL	0.019003352	0.119976455	0.01439435	-0.208905667	0.9180149
I:PINF	-0.002383977	0.119725733	0.014334251	1.101458062	4.288342512

We remark that the well known stylized facts about financial returns are once again confirmed: returns are not normally distributed. This is especially observable looking at skewness and kurtosis. If they were normal, these values would be 0 and 3 respectively. The stocks in our sample behave quite differently instead, and exhibit extraordinary statistics: as an example, Pininfarina (I:PINF) has a slightly asymmetric distribution of daily returns (skew=-0.55) but definitely heavy tails (kurt=95.22).

Exceptions aside, most securities overall tend to approach normality in the monthly statistics. This is to be expected, since data at sparser frequencies is less affected by noise.

In conclusion, all the evidence against normality means that we have to be cautious when using models that assume normality of returns.

1.1.2 Variance-Covariance Matrix

The variance-covariance matrix $\boldsymbol{\Sigma}$ provides us with information about co-movements between securities.

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & .. & \sigma_{2,N} \\ \dots & \dots & \dots & \dots \\ \sigma_{N,2} & \dots & \dots & \sigma_N^2 \end{bmatrix} \tag{1}$$

The matrix is understood to be symmetric and positive semidefinite in order to ensure invertibility. The diagonal is composed by the variances, while the rest of the matrix is populated by covariances. From this matrix we can also construct the correlation matrix, where each entry is normalized by the variance, so that the matrix is populated by values between -1 and +1 (representing respectively perfect negative correlation and perfect positive correlation).

There are several ways to compute the variance-covariance matrix, relying also on shrinkage methods. However, considered the scope of our project, we decided to stick to the sample covariance. Here, we present just a sample:

	I:LDO	I:ECK	I:LRZ	I:PIRL
I:LDO	0.137654146	0.018841401	0.042769522	0.052149758
I:ECK	0.018841401	0.256414764	0.017160375	0.017505362
I:LRZ	0.042769522	0.017160375	0.237998389	0.047663723
I:PIRL	0.052149758	0.017505362	0.047663723	0.125264065

To see the full covariance and correlation matrices, we point the reader to the attached spreadsheets '*covariance matrix.xlsx*' and '*correlation matrix.xlsx*'.

1.2 Stock Sample Picking(Q 3-4)

1.2.1 Stock selection

One of the pillars of portfolio optimization theory and in particular of the Capital Asset Pricing Model is the selection of securities sample. There can be followed several criteria.

Following the assignment hint, we inspect carefully the variance-covariance matrix. From the theory we have that the average return of a portfolio is the average of returns of the individual securities, while the variance can be lower than the average variance of the components. Defining w_1, w_2 the weights of securities 1 and 2 respectively, σ_1, σ_2 the standard deviations and $\rho_{1,2}$ the correlation between the two securities:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2} \tag{2}$$

To eventually minimize the portfolio volatility, we then want to identify those securities that are as little correlated as possible with each other, pairwise.

We go through the variance covariance matrix and identify those pairs whose correlation coefficient was under an arbitrary threshold of 0.07, which we find to be a good compromise between looking for weak comovements and still having a large enough pool to pick stocks from. At this stage we also decide to not consider in the loop the aforementioned stocks that either went delisted or missed data. The head of the output is as follows, while the full output can be found in the attachment '*pair correlations.docx*' or by running the code.

```
Pair I:ECK I:LRZ has correlation 0.06946535562810022
Pair I:ECK I:PINF has correlation 0.05059919411879345
Pair I:ECK I:CALT has correlation 0.06199489043529807
Pair I:ECK I:TRN has correlation 0.06051789968588873
...
Pair I:SAFI I:MON has correlation 0.04612681161826484
Pair I:SAFI I:FUL has correlation 0.043273798522091976
Pair I:SAFI I:ZUC has correlation 0.05633912288455402
Pair I:SAFI I:VIN has correlation 0.042259757731528784
```

Then we restrict once again our selection by picking those stocks that came up more often and by controlling for their historical performance. To avoid the distortion given by asymmetry of percentual returns, just in this step we looked at the average log-returns (while the rest of the analysis is conducted using percentual returns, as the optimization library is robust with respect to them).

We figure that while picking growing stocks is generally an appropriate choice, we are also required to construct a portfolio allowing short positions, so that some declining stocks could be also profitable over the timeframe of reference. Eventually, the optimizer will favour one kind of stock or the other, conditional on the constraints.

1.2.2 Statistics of selected stocks

We then present the descriptive statistics of the stocks we picked and the sector in which they operate.

Daily statistics for selected stocks

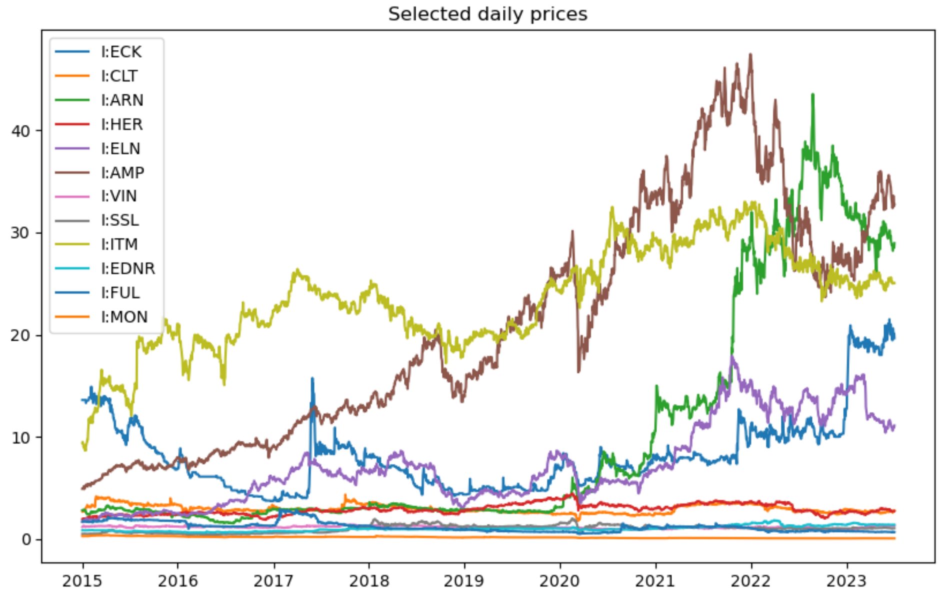
	Sector	Mean	SD	Variance	Skewness	Kurtosis
I:ECK	Energy	0.000651427	0.031898572	0.001017519	2.543398019	18.73465056
I:CLT	Food	0.000197181	0.019352048	0.000374502	2.785542138	27.3090626
I:ARN	Energy	0.001364262	0.025350024	0.000642624	1.667671836	12.30778965
I:HER	Utilities	0.000276584	0.015678451	0.000245814	-0.51385184	12.47392089
I:ELN	Biomedical	0.001136442	0.023664925	0.000560029	0.265945908	3.643639201
I:AMP	Biomedical	0.001060119	0.02013177	0.000405288	-0.392844048	6.042009356
I:VIN	Real Estate	0.000208336	0.017994476	0.000323801	0.382652238	3.795574282
I:SSL	Sports	0.000650783	0.024903742	0.000620196	0.276886515	9.665219793
I:ITM	Holding	0.000589089	0.017343106	0.000300783	2.213981003	33.35556366
I:EDNR	Energy	0.000321627	0.014250916	0.000203089	-0.354354712	13.4890771
I:FUL	Marketing	9.3448E-05	0.032785792	0.001074908	2.306216836	18.72592316
I:MON	Media	-0.000414356	0.025273103	0.00063873	2.346586076	22.72959091

Monthly statistics for selected stocks

	Sector	Mean	SD	Variance	Skewness	Kurtosis
I:ECK	Energy	0.025371239	0.298662767	0.089199448	7.56437303	68.29474565
I:CLT	Food	0.002493773	0.070685118	0.004996386	1.927755847	6.266449784
I:ARN	Energy	0.029788168	0.126579807	0.016022448	1.866077433	4.568616189
I:HER	Utilities	0.005497393	0.06467491	0.004182844	-0.574694998	0.942722073
I:ELN	Biomedical	0.027150833	0.132083401	0.017446025	0.173482983	0.209869035
I:AMP	Biomedical	0.022418118	0.084960238	0.007218242	-0.69363876	1.418310413
I:VIN	Real Estate	0.002513562	0.055275054	0.003055332	0.529580055	1.443709558
I:SSL	Sports	0.01567451	0.131609844	0.017321151	0.943099869	4.599984746
I:ITM	Holding	0.012114363	0.073379858	0.005384604	2.054092811	9.317986892
I:EDNR	Energy	0.006411902	0.056729758	0.003218265	0.229022794	2.595769964
I:FUL	Marketing	0.002840904	0.189936051	0.036075703	5.091597878	32.39660169
I:MON	Media	-0.011366609	0.094244093	0.008881949	1.838077129	9.00481348

We denote that also by looking at the sectors, we have a fairly diversified sample. Our algorithm is slightly biased towards the energy sector, probably because those securities have been profitable even through troubled times in the markets, which we do observe in the 2015-2023 period.

What follow are the plots for the price action of our selected sample.


Selected daily prices
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I:CLT
I:ARN
I:HER
I:ELN
I:AMP
I:VIN
I:SSL
I:ITM
I:EDNR
I:FUL
I:MON
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30
40
2015
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2017
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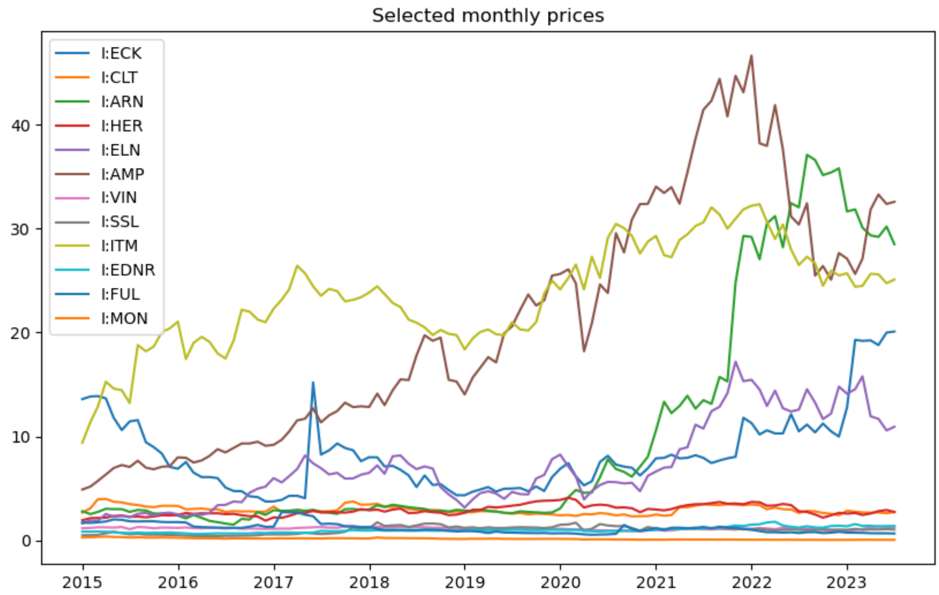
2020
2021
2022
2023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I:ECK
I:CLT
I:ARN
I:HER
I:ELN
I:AMP
I:VIN
I:SSL
I:ITM
I:EDNR
I:FUL
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2023

1.3 Mean-Variance Portfolio Optimization (Q 5-6-7)

1.3.1 Mean-Variance optimization: max Sharpe

For the MV and CAPM discussion we rely on the theoretical framework provided by [Markowitz, 1952], [Sharpe, 1964] and the Lecture Notes.

The investor derives from the portfolio utility:

$$U = \mu_p - \frac{\gamma}{2}\sigma_p^2 \tag{3}$$

where γ is a parameter representing the loss of utility linked with variance. In other words, it can be thought of as the investor's risk aversion.

Clearly, mean and variance of the portfolio depend on the securities composing them. We define w the vector Nx1 of the weight of each n-th security in the portfolio. The constraint is that $w = 1$: the investor has to spend his entire wealth and not more. Let then μ be the vector Nx1 of mean returns of securities and Σ the variance-covariance matrix. Then:

$$\mu_p = w'\mu\sigma_p^2 = w'\Sigma w \tag{4}$$

Given this structure, we can construct an efficient frontier and subsequently pick the portfolio we want among those. An efficient portfolio solves the following convex optimization routine:

$$\begin{cases} \min\limits_{w} \sigma_p^2 = \frac{1}{2}w'\Sigma w \\ s.t.\ \mu_p = w'\mu \\ w'1 = 1 \end{cases} \tag{5}$$

If we solve the problem through Lagrangian multipliers, we have a closed form of the portfolio weight vector:

$$w = \Sigma^{-1}(\mu 1)I^{-1}\begin{pmatrix}\mu_p \\ 1\end{pmatrix} \tag{6}$$

where $I = (\mu 1)'\Sigma^{-1}(\mu 1)$ is the fundamental information matrix.

We can remark from the theory that the portfolio variance is defined in matrix notation as:

$$\sigma_p^2 = (mu_p 1)I^{-1}\begin{pmatrix}\mu_p \\ 1\end{pmatrix} \tag{7}$$

and in analytical notation as a parabola:

$$\sigma_p^2 = \frac{1}{\Delta}(a - 2b\mu_p + c\mu_p^2) \tag{8}$$

where $\Delta = ac - b^2$. If we let μ_p vary we can then plot the entire frontier.

When we look for the optimal portfolio among all the feasible ones, then we want to define an objective function that puts in relation the investor's preferences in terms of required mean and tolerated risk. One option is fixing the mean and minimizing the risk, or fixing the risk and maximizing the return. Realistically, there might be other constraints, such as diversification by sector or lately, ESG rating requirements. This is to say that even if the mathematics of CAPM might seem to leave not much choice, in reality there is room for flexibility and portfolio manager skills.

The assignment consists in finding the optimal portfolio, so we decide to solve by maximizing the Sharpe ratio (SR):

$$\max_{w} \frac{E[R_p]}{\sigma_p} \tag{9}$$

Given the structure of investor utility, it intuitively makes sense to measure the performance of any given investment as a risk adjusted return. Clearly, the higher the SR, the better, as it means that the portfolio can generate extra returns without increasing the relative risk. In the domain of fund management, a SR above 1 can be considered good and a SR above 3 is exceptional.

