Homework

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Problem 1.

Let $P = \{(x, y, z) \in \mathbb{R}^3 | x = y\}$ and $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$X(u, v) = (u + v, u + v, u + v, uv)$$

where $U = \{(u, v) \in \mathbb{R}^2 | u > v\}$. Is X a coordinate patch of P? Does it cover the whole P?

Problem 2.

Consider a one-to-one, regular curve $\alpha(t) = (r(t), z(t))$, $t \in I$ and $r(t) > 0$. If we rotate the curve α about the z-axis, we obtain the surface of revolution S.

(a) Let $X : U \to S$ be given by

$$X(\theta, t) = (r(t) \cos\theta, r(t) \sin\theta, z(t)),$$

where $U = (-\pi, \pi) \times I$. Show that X is a coordinate patch, but it does NOT cover the whole surface S.

(b) By parameterizing S using two coordinate patches, show that S is a regular surface.

(c) Hence or otherwise, show that the torus is a regular surface. Write down a coordinate patch of the torus.

Problem 3.

Let $a = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a unit vector, $r > 0$ and

$$S := \left\{ x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \langle x, a \rangle^2 + r^2 = |x|^2 \right\}$$

(a) Show that S is a cylinder with radius r, and the axis of revolution is along the direction of a. (Hint: use Pythagoras Theorem)

(b) Let $F(x) = \langle x, a \rangle^2 + r^2 - \langle x, x \rangle$, show that

$$dF(x) = 2\langle x, a \rangle (a, b, c) - 2(x, y, z)$$

(c) Using (b), show that S is a regular surface.

(d) Let w be a vector satisfying $\langle w, a \rangle = 0$ and $\langle w, w \rangle = r^{-2}$. Show that the line $\alpha(t) := w + ta$ lies on the surface S.

(e) For any unit vector v, show that the line $\gamma(t) := w + tv$ lies on the surface S if and only if $v = \pm a$.

Problem 4.

Show that the two-sheeted cone $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ is not a surface.

Problem 5.

Let $\alpha : (-3, 0) \longrightarrow \mathbb{R}^2$ be defined by

$$\alpha(t) = \begin{cases} (0, -(t+2)) & t \in (-3, -1) \\ \text{a regular curve joining } p = (0, -1) \text{ to } q = (1/\pi, 0) & t \in [-1, -1/\pi] \\ (-t, -\sin\frac{1}{t}) & t \in (-1/\pi, 0) \end{cases}$$

It is possible to define the curve joining p to q so that all the derivatives of α are continuous at the corresponding points and α has no self-intersections. Let C be the trace of α.

(a) Is C a regular curve?

(b) Let a normal line to the plane $\mathbb{R}^2$ run through C so that it describes a "cylinder" S. Is S a regular surface?

Problem 6.

Let w be a tangent vector to a regular surface S at a point $p \in S$ and let $x(u, v)$ and $\bar{x}(\bar{u}, \bar{v})$ be two parametrizations at p. Suppose that the expressions of w in the bases associated to $x(u, v)$ and $\bar{x}(\bar{u}, \bar{v})$ are

$$w = \alpha_1 x_u + \alpha_2 x_v$$

and

$$w = \beta_1 \bar{x}_{\bar{u}} + \beta_2 \bar{x}_{\bar{v}}.$$

Show that the coordinates of w are related by

$$\beta_1 = \alpha_1 \frac{\partial \bar{u}}{\partial u} + \alpha_2 \frac{\partial \bar{u}}{\partial v}$$
$$\beta_2 = \alpha_1 \frac{\partial \bar{v}}{\partial u} + \alpha_2 \frac{\partial \bar{v}}{\partial v},$$

where $\bar{u} = \bar{u}(u,v)$ and $\bar{v} = \bar{v}(u,v)$ are the expressions of the change of coordinates.

Problem 7.

Recall the isoperemetric inequality, show that the equality holds if and only if the our curve is a circle.