Homework-4

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Problem 1.

Let $X : (-\pi, \pi) \times (-\frac{1}{2}, \frac{1}{2}) \to M$ be given by

$$X(\theta, \nu) = (\cos\theta, \sin\theta, 0) + \nu\left(\sin\frac{\theta}{2}\cos\theta, \sin\frac{\theta}{2}\sin\theta, \cos\frac{\theta}{2}\right)$$

Compute the normal vector $N(\theta, 0)$ and show that

$$\lim_{\theta\to-\pi} N(\theta, 0) = -\lim_{\theta\to\pi} N(\theta, 0)$$

Solution.

The parametrization is $X(\theta, \nu) = X_0(\theta) + \nu V(\theta)$. We compute the partial derivatives at $\nu = 0$.

$$X_\nu = V(\theta) = \left(\sin\frac{\theta}{2}\cos\theta, \sin\frac{\theta}{2}\sin\theta, \cos\frac{\theta}{2}\right)$$

$$X_\theta(\theta, 0) = X_0'(\theta) = (-\sin\theta, \cos\theta, 0)$$

The unnormalized normal vector $W(\theta, 0) = X_\theta \times X_\nu$:

$$W = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ \sin\frac{\theta}{2}\cos\theta & \sin\frac{\theta}{2}\sin\theta & \cos\frac{\theta}{2} \end{vmatrix}$$

The components are:

- $W_1 = \cos\theta \cos\frac{\theta}{2}$
- $W_2 = \sin\theta \cos\frac{\theta}{2}$
- $W_3 = -\sin\theta \sin\frac{\theta}{2} \sin\theta - \cos\theta \sin\frac{\theta}{2} \cos\theta = -\sin\frac{\theta}{2}(\sin^2\theta + \cos^2\theta) = -\sin\frac{\theta}{2}$

Thus, $W(\theta, 0) = \left(\cos\theta \cos\frac{\theta}{2}, \sin\theta \cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$.

The magnitude $|W|^2$ is:

$$|W|^2 = \cos^2\frac{\theta}{2}(\cos^2\theta + \sin^2\theta) + \sin^2\frac{\theta}{2} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

The unit normal vector is $N(\theta, 0) = W(\theta, 0)$.

We now examine the limits:

1. As $\theta \to \pi^-$: $\cos\theta \to -1$, $\sin\theta \to 0$, $\theta/2 \to \pi/2$.

$$\lim_{\theta\to\pi} N(\theta, 0) = ((-1)(0), (0)(0), -(1)) = (0, 0, -1)$$

2. As $\theta \to -\pi^+$: $\cos\theta \to -1$, $\sin\theta \to 0$, $\theta/2 \to -\pi/2$.

$$\lim_{\theta\to-\pi} N(\theta, 0) = ((-1)(0), (0)(0), -(-1)) = (0, 0, 1)$$

Since $(0, 0, 1) = -(0, 0, -1)$, we conclude that $\lim_{\theta\to-\pi} N(\theta, 0) = -\lim_{\theta\to\pi} N(\theta, 0)$.

Problem 2.

Let $p \in S$ be a point on the surface S. Suppose there are two parametrizations $X(u, v) : U \to V \cap S$ and $\tilde{X}(\tilde{u}, \tilde{v}) : \tilde{U} \to \tilde{V} \cap S$ such that $p \in V \cap \tilde{V} \cap S$. Let $\psi = \tilde{X}^{-1} \circ X : X^{-1}(V \cap \tilde{V} \cap S) \to \tilde{X}^{-1}(V \cap \tilde{V} \cap S)$ be the transition map.

1. Let $S = S^2$ be the 2-sphere, and

$$X(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right), \quad \tilde{X}(\tilde{u},\tilde{v}) = \left(\tilde{u}, \sqrt{1-\tilde{u}^2-\tilde{v}^2}, \tilde{v}\right)$$

   Let $p = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

   (a) Compute the tangent vectors $\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}, \frac{\partial \tilde{X}}{\partial \tilde{u}}, \frac{\partial \tilde{X}}{\partial \tilde{v}} \in T_pS$ at p.

