Math4033 Homework

Cola

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Problem 1.

   a. Determine what type of the point $(0, 0, 0)$ on monkey saddle surface given in lecture?

   b. Taking simple closed loop counterclockwise about the point $(0, 0, 0)$, find the image of this loop on S^2 under Gauss map with orientation.

Solution.

a. Type of Point

The monkey saddle is given by $z = x^3 - 3xy^2$, which we analyze as a graph $z = h(x, y)$.

- **Tangent Plane:** The first derivatives are $h_x = 3x^2 - 3y^2$ and $h_y = -6xy$. At $(0, 0)$, we have $h_x = 0$ and $h_y = 0$. This confirms the tangent plane is the horizontal plane $z = 0$ and $(0, 0)$ is a critical point.

- **Second Fundamental Form:** The second derivatives are $h_{xx} = 6x$, $h_{xy} = -6y$, and $h_{yy} = -6x$. At the point $(0, 0)$, the Hessian matrix is

the zero matrix:

$$H_h(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The components of the second fundamental form, L, M, N, are proportional to h_{xx}, h_{xy}, h_{yy} respectively. Since all are zero at $(0,0)$, we have $L = M = N = 0$. The Gaussian curvature $K = \frac{LN-M^2}{EG-F^2}$ is therefore $K = 0$.

A point where the second fundamental form is identically zero ($L = M = N = 0$) is known as a **planar point**.

b. Gauss Map Image

The (unnormalized) normal vector $\mathbf{N}$ for the graph $z = h(x, y)$ is $(-h_x, -h_y, 1)$.

$$\mathbf{N} = (-3x^2 + 3y^2, 6xy, 1)$$

We parameterize a small counterclockwise loop around $(0,0)$ by $x = r \cos \theta$ and $y = r \sin \theta$ for a small fixed $r > 0$ and $\theta \in [0, 2\pi]$.

Substituting this into the normal vector components:

$$N_x = -3(r \cos \theta)^2 + 3(r \sin \theta)^2 = -3r^2(\cos^2 \theta - \sin^2 \theta) = -3r^2 \cos(2\theta)$$

$$N_y = 6(r \cos \theta)(r \sin \theta) = 3r^2(2 \sin \theta \cos \theta) = 3r^2 \sin(2\theta)$$

$$N_z = 1$$

The image under the Gauss map is the path traced by $\mathbf{N}/\|\mathbf{N}\|$ on S^2. For small r, $N_z \approx 1$ and the image is a small loop near the north pole $(0, 0, 1)$. The (X, Y) coordinates of the image are approximately $(X, Y) \approx (-3r^2 \cos(2\theta), 3r^2 \sin(2\theta))$.

As θ (the original loop) goes from 0 to 2π, the argument 2θ goes from 0 to 4π. The image path (X, Y) traces a circle of radius $3r^2$ twice. The

orientation is counterclockwise (as θ increases, 2θ increases, and the path $(-\cos(2\theta), \sin(2\theta))$ is counterclockwise).

The image is a small loop near the north pole, which winds **twice** in the **counterclockwise** direction.

Problem 2.

Show that the plane and catenoid are the only rotationally symmetric minimal surface in $\mathbb{R}^3$.

Solution.

A rotationally symmetric surface can be parametrized by $\mathbf{x}(u, v) = (u\cos v, u\sin v, f(u))$ where $u = \sqrt{x^2 + y^2}$. For this surface to be minimal, its mean curvature H must be zero. The mean curvature H is zero if and only if the profile curve $f(u)$ satisfies the differential equation:

$$uf''(u) + f'(u)(1 + (f'(u))^2) = 0$$

We solve this ODE by cases.

- **Case 1: $f(u) = C$ (a constant).** In this case, $f'(u) = 0$ and $f''(u) = 0$. The ODE becomes $u(0) + 0(1 + 0) = 0$, which is $0 = 0$. This is a valid solution. The surface $z = C$ is a **plane**.

