Homework-5 – Answer Sheet

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Problem 1.

[Gauss Map Image] Describe the region of the unit sphere covered by the image of the Gauss map $N : S \to S^2$ of the following surfaces:

1. Paraboloid: $z = x^2 + y^2$
2. Hyperboloid: $x^2 + y^2 - z^2 = 1$
3. Catenoid: $x^2 + y^2 = \cosh^2 z$ ($\cosh z = \frac{e^z+e^{-z}}{2}$).

(You do not have to compute the Gauss map. Just describe the image of N in terms of the picture of S).

Solution.

The image of the Gauss map consists of all unit normal vectors to the surface, translated to the origin.

1. **Paraboloid:** The surface $z = x^2 + y^2$ is a bowl opening upwards. The normal vector at the origin $(0, 0, 0)$ is $(0, 0, 1)$, pointing to the North Pole of S^2. As we move away from the origin, the surface becomes steeper, and the normal vector tilts outwards, becoming more horizon-

tal. The z-component of the normal is always positive. As $(x, y) \to \infty$, the normal vector approaches the equatorial plane ($z = 0$) but never reaches it. Therefore, the image of the Gauss map is the **open northern hemisphere** ($z > 0$).

2. **Hyperboloid of one sheet:** The surface $x^2+y^2-z^2 = 1$ is narrowest at the "neck" circle $x^2 + y^2 = 1$ in the $z = 0$ plane. At this neck, the normal vectors are horizontal and point outwards from the z-axis, so their image covers the equator of S^2. As $|z|$ increases, the surface approaches the cone $x^2 + y^2 = z^2$. The normal vectors tilt away from the horizontal plane, pointing upwards in the upper half ($z > 0$) and downwards in the lower half ($z < 0$). As $|z| \to \infty$, the normal vectors approach (but never reach) the vertical directions. The limit angle of the cone is 45°, so the normal vectors are confined to a band around the equator. The image is an **open band around the equator**, specifically the region $-\frac{1}{\sqrt{2}} < z < \frac{1}{\sqrt{2}}$ on the sphere.

3. **Catenoid:** The catenoid has a neck at $z = 0$, where the tangent plane is vertical, so the normal vectors are horizontal, mapping to the equator of S^2. As $|z|$ increases, the surface flares out and becomes more horizontal. The normal vectors tilt towards the poles. As $|z| \to \infty$, the surface becomes almost horizontal, and the normal vectors approach the North and South Poles but never reach them. Thus, the image of the Gauss map is the **entire sphere except for the North and South Poles**.

Problem 2.

[Area of Torus] Find the area of the torus of revolution S defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2\},$$

where $a > r > 0$ are given positive constants.

Solution.

We can parametrize the torus by

$$X(u, v) = ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v), \quad u, v \in [0, 2\pi].$$

The partial derivatives are:

$$\begin{aligned} X_u &= (-(a + r\cos v)\sin u, (a + r\cos v)\cos u, 0) \\ X_v &= (-r\sin v\cos u, -r\sin v\sin u, r\cos v) \end{aligned}$$

The coefficients of the first fundamental form are:

$$\begin{aligned} E &= \langle X_u, X_u \rangle = (a + r\cos v)^2 \sin^2 u + (a + r\cos v)^2 \cos^2 u = (a + r\cos v)^2 \\ F &= \langle X_u, X_v \rangle = 0 \\ G &= \langle X_v, X_v \rangle = r^2 \sin^2 v \cos^2 u + r^2 \sin^2 v \sin^2 u + r^2 \cos^2 v = r^2 \end{aligned}$$

The area element is $dA = \sqrt{EG - F^2}\, du\, dv$.

$$\sqrt{EG - F^2} = \sqrt{(a + r\cos v)^2 r^2} = r(a + r\cos v) \quad (\text{since } a > r > 0).$$

The area A is the integral of the area element over the domain of parametrization:

$$
\begin{aligned}
A &= \int_0^{2\pi} \int_0^{2\pi} r(a + r\cos v)\, dv\, du \\
&= \int_0^{2\pi} [r(av + r\sin v)]_0^{2\pi}\, du \\
&= \int_0^{2\pi} r(a(2\pi) + r\sin(2\pi) - 0)\, du \\
&= \int_0^{2\pi} 2\pi ar\, du = 2\pi ar[u]_0^{2\pi} = 2\pi ar(2\pi) = 4\pi^2 ar.
\end{aligned}
$$

The area of the torus is $(2\pi a)(2\pi r)$.