Our choice is also economically motivated. If we were to introduce a risk-free rate at this stage and allow the investor to buy a mix of risky and riskless securities, we would find that the portfolio maximizing the Sharpe Ratio is also the portfolio connecting the Capital Market Line and the efficient frontier (i.e. tangency portfolio). However this is not required of us, so we will not run again the computations for the sake of simplicity.

We run the optimization both in the daily and monthly frequency. First we allow for weights to be negative (i.e. short positions are feasible). We obtain the following weights:

	Daily weights			Monthly weights
I:ECK	-0.01865		I:ECK	-0.00721
I:CLT	-0.30836		I:CLT	-0.18813
I:ARN	0.90248		I:ARN	0.39525
I:HER	-0.70313		I:HER	-0.15492
I:ELN	0.67947		I:ELN	0.31714
I:AMP	1		I:AMP	0.7588
I:VIN	-0.04911		I:VIN	0.23868
I:SSL	0.09774		I:SSL	0.095
I:ITM	0.49361		I:ITM	0.48396
I:EDNR	-0.04303		I:EDNR	0.0663
I:FUL	-0.29342		I:FUL	-0.00487
I:MON	-0.75761		I:MON	-1
Expected returns	86.80%		Expected returns	63.10%
Expected volatility	69.10%		Expected volatility	44.70%
Sharpe Ratio	1.23		Sharpe Ratio	1.37

On the daily data, we see that the optimizer strongly favours Amplifon and Alerion, allocating the biggest chunk of wealth to these two securities. It is unsurprising, given that their mean returns are among the best in the sample, while their variance is not particularly out of line. In a similar fashion, Monrif is the most shorted security, considered its persistent downtrend. As far as the portfolio goes, it has an expected annual return of 86,8%, an expected annual volatility of 69.1% and a Sharpe Ratio of 1.23.

On the monthly data, the long allocation still favours Amplifon, while it allocates fewer resources to Alerion. Overall we have less stocks that are shorted. Remarkably, Monrif gets shorted even more than in the daily portfolio, to capitalize once again on the definite negative mean returns. Overall, the portfolio nets an expected annual return of 63.1%, expected annual volatility of 44.7% and Sharpe ratio of 1.37. We denote then that the monthly portfolio is less volatile, as it is expected, given the overall better behaving returns on a larger timeframe. Even then, both the expected returns and the volatilities are quite high. This is a consequence of the quite aggressive sample composition, comprised of many small and medium capitalization stocks.

We would surely have a hard time trying to market these portfolios, given the kind of volatility, and this is assuming any risk manager would even approve it. Few to no investors could coupe with roughly 70% volatility. However, given that we are not provided any boundaries to operate within, we keep the sample composition that is robust with respect to the diversification principle applied to our variance-covariance matrix. Considering that specific risk is diversified with a sample of at least 30 securities, with some practitioners arguing that an even bigger sample would be better [Chen and Israelov, 2022], some degree of high volatility is still to be expected.

In real life, short positions come with a risk and might influence negatively the markets, amplifying market shocks and increasing the instability of financial institutions. The legislator then has devised a set of rules to curb short selling. Only professional individuals can assume short positions and they must be within certain value bounds.

It is then a worthwhile exercise to rerun the portfolio optimization routine, this time by imposing a non-negativity constraint on the weights: that is, we do not allow for short positions.

The new allocations and portfolio and portfolio performances are the following:

	Daily weights			Monthly weights
I:ECK	0		I:ECK	0
I:CLT	0		I:CLT	0
I:ARN	0.34106		I:ARN	0.30022
I:HER	0		I:HER	0
I:ELN	0.21414		I:ELN	0.09354
I:AMP	0.35237		I:AMP	0.50634
I:VIN	0		I:VIN	0
I:SSL	0		I:SSL	0
I:ITM	0.09243		I:ITM	0.0999
I:EDNR	0		I:EDNR	0
I:FUL	0		I:FUL	0
I:MON	0		I:MON	0
Expected returns	25.10%		Expected returns	25.60%
Expected volatility	23.80%		Expected volatility	23.50%
Sharpe Ratio	0.97		Sharpe Ratio	1

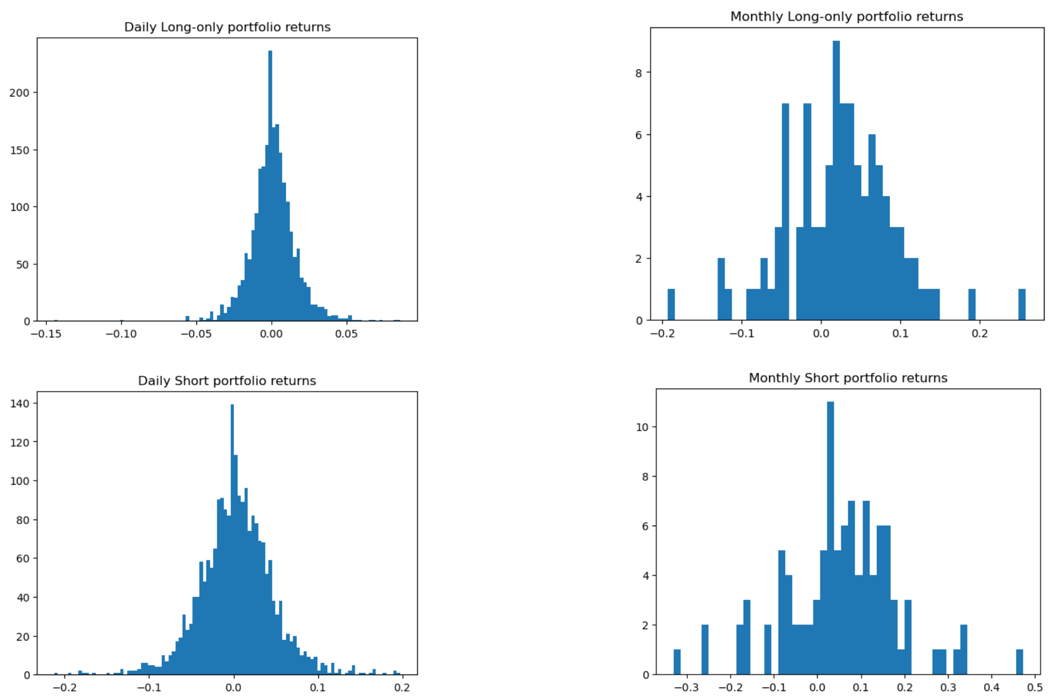
The selected stocks are quite similar to those who had the biggest positive weights in the unconstrained optimization. Most weights are negligible and have been rounded to zero, so that the only significant securities picked are Alerion, EL EN, Amplifon and ItalMobilare. In this case the daily and monthly portfolios achieve an almost identical performance: in both cases we have roughly 25% of expected annual returns, 23.5€ of expected annual volatility and 1 of Sharpe Ratio. The portfolio is overall less volatile than the short allowed one, which we would expect given the whole reasoning behind the constraint to begin with. However, the expected returns are also lower, since we cannot capitalize on downturns anymore and we cannot borrow resources to increase the wealth allocation of the winners either.

1.3.2 Descriptive statistics of the portfolios

We present as before the main statistics for all four portfolios devised until now.

	Mean	SD	Variance	Skewness	Kurtosis
Daily Long	0.001136657	0.014963678	0.000223912	-0.180660408	7.017777852
Monthly Long	0.024044127	0.067912099	0.004612053	-0.013973946	1.491595182
Daily Short	0.003412908	0.043499793	0.001892232	0.00601648	2.144784715
Monthly Short	0.055621238	0.128959408	0.016630529	-0.087056743	1.270533715

We also present the distribution of returns:



We underline that the mean returns of portfolios are generally one order of magnitude bigger than the mean returns of the single securities, while the variances are comparable, once again confirming that diversification in the MV framework is the key to obtain better returns while also lowering the risk.

Except for the unconstrained daily portfolio, we observe negative skewness which signifies heavy left tails on the distributions. The kurtosis on the constrained daily portfolio is also quite high and generally we do not have normality. This said, both monthly portfolios get closer to the 0 skew and 3 kurtosis typical of a normal distribution.

The distribution plots add one more piece of evidence against normality. Unfortunately, the monthly plots are quite crude as we do not have many observations. Anyhow, we can see outliers very out in the tails in all four plots and moderate asymmetries.

1.4 Italian Stock Market Efficient Frontier (Q8)

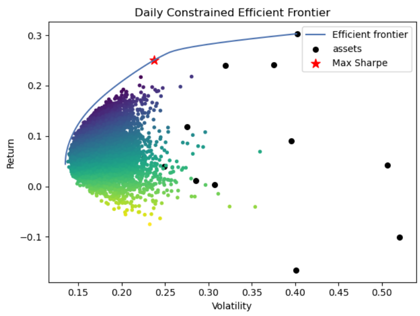
As defined by Markowitz, the efficient portfolio frontier is the set of portfolios that achieve the greatest return for any given level of volatility (risk). Considering the utility function we have defined, in which volatility is a malus component, no rational investor would settle for a portfolio that does not lie on the efficient frontier.

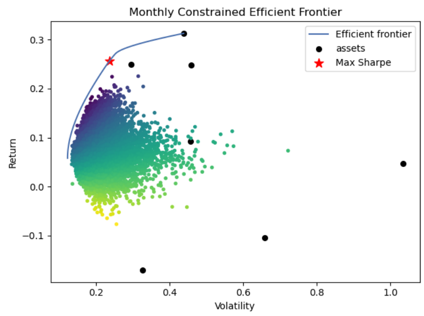
The efficient frontier typically takes the shape of a parabola, with the bottom half of it being fully dominated by the upper half.

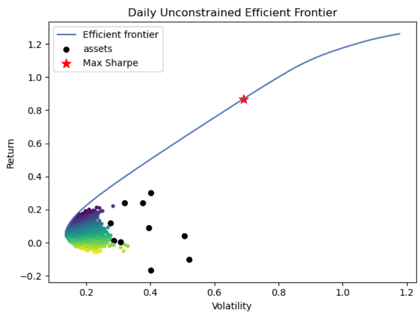
No singular asset usually lies on the efficient frontier, considering the diversification principle we have previously expanded upon.

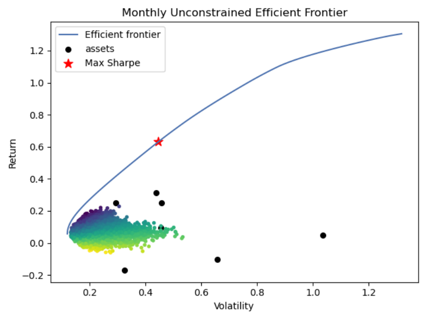
Thanks to the Separation Theorem, if we have two efficient portfolios, any linear combination of them will still be efficient and allows us to draw the entire efficient frontier.

We now present the plots of efficient frontiers for daily and monthly data, with long positions only and with shorts allowed.









The point plotted with a red star represents the portfolio having the maximum Sharpe ratio (i.e. the portfolios we found weights for in the previous points), while the cloud of points is a set of 10000 simulated suboptimal portfolios, with the gradient representing their Sharpe ratios (the closer to purple, the higher). Some of our assets lie in the bottom half of the parabola and they do not contribute to the constrained optimal portfolios. Lastly, we observe that one asset actually lies on the frontier. Alerion's exceptional mean returns compared with the volatility make the stock a favourite under the M-V framework.

1.5 FTSE Italia All Market vs Our Sample Portfolio (Q9)

As market index of reference we are provided with the FTSE Italia All Market. It is composed by all securities listed in the FTSE MIB, FTSE Italia Mid Cap and FTSE Italia Small Cap. In its total returns version, the quote incorporates dividend yields also. This choice is coherent with the task of benchmarking a portfolio, considering that the investor is entitled to all dividends of the securities he owns.

We have here extracted the main statistics for the index daily and monthly, comparing them with the portfolios.

Daily index and portfolios comparisons

	FITASHE (Daily)	Daily Long	Daily Short
Mean	0.000418251	0.001136657	0.003412908
SD	0.013778725	0.014963678	0.043499793
Variance	0.000189853	0.000223912	0.001892232
Skewness	-1.331955097	-0.180660408	0.00601648
Kurtosis	15.22110187	7.017777852	2.144784715

Monthly index and portfolios comparisons

	FITASHE (Mo)	Monthly Long	Monthly Short
Mean	0.005800819	0.024044127	0.055621238
SD	0.058340032	0.067912099	0.128959408
Variance	0.003403559	0.004612053	0.016630529
Skewness	-0.503169902	-0.013973946	-0.087056743
Kurtosis	2.82911905	1.491595182	1.270533715

We see once more confirmation that returns are not normally distributed, not even at market-wide level. This is definitely accentuated in the daily statistics, while the monthly ones get rather close, with a skewness that is almost 0 and a kurtosis that is almost 3. At both frequencies, returns have longer left tails. Overall the market has been growing during the sample, but the growth is not particularly remarkable when compared with other indices such as the SP500.

If we compare the market returns with the portfolio returns, we see across all cases that our portfolios achieve way better mean returns. Constrained portfolios have comparable standard deviations, while short portfolios are riskier, as it is expected. We remark however that our portfolios are comprised by relatively small capitalized securities, while the FTSE Italia All Share takes into account also large cap Italian firms, which are less volatile on average. This considered, then, the comparison is a good starting point, but a more accurate benchmark for our portfolio could be an index comprised of selected securities, such as the FTSE Italia Mid Cap.

1.6 Security Market Line: an Empirical Check(Q 10-11)

The concepts of β and Security Market Line (SML) allow us to define the equilibrium average return for each security in the market, assuming to have a market index to use as benchmark.

Assuming that the average of returns is a consistent estimator, $\mu_i = E[R_i]$, and defining a certain risk-free return R_f, we have:

$$E[R_i] = R_f + \beta_i(E[R_m] - R_f) \qquad \text{where} \qquad \beta_i = \frac{\sigma_{i,m}}{\sigma_m^2} = \rho_{i,m}\frac{\sigma_i}{\sigma_m} \tag{10}$$

The difference between market returns and risk free rate is the market premium and mathematically represents the slope of the Security Market Line and it is understood to be a common factor across all stocks in one economy. The β of any given security can be thought of as the risk, compared to the market, that it embeds. Then the expected return is composed of a risk free baseline, augmented by the risk premium, in proportion with the riskiness of the stocks picked.

