AMS 310 Spring 2018 **Fred Rispoli**

Practice Test 2 Solutions

1. When a balanced coin is flipped 10,000 times, find the lower bound of the probability that the proportion of heads is obtained will fall between .45 and .55.

μ=10,000(0.5) = 5,000

σ=SQRT(10,000(0.5)(0.5)) = 50

$\mu-k\sigma$=5,000−10(50)=4,500 and $\mu+k\sigma$=5,000+10(50)=5,500

4,500/10,000=0.45 and 5,500/10,000=0.55

Thus we have σ=50, we have k=10.

Applying Chebyshev's Inequality: $P(|X-\mu|<k\sigma)\geq 1-\frac{1}{k^2}$

$P(|X-5{,}000|<500)\geq 1-\frac{1}{100}=0.99$

2. Let $X_1, X_2, \ldots X_{100}$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{3x^2}{2} + x, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

a) Find the mean of X
b) Find the variance of X
c) Use the Central Limit Theorem to find the probability of P(0.7 < $\bar{X}$ < 0.75)

(a)

$$E(X_1) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 \left(\frac{3x^3}{2} + x^2\right) dx = \left[\frac{3x^4}{8} + \frac{x^3}{3}\right]_0^1 = \frac{3}{8} + \frac{1}{3} = \frac{17}{24}$$

(b)

$$E\left({X_1}^2\right) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 \left(\frac{3x^4}{2} + x^3\right) dx = \left[\frac{3x^5}{10} + \frac{x^4}{4}\right]_0^1 = \frac{3}{10} + \frac{1}{4} = \frac{11}{20}$$

$$Var(X_1) = E\left({X_1}^2\right) - [E(X_1)]^2 = \frac{11}{20} - \left(\frac{17}{24}\right)^2 = \frac{139}{2880} = 0.0483$$

(c) By CLT, $\bar{X}$ is approximately

$$N(\mu, \sigma^2/n) = N\left(\frac{17}{24}, {}^{(0.0483)}/_{100}\right) = \left(\frac{17}{24}, 0.000483\right).$$

$$P(0.7 < \bar{X} < 0.75) \approx P\left(\frac{0.7 - 0.708}{\sqrt{0.000483}} < Z < \frac{0.75 - 0.708}{\sqrt{0.000483}}\right) = P(-0.36 < Z < 1.91)$$

$$= \Phi(1.91) - \Phi(-0.36) = 0.9719 - 0.3594 = 0.6125$$

3. A quality control manager for a company that manufactures aluminum water pipes believes that the product lengths of one of the pipes produced can be modeled by a uniform probability distribution over the interval 29.50 to 30.05 feet. Set up the correct integral to determine the probability that a pipe produced has length:

a) Less than 29.75 feet

b) Between 29.75 and 29.90 feet

The pdf is f(x) = $\frac{1}{30.05-29.5}$ = $\frac{1}{0.55}$

a) P(X < 29.75) = $\int_{29.50}^{29.75} f(x)dx$

b) (29.75 < X < 29.90) = $\int_{29.75}^{29.90} f(x)dx$

4. The number of gallons of Gatorade consumed by a football team during a game follows a normal distribution with mean 20. The standard deviation is 3.

a) If a game is selected at random, find the probability that the number of gallons consumed will be greater than 23 gallons.

b) If a game is selected at random, find the probability that the number of gallons consumed will be between 22 and 25 gallons.

c) Find the 90th percentile.

a) z = $\frac{23-20}{3}$ = 1.0

P(X > 23) = 1 – P(X ≤ 23) = 1 – P(Z ≤ 1) = 1 – Φ(1) = 1 – .8413 = .1587

b) P(22 < X < 25) = P(.67 < Z < 1.67) = Φ(1.67) - Φ(.67) = .9525 - .7486 = .2039

z = $\frac{22-20}{3}$ = .67 z = $\frac{25-20}{3}$ = 1.67

c) Set F(x) = .90

Use the table in reverse to get Z = 1.645, then solve 1.645 = $\frac{x-20}{3}$ to get x= 24.935

5. Let X be a random variable with cdf

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{x-2}{2}, & 2 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

a) Find the pdf of X.

b) Find P($\frac{2}{3}$ < X < 3)

c) Find P(X > 3.5)

d) Find the 60th percentile

e) Find P(X=3)

