Mat4033 Supplement Problems for Second Fundamental Forms

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Problem 1.

Calculate the first and second fundamental forms of the pseudosphere (see Example 8) and check our computations of the principal curvatures and Gaussian curvature.

Solution.

We use the standard parametrization for the pseudosphere, which is the surface of revolution of a tractrix:

$$\mathbf{x}(u,v) = (u - \tanh u, \text{sech } u \cos v, \text{sech } u \sin v)$$

where $u > 0$ and $v \in [0, 2\pi)$.

Step 1: First Fundamental Form (I)

First, we compute the partial derivatives:

$$\begin{aligned}
\mathbf{x}_u &= \left(1 - \text{sech}^2 u, -\text{sech } u \tanh u \cos v, -\text{sech } u \tanh u \sin v\right) \\
&= \left(\tanh^2 u, -\text{sech } u \tanh u \cos v, -\text{sech } u \tanh u \sin v\right) \\
\mathbf{x}_v &= (0, -\text{sech } u \sin v, \text{sech } u \cos v)
\end{aligned}$$

Next, we compute the coefficients E, F, G:

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = (\tanh^4 u) + (\text{sech}^2 u \tanh^2 u \cos^2 v) + (\text{sech}^2 u \tanh^2 u \sin^2 v)$$

$$= \tanh^4 u + \text{sech}^2 u \tanh^2 u = \tanh^2 u(\tanh^2 u + \text{sech}^2 u) = \tanh^{\mathbf{2}} \mathbf{u}$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = 0 + (\text{sech}^2 u \tanh u \cos v \sin v) - (\text{sech}^2 u \tanh u \sin v \cos v)$$

$$= \mathbf{0}$$

$$G = \mathbf{x}_v \cdot \mathbf{x}_v = 0 + (\text{sech}^2 u \sin^2 v) + (\text{sech}^2 u \cos^2 v)$$

$$= \text{sech}^2 u(\sin^2 v + \cos^2 v) = \text{sech}^{\mathbf{2}} \mathbf{u}$$

The First Fundamental Form is $I = \begin{pmatrix} \tanh^2 u & 0 \\ 0 & \text{sech}^2 u \end{pmatrix}$.

Step 2: Second Fundamental Form (II)

The normal vector $\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$.

$$\mathbf{x}_u \times \mathbf{x}_v = \left(-\text{sech}^2 u \tanh u, -\text{sech } u \tanh^2 u \cos v, -\text{sech } u \tanh^2 u \sin v\right)$$

The magnitude is $\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{EG - F^2} = \sqrt{\tanh^2 u \text{sech}^2 u} = \tanh u \text{sech } u$.

$$\mathbf{N} = \frac{1}{\tanh u \text{sech } u}(\mathbf{x}_u \times \mathbf{x}_v) = (-\text{sech } u, -\tanh u \cos v, -\tanh u \sin v)$$

Now we compute the second partial derivatives:

$$\mathbf{x}_{uu} = \left(2 \tanh u \text{sech}^2 u, (\text{sech } u \tanh^2 u - \text{sech}^3 u) \cos v, (\text{sech } u \tanh^2 u - \text{sech}^3 u) \sin v\right)$$

$$\mathbf{x}_{uv} = (0, \text{sech } u \tanh u \sin v, -\text{sech } u \tanh u \cos v)$$

$$\mathbf{x}_{vv} = (0, -\text{sech } u \cos v, -\text{sech } u \sin v)$$

We compute the coefficients e, f, g:

$$e = \mathbf{N} \cdot \mathbf{x}_{uu} = -2 \tanh u \text{sech}^3 u - \tanh u(\text{sech } u \tanh^2 u - \text{sech}^3 u)$$

$$= -2 \tanh u \text{sech}^3 u - \tanh^3 u \text{sech } u + \tanh u \text{sech}^3 u$$

$$= -\tanh u \text{sech } u(\text{sech}^2 u + \tanh^2 u) = -\tanh \mathbf{u} \text{sech } \mathbf{u}$$

$$f = \mathbf{N} \cdot \mathbf{x}_{uv} = 0 - \tanh^2 u\text{sech } u \cos v \sin v + \tanh^2 u\text{sech } u \sin v \cos v$$
$$= \mathbf{0}$$

$$g = \mathbf{N} \cdot \mathbf{x}_{vv} = 0 + \tanh u\text{sech } u \cos^2 v + \tanh u\text{sech } u \sin^2 v$$
$$= \tanh \mathbf{u}\text{sech } \mathbf{u}$$

The Second Fundamental Form is $II = \begin{pmatrix} -\tanh u\text{sech } u & 0 \\ 0 & \tanh u\text{sech } u \end{pmatrix}$.

