CSE 150 – Foundations of Computer Science – Honors
Solutions for Sample Exam Problems
October 10, 2017

1. Let $P(n)$ be the following property:

$$\sum_{i=0}^{n} 3^i = \frac{1}{2}(3^{n+1} - 1).$$

We use the Well Ordering Principle to prove that $P(n)$ is true for all natural numbers $n \geq 0$.

The proof is by contradiction. Suppose $P(n)$ is not true for all natural numbers $n \geq 0$. Then the set C of all counterexamples to P is nonempty and, by the Well Ordering Principle, has a smallest element, say, k.

Now, observe that $P(0)$ is true, as

$$\sum_{i=0}^{0} 3^i = 3^0 = 1 = \frac{1}{2}(3^{0+1} - 1).$$

Thus 0 is not a counterexample to P and hence $k > 0$. Then $k - 1$ is also not a counterexample to C, as $k - 1 \geq 0$ and k is the smallest counterexample to C. Since $P(k-1)$ is true we have

$$\sum_{i=0}^{k-1} 3^i = \frac{1}{2}(3^k - 1).$$

But then

$$\begin{aligned}
\sum_{i=0}^{k} 3^i &= (\sum_{i=0}^{k-1} 3^i) + 3^k \\
&= \frac{1}{2}(3^k - 1) + 3^k \\
&= \frac{1}{2}(3^k - 1 + 2 * 3^k) \\
&= \frac{1}{2}(3^{k+1} - 1)
\end{aligned}$$

which indicates that $P(k)$ is true, and contradicts the assumption that k is a counterexample to P. We conclude that the set C of counterexamples to P is empty and, therefore, $P(n)$ is true for all natural numbers $n \geq 0$.

2. Tony, Bill, and Chris are suspects in a robbery case. They testify under oath as follows:

   Tony: *I am innocent.*
   Bill: *If Chris is guilty, then so is Tony.*
   Chris: *Bill did it, but Tony is innocent.*

   Assume that a person is either innocent or guilty, but not both. We use the variables p, q, and r to denote the following propositions:

   p: *Tony is guilty*
   q: *Bill is guilty*
   r: *Chris is guilty*

   (a) The above statements can then be expressed in propositional logic as follows: (i) [Tony] $\neg p$, (ii) [Bill] $r \to p$, and (iii) [Chris] $q \land \neg p$.

   We use this formalization to obtain answers to the following questions.

   (b) If all three are guilty, who told the truth and who lied?

   If p, q, and r are all true, then Tony and Chris lied (because their statements are false), but Bill told the truth.

   (c) If only Tony and Bill spoke the truth, who is innocent and who is guilty?

   In this case, $\neg p$ and $r \to p$ are true, but $q \land \neg p$ is false. From the first two formulas we obtain $\neg r$ by Modus Tollens. The negation of the third formula, $\neg(q \land \neg p)$, is equivalent to $\neg q \lor p$ by De Morgan's Law and Double Negation Elimination. The latter formula is equivalent to $q \to p$. Using Modus Tollens again we obtain $\neg q$. In sum, we conclude that all three suspects are innocent.

   (d) If Bill lied and Chris spoke the truth, who is innocent and who is guilty?

   In this case, $\neg(r \to p)$ and $q \land \neg p$ are true. The first formula is equivalent to $r \land \neg p$. Simplifying the two conjunctions, we obtain $\neg p$ and q and r, i.e., Tony is innocent and Bill and Chris are guilty.

3. Proof of $p \to r \vdash (q \to r) \to ((p \lor q) \to r)$:

1	$p \to r$	premise
2	$q \to r$	assumption
3	$p \lor q$	assumption
4	p	assumption
5	r	$\to$e 4,1
6	q	assumption
7	r	$\to$e 6,2
8	r	$\lor$e 3,4-5,6-7
9	$(p \lor q) \to r$	$\to$i 3-8
10	$(q \to r) \to ((p \lor q) \to r)$	$\to$i 2-9

4. We translate the following statements into predicate logic formulas, using a domain A and the predicates,

   Perfect(x): *x is perfect*
   Friend(x): *x is your friend*

   (a) No one is perfect.
   $\neg\exists x \in A.\, Perfect(x)$

   (b) Not everyone is perfect.
   $\neg\forall x \in A.\, Perfect(x)$

   (c) All your friends are perfect.
   $\forall x \in A.\, [Friend(x) \rightarrow Perfect(x)]$

   (d) None of your friends are perfect.
   $\neg\exists x \in A.\, [Friend(x) \wedge Perfect(x)]$