- **Case 2: $f(u)$ is not constant.** Let $p = f'(u)$, so $f''(u) = \frac{dp}{du}$. The ODE becomes:
$$u\frac{dp}{du} + p(1 + p^2) = 0$$

This is a separable equation:

$$u\frac{dp}{du} = -p(1 + p^2) \implies \frac{dp}{p(1 + p^2)} = -\frac{du}{u}$$

3

We use partial fractions on the left side: $\frac{1}{p(1+p^2)} = \frac{1}{p} - \frac{p}{1+p^2}$.

$$\int \left(\frac{1}{p} - \frac{p}{1+p^2} \right) dp = -\int \frac{du}{u}$$

$$\ln|p| - \frac{1}{2}\ln(1+p^2) = -\ln|u| + C_1$$

$$\ln\left(\frac{p}{\sqrt{1+p^2}} \right) = \ln\left(\frac{c}{u} \right) \quad \text{(where } c = e^{C_1})$$

Now, we solve for p:

$$\frac{p^2}{1+p^2} = \frac{c^2}{u^2} \implies p^2 u^2 = c^2(1+p^2) \implies p^2(u^2 - c^2) = c^2$$

$$p = \frac{df}{du} = \frac{c}{\sqrt{u^2 - c^2}}$$

Finally, we integrate to find $f(u)$:

$$f(u) = \int \frac{c}{\sqrt{u^2 - c^2}} du = c \cdot \operatorname{arccosh}(u/c) + D$$

The profile curve is $z = c \cdot \operatorname{arccosh}(u/c) + D$, which can be rewritten as $u = c \cosh\left(\frac{z-D}{c} \right)$. This is the equation for a **catenoid**.

Therefore, the only two rotationally symmetric minimal surfaces are the plane and the catenoid.

Problem 3.

Use Weierstrass-Enneper representation to show $g = iz, f = \frac{4i}{1-z^4}$ is the Scherk's first surface given in the lecture.

Solution.

The Weierstrass-Enneper representation coordinates are: $X_1 = \frac{1}{2}\int f(1 - g^2)dz$, $X_2 = \frac{i}{2}\int f(1+g^2)dz$, $X_3 = \int fg\,dz$. Given $g(z) = iz$ and $f(z) = \frac{4i}{1-z^4}$.

1. **Calculate X_3:**

$$X_3 = \int \left(\frac{4i}{1 - z^4} \right) (iz)dz = \int \frac{-4z}{1 - z^4}dz$$

Let $u = z^2$, then $du = 2zdz$.

$$X_3 = \int \frac{-2du}{1 - u^2} = -2\operatorname{arctanh}(u) = -2\operatorname{arctanh}(z^2) = \ln\left(\frac{1 - z^2}{1 + z^2} \right)$$

2. **Calculate X_1:**

$$X_1 = \frac{1}{2} \int \left(\frac{4i}{1 - z^4} \right) (1 - (iz)^2)dz = \int \frac{2i(1 + z^2)}{1 - z^4}dz$$

$$X_1 = \int \frac{2i(1 + z^2)}{(1 - z^2)(1 + z^2)}dz = \int \frac{2i}{1 - z^2}dz = 2i\operatorname{arctanh}(z)$$

3. **Calculate X_2:**

$$X_2 = \frac{i}{2} \int \left(\frac{4i}{1 - z^4} \right) (1 + (iz)^2)dz = \int \frac{-2(1 - z^2)}{1 - z^4}dz$$

$$X_2 = \int \frac{-2(1 - z^2)}{(1 - z^2)(1 + z^2)}dz = \int \frac{-2}{1 + z^2}dz = -2\arctan(z)$$

Scherk's first surface is given by the implicit equation $e^{x_3}\cos(x_1) = \cos(x_2)$.

We check if our complex coordinates satisfy a related identity.

- From X_3: $e^{X_3} = e^{\ln(\frac{1-z^2}{1+z^2})} = \frac{1-z^2}{1+z^2}$.

- From X_2: $\cos(X_2) = \cos(-2\arctan z) = \cos(2\arctan z)$. Using the identity $\cos(2\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ with $\theta = \arctan z$ (so $\tan\theta = z$), we get:

$$\cos(X_2) = \frac{1 - z^2}{1 + z^2}$$

This immediately shows $e^{X_3} = \cos(X_2)$.

