MAT4033 Differential Geometry

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November 11, 2025

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Foreward

Introduction

In this course we study curves and surfaces in $\mathbb{R}^3$

What are we interested in?

- How we decribe a curve?
  use the parametrization: $\alpha : I \to \mathbb{R}^3$

- How much information can we get from α?
  length, curvature, torsion

- If we know the curvature of a curve of evrey point, can we describe the curve?

- Some "global" problems: Suppose we have a closed curve in $\mathbb{R}^3$ of a given length, what is the largest possible area bounded by the curve?

- How do we describe a surface? More precisely, what kind of surface should we study?

- For a regular surface, how can we study the area and curvature of them?

- What is the "shortest" between two points of a surface?

- What is the relationship between geometry and topology on surfaces? (Gauss-Bonnet Theorem)

1 Local Theory of Curves

Definition 1.1 (regular curve). $\alpha : [c,d] \to \mathbb{R}^3$ is called a regular curve if $|\alpha'| \neq 0$, (α is taken to be differentiable (smooth))

Proposition 1.1. *Let $\alpha : [c,d] \to \mathbb{R}^3$ be a given curve with partition $P = \{c = t_0 < t_1 < \cdots < t_n = b\}$. Let $l(\alpha, P) := \sum |\alpha(t_{i+1}) - \alpha(t_i)|$, then we can have*

$$\int_c^d |\frac{d\alpha}{dt}| dt = \sup\{l(\alpha, P)|P \text{ any partition}\}$$

Proof. The proof is straight forward. $\qquad\qquad\square$

Proposition 1.2 (length is invariant under reparametriazation). *Given a differntiable curve $\alpha : [a,b] \to \mathbb{R}^3$ and a $g : [c,d] \to [a,b]$ with $\beta = \alpha \circ g$, then*

$$\int_a^b |\frac{d\alpha}{dt}| dt = \int_c^d |\frac{d\beta}{ds}| ds$$

Proof. also straight forward, by simple chain rule. $\qquad\qquad\square$

There are many ways to parametrize the curves. But we always want to pick one such that the *pointers* is unit.

Remark 1.1. The cusp is not a regular curve since $\alpha'(0) = 0$, but we allow self-intersections, in which case a regular curve may have different tangent at the same point.

1.1 Reparametrization

Consider $\alpha_1(t) = (\cos t, \sin t), \alpha_2(t) = (\cos(2t), \sin(2t))$, both give the same trace with different parameters.

Definition 1.2. A reparametrizastion of a curve $\alpha : (a,b) \to \mathbb{R}^3$ is a bijective function $g : (c,d) \to (a,b)$ such that g is C^∞ and g^{-1} is C^1. Given a raparametrization g, one can define a new "curve" $\beta : (c,d) \to \mathbb{R}^3$ by $\beta = \alpha \circ g$

Example 1.1 (nonexample). $g(s) = s^3$, we can check that g^{-1} is not continuously differentiable in any interval contain 0.

Proposition 1.3. *If α is a regular curve and g is a reparametrization, then $\beta(s) = \alpha(g(s))$ is also regular.*

Proof. left as an exercise. $\qquad\qquad\square$

Remark 1.2. To check whether the given smooth g is a reparametrization or not, one only needs to check if g' is nonzero or not by Implicit Function Theorem(IVT).

Proposition 1.4. *The tangent vector of a regular curve is invariant (possibly reverse its direction) under any parametrization.*

Proof. left as an exercise. $\qquad\qquad\square$

1.2 Arc-length

Definition 1.3. The length of the curve segment $[c,d] \subseteq I$ for $\alpha : I \to \mathbb{R}^3$ regular is iven by

$$\int_\alpha ds = \int_c^d |\alpha'(t)| dt$$

Proposition 1.5. *Reparametrization does not affect the arc-length*

Proof. left as an exercise $\square$

To develop the theory of curves, we always want our curves to be parametrized by arc-length(p.a.l), or in other words, it is more convenient to work with curves with unit "pointers". The following discussion shows that we can always assumed the curve to be a p.a.l curve.

Lemma 1.6. *Let $\alpha : [a,b] \to \mathbb{R}^3$ regular and $t_0 \in [a,b]$ consider $h(t) = \int_{t_0}^t |\alpha'(t)| dt$. Then h is a reparametrizaton.*

Proof. left as an exercise $\square$

Definition 1.4 (arc-length parametrization). Let $\alpha : [a,b] \to \mathbb{R}^3$ be regular curve and $t_0 \in I$ with $g = h^{-1}$ with h is given in the lemma above. Then $\beta = \alpha \circ g : [c,d] \to \mathbb{R}^{\Vdash}$ is parametrization by arc-length

Proposition 1.7. *β the curve defined as above, then $\beta'(s)$ is of unit length.*

Remark 1.3. In theory, there is a reparametrization by arc length for a regualr curves. However, it can be hard to explicitly find such a parameter.

- The formula s=h(t) may not have a closed formula

- Even we have a closed formula for h(t) in some cases, the inverse of g is hard to find.