Problem 3.

[Geodesic on Sphere] Consider the sphere parametrized by spherical coordinates:

$$X(u, v) = (\sin v \cos u, \sin v \sin u, \cos v)$$

with $-\pi < u < \pi$, $0 < v < \pi$. Find the length of the curve α given by $u = u_0$ and $a \leq v \leq b$ with $0 < a < b < \pi$. (That is $\alpha(t) = (\sin t \cos u_0, \sin t \sin u_0, \cos t)$, with $a \leq t \leq b$.) Let $\beta(t)$ be another curve joining $\alpha(a)$ to $\alpha(b)$ on the surface, i.e., $\beta(t) = X(u(t), v(t))$, $a \leq t \leq b$ with $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$. Show that $l(\beta) \geq l(\alpha)$.

Solution.

First, we compute the first fundamental form of the sphere. $X_u = (-\sin v \sin u, \sin v \cos u, 0)$ and $X_v = (\cos v \cos u, \cos v \sin u, -\sin v)$.

$$
\begin{aligned}
E &= \langle X_u, X_u \rangle = \sin^2 v \\
F &= \langle X_u, X_v \rangle = 0 \\
G &= \langle X_v, X_v \rangle = \cos^2 v + \sin^2 v = 1
\end{aligned}
$$

The curve α is given by $u(t) = u_0$ (constant) and $v(t) = t$ for $a \le t \le b$. The velocity vector is $\alpha'(t) = X_u \frac{du}{dt} + X_v \frac{dv}{dt} = 0 \cdot X_u + 1 \cdot X_v = X_v$. The speed is $||\alpha'(t)|| = \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2} = \sqrt{G} = 1$. The length of α is

$$l(\alpha) = \int_a^b ||\alpha'(t)||\, dt = \int_a^b 1\, dt = b - a.$$

Now, let $\beta(t) = X(u(t), v(t))$ be any other curve with $\beta(a) = \alpha(a)$ and $\beta(b) = \alpha(b)$. This means $u(a) = u_0, v(a) = a$ and $u(b) = u_0, v(b) = b$. The velocity vector of β is $\beta'(t) = X_u u'(t) + X_v v'(t)$. The squared speed is $||\beta'(t)||^2 = E(u')^2 + 2Fu'v' + G(v')^2 = \sin^2(v(t))(u'(t))^2 + (v'(t))^2$. The length of β is

$$l(\beta) = \int_a^b \sqrt{\sin^2(v(t))(u'(t))^2 + (v'(t))^2}\, dt.$$

Since $\sin^2(v(t))(u'(t))^2 \ge 0$, we have:

$$\sqrt{\sin^2(v(t))(u'(t))^2 + (v'(t))^2} \ge \sqrt{(v'(t))^2} = |v'(t)|.$$

Therefore,

$$l(\beta) \ge \int_a^b |v'(t)|\, dt.$$

By the fundamental theorem of calculus, $\int_a^b v'(t)\, dt = v(b) - v(a)$. Also, the integral of the absolute value is greater than or equal to the absolute value of the integral:

$$\int_a^b |v'(t)|\, dt \ge \left| \int_a^b v'(t)\, dt \right| = |v(b) - v(a)|.$$

Since $v(a) = a$ and $v(b) = b$, we have $|v(b) - v(a)| = |b - a| = b - a$ (as $b > a$). Combining the inequalities, we get

$$l(\beta) \ge b - a = l(\alpha).$$

Equality holds if and only if $u'(t) = 0$ for all t (so $u(t) = u_0$) and $v'(t) \ge 0$. This means β must be the same curve as α. Thus, the meridian arc is the shortest path between its endpoints.

Problem 4.

[Mean Curvature Formula] Show that the mean curvature of $p \in S$ is given by the formula

$$H = \frac{1}{2}\left(\frac{Eg - 2Ff + Ge}{EG - F^2}\right)$$

Solution.