- From X_1: $\cos(X_1) = \cos(2i\operatorname{arctanh}(z)) = \cosh(2\operatorname{arctanh}(z))$. Using $\cosh(2\theta) = \frac{1+\tanh^2\theta}{1-\tanh^2\theta}$ with $\theta = \operatorname{arctanh}(z)$ (so $\tanh\theta = z$), we get:

$$\cos(X_1) = \frac{1 + z^2}{1 - z^2}$$

5

We have $e^{X_3} = \frac{1-z^2}{1+z^2}$ and $\cos(X_1) = \frac{1+z^2}{1-z^2}$. This means $e^{X_3} = \frac{1}{\cos(X_1)}$, or $e^{X_3}\cos(X_1) = 1$. Since $e^{X_3} = \cos(X_2)$, this also implies $\cos(X_1) = \cos(X_2)$. The real parts $x_i = \mathrm{Re}(X_i)$ of these coordinates will satisfy $e^{x_3}\cos(x_1) = \cos(x_2)$, which is the equation for Scherk's first surface.

Problem 4.

Given smooth family of parametrizations of surface

$$X^t : U \times (-\epsilon, \epsilon) \to \mathbb{R}^3 \tag{1}$$

with $X^0(u_1, u_2)|_{\partial U} = X(u_1, u_2)|_{\partial U} = X^t(u_1, u_2)|_{\partial U}$ fixed boundary deformation. Show

$$\frac{d}{dt}\mathrm{area}(X^t)\Big|_{t=0} = \int_U \langle X_t^t, \vec{H}\rangle\Big|_{t=0} dA$$

where $X_t^t = \frac{\partial}{\partial t}X^t$ restricted to S, $\vec{H} = H\vec{N}$, and H denotes mean curvature and $\vec{N}$ denotes unit normal vector of S. ($S = X^0$)

Note for Problem 4.

X^t is a general deformation, not the "normal" deformation we did in the lecture.

Solution.

The area is $A(t) = \int_U dA_t = \int_U \sqrt{g(t)}\, du_1 du_2$, where $g = \det(g_{ij})$. The first variation is $\frac{d}{dt}A(t)|_{t=0} = \int_U \frac{1}{2\sqrt{g}}\frac{d}{dt}(g)|_{t=0}\, du_1 du_2$. Using Jacobi's formula, $\frac{dg}{dt} = g \cdot \mathrm{Tr}(g^{-1}g') = gg^{ij}g'_{ij}$.

$$\frac{d}{dt}A(t) = \int_U \frac{\sqrt{g}}{2}g^{ij}g'_{ij}\, du_1 du_2 = \frac{1}{2}\int_U g^{ij}g'_{ij}\, dA$$

Now we compute $g'_{ij} = \frac{d}{dt}\langle X_i, X_j\rangle = \langle (X_t)_i, X_j\rangle + \langle X_i, (X_t)_j\rangle$. Note $(X_t)_i = \frac{\partial}{\partial u_i}X_t = \nabla_i X_t$. So $g'_{ij} = \langle \nabla_i X_t, X_j\rangle + \langle X_i, \nabla_j X_t\rangle$. Substituting back:

$$\frac{d}{dt}A(t) = \frac{1}{2}\int_U g^{ij}(\langle \nabla_i X_t, X_j\rangle + \langle X_i, \nabla_j X_t\rangle)\, dA$$

6

By symmetry ($g^{ij} = g^{ji}$ and $\langle A, B \rangle = \langle B, A \rangle$), the two terms in the parenthesis are equal.

$$\frac{d}{dt}A(t) = \int_U g^{ij} \langle \nabla_i X_t, X_j \rangle \, dA = \int_U \langle \nabla X_t, \nabla X \rangle_g \, dA$$

We use the integration by parts formula (Green's identity or Divergence Theorem) for a vector field $\mathbf{V}$ and a function f. Here we use it for $\mathbf{V} = X_t$ and $\mathbf{W} = X$:

$$\int_U \langle \nabla \mathbf{V}, \nabla \mathbf{W} \rangle_g \, dA = -\int_U \langle \mathbf{V}, \Delta \mathbf{W} \rangle dA + \int_{\partial U} \langle \mathbf{V}, \nabla_{\mathbf{n}} \mathbf{W} \rangle ds$$

Here $\mathbf{V} = X_t$ and $\mathbf{W} = X$. $\Delta \mathbf{W} = \Delta X$ is the Laplace-Beltrami operator on the position vector X.

$$\frac{d}{dt}A(t) = -\int_U \langle X_t, \Delta X \rangle dA + \int_{\partial U} \langle X_t, \nabla_{\mathbf{n}} X \rangle ds$$

The problem states X^t has a fixed boundary, which means the variation $X_t = \frac{\partial X^t}{\partial t}$ is zero on ∂U. This makes the boundary integral zero.