In the next two points we will elaborate on the betas and the Security Market Line plots in our specific case

1.6.1 Beta of securities and portfolio

From the definition of beta, we have that for the market $\beta_m = 1$ and trivially any security with such a coefficient replicates exactly the market returns. In all other cases we can operate the following classification:

- $\beta > 1$: the security is said to be ”aggressive” and its volatility is greater than that of the market. For this reason the investor expects to be compensated more than the market risk premium.
- $\beta < 1$: the security is ”defensive” and its volatility is lower than that of the market, so that the compensation follows
- $\beta < 0$: the security is a ”hedger” and it acts counter cyclically, lowering the overall riskiness of a portfolio in which they belong. Given the safe nature, their compensation is even smaller than the risk free rate.

We choose the FTSE Italia All Share as our market of reference and calculated the betas on daily and monthly data of each security and of each portfolio found.

Security	Beta daily	Beta monthly
I:ECK	0.297940748	0.683024602
I:CLT	0.399765257	0.487835253
I:ARN	0.429693089	0.523879251
I:HER	0.641404211	0.616664052
I:ELN	0.786438845	1.310397993
I:AMP	0.640364543	0.56674901
I:VIN	0.123082911	0.202506745
I:SSL	0.573530052	1.083660945
I:ITM	0.504499691	0.527289411
I:EDNR	0.448841285	0.381022717
I:FUL	0.354071896	0.326413073
I:MON	0.308196348	0.842292253
Constrained	0.587234694	0.619500176
Unconstrained	0.925042657	0.448305105

Our sample securities in the daily frequency appear to be conservative in their beta estimates and most of them are around the 0.5 mark. We do not have any hedging security in our sample, so that our risk cannot be mitigated to the fullest.

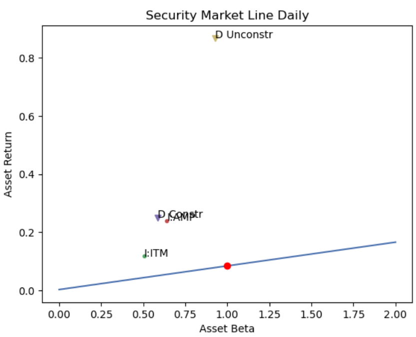
In the monthly data however we observe that several securities have a higher beta and some of those even tipped above one, such as EL EN and Società Sportiva Lazio. We could attribute these differences to changes that appear more abrupt in the monthly data than in the daily.

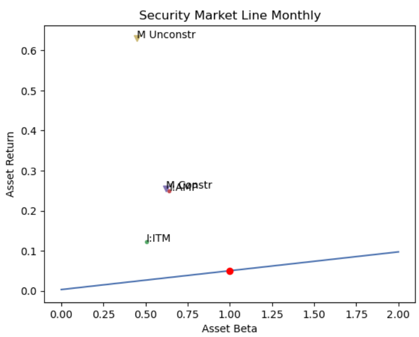
The portfolio betas are defensive. We see that in the constrained situation the monthly beta is higher than the daily beta, as a consequence of what we have just observed. The daily unconstrained portfolio exhibits the highest beta, at 0.925, which means that it almost replicates the market volatility. We were expecting this portfolio to behave in this way.

1.6.2 Security Market Line

The Security Market Line allows us to plot the expected return of any given security as a function of its beta times the market risk premium. Since we have just computed all the betas, we can figure out whether our securities and our portfolios are underperforming or overperforming, according to this one metric.

Keeping in mind all the introduction about the SML, we here present the plots using daily and monthly data.





We arbitrarily picked ITM (ItalMobiliare) and ELN (EL EN) as securities to compare against the SML. We also plotted the constrained and unconstrained portfolios. The red dot represents the market ($\beta = 1$, r_m).

The daily and monthly plots are similar looking. That is because when we annualize the returns, the results are comparable (rightly so). We notice that ITM is slightly overvalued, but it is still within range of its worth according to the theory. EL EN and the constrained portfolio are instead mildly overvalued. Coincidentally, they happen to have overlapping beta and returns, as we could have predicted looking at all the descriptive data presented until now.

What can be thought of as an outlier really is the unconstrained portfolio. Despite it having a relatively low beta, it historically outperformed the market.

1.6.3 Concluding remarks on CAPM

We want to remark once again that this analysis has to be taken with a pinch of salt. However popular, the CAPM theory relies on a set of strong assumptions. Market efficiency is not verified empirically, at least on this set of securities. It is quite unrealistic to think of the beta as the only measure of risk. We conjecture that if we could look at more factors that determine the price of any one security, we could figure out a fair value that is more coherent with our empirical results. Lastly, it has been shown that CAPM weights are sensitive to sampling errors: even small differences in estimated returns and volatility can lead to contrasting optimal allocations.

In the next two points, we take a Bayesian approach that hopefully adjusts part of CAPM pitfalls.

1.7 Black Litterman vs Standard Mean-Variance (Q12)

The Black - Litterman asset allocation model has been devised in 1990 by Fischer Black and Robert Litterman at Goldman Sachs [Black and Litterman, 1991]. The driving need behind the new model are the highly concentrated and error sensitive weights resulting from the CAPM estimation we have seen until now. In more than one paper, Litterman observes results that we have also seen until now: large long and short positions in unconstrained optimization and only a handful of securities used in constrained optimization.

The reasoning behind the model lies in obtaining a better estimate of returns, upon which to build the efficient frontier. This is done through Bayesian reasoning. We leave the mathematical details for the next question, but we anticipate that the main intuition consists in updating the prior knowledge of the world with a set of evidence or views in order to obtain a posterior estimate of the world. We are then able to integrate in one elegant framework what the data suggest and what the experience of the manager tells.

We follow the notation of [Idzorek, 2019] in defining the posterior estimate of returns:

$$E[R] = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q] \tag{11}$$

where we have K views and N assets:

- E[R] is the posterior return vector (N x 1)
- τ is a scalar
- Σ prior covariance matrix (N x N)
- P is the matrix that links together the different assets in the views (K x N)
- Ω is the uncertainty of views matrix (K x K)
- Π is the prior return vector (N x 1)
- Q is the view vector (K x 1)

The posterior variance-covariance matrix is instead defined as:

$$\sigma_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1} \tag{12}$$

Our job is then to define each of those quantities. As far as the prior returns Π and covariances Σ go, we have decided to settle for the historical data. We are aware that it is an imprecise assumption and that the literature on the Black - Litterman model favours the Implied Equilibrium Return vector. However, we do not possess sufficient data nor sophistication to obtain it and we will assume some consistency in the historical means.

There is an active discussion around the choice of τ and it typically settles with values close to 0. Since we will specify the uncertainty, however, this scalar is mathematically irrelevant, so we do not have to take a position.

As far as the confidence in our views goes, we have decided to take a conservative approach. We assess that we do not possess the necessary investment experience nor the necessary data to place a number on how confident we are in our views and not incur into biases. What we decide to do, then, is to use an uncertainty matrix that is proportional to the variance of each security. It is reasonable to our eyes that if a stock is more volatile, we will have a harder time being right in our views. Then:

$$\Omega = \tau \cdot P \Sigma P' \tag{13}$$

Finally, the key of this model, we list here our views:

- We expect BeeWize (I:FUL) to keep losing ground, so we set the views for an annualized -15% expected return. We see that the company is in a dire financial situation, with losses getting bigger and bigger in the last two years. The technical indicators also suggest "Strong Sell".
- We forecast Edison (I:EDNR) to do slightly better than Alerion (I:ARN). We believe the former company to be better placed in operating structural changes in the renewable energies sector, considering its bigger dimension and better contracts already in place. On the flip side, despite being impressed with Alerion's past performance, we are not convinced that it can keep the growth going for long. In light of this, we expect Edison to beat Alerion a modest +3% annualized.
- We believe that Amplifon (I:AMP) can be a leader player in the Italian hearing devices sector. The Italian population is growing older and older, so we can expect the demand for their devices to grow. Their financials are solid and show a slowly and steadily growing net profit. In the last years their stock has been soaring at an impressive pace. Despite not being sure that the security can keep growing *that* fast, we are convinced that a +10% annualized return is to be expected in the long term.
- We think that ItalMobiliare (I:ITM) will do better than Vianini (I:VIN). The former is a holding with a well diversified investment choice that can turn out profitable. Analysts moreover have suggested a Buy position. Vianini, however, is in the construction and real estate sector. We conjecture that demand has already been saturated following the government grants of the past years, so that future performance will be disappointing. Overall, we set that ItalMobiliare outperforms Vianini at about +4% annualized.

The new constrained allocation is then:

	Daily weights			Monthly weights
I:ECK	0		I:ECK	0
I:CLT	0		I:CLT	0
I:ARN	0.24646		I:ARN	0.20585
I:HER	0		I:HER	0
I:ELN	0.32447		I:ELN	0.14212
I:AMP	0.30969		I:AMP	0.48356
I:VIN	0		I:VIN	0
I:SSL	0		I:SSL	0
I:ITM	0.0976		I:ITM	0.052
I:EDNR	0.02178		I:EDNR	0.11647
I:FUL	0		I:FUL	0
I:MON	0		I:MON	0
Expected return	17.30%		Expected return	15.70%
Expected volatility	23.70%		Expected volatility	22.20%
Sharpe Ratio	0.64		Sharpe Ratio	0.62

We can compare this portfolio with the basic M-V constrained case. We observe overall comparable allocations and performances both on daily and monthly data. In our specific case, we still obtain somewhat concentrated allocations. We can put this again on our sample that has some clear winners in it. We have a new security being allocated: Edison. Considering the positive view on it, beating even Alerion, we expected this. The still low weight assigned however is reflective of the high uncertainty around that one forecast. Having provided a relatively more conservative, yet positive, view of Amplifon's growth, we see that its weight in the portfolio has been mildly reduced. Providing a negative view on BeeWize did not change the constrained allocation to that security: the optimizer already had discarded the stock. Overall the annualized expected return has decreased, considering we are allocating fewer funds to some (historically) exceptionally performing stocks. The volatility is however comparable, so that the Sharpe Ratio declined from ≈ 1 to ≈ 0.6. Considering the adage that past performance does not necessarily extend in the future, we can expect these returns to be more realistic, but sensible portfolio management would need to curb the volatility.

What follows now is the unconstrained allocation:

	Daily weights			Monthly weights
I:ECK	-0.02169		I:ECK	-0.00923
I:CLT	-0.42464		I:CLT	-0.17238
I:ARN	0.67889		I:ARN	0.18305
I:HER	-0.99737		I:HER	-0.14236
I:ELN	0.93334		I:ELN	0.32175
I:AMP	1		I:AMP	0.55822
I:VIN	0.08545		I:VIN	0.41666
I:SSL	0.1359		I:SSL	0.08225
I:ITM	0.55989		I:ITM	0.40203
I:EDNR	0.53102		I:EDNR	0.38283
I:FUL	-0.48078		I:FUL	-0.02283
I:MON	-1		I:MON	-1
Expected return	80.70%		Expected return	44.60%
Expected volatility	80.60%		Expected volatility	39.20%
Sharpe Ratio	0.98		Sharpe Ratio	1.09

Similar observations as the constrained allocation also apply here. The only noticeable difference between the M-V allocation and the B-L is that now Edison gets ≈ 0.5 of the wealth, while before it had a moderate negative weight. The allocation still is not as smoothed as we would have hoped, as we still see assets fully bought (I:AMP in the daily) and assets fully sold (I:MON in the monthly). The performance drops too, from $\approx$ 1.3 SR to $\approx$ 1.

We also present the descriptive statistics for all four Black - Litterman portfolios.

	Daily Long BL	Monthly Long BL	Daily Short BL	Monthly Short BL
Mean	0.001097786	0.022207817	0.00364956	0.046219371
SD	0.014685952	0.063265032	0.049789053	0.110974638
Variance	0.000215677	0.004002464	0.00247895	0.01231537
Skewness	-0.358549272	-0.427361609	-0.310239728	-0.472364574
Kurtosis	6.388120925	1.704359781	2.276740714	1.555611158

All the comparisons about expected mean and volatility apply here as well, since all that changes is just the timeframe.

To conclude this section, we remark that despite our seemingly unsatisfactory results, the Black Litterman model is an extremely powerful tool. We believe that if we expressed more views and were more confident in those, we would achieve a portfolio allocation that better exploits the BL posteriors. However, we want to err on the side of caution and we prefer to remain conservative than to be confidently wrong in our forecasts.

1.8 Bayesian Asset Allocation (Q13)

For the Bayesian theoretical framework, we rely on the Lecture Notes. The Bayesian approach consists in blending one's experience, called prior, and evidence, typically observed data, to obtained an updated view of the world, called posterior. In investment terms, the portfolio manager has his own set of beliefs that refines according to what happens in the markets. We define the Bayes theorem in general terms as:

$$f_{po}(\theta|Y_T) = \frac{f(Y_T|\theta)f_{pr}(\theta)}{f(Y_T)} \tag{14}$$

where:

- $f_{po}(\theta|Y_t)$ is the posterior density
- $f_{pr}(\theta) = \pi(\theta)$ is the prior density of data
- $f(Y_T) = \int_\Theta f(Y_T, \theta)d\theta$ is the marginal density of data
- θ is the set of parameters
- Y_T is the set of observations

$f(Y_T|\theta)$ is a conditional likelihood. By definition of likelihood function we have:

$$\mathscr{L}(\theta|Y_T) = \prod_{t=1}^{T} f(Y_T|\theta) \implies f(Y_T|\theta) \propto \mathscr{L}(\theta|Y_T) \tag{15}$$

and this allows us to rewrite Bayes theorem in terms of likelihoods and simplify it as:

$$f_{po}(\theta|Y_T) = \frac{\mathscr{L}(\theta|Y_T)\pi(\theta)}{f(Y_T)} \propto \mathscr{L}(\theta|Y_T)\pi(\theta) \propto f(Y_T|\theta)\pi(\theta) \tag{16}$$

considering that we do not need the normalizing pdf because we know the prior and conditional densities.