The mean curvature H is defined as half the trace of the shape operator (Weingarten map) dN_p. The matrix of the shape operator with respect to the basis $\{X_u, X_v\}$ is given by $W = I^{-1}II$, where I and II are the matrices of the first and second fundamental forms:

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad II = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The inverse of the first fundamental form matrix is:

$$I^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

Now we compute the matrix product $W = I^{-1}II$:

$$\begin{aligned} W &= \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} e & f \\ f & g \end{pmatrix} \\ &= \frac{1}{EG - F^2}\begin{pmatrix} Ge - Ff & Gf - Fg \\ -Fe + Ef & -Ff + Eg \end{pmatrix}. \end{aligned}$$

The trace of a matrix is the sum of its diagonal elements.

$$\mathrm{Tr}(W) = \frac{1}{EG - F^2}((Ge - Ff) + (-Ff + Eg)) = \frac{Ge - 2Ff + Eg}{EG - F^2}.$$

The mean curvature is $H = \frac{1}{2}\mathrm{Tr}(W)$.

$$H = \frac{1}{2}\left(\frac{Eg - 2Ff + Ge}{EG - F^2}\right).$$

This completes the proof.

Problem 5.

[Ellipsoid Curvature] Compute the first and second fundamental form of the ellipsoid S

$$X(\theta, \phi) = (a \sin\theta \cos\phi, a \sin\theta \sin\phi, c \cos\theta)$$

Hence find its Gaussian curvature K and mean curvature H. Moreover, verify that

$$\int_S K\, dA = \iint_U K\sqrt{EG - F^2}\, dudv = 4\pi.$$

Solution.

Let's use (u, v) for (θ, ϕ). $X(u, v) = (a \sin u \cos v, a \sin u \sin v, c \cos u)$. **First Fundamental Form:** $X_u = (a \cos u \cos v, a \cos u \sin v, -c \sin u)$ $X_v = (-a \sin u \sin v, a \sin u \cos v,$ $E = \langle X_u, X_u\rangle = a^2 \cos^2 u \cos^2 v + a^2 \cos^2 u \sin^2 v + c^2 \sin^2 u = a^2 \cos^2 u + c^2 \sin^2 u$. $F = \langle X_u, X_v\rangle = 0$. $G = \langle X_v, X_v\rangle = a^2 \sin^2 u \sin^2 v + a^2 \sin^2 u \cos^2 v = a^2 \sin^2 u$. $\sqrt{EG - F^2} = a \sin u\sqrt{a^2 \cos^2 u + c^2 \sin^2 u}$.

Second Fundamental Form: $X_u \times X_v = (-ac \sin^2 u \cos v, -ac \sin^2 u \sin v, a^2 \sin u \cos u)$. $|X_u \times X_v| = \sqrt{EG - F^2}$. $N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{\sqrt{a^2 \cos^2 u + c^2 \sin^2 u}}(-c \sin u \cos v, -c \sin u \sin v, a \cos u)$. $X_{uu} = (-a \sin u \cos v, -a \sin u \sin v, -c \cos u)$. $X_{uv} = (-a \cos u \sin v, a \cos u \cos v, 0)$. $X_{vv} = (-a \sin u \cos v, -a \sin u \sin v, 0)$. $e = \langle X_{uu}, N\rangle = \frac{ac \sin^2 u + ac \cos^2 u}{\sqrt{a^2 \cos^2 u + c^2 \sin^2 u}} = \frac{ac}{\sqrt{a^2 \cos^2 u + c^2 \sin^2 u}}$. $f = \langle X_{uv}, N\rangle = 0$. $g = \langle X_{vv}, N\rangle = \frac{ac \sin^2 u}{\sqrt{a^2 \cos^2 u + c^2 \sin^2 u}}$.

Curvatures: Let $W = \sqrt{a^2 \cos^2 u + c^2 \sin^2 u}$. $K = \frac{eg - f^2}{EG - F^2} = \frac{eg}{EG} = \frac{a^2 c^2 \sin^2 u / W^2}{(a^2 \cos^2 u + c^2 \sin^2 u) a^2 \sin^2 u} = \frac{c^2}{W^4} = \frac{c^2}{(a^2 \cos^2 u + c^2 \sin^2 u)^2}$. $H = \frac{Eg + Ge}{2(EG - F^2)} = \frac{1}{2}(\frac{g}{G} + \frac{e}{E}) = \frac{1}{2}\left(\frac{ac \sin^2 u / W}{a^2 \sin^2 u} + \frac{ac/W}{a^2 \cos^2 u + c^2 \sin^2 u}\right) = \frac{c(2a^2 \cos^2 u + (a^2 + c^2) \sin^2 u)}{2a(a^2 \cos^2 u + c^2 \sin^2 u)^{3/2}}$.

