AMS 310 Homework 4 Solutions
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Chapter 5

1. Two electronic traded funds (ETF's) which we call ETF A and ETF B have the following distribution for their annual rate of return. Let X and Y be the rate of returns for the ETFs.

ETF X		ETF Y	
Rate of Return x,	**Probability**	**Rate of Return, y**	**Probability**
+30%	.25	+15%	.40
+10%	.30	+5%	.25
+5%	.20	0%	.20
-10%	.25	-5%	.15

a) Assuming the ETF return probabilities are independent, find the joint probability distribution f(x,y).
b) If $40,000 is invested in ETF X, what is the probability of losing money? What if it is all $40,000 is invested in ETF Y? Which investment is safer? Explain.
c) Find the 4 by 4 payoff matrix when $10,000 is invested in ETF X and $30,000 is invested in ETF Y.
d) What is the probability of a loss when $10,000 is invested in ETF X and $30,000 is invested in ETF Y?

a)

			Y		
	f(x,y)	**+15%**	**+5%**	**0%**	**-5%**
	+30%	.10	.0625	.05	.0375
X	+10%	.12	.075	.06	.045
	+5%	.08	.05	.04	.03
	-10%	.10	.0625	.05	.0375

b) The chance of losing money if $40,000 invested in X is .25, the chance of losing money if $40,000 invested in Y is .15

c)

		4,500	1,500	0	-1,500
	Payoff	**+15%**	**+5%**	**0%**	**-5%**
(+3,000)	+30%	7,500	4,500	3,000	1,500
(+1,000)	+10%	5,500	2,500	1,000	**-500**
(+500)	+5%	5,000	2,000	500	**-1000**
(-1,000)	-10%	3,500	500	**-1000**	**-2500**

Payoff when X increases 30% and Y increases 15% = (.30)(10,000) + (.15)(30,000) = 7,500

d) Probability of a loss is .05 + .045 + .03 + .0375 = .1625 = 16.25%

2. Suppose the random variables and have joint pdf

$$f(x,y) = 15xy^2, 0 < y < x < 1.$$

a) Find the marginal pdf $f_1(x)$ of X.

b) Find the conditional pdf f $f_2(y|x)$.

c) Find $P(Y > \frac{1}{3}|X = x)$ for any x > 1/3.

d) Are X and Y independent? Justify your answer.

(a) $f_1(x) = \int f(x,y)dy = \int 15xy^2dy = [5xy^3]_{y=0}^{y=x} = 5x^4$, $\quad 0 < x < 1$

(b) $f_2(y|x) = f(x,y)f_1(x) = 15xy^2 / 5x^4 = 3y^2/x^3$, $\quad 0 < y < x$

(c) $P(Y > 1/3 \mid X = x) = \int f_2(y|x)dy = \int 3y^2x^3 dy = 1 - 1/(27x^3)$, for $x > 1/3$

(d) They are not independent because $f_2(y|x)$ includes x from the answer to (b).

3. Let and be independent random variables representing the lifetime (in 100 hours) of Type A and Type B light bulbs, respectively. Both variables have exponential distributions, and the mean of X is 2 and the mean of Y is 3.

a) Find the joint pdf $f(x, y)$ of X and Y.

b) Find the conditional pdf $f_2(y|x)$ of Y given X = x.

c) Find the probability that a Type A bulb lasts at least 300 hours and a Type B bulb lasts at least 400 hours.

d) Given that a Type B bulb fails at 300 hours, find the probability that a Type A bulb lasts longer than 300 hours.

e) What is the expected total lifetime of two Type A bulbs and one Type B bulb?

f) What is the variance of the total lifetime of two Type A bulbs and one Type B bulb?