(a) $f(x)=F'(x)=\{\quad 1/2,\quad 2\leq x\leq 4$
0, otherwise

(b) $P(2/3<X<3) = F(3)-F(2/3)=1/2-0=1/2$

(c) $P(X > 3.5) = 1-F(3.5)=1-3/4=1/4$

(d) $0.6 = F(x)=(x-2)/2\Rightarrow x-2 = 1.2 \Rightarrow x=3.2$

(e) $P(X=3) = 0$

6. The random variables X and Y have the following joint probability distribution.

		X			
f(x,y)		1	2	3	4
	1	.1	.05	.04	.01
Y	2	.2	.05	.15	.04
	3	.15	.05	.10	.06

a) Find $P(X + Y \leq 5)$.

b) Find the marginal probability distributions $f_1(x)$ and $f_2(y)$.

c) Find $P(X < 2|Y = 3)$.

d) Are X and Y independent? Thoroughly explain your answer.

e) Find E(X) and Var(X)

f) Find the correlation coefficient of X and Y.

a) The X,Y combinations that sum to 5 or less are in red below.

		X			
f(x,y)		1	2	3	4
	1	.1	.05	.04	.01
Y	2	.2	.05	.15	.04
	3	.15	.05	.10	.06

$P(X + Y \leq 5)$ = .1 + .05 + .04 + .01 + .2 + .05 + .15 + .15 + .05 = .80

b)

X	1	2	3	4
$f_1(x)$	.45	.15	.29	.11

Y	1	2	3
$f_2(y)$	.2	.44	.36

c) $P(X < 2|Y = 3)$ = .15/.36

d) Not independent f(1,1) = .1 $\neq$ $f_1(1)\, f_2(1)$ = (.45)(.2) = .09

e) E(X) = 1(.45) + 2(.15) + 3(.29) + 4(.11) = .45 + .30 + .87 + .44 = 2.06

X^2	1	4	9	16
$f_1(x)$	.45	.15	.29	.11

$E(X^2) = 1(.45) + 4(.15) + 9(.29) + 16(.11) = .45 + .60 + 2.61 + 1.76 = 5.42$

$Var(X) = E(X^2) - E(X)^2 = 5.42 – 2.06^2 = 1.176$

$\sigma_X = SQRT(1.176) = 1.084$

f) Use $\boldsymbol{\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}}$

$E(Y) = 1(.2) + 2(.44) + 3(.36) = 2.06$

Y^2	1	4	9
$f_2(y)$	.2	.44	.36

$E(Y^2) = 1(.2) + 4(.44) + 9(.36) = 5.20$

$Var(Y) = E(Y^2) - E(Y)^2 = 5.02 – 1.98^2 = 0.5344$

$\sigma_X = SQRT(1.176) = 0.731$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Thus we need to know E(XY), the procedure is the following:

P(XY=1)=P(X=1,Y=1)=0.1
P(XY=2)=P(X=1,Y=2)+ P(X=2,Y=1)=0.2+0.05=0.25
P(XY=3)=P(X=1,Y=3)+ P(X=3,Y=1)=0.04+0.15=0.19
P(XY=4)=P(X=4,Y=1)+ P(X=2,Y=2)=0.01+0.05=0.06
P(XY=6)=P(X=2,Y=3)+ P(X=3,Y=2)=0.05+0.15=0.2
P(XY=8)=P(X=4,Y=2)=0.04
P(XY=9)=P(X=3,Y=3)=0.1
P(XY=12)=P(X=4,Y=3)=0.06

E(XY)=1(0.1)+2(0.25)+3(0.19)+4(0.06)+6(0.2)+8(0.04)+9(0.1)+12(0.06)=4.55

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_x \sigma_y} = \frac{4.55 - 1.98 * 2.06}{1.084 * 1.10} = 0.1266$$

7. Suppose the random variables X and Y have joint pdf as follows:

f(x,y) = $\frac{4}{7}\left(x^2 + \frac{xy}{3}\right)$, $0 < x < 1,\ 0 < y < 3$

a) Find the marginal pdf $f_1(x)$ of X, and $f_2(y)$ of Y.

$$f_1(x) = \int_0^3 f(x,y)dy = \int_0^3 \frac{4}{7}\left(x^2 + \frac{xy}{3}\right)dy = \frac{12}{7}x^2 + \frac{6}{7}x$$

$$f_2(y) = \int_0^1 f(x,y)dx = \int_0^1 \frac{4}{7}\left(x^2 + \frac{xy}{3}\right)dx = \frac{12}{7}x^2 + \frac{6}{7}x = \frac{4}{21} + \frac{2}{21}y$$

b) Find the cdf of X and Y.