We are provided informative priors for the mean. We are told to assume conjugate prior normally distributed, such that mean and variance are constructed starting from the sample parameters μ_h and Σ_h:

$$\pi(\mu) \sim \mathcal{N}(\mu_{pr}, \Sigma_{pr}) \tag{17}$$

where:

- $\mu_{pr} = \mu_h + \sigma_h$
- $\Sigma_{pr} = 2 \cdot \Sigma_h$

Having defined the prior, we apply the approximation of Bayes' theorem $f_{po}(\mu|Y) = f(Y|\mu)\pi(\mu)$ and obtain that for conjugate normal priors, also the posterior pdf for returns is normally distributed $\mathcal{N}(\mu_{po}, \Sigma_{po})$ and the parameters are obtained:

$$\begin{aligned} \mu_{po} &= [T\Sigma_h^{-1} + \Sigma_{pr}^{-1}]^{-1}[T\Sigma_h^{-1}\mu_h + \Sigma_{pr}^{-1}\mu_{pr}] \\ \Sigma_{po} &= [T\Sigma_h^{-1} + \Sigma_{pr}^{-1}]^{-1} \end{aligned} \tag{18}$$

We assume also that returns are normally distributed $\implies f(r_t|\mu_h, \Sigma_h) \sim \mathcal{N}(\mu_h, \Sigma_h)$. What this means is that, similarly for $t+1$ return estimations, we still have normal predictive density $f(r_{t+1}|\mu_h, \Sigma_h) \sim \mathcal{N}(\mu_h, \Sigma_h)$. In particular, through integration we obtain the result:

$$f(r_{t+1}|Y\Sigma_h) \sim \mathcal{N}(\mu_{po}, \Sigma_{po} + \Sigma_h) \tag{19}$$

Throughout the computation we use annualized returns and variance-covariance matrices, so that we're coherent with the portfolio metrics we have provided until now. Our t+1 estimate will then refer to the predictions for the following year. The estimated returns and variance covariance matrix are included in the spreadsheets '*mean predictive.xlsx*' and '*variance covariance predictive.xlsx*' respectively.

Lastly, we provide the tables for constrained allocation.

	Daily weights			Monthly weights
I:ECK	0		I:ECK	0
I:CLT	0		I:CLT	0
I:ARN	0.33343		I:ARN	0.29695
I:HER	0		I:HER	0
I:ELN	0.21194		I:ELN	0.09723
I:AMP	0.34265		I:AMP	0.49255
I:VIN	0		I:VIN	0
I:SSL	0.00495		I:SSL	0
I:ITM	0.10704		I:ITM	0.11327
I:EDNR	0		I:EDNR	0
I:FUL	0		I:FUL	0
I:MON	0		I:MON	0
Expected return	26.20%		Expected return	26.80%
Expected volatility	24.40%		Expected volatility	24.30%
Sharpe Ratio	0.99		Sharpe Ratio	1.02

The differences between this allocation and the standard M-V ones are negligible. The only notable difference is that now SS Lazio (I:SSL) gets assigned a non-zero weight in the daily data allocation, albeit negligibly low. The expected performance is slightly better too, since it is probably influenced by the overall optimistic posterior returns we have obtained. We hypothesize that results are comparable because the posterior VCV matrix contains terms one order of magnitude smaller than the sample VCV, so the predictive VCV is not far from the starting point. We suppose that if we try different priors the result could be appreciably different. For the sake of completeness, we present also the unconstrained allocations and descriptive statistics of all four portfolios. By the same token of what we just discussed, also those results will be similar to the standard M-V case, so the same commentary applies.

	Daily weights			Monthly weights
I:ECK	0.00273		I:ECK	0.00468
I:CLT	-0.26614		I:CLT	-0.16518
I:ARN	0.81701		I:ARN	0.3878
I:HER	-0.65609		I:HER	-0.16929
I:ELN	0.60657		I:ELN	0.32124
I:AMP	0.94777		I:AMP	0.73069
I:VIN	-0.02867		I:VIN	0.22279
I:SSL	0.10104		I:SSL	0.11362
I:ITM	0.43582		I:ITM	0.48197
I:EDNR	-0.06444		I:EDNR	0.05841
I:FUL	-0.24156		I:FUL	0.01327
I:MON	-0.65403		I:MON	-1
Expected return	80.80%		Expected return	63.60%
Expected volatility	64.00%		Expected volatility	45.80%
Sharpe Ratio	1.2		Sharpe Ratio	1.35

	Daily Long Bay	Monthly Long Bay	Daily Short Bay	Monthly Short Bay
Mean	0.001125271	0.02389971	0.003120754	0.055389163
SD	0.014773556	0.067375144	0.039296174	0.127087034
Variance	0.000218258	0.00453941	0.001544189	0.016151114
Skewness	-0.204669068	-0.005573955	0.022503183	-0.075426691
Kurtosis	7.122017561	1.47131763	2.101285938	1.28882659

1.9 Global Minimum Variance Portfolio: Overview (Q14)

We now return to the CAPM discussion of Q5 and onward. When optimizing in that stage, we opted for the weights that maximized Sharpe ratio. Another economic relevant choice is the Global Minimum Variance Portfolio.

As the name suggests, the GMVP is the portfolio exhibiting the lowest variance among those on the efficient frontier. We recall that the EF is parabolic and from the plots we can infer that the GMVP is the one placed on the apex of the plot. To obtain the coordinates then we must solve the derivative:

$$\frac{\partial \sigma_p^2}{\partial \mu_p} = \frac{1}{\Delta}(c\mu_p - b) = 0 \implies \mu_{mv} = \frac{b}{c} \tag{20}$$

and we can substitute it into the expression of the portfolio variance to obtain that:

$$\sigma_{MV}^2 = \frac{1}{\Delta}(a - \frac{2b^2}{c} + c\frac{b^2}{c^2}) = \frac{1}{c} \tag{21}$$

where throughout the algebra, a, b, c, Δ are those defined previously in the general CAPM discussion. The GMVP is then unique, regardless of long or short constraints.

Finally, to find the weights we can use the generic equation using the newfound coordinates. Through some algebra, we obtain

$$w_{MV} = \frac{1}{\Delta}\Sigma^{-1}(c\mu_{MV} - b)\mu_M V + \frac{\Sigma^{-1}1}{\Delta}(a - b\mu) = \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1} \tag{22}$$

The GMVP has the property that the covariance with every other efficient portfolio is equal to the variance of GMVP itself.

Having provided a theoretical introduction, we present here the weights and performance of the GMVP on our sample.

	Daily weights			Monthly weights
I:ECK	0.04958		I:ECK	0
I:CLT	0.09639		I:CLT	0.12485
I:ARN	0.03371		I:ARN	0.02295
I:HER	0.10775		I:HER	0.09283
I:ELN	0.00282		I:ELN	0
I:AMP	0.04637		I:AMP	0.07097
I:VIN	0.20122		I:VIN	0.269
I:SSL	0.04142		I:SSL	0
I:ITM	0.11747		I:ITM	0.07649
I:EDNR	0.19565		I:EDNR	0.31764
I:FUL	0.03457		I:FUL	0.02526
I:MON	0.07306		I:MON	0
Expected return	4.40%		Expected return	5.80%
Expected volatility	13.50%		Expected volatility	12.20%
Sharpe Ratio	0.18		Sharpe Ratio	0.31

We include only the constrained weights to avoid clutter. The unconstrained optimiza-

	Daily GMV	Monthly GMV
Mean	0.000353407	0.006807719
SD	0.008510109	0.035234229
Variance	0.00007	0.001241451
Skewness	-1.181260217	0.013504351
Kurtosis	14.92977495	0.197359534

tion outputs different weights but the same performance, as expected by the theoretical discussion, some small rounding differences aside.

We see that unlike the maximizing Sharpe allocation, every asset in the daily data and almost every asset in the monthly data has a non-zero weight. The intuition is that if we want to minimize volatility, we do so by increasing the diversification. Since the sample we picked was well specified with respect to low correlations, we are not surprised to see how each securities has its place in the composition.

The annualized returns and volatility are considerably lower than from the max Sharpe portafolios, at 4.4% and 13.5% on the daily and 5.8% and 12.2% on the monthly respectively. This risk profile might be more suitable for an investor, however the return is not particularly attractive and the low Sharpe ratios reflect this performance.

With respect to the descriptive statistics table, we denote that the daily data is quite erratic, probably given the noisy nature of high frequency data. Monthly data is instead quite regular, with a definitely small kurtosis that indicates a flat-ish kind of distribution.

1.10 Why so Many 'Optimal' Portfolios? (Q15)

We have seen until now four different portfolio composition philosophies: the max Sharpe and the Global Minimum Variance in the M-V plane using only sample data and the Bayesian approach in its pure and Black-Litterman applications.

The maximum Sharpe let us achieve an overall good Sharpe, but the portfolio composition is way too aggressive and concentrated, skewed by the presence of some great performers in the sample. This resulted in almost unacceptable yearly volatility levels. These results are further influenced by the high sensitivity to estimation error in the M-V model. Considering that past does not necessarily repeat itself, one has to ask himself how likely it is for the time series to keep evolving in that direction. This is all the more important considering some of our securities have had exceptionally good performances: there is only so much growth any company can achieve consistently. We then computed the Global Minimum Volatility portfolio, which indeed curbed the volatility issue, but at the cost of performance. Even then, the same concerns about parameter estimation persist.

Lastly, we applied a Bayesian reasoning to our portfolio composition. In other words, we estimate our parameters taking a mix of sample data and views about the market. In the pure Bayesian model, the views are provided comprehensively on the distribution of mean returns, while in the Black - Litterman model views are expressed specifically with respect to securities and their performances, both in absolute and relative terms. This approach then surely fixes the high reliance on data that we have in the standard M-V situation, and ideally should solve the concentration issue we reported. However we also observe a drop in portfolio performance and we must ask ourselves whether our views are reliable at all.

The idea is then to take an average of all the resulting positions. Hopefully, the resulting portfolio can capitalize on the advantages of each model, while smoothing out the drawbacks. In selecting the relative importance for each method, we follow the hint of 0.25 each, which means we do not favour any one model in particular. We run this on both constrained and unconstrained, daily and monthly datasets, to get a better picture.

	Daily weights			Monthly weights
I:ECK	0.012393894		I:ECK	0
I:CLT	0.024098287		I:CLT	0.031212119
I:ARN	0.238664116		I:ARN	0.20649332
I:HER	0.026936333		I:HER	0.023206828
I:ELN	0.18834255		I:ELN	0.0832225
I:AMP	0.262769788		I:AMP	0.388355761
I:VIN	0.050303953		I:VIN	0.067251159
I:SSL	0.011592765		I:SSL	0
I:ITM	0.103634866		I:ITM	0.085415
I:EDNR	0.054357989		I:EDNR	0.108528487
I:FUL	0.008641758		I:FUL	0.006314825
I:MON	0.0182662		I:MON	0
Expected return	19.60%		Expected return	20%
Expected volatility	19.70%		Expected volatility	19%
Sharpe Ratio	0.89		Sharpe Ratio	0.95

Overall, we are satisfied with this composition and we believe that mixing strategies is the best way to go in our specific case. Compared to the standard M-V, we obtain comparable performances, but we can appreciate a series of improvements. Thanks to the presence of the GMV weights, annualized volatility has been adjusted. Weights are now more distributed: all securities in the daily (almost all in the monthly) get allocated some wealth and no stock dominates over the rest. Two of the main shortcomings of M-V have then been fixed. We do not forget also the role of Black - Litterman and Bayesian, that ensure that our newly recalculated portfolio is also more robust against estimation error and look-back bias.

We now look at the unconstrained portfolio.

	Daily weights			Monthly weights
I:ECK	0.002991394		I:ECK	-0.003781769
I:CLT	-0.225686713		I:CLT	-0.098846556
I:ARN	0.608021616		I:ARN	0.248124919
I:HER	-0.562211167		I:HER	-0.087590381
I:ELN	0.55555005		I:ELN	0.232306366
I:AMP	0.748534788		I:AMP	0.532929737
I:VIN	0.052221453		I:VIN	0.284304552
I:SSL	0.094025265		I:SSL	0.068765318
I:ITM	0.401697366		I:ITM	0.36435586
I:EDNR	0.154800489		I:EDNR	0.207791856
I:FUL	-0.245298242		I:FUL	0.002868775
I:MON	-0.5846438		I:MON	-0.751231178
Expected return	66.30%		Expected return	45.90%
Expected volatility	52.90%		Expected volatility	32.50%
Sharpe Ratio	1.22		Sharpe Ratio	1.35

What we said for the constrained case, also applies here. We remark that in this case the drops in Sharpe Ratios are almost minimal, while the reduction in volatility compared to the standard M-V is substantial, both at daily and monthly level. Asset allocation is way more balanced too, as no asset now has full weight (positive or negative). The unconstrained portfolios seem to have benefit even more from the mixing of which we have discussed previously.

Lastly, we show descriptive statistics tables for all four portfolios.

	Daily Long Tot	Monthly Long Tot	Daily Short Tot	Monthly Short Tot
Mean	0.00092828	0.019239838	0.002634157	0.040907673
SD	0.012398167	0.054813796	0.033305226	0.093941059
Variance	0.000153715	0.003004552	0.001109238	0.008824923
Skewness	-0.52587163	-0.231317169	-0.114390405	-0.150293873
Kurtosis	9.304488641	1.260173018	2.158723481	1.230637002

We denote that all four portfolios are slightly negatively skewed, so that the left tail is slightly longer, but by an overall negligible factor. Except for the daily constrained portfolio, all other distributions have a kurtosis below 3, so compared to a standard normal they have thinner tails. We think this is an overall appreciable feature in returns.

2 Asset Allocation with Endogenous Labor Income: The Case of Incomplete Markets

This research ([Viceira, 2001]) is focused on optimal consumption and portfolio decisions in presence of flexible labor supply, with risky working wage not perfectly correlated with stock returns.
The aim of the research is to provide a closed form characterisation of optimal consumption and portfolio policies, showing how the choice of leisure/work can have dramatic effects on portfolio allocation.

The nontradability nature of labor income affects the optimal allocation of financial wealth between risk and riskless assets. [?] considered a framework in which an investor must choose every period her work supply as well as how to allocate her portfolio, but,in order to find a tractable analytical solution, they restricted their focus on either deterministic exogenous wages or stochastic wages perfectly correlated with risky return, de facto allowing for a perfect hedge of labor income risk.

This paper generalizes [Merton and Samuelson, 1991] to an incomplete market model, as in [Viceira, 2001], where it analyzed the possibility of a flexible labour supply, showing how investors can react to negative shocks to their financial wealth by increasing their labor supply, which acts like a buffer to financial losses.
However, how this affects both consumption and portfolio policies was not investigated, let alone the relation between labor income risk and equity premium puzzle.