1.3 Frenet Frames of Plane curves

In this section, we assume α to be p.a.l, and alpha is a plane curve mapped from an interval to $\mathbb{R}^2$. Recall we have $\alpha'(s) = t(s)$

Definition 1.5. The normal vector $n(s)$ of a plane curve at t=s is the vector $n(s) := J(t(s))$, where J is the 90 degree counterclockwise rotation. The set {t(s),n(s)} forms an ordered, oriented basis of $\mathbb{R}^{\Vdash}$ which is orthonormal. It is called the Frenet Frame of α.

Remark 1.4. We will have a slightly different definition of normal vector for general space curve.

Note that $\langle t(s), t(s) \rangle = 1 \Rightarrow \langle t'(s), t(s) \rangle$, which implies $t'(s) = k(s)n(s)$ for some scalar $k(s)$

Definition 1.6. The curvature of α at s is the value k(s)

Remark 1.5. • $k(s)$ records the change of t(s)

- $k(s) > 0$, $t(s)$ is turning towards n(s)

- $k(s) < 0$, $t(s)$ is turning away form n(s)
For space curves, it is not natural to define the ordered basis in this manner.

Example 1.2 (circle). $\alpha(s) = u + r(\cos(\frac{s}{r}), \sin(\frac{s}{r}))$, do the computation we will see the curvature of $\alpha(s)$ is $\frac{1}{r}$

Note that if we raparametrize the curve with -t then the curvature will change its sign.

The following Proposition provides us a formula to compute the curvature when our curve is not parametrized by arc length initially. (Also, one can show that curvature is independent of choice (up to a sign) of parameter using the formula)

Proposition 1.8 (Formula to compute the curvature). *Let $\alpha : I \to \mathbb{R}^3$ regular plane curve with $\beta = \alpha \circ g$ p.a.l, then the curvature of α (whcih is defined to be the curvature of β is*

$$k_\alpha(t) = \frac{det|\alpha'(t), \alpha''(t)|}{|\alpha'(t)|^3}$$

Proof. left as an exercise (Hint: by direct computation and chain rule, it takes about 20-30 minutes work) $\qquad\square$

It seems that for a regular plane curve, the configuration of the curve is uniquely determined by its curvature. In fact we have the following theorem

Theorem 1.9 (Fundamental theorem of Plane curve).   • *For any C^∞ function $k : I \to \mathbb{R}^3$, there exists a p.a.l curve $\alpha : I \to \mathbb{R}^3$ such that $k_\alpha(s) = k(s)$ and $\alpha(0) = x_0$*

    • *The curve satisfy the $k_\alpha(s) = k(s)$ is unique up to a rigid motion.*

You may wonder what is a rigid motion, We have the following definition

Definition 1.7 (rigid motion). We say a map $\mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion if it is of the form $x \to Ax + b$ with $A \in O(n)$. If A is in the special orthogonal group, we called the rigid motion orientation preserving.

In fact, one can show that the isometry in the Euclidean space is exactly all the rigid motion.

Proof. $\qquad\square$

1.4   Space Curve

Definition 1.8 (curvature).

Remark 1.6. n(s) is simply along the same direction of t' unlike the case of plane curves.

The Frenet-Serret Frames of a space curve is the orthonormal basis $\{t, n, b\}$, where b is the derivative of n.

As in the plane curve, we want to study the derivatives of $\{t, n, b\}$ where b is defined to be the cross product of t and n.

- $\langle n', n \rangle = 0 \Rightarrow n' = at + kb$

- $\langle b', b \rangle = 0 \Rightarrow b' = ct + dn$

- and then note that $b = t \times n \Rightarrow b' \perp b, b' \perp t$

- hence we have $b'(s) = \tau(s)n(s)$

Definition 1.9 (torsion). Let α regular p.a.l curve, with $k(s) \neq 0$, the torsion of α is given by the above discussion.

Remark 1.7. $\tau(s)$ tells us how far the curve is away from a plane curve.

Proposition 1.10 (Frenet Serret Frame). *Like the plane curve case, we have a formula to connect*

t, n, b and the derivative t', n', b'

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\kappa(s)$ is the curvature of the curve,$\tau(s)$ is the torsion of the curve.

Remark 1.8. Note that there is two conventions on the sign of the torsion τ

Definition 1.10. If α is any regular cuve, then the Frennet-Serret Frame is obtained by that of $\beta = \alpha \circ s($ (the arc-length reparamatrization)

Theorem 1.11. *Let α be regular curve p.a.l with $\kappa(s) \neq 0$ then the following are equivalent*

1. *α is a plane curve*

2. *b is a constant*

3. *$\tau(s) = 0$*

Proof.
- $(2) \Leftrightarrow (3)$ follows from the frenet serret equation.

- $(1) \Leftrightarrow (2)$ Suppose α be a plane curve, then b(s) is clearly a constant. Conversely, if b is a constant, then $b' = 0$, consider $\langle \alpha(s) - \alpha(0), b \rangle' = 0$ now since b is a constant, then α is indeed a plane curve. $\square$

We end by applying the Frenet Serret Equation.

