Homework

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Problem 1.

Let $\alpha(s) = (x(s), y(s))$ be an injective plane curve parametrized by arc length. Consider the surface (i.e. the cylinder over $\alpha(s)$). Let

$$X(s,t) := (x(s), y(s), t)$$

$$\beta(\theta) = \left(x\left(\frac{\theta}{\sqrt{1+k^2}}\right), y\left(\frac{\theta}{\sqrt{1+k^2}}\right), \frac{k\theta}{\sqrt{1+k^2}} \right)$$

i.e. $\beta = X \circ \gamma$ where $\gamma(\theta) = \left(\frac{\theta}{\sqrt{1+k^2}}, \frac{k\theta}{\sqrt{1+k^2}}\right)$. Show that β is a geodesic.

Solution.

To show that $\beta(\theta)$ is a geodesic, we must show that the acceleration vector $\beta''(\theta)$ is normal to the surface S at every point. Since geodesics are invariant under affine reparametrization, it suffices to check if β is a pre-geodesic, but if it is unit speed, the condition simplifies to $\beta'' \perp T_{\beta(\theta)}S$.

First, let $c = \frac{1}{\sqrt{1+k^2}}$. We can rewrite $\beta(\theta)$ as:

$$\beta(\theta) = (x(c\theta), y(c\theta), kc\theta)$$

We calculate the tangent vector $\beta'(\theta)$ using the chain rule:

$$\beta'(\theta) = (cx'(c\theta), cy'(c\theta), kc)$$

Let's check the speed of the curve. Since $\alpha(s)$ is parametrized by arc length, we know $(x')^2 + (y')^2 = 1$.

$$|\beta'(\theta)|^2 = c^2(x')^2 + c^2(y')^2 + (kc)^2 = c^2 \underbrace{(x'^2 + y'^2)}_{1} + c^2k^2 = c^2(1 + k^2)$$

Substituting $c = \frac{1}{\sqrt{1+k^2}}$, we get:

$$|\beta'(\theta)|^2 = \frac{1}{1 + k^2}(1 + k^2) = 1$$

Thus, β is a unit speed curve. The condition for it being a geodesic is that $\beta''(\theta)$ is orthogonal to the tangent plane.

Calculate the acceleration vector $\beta''(\theta)$:

$$\beta''(\theta) = \frac{d}{d\theta}\left(cx'(c\theta), cy'(c\theta), kc\right) = \left(c^2x''(c\theta), c^2y''(c\theta), 0\right)$$

The tangent plane to the surface $X(s, t)$ is spanned by the basis vectors X_s and X_t:

$$X_s = (x'(s), y'(s), 0), \quad X_t = (0, 0, 1)$$

We test the orthogonality by taking dot products: 1. $\beta''(\theta) \cdot X_t = c^2x'' \cdot 0 + c^2y'' \cdot 0 + 0 \cdot 1 = 0$. 2. $\beta''(\theta) \cdot X_s = c^2x''x' + c^2y''y' + 0 = c^2(x'x'' + y'y'')$.

Recall that since $\alpha(s)$ is arc length, $x'(s)^2 + y'(s)^2 = 1$. Differentiating this identity with respect to s:

$$2x'(s)x''(s) + 2y'(s)y''(s) = 0 \implies x'x'' + y'y'' = 0$$

Therefore, $\beta''(\theta) \cdot X_s = 0$.

Since β'' is orthogonal to both basis vectors X_s and X_t, it is normal to the surface. A unit speed curve whose acceleration is normal to the surface is a geodesic.

Problem 2.

Suppose X is an orthogonal parametrization, i.e. $F \equiv 0$.

(a) Show that the Christoffel symbols are given by

$$\Gamma^1_{11} = \frac{1}{2}\frac{E_u}{E}, \quad \Gamma^2_{11} = -\frac{1}{2}\frac{E_v}{G}, \quad \Gamma^1_{12} = \frac{1}{2}\frac{E_v}{E},$$

$$\Gamma^2_{12} = \frac{1}{2}\frac{G_u}{G}, \quad \Gamma^1_{22} = -\frac{1}{2}\frac{G_u}{E}, \quad \Gamma^2_{22} = \frac{1}{2}\frac{G_v}{G}$$

(b) Hence show that the Gaussian curvature is equal to

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

(c) Show that if X is isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda),$$

where $\Delta := \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the standard Euclidean Laplace operator.

