MAT4033 Differential Geometry

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October 14, 2025

Contents

Afterward

Introduction

In this course we study curves and surfaces in $\mathbb{R}^{\nvDash}$

What are we interested in?

- How we decribe a curve?
  use the parametrization: $\alpha : I \to \mathbb{R}^3$
- How much information can we get from α?
  length, curvature, torsion
- If we know the curvature of a curve of evrey point, can we describe the curve?
- Some "global" problems: Suppose we have a closed curve in $\mathbb{R}^3$ of a given length, what is the largest possible area bounded by the curve?
- How do we describe a surface? More precisely, what kind of surface should we study?
- For a regular surface, how can we study the area and curvature of them?
- What is the "shortest" between two points of a surface?
- What is the relationship between geometry and topology on surfaces? (Gauss-Bonnet Theorem)

1 Local Theory of Curves

Definition 1.1 (regular curve)**.** $\alpha : [c,d] \to \mathbb{R}^3$ is called a regular curve if $|\alpha'| \neq 0$, (α is taken to be differentiable (smooth))

Proposition 1.1. *Let $\alpha : [c,d] \to \mathbb{R}^3$ be a given curve with partition $P = \{c = t_0 < t_1 < \cdots < t_n = b\}$. Let $l(\alpha, P) := \sum |\alpha(t_{i+1}) - \alpha(t_i)|$, then we can have*

$$\int_c^d |\frac{d\alpha}{dt}| dt = \sup\{l(\alpha, P) | P \textit{ any partition}\}$$

Proof. The proof is straight forward. □

Proposition 1.2 (length is invariant under reparametriazation)**.** *Given a differntiable curve $\alpha : [a,b] \to \mathbb{R}^3$ and a $g : [c,d] \to [a,b]$ with $\beta = \alpha \circ g$, then*

$$\int_a^b |\frac{d\alpha}{dt}| dt = \int_c^d |\frac{d\beta}{ds}| ds$$

Proof. also straight forward, by simple chain rule. □

There are many ways to parametrize the curves. But we always want to pick one such that the *pointers* is unit.

Remark 1.1. The cusp is not a regular curve since $\alpha'(0) = 0$, but we allow self-intersections, in which case a regular curve may have different tangent at the same point.

1.1 Reparametrization

Consider $\alpha_1(t) = (\cos t, \sin t), \alpha_2(t) = (\cos(2t), \sin(2t))$, both give the same trace with different parameters.

Definition 1.2. A reparametrizastion of a curve $\alpha : (a,b) \to \mathbb{R}^3$ is a bijective function $g : (c,d) \to (a,b)$ such that g is C^∞ and g^{-1} is C^1. Given a raparametrization g, one can define a new "curve" $\beta : (c,d) \to \mathbb{R}^3$ by $\beta = \alpha \circ g$

Example 1.1 (nonexample)**.** $g(s) = s^3$, we can check that g^{-1} is not continuously differentiable in any interval contain 0.

Proposition 1.3. *If α is a regular curve and g is a reparametrization, then $\beta(s) = \alpha(g(s))$ is also regular.*

Proof. left as an exercise. □

Remark 1.2. To check whether the given smooth g is a reparametrization or not, one only needs to check if g' is nonzero or not by Implicit Function Theorem(IVT).

Proposition 1.4. *The tangent vector of a regular curve is invariant (possibly reverse its direction) under any parametrization.*

Proof. left as an exercise. □

1.2 Arc-length

Definition 1.3. The length of the curve segment $[c, d] \subseteq I$ for $\alpha : I \to \mathbb{R}^3$ regular is iven by

$$\int_\alpha ds = \int_c^d |\alpha'(t)|dt$$

Proposition 1.5. *Reparametrization does not affect the arc-length*

Proof. left as an exercise ☐

To develop the theory of curves, we always want our curves to be parametrized by arc-length(p.a.l), or in other words, it is more convenient to work with curves with unit "pointers". The following discussion shows that we can always assumed the curve to be a p.a.l curve.

Lemma 1.6. *Let $\alpha : [a, b] \to \mathbb{R}^3$ regular and $t_0 \in [a, b]$ consider $h(t) = \int_{t_0}^t |\alpha'(t)|dt$. Then h is a reparametrizaton.*

Proof. left as an exercise ☐

Definition 1.4 (arc-length parametrization)**.** Let $\alpha : [a, b] \to \mathbb{R}^3$ be regular curve and $t_0 \in I$ with $g = h^{-1}$ with h is given in the lemma above. Then $\beta = \alpha \circ g : [c, d] \to \mathbb{R}^{\nvDash}$ is parametrization by arc-length

Proposition 1.7. *β the curve defined as above, then $\beta'(s)$ is of unit length.*

Remark 1.3. In theory, there is a reparametrization by arc length for a regualr curves. However, it can be hard to explicitly find such a parameter.

- The formula s=h(t) may not have a closed formula
- Even we have a closed formula for h(t) in some cases, the inverse of g is hard to find.

1.3 Frenet Frames of Plane curves

In this section, we assume α to be p.a.l, and alpha is a plane curve mapped from an interval to $\mathbb{R}^2$. Recall we have $\alpha'(s) = t(s)$

Definition 1.5. The normal vector $n(s)$ of a plane curve at t=s is the vector $n(s) := J(t(s))$, where J is the 90 degree counterclockwise rotation. The set {t(s),n(s)} forms an ordered, oriented basis of $\mathbb{R}^{\nvDash}$ which is orthonormal. It is called the Frenet Frame of α.

Remark 1.4. We will have a slightly different definition of normal vector for general space curve.