Example 1.3. α is a straight line if and only if $\exists x_0 \in \mathbb{R}^3$, such that every tangent line to α passes through x_o

Proof. left as homework $\square$

Example 1.4. Let the normal plane of $\alpha(s)$ be the plane perpendicular to $t(s)$. Let α be a regular curve p..a.l such that every normal plane of $\alpha(s)$ passes through a fixed point of x_0. Then $\alpha(s)$ lie s on a sphere.

Proof. The normal plane of $\alpha(s)$ has the equation $\langle t(s), x - \alpha(s) \rangle = 0$, since the x_0 lies on the plane and hence we have $\langle t(s), x_0 - \alpha(s) \rangle = 0$ Consider taking the derivative of $\langle x_0 - \alpha(s), x_0 - \alpha(s) \rangle$, eh derivative is zero andhence this is a constatn, andthen we are done. $\square$

1.5 Fundamental Theorem of Curves

Recall that all plane curves are determined by its curvature up to orientation preserving its rigid motion. The same goes for regular space curves $\alpha(s)$ with $\kappa(s) > 0$

Theorem 1.12. *Given two functions $\kappa(s), \tau(s)$ defined on an interval, with $\kappa(s) > 0$, $x_0 \in \mathbb{R}^3$ fixed.$\{D, E, F\}$ be an orthonormal basis of $\mathbb{R}^3$, then there exists a p.a.l curve such that the curvature and torsion of α is exactly the given function $\kappa(s)$ $\tau(s)$. What's more, α is completely determined by its curvature and torsion, in the sense that any curve with the same curvature and torsion differs only be a rigid motion.*

Proof. For detailed proof, see Do carmo's appendix. Here we sketch the proof.

- Existence:

1. First construct the differential equation according to the Frenet Serret Formula, and get a candidate for the tangent vector, normal vector, and the binormal vector.

2. Now check that the t, n, b we get from the above equation is orthonormal for all s. This is equivalent to $\langle t, t \rangle, \langle t, n \rangle, \langle t, b \rangle, \langle n, n \rangle, \langle n, b \rangle, \langle b, b \rangle$ is equal to 1,0,0,1,0,1. To conclude this, we consider another differential equation and use the uniqueness of the ode.

3. Then we consider the curve $\alpha(s) = x_0 + \int t(s)ds$, and check it indeed have the two given function as curvature and torsion.

- Uniqueness: we move one of the curve α' by rigid motion to α with the same starting point and the same initial tangent vector, and then we are done by the uniqueness of ode.

$\square$

2 Global Theory of Curves

In this section we would like to study the theory of plane curves globally.

An important question in geometry is the following. Suppose we know the local property of a curve, can we say some thing about the global properties of it?

2.1 Hopf's Umlaufsatz

Definition 2.1. The rotation index of $\alpha : I \to \mathbb{R}^3$ with period is given by

$$\frac{\theta(a) - \theta(0)}{2\pi} = \frac{1}{2\pi} \int_0^a \theta'(s)ds$$

. where $\theta(s)$ is defined by $t(s) = (\cos(\theta(s)), \sin(\theta(s)))$.

Theorem 2.1 (Hopf's Umlaufsatz). *Suppose α is a simple closed curve, then its rotation index must be equal to ± 1. Moreover, since we have $\theta'(s) = \kappa(s)$, we have $\int \kappa(s)ds = \pm 2\pi$.*

Proof. Let the curve $\alpha : [0, L] \to \mathbb{R}^2$ be a simple closed curve parameterized by arc length s. The unit tangent vector is $T(s) = \alpha'(s)$, and its angle with a fixed axis is $\theta(s)$. The total curvature is the net change in this angle, $\theta(L) - \theta(0)$. Since the curve is closed, $T(L) = T(0)$, which implies $\theta(L) - \theta(0) = 2\pi n$ for some integer n. We will show that $n = \pm 1$.

As suggested by the proof sketch, we introduce the unit secant vector field, which maps two points on the curve to the unit vector connecting them:

$$u(s_1, s_2) = \frac{\alpha(s_2) - \alpha(s_1)}{||\alpha(s_2) - \alpha(s_1)||}$$

This map is smooth provided $s_1 \neq s_2$, which is guaranteed for a simple curve. Let $\phi(s_1, s_2)$ be the angle of this secant vector.

This angle function is defined on the domain $[0, L] \times [0, L]$ excluding the diagonal. We consider the triangular region $D = \{(s_1, s_2) \mid 0 \leq s_1 \leq s_2 \leq L\}$. Since ϕ is a smooth, well-defined function on the interior of D, its exterior derivative $d\phi$ is an exact 1-form. By Green's theorem (or Stokes' theorem for forms), the integral of $d\phi$ around the boundary ∂D must be zero, provided we handle the singular diagonal edge properly.

$$\oint_{\partial D} d\phi = 0$$

The boundary ∂D consists of three segments. A careful limiting argument shows that the integral of $d\phi$ over the boundary is the sum of the changes in angle along the limiting paths.

1. **The diagonal path ($s_1 = s_2$):** As $s_2 \to s_1$, the secant vector $u(s_1, s_2)$ limits to the tangent vector $T(s_1)$. Thus, the angle $\phi(s, s)$ corresponds to the tangent angle $\theta(s)$. The change in angle along this path is the total curvature: $\Delta\phi_{diag} = \theta(L) - \theta(0)$.