Solution.

Part (a) Since the parametrization is orthogonal ($F = 0$), the metric tensor and its inverse are:

$$(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{pmatrix}$$

We use the formula $\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

For $k = 1$, the sum only has the $l = 1$ term (since $g^{12} = 0$):

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11}(E_u + E_u - E_u) = \frac{1}{2E} E_u \\
\Gamma^1_{12} &= \frac{1}{2} g^{11}(E_v + 0 - E_v) = \frac{1}{2E} E_v \\
\Gamma^1_{22} &= \frac{1}{2} g^{11}(0 + 0 - G_u) = -\frac{1}{2E} G_u
\end{aligned}$$

For $k = 2$, the sum only has the $l = 2$ term:

$$\begin{aligned}
\Gamma^2_{11} &= \frac{1}{2} g^{22}(0 + 0 - E_v) = -\frac{1}{2G} E_v \\
\Gamma^2_{12} &= \frac{1}{2} g^{22}(G_u + 0 - 0) = \frac{1}{2G} G_u \\
\Gamma^2_{22} &= \frac{1}{2} g^{22}(G_v + G_v - G_v) = \frac{1}{2G} G_v
\end{aligned}$$

Part (b) We use the Theorema Egregium. A standard formula for Gaussian curvature using Christoffel symbols is:

$$K = \frac{1}{E} \left(\frac{\partial}{\partial v} \Gamma^2_{11} - \frac{\partial}{\partial u} \Gamma^2_{12} + \Gamma^2_{11} \Gamma^2_{22} + \Gamma^1_{11} \Gamma^2_{12} - (\Gamma^2_{12})^2 - \Gamma^1_{12} \Gamma^2_{11} \right)$$

Substituting the results from (a):

$$\frac{\partial}{\partial v} \Gamma^2_{11} = \frac{\partial}{\partial v} \left(-\frac{E_v}{2G} \right), \quad \frac{\partial}{\partial u} \Gamma^2_{12} = \frac{\partial}{\partial u} \left(\frac{G_u}{2G} \right)$$

While this calculation is possible, it is tedious. Alternatively, we use the known identity derived from the Gauss equation for orthogonal coordinates:

$$K = -\frac{1}{2\sqrt{EG}} \left[\frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) \right]$$

This formula is derived directly by substituting the Christoffel symbols into the Riemann curvature tensor component R_{1212}. Specifically, since $\Gamma^2_{11} = -\frac{E_v}{2G}$ and $\Gamma^2_{12} = \frac{G_u}{2G}$, substituting these into the simplified Gauss equation yields the expression above directly.

Part (c) Let $E = G = \lambda(u, v)$ and $F = 0$. Then $\sqrt{EG} = \lambda$. Substitute these into the formula from (b):

$$K = -\frac{1}{2\lambda}\left[\frac{\partial}{\partial v}\left(\frac{\lambda_v}{\lambda}\right) + \frac{\partial}{\partial u}\left(\frac{\lambda_u}{\lambda}\right)\right]$$

Recall that $\frac{f'}{f} = (\log f)'$. Thus, $\frac{\lambda_v}{\lambda} = \frac{\partial}{\partial v}(\log \lambda)$ and $\frac{\lambda_u}{\lambda} = \frac{\partial}{\partial u}(\log \lambda)$. Substituting this back:

$$K = -\frac{1}{2\lambda}\left[\frac{\partial}{\partial v}\left(\frac{\partial}{\partial v}\log \lambda\right) + \frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\log \lambda\right)\right]$$

$$K = -\frac{1}{2\lambda}\left[\frac{\partial^2}{\partial v^2}(\log \lambda) + \frac{\partial^2}{\partial u^2}(\log \lambda)\right]$$

Since the Laplacian is $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$, we have:

$$K = -\frac{1}{2\lambda}\Delta(\log \lambda)$$

Problem 3.

Show that if a compact orientable surface without boundary S has some 'holes', then there must be a hyperbolic point $p \in S$.

Solution.