Note that $\langle t(s), t(s)\rangle = 1 \Rightarrow \langle t'(s), t(s)\rangle$, which implies $t'(s) = k(s)n(s)$ for some scalar $k(s)$

Definition 1.6. The curvature of α at s is the value k(s)

Remark 1.5.
- $k(s)$ records the change of t(s)
- $k(s) > 0$, $t(s)$ is turning towards n(s)
- $k(s) < 0$, $t(s)$ is turning away form n(s)

For space curves, it is not natural to define the ordered basis in this manner.

Example 1.2 (circle)**.** $\alpha(s) = u + r(\cos(\frac{s}{r}), \sin(\frac{s}{r}))$, do the computation we will see the curvature of $\alpha(s)$ is $\frac{1}{r}$

Note that if we raparametrize the curve with -t then the curvature will change its sign.

The following Proposition provides us a formula to compute the curvature when our curve is not parametrized by arc length initially. (Also, one can show that curvature is independent of choice (up to a sign) of parameter using the formula)

Proposition 1.8 (Formula to compute the curvature)**.** *Let $\alpha : I \to \mathbb{R}^3$ regular plane curve with $\beta = \alpha \circ g$ p.a.l, then the curvature of α (whcih is defined to be the curvature of β is*

$$k_\alpha(t) = \frac{det|\alpha'(t), \alpha''(t)|}{|\alpha'(t)|^3}$$

Proof. left as an exercise (Hint: by direct computation and chain rule, it takes about 20-30 minutes work) ☐

It seems that for a regular plane curve, the configuration of the curve is uniquely determined by its curvature. In fact we have the following theorem

Theorem 1.9 (Fundamental theorem of Plane curve)**.**
- *For any C^∞ function $k : I \to \mathbb{R}^3$, there exists a p.a.l curve $\alpha : I \to \mathbb{R}^3$ such that $k_\alpha(s) = k(s)$ and $\alpha(0) = x_0$*
- *The curve satisfy the $k_\alpha(s) = k(s)$ is unique up to a rigid motion.*

You may wonder what is a rigid motion, We have the following definition

Definition 1.7 (rigid motion)**.** We say a map $\mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion if it is of the form $x \to Ax + b$ with $A \in O(n)$. If A is in the special orthogonal group, we called the rigid motion orientation preserving.

In fact, one can show that the isometry in the Euclidean space is exactly all the rigid motion.

Proof. ☐

1.4 Space Curve

Definition 1.8 (curvature)**.**

Remark 1.6. n(s) is simply along the same direction of t' unlike the case of plane curves.

The Frenet-Serret Frames of a space curve is the orthonormal basis $\{t, n, b\}$, where b is the derivative of n.

As in the plane curve, we want to study the derivatives of $\{t, n, b\}$ where b is defined to be the cross product of t and n.

- $\langle n', n\rangle = 0 \Rightarrow n' = at + kb$
- $\langle b', b\rangle = 0 \Rightarrow b' = ct + dn$
- and then note that $b = t \times n \Rightarrow b' \perp b, b' \perp t$
- hence we have $b'(s) = \tau(s)n(s)$

Definition 1.9 (torsion)**.** Let α regular p.a.l curve, with $k(s) \neq 0$, the torsion of α is given by the above discussion.

Remark 1.7. $\tau(s)$ tells us how far the curve is away from a plane curve.

Proposition 1.10 (Frenet Serret Frame)**.** *Like the plane curve case, we have a formula to connect*

t, n, b and the derivative t', n', b'

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\kappa(s)$ is the curvature of the curve,$\tau(s)$ is the torsion of the curve.

Remark 1.8. Note that there is two conventions on the sign of the torsion τ

Definition 1.10. If α is any regular cuve, then the Frennet-Serret Frame is obtained by that of $\beta = \alpha \circ s($ (the arc-length reparamatrization)

Theorem 1.11. *Let α be regular curve p.a.l with $\kappa(s) \neq 0$ then the following are equivalent*

1. *α is a plane curve*
2. *b is a constant*
3. *$\tau(s) = 0$*

Proof.

- (2) $\Leftrightarrow$ (3) follows from the frenet serret equation.
- (1) $\Leftrightarrow$ (2) Suppose α be a plane curve, then b(s) is clearly a constant. Conversely, if b is a constant, then $b' = 0$, consider $\langle \alpha(s) - \alpha(0), b\rangle' = 0$ now since b is a constant, then α is indeed a plane curve. $\square$

We end by applying the Frenet Serret Equation.

Example 1.3. α is a straight line if and only if $\exists x_0 \in \mathbb{R}^3$, such that every tangent line to α passes through x_o

Proof. left as homework $\square$

Example 1.4. Let the normal plane of $\alpha(s)$ be the plane perpendicular to $t(s)$. Let α be a regular curve p..a.l such that every normal plane of $\alpha(s)$ passes through a fixed point of x_0. Then $\alpha(s)$ lie s on a sphere.

Proof. The normal plane of $\alpha(s)$ has the equation $\langle t(s), x - \alpha(s)\rangle = 0$, since the x_0 lies on the plane and hence we have $\langle t(s), x_0 - \alpha(s)\rangle = 0$ Consider taking the derivative of $\langle x_0 - \alpha(s), x_0 - \alpha(s)\rangle$, eh derivative is zero andhence this is a constatn, andthen we are done. $\square$

1.5 Fundamental Theorem of Curves

Recall that all plane curves are determined by its curvature up to orientation preserving its rigid motion. The same goes for regular space curves $\alpha(s)$ with $\kappa(s) > 0$

Theorem 1.12. *Given two functions $\kappa(s), \tau(s)$ defined on an interval, with $\kappa(s) > 0$, $x_0 \in \mathbb{R}^3$ fixed.$\{D, E, F\}$ be an orthonormal basis of $\mathbb{R}^3$, then there exists a p.a.l curve such that the curvature and torsion of α is exactly the given function $\kappa(s)$ $\tau(s)$. What's more, α is completely determined by its curvature and torsion, in the sense that any curve with the same curvature and torsion differs only be a rigid motion.*

Proof. For detailed proof, see Do carmo's appendix. Here we sketch the proof.