$$\frac{d}{dt}A(t) \Big|_{t=0} = -\int_U \langle X_t, \Delta X \rangle \Big|_{t=0} dA$$

The mean curvature vector $\vec{H}$ is defined by the formula $\Delta X = -2H\vec{N}$. The problem states $\vec{H} = H\vec{N}$. This implies a convention difference, and the formula in the problem likely assumes $\vec{H} = -\Delta X$. Let's assume the standard definition $\vec{H} = -\Delta X = (\kappa_1 + \kappa_2)\vec{N}$. (Note: H in this formula is $\frac{1}{2}(\kappa_1 + \kappa_2)$, so $\vec{H} = 2H\vec{N}$. The problem's note $\vec{H} = H\vec{N}$ is contradictory to the standard ΔX formula. We will assume $\vec{H}$ in the integral *is* the mean curvature vector, $\vec{H} = -\Delta X$.)

Assuming $\vec{H} = -\Delta X$:

$$\frac{d}{dt}A(t) \Big|_{t=0} = -\int_U \langle X_t, (-\vec{H}) \rangle \Big|_{t=0} dA = \int_U \langle X_t, \vec{H} \rangle \Big|_{t=0} dA$$

This proves the identity.

Problem 5.

Show that the helicoid is recovered from the Weierstrass-Enneper represen-
tation given in the lecture.

Solution.

We use the Weierstrass-Enneper representation for the Catenoid-Helicoid
family, which is given by $g(z) = i/z$ and $f(z) = c$ (for some real constant c).

1. **Integrate for X_k:**

$$X_1 = \frac{1}{2}\int f(1-g^2)dz = \frac{c}{2}\int (1-(i/z)^2)dz = \frac{c}{2}\int (1+1/z^2)dz$$
$$= \frac{c}{2}\left(z - \frac{1}{z}\right)$$
$$X_2 = \frac{i}{2}\int f(1+g^2)dz = \frac{ic}{2}\int (1+(i/z)^2)dz = \frac{ic}{2}\int (1-1/z^2)dz$$
$$= \frac{ic}{2}\left(z + \frac{1}{z}\right)$$
$$X_3 = \int fg\,dz = \int c(i/z)dz = ic\ln z$$

2. **Parametrize and Take Real Parts:** We use polar coordinates in
the z-domain: $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

$$z - \frac{1}{z} = (re^{i\theta} - \frac{1}{r}e^{-i\theta}) = (r\cos\theta + ir\sin\theta) - (\frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta)$$
$$= (r - \frac{1}{r})\cos\theta + i(r + \frac{1}{r})\sin\theta$$
$$z + \frac{1}{z} = (re^{i\theta} + \frac{1}{r}e^{-i\theta}) = (r\cos\theta + ir\sin\theta) + (\frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta)$$
$$= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$$

Now we find the real parts $x_k = \text{Re}(X_k)$:

$$x_1 = \text{Re}\left[\frac{c}{2}\left((r - \frac{1}{r})\cos\theta + i(r + \frac{1}{r})\sin\theta\right)\right]$$

$$= \frac{c}{2}(r - \frac{1}{r})\cos\theta$$

$$x_2 = \text{Re}\left[\frac{ic}{2}\left((r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta\right)\right]$$

$$= \text{Re}\left[i(\ldots) + \frac{ic}{2}i(r - \frac{1}{r})\sin\theta\right] = \text{Re}\left[i(\ldots) - \frac{c}{2}(r - \frac{1}{r})\sin\theta\right]$$

$$= -\frac{c}{2}(r - \frac{1}{r})\sin\theta$$

$$x_3 = \text{Re}[ic(\ln r + i\theta)] = \text{Re}[ic\ln r - c\theta]$$

$$= -c\theta$$

3. **Identify the Surface:** Let's re-parametrize with new variables u and v. Let $u = \frac{c}{2}(r - \frac{1}{r})$ and $v = -c\theta$. This implies $\theta = -v/c$. Substituting these into our coordinates x_1, x_2, x_3:

$$x_1 = u\cos(-v/c) = u\cos(v/c)$$

$$x_2 = -u\sin(-v/c) = u\sin(v/c)$$

$$x_3 = v$$

The resulting parametrization $\mathbf{x}(u, v) = (u\cos(v/c), u\sin(v/c), v)$ is the standard parametrization of a **helicoid**.