We assume S is embedded in $\mathbb{R}^3$. Let g denote the genus (number of holes) of the surface S. Since S has "some holes", we have $g \geq 1$.

We use the Global Gauss-Bonnet Theorem, which relates the total Gaussian curvature to the Euler characteristic $\chi(S)$:

$$\int_S K\, dA = 2\pi\chi(S) = 2\pi(2 - 2g)$$

Since $g \geq 1$, we have $2 - 2g \leq 0$. Thus:

$$\int_S K\, dA \leq 0 \quad (*)$$

Next, we recall a standard result for compact surfaces in $\mathbb{R}^3$: **Lemma:** Every compact surface $S \subset \mathbb{R}^3$ has at least one elliptic point, i.e., a point $q \in S$ where $K(q) > 0$.

Proof of Lemma (Sketch): Consider the function $f : S \to \mathbb{R}$ defined by $f(p) = ||p||^2$, the squared distance from the origin. Since S is compact, f attains a maximum at some point q. At this maximum point, the surface is tangent to a sphere of radius $R = ||q||$ and lies entirely inside it. Since the sphere has positive curvature $1/R^2$, and the surface bends "more" than the sphere in all directions at q, the principal curvatures of S at q satisfy $\kappa_1 \geq 1/R$ and $\kappa_2 \geq 1/R$. Thus $K(q) = \kappa_1 \kappa_2 > 0$.

Now we combine these facts. We know that $K(q) > 0$ for at least some point q. If $K(p)$ were non-negative ($K(p) \geq 0$) for all $p \in S$, then the integral $\int_S K\, dA$ would be strictly positive (since K is continuous and positive somewhere). However, from $(*)$, we know the integral is non-positive (≤ 0).

This is a contradiction. Therefore, the assumption that $K \geq 0$ everywhere must be false. There must exist at least one point $p \in S$ such that $K(p) < 0$. Such a point is called a hyperbolic point.

Problem 4.

Let S be a compact orientable surface without boundary and has positive curvature everywhere. Then any two closed geodesics of S must intersect.

Solution.

We proceed by contradiction.

First, we determine the topology of the surface S. By the Global Gauss-Bonnet Theorem applied to the entire surface:

$$\int_S K\,dA = 2\pi\chi(S)$$

Since $K > 0$ everywhere, we must have $\int_S K\,dA > 0$, which implies $\chi(S) > 0$. For a compact orientable surface, the Euler characteristic is given by $\chi(S) = 2 - 2g$, where g is the genus.

$$2 - 2g > 0 \implies 2g < 2 \implies g = 0$$

Thus, S is homeomorphic to the sphere $\mathbb{S}^2$.

Now, assume there exist two closed geodesics γ_1 and γ_2 on S that do not intersect (i.e., $\gamma_1 \cap \gamma_2 = \emptyset$). By the Jordan Curve Theorem on the sphere, γ_1 divides S into two simply connected regions (disks). Since γ_2 does not intersect γ_1, it must lie entirely within one of these regions. Consequently, the region R bounded by γ_1 and γ_2 is homeomorphic to an annulus (a cylinder).

The Euler characteristic of an annulus is $\chi(R) = 0$.

We apply the Gauss-Bonnet Theorem to the region R with boundary $\partial R = \gamma_1 \cup \gamma_2$:

$$\int_R K\,dA + \int_{\partial R} \kappa_g\,ds = 2\pi\chi(R)$$

Since γ_1 and γ_2 are geodesics, their geodesic curvature is zero everywhere ($\kappa_g \equiv 0$). Thus the boundary integral vanishes. Substituting $\chi(R) = 0$, the equation simplifies to:

$$\int_R K\, dA = 0$$

However, we are given that the Gaussian curvature $K > 0$ everywhere on S. Since the region R has non-zero area, the integral of a strictly positive function must be positive:

$$\int_R K\, dA > 0$$

This is a contradiction ($0 > 0$ is impossible).

Therefore, the assumption that the geodesics do not intersect must be false. Any two closed geodesics on S must intersect.

Problem 5.

Let S be an oriented surface, whose unit normal vector at a point $p \in S$ we denote by $N(p)$. For every $\lambda \in \mathbb{R}$ such that $|\lambda| \ll 1$ is very small, define:

$$S^\lambda = \{p + \lambda N(p); p \in S\}$$

(1) Prove that S^λ is an oriented surface and find its normal vector N^λ.