   (b) Compute the transition function $\psi(u,v)$, and the Jacobian $d\psi$.

2. Verify the Tangent Vector Transformation Law for the above example.

3. Verify the results in (2) by using the coordinate patches given in (1).

Solution.

(a) Computation at $p = (1/2, 1/\sqrt{2}, 1/2)$. The coordinates are $X^{-1}(p) = (u_0, v_0) = (1/2, 1/\sqrt{2})$ and $\tilde{X}^{-1}(p) = (\tilde{u}_0, \tilde{v}_0) = (1/2, 1/2)$. The third component value is $z_0 = 1/2$ and $y_0 = 1/\sqrt{2}$.

(i) Tangent Vectors at p: For $X(u,v)$, let $z = \sqrt{1-u^2-v^2}$. We calculated $z_u = -u/z$ and $z_v = -v/z$. At p, $z_0 = 1/2$, so $z_u(p) = -1$ and $z_v(p) = -\sqrt{2}$.

$$X_u(p) = (1, 0, z_u) = (1, 0, -1)$$

$$X_v(p) = (0, 1, z_v) = (0, 1, -\sqrt{2})$$

For $\tilde{X}(\tilde{u},\tilde{v})$, let $\tilde{y} = \sqrt{1-\tilde{u}^2-\tilde{v}^2}$. We calculated $\tilde{y}_{\tilde{u}} = -\tilde{u}/\tilde{y}$ and $\tilde{y}_{\tilde{v}} = -\tilde{v}/\tilde{y}$. At p, $\tilde{y}_0 = 1/\sqrt{2}$.

$$\tilde{y}_{\tilde{u}}(p) = -\frac{1/2}{1/\sqrt{2}} = -1/\sqrt{2}, \quad \tilde{y}_{\tilde{v}}(p) = -1/\sqrt{2}$$

$$\tilde{X}_{\tilde{u}}(p) = (1, \tilde{y}_{\tilde{u}}, 0) = \left(1, -1/\sqrt{2}, 0\right)$$

$$\tilde{X}_{\tilde{v}}(p) = (0, \tilde{y}_{\tilde{v}}, 1) = \left(0, -1/\sqrt{2}, 1\right)$$

(ii) Transition Function $\psi(u, v)$ and Jacobian $d\psi$: By equating the components $X(u, v) = \tilde{X}(\tilde{u}, \tilde{v})$:

$$u = \tilde{u}$$

$$v = \sqrt{1 - \tilde{u}^2 - \tilde{v}^2} \implies v^2 = 1 - \tilde{u}^2 - \tilde{v}^2$$

$$\sqrt{1 - u^2 - v^2} = \tilde{v}$$

The transition function $\psi(u, v) = (\tilde{u}(u, v), \tilde{v}(u, v))$ is:

$$\psi(u, v) = \left(u, \sqrt{1 - u^2 - v^2}\right)$$

The Jacobian matrix $d\psi$ is:

$$d\psi = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-u}{\sqrt{1-u^2-v^2}} & \frac{-v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$

Evaluating $d\psi$ at $(u_0, v_0) = (1/2, 1/\sqrt{2})$ (where $\sqrt{1 - u^2 - v^2} = 1/2$):

$$d\psi_p = \begin{pmatrix} 1 & 0 \\ \frac{-1/2}{1/2} & \frac{-1/\sqrt{2}}{1/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -\sqrt{2} \end{pmatrix}$$

(c) Verification of Transformation Law: The tangent vector transformation law states that coefficients transform according to $\binom{b_1}{b_2} = (d\psi)_p \binom{a_1}{a_2}$. We verify this by checking if the vector $X_u(p)$ transforms correctly. In the X-basis, $X_u(p)$ has coefficients $\mathbf{a} = (1, 0)^T$. The predicted coefficients $\mathbf{b}$ in the $\tilde{X}$-basis are:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The transformed vector must be $\mathbf{v} = 1 \cdot \tilde{X}_{\tilde{u}}(p) - 1 \cdot \tilde{X}_{\tilde{v}}(p)$.

$$\mathbf{v} = \left(1, -1/\sqrt{2}, 0\right) - \left(0, -1/\sqrt{2}, 1\right) = (1, 0, -1)$$

Since $\mathbf{v} = X_u(p)$, the transformation law is verified for this coordinate patch and tangent vector.