2. **The 'start' path ($s_1 = 0$):** This path corresponds to the secant vector $u(0, s_2)$ from the fixed starting point $\alpha(0)$ to a moving point $\alpha(s_2)$. Let the total change in its angle be $\Delta\phi_{start} = \phi(0, L) - \phi(0, 0)$.

3. **The 'end' path ($s_2 = L$):** This path corresponds to the secant vector $u(s_1, L)$ from a moving point $\alpha(s_1)$ to the fixed end point $\alpha(L) = \alpha(0)$. This vector is the opposite of the 'start' vector: $u(s_1, L) = -u(0, s_1)$. Thus their angles are related by $\phi_{end}(s_1) = \phi_{start}(s_1) \pm \pi$. The total change in angle along this path is $\Delta\phi_{end} = \Delta\phi_{start}$.

The relationship between these changes in angle, derived from the boundary integral being zero, is:

$$\theta(L) - \theta(0) = \Delta\phi_{start} + \Delta\phi_{end}$$

Substituting $\Delta\phi_{end} = \Delta\phi_{start}$, we get:

$$\theta(L) - \theta(0) = 2\Delta\phi_{start}$$

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Now we must evaluate $\Delta\phi_{start}$. This quantity represents the total rotation of the secant line anchored at $\alpha(0)$ as its other end traverses the entire curve. Since the curve α is simple and closed, it encloses a region. The secant vector starts in the direction of the tangent $T(0)$ and, as it sweeps through the interior of the curve, ends by pointing in the opposite direction, $-T(0)$. This corresponds to a net rotation of exactly half a circle. Therefore, the change in angle is:

$$\Delta\phi_{start} = \pm\pi$$

Substituting this result into our equation gives the final result:

$$\int_0^L \kappa(s)\,ds = \theta(L) - \theta(0) = 2(\pm\pi) = \pm 2\pi$$

This shows that the rotation index n must be either $+1$ or -1. $\qquad\square$
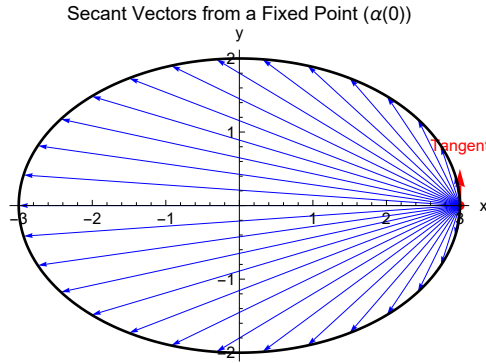

Figure 1: Illustration of secant vectors from a fixed point on a closed curve, visualizing the $\Delta\phi_{start}$ component in the proof of Hopf's Umlaufsatz.

Remark 2.1. The theorem can be viewed as a 1-dimensional version of Gauss-Bonnet Formula.

2.2 Isoperimetric Inequality

Theorem 2.2 (Isoperimetric Inequality)**.** *Let α be a simple closed plane curve. Let A be the area bounded by curves, then $L^2 \geq 4\pi A$*

Proof. The proof compares the area of the given curve $\mathcal{C}$ to that of a circle $\mathcal{S}^1$ by relating their parametrizations.

Step 1: The Setup. Let $\mathcal{C}$ be a simple closed curve of length L enclosing area A. We can parameterize $\mathcal{C}$ by arc length, $\gamma(t) = (x(t), y(t))$ for $t \in [0, L]$. A key property of this parametrization is that the tangent vector has unit length:

$$(x'(t))^2 + (y'(t))^2 = 1$$

We construct a circle $\mathcal{S}^1$ of an arbitrary radius r centered at the origin. We then define a new curve, $\beta(t) = (x(t), \tilde{y}(t))$, which traces this circle using the same x-coordinate as $\gamma(t)$. To ensure $\beta(t)$ remains on the circle, we must have $\tilde{y}(t) = \pm\sqrt{r^2 - x(t)^2}$.

Step 2: Combining the Area Formulas. Using Green's Theorem, we can express the areas as line integrals:

- The area of $\mathcal{C}$ is $A = \int_0^L x(t)y'(t)\,dt$.

- The area of the circle $\mathcal{S}^1$ is $\pi r^2 = -\int_0^L \tilde{y}(t)x'(t)\,dt$.