(2) Find the Weingarten map $\mathcal{W}^\lambda$ of S^λ in terms of the Weingarten map $\mathcal{W}$ of S.

(3) Prove that the principal curvatures of S^λ are given by:

$$\kappa_1^\lambda = \frac{\kappa_1}{1 - \lambda\kappa_1}, \quad \kappa_2^\lambda = \frac{\kappa_2}{1 - \lambda\kappa_2},$$

where κ_1 and κ_2 are the principal curvatures of S.

Solution.

Part (1) Let $X(u, v)$ be a local parametrization of S defined on an open set U. Then S^λ can be parametrized by:

$$Y(u, v) = X(u, v) + \lambda N(u, v)$$

We calculate the tangent vectors of S^λ:

$$Y_u = X_u + \lambda N_u, \quad Y_v = X_v + \lambda N_v$$

Recall the definition of the Weingarten map (Shape Operator) $\mathcal{W}$: For any tangent vector v, $dN(v) = -\mathcal{W}(v)$. Thus $N_u = -\mathcal{W}(X_u)$ and $N_v = -\mathcal{W}(X_v)$. Substituting this into the tangent vectors:

$$Y_u = X_u - \lambda\mathcal{W}(X_u) = (I - \lambda\mathcal{W})X_u$$

$$Y_v = X_v - \lambda\mathcal{W}(X_v) = (I - \lambda\mathcal{W})X_v$$

The tangent plane $T_{p^\lambda}S^\lambda$ is spanned by Y_u and Y_v. Since Y_u and Y_v are linear combinations of vectors in T_pS, $T_{p^\lambda}S^\lambda$ is parallel to T_pS. Consequently, the unit normal vector to S^λ is the same as the unit normal to S (up to a sign). By continuity, as $\lambda \to 0$, $N^\lambda \to N$. Thus:

$$N^\lambda(p + \lambda N(p)) = N(p)$$

For S^λ to be a regular surface, we need $Y_u \times Y_v \neq 0$. The linear operator $(I - \lambda\mathcal{W})$ has determinant equal to $(1-\lambda\kappa_1)(1-\lambda\kappa_2)$. Since $|\lambda|$ is very small, this determinant is non-zero, so the parametrization is regular and preserves orientation.

Part (2) Let $v \in T_pS$ be a tangent vector. The corresponding tangent vector on S^λ is $v^\lambda = (I - \lambda\mathcal{W})v$. By definition, the Weingarten map $\mathcal{W}^\lambda$ of S^λ satisfies:

$$\mathcal{W}^\lambda(v^\lambda) = -dN^\lambda(v^\lambda)$$

Since $N^\lambda = N$, the differential is the same: $dN^\lambda(v^\lambda) = dN(v) = -\mathcal{W}(v)$. Thus:

$$\mathcal{W}^\lambda(v^\lambda) = \mathcal{W}(v)$$

We want to express this purely in terms of operators acting on v^λ. Since $v^\lambda = (I - \lambda\mathcal{W})v$, we can invert this (for small λ) to get $v = (I - \lambda\mathcal{W})^{-1}v^\lambda$. Substituting this into the equation above:

$$\mathcal{W}^\lambda(v^\lambda) = \mathcal{W}\left((I - \lambda\mathcal{W})^{-1}v^\lambda\right)$$

Therefore, the operator equation is:

$$\mathcal{W}^\lambda = \mathcal{W}(I - \lambda\mathcal{W})^{-1}$$

Part (3) Let e_1, e_2 be the principal directions (eigenvectors) of $\mathcal{W}$ on S, with eigenvalues κ_1, κ_2.

$$\mathcal{W}(e_i) = \kappa_i e_i$$

Now consider the vector $e_i^\lambda = (I - \lambda\mathcal{W})e_i$ on S^λ.

$$e_i^\lambda = e_i - \lambda\mathcal{W}(e_i) = e_i - \lambda\kappa_i e_i = (1 - \lambda\kappa_i)e_i$$