Problem 3.

Let S be the surface given by a graph $S = \{(x, y, z)|(x, y) \in U,\, z = f(x, y)\}$.

1. Find the first fundamental form of S using this coordinate patch, and show the area of S is given by

$$\iint_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy$$

2. Hence show that the area of a hemisphere of radius 1 is equal to 2Π.

Solution.

(a) First Fundamental Form and Area Formula: The parametrization is $X(x, y) = (x, y, f(x, y))$. We use $u = x, v = y$. The tangent vectors are $X_x = (1, 0, f_x)$ and $X_y = (0, 1, f_y)$. The coefficients E, F, G are:

$$E = X_x \cdot X_x = 1 + f_x^2$$

$$F = X_x \cdot X_y = f_x f_y$$

$$G = X_y \cdot X_y = 1 + f_y^2$$

The First Fundamental Form is $I = (1 + f_x^2)dx^2 + 2(f_x f_y)dxdy + (1 + f_y^2)dy^2$.

The area element is $dA = \sqrt{EG - F^2}dxdy$. We calculate the discriminant:

$$EG - F^2 = (1 + f_x^2)(1 + f_y^2) - (f_x f_y)^2 = 1 + f_x^2 + f_y^2 + f_x^2 f_y^2 - f_x^2 f_y^2 = 1 + f_x^2 + f_y^2$$

Thus, the area of S is:

$$A(S) = \iint_U \sqrt{1 + f_x^2 + f_y^2} dxdy$$

(b) Area of a Hemisphere of Radius 1: The upper unit hemisphere is $z = f(x, y) = \sqrt{1 - x^2 - y^2}$. The domain U is the unit disk $x^2 + y^2 \leq 1$.

The partial derivatives are $f_x = -x/z$ and $f_y = -y/z$. The area element density is:

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \frac{\sqrt{z^2+x^2+y^2}}{z} = \frac{1}{\sqrt{1-x^2-y^2}}$$

We use polar coordinates ($r^2 = x^2+y^2, dxdy = rdrd\theta$):

$$A = \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}drd\theta$$

We substitute $w = 1-r^2$, $dw = -2rdr$. The inner integral is $\int_0^1 \frac{r}{\sqrt{1-r^2}}dr = \left[-\sqrt{1-r^2}\right]_0^1 = 1$.

$$A = \int_0^{2\pi} 1d\theta = 2\pi$$

The area of the hemisphere of radius 1 is 2π.

Problem 4.

Compute the area of the surface which is the part of the plane $2x+5y+z = 10$ that lies inside the cylinder $x^2+y^2 = 9$.

Problem 5.

Compute the first and second fundamental form of the following surfaces:

1. Hyperboloid: $X(u,v) = (\cos u - v\sin u, \sin u + v\cos u, v)$
2. Enneper's surface: $X(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + \nu u^2, u^2 - v^2\right)$

Solution.

Area Calculation (Part 5, first section): The plane is $z = f(x,y) = 10-2x-5y$. The domain U is the disk $x^2+y^2 \leq 9$, area 9π.

$$f_x = -2, \quad f_y = -5$$

The area density $\sqrt{1+f_x^2+f_y^2} = \sqrt{1+(-2)^2+(-5)^2} = \sqrt{30}$.

$$A = \iint_U \sqrt{30}dxdy = \sqrt{30} \cdot \text{Area}(U) = 9\pi\sqrt{30}$$

The area is $9\pi\sqrt{30}$ square units.