2.1 The Model

2.1.1 Preferences

In this model, the investor gets utility from both consumption and leisure, having in each period a time endowment (normalized to 1) which can split between working and leisure, if she is employed, while during the retirement all her time endowment is allocated toward leisure, earning no income.Practically, her preferences take the following time- and state-separable form:

$$u(C_t, N_t) = \frac{1}{1-\gamma}[C_t^{\theta}(1-N_t)^{1-\theta}]^{1-\gamma} \tag{23}$$

where C_t is consumption good and $N_t \leq 1$ is the amount of time spent working at time t. θ captures the relative importance of leisure $(1-N_t)$ to the investor and γ is a parameter which represents the coefficient of relative risk aversion over the composite good $C_t^{\theta}(1-N_t)^{1-\theta}$. Note that the coefficient of relative risk aversion over C_t depends on both preference parameters θ and γ, given by:

$$\hat{\gamma} = -\frac{C \cdot u_{CC}}{u_C} = 1 + \theta(\gamma - 1) \tag{24}$$

2.1.2 Employment and Wage Process

Following [Viceira, 2001], the investor deals with an external uncertainty over her employment status in each period, being employed with probability p_e, while she enters retirement with probability $(1-p_e)$. being this probability constant over time, the number of employed periods is $\frac{1}{1-p_e}$.
In the employment case, the investor can decide how much she wants to work, having in mind that the wage rate (H_t) is stochastic, and subject both to temporary and permanent shocks. It is possible to model the log total wage as a composed by a random-walk with drift with a transitory shock, while the log permanent part of the wage being an AR(1) process such that:

$$\begin{aligned} ln(H_{t+1}) &= ln(H_{t+1,p}) + (\epsilon_{t+1} - \frac{1}{2}\sigma_\epsilon^2) \\ ln(H_{t+1,p}) &= g + ln(H_{t,p}) + \xi_{t+1} \end{aligned} \tag{25}$$

with $\epsilon_{t+1} \sim N(0, \sigma_\epsilon^2)$ and $\xi_{t+1} \sim N(0, \sigma_\xi^2)$ serially uncorrelated but cross-sectionally correlated, with $Cov(\epsilon_{t+1}, \xi_{t+1}) = \sigma_{\epsilon,\xi} > 0$; lastly, $g \geq 0$ is the expected growth rate of the log wage.

2.1.3 Securities

In this model there are just two financial assets available to the investor, a risky asset (e.g a stock) and a risk-free one, whose log return are denoted by $r_{1,t+1}$ and r_f. the excess log risky return is made of the risk premium μ and a noise u:

$$r_{1,t+1} - r_f = \mu + u_{t+1} \tag{26}$$

In order to provide the stock a potential hedging role against wage fluctuations, it is possible to model the noise u_t, in such a way that is correlated with both the permanent and total wage shocks , namely $Cov(u_t, \epsilon_t) = \sigma_{u,\epsilon} > 0$ and $Cov(u_t, \xi_t) = \sigma_{u,\xi} > 0$.

2.1.4 Optimality Conditions

The investor has to deal with the canonical consumption-investment problem, extended with a labor supply decision, given the flexibility assumption of the work supply. At any time t, she inherits the wealth $Wt-1$ from the last period and has to choose how to allocate her resources between consumption and investment as well as her time between working and leisure, in such a way to maximize her utility (in a life-time sense). Formally, she has to solve:

$$\max_{C_t, N_t, \alpha_t} E\left[\sum_{t=0}^{\infty} \beta^t u(C_t, N_t)\right] \tag{27}$$

under the inter-temporal budget constraint:

$$W_t + 1 = (W_t + N_t H_t - C_t) R_{p,t+1} \tag{28}$$

where β is the discount factor , H_t is the real wage rate, $R_{p,t+1}$ is the return on the portfolio.Clearly, the expectation is taken w.r.t. all the stochastic shocks of the model u,ϵ , ξ. In addition, the budget constraint is different from most of the literature , since the agent has now the possibility to chose the optimal level of working time. Having in mind that α_t is the weight of the risky asset invested in the portfolio,the gross total return would be:

$$\mathbb{R}_{p,t+1} = \alpha_t R_{1,t} + (1-\alpha_t)R_f \quad \text{or, equivalently} \quad R_{p,t+1} = \alpha(R_{t+1} - R_f) + R_f \tag{29}$$

Since there are stochastic shocks and two possible state (employed, retired), the intertemporal first-order conditions are represented by a pair of Euler equations, one for each state. in employment:

$$E\left\{\beta\left[p_e\frac{u_C(C_{e,t+1},N_{t+1})}{u_C(C_{e,t},N_t)} + (i-p_e)\frac{u_C(C_{r,t+1},0)}{u_C(C_{r,t},N_t)}\right]R_{i,t}\right\} = 1 \tag{30}$$

while in the retirement (when $p_e = 0$ and $N_t = 0 \ \forall$ t):

$$E\left\{\beta\left[\frac{u_C(C_{r,t+1},0)}{u_C(C_{r,t},0)}\right]R_{i,t}\right\} = 1 \tag{31}$$

$i = 1, f$ and "e","r" stands for employed or retired. Notice that since N_t is endogenous it appears both in the Euler equation 30 and the budget constraint 28.
The optimum choice of N_t is determined by the following first order condition with the trade-off between leisure and consumption.[3]:

$$u_C(C_t,N_t) = \frac{u_L(C_t,N_t)}{H_t} = -\frac{u_N(C_t,N_t)}{H_t} \tag{32}$$

2.1.5 The Approximation Framework

Starting from the budget constraint in the employment state given by $W_{e,t+1} = (W_{e,t} + N_t H_t - C_t)R_{e,p,t+1}$, we can divide by $H_{p,t+1}$ take the logs, and finally take the first-order Taylor expansion around $E(ln(W_{e,t}/H_{p,t}))$, $E(ln(C_t/H_{p,t}))$ and $(E(ln(N_t) + \sigma^2_\epsilon/2))$ of the right-hand side of the equation will lead to:

$$w_{e,t+1} - h_{e,t+1} \approx k_e + \lambda_{e,w}(w_{e,t} - h_{p,t}) + \lambda_{e,c}(c_t - h_{p,t}) + \lambda_{e,n}n_t + \lambda_{\epsilon,n}\epsilon_t - \Delta h_{p,t+1} + r_{e,p,t+1} \tag{33}$$

where all λ_i are constants deriving from this loglinearization procedure.
Moving to the loglinearization procedure of portfolio ,it is possible to apply Ito's lemma since the risky asset has been modeled as a Geometric Brownian Motion and the risk free asset as a simple ODE, with the following Q-dynamics:

$$\frac{dS_{1,t}}{S_{1,t}} = \mu dt + \sigma dW_t \tag{34}$$

$$\frac{dS_{0,t}}{S_{0,t}} = r_f dt \tag{35}$$

[3] since $N_t = 1 - L_t$

Given the fact that the goal is to deal with log returns we have to find the log dynamic [4]:

$$d(lnS_{1,t}) = \frac{dS_{1,t}}{S_{1,t}} - \frac{1}{2}\left(\frac{dS_{1,t}}{S_{1,t}}\right)^2 \tag{36}$$

$$d(lnS_{0,t}) = \frac{dS_{0,t}}{S_{0,t}} \tag{37}$$

Then, to recover the log-dynamic of the portfolio from the linear one:

$$\frac{dV_t}{V_t} = \alpha_t \frac{dS_{1,t}}{S_{1,t}} + (1-\alpha_t)\frac{dS_{0,t}}{S_{0,t}} = \alpha_t(\mu_t dt + \sigma dW_t) + (1-\alpha_t)r_f dt \implies \left(\frac{dV_t}{V_t}\right)^2 = \alpha_t^2\sigma^2 dt \tag{38}$$

Now terms must be rearranged:

$$\begin{aligned} \frac{dV_t}{V_t} &= \alpha_t \frac{dS_{1,t}}{S_{1,t}} + (1-\alpha_t)\frac{dS_{0,t}}{S_{0,t}} \\ &= \alpha_t(d(lnS_{1,t}) + \frac{1}{2}\sigma^2 dt) + (1-\alpha_t)d(lnS_{0,t}) \\ &= \alpha_t(\underbrace{d(lnS_{1,t})}_{r_{1,t}} - \underbrace{d(lnS_{0,t})}_{\frac{dS_{0,t}}{S_{0,t}}=r_f}) + d(lnS_{0,t}) + \frac{1}{2}\sigma^2\alpha_t \end{aligned} \tag{39}$$

Finally:

$$r_{p,t+1} = d(lnV_t) = \alpha_{1,t+1}(r_{1,t+1} - r_f) + r_f + \frac{1}{2}\sigma_u^2\alpha_t(1-\alpha_t) \tag{40}$$

Moving to the Euler equation for the employment state 30, it is possible to decompose it as follows:

$$E\left\{\beta\left[p_e\frac{u_C(C_{e,t+1},N_{t+1})}{u_C(C_{e,t},N_t)} + (i-p_e)\frac{u_C(C_{r,t+1},0)}{u_C(C_{r,t},N_t)}\right]R_{i,t}\right\} = E\left\{\beta(T_1+T_2)R_{1,t}\right\} \tag{41}$$

Focusing on the first component and substitute in the expression for the marginal utilities, it can be derived:

$$E\left\{\beta T_1 R_{1,t}\right\} = p_e E\left[\exp(ln\beta + r_{i,t+1} + (1-\theta)(1-\gamma)ln\left(\frac{1-N_{t+1}}{1-N_t}\right) - \hat{\gamma}\,\Delta c_{e,t+1}\right] \tag{42}$$

Now, putting $k_1 = (1-\theta)(1-\gamma)$ and noticing that $\forall\ n_t\ \exists Q_n$ s.t. $ln\left(\frac{1-N_{t+1}}{1-N_t}\right) \approx Q_n(n_{t+1}-n_t)$:

$$E\left\{\beta T_1 R_{1,t}\right\} = p_e E\left[\exp(ln\beta + r_{i,t+1} + k_1 Q_n\,(n_{t+1}-n_t) - \hat{\gamma}\,\Delta c_{e,t+1}\right] \tag{43}$$

Performing a second-order Taylor expansion to approximate the inner part of the expectation, around $E_t(r_{i,t+1})$, $E_t(\Delta n_{t+1})$ and $E_t(\Delta c_{e,t+1})$,it is possible to derive the expression for the first part:

$$\begin{aligned} E\left\{\beta T_1 R_{1,t}\right\} = p_e[1 + ln\beta + E_t(r_{i,t+1} + k_1 Q_n E_t(\Delta n_{t+1} - \hat{\gamma}E_t(\Delta c_{e,t+1})]+ \\ + \frac{p_e}{2}Var_t(r_{i,t+1} + k_1 Q_n \Delta n_{t+1} - \hat{\gamma}\,\Delta c_{e,t+1}) \end{aligned} \tag{44}$$

[4] in general $d(lnX_t) = \frac{dX_t}{X_t} - \frac{1}{2}\left(\frac{dX_t}{X_t}\right)^2$

Following the same derivation,it can also be retrieved the following expression for the second component:

$$\begin{aligned}E\left\{\beta T_2 R_{1,t}\right\} &= (1-p_e)E\left[\exp(ln\beta + r_{i,t+1} + k_1(1-e^{n_t}) - \hat{\gamma}(c_{r,t+1} - c_{e,t})\right] \\ &= (1-p_e)[1 + ln\beta - k_1(ln(1-\exp(E_t(n_t) - Q_n E(n_t))]+ \\ &+ (1-p_e)\left[E(r_{i,t+1}) - k_1 Q_n n_t - \hat{\gamma}(c_{r,t+1} - c_{e,t}) + \frac{Var(r_{i,t+1} - \hat{\gamma}(c_{r,t+1} - c_{e,t}))}{2}\right]\end{aligned} \tag{45}$$

Both 44 and 56 contain the choice variable n_t, thus using 54 it can be obtained:

$$\Delta n_{t+1} = Q_{n,2}(\Delta c_{e,t+1} - \Delta h_{p,t+1}) + Q_{n,3}\Delta\epsilon_t + 1 \tag{46}$$

$$E_t(\Delta n_{t+1}) = Q_{n,2}(E(\Delta c_{e,t+1}) - g) - Q_{n,3}\epsilon_t \tag{47}$$

Therefore the previously derived two elements can be rewritten as:

$$\begin{aligned}E\left\{\beta T_1 R_{1,t}\right\} &= p_e[1 + ln\beta - k_1 g + E_t(r_{i,t+1}) - \gamma E_t(\Delta c_{e,t+1}) + k_1\epsilon_t]+ \\ &+ \frac{p_e}{2}\underbrace{Var_t(r_{i,t+1} - \gamma\Delta c_{e,t+1} - k_1\xi_{t+1} - k_1\epsilon_{t+1})}_{\bar{V}_1}\end{aligned} \tag{48}$$

Analogously:

$$\begin{aligned}E\left\{\beta T_2 R_{1,t}\right\} &= (1-p_e)\left[1 + ln\beta + k_2 + E(r_{i,t+1} - \hat{\gamma}E_t(c_{r,t+1} - c_{e,t}) - k_1(c_{e,t} - h_{p,t} + k_1\epsilon_t)\right] + \\ &+ \frac{1-p_e}{2}\underbrace{Var_t(r_{i,t+1} - \hat{\gamma}(c_{r,t+1} - c_{e,t}))}_{\bar{V}_2}\end{aligned} \tag{49}$$

Where the following properties have been used: $Q_n Q_{n,2} = 1$, $Q_n Q_{n,3} = -1$, $k_1 - \hat{\gamma} = -\gamma$, and $k_2 = k_1[\frac{1}{2}\gamma_\epsilon^2 - ln(\theta/(1-\theta))]$.
Combining these two results, the final expression for the log-linearized Euler equation in the retirement state is:

$$\begin{aligned}&\bar{V}_1 + \bar{V}_2 + p_e[1 + ln\beta - k_1 g + E_t(r_{i,t+1}) - \gamma E_t(\Delta c_{e,t+1}) + k_1\epsilon_t]+ \\ &+ (1-p_e)\left[ln\beta + k_2 + E(r_{i,t+1}) - \hat{\gamma}E_t(c_{r,t+1} - c_{e,t}) - k_1(c_{e,t} - h_{p,t}) + k_1\epsilon_t)\right] = 0\end{aligned} \tag{50}$$

Similarly, for the retirement state we have:

$$ln\beta - \hat{\gamma}E_t(\Delta c_{r,t+1}) + E_t(r_{i,t+1}) + \frac{Var_t(\hat{\gamma}\Delta c_{r_t+1} - r_{i,t+1})}{2} = 0 \tag{51}$$

Finally, we can also rewrite the first order condition of consumption-leisure trade-off 32 as:

$$ln\left(\frac{\theta}{1-\theta}\right) + ln(1-\exp(n_t)) \tag{52}$$

Remembering the properties of Q_n , $ln(1-\exp(n_t)) \approx ln(1-\exp(E(n_t)) - Q_n E(n_t) + Q_n n_t$, we can substitute this in the previous representation to obtain:

$$ln\left(\frac{\theta}{1-\theta}\right) + ln(1 - \exp(n_t)) = c_t + h_{p,t} - \epsilon_t + \frac{\sigma_\epsilon^2}{2} \tag{53}$$

Therefore, this condition determines an explicit a relationship between current labor supply, wage and actual consumption, depending just on θ instead of γ, because the latter affects optimal decisions for the composite good formed by C_t and $1 - N_t$ but the mix between C_t and N_t is driven exclusively by θ. To go further ,42 can be approximated to retrieve an explicit form of the labour supply, arriving to:

$$n_t = Q_{n,1} + Q_{n,2}(c_t - h_{p,t}) + Q_{n,3}\epsilon_t \tag{54}$$

with $Q_{n,3} > 0 > Q_{n,2}$, that explain how the investor reduces her labor supply whenever its consumption to wage ratio increases, due to a lower marginal utility of consumption.