Step 3: Curvatures

Gaussian Curvature (K):

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(-\tanh u\text{sech } u)(\tanh u\text{sech } u) - 0^2}{(\tanh^2 u)(\text{sech}^2 u) - 0^2} = \frac{-\tanh^2 u\text{sech}^2 u}{\tanh^2 u\text{sech}^2 u} = \mathbf{-1}$$

Principal Curvatures (κ_1, κ_2): The principal curvatures are the eigenvalues of the shape operator $S = I^{-1}II$.

$$S = \begin{pmatrix} \tanh^2 u & 0 \\ 0 & \text{sech}^2 u \end{pmatrix}^{-1} \begin{pmatrix} -\tanh u\text{sech } u & 0 \\ 0 & \tanh u\text{sech } u \end{pmatrix}$$

$$S = \begin{pmatrix} 1/\tanh^2 u & 0 \\ 0 & 1/\text{sech}^2 u \end{pmatrix} \begin{pmatrix} -\tanh u\text{sech } u & 0 \\ 0 & \tanh u\text{sech } u \end{pmatrix}$$

$$S = \begin{pmatrix} -\frac{\text{sech } u}{\tanh u} & 0 \\ 0 & \frac{\tanh u}{\text{sech } u} \end{pmatrix} = \begin{pmatrix} -1/\sinh u & 0 \\ 0 & \sinh u \end{pmatrix}$$

The principal curvatures are $\kappa_\mathbf{1} = \mathbf{-1}/\sinh \mathbf{u}$ and $\kappa_\mathbf{2} = \sinh \mathbf{u}$. We check $K = \kappa_1 \kappa_2 = (-1/\sinh u)(\sinh u) = -1$, which is correct.

Problem 2.

Show that a ruled surface has Gaussian curvature $K \leq 0$.

Solution.

What is a Ruled Surface?

A ruled surface is a surface that can be parametrized by

$$\mathbf{x}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u)$$

where $\boldsymbol{\alpha}(u)$ is a base curve (the directrix) and $\boldsymbol{\beta}(u)$ is a vector field defining the direction of the line (the ruling) at each point $\boldsymbol{\alpha}(u)$.

Solution

We need to show that $K \leq 0$. The formula for Gaussian curvature is

$$K = \frac{eg - f^2}{EG - F^2}$$

For a regular surface, the denominator $EG - F^2 > 0$. Thus, we only need to show that the numerator $eg - f^2 \leq 0$.

1. **Find Second Partial Derivatives**

$$\mathbf{x}_u = \boldsymbol{\alpha}'(u) + v\boldsymbol{\beta}'(u)$$
$$\mathbf{x}_v = \boldsymbol{\beta}(u)$$

The second partial derivatives are:

$$\mathbf{x}_{uu} = \boldsymbol{\alpha}''(u) + v\boldsymbol{\beta}''(u)$$
$$\mathbf{x}_{uv} = \boldsymbol{\beta}'(u)$$
$$\mathbf{x}_{vv} = \mathbf{0}$$

2. Find the Second Fundamental Form (II) The coefficients e, f, g are the dot products of the second partials with the unit normal vector $\mathbf{N} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$.

$$\begin{aligned}
e &= \langle \mathbf{N}, \mathbf{x}_{uu} \rangle = \langle \mathbf{N}, \boldsymbol{\alpha}''(u) + v\boldsymbol{\beta}''(u) \rangle \\
f &= \langle \mathbf{N}, \mathbf{x}_{uv} \rangle = \langle \mathbf{N}, \boldsymbol{\beta}'(u) \rangle \\
g &= \langle \mathbf{N}, \mathbf{x}_{vv} \rangle = \langle \mathbf{N}, \mathbf{0} \rangle = \mathbf{0}
\end{aligned}$$