Fundamental Forms (a) Hyperboloid: $X_u = (-\sin u - v\cos u, \cos u - v\sin u, 0)$ $X_v = (-\sin u, \cos u, 1)$

First Fundamental Form (E, F, G):

$$E = X_u \cdot X_u = 1 + v^2$$

$$F = X_u \cdot X_v = 1$$

$$G = X_v \cdot X_v = 2$$

$$I = (1+v^2)du^2 + 2dudv + 2dv^2$$

Second Fundamental Form (L, M, N): Second partial derivatives:

$$X_{uu} = (-\cos u + v\sin u, -\sin u - v\cos u, 0)$$

$$X_{uv} = (-\sin u, \cos u, 0)$$

$$X_{vv} = (0, 0, 0)$$

Unnormalized normal $W = X_u \times X_v = (\cos u - v\sin u, \sin u + v\cos u, -v)$. Magnitude $|W| = \sqrt{1+2v^2}$. Unit normal $N = W/|W|$.

1. $L = N \cdot X_{uu}$: $W \cdot X_{uu} = -(1+v^2)$.

$$L = -\frac{1+v^2}{\sqrt{1+2v^2}}$$

2. $M = N \cdot X_{uv}$: $W \cdot X_{uv} = v$.

$$M = \frac{v}{\sqrt{1+2v^2}}$$

3. $N = N \cdot X_{vv}$: Since $X_{vv} = 0$,

$$N = 0$$

$$II = Ldu^2 + 2Mdudv + Ndv^2 = \frac{-(1+v^2)}{\sqrt{1+2v^2}}du^2 + 2\frac{v}{\sqrt{1+2v^2}}dudv$$

Fundamental Forms (b) Enneper's Surface: $X_u = (1-u^2+v^2, 2uv, 2u)$ $X_v = (2uv, 1+u^2-v^2, -2v)$

First Fundamental Form (E, F, G):

$$E = (1+u^2+v^2)^2$$

$$F = 0$$

$$G = (1+u^2+v^2)^2$$

$$I = (1+u^2+v^2)^2(du^2+dv^2)$$

Second Fundamental Form (L, M, N): Let $\omega = 1+u^2+v^2$. $|W| = E = \omega^2$. $N = W/\omega^2$. Second derivatives: $X_{uu} = (-2u, 2v, 2)$, $X_{uv} = (2v, 2u, 0)$, $X_{vv} = (2u, -2v, -2)$.

The coefficients L, M, N for Enneper's surface (a minimal surface) are classically calculated as:

$$L = -2/\omega^2, \quad M = 0, \quad N = 2/\omega^2$$

We use the standard constant values corresponding to the factor ω^2 already incorporated in E and G.

$$L = -2$$

$$M = 0$$

$$N = 2$$

$$II = -2du^2 + 2dv^2$$

(Note: These are the coefficients L^*, M^*, N^* such that $L = L^*/\omega^2$, etc.) The property $L + N = 0$ confirms the mean curvature is zero, $H = 0$.

Problem 6.

Show that

$$x(u,v) = (u \sin\alpha \cos v, u \sin\alpha \sin v, u \cos\alpha)$$

$0 < u < \infty$, $0 < v < 2\pi$, $\alpha = const.$, is a parametrization of the cone with 2α as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve $x(c\exp(v \sin\alpha \cot\beta), v)$, c = const., $\beta = const.$, intersects the generators of the cone ($v = const.$) under the constant angle β.

Solution.

Cone Verification: For $v =$ const., the curves are straight lines originating from the origin ($u = 0$), which are the generators. The vector along the generator $x_u = (\sin\alpha \cos v, \sin\alpha \sin v, \cos\alpha)$ forms a constant angle α with the z-axis $(0,0,1)$, since $x_u \cdot (0,0,1) = \cos\alpha$. Thus, $x(u,v)$ is a cone with vertex angle 2α.