- Existence:

1. First construct the differential equation according to the Frenet Serret Formula, and get a candidate for the tangent vector, normal vector, and the binormal vector.
2. Now check that the t, n, b we get from the above equation is orthonormal for all s. This is equivalent to $\langle t, t\rangle, \langle t, n\rangle, \langle t, b\rangle, \langle n, n\rangle, \langle n, b\rangle, \langle b, b\rangle$ is equal to 1,0,0,1,0,1. To conclude this, we consider another differential equation and use the uniqueness of the ode.
3. Then we consider the curve $\alpha(s) = x_0 + \int t(s)ds$, and check it indeed have the two given function as curvature and torsion.

- Uniqueness: we move one of the curve α' by rigid motion to α with the same starting point and the same initial tangent vector, and then we are done by the uniqueness of ode.

$\square$

2 Global Theory of Curves

In this section we would like to study the theory of plane curves globally.

An important question in geometry is the following. Suppose we know the local property of a curve, can we say some thing about the global properties of it?

2.1 Hopf's Umlaufsatz

Definition 2.1. The rotation index of $\alpha : I \to \mathbb{R}^3$ with period is given by

$$\frac{\theta(a) - \theta(0)}{2\pi} = \frac{1}{2\pi} \int_0^a \theta'(s) ds$$

. where $\theta(s)$ is defined by $t(s) = (\cos(\theta(s)), \sin(\theta(s)))$.

Theorem 2.1 (Hopf's Umlaufsatz)**.** *Suppose α is a simple closed curve, then its rotation index must be equal to ± 1. Moreover, since we have $\theta'(s) = \kappa(s)$, we have $\int \kappa(s)ds = \pm 2\pi$.*

Proof. Let the curve $\alpha : [0, L] \to \mathbb{R}^2$ be a simple closed curve parameterized by arc length s. The unit tangent vector is $T(s) = \alpha'(s)$, and its angle with a fixed axis is $\theta(s)$. The total curvature is the net change in this angle, $\theta(L)-\theta(0)$. Since the curve is closed, $T(L) = T(0)$, which implies $\theta(L)-\theta(0) = 2\pi n$ for some integer n. We will show that $n = \pm 1$.

As suggested by the proof sketch, we introduce the unit secant vector field, which maps two points on the curve to the unit vector connecting them:

$$u(s_1, s_2) = \frac{\alpha(s_2) - \alpha(s_1)}{||\alpha(s_2) - \alpha(s_1)||}$$

This map is smooth provided $s_1 \neq s_2$, which is guaranteed for a simple curve. Let $\phi(s_1, s_2)$ be the angle of this secant vector.

This angle function is defined on the domain $[0, L] \times [0, L]$ excluding the diagonal. We consider the triangular region $D = \{(s_1, s_2) \mid 0 \leq s_1 \leq s_2 \leq L\}$. Since ϕ is a smooth, well-defined function on the interior of D, its exterior derivative $d\phi$ is an exact 1-form. By Green's theorem (or Stokes' theorem for forms), the integral of $d\phi$ around the boundary ∂D must be zero, provided we handle the singular diagonal edge properly.

$$\oint_{\partial D} d\phi = 0$$

The boundary ∂D consists of three segments. A careful limiting argument shows that the integral of $d\phi$ over the boundary is the sum of the changes in angle along the limiting paths.

1. **The diagonal path ($s_1 = s_2$):** As $s_2 \to s_1$, the secant vector $u(s_1, s_2)$ limits to the tangent vector $T(s_1)$. Thus, the angle $\phi(s, s)$ corresponds to the tangent angle $\theta(s)$. The change in angle along this path is the total curvature: $\Delta\phi_{diag} = \theta(L) - \theta(0)$.

2. **The 'start' path ($s_1 = 0$):** This path corresponds to the secant vector $u(0, s_2)$ from the fixed starting point $\alpha(0)$ to a moving point $\alpha(s_2)$. Let the total change in its angle be $\Delta\phi_{start} = \phi(0, L) - \phi(0, 0)$.

3. **The 'end' path ($s_2 = L$):** This path corresponds to the secant vector $u(s_1, L)$ from a moving point $\alpha(s_1)$ to the fixed end point $\alpha(L) = \alpha(0)$. This vector is the opposite of the 'start' vector: $u(s_1, L) = -u(0, s_1)$. Thus their angles are related by $\phi_{end}(s_1) = \phi_{start}(s_1) \pm \pi$. The total change in angle along this path is $\Delta\phi_{end} = \Delta\phi_{start}$.

The relationship between these changes in angle, derived from the boundary integral being zero, is:

$$\theta(L) - \theta(0) = \Delta\phi_{start} + \Delta\phi_{end}$$

Substituting $\Delta\phi_{end} = \Delta\phi_{start}$, we get:

$$\theta(L) - \theta(0) = 2\Delta\phi_{start}$$

Now we must evaluate $\Delta\phi_{start}$. This quantity represents the total rotation of the secant line anchored at $\alpha(0)$ as its other end traverses the entire curve. Since the curve α is simple and closed, it encloses a region. The secant vector starts in the direction of the tangent $T(0)$ and, as it sweeps through the interior of the curve, ends by pointing in the opposite direction, $-T(0)$. This corresponds to a net rotation of exactly half a circle. Therefore, the change in angle is:

$$\Delta\phi_{start} = \pm\pi$$

Substituting this result into our equation gives the final result:

$$\int_0^L \kappa(s)\, ds = \theta(L) - \theta(0) = 2(\pm\pi) = \pm 2\pi$$

This shows that the rotation index n must be either $+1$ or -1. □

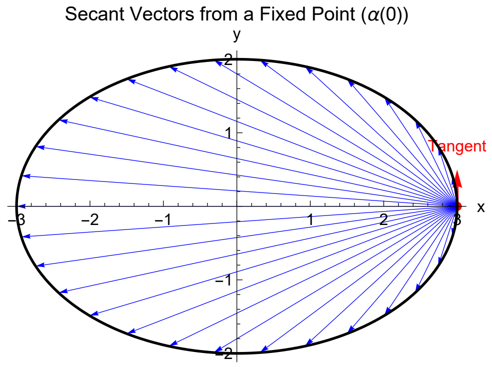

Figure 1: Illustration of secant vectors from a fixed point on a closed curve, visualizing the $\Delta\phi_{start}$ component in the proof of Hopf's Umlaufsatz.

Remark 2.1. The theorem can be viewed as a 1-dimensional version of Gauss-Bonnet Formula.

2.2 Isoperimetric Inequality

Theorem 2.2 (Isoperimetric Inequality)**.** *Let α be a simple closed plane curve. Let A be the area bounded by curves, then $L^2 \geq 4\pi A$*

Proof. The proof compares the area of the given curve $\mathcal{C}$ to that of a circle $\mathcal{S}^1$ by relating their parametrizations.

Step 1: The Setup. Let $\mathcal{C}$ be a simple closed curve of length L enclosing area A. We can parameterize $\mathcal{C}$ by arc length, $\gamma(t) = (x(t), y(t))$ for $t \in [0, L]$. A key property of this parametrization is that the tangent vector has unit length:

$$(x'(t))^2 + (y'(t))^2 = 1$$

We construct a circle $\mathcal{S}^1$ of an arbitrary radius r centered at the origin. We then define a new curve, $\beta(t) = (x(t), \tilde{y}(t))$, which traces this circle using the same x-coordinate as $\gamma(t)$. To ensure $\beta(t)$ remains on the circle, we must have $\tilde{y}(t) = \pm\sqrt{r^2 - x(t)^2}$.

Step 2: Combining the Area Formulas. Using Green's Theorem, we can express the areas as line integrals:

- The area of $\mathcal{C}$ is $A = \int_0^L x(t)y'(t)\, dt$.
- The area of the circle $\mathcal{S}^1$ is $\pi r^2 = -\int_0^L \tilde{y}(t)x'(t)\, dt$.

Adding these two expressions for area yields:

$$A + \pi r^2 = \int_0^L \left(x(t)y'(t) - \tilde{y}(t)x'(t)\right)\, dt$$

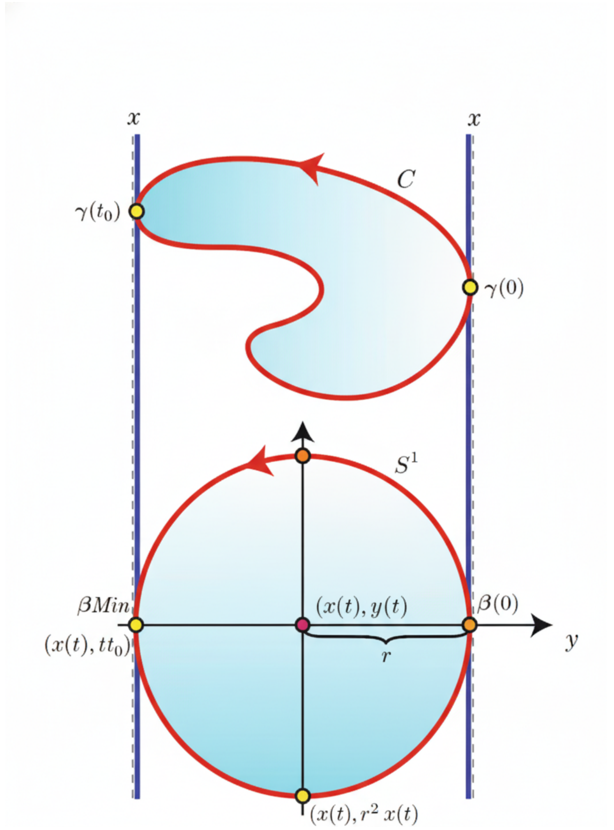

Figure 2: Geometric construction for the proof of the Isoperimetric Inequality. Curve C (red) and constructed circle S^1 (blue) sharing x-coordinates.

Step 3: Applying the Cauchy-Schwarz Inequality. The integrand can be interpreted as the dot product of two vectors, $\mathbf{v}_1(t) = (x(t), -\tilde{y}(t))$ and $\mathbf{v}_2(t) = (y'(t), x'(t))$. We apply the Cauchy-Schwarz inequality for integrals, which states $\int \mathbf{v}_1 \cdot \mathbf{v}_2 \, dt \leq \int ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \, dt$.

$$A + \pi r^2 \leq \int_0^L \sqrt{x(t)^2 + (-\tilde{y}(t))^2} \cdot \sqrt{y'(t)^2 + x'(t)^2} \, dt$$

The terms in the integral simplify significantly:

- Since $\beta(t)$ is on the circle, $\sqrt{x(t)^2 + \tilde{y}(t)^2} = \sqrt{r^2} = r$.
- Since $\gamma(t)$ is parameterized by arc length, $\sqrt{y'(t)^2 + x'(t)^2} = 1$.