Integral of K: $\int_S K dA = \int_0^{2\pi}\int_0^{\pi} K\sqrt{EG}\, du\, dv$. $K\sqrt{EG} = \frac{c^2}{(a^2 \cos^2 u + c^2 \sin^2 u)^2} \cdot a \sin u\sqrt{a^2 \cos^2 u + c^2 \sin^2 u} = \frac{ac^2 \sin u}{(a^2 \cos^2 u + c^2 \sin^2 u)^{3/2}}$. The integral is $2\pi \int_0^{\pi} \frac{ac^2 \sin u}{(a^2 \cos^2 u + c^2 \sin^2 u)^{3/2}}\, du$.

Let $w = \cos u$, $dw = -\sin u\, du$. The bounds change from $[0, \pi]$ to $[1, -1]$.

$$\int_S K dA = 2\pi \int_1^{-1} \frac{ac^2}{(a^2w^2 + c^2(1-w^2))^{3/2}}(-dw) = 2\pi ac^2 \int_{-1}^{1} \frac{dw}{(c^2 + (a^2-c^2)w^2)^{3/2}}.$$

Using the standard integral $\int \frac{dx}{(A+Bx^2)^{3/2}} = \frac{x}{A\sqrt{A+Bx^2}}$, we have:

$$\int_S K dA = 2\pi ac^2 \left[\frac{w}{c^2\sqrt{c^2 + (a^2-c^2)w^2}}\right]_{-1}^{1} = 2\pi a \left[\frac{w}{\sqrt{a^2w^2 + c^2(1-w^2)}}\right]_{-1}^{1}.$$

$$= 2\pi a \left(\frac{1}{\sqrt{a^2}} - \frac{-1}{\sqrt{a^2}}\right) = 2\pi a \left(\frac{1}{a} + \frac{1}{a}\right) = 2\pi a \left(\frac{2}{a}\right) = 4\pi.$$

This is consistent with the Gauss-Bonnet theorem, as the ellipsoid is homeomorphic to a sphere, for which the Euler characteristic $\chi = 2$, and $\int_S K dA = 2\pi\chi = 4\pi$.

Problem 6.

[Curvature of Paraboloids] Calculate the mean curvature H and Gauss curvature K of the following surfaces:

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 | z = x^2 + y^2\},$$

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 | z = x^2 - y^2\}$$

with respect to the "upward" (toward positive z-axis) pointing unit normal N. Express the second fundamental form of each surface at $p = (0, 0, 0)$ as a diagonal matrix. What are the principal curvatures and principal directions? Sketch the surfaces near $(0, 0, 0)$.

Solution.

For a surface defined by a graph $z = f(x, y)$, let $p = f_x, q = f_y, r = f_{xx}, s = f_{xy}, t = f_{yy}$. The curvatures are $K = \frac{rt-s^2}{(1+p^2+q^2)^2}$ and $H = \frac{(1+q^2)r-2pqs+(1+p^2)t}{2(1+p^2+q^2)^{3/2}}$.

For $S_1 : z = x^2 + y^2$ **(Elliptic Paraboloid)** $p = 2x, q = 2y, r = 2, s = 0, t = 2$. $K = \frac{2\cdot 2 - 0^2}{(1+4x^2+4y^2)^2} = \frac{4}{(1+4(x^2+y^2))^2}$. $H = \frac{(1+4y^2)2-0+(1+4x^2)2}{2(1+4x^2+4y^2)^{3/2}} = \frac{4+8(x^2+y^2)}{2(1+4(x^2+y^2))^{3/2}} = \frac{2(1+2(x^2+y^2))}{(1+4(x^2+y^2))^{3/2}}$.

For $S_2 : z = x^2 - y^2$ **(Hyperbolic Paraboloid)** $p = 2x, q = -2y, r = 2, s = 0, t = -2$. $K = \frac{2(-2)-0^2}{(1+4x^2+4y^2)^2} = \frac{-4}{(1+4(x^2+y^2))^2}$. $H = \frac{(1+4y^2)2-0+(1+4x^2)(-2)}{2(1+4x^2+4y^2)^{3/2}} = \frac{2+8y^2-2-8x^2}{2(1+4(x^2+y^2))^{3/2}} = \frac{4(y^2-x^2)}{(1+4(x^2+y^2))^{3/2}}$.