$F_1(x) = \int_0^x f_1(t)dt = \frac{4}{7}x^3 + \frac{3}{7}x^2$, for $0 \le x \le 1$

Thus $F_1(x) = \begin{cases} 0, & x < 0 \\ \frac{4}{7}x^3 + \frac{3}{7}x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$

$F_2(y) = \int_0^y f_2(t)dt = \frac{4}{21}y + \frac{1}{21}y^2$, for $0 \le y \le 3$

Thus $F_2(y) = \begin{cases} 0, & y < 0 \\ \frac{4}{21}y + \frac{1}{21}y^2, & 0 \le y \le 3 \\ 1, & y > 3 \end{cases}$

c) Find P(Y < 2)

$$P(Y < 2) = F_2(2) = \frac{12}{21}$$

d) Find P($X > \frac{1}{2}, Y < 1$)

$$P\left(X > \frac{1}{2}, Y < 1\right) = \int_{\frac{1}{2}}^1 \int_0^1 f(x,y)dydx = \frac{17}{84}$$

e) Find the conditional pdf $f_2(y \mid x)$

$$f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{4}{7}\left(x^2+\frac{xy}{3}\right)}{\frac{12}{7}x^2+\frac{6}{7}x} = \frac{6x+2y}{18x+9}$$ (after simplifying)

8. Let X and Y be independent random variables representing the lifetime (in 100 hours) of type A and B lightbulbs, respectively. Both variables have exponential distributions, and the mean of X is 2 and the mean of Y is 3.

a) Find the joint pdf f(x,y) of X and Y.

b) Find the conditional pdf $f_2(y \mid x)$

c) Given the probability that a type A bulb fails at 200 hours, find the probability that a type B bulb fails lasts longer than 300 hours.

a) By independence, $f(x,y)=f_1(x)f_2(y)=\{(1/6)e^{-x/2-y/3}, x \geq 0, y \geq 0$
0, otherwise

b) By independence, $f_2(y|x)=f_2(y)=1/3e^{-y/3}$, $y \geq 0$

c) By independence, $P(Y>3|X=2)=P(Y>3)=1-F_2(3)= 1 - (1- e^{-3/3}) = e^{-1}= 0.3679$

9. Three random variables X, Y and Z are independent. Each has a binomial distribution with success probability 0.3 and 5 trials.

a) Find the joint probability distribution function f(x,y,z).

b) Find the probability P(Z > Y > X > 1).

a) X, Y and Z are all binomial(5, 0.3), so

f(x,y,z) = $(\binom{5}{x}0.3^x .7^{5-x})\ (\binom{5}{y}0.3^y .7^{5-y})\ (\binom{5}{z}0.3^z .7^{5-z})$

b) Add the probabilities for (x,y,z) in {(2,3,4), (2,3,5), (2,4,5), (3,4,5)}

10. The weight of adult bottlenose dolphins was found to follow a normal distribution with a mean of 550 pounds and a standard deviation of 50 pounds.

a) What percentage of bottlenose dolphins weigh from 400 to 600 pounds?

b) If $\bar{X}$ represents the mean weight of a random sample of 9 adult dolphins, what is

P(500 < $\bar{X}$ < 580)?

c) In a random sample of 9 adult bottlenose dolphins, what is the probability that 5 of them are heavier than 560 pounds?

(a) Let X be the weight of an adult bottlenose dolphin. Then

$$P(400 < X < 600) = P\left(\frac{400-550}{50} < Z < \frac{600-550}{50}\right) = P(-3 < Z < 1)$$
$$= \Phi(1) - \Phi(-3) = 0.8413 - 0.0013 = 0.84$$

(b)

$$P(500 < \bar{X} < 580) = P\left(\frac{500-550}{\frac{50}{\sqrt{9}}} < Z < \frac{580-550}{\frac{50}{\sqrt{9}}}\right) = P(-3 < Z < 1.8)$$
$$= \Phi(1.8) - \Phi(-3) = 0.9641 - 0.0013 = 0.9628$$

(c)

$$P(X > 560) = P\left(Z > \frac{560-550}{50}\right) = P(Z > 0.2) = 1 - \Phi(0.2) = 1 - 0.5793 = 0.4207$$

Let Y be the number of dolphins which are heavier than 600 pounds out of the sample of 9. Then $Y \sim \text{Bin}(9, 0.4207)$.

$$P(Y = 5) = \binom{9}{5}(0.4207)^5(0.5793)^4 = 0.1870$$

11. If the variance of a normal population is 4, what is the probability that the variance of a random sample of size 10 exceeds 6.526?

a) Find the probability using the distribution table.

b) Not on test but just for fun, find the probability using R or Excel

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{9S^2}{4} \text{ is distributed as } \chi^2 \text{ with 9 degrees of freedom.}$$

(a) $\chi^2 = \frac{9(6.526)}{4} = 14.6835$

$$P(S^2 > 6.526) = P(\chi^2 > 14.6835) = 0.10$$

(b)

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>1-pchisq(9*6.526/4, 9)
0.1000047
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