Adding these two expressions for area yields:

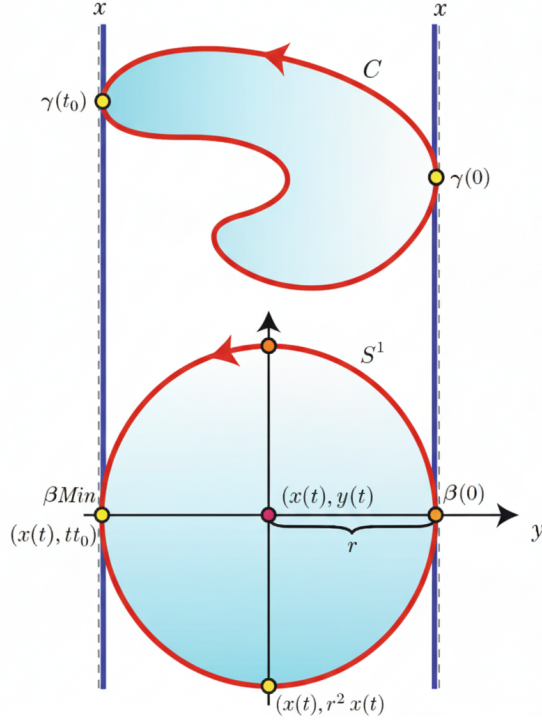$$A + \pi r^2 = \int_0^L \left(x(t)y'(t) - \tilde{y}(t)x'(t) \right)\,dt$$

Figure 2: Geometric construction for the proof of the Isoperimetric Inequality. Curve C (red) and constructed circle S^1 (blue) sharing x-coordinates.

Step 3: Applying the Cauchy-Schwarz Inequality. The integrand can be interpreted as the dot product of two vectors, $\mathbf{v}_1(t) = (x(t), -\tilde{y}(t))$ and $\mathbf{v}_2(t) = (y'(t), x'(t))$. We apply the Cauchy-Schwarz inequality for integrals, which states $\int \mathbf{v}_1 \cdot \mathbf{v}_2 \, dt \leq \int ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \, dt$.

$$A + \pi r^2 \leq \int_0^L \sqrt{x(t)^2 + (-\tilde{y}(t))^2} \cdot \sqrt{y'(t)^2 + x'(t)^2} \, dt$$

The terms in the integral simplify significantly:

- Since $\beta(t)$ is on the circle, $\sqrt{x(t)^2 + \tilde{y}(t)^2} = \sqrt{r^2} = r$.

- Since $\gamma(t)$ is parameterized by arc length, $\sqrt{y'(t)^2 + x'(t)^2} = 1$.

Substituting these back, the inequality becomes:

$$A + \pi r^2 \leq \int_0^L r \cdot 1 \, dt = rL$$

Step 4: Finding the Optimal Bound. We have derived the relation $A \leq rL - \pi r^2$, which must hold for any choice of $r > 0$. To find the tightest possible bound for A, we find the value of r that maximizes the function $f(r) = rL - \pi r^2$. Taking the derivative with respect to r and setting it to zero gives:

$$f'(r) = L - 2\pi r = 0 \implies r = \frac{L}{2\pi}$$

This is the radius of a circle with perimeter L. Substituting this optimal radius back into our inequality yields:

$$A \leq \left(\frac{L}{2\pi}\right) L - \pi \left(\frac{L}{2\pi}\right)^2$$

$$A \leq \frac{L^2}{2\pi} - \pi \left(\frac{L^2}{4\pi^2}\right)$$

$$A \leq \frac{L^2}{2\pi} - \frac{L^2}{4\pi}$$

$$A \leq \frac{L^2}{4\pi}$$

Rearranging this gives the isoperimetric inequality: $4\pi A \leq L^2$.

The Equality Case. Equality holds if the Cauchy-Schwarz inequality becomes an equality, which occurs if and only if the vectors $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are parallel for all t. This means $\mathbf{v}_1(t) = k\,\mathbf{v}_2(t)$ for some scalar k. The magnitudes must also be equal, so $|k| \cdot \|\mathbf{v}_2\| = \|\mathbf{v}_1\|$, which gives $|k| \cdot 1 = r$, so $k = \pm r$.

This implies that $x(t) = \pm r y'(t)$ and $-\tilde{y}(t) = \pm r x'(t)$. For the parametrizations to match, the original curve $\gamma(t)$ must itself be a circle of radius r. Since the optimal radius was $r = L/(2\pi)$, equality holds if and only if $\mathcal{C}$ is a circle. $\qquad\square$

2.3 Four vertex Theorem (Covered in Tutorial)

We begin by defining the key concepts required for the theorem and its proof. Let $\gamma(t) = (x(t), y(t))$ be a regular plane curve.

Definition 2.2 (Signed Curvature). For a regular plane curve $\gamma(t)$ parameterized by arc length s, its velocity vector is $\mathbf{v}(s) = \gamma'(s)$ and its acceleration is $\mathbf{a}(s) = \gamma''(s)$. The signed curvature, $k_s(s)$, is defined by the relation $\mathbf{a}(s) = k_s(s) R_{90}(\mathbf{v}(s))$, where R_{90} is a 90-degree counter-clockwise rotation. For a general parameter t, the curvature equations are:

$$x''(t) = -k_s(t) y'(t) \tag{1}$$
$$y''(t) = k_s(t) x'(t) \tag{2}$$

Definition 2.3 (Vertex). A point $\gamma(t)$ on the trace of a regular plane curve is called a **vertex** if the signed curvature function k_s has a local maximum or a local minimum at t.

Definition 2.4 (Convex Curve). A simple closed plane curve is called **convex** if its trace lies entirely on one closed side of each of its tangent lines.

Now we state the four vertex theorem.