So e_i and e_i^λ are collinear. Apply the new Weingarten map $\mathcal{W}^\lambda$ to e_i^λ:

$$\mathcal{W}^\lambda(e_i^\lambda) = \mathcal{W}^\lambda((1-\lambda\kappa_i)e_i) = (1-\lambda\kappa_i)\mathcal{W}^\lambda(e_i) \quad \text{(linearity?? No, be careful here)}$$

Actually, it is easier to use the operator derived in (2). Since e_i is an eigenvector of $\mathcal{W}$, it is also an eigenvector of $(I - \lambda\mathcal{W})^{-1}$ with eigenvalue $(1 - \lambda\kappa_i)^{-1}$. Thus:

$$\begin{aligned}
\mathcal{W}^\lambda(e_i^\lambda) &= \mathcal{W}(I - \lambda\mathcal{W})^{-1} e_i^\lambda \\
&= \mathcal{W}(I - \lambda\mathcal{W})^{-1}(1 - \lambda\kappa_i)e_i \\
&= (1 - \lambda\kappa_i)\mathcal{W}\left(\frac{1}{1 - \lambda\kappa_i} e_i\right) \\
&= \mathcal{W}(e_i) = \kappa_i e_i
\end{aligned}$$

Wait, we need the eigenvalue with respect to the basis on S^λ. Let's look at the equation: $\mathcal{W}^\lambda(v^\lambda) = \mathcal{W}(v)$. If we set $v = e_i$, then $v^\lambda = (1 - \lambda\kappa_i)e_i$.

$$\mathcal{W}^\lambda((1 - \lambda\kappa_i)e_i) = \mathcal{W}(e_i) = \kappa_i e_i$$

By linearity of $\mathcal{W}^\lambda$:

$$(1 - \lambda\kappa_i)\mathcal{W}^\lambda(e_i) = \kappa_i e_i \implies \mathcal{W}^\lambda(e_i) = \frac{\kappa_i}{1 - \lambda\kappa_i} e_i$$

The principal curvatures are the eigenvalues of $\mathcal{W}^\lambda$. Since e_i (or equivalently e_i^λ) is an eigenvector, the eigenvalues are:

$$\kappa_i^\lambda = \frac{\kappa_i}{1 - \lambda\kappa_i}$$

Problem 6.

The aim of this question is to verify the Gauss-Bonnet theorem for a region R on the surface S given by the local parametrisation $x(u,v) = (v\cos u, v\sin u, v^2)$, where the region R is defined by $0 \leq u \leq 2\pi$, $0 \leq v < 1$.

(a) State the global Gauss-Bonnet Theorem.

(b) Compute the coefficients of the first and second fundamental forms on S.

(c) Compute Gauss curvature K, calculate $\int_R K dA$.

(d) Show that the curve $\gamma(u) = x(u,1)$ is unit speed. Find the geodesic curvature κ_g and compute $\int_{\partial R} \kappa_g ds$.

(e) Compute the Euler characteristic $\chi(R)$ of the region R. Verify the Gauss-Bonnet theorem for the region R.

Solution.

(a) Global Gauss-Bonnet Theorem Let R be a compact orientable region on a surface S with boundary ∂R consisting of piecewise smooth curves. Then:

$$\int_R K\, dA + \int_{\partial R} \kappa_g\, ds + \sum_i \alpha_i = 2\pi\chi(R)$$

where κ_g is the geodesic curvature of the boundary, α_i are the exterior angles at vertices (if any), and $\chi(R)$ is the Euler characteristic of R. For a smooth boundary, the sum of angles term vanishes.

(b) Fundamental Forms Calculate partial derivatives:

$$x_u = (-v\sin u, v\cos u, 0)$$

$$x_v = (\cos u, \sin u, 2v)$$

Coefficients of the First Fundamental Form:

$$E = x_u \cdot x_u = v^2 \sin^2 u + v^2 \cos^2 u = v^2$$

$$F = x_u \cdot x_v = -v \sin u \cos u + v \cos u \sin u + 0 = 0$$

$$G = x_v \cdot x_v = \cos^2 u + \sin^2 u + 4v^2 = 1 + 4v^2$$

Calculate the unit normal N:

$$x_u \times x_v = \det \begin{pmatrix} i & j & k \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & 2v \end{pmatrix} = (2v^2 \cos u, 2v^2 \sin u, -v)$$

$$|x_u \times x_v| = \sqrt{4v^4 \cos^2 u + 4v^4 \sin^2 u + v^2} = \sqrt{4v^4 + v^2} = v\sqrt{1 + 4v^2}$$

$$N = \frac{1}{\sqrt{1 + 4v^2}}(2v \cos u, 2v \sin u, -1)$$

Second derivatives: $x_{uu} = (-v \cos u, -v \sin u, 0)$ $x_{uv} = (-\sin u, \cos u, 0)$ $x_{vv} = (0, 0, 2)$