3. Calculate K Now we substitute $g = 0$ into the formula for K:

$$K = \frac{e \cdot (0) - f^2}{EG - F^2} = \frac{-f^2}{EG - F^2}$$

4. Conclusion Let's analyze this result:

- The numerator is $-f^2$. Since f is a real-valued scalar (from a dot product), f^2 must be non-negative ($f^2 \geq 0$). Therefore, $\mathbf{-f^2 \leq 0}$.
- The denominator $EG - F^2$ is the squared magnitude of the normal vector, which must be positive for a regular surface ($\mathbf{EG - F^2 > 0}$).

Since $K = \frac{\text{(a value } \leq 0)}{\text{(a value } > 0)}$, the result K must be less than or equal to zero. Thus, $K \leq 0$ for any ruled surface.

Problem 3.

Let $\kappa_n(\theta)$ denote the normal curvature in the direction making angle θ with the first principal direction.

a. Show that $H = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) d\theta$.

b. Show that $H = \frac{1}{2}(\kappa_n(\theta) + \kappa_n(\theta + \frac{\pi}{2}))$ for any θ.

c. (More challenging) Show that, more generally, for any θ and $m \geq 3$, we have

$$H = \frac{1}{m}\left(\kappa_n(\theta) + \kappa_n\left(\theta + \frac{2\pi}{m}\right) + \cdots + \kappa_n\left(\theta + \frac{2\pi(m-1)}{m}\right)\right).$$

Solution.

All three parts rely on Euler's Formula for normal curvature and the definition of Mean Curvature.

- **Euler's Formula:** $\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$, where κ_1, κ_2 are principal curvatures.
- **Mean Curvature:** $H = \frac{1}{2}(\kappa_1 + \kappa_2)$.

a. Show that $H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)d\theta$

1. Substitute Euler's Formula into the integral:

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} (\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta)d\theta$$

2. Split the integral:

$$= \frac{1}{2\pi}\left[\kappa_1 \int_0^{2\pi} \cos^2\theta d\theta + \kappa_2 \int_0^{2\pi} \sin^2\theta d\theta\right]$$

3. Use the identities $\int_0^{2\pi} \cos^2\theta d\theta = \pi$ and $\int_0^{2\pi} \sin^2\theta d\theta = \pi$:

$$= \frac{1}{2\pi}\left[\kappa_1(\pi) + \kappa_2(\pi)\right] = \frac{\pi}{2\pi}(\kappa_1 + \kappa_2)$$

4. This simplifies to $\frac{1}{2}(\kappa_1 + \kappa_2)$, which is the definition of H.

b. Show that $H = \frac{1}{2}(\kappa_n(\theta) + \kappa_n(\theta + \frac{\pi}{2}))$

1. Write out the two terms using Euler's Formula:

$$\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$
$$\kappa_n(\theta + \frac{\pi}{2}) = \kappa_1 \cos^2(\theta + \frac{\pi}{2}) + \kappa_2 \sin^2(\theta + \frac{\pi}{2})$$

2. Use the identities $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta + \frac{\pi}{2}) = \cos\theta$:

$$\kappa_n(\theta + \frac{\pi}{2}) = \kappa_1(-\sin\theta)^2 + \kappa_2(\cos\theta)^2 = \kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta$$

3. Add the two terms:

$$\begin{aligned}\kappa_n(\theta) + \kappa_n(\theta + \frac{\pi}{2}) &= (\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta) + (\kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta) \\ &= \kappa_1(\cos^2\theta + \sin^2\theta) + \kappa_2(\sin^2\theta + \cos^2\theta) \\ &= \kappa_1(1) + \kappa_2(1) = \kappa_1 + \kappa_2\end{aligned}$$

4. Substitute this back into the expression:

$$\frac{1}{2}(\kappa_n(\theta) + \kappa_n(\theta + \frac{\pi}{2})) = \frac{1}{2}(\kappa_1 + \kappa_2) = H$$

c. Show that $H = \frac{1}{m}(\sum_{k=0}^{m-1} \kappa_n(\theta + \frac{2\pi k}{m}))$

