AMS 310 Homework 3 Solutions
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Chapter 4

1. Let X be a random variable with pdf $f(x) = \frac{1}{2}, 0 < x < 2$.

a) Find the cdf F(x).

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

b) Find the mean of X.

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 \frac{1}{2}xdx = \frac{1}{2}\int_0^2 xdx = \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_0^2 = \frac{1}{2}(2-0) = \frac{1}{2}(2) = 1$$

c) Find the variance of X.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 \frac{1}{2}x^2dx = \frac{1}{2}\int_0^2 x^2dx = \frac{1}{2}\left(\frac{x^3}{3}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{8}{3}-0\right) = \frac{8}{6} = \frac{4}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{4}{3} - (1)^2 = \frac{1}{3}$$

d) Find F (1.4).

$$F(1.4) = \frac{1.4}{2} = 0.7$$

e) Find $P(\frac{1}{2} < X < 1)$.

$$P\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^{1} f(x)dx = \frac{x}{2}\Big|_{1/2}^{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

f) Find $P(X > 3)$.

$$P(X > 3) = 1$$

g) Find the 35th percentile.

Solve $F(X) = 0.35$

$$\frac{x}{2} = 0.35$$

$$x = 0.7$$

2. Suppose the weight (in lbs) of an adult male sheep in a pasture is distributed as N(100, 225)

a) Find the probability that the weight of a sheep is over 120 pounds.

$$P(X > 120) = 1 - P(X \leq 120) = 1 - P\left(Z \leq \frac{120 - 100}{15}\right) = 1 - \phi(1.333) = 1 - 0.9082 = 0.0918$$

b) What value of weight separates the heaviest 10% of all the sheep in the pasture from the other 90%? Use the Normal Table.

We find the 90^{th} Percentile

$$\phi(z) = 0.90$$

$$z = 1.285\ (from\ table)$$

$$z = \frac{x - 100}{15}$$

$$15z = x - 100$$

$$x = 100 + 15z = 100 + 15(1.285) = 119.28\ lbs$$

3. The lifetime of the timing belt of a certain make of cars is normally distributed with mean 125,000 miles and standard deviation 10,000 miles.

a) Find the probability that a timing belt lasts until the car runs 140,000 miles.

$$P(X \geq 140000) = 1 - P(X < 140000) = 1 - P\left(Z < \frac{140000 - 125000}{10000}\right) = 1 - \phi(1.5) = 1 - 0.9332 = 0.0668$$

b) The auto maker recommends that owners have the timing belt replaced when the mileage reaches 90,000 miles. What is the probability that the timing belt fails before the car reaches the manufacturer's recommended mileage?

$$P(X < 90000) = P\left(Z < \frac{90000 - 125000}{10000}\right) = \phi(-3.5) = 1 - 0.99977 = 0.00023$$

c) An owner of this type of car wants to take a chance and replace the timing belt at the 1^{st} percentile of the distribution. What should be the mileage of the car when he has the timing belt replaced? Use the Normal Table.

$$\phi(z) = 0.01$$

$$z = -2.325\ (from\ table)$$

$$z = \frac{x - 125000}{10000}$$

$$10000z = x - 125000$$

$$x = 10000(-2.325) + 125000 = 101750\ mi$$

4. The time (in minutes) that it takes a mechanic to change oil has an exponential distribution with mean 20.

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & x > 0 \\ 0, & otherwise \end{cases} \qquad F(x) = \begin{cases} 1 - e^{-x/20}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

a) Find P(X < 25), P(X > 15), and P(15 < X < 25)

$$P(X < 25) = 1 - e^{-25/20} = 0.7135$$

$$P(X > 15) = 1 - P(X \le 15) = 1 - \left(1 - e^{-15/20}\right) = 1 - 0.5276 = 0.4724$$

$$P(15 < X < 25) = P(X < 25) - P(X \le 15) = 0.7135 - 0.5276 = 0.1851$$

b) Find the 40th percentile

$$F(x) = 0.40$$
$$1 - e^{-x/20} = 0.40$$
$$-e^{-x/20} = -0.6$$
$$e^{-x/20} = 0.6$$
$$\frac{-x}{20}\ln(e) = \ln(0.6)$$
$$\frac{-x}{20} = \ln(0.6)$$
$$x = -20\ln(0.6) = 10.22\ min$$