Substituting these back, the inequality becomes:

$$A + \pi r^2 \leq \int_0^L r \cdot 1 \, dt = rL$$

Step 4: Finding the Optimal Bound. We have derived the relation $A \leq rL - \pi r^2$, which must hold for any choice of $r > 0$. To find the tightest possible bound for A, we find the value of r that maximizes the function $f(r) = rL - \pi r^2$. Taking the derivative with respect to r and setting it to zero gives:

$$f'(r) = L - 2\pi r = 0 \implies r = \frac{L}{2\pi}$$

This is the radius of a circle with perimeter L. Substituting this optimal radius back into our inequality yields:

$$\begin{aligned}
A &\leq \left(\frac{L}{2\pi}\right) L - \pi \left(\frac{L}{2\pi}\right)^2 \\
A &\leq \frac{L^2}{2\pi} - \pi \left(\frac{L^2}{4\pi^2}\right) \\
A &\leq \frac{L^2}{2\pi} - \frac{L^2}{4\pi} \\
A &\leq \frac{L^2}{4\pi}
\end{aligned}$$

Rearranging this gives the isoperimetric inequality: $4\pi A \leq L^2$.

The Equality Case. Equality holds if the Cauchy-Schwarz inequality becomes an equality, which occurs if and only if the vectors $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are parallel for all t. This means $\mathbf{v}_1(t) = k\,\mathbf{v}_2(t)$ for some scalar k. The magnitudes must also be equal, so $|k| \cdot ||\mathbf{v}_2|| = ||\mathbf{v}_1||$, which gives $|k| \cdot 1 = r$, so $k = \pm r$.

This implies that $x(t) = \pm r y'(t)$ and $-\tilde{y}(t) = \pm r x'(t)$. For the parametrizations to match, the original curve $\gamma(t)$ must itself be a circle of radius r. Since the optimal radius was $r = L/(2\pi)$, equality holds if and only if $\mathcal{C}$ is a circle. □

2.3 Four vertex Theorem (Covered in Tutorial)

We begin by defining the key concepts required for the theorem and its proof. Let $\gamma(t) = (x(t), y(t))$ be a regular plane curve.

Definition 2.2 (Signed Curvature)**.** For a regular plane curve $\gamma(t)$ parameterized by arc length s, its velocity vector is $\mathbf{v}(s) = \gamma'(s)$ and its acceleration is $\mathbf{a}(s) = \gamma''(s)$. The signed curvature, $k_s(s)$, is defined by the relation $\mathbf{a}(s) = k_s(s) R_{90}(\mathbf{v}(s))$, where R_{90} is a 90-degree counter-clockwise rotation. For a general parameter t, the curvature equations are:

$$x''(t) = -k_s(t) y'(t) \tag{1}$$

$$y''(t) = k_s(t) x'(t) \tag{2}$$

Definition 2.3 (Vertex)**.** A point $\gamma(t)$ on the trace of a regular plane curve is called a **vertex** if the signed curvature function k_s has a local maximum or a local minimum at t.

Definition 2.4 (Convex Curve)**.** A simple closed plane curve is called **convex** if its trace lies entirely on one closed side of each of its tangent lines.

Now we state the four vertex theorem.

Theorem 2.3 (The Four Vertex Theorem)**.** *Every simple closed plane curve has at least four vertices.*

Note: The following proof addresses the special case where the curve is convex.

The proof relies on the following geometric property of convex curves.

Lemma 2.4. *Let C be the trace of a simple closed convex plane curve, and let L be a line. If L intersects C in at least three points, then C must contain the entire line segment of L between any pair of these points.*

A direct consequence is that if a line is not part of the curve, it can intersect a convex curve at most twice.

Proof of Theorem 2.3 for Convex Curves. The proof is by contradiction.

Step 1: The Contradiction Hypothesis. Assume there exists a simple, closed, convex curve γ with fewer than four vertices. Since the continuous curvature function k_s on a closed curve must attain a global maximum and a global minimum, the curve must have at least two vertices. As local maxima and minima must alternate, the number of vertices must be even. Therefore, our assumption implies that γ has exactly two vertices:

- A point p where k_s attains its global maximum.
- A point q where k_s attains its global minimum.

Step 2: Geometric Setup. Let L be the line passing through the points p and q. By Lemma 2.4, since the curve is convex and not a line segment, L can intersect the curve only at points p and q.

We can apply a rigid motion (translation and rotation) to the curve to simplify our coordinate system without changing its geometric properties. We set up the coordinates such that:

- The point p is at the origin, $p = (0, 0)$.
- The line L is the x-axis.
- The point q is on the positive x-axis, $q = (a, 0)$ for some $a > 0$.

Let the curve be parameterized by $\gamma(t) = (x(t), y(t))$ for $t \in [0, l]$.

Step 3: Analysis of the Functions. From our setup, the entire curve lies on one side of the x-axis. We can assume without loss of generality that $y(t) \geq 0$ for all $t \in [0, l]$. Furthermore, for a positively oriented (counter-clockwise) simple closed convex curve, the signed curvature $k_s(t)$ is strictly positive.

Therefore, the product of these two functions, $y(t)k_s(t)$, must be non-negative for all t, and it is only zero at the points p and q.

Step 4: The Contradiction. We analyze the integral of the product $y(t)k_s(t)$ over the entire curve.

1. **The integral must be positive.** Since $y(t)k_s(t) \geq 0$ for all t and is not identically zero, its integral must be strictly positive:
$$\int_0^l y(t)k_s(t)\, dt > 0$$
2. **The integral must be zero.** However, as stated in the textbook's argument, integrating by parts and using the curvature equations from (1) and (2) shows that the integral is zero. A key part of such a derivation involves using the fact that for a closed curve, the integral of an exact derivative is zero. For example, using equation (2), $y''(t) = k_s(t)x'(t)$, we know:
$$\int_0^l k_s(t)x'(t)\, dt = \int_0^l y''(t)\, dt = [y'(t)]_0^l = 0$$
The textbook asserts that a similar line of reasoning (though not fully detailed) leads to the conclusion that our original integral is zero.