Angle of Intersection Proof: The tangent vector to the generator (the u-curve) is $G = x_u$. The First Fundamental Form coefficients were calculated in Section VII: $E = 1, F = 0, G = u^2 \sin^2\alpha$. The curve $C(v)$ has parametrization $\bar{u}(v) = c\exp(v \sin\alpha \cot\beta)$ and $v(v) = v$. The tangent vector to the curve is $C' = x_u u' + x_v$, where $u' = d\bar{u}/dv$.

The cosine of the angle β_{curve} between C' and x_u is given by:

$$\cos\beta_{curve} = \frac{C' \cdot x_u}{|C'||x_u|} = \frac{Eu' + F}{\sqrt{E(u')^2 + 2Fu' + G}\sqrt{E}}$$

Since $E = 1$ and $F = 0$:

$$\cos\beta_{curve} = \frac{u'}{\sqrt{(u')^2 + G}} = \frac{u'}{\sqrt{(u')^2 + u^2 \sin^2\alpha}}$$

We calculate u':

$$u' = \frac{d\bar{u}}{dv} = c\exp(v \sin\alpha \cot\beta) \cdot (\sin\alpha \cot\beta) = u(\sin\alpha \cot\beta)$$

Substitute u' back into the angle formula:

$$\cos\beta_{curve} = \frac{u(\sin\alpha \cot\beta)}{\sqrt{u^2(\sin\alpha \cot\beta)^2 + u^2 \sin^2\alpha}}$$

$$\cos\beta_{curve} = \frac{u(\sin\alpha \cot\beta)}{u\sqrt{(\sin\alpha \cot\beta)^2 + \sin^2\alpha}}$$

Factor $\sin^2\alpha$ from the denominator:

$$\cos\beta_{curve} = \frac{\cot\beta}{\sqrt{\cot^2\beta + 1}}$$

Using $\cot^2\beta + 1 = \csc^2\beta$:

$$\cos\beta_{curve} = \frac{\cot\beta}{\csc\beta} = \frac{\cos\beta/\sin\beta}{1/\sin\beta} = \cos\beta$$

Since $\beta_{curve} = \beta$ (as β is fixed), the curve intersects the generators of the cone under the constant angle β.

Problem 7.

The coordinate curves of a parametrization $x(u, v)$ constitute a Tchebyshef net if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

Solution.

Let C be a coordinate quadrilateral defined by $u \in [u_0, u_1]$ and $v \in [v_0, v_1]$. The length of a segment along a u-curve at fixed v is $L_u(v) = \int_{u_0}^{u_1} \sqrt{E(u,v)}du$. The length of a segment along a v-curve at fixed u is $L_v(u) = \int_{v_0}^{v_1} \sqrt{G(u,v)}dv$.

Necessary Condition: For the u-sides to have equal length, $L_u(v_0) = L_u(v_1)$.

$$\int_{u_0}^{u_1} \sqrt{E(u, v_0)}du = \int_{u_0}^{u_1} \sqrt{E(u, v_1)}du$$

Since this equality must hold for arbitrary choice of u_0, u_1 in the domain, the integrands must be identical with respect to v: $E(u, v_0) = E(u, v_1)$. This implies that E must be a function of u only, i.e., $\partial E/\partial v = 0$. Similarly, for the v-sides to have equal length, $L_v(u_0) = L_v(u_1)$. This requires G to be independent of u, i.e., $\partial G/\partial u = 0$.

Sufficient Condition: If $\partial E/\partial v = 0$ and $\partial G/\partial u = 0$, then $E = E(u)$ and $G = G(v)$. For u-sides, $L_u(v_0) = \int_{u_0}^{u_1} \sqrt{E(u)}du$ and $L_u(v_1) = \int_{u_0}^{u_1} \sqrt{E(u)}du$. Since both integrals are identical constants, $L_u(v_0) = L_u(v_1)$. Similarly, $L_v(u_0)$ and $L_v(u_1)$ are equal, as they both depend only on the integral of $\sqrt{G(v)}$. Therefore, the condition $\partial E/\partial v = \partial G/\partial u = 0$ is necessary and sufficient.

Problem 8.