2.2 Road to the Explicit Solution

The strategy to solve the problem followed by the authors mimick the game-theoretical approach called 'Backward Induction', which consists in starting from the end and proceding towards the start,firstly solving the consumption-investment problem faced by the investor in the retirement state, then given this solution, switching to the employed state.
Additionally, two crucial assumptions were made:

$$\alpha_{e,t} = \alpha_t \qquad \forall t \tag{55}$$

$$c_{e,t} - h_{p,t} = b_{e,0} + b_{e,1}(w_{e,t} - h_{p,t}) + b_{e,2} \tag{56}$$

Namely, that the optimal stock allocation during the employment state is constant (not time varying) and that the consumption-permanent wage ratio is a linear functional of the wealth to permanent wage ratio (given $b_{e,i}$ constants).

2.2.1 Consumption and Labor Supply

The investor's optimization problem in the retirement state is the standard problem with no labor income and constant investment opportunities, for which [Merton and Samuelson, 1991], [Merton, 1971] and [Viceira, 2001] provided solutions in different contexts, both providing the condition for which the consumption-wealth ratio in the retirement state is constant over time:

$$c_{r,t} - wr, t = [k_r - ln\beta/\hat{\gamma} + (1 - 1/\hat{\gamma}E(r_{r,p,t+1} - (1 - \hat{\gamma}^2 Var(r_{r,p,t+1})/2\hat{\gamma}]\lambda_{r,c} \equiv b_{r,0} \tag{57}$$

Differently, in the employed state the log consumption-wage ratio is not constant (this is the 2nd crucial assumption), with $0 < \frac{d(c_{t,e} - h_{p,t})}{d(w_{e,t} - h_{p,t})} = b_{e,1} < \frac{\hat{\gamma}}{\gamma} \leq 1 \; \forall \gamma > 1$ and $0 < \frac{d(c_{t,e} - h_{p,t})}{d\epsilon_t} = b_{e,2} < 1$.
The optimal consumption choice has to satisfy the Euler equation 50; using the identity

$\Delta c_{e,t+1} = (c_{e,t+1} - h_{p,t+1}) - (c_{e,t} - h_{p,t}) + (h_{p,t+1} - h_{p,t})$ and similarly for the retirement state, it can be stated that:

$$E_t(\Delta c_{e,t+1}) = E_t(c_{e,t+1} - h_{p,t+1}) - (c_{e,t} - h_{p,t}) + g \tag{58}$$

$$E_t(c_{r,t+1} - c_{e,t}) = E_t(c_{r,t+1} - h_{p,t+1}) - (c_{e,t} - h_{p,t}) + g \tag{59}$$

Plugging in this expression into 50, with $i = p$:

$$\begin{aligned} &\underbrace{\gamma p_e E_t(c_{e,t+1} - h_{p,t+1}) + \hat{\gamma}(1-p_e)E_t(c_{r,t+1} - h_{e,t+1})}_{LHS} \\ &= \bar{V} + p_e(\gamma(c_{e,t} - h_{p,t}) - \gamma g + ln\beta + E_t(r_{p,t+1} - k_1 g + k_1\epsilon_t) + (1-p_e)) + \\ &+ (1-p_e)((\hat{\gamma} - k_1)(c_{e,t} - h_{p,t}) - \hat{\gamma} g + ln\beta + k_2 + E_t(r_{p,t+1}) + k_1\epsilon_t) \end{aligned} \tag{60}$$

Now LHS and RHS must be evaluated as functions of $w_t, h_{p,t}, \epsilon_t$.
For LHS:

$$\begin{aligned} &\gamma p_e E_t(c_{e,t+1} - h_{p,t+1}) + \hat{\gamma}(1-p_e)E_t(c_{r,t+1} - h_{e,t+1}) \\ &= \gamma p_e(b_{0,e} + b_{1,e}E_t(w_{e,t+1} - h_{p,t+1})) + \hat{\gamma}(1-p_e)(b_{0,r} + E_t(w_{e,t+1} - h_{p,t+1})) \\ &= [(p_e\gamma b_{0,e} + (1-p_e)\hat{\gamma} b_{0,r}) + (p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma})\cdot \\ &\cdot (k_e + \bar{N}\lambda_{e,n}Q_{n,1} - g + E_t(r_{p,t+1}) - (\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})b_{0,e}))] \\ &+ (p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma})[\lambda_{e,w} - (\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})b_{e,1}](w_{e,t} - h_{p,t}) \\ &+ (p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma})[-(\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})b_{e,2} + (\bar{N}\lambda_{e,n} + \bar{N}\lambda_{e,n}Q_{n,3})]\epsilon_t \end{aligned} \tag{61}$$

Plugging in 56 into RHS:

$$\begin{aligned} &\underbrace{[-(p_e\gamma + (1-p_e)\hat{\gamma})g + \gamma b_{0,e} + ln\beta + E_t(r_{p,t+1}) - p_e k_1 g + (1-p_e)k_2 + \bar{V}]}_{Z} + \\ &+ \gamma b_{1,e}(w_{e,t} - h_{p,t}) + (k_1 + \gamma b_{e,2})\epsilon_t \end{aligned} \tag{62}$$

Moreover the coefficients in these two expressions must be equated, in such a way that:

$$\begin{cases} (p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma})[\lambda_{e,w} - (\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})b_{e,1}] = \gamma b_{e,1} \\ (p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma})[-(\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})b_{e,2} + (\bar{N}\lambda_{e,n} + \bar{N}\lambda_{e,n}Q_{n,3}) = k_1 + \gamma b_{e,2} \\ Z = (p_e\gamma b_{0,e} + (1-p_e)\hat{\gamma} b_{0,r}) + (p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma})\cdot \\ \quad \cdot (k_e + \bar{N}\lambda_{e,n}Q_{n,1} - g + E_t(r_{p,t+1}) - (\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})b_{0,e})) \end{cases} \tag{63}$$

The I equation pins down $b_{e,1}$, which can be used then to solve equation II for $b_{e,2}$ as a function of $b_{1,e}$, and finally $b_{0,e}$.Thus the following quadratic form appears:

$$Ab_{e,1}^2 + Bb_{e,1} + C = 0 \tag{64}$$

This Equation has two roots, but the negative one will imply that consumption is always decreasing in wealth for any level of wealth, so despite being mathematically correct is economically intractable, therefore:

$$b_{e,2} = \frac{(p_e \gamma b_{e,1} + (1-p_e)\hat{\gamma})(\bar{N}\lambda_{e,n} + \bar{N}\lambda_{e,n}Q_{n,3}) - k_1}{\gamma + (p_e \gamma b_{e,1} + (1-p_e)\hat{\gamma})(\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2})} \tag{65}$$

Therefore,

$$b_{0,2} = \frac{K_1 - K_2}{p_e \gamma b_{e,1} + (1-p_e)\hat{\gamma})(\lambda_{e,c} - \bar{N}\lambda_{e,n}Q_{n,2}) + \gamma(1-p_e)} \tag{66}$$

With

$$K_1 = (p_e \gamma b_{e,1} + (1-p_e)\hat{\gamma})(k_e + \bar{N}\lambda_{e,n}Q_{n,1} - g + E_t(r_{p,t+1})) + (1-p_e)\hat{\gamma} b_{0,r} \tag{67}$$

$$K_2 = -(p_e \gamma + (1-p_e)\hat{\gamma})g + ln\beta + E_t(r_{p,t+1}) - p_e k_1 g + (1-p_e)k_2 + \bar{V} \tag{68}$$

Notice that after checking 61 and 62, it is possible to observe that the wealth process has an autoregressive component coefficient which is equal to:

$$\frac{\gamma b_{e,1}}{p_e \gamma b_{e,1} + (1-p_e)\hat{\gamma}} \tag{69}$$

Since the wealth-wage ratio has to be stationary, this coefficient must be strictly lower than 1, therefore:

$$0 < b_{e,1} < \frac{\hat{\gamma}}{\gamma} \tag{70}$$

The upper bound for $b_{e,1}$ implies, that shocks to financial wealth w_t are not fully absorbed into consumption like in the retirement state, because of two mainly two reasons for this: Firstly, the labor income provides a buffer to financial losses, secondly in case of negative shocks to both stock returns and wages, the investor can still increase her work supply to offset the wealth reduction. This gives us also another insight about the parameter $b_{e,1}$, which should be smaller in presence of exogeneity of the work income. In addition, we have to discuss the effect of g , the wage growth rate, since it has a pair of consequences: on one hand there is a positive income effect on consumption from a greater income, on the other hand, there is also a substitution effect, given that when the wage rise it is optimal to provide more work and spend less, but having $b_{1,e}$ an upper bound determines that the income effect dominates the substitution effect.

Turning to the optimal labor supply choice, we can see that , thanks to the Q approximation we can rewrite:

$$n_t = J_1 + J_2(w_{e,t} - h_{p,t}) + J_3\epsilon_t \qquad J_2 = b_{e,1}Q_{n,2} < 0 < (1-b_{e,1})Q_{n,3} = J_3 \tag{71}$$

This labor supply rule gives us a sensible and useful insights, that work effort responds positively both to permanent and transitory wage shocks, creating a substitution effect which dominates the income effect, resulting in taking advantage of the higher wage.

2.2.2 The Portfolio Choice

Regarding the retirement state, the optimal portfolio allocation for the stock is a well known result from Samuelson(1969) and Merton (1969,1971,1973) and is given by:

$$\alpha_{r,t} = \frac{\mu + \sigma_u^2/2}{\hat{\gamma}\sigma_u^2} \tag{72}$$

In this case the allocation is proportional to the Sharpe ratio and inversely proportional to stock volatility and the relative risk aversion. As [Viceira, 2001] noted, the relative risk aversion of the value function is equal to the product of the relative risk aversion of the utility over consumption times the elasticity of wealth w.r.t. consumption, but in the retirement state this last one element is equal to one, so the relative risk aversion of both utilities is the same, since in the retirement state the investor still derives utility from leisure, but the only difference is that she is forced to allocate all of its time to leisure.
To derive the optimal portfolio policy in the employmennt state, first we want to take the difference of the two Euler equation 30 when $i = 1$ and $i = 1$ so to derive an expression for the excess return of the stock:

$$0 = \underbrace{E_t(r_{1,t+1} - r_f) + \frac{Var_t(r_{1,t+1})}{2}}_{\text{Y}} + p_e Cov_t(r_{1,t+1}, -\gamma \Delta c_{e,t+1} - k_1 \xi_{t+1} - k_1 \epsilon_{t+1}) + \tag{73}$$

$$+ (1 - p_e) Cov_t(r_{1,t+1}, \hat{\gamma}(c_{r,t+1} - c_{e,t})) \tag{74}$$

Then

$$Y = p_e \underbrace{Cov_t(r_{1,t+1}, \gamma \Delta c_{e,t+1})}_{\text{A}} + (1 - p_e) \underbrace{Cov_t(r_{1,t+1}, \hat{\gamma}(c_{r,t+1} - c_{e,t}))}_{\text{B}} + p_e k_1 \sigma_{u\xi} + p_e k_1 \sigma_{u\epsilon} \tag{75}$$

Now we want to retrieve the two covariance terms A and B: for the first one we substitute the assumption regarding consumption policy 56 into this identity:

$$\Delta c_{e,t+1} = (c_{e,t+1} - h_{p,t+1}) - (c_{e,t} - h_{p,t}) + (h_{p,t+1} - h_{p,t}) \tag{76}$$

now using the log linearized budget constraint 33 and the expression 54 regarding the optimal labor supply policy, it is possible to get:

$$\begin{aligned} \Delta c_{e,t+1} &= b_{e,1}(-\Delta h_{p,t+1} + r_{p,t+1}) + b_{e,2}\epsilon_{t+1} + \xi_{t+1} \\ &= b_{e,1} r_{p,t+1} + b_{e,2}\epsilon_{t+1} + (1 - b_{e,1})\xi_{t+1} \end{aligned} \tag{77}$$

Since $\Delta h_{p,t+1} = g + \xi_{t+1}$ and every t indexed term as well as constants cancel out with the Cov operator.
Lastly, given the portfolio return expression 40, the following expression can be recovered:

$$\begin{aligned} A &= Cov_t(r_{1,t+1}, b_{e,1} r_{p,t+1} + b_{e,2}\epsilon_{t+1} + (1 - b_{e,1})\xi_{t+1}) \\ &= b_{e,1}\alpha_{e,t}\sigma_u^2 + b_{e,2}\sigma_{u\epsilon} + (1 - b_{e,1})\sigma_{u\xi} \end{aligned} \tag{78}$$