Theorem 2.3 (The Four Vertex Theorem). *Every simple closed plane curve has at least four vertices.*

Note: The following proof addresses the special case where the curve is convex.

The proof relies on the following geometric property of convex curves.

Lemma 2.4. *Let C be the trace of a simple closed convex plane curve, and let L be a line. If L intersects C in at least three points, then C must contain the entire line segment of L between any pair of these points.*

A direct consequence is that if a line is not part of the curve, it can intersect a convex curve at most twice.

Proof of Theorem 2.3 for Convex Curves. The proof is by contradiction.

Step 1: The Contradiction Hypothesis. Assume there exists a simple, closed, convex curve γ with fewer than four vertices. Since the continuous curvature function k_s on a closed curve must attain a global maximum and a global minimum, the curve must have at least two vertices. As local maxima and minima must alternate, the number of vertices must be even. Therefore, our assumption implies that γ has exactly two vertices:

- A point p where k_s attains its global maximum.

- A point q where k_s attains its global minimum.

Step 2: Geometric Setup. Let L be the line passing through the points p and q. By Lemma 2.4, since the curve is convex and not a line segment, L can intersect the curve only at points p and q.

We can apply a rigid motion (translation and rotation) to the curve to simplify our coordinate system without changing its geometric properties. We set up the coordinates such that:

- The point p is at the origin, $p = (0,0)$.

- The line L is the x-axis.

- The point q is on the positive x-axis, $q = (a, 0)$ for some $a > 0$.

Let the curve be parameterized by $\gamma(t) = (x(t), y(t))$ for $t \in [0, l]$.

Step 3: Analysis of the Functions. From our setup, the entire curve lies on one side of the x-axis. We can assume without loss of generality that $y(t) \geq 0$ for all $t \in [0, l]$. Furthermore, for a positively oriented (counter-clockwise) simple closed convex curve, the signed curvature $k_s(t)$ is strictly positive.

Therefore, the product of these two functions, $y(t)k_s(t)$, must be non-negative for all t, and it is only zero at the points p and q.

Step 4: The Contradiction. We analyze the integral of the product $y(t)k_s(t)$ over the entire curve.

1. **The integral must be positive.** Since $y(t)k_s(t) \geq 0$ for all t and is not identically zero, its integral must be strictly positive:
$$\int_0^l y(t)k_s(t)\, dt > 0$$

2. **The integral must be zero.** However, as stated in the textbook's argument, integrating by parts and using the curvature equations from (1) and (2) shows that the integral is zero. A key part of such a derivation involves using the fact that for a closed curve, the integral of an exact derivative is zero. For example, using equation (2), $y''(t) = k_s(t)x'(t)$, we know:

$$\int_0^l k_s(t)x'(t)\, dt = \int_0^l y''(t)\, dt = [y'(t)]_0^l = 0$$

The textbook asserts that a similar line of reasoning (though not fully detailed) leads to the conclusion that our original integral is zero.

We have reached a contradiction: the integral cannot be both strictly positive and equal to zero. Thus, our initial assumption that the curve has fewer than four vertices must be false. This completes the proof. □

3 Local Theory of Surfaces

3.1 Introduction

Definition 3.1. A **parameterized surface** is a function $X : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ where Z is differentiable. We say Z is **regular** if the Jacobian matrix

$$dX = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank 2 for all points in U. In coordinates:

$$X(u, v) = (x(u, v), y(u, v), z(u, v))$$

Equivalently, $Z : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is an embedding. The image $Z(U)$ is called the **trace** of Z.

Remark 3.1. For regular curves $\alpha : I \to \mathbb{R}^3$, we require $\alpha' \neq 0$. For surfaces, the analogous condition is that the differential has full rank.

Example 3.1. Consider the parameterization:

$$X : (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$$

$$X(\theta, \varphi) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi)$$

This parameterization Z is regular because

$$\frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \varphi} \neq 0$$

and the Jacobian matrix has rank 2.

As in the case of curves. We wish to parametrize a surface to do Calculus on surfaces. Of course we need two variables rather than 1 variable in the curve case.

Remark 3.2 (Problem of the parametrized surface). The above example of S^2 reveals that in order to develop a unified theory for studying surfaces, it is necessary to use more than one coordinate patch. Now we would like to introduce the concept of regular surface.

Definition 3.2. A subset $S \subset \mathbb{R}^3$ is a **regular surface** if for every point $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$ of p (which can be taken as an open ball centered at p with radius $r > 0$) and a map $X : U \to V \cap S$ where $U \subset \mathbb{R}^2$ is open and connected, such that:

1. X is differentiable:

    $$X(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U$$

    has partial derivatives of all orders.

2. X is bijective with continuous inverse X^{-1}, i.e., X is a homeomorphism.

3. (Regularity) For $q \in U$ with $X(q) = p$, the Jacobian matrix

    $$dX_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

    has rank 2 for all $q \in U$.

Remark 3.3. Conditions (2) and (3) together guarantee that X is an embedding from U to S, ensuring the existence of a tangent plane at each point of S. This is analogous to the requirement $\alpha' \neq 0$ for

regular curves.