We have reached a contradiction: the integral cannot be both strictly positive and equal to zero. Thus, our initial assumption that the curve has fewer than four vertices must be false. This completes the proof. □

3 Local Theory of Surfaces

3.1 Introduction

Definition 3.1. A **parameterized surface** is a function $X : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ where Z is differentiable. We say Z is **regular** if the Jacobian matrix

$$dX = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank 2 for all points in U. In coordinates:

$$X(u,v) = (x(u,v), y(u,v), z(u,v))$$

Equivalently, $Z : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is an embedding. The image $Z(U)$ is called the **trace** of Z.

Remark 3.1. For regular curves $\alpha : I \to \mathbb{R}^3$, we require $\alpha' \neq 0$. For surfaces, the analogous condition is that the differential has full rank.

Example 3.1. Consider the parameterization:

$$X : (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$$

$$X(\theta, \varphi) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$$

This parameterization Z is regular because

$$\frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \varphi} \neq 0$$

and the Jacobian matrix has rank 2.

As in the case of curves. We wish to parametrize a surface to do Calculus on surfaces. Of course we need two variables rather than 1 variable in the curve case.

Remark 3.2 (Problem of the parametrized surface). The above example of S^2 reveals that in order to develop a unified theory for studying surfaces, it is necessary to use more than one coordinate patch. Now we would like to introduce the concept of regular surface.

Definition 3.2. A subset $S \subset \mathbb{R}^3$ is a **regular surface** if for every point $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$ of p (which can be taken as an open ball centered at p with radius $r > 0$) and a map $X : U \to V \cap S$ where $U \subset \mathbb{R}^2$ is open and connected, such that:

1. X is differentiable:

   $$X(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U$$

   has partial derivatives of all orders.

2. X is bijective with continuous inverse X^{-1}, i.e., X is a homeomorphism.

3. (Regularity) For $q \in U$ with $X(q) = p$, the Jacobian matrix

   $$dX_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

   has rank 2 for all $q \in U$.

Remark 3.3. Conditions (2) and (3) together guarantee that X is an embedding from U to S, ensuring the existence of a tangent plane at each point of S. This is analogous to the requirement $\alpha' \neq 0$ for regular curves.

Example 3.2. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere. We can cover S^2 using 6 coordinate patches. For example, let

$$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

and define the coordinate patch for the upper hemisphere:

$$X(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right), \quad (x, y) \in U$$

And then we can do the same for other coordinates.

Example 3.3. Suppose $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a smooth map. Then the graph of f defined as

$$S = \mathrm{graph}(f) = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in U, z = f(x, y)\}$$

is a regular surface.
<u>Homework</u>: Check the graph is indeed a regular surface.

The above example shows that graph is a regular surface, and indeed the converse is in some sense true. Every regular surface is Locally a graph.

Proposition 3.1. *Any regular surface S is locally a graph.*

Proof. Let $p \in S$ be any point and let $X : U \to V \cap S$ be the coordinate patch given in the definition of regular surfaces, with $X(q) = p$. By the regularity condition, the differential

$$dX_q = \begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \end{pmatrix}$$

has rank 2. This means that among the three possible 2×2 Jacobian matrices:

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

at least one has nonzero determinant.

Without loss of generality, assume the first Jacobian matrix has nonzero determinant. Then by the inverse function theorem, the projection

$$\pi \circ X : (u, v) \mapsto (x(u, v), y(u, v))$$

has a differentiable inverse in some neighborhood of q.

Let $g = (\pi \circ X)^{-1}$ be this inverse mapping, which sends $(x, y) \mapsto (u(x, y), v(x, y))$. Then we have

$$(x, y) \to (u, v) \to (x, y, z(x(u, v), y(u, v)))$$

By the formulas we see that indeed it is locally a graph. □

Remark 3.4. The above proof is a routine technique to apply the inverse function/implicit function theorem in geometry.

Example 3.4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth map and $a \in \mathbb{R}$ be such that for all $p \in f^{-1}(a)$, the differential $(df)_p \neq 0$ (i.e., a is a regular value of f). Then $f^{-1}(a) \subset \mathbb{R}^3$ is a regular surface.

Proof. For any $p \in f^{-1}(a)$, suppose without loss of generality that $\left(\frac{\partial f}{\partial z}\right)_p \neq 0$.
Define the function $F : \mathbb{R}^3 \to \mathbb{R}$ by:

$$F(x) = f(x) - a$$

Then $F(p) = 0$.

Since $\left(\frac{\partial F}{\partial z}\right)_p = \left(\frac{\partial f}{\partial z}\right)_p \neq 0$, by the implicit function theorem, there exists a smooth function $g : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that:

$$F(u, g(u)) = 0 \quad \forall u \in U$$

where U is an open neighborhood in $\mathbb{R}^2$.
This means we can express a neighborhood of $p \in f^{-1}(a)$ as:

$$W = \{(u, g(u)) \mid u \in U\}, \quad \text{with } p = (u_0, g(u_0))$$

Therefore, there exists a neighborhood V such that:

$$V \cap f^{-1}(a) = \{(x, y, z) \mid (x, y) \in U, z = g(x, y)\}$$

This shows that every point $p \in f^{-1}(a)$ is locally a graph, hence by the previous proposition about graphs of functions being regular surfaces, $f^{-1}(a)$ is a regular surface. □

Example 3.5. Consider the ellipsoid defined by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a, b, c > 0$ are constants.