Following the same reasoning it can been get that:

$$B = \alpha_{e,t}\sigma_u^2 \tag{79}$$

Collecting the two results 79 and 78 into 74 yields:

$$Y = p_e\gamma(b_{e,1}\alpha_{e,t}\sigma_u^2 + b_{e,2}\sigma_{u,\epsilon}) + (1-p_e)\hat{\gamma}\alpha_{e,t}\sigma_u^2 + p_e k_1(\sigma_{u\xi} + \sigma_{u\epsilon}) \tag{80}$$

Solving this for $\alpha_{e,t}$, we will retrieve the solution to our problem:

$$\boxed{\alpha_{e,t} = \frac{1}{p_e\gamma b_{e,1} + (1-p_e)\hat{\gamma}}\left[\frac{\mu + \sigma_u^2/2}{\sigma_u^2} - p_e(k_1 + \gamma b_{e,2})\frac{\sigma_{u\epsilon}}{\sigma_u^2} - p_e(k_1 + \gamma(1-b_{e,1}))\frac{\sigma_{u\xi}}{\sigma_u^2}\right]} \tag{81}$$

2.2.3 Consequences

If we analyse this final result, we can see that the optimal asset allocation policy is composed by three elements: the first is quite similar with the optimal rule in the retirement state , reflecting the speculative reason for stocks demand. Additionally, we have two terms linearly related with the covariance of stock returns with permanent and transitory shocks to wage times the probability of being employed in the next period (that acts like a weight). These two components stress the fact that stocks can serve an extra role as a hedging strategy for wage fluctuations.
Note that with flexible labor supply, this result is not obvious, since there are two contrasting effects on hedging stock demand: firstly, if the correlation is positive, the investor can insure her consumption against wage fluctuations by reducing her stock exposition (this behavior is captured by the factor $-p_e\gamma(1-b_{e,1}) < 0$ in front of $\sigma_{u\xi}$.
Secondly,in this model there is a second effect that acts in opposite directions,namely the labour supply flexibility,thanks to which the investor can avail herself of an extra buffer in times of financial downturns simply by working more to offset losses. This makes possible to take more aggressive positions with larger exposure to risky stocks.
Ultimately, The insurance effect always dominates the substitution effect between labor and leisure, and the overall hedging demand for stock gets the classic negative sloped shape.
Intuitively, the insurance effect dominates because by supplying more labor, the investor is exposing herself more toward the wage risk and, in presence of positive correlation between financial and wage shocks, this will end up in an increase of hedging desire.
Therefore the introduction of labor/leisure choice exacerbates the difference of portfolio allocations between the employment and retirement state, stressing the importance of labor supply flexibility in the life-cycle pattern of asset allocation.

2.3 Asset Allocation in Different Settings

2.3.1 Exogenous vs Endogenous Labor Supply

In order to reflect on the magnitude of the paper assumptions, let us now consider two investors, one who is free to choose how many hours to work (so in case of Endogenous Work

Supply), while the other one has a fixed time that must work per week (therefore in case of Exogenous Labor Supply). In the figure below, we have on the Z-axis the exposure towards stock, with on the X-Y plane the Time Horizon and Risk Aversion:

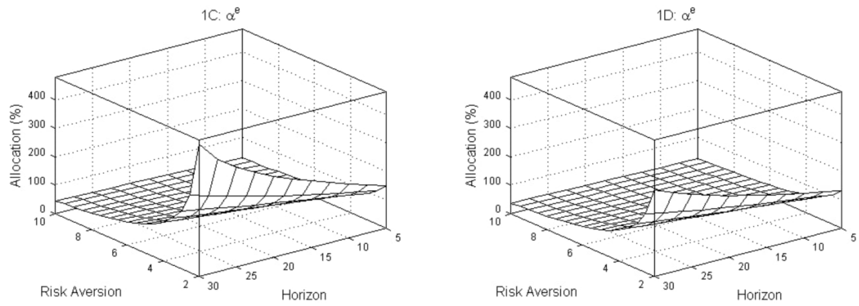


Figure 1: Endogenous Labor Supply (Left) vs Exogenous Case (Right)

As shown in the graphs α_e declines as we consider an increasing risk aversion and shorter expected retirement horizons. For this latter effect is as though a young worker with longer expected retirement horizon has more human capital, that can be treated as a risk-free asset, and therefore the more you have it, the more you can expose towards risky position in order to reach the optimal allocation.Moreover, the figure shows that α_e is systematically larger when the labor income is endogenous (when you can decide how much to work).
When the labor supply decision is endogenous, the possibility to work more becomes an extra buffer to financial downfalls,therefore financial wealth shocks have less negative impact on consumption, therefore the elasticity of consumption w.r.t. financial wealth $b_{e,1}$ is smaller in case of endogeneity of work supply.

2.3.2 Stock Return and Wage Shock Correlation

In the first place, it must be highlighted the fact that the magnitude of the correlation is not enough to compensate for the horizon effect at any horizon.
The following figure shows the hedging ratio(the ratio between the hedging demand and the total demand for stocks) both in presence of endogeneity or in absence:

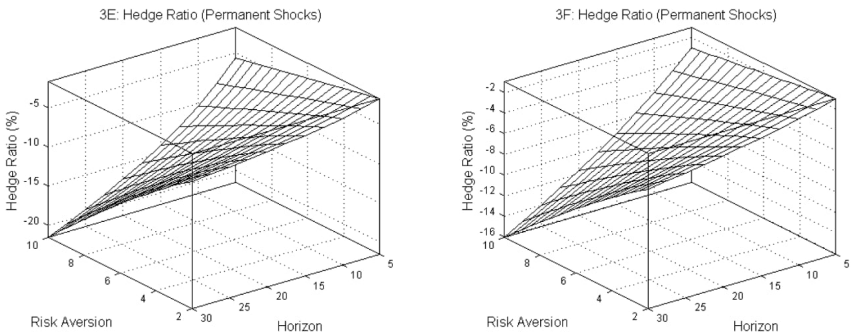


Figure 2: Endogenous Labor Supply (Left) vs Exogenous Case (Right)

Interestingly, we can see that when the expected retirement horizon shortens, the impact of a permanent shock on wages is reduced since the investor has a shorter period to receive labor income.Therefore, hedging demand against permanent shocks declines as the retirement horizon shortens.

2.3.3 Consumption-Wage Ratio and Labor Supply

Moving to consumption e working decisions, it is possible to plot both of these variables as functions of time horizon and risk tolerance,always differentiating for exogeneity and endogeneity of the work supply.For consumption we have:

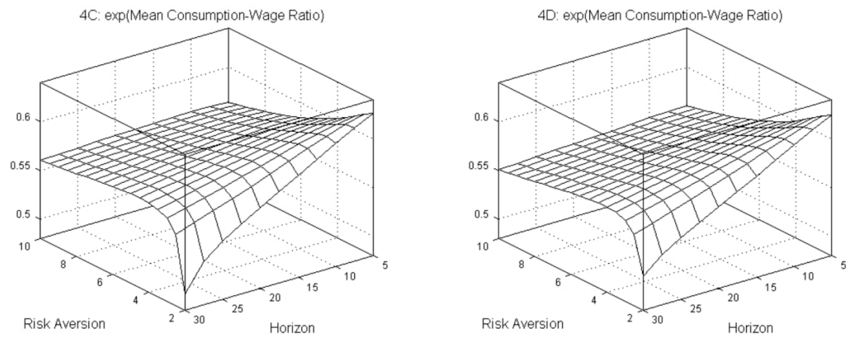


Figure 3: Endogenous Labor Supply (Left) vs Exogenous Case (Right)

For long expected retirement horizons, the mean consumption to wealth ratio increases with $\hat{\gamma}$, but for short retirement horizons it decreases. Additionally, it increases as expected horizon decreases for any $\hat{\gamma}$. That is the case since $\hat{\gamma}$ captures a wide range of phenomena: when we consider larger values we are taking into account an investor which is not so willing to intertemporally exchange consumption, therefore this tends to push up current consumption. On the other hand we are also selecting a more financially prudent individual, who wants more risk-free assets,but lower returns reduce sustainable consumption,whereas lower volatility reduces precautionary savings. Depending on which effects dominates, the optimal consumption-wage ratio can show different patterns. However at long expected horizon, the substitution effect dominates for risk tolerant individuals because the precautionary savings motive is weak. As $\hat{\gamma}$ increases, precautionary savings start to become more prominent. Regarding the work supply decision:

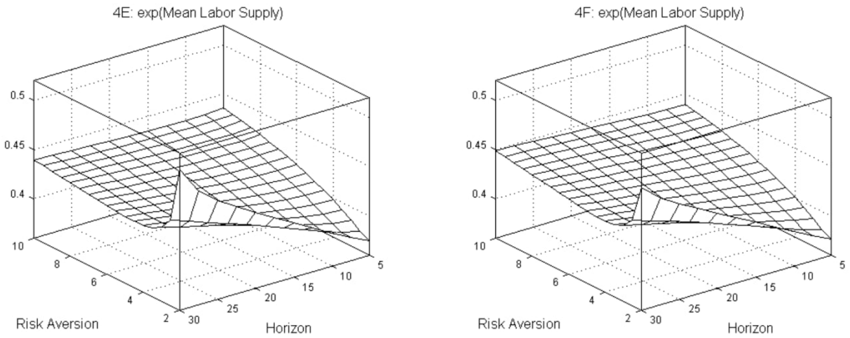


Figure 4: Endogenous Labor Supply (Left) vs Exogenous Case (Right)

In deciding how much to work, investors has to balance the benefits deriving from labor income that can be translated in more consumption (both current and future) and the disutility coming from having less leisure time.
For example when consumption increases, its marginal utility decreases, and this implies that the investor can move toward an higher utility level trading consumption for leisure, which drives down the labor supply, therefore , as investor builds up wealth, she can afford to work less.
If we now take into account the correlation between wages and financial shocks, we have that wages tends to fall when stock underperform, therefore the mean labor supply will be higher in order to offset this two negative effects.
Furthermore it is also possible to analyse the elasticity of the labor supply w.r.t risk aversion and time horizon: having in mind 71, the figure 3 shows that the elasticity of labor supply to financial shocks (J_2) is larger for investors with shorter expected retirement horizons. Intutively, investors with short expected horizons have a more urgent need to build up financial wealth and react to adverse shocks to their financial wealth by working more. In addition to this, if we consider that (1-J_2) is the elasticity of labor supply w.r.t. permanent wage shocks, we can also see that this is smaller for investors with shorter time horizons. For example, a permanent increase in the wage rate is less valuable to investors with short retirement horizons, and they are less willing to give up leisure after a shock.

2.4 Conclusion and Comments

This paper analyzed the effects on portfolio choice and savings of risky labor income when investors are able to adjust their labor supply in response to shocks to their wages and financial wealth. Previously, [Merton and Samuelson, 1991] showed that with riskless wage income, ignoring investors willingness to change her labor supply in response to financial shocks can understate their propensity to invest in risky assets. This Result holds in this paper too, and also when wage risk is correlated with financial one, the optimal portfolio allocation to stocks is systematically smaller w.r.t the same one with an extra protection for consumption against unexpected negative shocks to wage and/or financial wealth.

Overall, this paper layed foundations for a whole new branch of microeconomic research, whose aim is to investigate the asset allocation choices of an individual in a comprehensive setting rich of traid offs, starting from consumption and saving, through labor supply choice. It would be useful also to generalize such findings, modifying or removing at all some restriction,adjusting with different utility functions, or taking into account a more updated dynamic for stock.

Regarding the risky asset, we can also say that a multi-asset approach is severely needed in the economic research,capturing finer and finer distictions in risk tolerance,providing a gamma of sources of risk

Another critical point is the presence of a ’representative agent’, which is a choice no more updated, given the current development of agent based modeling (which is more and more possible, given the availability of computational power).

Lastly it can be captured also the role of information, as well as the decision making in general, embedding then behavioral economics elements.

A Daily and monthly returns

Daily returns statistics

	Mean	Standard Deviation	Variance	Skewness	Kurtosis
I:LDO	0.000406336	0.023371919	0.000546247	-0.334182707	11.76002227
I:ECK	0.000651427	0.031898572	0.001017519	2.543398019	18.73465056
I:LRZ	0.000223826	0.030731711	0.000944438	1.037096265	11.64504118
I:PIRL	-9.94865E-06	0.022295283	0.00049708	-0.041668302	6.072628837
I:STL	0.000833429	0.024366738	0.000593738	-0.451300597	5.447452701
I:PINF	0.000270955	0.03538062	0.001251788	-0.550216643	95.226757
I:BRE	0.000588584	0.019401951	0.000376436	0.128629922	3.770013554
I:ISP	0.000227052	0.021177967	0.000448506	-0.762730657	12.12068032
I:ILLB	6.63137E-05	0.019296103	0.00037234	0.042382777	6.822685645
I:UCG	0.000294882	0.02725384	0.000742772	-0.034070261	7.081486198
I:BANC	0.000333629	0.019524914	0.000381222	0.023582906	8.983822709
I:BPE	0.000280525	0.028822651	0.000830745	0.167430269	7.527691833
I:FCBK	0.000637979	0.020472488	0.000419123	-0.123163633	3.542109838
I:CPR	0.000841698	0.015968581	0.000254996	-0.250510359	8.497894375
I:SPAC	-0.000298307	0.02434616	0.000592736	0.363005626	6.996302511
I:CALT	0.000478005	0.01743403	0.000303945	0.129018943	4.642195631
I:AST	-0.000408282	0.03626493	0.001315145	0.298587596	28.05753195
I:ENEL	0.000369039	0.01575672	0.000248274	-1.122542566	14.0454538
I:ARN	0.001364262	0.025350024	0.000642624	1.667671836	12.30778965
I:A2A	0.000453709	0.01643298	0.000270043	-0.878119622	10.62039325
I:TRN	0.000425181	0.013914127	0.000193603	-0.651073936	8.27335006
I:ACE	0.000258244	0.015818789	0.000250234	-0.25663025	6.837786792
I:DEA	0.000112076	0.01635511	0.00026749	1.726547604	51.81701321
I:BMED	0.000391517	0.019772156	0.000390938	-0.436009767	5.724190543
I:BIM	-0.000513388	0.045959927	0.002112315	1.157220416	18.79386618
I:TIPS	0.000694626	0.016222171	0.000263159	0.229758214	10.74182015
I:MB	0.000446684	0.020924582	0.000437838	-0.860052904	12.01426231
I:EQUI	0.000276856	0.016102041	0.000259276	0.909269343	11.69814909
I:ANI	0.000204591	0.023873505	0.000569944	0.115670959	5.091296274
I:TITR	-0.000162698	0.023977832	0.000574936	0.136715231	16.89246871
I:TIT	-0.000251806	0.02431619	0.000591277	0.549174318	16.36421165
I:ENV	0.00014469	0.018494003	0.000342028	0.829050686	15.31521548
I:VAL	-5.53052E-05	0.017637841	0.000311093	0.692251626	6.943694879
I:CLT	0.000197181	0.019352048	0.000374502	2.785542138	27.3090626
I:HER	0.000276584	0.015678451	0.000245814	-0.51385184	12.47392089
I:IRE	0.000416919	0.016270844	0.00026474	-0.513376178	6.121101387
I:IG	0.00029543	0.015194481	0.000230872	-0.5298109	6.267641742