a) By independence, $f(x,y) = f_1(x)f_2(y) = \begin{cases} (1/6)e^{-x/2-y/3}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

b) By independence, $f_2(y|x) = f_2(y) = (1/3)e^{-y/3}$, $y \geq 0$

c) By independence, $P(X > 3, Y > 4) = P(X > 3)P(Y > 4) = (1 - F_1(3))(1 - F_2(4)) = (e^{-1})(e^{-4/3}) = 0.0588$

d) By independence, $P(X > 3|Y = 3) = P(X > 3) = 1 - F_1(3) = e^{-3/2} = 0.2231$

e) $E(2X + Y) = 2E(X) + E(Y) = 2 \cdot 2 + 3 = 7$, so 700 hours

f) $Var(2X+Y) = 4Var(X) + Var(Y) = 4 \cdot 2^2 + 3^2 = 25$, or 2,500 hours

4. Random variable and have the following joint probability distribution.

$f(x,y)$		$x=1$	$x=2$	$x=3$
y	1	0.1	0.1	0.15
	2	0.05	0.05	0.2
	3	0.15	0.1	0.1

a) Find $P(X+Y \leq 4)$.

b) Find the marginal probability distributions $f_1(x)$ and $f_2(y)$.

c) Find $P(X < 2|Y = 2)$.

d) Are X and Y independent?

e) Find E(X) and E(Y).

f) Find Var(X) and Var(Y).

g) Find the correlation coefficient of X and Y.

a) $P(X+Y \leq 4) = f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(3,1)=0.1+0.05+0.15+0.1+0.05+0.15 = 0.6$

(b) $f_1(x)=$ $f(1,1) + f(1,2) + f(1,3) = 0.1+0.05 +0.15 = 0.3$, for $x=1$

$f(2,1)+f(2,2)+f(2,3) = 0.1+ 0.05+0.1 = 0.25$, for $x = 2$

$(3,1)+(3,2)+f(3,3) = 0.15 + 0.2 + 0.1 = 0.45$, for $x =3$

$f_2(y)=$ $f(1,1) + f(2,1) + f(3,1) = 0.1 + 0.1 + 0.15 = 0.35$, for $y = 1$

$f(1,2) + f(2,2) + f(3,2) = 0.05 +0.05 + 0.2 = 0.3$, for $y =2$

$f(1,3)+f(2,3)+f(3,3)=0.15+0.1+0.1 = 0.35$, for $y=3$

c) $P(X < 2|Y = 2) = P(X = 1|Y = 2) = f(1,2)/f_2(2) = 0.05/0.3 = 1/6$

d) X and Y are not independent because from part c, $f_1(1|2)=1/6\neq 0.3=f_1(1)$

e) $E(X) = \Sigma x f_1(x) = 0.3+2(0.25)+3(0.45) = 2.15$

$E(Y) = \Sigma y f_2(y) = 0.35+2(0.3)+3(0.35) = 2$

f) $E(X^2) = 5.35$ and Var (X) = 0.7275, $E(Y^2) = 4.7$ and Var (Y) = 0.7

g) $E(XY) = 4.2$ $Cov(X,Y)=E(XY) - E(X)E(Y)=4.2 - (2.15)(2) = -0.1$

$\rho = (X,)/\sigma_X\sigma_Y = -0.1/\mathrm{SQRT}(0.7275)(0.7)) = -0.14$

5. (Note this problem originally had an error. The corrected version and solution is given here)

.Let the random variable X and Y have joint pdf

$$f(x,y) = \frac{3}{4}(x^2 + 3y^2), 0 < x < 1, 0 < y < 1$$

a) Find the marginal densities of X and Y.
b) Find the (marginal) cdf of X and (marginal) cdf of Y.
c) Determine the conditional pdf's $f_1(x|y)$ and $f_2(y|x)$.
d) Find E(X) and E(Y).
e) Find Var (X) and Var (Y).
f) Find Cov (X, Y).
g) Find $P(Y < 1/3|X = \frac{1}{3})$.
h) Find $E(Y|X = \frac{1}{2})$.