Example 3.2. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere. We can cover S^2 using 6 coordinate patches. For example, let

$$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

and define the coordinate patch for the upper hemisphere:

$$X(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right), \quad (x, y) \in U$$

And then we can do the same for other coordinates.

Example 3.3. Suppose $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a smooth map. Then the graph of f defined as

$$S = \operatorname{graph}(f) = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in U, z = f(x, y)\}$$

is a regular surface.
Homework: Check the graph is indeed a regular surface.

The above example shows that graph is a regular surface, and indeed the converse is in some sense true. Every regular surface is Locally a graph.

Proposition 3.1. *Any regular surface S is locally a graph.*

Proof. Let $p \in S$ be any point and let $X : U \to V \cap S$ be the coordinate patch given in the definition of regular surfaces, with $X(q) = p$. By the regularity condition, the differential

$$dX_q = \begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \end{pmatrix}$$

has rank 2. This means that among the three possible 2×2 Jacobian matrices:

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

at least one has nonzero determinant.

Without loss of generality, assume the first Jacobian matrix has nonzero determinant. Then by the inverse function theorem, the projection

$$\pi \circ X : (u, v) \mapsto (x(u, v), y(u, v))$$

has a differentiable inverse in some neighborhood of q.

Let $g = (\pi \circ X)^{-1}$ be this inverse mapping, which sends $(x, y) \mapsto (u(x, y), v(x, y))$. Then we have

$$(x, y) \to (u, v) \to (x, y, z(x(u, v), y(u, v)))$$

By the formulas we see that indeed it is locally a graph. $\qquad \square$

Remark 3.4. The above proof is a routine technique to apply the inverse function/implicit function theorem in geometry.

Example 3.4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth map and $a \in \mathbb{R}$ be such that for all $p \in f^{-1}(a)$, the differential $(df)_p \neq 0$ (i.e., a is a regular value of f). Then $f^{-1}(a) \subset \mathbb{R}^3$ is a regular surface.

Proof. For any $p \in f^{-1}(a)$, suppose without loss of generality that $\left(\frac{\partial f}{\partial z}\right)_p \neq 0$.

Define the function $F : \mathbb{R}^3 \to \mathbb{R}$ by:
$$F(x) = f(x) - a$$

Then $F(p) = 0$.

Since $\left(\frac{\partial F}{\partial z}\right)_p = \left(\frac{\partial f}{\partial z}\right)_p \neq 0$, by the implicit function theorem, there exists a smooth function $g : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that:

$$F(u, g(u)) = 0 \quad \forall u \in U$$

where U is an open neighborhood in $\mathbb{R}^2$.

This means we can express a neighborhood of $p \in f^{-1}(a)$ as:

$$W = \{(u, g(u)) \mid u \in U\}, \quad \text{with } p = (u_0, g(u_0))$$

Therefore, there exists a neighborhood V such that:

$$V \cap f^{-1}(a) = \{(x, y, z) \mid (x, y) \in U, z = g(x, y)\}$$

This shows that every point $p \in f^{-1}(a)$ is locally a graph, hence by the previous proposition about graphs of functions being regular surfaces, $f^{-1}(a)$ is a regular surface. $\qquad\square$

Example 3.5. Consider the ellipsoid defined by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a, b, c > 0$ are constants.

Proof. Define the function $F : \mathbb{R}^3 \to \mathbb{R}$ by:

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Then:

1. F is obviously differentiable (in fact, smooth) on $\mathbb{R}^3$

2. The differential of F is given by:

$$dF = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$$

3. For any point $w = (w_1, w_2, w_3) \in F^{-1}(0)$ (i.e., on the ellipsoid), we have:

$$dF(w) = \left(\frac{2w_1}{a^2}, \frac{2w_2}{b^2}, \frac{2w_3}{c^2}\right)$$

   Since w lies on the ellipsoid, at least one coordinate must be nonzero, hence $dF(w) \neq 0$. Therefore, 0 is a regular value of F.

By the regular value theorem, $F^{-1}(0)$ is a regular surface. $\qquad\square$

Remark 3.5. For the sphere S^2, it is possible to cover the surface using only 2 coordinate patches. See Streographic Peojection.

Example 3.6 (Stereographic Projection)**.** See Homework.

3.2   Change of Coordinates

Recall that for curves, we may have different parametrizations. The same holds for regular surfaces. We say that two parametrizations describe the same surface even if they use different coordinate systems.

Proposition 3.2. *Let $p \in S$ be a point in the image of two parametrizations:*

$$X : U \to S \quad and \quad Y : V \to S$$

Let $W = X(U) \cap Y(V)$ be the overlapping region. Then the transition map

$$\varphi = X^{-1} \circ Y : Y^{-1}(W) \to X^{-1}(W)$$

defined by $(u, v) \mapsto (u', v')$ is a diffeomorphism.