I:ELN	0.001136442	0.023664925	0.000560029	0.265945908	3.643639201
I:AMP	0.001060119	0.02013177	0.000405288	-0.392844048	6.042009356
I:DLG	0.000346653	0.020831716	0.00043396	0.205783586	3.936180746
I:BOR	0.00068163	0.026507554	0.00070265	5.097964011	164.3415581
I:CNHI	0.000625406	0.022513985	0.00050688	-0.387618374	4.645736679
I:FD	0.000149144	0.031715962	0.001005902	2.659886487	22.38892915
I:IP	0.00084001	0.019313417	0.000373008	-0.123073835	2.629585619
I:IKG	0.000765686	0.022089601	0.00048795	1.072803732	10.50960943
I:ENAV	0.000155342	0.015416174	0.000237658	1.144578638	17.85699669
I:PST	0.000352846	0.017686033	0.000312796	-1.255482797	16.60338286
I:CASS	0.000276211	0.020478275	0.00041936	3.326865944	63.37882692
I:RCS	0.000205472	0.025356163	0.000642935	1.05877779	11.71146246
I:CAI	-0.000263186	0.020894866	0.000436595	0.460281363	5.103383044
I:MON	-0.000414356	0.025273103	0.00063873	2.346586076	22.72959091
I:GAMB	1.8772E-05	0.033015805	0.001090043	3.060998014	24.97466596
I:UNI	0.000309042	0.021330587	0.000454994	-0.280203586	8.991717646
I:G	0.000178481	0.015704976	0.000246646	-0.73719005	12.08020652
I:SRG	0.000263904	0.014664304	0.000215042	-1.189560029	16.77614492
I:ENI	0.00013191	0.018069806	0.000326518	-0.968746071	17.03314625
I:TOD	4.38368E-06	0.022711506	0.000515812	0.824776825	13.36916101
I:REC	0.000700714	0.017312845	0.000299735	0.045201518	12.55555926
I:RN	0.000558839	0.034194627	0.001169273	1.425939579	9.653294943
I:BRI	0.000168244	0.023384014	0.000546812	0.559861278	7.205170031
I:FUL	9.3448E-05	0.032785792	0.001074908	2.306216836	18.72592316
I:AISW	0.000783383	0.029700475	0.000882118	1.858959491	14.58101051
I:AGL	0.000341915	0.022684112	0.000514569	1.328083291	32.1540987
I:JUVE	0.000707791	0.026326142	0.000693066	0.257384527	8.05655578
I:SSL	0.000650783	0.024903742	0.000620196	0.276886515	9.665219793
I:CLE	-0.000683086	0.030337653	0.000920373	1.528530961	14.39402434
I:B	-0.00034609	0.023067472	0.000532108	1.139049226	10.68161542
I:CEM	0.000384923	0.02081972	0.000433461	0.177572774	2.66130856
I:US	0.000133991	0.015831668	0.000250642	-0.172439347	5.006213563
I:BZU	0.000540258	0.019821658	0.000392898	-0.034719409	4.400473151
I:CE	0.000234255	0.018277169	0.000334055	-0.035803251	2.903421915
I:DAN	0.000235955	0.020711294	0.000428958	0.58769925	9.153347791
I:ITM	0.000589089	0.017343106	0.000300783	2.213981003	33.35556366
I:ZUC	0.00030126	0.036058023	0.001300181	2.938235013	48.01214611
I:IPG	6.13658E-05	0.024567417	0.000603558	-0.191832448	11.05010448
I:VIN	0.000208336	0.017994476	0.000323801	0.382652238	3.795574282
I:EDNR	0.000321627	0.014250916	0.000203089	-0.354354712	13.4890771
I:RAT	0.000301519	0.020151874	0.000406098	0.943685971	12.83185452

I:GAB	0.000442783	0.028442775	0.000808991	1.228546677	7.867968086
I:MS	-0.000154492	0.023925108	0.000572411	1.164558439	21.90285025
I:ERG	0.000638571	0.017299791	0.000299283	-0.149560227	12.54591891
I:CMB	0.000639408	0.01787161	0.000319394	0.149451188	3.629554294
I:SAB	0.000277969	0.018609224	0.000346303	0.441720182	5.074727454
I:BE	7.11578E-05	0.023430837	0.000549004	1.974863553	14.32842202
I:SOL	0.00077313	0.017558175	0.00030829	0.38087579	1.789594462
I:DAL	0.000179127	0.024191741	0.00058524	0.423539088	7.42909372
I:BSS	0.000478842	0.02735029	0.000748038	-0.150991177	5.811314738
I:SAFI	-0.000317676	0.028694902	0.000823397	0.432305971	10.20825147

Monthly returns statistics

	Mean	Standard Deviation	Variance	Skewness	Kurtosis
I:LDO	0.009081459	0.110900047	0.01229882	0.199375341	2.783640575
I:ECK	0.025371239	0.298662767	0.089199448	7.56437303	68.29474565
I:LRZ	0.006059619	0.169457622	0.028715886	3.573282226	22.04243009
I:PIRL	-0.001640596	0.094643305	0.008957355	-0.43077611	0.917258478
I:STL	0.019003352	0.119976455	0.01439435	-0.208905667	0.9180149
I:PINF	-0.002383977	0.119725733	0.014334251	1.101458062	4.288342512
I:BRE	0.013266378	0.095181468	0.009059512	-0.012892604	-0.242609807
I:ISP	0.004823466	0.096086696	0.009232653	-0.398136203	2.061414563
I:ILLB	-0.001818	0.094500668	0.008930376	-0.996065018	2.691811711
I:UCG	0.006076151	0.121318589	0.0147182	-0.204169768	1.736319643
I:BANC	0.007512713	0.090725057	0.008231036	-0.64957085	0.928547728
I:BPE	0.00535626	0.128529631	0.016519866	0.400285219	0.476940979
I:FCBK	0.013305017	0.087322517	0.007625222	-0.080724018	-0.021479362
I:CPR	0.017941534	0.068321586	0.004667839	-0.373065229	0.535194434
I:SPAC	-0.008455326	0.098983337	0.009797701	0.057502406	1.350172002
I:CALT	0.009895142	0.074974784	0.005621218	-0.152915638	0.849689639
I:AST	-0.009942455	0.148800572	0.02214161	-0.417972028	6.254001619
I:ENEL	0.007172689	0.063889115	0.004081819	-0.058872216	0.812602773
I:ARN	0.029788168	0.126579807	0.016022448	1.866077433	4.568616189
I:A2A	0.009654419	0.072350618	0.005234612	-0.950409847	2.019843782
I:TRN	0.008325517	0.047759165	0.002280938	-0.092133633	-0.480066944
I:ACE	0.005886618	0.076748563	0.005890342	-0.286044596	0.124452608
I:DEA	0.002867134	0.082804909	0.006856653	0.770655276	3.264088997
I:BMED	0.008585042	0.089880713	0.008078543	-0.455843748	2.371492453
I:BIM	-0.019851715	0.165637753	0.027435865	0.870033191	3.336793129
I:TIPS	0.014444341	0.064983795	0.004222894	-0.103846701	-0.062904732
I:MB	0.009442896	0.093298921	0.008704689	-0.710117053	2.245517159

I:EQUI	0.005097261	0.066715915	0.004451013	0.048576717	2.163392717
I:ANI	0.004633178	0.110065384	0.012114389	0.069175606	0.905163049
I:TITR	-0.004798648	0.102505297	0.010507336	0.699612143	1.493415258
I:TIT	-0.006428702	0.108016997	0.011667672	1.227989096	4.178957428
I:ENV	0.001346976	0.062679624	0.003928735	0.038799105	0.941785253
I:VAL	-0.001660826	0.076644596	0.005874394	0.983559888	2.923399893
I:CLT	0.002493773	0.070685118	0.004996386	1.927755847	6.266449784
I:HER	0.005497393	0.06467491	0.004182844	-0.574694998	0.942722073
I:IRE	0.009251758	0.07686501	0.00590823	-0.509126363	0.61143664
I:IG	0.008049646	0.0607631	0.003692154	-0.20574616	0.426363968
I:ELN	0.027150833	0.132083401	0.017446025	0.173482983	0.209869035
I:AMP	0.022418118	0.084960238	0.007218242	-0.69363876	1.418310413
I:DLG	0.007115774	0.093886496	0.008814674	-0.069388141	0.101221054
I:BOR	0.01803807	0.168752641	0.028477454	4.247416741	27.85373064
I:CNHI	0.012942652	0.097506028	0.009507426	-0.204735031	2.125835068
I:FD	0.005941932	0.201380451	0.040554086	5.032968737	35.71908877
I:IP	0.018712983	0.092840106	0.008619285	-0.547513851	0.111474624
I:IKG	0.016044747	0.098721224	0.00974588	0.939727855	2.115911384
I:ENAV	0.003043153	0.066133756	0.004373674	-0.170232826	1.828402772
I:PST	0.00681995	0.069351064	0.00480957	-0.321749739	0.975163942
I:CASS	0.007661202	0.113080552	0.012787211	1.013539624	4.967156342
I:RCS	0.004407063	0.120888619	0.014614058	1.052471962	2.637892053
I:CAI	-0.005694065	0.09819158	0.009641586	0.347199303	0.538350109
I:MON	-0.011366609	0.094244093	0.008881949	1.838077129	9.00481348
I:GAMB	0.004041358	0.22199695	0.049282646	5.491515853	41.95842488
I:UNI	0.00662582	0.096729114	0.009356521	-0.524247437	1.837071513
I:G	0.003916454	0.07316679	0.005353379	-0.198824107	2.008081192
I:SRG	0.004809062	0.051881573	0.002691698	-0.215790359	-0.438754604
I:ENI	0.00202916	0.0759843	0.005773614	0.638264851	2.882873117
I:TOD	-0.000339144	0.110389014	0.012185734	1.086286537	3.286346502
I:REC	0.014377389	0.069641154	0.00484989	0.048516631	0.866181689
I:RN	0.011859325	0.161487409	0.026078183	1.436088228	3.986473569
I:BRI	0.001945936	0.090794733	0.008243683	-0.093105813	1.225608699
I:FUL	0.002840904	0.189936051	0.036075703	5.091597878	32.39660169
I:AISW	0.018819722	0.161254219	0.026002923	2.376395433	13.78794037
I:AGL	0.00748046	0.110512901	0.012213101	1.477028248	10.63492083
I:JUVE	0.017766623	0.149470523	0.022341437	1.329077057	5.840026243
I:SSL	0.01567451	0.131609844	0.017321151	0.943099869	4.599984746
I:CLE	-0.017145574	0.115393635	0.013315691	0.749972676	2.391781436
I:B	-0.011464987	0.05836944	0.003406992	1.178697812	4.901278986
I:CEM	0.008508878	0.09763102	0.009531816	0.676659704	0.837410169

I:US	0.002797886	0.071704941	0.005141599	-0.103042506	0.810899628
I:BZU	0.010629817	0.077544242	0.00601311	-0.121345018	-0.302881372
I:CE	0.00464024	0.080257233	0.006441223	0.505178258	3.31920697
I:DAN	0.004352683	0.085563674	0.007321142	-0.11697665	0.732861177
I:ITM	0.012114363	0.073379858	0.005384604	2.054092811	9.317986892
I:ZUC	-0.003297006	0.098243077	0.009651702	1.631834677	5.719456062
I:IPG	0.000356755	0.108325607	0.011734437	0.762902924	1.422899712
I:VIN	0.002513562	0.055275054	0.003055332	0.529580055	1.443709558
I:EDNR	0.006411902	0.056729758	0.003218265	0.229022794	2.595769964
I:RAT	0.003974814	0.060197047	0.003623684	0.217624469	1.623430331
I:GAB	0.01428268	0.178887103	0.032000595	2.146799976	7.152488436
I:MS	-0.003758338	0.1200655	0.014415724	3.240226695	20.45100532
I:ERG	0.012894496	0.069012608	0.00476274	0.143738385	1.593918394
I:CMB	0.013932852	0.081275735	0.006605745	-0.084361846	3.619976801
I:SAB	0.007044828	0.096233599	0.009260906	0.200365726	0.245683479
I:BE	-0.000404692	0.093270709	0.008699425	1.415495473	4.329549735
I:SOL	0.015557464	0.06361303	0.004046618	0.299077281	-0.460984909
I:DAL	0.003568891	0.111281546	0.012383583	0.273132358	-0.187823581
I:BSS	0.010685144	0.12805411	0.016397855	-0.190619176	0.225876489
I:SAFI	-0.006263896	0.137955103	0.01903161	1.006199783	3.357588674

References

[Black and Litterman, 1991] Black, F. and Litterman, R. B. (1991). Asset Allocation: Combining Investor Views with Market Equilibrium. *The Journal of Fixed Income*, 1(2):7–18.

[Chen and Israelov, 2022] Chen, Y. and Israelov, R. (2022). How Many Stocks Should You Own?

[Idzorek, 2019] Idzorek, T. (2019). A Step-By-Step Guide to the Black-Litterman Model Incorporating User-specified Confidence Levels. *SSRN Electronic Journal*.

[Markowitz, 1952] Markowitz, H. (1952). Portfolio Selection.

[Merton and Samuelson, 1991] Merton, C. and Samuelson, F. (1991). Labor supply flexibility and portfolio choice in a life cycle model.

[Merton, 1971] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413.

[Sharpe, 1964] Sharpe, W. F. (1964). CAPITAL ASSET PRICES: A THEORY OF MARKET EQUILIBRIUM UNDER CONDITIONS OF RISK*. *The Journal of Finance*, 19(3):425–442.

[Viceira, 2001] Viceira, L. (2001). Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *Journal of Finance*, 56(2):433–470.