1. Let $S = \sum_{k=0}^{m-1} \kappa_n\left(\theta + \frac{2\pi k}{m}\right)$. We substitute Euler's Formula:

$$S = \sum_{k=0}^{m-1} \left[\kappa_1 \cos^2\left(\theta + \frac{2\pi k}{m}\right) + \kappa_2 \sin^2\left(\theta + \frac{2\pi k}{m}\right)\right]$$

2. Split the sum:

$$S = \kappa_1 \underbrace{\left[\sum_{k=0}^{m-1} \cos^2\left(\theta + \frac{2\pi k}{m}\right)\right]}_{A} + \kappa_2 \underbrace{\left[\sum_{k=0}^{m-1} \sin^2\left(\theta + \frac{2\pi k}{m}\right)\right]}_{B}$$

3. We evaluate the sums A and B.

$$A + B = \sum_{k=0}^{m-1} \left[\cos^2(\dots) + \sin^2(\dots)\right] = \sum_{k=0}^{m-1} 1 = m$$

$$A - B = \sum_{k=0}^{m-1} \left[\cos^2(\dots) - \sin^2(\dots)\right] = \sum_{k=0}^{m-1} \cos\left(2\theta + \frac{4\pi k}{m}\right)$$

This is the sum of cosines of angles in arithmetic progression, which is the real part of $\sum_{k=0}^{m-1} e^{i(2\theta+4\pi k/m)}$. For $m \geq 3$, this sum is 0. 4. We have a system of equations:

- $A + B = m$
- $A - B = 0 \implies A = B$

This gives $2A = m \implies A = m/2$ and $B = m/2$. 5. Substitute this back into the expression for S:

$$S = \kappa_1 \left(\frac{m}{2}\right) + \kappa_2 \left(\frac{m}{2}\right) = \frac{m}{2}(\kappa_1 + \kappa_2)$$

6. Finally, we find the average:

$$\frac{1}{m} S = \frac{1}{m}\left[\frac{m}{2}(\kappa_1 + \kappa_2)\right] = \frac{1}{2}(\kappa_1 + \kappa_2) = H$$

Problem 4.

(Surfaces of Revolution with Constant Curvature.) A surface is given as a surface of revolution $(\phi(v)\cos u, \phi(v)\sin u, \psi(v))$ with constant Gaussian curvature K. To determine the functions ϕ and ψ, choose the parameter v in such a way that $(\varphi')^2 + (\psi')^2 = 1$ (geometrically, this means that v is the arc length of the generating curve $(\phi(v), \psi(v))$).

a. Show that φ satisfies $\varphi'' + K\varphi = 0$ and ψ is given by $\psi = \int \sqrt{1 - (\varphi')^2} dv$; thus, $0 < u < 2\pi$, and the domain of v is such that the last integral makes sense.

b. All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane xOy are given by

$$\varphi(v) = C\cos v$$
$$\psi(v) = \int_0^v \sqrt{1 - C^2\sin^2 t}\,dt,$$

where C is a constant ($C = \varphi(0)$). Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane for the cases $C = 1$, $C > 1$, $C < 1$. Observe that $C = 1$ gives a sphere.

c. All surfaces of revolution with constant curvature $K = -1$ may be given by one of the following types:

1. $\varphi(v) = C\cosh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 t}\,dt$
2. $\varphi(v) = C\sinh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 t}\,dt$
3. $\varphi(v) = e^v, \quad \psi(v) = \int_0^v \sqrt{1 - e^{2t}}\,dt$

Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane.

d. The surface of type 3 in part c is the pseudosphere of Exercise 6.

e. The only surfaces of revolution with $K \equiv 0$ are the right circular cylinder, the right circular cone, and the plane.

Solution.

a. Deriving the Fundamental Equations

We are given $\mathbf{x}(u, v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v))$ and $(\phi')^2 + (\psi')^2 = 1$.

1. First Fundamental Form:

$$\mathbf{x}_u = (-\phi \sin u, \phi \cos u, 0)$$
$$\mathbf{x}_v = (\phi' \cos u, \phi' \sin u, \psi')$$

$E = \mathbf{x}_u \cdot \mathbf{x}_u = \phi^2$

$F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$

$G = \mathbf{x}_v \cdot \mathbf{x}_v = (\phi')^2(\cos^2 u + \sin^2 u) + (\psi')^2 = (\phi')^2 + (\psi')^2 = 1$

So, $EG - F^2 = \phi^2$.