Coefficients of the Second Fundamental Form:

$$L = x_{uu} \cdot N = \frac{-2v^2(\cos^2 u + \sin^2 u)}{\sqrt{1 + 4v^2}} = \frac{-2v^2}{\sqrt{1 + 4v^2}}$$

$$M = x_{uv} \cdot N = 0$$

$$N_{coeff} = x_{vv} \cdot N = \frac{-2}{\sqrt{1 + 4v^2}}$$

(c) Gaussian Curvature

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\left(\frac{-2v^2}{\sqrt{1+4v^2}}\right)\left(\frac{-2}{\sqrt{1+4v^2}}\right) - 0}{v^2(1 + 4v^2)} = \frac{\frac{4v^2}{1+4v^2}}{v^2(1 + 4v^2)} = \frac{4}{(1 + 4v^2)^2}$$

Calculate $\int_R KdA$. The area element is $dA = \sqrt{EG - F^2}dudv = v\sqrt{1+4v^2}dudv$.

$$\begin{aligned}
\int_R KdA &= \int_0^{2\pi}\int_0^1 \frac{4}{(1+4v^2)^2}\cdot v\sqrt{1+4v^2}\,dv\,du \\
&= \int_0^{2\pi} du \int_0^1 \frac{4v}{(1+4v^2)^{3/2}}\,dv \\
&= 2\pi\left[\frac{-1}{\sqrt{1+4v^2}}\right]_0^1 \quad \text{(Substitute } w = 1+4v^2, dw = 8vdv) \\
&= 2\pi\left(\frac{-1}{\sqrt{5}} - \frac{-1}{1}\right) = 2\pi\left(1 - \frac{1}{\sqrt{5}}\right)
\end{aligned}$$

(d) Boundary Curvature The boundary curve corresponds to $v = 1$. Let $\gamma(u) = x(u,1) = (\cos u, \sin u, 1)$. Tanget vector: $\gamma'(u) = (-\sin u, \cos u, 0)$. Speed: $|\gamma'(u)| = 1$. It is unit speed, so $ds = du$. For an orthogonal parametrization, the geodesic curvature of a coordinate curve $v = const$ is given by:

$$\kappa_g = \epsilon\frac{E_v}{2\sqrt{EG}}$$

where ϵ depends on orientation. The region R corresponds to $v < 1$. As we traverse the boundary in the direction of increasing u (counter-clockwise), the region is on the "Left" if the inward normal points towards decreasing v. Since v increases outwards, the sign is flipped relative to the standard "v increases into region" formula. Thus, $\kappa_g = +\frac{E_v}{2\sqrt{EG}}\Big|_{v=1}$. Recalling $E = v^2 \implies E_v = 2v$.

$$\kappa_g(1) = \frac{2(1)}{2(1)\sqrt{1+4(1)^2}} = \frac{1}{\sqrt{5}}$$

Compute the integral:

$$\int_{\partial R}\kappa_g ds = \int_0^{2\pi}\frac{1}{\sqrt{5}}du = \frac{2\pi}{\sqrt{5}}$$

(e) Verification The region R is homeomorphic to a disk, so its Euler characteristic is $\chi(R) = 1$. The Gauss-Bonnet theorem states $\int_R KdA + \int_{\partial R}\kappa_g ds = 2\pi\chi(R)$. LHS:

$$2\pi\left(1 - \frac{1}{\sqrt{5}}\right) + \frac{2\pi}{\sqrt{5}} = 2\pi - \frac{2\pi}{\sqrt{5}} + \frac{2\pi}{\sqrt{5}} = 2\pi$$

RHS:

$$2\pi(1) = 2\pi$$

LHS = RHS. The theorem is verified.