Seminar 6: Curvature

Cola Ke

The Chinese University of Hong Kong (Shenzhen)

November 9, 2025

Outlines

Notation for Surfaces

- **Local Chart:**
  - $X : U \to V \cap S \subseteq S$
  - $(u, v) \mapsto X(u, v)$
- **Tangent Space:**
  - $p \in S$. $T_pM := \text{span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$
  - We can identify $T_pM = \text{span}\{X_u, X_v\}$, which is a plane in $\mathbb{R}^3$.

First Fundamental Form

Definition (First Fundamental Form)

$I_p : T_pM \times T_pM \to \mathbb{R}$

$$(v, w) \mapsto \langle v, w \rangle_{\mathbb{R}^3}$$

In local coordinates, this is given by the matrix:

$$I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

where:

- $E = \langle X_u, X_u \rangle$
- $F = \langle X_u, X_v \rangle$
- $G = \langle X_v, X_v \rangle$

Gauss Map & Shape Operator

Definition (Gauss Map)

Given $S \subseteq \mathbb{R}^3$ a regular surface with a smooth normal vector $N_p \perp T_pM$ for all $p \in S$.

- The map $N : S \to S^2$ is called the Gauss map.
- Rmk: Not every S has such a N.
- In a coordinate chart: $N = \frac{X_u \times X_v}{||X_u \times X_v||}$
- We use the Gauss map to detect curvature.

Definition (Shape Operator)

Given $N : S \to S^2$ at $p \in S$:

$$(-dN_p) : T_pS \to T_{N(p)}S^2 = T_pS$$

We call $-dN_p$ the **Shape Operator**.

Second Fundamental Form

Definition (Second Fundamental Form)

$\Pi_p : T_pM \times T_pM \to \mathbb{R}$

$$\Pi_p(v, w) = \langle -dN_p(v), w \rangle$$

This is a symmetric, bilinear form.

- $-dN_p$ is self-adjoint, so it has real eigenvalues λ_1, λ_2 and orthonormal eigenvectors e_1, e_2 (the principal directions).
- **Gaussian Curvature:** $K := \det(-dN_p) = \lambda_1 \lambda_2$
- **Mean Curvature:** $H := \frac{1}{2}\mathrm{tr}(-dN_p) = \frac{\lambda_1 + \lambda_2}{2}$

Second Fundamental Forms

Computation in Local Charts

$I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, $\Pi_p = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

- $e = \langle -N_u, X_u \rangle = \langle N, X_{uu} \rangle$
- $f = \langle -N_v, X_u \rangle = \langle N, X_{uv} \rangle$
- $g = \langle -N_v, X_v \rangle = \langle N, X_{vv} \rangle$

Claim: $-dN_p = I_p^{-1} \Pi_p$

$$K = \det(-dN_p) = \frac{\det \Pi_p}{\det I_p} = \frac{eg - f^2}{EG - F^2}$$

∇_X Generalised to Tensors

We extend $\nabla_X : \Gamma(TM) \to \Gamma(TM)$ to act on any (r,s)-tensor field by generalizing the Leibniz rule.

$$\nabla_X[T(\alpha^1, \dots, \alpha^r, Y_1, \dots, Y_s)] =$$

- $(\nabla_X T)(\alpha^1, \dots, \alpha^r, Y_1, \dots, Y_s)$
- $+T(\nabla_X \alpha^1, \dots, \alpha^r, Y_1, \dots, Y_s)$
- $+T(\alpha^1, \nabla_X \alpha^2, \dots, \alpha^r, Y_1, \dots, Y_s) + \dots$
- $+T(\alpha^1, \dots, \alpha^r, \nabla_X Y_1, \dots, Y_s) + \dots$

A specific Example is to define $\nabla_X \omega$ (for ω a one-form) to fit the Leibniz rule:

$$X(\omega(Y)) = \nabla_X[\omega(Y)] = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$$

Covariant Derivative in Local Charts

Given Θ a (2,1) tensor:

$$\Theta = \Theta^{\mu\nu}_{\rho} \frac{\partial}{\partial x^{\mu}} \otimes \frac{\partial}{\partial x^{\nu}} \otimes dx^{\rho}$$

Then $\nabla\Theta$ is a (2,2) tensor. To compute the component $(\nabla_k \Theta)^{ij}_{m}$:

$$\begin{aligned}
(\nabla_k \Theta)^{ij}_{m} :=& (\nabla_{\frac{\partial}{\partial x^k}} \Theta)(dx^i, dx^j, \frac{\partial}{\partial x^m}) \\
=& \frac{\partial}{\partial x^k} \Theta(dx^i, dx^j, \frac{\partial}{\partial x^m}) \\
& - \Theta(\nabla_{\frac{\partial}{\partial x^k}} dx^i, dx^j, \frac{\partial}{\partial x^m}) \\
& - \Theta(dx^i, \nabla_{\frac{\partial}{\partial x^k}} dx^j, \frac{\partial}{\partial x^m}) \\
& - \Theta(dx^i, dx^j, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^m})
\end{aligned}$$