(a)

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{3}{4}(x^2 + 3y^2)dy = \frac{3(x^2y + y^3)}{4}\bigg|_{y=0}^{1} = \frac{3(x^2+1)}{4},$$

$$0 < x < 1$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \frac{3}{4}(x^2 + 3y^2)dx = \left[\frac{x^3}{4} + \frac{9xy^2}{4}\right]_{x=0}^{1} = \frac{9y^2+1}{4},$$

$$0 < y < 1$$

(b)

$$F_1(x) = \begin{cases} 0 & x \le 0 \\ \int_{-\infty}^{x} f_1(t)dt = \int_0^x \frac{3(t^2+1)}{4}dt = \left[\frac{t^3}{4} + \frac{3t}{4}\right]_0^x = \frac{x^3+3x}{4}, & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F_2(y) = \begin{cases} 0 & y \le 0 \\ \int_{-\infty}^{y} f_2(t)dt = \int_0^y \frac{9t^2+1}{4}dt = \left[\frac{3t^3}{4} + \frac{t}{4}\right]_0^y = \frac{3y^3+y}{4}, & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$$

(c)

$$f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\frac{3}{4}(x^2+3y^2)}{\frac{9y^2+1}{4}} = \frac{3(x^2+3y^2)}{9y^2+1}, \qquad 0<x<1, 0<y<1$$

$$f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{3}{4}(x^2+3y^2)}{\frac{3(x^2+1)}{4}} = \frac{x^2+3y^2}{x^2+1}, \qquad 0<y<1, 0<x<1$$

(d)

$$E(X) = \int_{-\infty}^{\infty} x f_1(x)dx = \int_0^1 \frac{3(x^3+x)}{4}dx = \left[\frac{3x^4}{16}+\frac{3x^2}{8}\right]_0^1 = \frac{3}{16}+\frac{3}{8} = \frac{9}{16}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_2(y)dy = \int_0^1 \frac{9y^3+y}{4}dy = \left[\frac{9y^4}{16}+\frac{y^2}{8}\right]_0^1 = \frac{9}{16}+\frac{1}{8} = \frac{11}{16}$$

(e)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_1(x)dx = \int_0^1 \frac{3(x^4+x^2)}{4}dx = \left[\frac{3x^5}{20}+\frac{x^3}{4}\right]_0^1 = \frac{3}{20}+\frac{1}{4} = \frac{2}{5}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{5} - \left(\frac{9}{16}\right)^2 = \frac{107}{1280}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_2(y)dy = \int_0^1 \frac{9y^4+y^2}{4}dy = \left[\frac{9y^5}{20}+\frac{y^3}{12}\right]_0^1 = \frac{9}{20}+\frac{1}{12} = \frac{8}{15}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{8}{15} - \left(\frac{11}{16}\right)^2 = \frac{233}{3840}$$

(f)

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^1\int_0^1 \frac{3}{4}(x^3y+3xy^3)dxdy$$

$$= \int_0^1 \left(\left[\frac{3x^4y}{16}+\frac{9x^2y^3}{8}\right]_{x=0}^1\right)dy = \int_0^1 \left(\frac{3y}{16}+\frac{9y^3}{8}\right)dy$$

$$= \left[\frac{3}{32}(y^2+3y^4)\right]_0^1 = \frac{3(1+3)}{32} = \frac{3}{8}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{3}{8} - \frac{9}{16}\cdot\frac{11}{16} = -\frac{3}{256}$$

(g)

$$P\left(Y < \frac{1}{3} \middle| X = \frac{1}{3}\right) = \int_{-\infty}^{1/3} f_2(y|1/3)dy = \int_{0}^{1/3} \frac{(1/9) + 3y^2}{(1/9) + 1} dy = \int_{0}^{1/3} \frac{27y^2 + 1}{10} dy$$
$$= \left[\frac{9y^3 + y}{10}\right]_0^{1/3} = \frac{(1/3) + (1/3)}{10} = \frac{1}{15}$$