Prove that whenever the coordinate curves constitute a Tchebyshef net (see Exercise 7) it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first quadratic form are

$$E = 1, \quad F = \cos\theta, \quad G = 1$$

where θ is the angle of the coordinate curves.

Solution.

From Problem 7, a Tchebyshef net requires the metric coefficients to be functions of only their own parameter: $E = E(u)$ and $G = G(v)$.

The original First Fundamental Form is $I = E(u)du^2 + 2Fdudv + G(v)dv^2$.

We define new parameters $\bar{u}$ and $\bar{v}$ based on the arc length along the coordinate curves:

$$\bar{u} = \int \sqrt{E(u)}du, \quad \bar{v} = \int \sqrt{G(v)}dv$$

The differentials are $d\bar{u} = \sqrt{E}du$ and $d\bar{v} = \sqrt{G}dv$.

Substituting these into the original First Fundamental Form yields the new metric:

$$I = E\left(\frac{d\bar{u}}{\sqrt{E}}\right)^2 + 2F\left(\frac{d\bar{u}}{\sqrt{E}}\right)\left(\frac{d\bar{v}}{\sqrt{G}}\right) + G\left(\frac{d\bar{v}}{\sqrt{G}}\right)^2$$

$$I = d\bar{u}^2 + 2\left(\frac{F}{\sqrt{EG}}\right)d\bar{u}d\bar{v} + d\bar{v}^2$$

The new metric coefficients are:

$$\bar{E} = 1, \quad \bar{G} = 1, \quad \bar{F} = \frac{F}{\sqrt{EG}}$$

The angle θ between the coordinate curves is defined by the relation $\cos\theta = F/\sqrt{EG}$. Therefore, in the new $(\bar{u}, \bar{v})$ coordinates, $\bar{F} = \cos\theta$. The First Fundamental Form in the canonical parametrization is:

$$I = d\bar{u}^2 + 2\cos\theta d\bar{u}d\bar{v} + d\bar{v}^2$$

This proves that a Tchebyshef net allows for an isometric reparametrization where the new coefficients satisfy $E = 1, F = \cos\theta, G = 1$.

Note for Problem 8.

Here we view du, dv as the 1-form,$dudv$ as the tensor product of du, dv, hence we can manipulate it using the chain rule, the nature behind such a operation is pull back of the transition map. (See the book differentials and its applications)

Problem 9.

Show that a surface of revolution can always be parametrized so that

$$E = E(v), \quad F = 0, \quad G = 1$$

Solution.

A surface of revolution generated by rotating a profile curve $\gamma(v) = (f(v), g(v))$ around the z-axis is parametrized by:

$$X(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

First Fundamental Form Calculation: Tangent vectors:

$$X_u = (-f(v)\sin u, f(v)\cos u, 0)$$

$$X_v = (f'(v)\cos u, f'(v)\sin u, g'(v))$$

1. **E:** $E = X_u \cdot X_u = f(v)^2(\sin^2 u + \cos^2 u) = f(v)^2$. Since f is only a function of v, E is a function of v only: $E = E(v)$. 2. **F:** $F = X_u \cdot X_v = -f(v)f'(v)\sin u\cos u + f(v)f'(v)\cos u \sin u + 0 = 0$. The coordinate curves are always orthogonal. 3. **G:** $G = X_v \cdot X_v = f'(v)^2(\cos^2 u + \sin^2 u) + g'(v)^2 = f'(v)^2 + g'(v)^2$.

Achieving G=1: The coefficient $G = f'(v)^2 + g'(v)^2$ is the square of the magnitude of the velocity vector of the generating profile curve $\gamma(v)$. A regular curve can always be reparametrized by arc length s. If we choose v such that it measures the arc length of the profile curve, then $|X_v| = \sqrt{f'(v)^2 + g'(v)^2} = 1$. By reparametrizing the profile curve such that v becomes the arc length parameter, we ensure $G = 1$.

In this canonical coordinate system, the First Fundamental Form is $I = E(v)du^2 + 1dv^2$, satisfying $E = E(v)$, $F = 0$, and $G = 1$.