Covariant Derivatives in Local Charts

Using $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^m} = \Gamma^{\sigma}_{km} \frac{\partial}{\partial x^\sigma}$ and $(\nabla_{\frac{\partial}{\partial x^k}} dx^\beta) = -\Gamma^{\beta}_{\gamma k} dx^\gamma$:

$$(\nabla_k \Theta)^{ij}_m = \partial_k \Theta^{ij}_m + \Gamma^i_{k\sigma} \Theta^{\sigma j}_m + \Gamma^j_{k\sigma} \Theta^{i\sigma}_m - \Gamma^{\sigma}_{km} \Theta^{ij}_\sigma$$

Riemann Curvature Tensor: Motivation

- **Motivation:** To detect the curvature on a manifold.
- Parallel transporting a vector along different paths may result in different vectors.
- It is natural to detect curvature by studying the non-commutativity of the covariant derivative:

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X$$

Definition (Riemann Curvature tensor)

$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Why R is a Tensor

- **Question:** Why the extra term $-\nabla_{[X,Y]}Z$?
- **Answer:** We add this term to make R a tensor (i.e., $C^\infty(M)$-linear in all arguments).
- Consider the non-tensorial part:

$$\nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z$$

$$\begin{aligned} &= f(\nabla_X\nabla_Y Z) - \nabla_Y(f\nabla_X Z) \\ &= f(\nabla_X\nabla_Y Z) - (Yf)(\nabla_X Z) - f(\nabla_Y\nabla_X Z) \\ &= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - (Yf)(\nabla_X Z) \end{aligned}$$

- This has an extra non-linear term $-(Yf)(\nabla_X Z)$.
- The term $-\nabla_{[fX,Y]}Z = -\nabla_{f[X,Y]-(Yf)X}Z = -f\nabla_{[X,Y]}Z + (Yf)\nabla_X Z$
- This second part $(Yf)\nabla_X Z$ precisely cancels the non-tensorial term, making $R(fX,Y)Z = fR(X,Y)Z$.

Curvature Tensor in Local Charts

- As R is a (1,3) tensor , we can compute its components.
- Since $[\partial_k, \partial_j] = 0$, the definition simplifies:

$$R(\partial_k, \partial_j)\partial_l = \nabla_k \nabla_j \partial_l - \nabla_j \nabla_k \partial_l$$

- Writing $R(\partial_k, \partial_j)\partial_l = R^i{}_{lkj}\partial_i$, we can expand to find:

$$R^i{}_{lkj} = \partial_k \Gamma^i_{lj} - \partial_j \Gamma^i_{lk} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{jm}\Gamma^m_{lk}$$

The (0,4) Curvature Tensor

We can "lower the index" to create a (0,4) tensor:

$$R(X, Y, Z, T) := \langle R(X, Y)Z, T \rangle$$

In local coordinates:

$$R_{ilkj} = g_{mi} R^m{}_{lkj}$$

Symmetries of R and Bianchi Identity

The (0,4) tensor R_{ijks} has the following symmetries:

- $R_{ijks} = -R_{jiks}$
- $R_{ijks} = -R_{ijsk}$
- $R_{ijks} = R_{ksij}$

Theorem (First Bianchi Identity)

$$R(X, Y, Z, T) + R(Y, Z, X, T) + R(Z, X, Y, T) = 0$$

In local charts, this is the cyclic sum:

$$R_{ijks} + R_{ikjs} + R_{kijs} = 0$$

Sectional Curvature

Definition (Sectional Curvature)

- Let $p \in M$ and σ be a 2-dim plane in T_pM.
- Let $\{X, Y\}$ be a basis for σ.
- The sectional curvature $K(\sigma)$ is:

$$K(\sigma) := K(X,Y) := \frac{\langle R(X,Y)Y, X\rangle}{|X \wedge Y|^2} = \frac{R(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

Lemma (13.3)

The sectional curvature $K(\sigma)$ for all planes σ completely determines the Riemann Curvature tensor R.