(h)

$$E\left(Y \middle| X = \frac{1}{2}\right) = \int_{-\infty}^{\infty} y f_2(y|1/2)dy = \int_{0}^{1} \frac{(1/4)y + 3y^3}{(1/4) + 1} dy = \int_{0}^{1} \frac{12y^3 + y}{5} dy$$
$$= \left[\frac{3y^4}{5} + \frac{y^2}{10}\right]_0^1 = \frac{3}{5} + \frac{1}{10} = \frac{7}{10}$$

6. Suppose that the joint probability density of two random variables X_1 and X_2 is given by:

$$f(x_1,x_2) = \begin{cases} 6e^{-2x_1-3x_2} & for\ x_1 > 0\ and\ x_2 > 0, \\ 0 & otherwise \end{cases}$$

Find the following probabilities. Show all steps.

a) $P(1 \leq X_1 \leq 2, 2 \leq X_2 \leq 3)$

b) $P(X_1 < 2, X_2 > 2)$.

a)

$$P(1 \leq X_1 \leq 2, 2 \leq X_2 \leq 3)$$
$$= \int_1^2 \int_2^3 6e^{-2x_1-3x_2}\, dx_2 dx_1 = \int_1^2 [-2e^{-2x_1-3x_2}]_{x_2=2}^{3}\, dx_1$$
$$= -2\int_1^2 (e^{-2x_1-9} - e^{-2x_1-6})\, dx_1 = -2[\frac{e^{-2x_1-9} + e^{-2x_1-6}}{-2}]_{x_1=1}^{2}$$
$$= 0.0003$$

b)

$$P(X_1 < 2, X_2 > 2) = \int_0^2 \int_2^\infty 6e^{-2x_1-3x_2}\, dx_2 dx_1 = \int_0^2 [-2e^{-2x_1-3x_2}]_{x_2=2}^{\infty}\, dx_1$$

$$= -2\int_0^2 (-e^{-2x_1-6})\, dx_1 = -[e^{-2x_1-6}]_{x_1=0}^{2} = -(e^{-10} - e^{-6}) = 0.002$$

7. Given the joint pdf from problem 6, find the marginal density $f_1(x_1)$.

$$f_1(x) = \int_0^\infty 6e^{-2x-3y}\, dy = 2e^{-2x}$$

8. Suppose that the joint probability density of two random variables X_1 and X_2 is given by:

$$f(x_1,x_2) = \begin{cases} \frac{2}{3}(x_1 + 2x_2) \; for \; 0 < x_1 < 1 \; and \; 0 < x_2 < 1 \\ 0 \qquad\qquad otherwise \end{cases}$$

Find the conditional density $f_1(x_1 | x_2)$. Hint, first find the marginal density $f_2(x_2)$.

$$f_1(x_1) = \int_0^1 \frac{2}{3}(x_1 + 2x_2)dx = \frac{1}{3}(4x_2 + 1)$$

$$f_2(x_1|x_2) = \frac{f(x_1,x_2)}{f_2(x_2)} = \frac{2(x_1+2x_2)}{4x_2+1}$$

9. Two random variables X and Y are independent. Each has a binomial distribution with success probability 0.4 and 2 trials.

a) Find the joint probability distribution function f(x,y).

b) Give the joint probabilities using a table. Hint, the size of the tables is 3 by 3.

c) Find the probability P(Y > X).

a) X and Y are all binomial(2,.4), so $f(x,y) = (\binom{2}{x}0.4^x 0.6^{2-x})\,(\binom{2}{y}0.4^y 0.6^{2-y})$

b)

	f(x,y)	Y: 0	1	2
X	0	0.36^2	(0.36)(0.48)	(0.36)(0.16)
	1	(0.48)(0.36)	0.48^2	(0.48)(0.16)
	2	(0.16)(0.36)	(0.48)(0.36)	0.16^2

c) Add the probabilities for p(x,y) for (x,y) in {(0,1), (0,2), (1,2)} , i.e. find

P(0,1) + P(0,2) + P(1,2) = (.36)(.48) + (.36)(.16) + (.48)(.48)

0.3072