2. Second Fundamental Form: The normal vector is $\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{1}{\phi}(\phi\psi' \cos u, \phi\psi' \sin u, -\phi\phi') = (\psi' \cos u, \psi' \sin u, -\phi')$. The second partials are:

$$\mathbf{x}_{uu} = (-\phi \cos u, -\phi \sin u, 0)$$
$$\mathbf{x}_{vv} = (\phi'' \cos u, \phi'' \sin u, \psi'')$$

The coefficients e, f, g are: $e = \mathbf{N} \cdot \mathbf{x}_{uu} = -\phi\psi' \cos^2 u - \phi\psi' \sin^2 u = -\phi\psi'$

$f = \mathbf{N} \cdot \mathbf{x}_{uv} = 0$ (as $\mathbf{x}_{uv}$ is orthogonal to $\mathbf{N}$)

$g = \mathbf{N} \cdot \mathbf{x}_{vv} = \psi'\phi''(\cos^2 u + \sin^2 u) - \phi'\psi'' = \psi'\phi'' - \phi'\psi''$

3. Simplify g and find K: From $(\phi')^2 + (\psi')^2 = 1$, we differentiate w.r.t v: $2\phi'\phi'' + 2\psi'\psi'' = 0 \implies \psi'\psi'' = -\phi'\phi''$. Substitute $\psi'' = -\frac{\phi'\phi''}{\psi'}$ into g:

$$g = \psi'\phi'' - \phi'\left(-\frac{\phi'\phi''}{\psi'}\right) = \frac{(\psi')^2\phi'' + (\phi')^2\phi''}{\psi'} = \frac{\phi''((\psi')^2 + (\phi')^2)}{\psi'} = \frac{\phi''(1)}{\psi'} = \frac{\phi''}{\psi'}$$

Now, we compute K:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(-\phi\psi')(\frac{\phi''}{\psi'}) - 0^2}{\phi^2} = \frac{-\phi\phi''}{\phi^2} = -\frac{\phi''}{\phi}$$

This gives the differential equation $\phi'' + \mathbf{K}\phi = \mathbf{0}$. From the arc-length condition, $(\psi')^2 = 1 - (\phi')^2$, so $\psi(\mathbf{v}) = \int \sqrt{\mathbf{1 - (\phi')^2}}\mathbf{dv}$.

b. Case $K = 1$

The equation is $\phi'' + \phi = 0$, with general solution $\phi(v) = C_1 \cos v + C_2 \sin v$. As given in the problem, we take the solution $\phi(\mathbf{v}) = \mathbf{C}\cos\mathbf{v}$. This gives $\phi'(v) = -C \sin v$. $\psi(v)$ becomes $\psi(\mathbf{v}) = \int_{\mathbf{0}}^{\mathbf{v}} \sqrt{\mathbf{1} - \mathbf{C^2}\sin^{\mathbf{2}}\mathbf{t}}\mathbf{dt}$.

- **C=1:** $\phi = \cos v$, $\psi = \int_0^v \cos t dt = \sin v$. The curve $(\cos v, \sin v)$ is a circle, which generates a **sphere**.
- **C $<$ 1:** The term $1 - C^2 \sin^2 t$ is always positive. This generates a **spindle** (prolate ellipsoid).
- **C $>$ 1:** The domain is restricted by $1 - C^2 \sin^2 t \geq 0$, or $|\sin v| \leq 1/C$. This generates a **barrel**.

c. Case $K = -1$

The equation is $\phi'' - \phi = 0$, with general solution $\phi(v) = C_1 e^v + C_2 e^{-v}$ (or $A \cosh v + B \sinh v$). The three types listed are special cases of this solution:

1. $\phi(\mathbf{v}) = \mathbf{C}\cosh\mathbf{v} \implies \phi' = C \sinh v \implies \psi(\mathbf{v}) = \int \sqrt{\mathbf{1} - \mathbf{C^2}\sinh^{\mathbf{2}}\mathbf{v}}\mathbf{dv}$. Domain is restricted.
2. $\phi(\mathbf{v}) = \mathbf{C}\sinh\mathbf{v} \implies \phi' = C \cosh v \implies \psi(\mathbf{v}) = \int \sqrt{\mathbf{1} - \mathbf{C^2}\cosh^{\mathbf{2}}\mathbf{v}}\mathbf{dv}$. Domain is trivial (or non-existent).
3. $\phi(\mathbf{v}) = \mathbf{e^v} \implies \phi' = e^v \implies \psi(\mathbf{v}) = \int \sqrt{\mathbf{1} - \mathbf{e^{2v}}}\mathbf{dv}$. Domain is $1 - e^{2v} \geq 0 \implies v \leq 0$.

d. Type 3 is the Pseudosphere

The generating curve for Type 3 is $\gamma(v) = (e^v, \int \sqrt{1 - e^{2v}} dv)$. This is the definition of a **tractrix**, a curve with a constant tangent length (of 1) to its asymptote (the z-axis). The surface of revolution of a tractrix is the **pseudosphere**.

e. Case $K = 0$

The equation is $\phi'' = 0$, so $\phi(v) = C_1 v + C_2$ and $\phi' = C_1$. $\psi' = \sqrt{1 - C_1^2}$. Let $C_3 = \sqrt{1 - C_1^2}$ (a constant, requires $C_1^2 \le 1$). $\psi(v) = C_3 v + C_4$. The generating curve $(\phi(v), \psi(v))$ is a line.

- **$C_1 = 0$:** $\phi = C_2$ (constant), $\psi = v + C_4$ (vertical line). Generates a **cylinder**.
- **$C_1^2 = 1$:** $\phi = \pm v + C_2$, $\psi = C_4$ (horizontal line). Generates a **plane**.
- **$0 < C_1^2 < 1$:** $\phi = C_1 v + C_2$, $\psi = C_3 v + C_4$ (slanted line). Generates a **cone**.

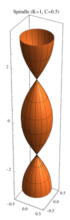

Figure 1: A detailed diagram of the sprindle shape.

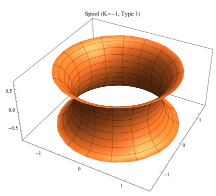

Figure 2: The geometry of the spool shape.

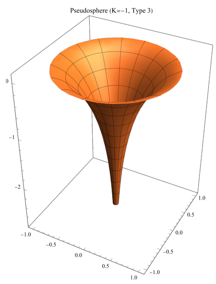

Figure 3: Illustration of the pseudosphere.

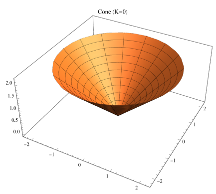

Figure 4: The standard cone geometry.

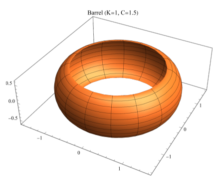

Figure 5: The three-dimensional representation of a barrel shape.

Problem 5.

1. Suppose $\alpha : I \to \mathbb{R}^3$ is a p.a.l (parametrized by arc length) spatial curve that attains its maximum norm at p i.e., $|p| = \max_{x \in \alpha(I)} |x|$. Prove that the curvature k of α at p is greater than or equal to $1/|p|$ (that is, $k \geq 1/|p|$), that is, the curve must be more "curving" than its circumscribed sphere (namely, $B_{|p|}(0)$) so as to curb back before penetrating it. (Hint: use the second order derivative test).

2. Suppose S is a regular surface that attains its maximum norm at p, prove that all the principal curvature(s) of S at p has magnitude greater than or equal to $1/|p|$ (i.e. $|\kappa_i| \geq 1/|p|$) and hence the magnitude of the Gaussian curvature $|K|$ greater than or equal to $1/|p|^2$ that is, the surface must be more "curving" than its circumscribed sphere (namely, $B_{|p|}(0)$) so as to curb back before penetrating it.

3. Hence, or otherwise, show no compact (closed and bounded) regular surface is minimal since such a surface must contain a point p with maximum norm.

Solution.

I have done this in the previous problem set.

Problem 6.

(From "Excersice Left in Lecture") Check the formula for Mean Curvature H:

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}$$

Solution.

I have done this in the last problem set.