Proof. Define the function $F : \mathbb{R}^3 \to \mathbb{R}$ by:

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Then:

1. F is obviously differentiable (in fact, smooth) on $\mathbb{R}^3$
2. The differential of F is given by:
$$dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$$
3. For any point $w = (w_1, w_2, w_3) \in F^{-1}(0)$ (i.e., on the ellipsoid), we have:
$$dF(w) = \left(\frac{2w_1}{a^2}, \frac{2w_2}{b^2}, \frac{2w_3}{c^2}\right)$$
Since w lies on the ellipsoid, at least one coordinate must be nonzero, hence $dF(w) \neq 0$. Therefore, 0 is a regular value of F.

By the regular value theorem, $F^{-1}(0)$ is a regular surface. □

Remark 3.5. For the sphere S^2,it is possible to cover the surface using only 2 coordinate patches. See Streographic Peojection.

Example 3.6 (Stereographic Projection)**.** See Homework.

3.2 Change of Coordinates

Recall that for curves, we may have different parametrizations. The same holds for regular surfaces. We say that two parametrizations describe the same surface even if they use different coordinate systems.

Proposition 3.2. *Let $p \in S$ be a point in the image of two parametrizations:*

$$X : U \to S \quad \text{and} \quad Y : V \to S$$

Let $W = X(U) \cap Y(V)$ be the overlapping region. Then the transition map

$$\varphi = X^{-1} \circ Y : Y^{-1}(W) \to X^{-1}(W)$$

defined by $(u, v) \mapsto (u', v')$ is a diffeomorphism.

Proof. Since both X and Y are regular parametrizations, they are homeomorphisms onto their images and their differentials have full rank. Therefore:

1. φ is well-defined as a map between open subsets of $\mathbb{R}^2$
2. φ is a homeomorphism because it is the composition of homeomorphisms
3. φ is differentiable because both X and Y are differentiable and regular
4. The inverse $\varphi^{-1} = Y^{-1} \circ X$ is similarly differentiable

The Jacobian matrix of the transition map is given by:

$$d\varphi = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{pmatrix}$$

This matrix is invertible everywhere because φ is a diffeomorphism. □

Remark 3.6. This property is fundamental in manifold theory. It ensures that geometric properties defined using different coordinate systems are consistent. The transition maps allow us to define a smooth structure on the surface S by specifying how different coordinate patches relate to each other. This is exactly the property we aim to generalize in the theory of smooth manifolds, where we use charts and transition maps to define the smooth structure.

3.3 Tangent plane

Definition 3.3. Let $\mathbf{x} : U \to$ VAS be a coordinate patch of a regular surface at a pt $\mathbf{x}(p) = p \in S$. Then the tangent plane at $p \in S$ is defined as

$$T_pS = \operatorname{Im}\left((\mathrm{d}\mathbf{x})_p\right) = \operatorname{span}\left\{\frac{\partial \mathbf{x}}{\partial u}(p), \frac{\partial \mathbf{x}}{\partial v}(p)\right\}.$$

Definition 3.4. The normal vector of a pt p on a coordinate patch $\mathbf{x} : U \to$ VAS is defined as

$$N(p) = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left\|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right\|} \quad (p)$$

<u>Problem:</u> although the tangent plane is well-defined, but $N(p)$ may be affected by our choice of coordinates. (the sign).

Definition 3.5. A tangent vector at $p \in S$ is the tangent vector $\alpha'(0)$ of a regular curve $\alpha : (-\varepsilon, \varepsilon) \to S \subseteq \mathbb{R}^3$ with $\alpha(0) = p$.

Proposition 3.3. *The set of tangent vectors at $p \in S$ is precisely equal to T_pS.*

Proof. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be regular with $\alpha(0) = p$. Recall that $\exists\ V$ of $p \in S$ such that by restricting VAS to $V \cap S$, we have $\mathbf{y} : U' \to V \cap S$ such that $\mathbf{y}(u', v') = (u', v', f(u', v'))$ or $(u', g(u', v'), v')$ (locally a graph).

we look at the first case. w.l.o.g. by making ε small enough. assume $\alpha(-\varepsilon, \varepsilon) \subseteq V \cap S$.

Let β be given by $\beta : (-\varepsilon, \varepsilon) \to U' \subseteq \mathbb{R}^2$ where $\mathbf{y}^{-1}$ is differentiable. ($\mathbf{y}^{-1}$ is just the projection map) Then $\alpha = \mathbf{y} \circ \beta = \mathbf{x} \circ (\mathbf{x}^{-1} \circ \mathbf{y} \circ \beta)$

we've seen in prop that $\mathbf{x}^{-1} \circ \mathbf{y}$ is differentiable so $\alpha'(0) = (\mathrm{d}\mathbf{x})_q \cdot (\mathrm{d}(\mathbf{x}^{-1} \circ \mathbf{y} \circ \beta))(0)$

$$= \left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\right)_q \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow \alpha'(0) \in T_pS$. We've just seen that $\alpha'(0) \in T_pS$.

Now we must to see $\mathbf{w} \in T_pS \Rightarrow \mathbf{w} = \alpha'(0)$ for some α. Suppose $\mathbf{w} = \left(\frac{\partial \mathbf{x}}{\partial u}(p), \frac{\partial \mathbf{x}}{\partial v}(p)\right) \begin{pmatrix} a \\ b \end{pmatrix} \in T_pS$. Then let $\beta : (-\varepsilon, \varepsilon) \to U$ by $\beta(t) = p + t \begin{pmatrix} a \\ b \end{pmatrix}$. and $\alpha = \mathbf{x} \circ \beta$. Then $\alpha(0) = \mathbf{x}(\beta(0)) = \mathbf{x}(p) = p$. $\alpha'(0) = (\mathrm{d}\mathbf{x})_p \cdot \mathrm{d}\beta_0 = \left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\right) \begin{pmatrix} a \\ b \end{pmatrix} \in T_pS$. □

Example 3.7. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ smooth. $S = f^{-1}(a)$. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$. Then

$$f(\alpha(t)) = f(x(t), y(t), z(t)) = a.$$

Differentiating with respect to t:

$$\Rightarrow (\mathrm{d}f)_p \cdot (\mathrm{d}\alpha)_0 = 0.$$

$$\Rightarrow (\mathrm{d}f)_p \cdot \alpha'(0) = 0.$$

$\Rightarrow (\mathrm{d}f)_p$ is always normal to T_pS.