At the origin $p = (0, 0, 0)$**:** For both surfaces, $x = y = 0$, so $p = q = 0$. At $p = (0, 0, 0)$, $X_x = (1, 0, 0), X_y = (0, 1, 0)$, so $E = 1, F = 0, G = 1$. The normal is $N = (0, 0, 1)$. The coefficients of the second fundamental form are $e = \langle X_{xx}, N\rangle = r$, $f = s$, $g = t$. The matrix of the second fundamental form is $II = \begin{pmatrix} r & s \\ s & t \end{pmatrix}$. Since I is the identity matrix, the principal curvatures are the eigenvalues of II.

For S_1 **at** $(0, 0, 0)$**:** $r = 2, s = 0, t = 2$. $II = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. This is already diagonal. The principal curvatures are $\kappa_1 = 2, \kappa_2 = 2$. Since the curvatures are equal, this is an umbilical point. Every tangent vector is a principal direction. $K = \kappa_1\kappa_2 = 4$, $H = (\kappa_1 + \kappa_2)/2 = 2$. Near the origin, S_1 is shaped like a bowl opening upwards.

For S_2 **at** $(0, 0, 0)$**:** $r = 2, s = 0, t = -2$. $II = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. This is already diagonal. The principal curvatures are $\kappa_1 = 2, \kappa_2 = -2$. The principal directions are the eigenvectors, which are $(1, 0)$ (the x-direction) and $(0, 1)$ (the y-direction). $K = \kappa_1\kappa_2 = -4$, $H = (\kappa_1 + \kappa_2)/2 = 0$. Near the origin, S_2 is shaped like a saddle.

Problem 7.

[Theorem of Beltrami-Enneper] Prove that the absolute value of the torsion

τ at a point of an asymptotic curve, whose curvature is nowhere zero, is given by

$$|\tau| = \sqrt{-K},$$

where K is the Gaussian curvature of the surface at the given point.

Solution.

Let $\alpha(s)$ be an asymptotic curve parametrized by arc length s. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be its Frenet-Serret frame. By definition, an asymptotic curve has zero normal curvature, $k_n = \langle \alpha''(s), N \rangle = 0$. Since $\alpha''(s) = k(s)\mathbf{n}(s)$ and $k \neq 0$, the principal normal $\mathbf{n}$ of the curve must be orthogonal to the surface normal N. This implies $\mathbf{n}$ lies in the tangent plane T_pS. Since $\mathbf{n}$ is also orthogonal to $\mathbf{t}$, the Darboux frame vector $\mathbf{g} = N \times \mathbf{t}$ must be $\pm\mathbf{n}$. Let's choose the orientation so that $\mathbf{g} = \mathbf{n}$. The binormal of the curve is $\mathbf{b} = \mathbf{t} \times \mathbf{n} = \mathbf{t} \times \mathbf{g} = \mathbf{t} \times (N \times \mathbf{t}) = (\mathbf{t} \cdot \mathbf{t})N - (\mathbf{t} \cdot N)\mathbf{t} = N$. The torsion is given by the Frenet-Serret formula $\mathbf{b}'(s) = -\tau(s)\mathbf{n}(s)$. Differentiating $\mathbf{b} = N$ along the curve, we get $\mathbf{b}'(s) = \frac{d}{ds}N(\alpha(s)) = dN_{\alpha(s)}(\alpha'(s)) = dN(\mathbf{t})$. So, $dN(\mathbf{t}) = -\tau\mathbf{n}$. The Gaussian curvature is $K = \det(dN)$. Let's compute this determinant in an orthonormal basis of T_pS. Let $\{\mathbf{t}, \mathbf{n}\}$ be an orthonormal basis for T_pS (since $\mathbf{n} \in T_pS$). $dN(\mathbf{t}) = -\tau\mathbf{n} = 0 \cdot \mathbf{t} - \tau \cdot \mathbf{n}$. Now we need to find $dN(\mathbf{n})$. From the property that dN is self-adjoint: $\langle dN(\mathbf{t}), \mathbf{n} \rangle = \langle \mathbf{t}, dN(\mathbf{n}) \rangle$. $\langle -\tau\mathbf{n}, \mathbf{n} \rangle = -\tau$. So $\langle \mathbf{t}, dN(\mathbf{n}) \rangle = -\tau$. Let $dN(\mathbf{n}) = c_1\mathbf{t} + c_2\mathbf{n}$. Then $\langle \mathbf{t}, c_1\mathbf{t} + c_2\mathbf{n} \rangle = c_1$. So $c_1 = -\tau$. The matrix of dN in the basis $\{\mathbf{t}, \mathbf{n}\}$ is

$$[dN] = \begin{pmatrix} 0 & -\tau \\ -\tau & c_2 \end{pmatrix}.$$

The Gaussian curvature is the determinant of this matrix:

$$K = \det([dN]) = (0)(c_2) - (-\tau)(-\tau) = -\tau^2.$$

Therefore, $\tau^2 = -K$. This requires $K \leq 0$, which is always true for a surface admitting asymptotic curves. Taking the square root, we get $|\tau| = \sqrt{-K}$.