Ricci and Scalar Curvature

Definition (Ricci Curvature)

The Ricci tensor is a (0,2) tensor defined as the trace of the Riemann tensor. Let $\{e_i\}$ be an orthonormal basis.

$$Ric(X,Y) = \sum_{i=1}^{n} R(e_i, X, e_i, Y)$$

In local charts:

$$R_{ij} = g^{kl} R_{kilj}$$

The Ricci curvature in the direction X (unit vector) is:

$$Ric(X,X) = \sum_{i=1}^{n} R(e_i, X, e_i, X)$$

Ricci and Scalar Curvature

Definition (Scalar Curvature)

The scalar curvature S is the trace of the Ricci tensor:

$$S = \mathrm{tr}_g(Ric)$$

In local charts:

$$S = g^{ij} R_{ij}$$

Historical Remarks

- For a surface $S \subseteq \mathbb{R}^3$, there is only one sectional curvature, K.
- In this 2D case: $Ric = Kg$ and $S = 2K$.
- This K is exactly the Gaussian curvature $K = \frac{eg-f^2}{EG-F^2}$ we defined earlier.
- **Theorema Egregium:** Gauss also showed (assuming $F = 0$):

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right)$$

- This shows K is determined *only* by the metric (E, F, G).
- This is the origin of **Intrinsic Geometry**.

General Relativity

- In Einstein's General Relativity, spacetime is a 4-manifold.
- Particles move along "straight" lines (geodesics).
- **"The curvature of space-time tells matters how to move."**
- **"Matter tells space how to curve."**
- This is described by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}\Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- The left side is geometry (curvature), and the right side is matter (stress-energy tensor).
- Gravity is now described by the curvature of the space instead of mysterious distant action.

The Classical Limit

The goal is to show that Einstein's theory of gravity contains Newton's theory. We do this by applying two "classical" approximations:

1. Weak Gravitational Field

The gravitational field is weak, so spacetime is "almost flat". We can write the metric $g_{\mu\nu}$ as the flat Minkowski metric $\eta_{\mu\nu}$ plus a small perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

2. Low Velocity (Non-Relativistic Matter)

Particles are moving much slower than the speed of light ($v \ll c$).

- This implies that pressure and momentum are negligible compared to mass-energy.
- The Stress-Energy Tensor $T_{\mu\nu}$ is dominated by its first component, the mass density ρ:

$$T_{00} \approx \rho c^2 \quad \text{and all other } T_{\mu\nu} \approx 0$$

The Two Equations

We need to show that the EFE simplifies into Poisson's equation under our approximations.
Poisson Equation can be viewed as the equivalent description of the Newtonian Gravity Law

Einstein's Field Equation

This describes how matter-energy curves spacetime. (Using S for the scalar curvature as in your notes).

$$R_{\mu\nu} - \frac{1}{2} S g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Newton's (Poisson's) Equation

This describes the Newtonian potential Φ generated by a mass density ρ.

$$\nabla^2 \Phi = 4\pi G \rho$$

Connecting Potential and Geometry

How do we get from the metric $g_{\mu\nu}$ to the potential Φ?

- We look at the **geodesic equation** (how particles move) in the same low-velocity limit.
- The geodesic equation $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = 0$ simplifies to:

$$\frac{d^2\mathbf{x}}{dt^2} \approx -\nabla\Phi \quad \text{(Newton's Law)}$$

- This comparison reveals a direct link between the Newtonian potential Φ and the g_{00} component of the metric:

The Key Link

$$g_{00} \approx -\left(1 + \frac{2\Phi}{c^2}\right) \quad \text{or} \quad h_{00} \approx -\frac{2\Phi}{c^2}$$

The Newtonian potential is just the small perturbation in the time-time component of the metric.

Recovering Newtonian Gravity

Now we apply our approximations to the 00-component of the EFE:

EFE 00-Component

$$R_{00} \approx \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}Tg_{00}\right)$$

(Using the trace-reversed form, where T is the trace of $T_{\mu\nu}$)

Left Side (Geometry):

- R_{00} simplifies in a static, weak field.
- Using $g_{00} \approx -1 - 2\Phi/c^2$:
- $R_{00} \approx -\frac{1}{2}\nabla^2 g_{00}$
- $R_{00} \approx \frac{1}{c^2}\nabla^2\Phi$

Right Side (Matter):

- $T_{00} \approx \rho c^2$
- $T \approx T^0_0 \approx -\rho c^2$
- $\approx \frac{8\pi G}{c^4}\left(\rho c^2 - \frac{1}{2}(-\rho c^2)(-1)\right)$
- $\approx \frac{4\pi G\rho}{c^2}$

The Result

Equating the two sides:

$$\frac{1}{c^2}\nabla^2\Phi \approx \frac{4\pi G\rho}{c^2}$$

Canceling c^2, we recover Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho$$