Question: we define $N(p)$ on a coordinate patch. can we choose coordinates patch appropriately so that $N : S \to \mathbb{R}^3$ is a smooth function?

Remark 3.7. When we move from local to global, it is always determined by the "topology" property.

3.4 First Fundamental Form

After defining the tangent plane, we wil study the inner product structure so that we can compute the area of the surface.

Definition 3.6. Let V be a vector space. We study $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ an inner product if it satisfies the following:

1. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ (symmetric).
2. $\langle a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w} \rangle = a\langle \mathbf{v}_1, \mathbf{w} \rangle + b\langle \mathbf{v}_2, \mathbf{w} \rangle$ (linearity).
3. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0 \quad \forall \mathbf{v} \in V$. equality holds if and only if $\mathbf{v} = \mathbf{0}$.

Definition 3.7. The quadratic form corresponding to an inner product is a map $q : V \to \mathbb{R}$ defined by

$$\mathbf{v} \to \langle \mathbf{v}, \mathbf{v} \rangle.$$

Definition 3.8. The **First Fundamental Form** of S is given by the quadratic form $I_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle$ for $\mathbf{v} \in T_pS$.

In order to understand I_p better, we need to choose a basis of T_pS and express I_p using matrices and vectors.

Let $\mathbf{x} : U \to$ VAS be a coordinate patch with $\mathbf{x}(p) = p$. Then $\mathcal{B} = \left\{\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}(p), \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}(p)\right\}$

$$\begin{aligned} \text{with } E_p &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p \\ F_p &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p \\ G_p &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p \end{aligned}$$

(We drop the subscript p if it does not cause confusion)

Then for all $\mathbf{v}$, $I_p(\mathbf{v}) = (a, b)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$.

More precisely, $\langle \mathbf{v}, \mathbf{w} \rangle_p = (a, b)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$, where $\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{w} = c\mathbf{x}_u + d\mathbf{x}_v$.

Therefore, in a coordinate patch IFF refers to $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

Example 3.8. 1. **Cylinder:**

$$\mathbf{x}(u, v) = (\cos u, \sin u, v)$$

$$\mathbf{x}_u = (-\sin u, \cos u, 0)$$

$$\mathbf{x}_v = (0, 0, 1)$$

$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1. \quad F = 0. \quad G = 1.$

$$\Rightarrow I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. **Plane:**

$$\mathbf{x}(u, v) = \mathbf{x}_0 + u\mathbf{w}_1 + v\mathbf{w}_2$$

(Assuming $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthonormal, $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_{ij}$)

$$I_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.

$$\mathbf{x}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

$$\mathbf{x}_u = \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right), \quad \mathbf{x}_v = \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\right)$$

$$I_p = \begin{pmatrix} \frac{1-v^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & \frac{1-u^2}{1-u^2-v^2} \end{pmatrix}$$

(Note: This is the IFF for the upper hemisphere in Cartesian coordinates.) $\tilde{\mathbf{x}}_\varphi = (-\sin\phi \sin\varphi, \sin\phi\cos\varphi, 0)$.

$$I_p = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix} \quad \text{different expression of } I_p \text{ for the same surface } S$$

Question: Suppose we have 2 parameterizations

$$\mathbf{x} : U \to V \cap S, \quad \tilde{\mathbf{x}} : \tilde{U} \to V \cap S.$$

Note that

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}, \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\right)^T \begin{pmatrix} \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \\ \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \end{pmatrix} = (\mathrm{d}\tilde{\mathbf{x}})^T \mathrm{d}\tilde{\mathbf{x}}.$$

Suppose $\Psi = \mathbf{x}^{-1} \circ \tilde{\mathbf{x}}$ is the transition function

$$\Psi(\tilde{u}, \tilde{v}) = (u(\tilde{u}, \tilde{v}), v(\tilde{u}, \tilde{v})).$$

$$\Rightarrow \tilde{\mathbf{x}} = \mathbf{x} \circ \Psi. \quad \text{so}$$

$$\mathrm{d}\tilde{\mathbf{x}} = \mathrm{d}\mathbf{x}\mathrm{d}\Psi, \quad \text{and hence}$$

$$\begin{aligned} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} &= (\mathrm{d}\mathbf{x}\mathrm{d}\Psi)^T (\mathrm{d}\mathbf{x}\mathrm{d}\Psi) \\ &= (\mathrm{d}\Psi)^T (\mathrm{d}\mathbf{x})^T (\mathrm{d}\mathbf{x})(\mathrm{d}\Psi) \\ &= (\mathrm{d}\Psi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} (\mathrm{d}\Psi) \end{aligned}$$

composition by **Jacobian matrix**.

Example 3.9. Back to the sphere example.

$$\Psi(\phi, \varphi) = (\sin\phi\cos\varphi, \sin\phi\sin\varphi)$$

$\mathrm{d}\Psi = \begin{pmatrix} \cos\phi\cos\varphi & -\sin\phi\sin\varphi \\ \cos\phi\sin\varphi & \sin\phi\cos\varphi \end{pmatrix}$ and

$$(\mathrm{d}\Psi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \mathrm{d}\Psi = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \sin^2\phi \end{pmatrix}$$

Corollary 3.4.

$$\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = \sqrt{EG - F^2} \;\det\left(\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}\right)$$

Remark 3.8. We are always working in the orientable cases.