Problem 8.

[Curvature of Intersection] If the surface S_1 intersects the surface S_2 along the regular curve C, then the curvature k of C at $p \in C$ is given by

$$k^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\theta,$$

where λ_1 and λ_2 are the normal curvatures at p, along the tangent line to C, of S_1 and S_2, respectively, and θ is the angle made up by the normal vectors of S_1 and S_2 at p.

Solution.

Let C be parametrized by arc length s. Let $p = C(s)$. Let $\mathbf{t}$ be the tangent vector, $\mathbf{n}$ the principal normal, and k the curvature of C at p. The acceleration vector is $C''(s) = k\mathbf{n}$. Let N_1 and N_2 be the unit normal vectors to S_1 and S_2 at p. The normal curvature of S_i in the direction $\mathbf{t}$ is $\lambda_i = \langle C''(s), N_i\rangle = k\langle \mathbf{n}, N_i\rangle$. The tangent vector $\mathbf{t}$ is orthogonal to both N_1 and N_2, so N_1 and N_2 lie in the normal plane of the curve C. The principal normal $\mathbf{n}$ also lies in the normal plane. Since the normal plane is two-dimensional, $\mathbf{n}$ must be a linear combination of N_1 and N_2 (assuming they are not collinear, i.e., $\sin\theta \neq 0$). So, we can write $\mathbf{n} = aN_1 + bN_2$ for some scalars a, b. From the definition of normal curvature: $\lambda_1 = k\langle aN_1 + bN_2, N_1\rangle = k(a\langle N_1, N_1\rangle + b\langle N_2, N_1\rangle) = k(a + b\cos\theta)$. $\lambda_2 = k\langle aN_1 + bN_2, N_2\rangle = k(a\langle N_1, N_2\rangle + b\langle N_2, N_2\rangle) = k(a\cos\theta + b)$. We have a linear system for a and b:

$$a + b\cos\theta = \lambda_1/k$$
$$a\cos\theta + b = \lambda_2/k$$

Solving this system (e.g., using Cramer's rule or substitution) gives:

$$a = \frac{\lambda_1 - \lambda_2\cos\theta}{k(1-\cos^2\theta)} = \frac{\lambda_1 - \lambda_2\cos\theta}{k\sin^2\theta}$$

$$b = \frac{\lambda_2 - \lambda_1\cos\theta}{k(1-\cos^2\theta)} = \frac{\lambda_2 - \lambda_1\cos\theta}{k\sin^2\theta}$$

Since $\mathbf{n}$ is a unit vector, $\langle \mathbf{n}, \mathbf{n}\rangle = 1$. $1 = \langle aN_1 + bN_2, aN_1 + bN_2\rangle = a^2\langle N_1, N_1\rangle + b^2\langle N_2, N_2\rangle + 2ab\langle N_1, N_2\rangle = a^2 + b^2 + 2ab\cos\theta$. Substitute the expressions for a and b: $k^2\sin^4\theta = (\lambda_1 - \lambda_2\cos\theta)^2 + (\lambda_2 - \lambda_1\cos\theta)^2 + 2(\lambda_1 - \lambda_2\cos\theta)(\lambda_2 - \lambda_1\cos\theta)\cos\theta$. Expanding the right hand side: RHS $= (\lambda_1^2 - 2\lambda_1\lambda_2\cos\theta + \lambda_2^2\cos^2\theta) + (\lambda_2^2 - 2\lambda_1\lambda_2\cos\theta + \lambda_1^2\cos^2\theta) + 2(\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)\cos\theta + \lambda_1\lambda_2\cos^2\theta)\cos\theta = (\lambda_1^2+\lambda_2^2)(1+\cos^2\theta) - 4\lambda_1\lambda_2\cos\theta + 2\lambda_1\lambda_2\cos\theta - 2(\lambda_1^2+\lambda_2^2)\cos^2\theta + 2\lambda_1\lambda_2\cos^3\theta = (\lambda_1^2+\lambda_2^2)(1-\cos^2\theta) - 2\lambda_1\lambda_2\cos\theta(1-\cos^2\theta)$ $= ((\lambda_1^2+\lambda_2^2) - 2\lambda_1\lambda_2\cos\theta)\sin^2\theta$. So, $k^2\sin^4\theta = (\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta)\sin^2\theta$. Dividing by $\sin^2\theta$ (which is non-zero), we get the desired result:

$$k^2\sin^2\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta.$$

Problem 9.