5. A random variable X is uniformly distributed over the interval (48,96). Find the following probabilities. You can use R just be sure to copy and paste the code and the answer from R into your paper.

a) P(50 < X < 70)

```
> punif(70,48,96)-punif(50,48,96)
[1] 0.4166667
```

b) P(X < 75)

```
> punif(75,48,96)
[1] 0.5625
```

c) P(X > 90)

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> 1-punif(90,48,96)
[1] 0.125
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d) P(X < 60 or X > 80)

$X < 60\ \&\ X > 80$ *are mutually exclusive, so* $P(X < 60\ \cup\ X > 80) = P(X < 60) + P(X > 80)$

```
> punif(60,48,96)+(1-punif(80,48,96))
[1] 0.58333336.
```

6. Use R to answer the following. Copy and paste the code and the answer from R into your paper. The length of life in years, T, of a heavily used terminal in a student computer laboratory is exponentially distributed with mean $\lambda = 0.5$ years.

Find the probability that:

a) Find the probability that the terminal lasts less than 1 year.

```
> pexp(1,2)
[1] 0.8646647
```

b) Find the probability that the terminal lasts between 1 year and 2 years.

```
> pexp(2,2)-pexp(1,2)
[1] 0.1170196
```

c) Find the 90th percentile for life length of the terminal.

> qexp(0.9,2)

[1] 1.151293 (years)

7. Suppose that the annual amount of rainfall (in million tons) accumulated in a lake follows a gamma distribution with $\alpha = 3$ and $\beta = 5$.

a) Find the expected annual rainfall accumulated in this lake.

$$E(X) = \alpha\beta = 3 * 5 = 15\ in.$$

b) Find the standard deviation of the annual amount of rainfall in this lake.

$$Var(X) = \alpha\beta^2 = 3 * 5^2 = 75$$
$$\sigma^2 = 75$$
$$\sigma = 8.66\ in.$$

c) Find P(X < 4), use R.

> pgamma(4,3,1/5)

[1] 0.0474226

d) Find P(2 < X < 5, use R.

> pgamma(5,3,1/5)-pgamma(2,3,1/5)

[1] 0.07237507

8. Suppose that a distribution is Beta(1,1).

$$\alpha = 1, \beta = 1$$
$$for\ any\ positive\ integer\ n, \qquad \Gamma(n) = (n-1)!$$
$$\Gamma(\alpha + \beta) = \Gamma(2) = 1! = 1$$
$$\Gamma(a) = \Gamma(\beta) = \Gamma(1) = 0! = 1$$

a) What if the pdf of this distribution?

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$
$$f(x) = \begin{cases} \dfrac{1}{1 * 1} x^{1-1}(1-x)^{1-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$
$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

b) Write another name for this distribution.

Uniform

c) Find the cdf.

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$
$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x-0}{1-0}, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

9. Suppose that X is a random variable that follows a Beta(1,4) distribution. Use R to find the following probabilities. Copy and paste the code and the answer into your document.

a) P(X < ½)

```
> pbeta(1/2,1,4)
[1] 0.9375
```

b) P(X > 5/8)

```
> 1-pbeta(5/8,1,4)
[1] 0.01977539
```

10. In a certain country, 20% of the female adult population smoke regularly. In a random sample of 800 adults, what is the probability that:

a) less than 150 are smokers? Use the normal approximation to the binomial.

$$P(X < 150) = P\left(Z \leq \frac{149.5 - 160}{11.31}\right) = \phi(-0.93) = 0.1762$$

b) 200 or more are smokers? Use the normal approximation to the binomial.

$$P(X \geq 200) = P\left(Z \geq \frac{199.5 - 160}{11.31}\right) = P(Z \geq 3.49) = 1 - \phi(3.49) = 1 - 0.9998 = 0.0002$$

c) Find answers to (a) and (b) using the binomial distribution and R. Copy and paste the code and the answer into your document.

```
> pbinom(150,800,0.2)
[1] 0.201268

> 1-pbinom(200,800,0.2)
[1] 0.0002443924
```