[Self-adjointness of the Shape Operator] Given a surface S parametrized by $X(u, v)$, the shape operator (or Weingarten map) $dN_p : T_pS \to T_pS$ at a point p can be viewed through its coefficients in the basis $\{X_u, X_v\}$ of the tangent plane T_pS. The property of the shape operator being **self-adjoint** with respect to the first fundamental form is equivalent to the following equality holding for any vectors $v_1, v_2 \in T_pS$:

$$\langle dN_p(v_1), v_2 \rangle = \langle v_1, dN_p(v_2) \rangle$$

In terms of the basis vectors, show that the condition $\langle dN_p(X_u), X_v \rangle = \langle X_u, dN_p(X_v) \rangle$ is equivalent to the following relation involving the partial derivatives of the unit normal vector N:

$$\langle N_u, X_v \rangle = \langle X_u, N_v \rangle$$

Solution.

First, we show that the condition holding for all vectors v_1, v_2 is equivalent to it holding for the basis vectors $\{X_u, X_v\}$. One direction is trivial: if it holds for all vectors, it must hold for the basis vectors. For the other direction, assume $\langle dN_p(X_i), X_j \rangle = \langle X_i, dN_p(X_j) \rangle$ for $i, j \in \{u, v\}$. Let $v_1 = aX_u + bX_v$ and $v_2 = cX_u + dX_v$. By linearity of dN_p and bilinearity of the inner product $\langle \cdot, \cdot \rangle$:

$$\begin{aligned}
\langle dN_p(v_1), v_2 \rangle &= \langle dN_p(aX_u + bX_v), cX_u + dX_v \rangle \\
&= \langle adN_p(X_u) + bdN_p(X_v), cX_u + dX_v \rangle \\
&= ac\langle dN_p(X_u), X_u \rangle + ad\langle dN_p(X_u), X_v \rangle + bc\langle dN_p(X_v), X_u \rangle + bd\langle dN_p(X_v), X_v \rangle
\end{aligned}$$

Using the assumption, this becomes:

$$\begin{aligned}
&= ac\langle X_u, dN_p(X_u) \rangle + ad\langle X_u, dN_p(X_v) \rangle + bc\langle X_v, dN_p(X_u) \rangle + bd\langle X_v, dN_p(X_v) \rangle \\
&= \langle aX_u + bX_v, cdN_p(X_u) + ddN_p(X_v) \rangle \\
&= \langle v_1, dN_p(v_2) \rangle
\end{aligned}$$

So the condition for all vectors is equivalent to the condition on the basis vectors.

Next, we show the equivalence with the relation involving partial derivatives of N. The differential map dN_p applied to a basis vector X_u is defined as the directional derivative of the vector field N in the direction of X_u. For a parametrized surface, this is simply the partial derivative with respect to the parameter u.

$$dN_p(X_u) = N_u \quad \text{and} \quad dN_p(X_v) = N_v$$

Substituting these into the condition for the basis vectors:

$$\langle dN_p(X_u), X_v\rangle = \langle X_u, dN_p(X_v)\rangle$$

becomes

$$\langle N_u, X_v\rangle = \langle X_u, N_v\rangle$$

This establishes the required equivalence.

Note for Problem 9.

This property, $\langle N_u, X_v\rangle = \langle X_u, N_v\rangle$, is always true for a C^2 parametrization. It follows from differentiating $\langle N, X_u\rangle = 0$ with respect to v and $\langle N, X_v\rangle = 0$ with respect to u, which gives $\langle N_v, X_u\rangle + \langle N, X_{uv}\rangle = 0$ and $\langle N_u, X_v\rangle + \langle N, X_{vu}\rangle = 0$. Since $X_{uv} = X_{vu}$ for a C^2 surface (Clairaut's theorem), the property holds. This confirms that the shape operator is indeed always self-adjoint.