Proof. Since both X and Y are regular parametrizations, they are homeomorphisms onto their images and their differentials have full rank. Therefore:
1. φ is well-defined as a map between open subsets of $\mathbb{R}^2$
2. φ is a homeomorphism because it is the composition of homeomorphisms
3. φ is differentiable because both X and Y are differentiable and regular
4. The inverse $\varphi^{-1} = Y^{-1} \circ X$ is similarly differentiable

The Jacobian matrix of the transition map is given by:

$$d\varphi = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{pmatrix}$$

This matrix is invertible everywhere because φ is a diffeomorphism. $\square$

Remark 3.6. This property is fundamental in manifold theory. It ensures that geometric properties defined using different coordinate systems are consistent. The transition maps allow us to define a smooth structure on the surface S by specifying how different coordinate patches relate to each other. This is exactly the property we aim to generalize in the theory of smooth manifolds, where we use charts and transition maps to define the smooth structure.

3.3 Tangent plane

Definition 3.3. Let $\mathbf{x} : U \to$ VAS be a coordinate patch of a regular surface at a pt $\mathbf{x}(p) = p \in S$. Then the tangent plane at $p \in S$ is defined as

$$T_p S = \text{Im}\,((\mathrm{d}\mathbf{x})_p) = \text{span}\left\{ \frac{\partial \mathbf{x}}{\partial u}(p), \frac{\partial \mathbf{x}}{\partial v}(p) \right\}.$$

Definition 3.4. The normal vector of a pt p on a coordinate patch $\mathbf{x} : U \to$ VAS is defined as

$$N(p) = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left\| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right\|} \quad (p)$$

Problem: although the tangent plane is well-defined, but $N(p)$ may be affected by our choice of coordinates. (the sign).

Definition 3.5. A tangent vector at $p \in S$ is the tangent vector $\alpha'(0)$ of a regular curve $\alpha : (-\varepsilon, \varepsilon) \to S \subseteq \mathbb{R}^3$ with $\alpha(0) = p$.

Proposition 3.3. *The set of tangent vectors at $p \in S$ is precisely equal to $T_p S$.*

Proof. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be regular with $\alpha(0) = p$. Recall that $\exists\, V$ of $p \in S$ such that by restricting VAS to $V \cap S$, we have $\mathbf{y} : U' \to V \cap S$ such that $\mathbf{y}(u', v') = (u', v', f(u', v'))$ or $(u', g(u', v'), v')$ (locally a graph).

we look at the first case. w.l.o.g. by making ε small enough. assume $\alpha(-\varepsilon, \varepsilon) \subseteq V \cap S$.

Let β be given by $\beta : (-\varepsilon, \varepsilon) \to U' \subseteq \mathbb{R}^2$ where $\mathbf{y}^{-1}$ is differentiable. ($\mathbf{y}^{-1}$ is just the projection map) Then $\alpha = \mathbf{y} \circ \beta = \mathbf{x} \circ (\mathbf{x}^{-1} \circ \mathbf{y} \circ \beta)$

we've seen in prop that $\mathbf{x}^{-1} \circ \mathbf{y}$ is differentiable so $\alpha'(0) = (\mathrm{d}\mathbf{x})_q \cdot (\mathrm{d}(\mathbf{x}^{-1} \circ \mathbf{y} \circ \beta))(0)$

$$= \left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \right)_q \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow \alpha'(0) \in T_p S$. We've just seen that $\alpha'(0) \in T_p S$.

Now we must to see $\mathbf{w} \in T_p S \Rightarrow \mathbf{w} = \alpha'(0)$ for some α. Suppose $\mathbf{w} = \left(\frac{\partial \mathbf{x}}{\partial u}(p), \frac{\partial \mathbf{x}}{\partial v}(p) \right) \begin{pmatrix} a \\ b \end{pmatrix} \in T_p S$.

Then let $\beta : (-\varepsilon, \varepsilon) \to U$ by $\beta(t) = p + t \begin{pmatrix} a \\ b \end{pmatrix}$. and $\alpha = \mathbf{x} \circ \beta$. Then $\alpha(0) = \mathbf{x}(\beta(0)) = \mathbf{x}(p) = p$.

$\alpha'(0) = (\mathrm{d}\mathbf{x})_p \cdot \mathrm{d}\beta_0 = \left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \right) \begin{pmatrix} a \\ b \end{pmatrix} \in T_p S.$ □

Example 3.7. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ smooth. $S = f^{-1}(a)$. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$. Then

$$f(\alpha(t)) = f(x(t), y(t), z(t)) = a.$$

Differentiating with respect to t:

$$\Rightarrow (\mathrm{d}f)_p \cdot (\mathrm{d}\alpha)_0 = 0.$$
$$\Rightarrow (\mathrm{d}f)_p \cdot \alpha'(0) = 0.$$

$\Rightarrow (\mathrm{d}f)_p$ is always normal to $T_p S$.

Question: we define $N(p)$ on a coordinate patch. can we choose coordinates patch appropriately so that $N : S \to \mathbb{R}^3$ is a smooth function?

Remark 3.7. When we move from local to global, it is always determined by the "topology" property.

3.4   First Fundamental Form

After defining the tangent plane, we wil study the inner product structure so that we can compute the area of the surface.

Definition 3.6. Let V be a vector space. We study $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ an inner product if it satisfies the following:

1. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ (symmetric).

2. $\langle a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w} \rangle = a\langle \mathbf{v}_1, \mathbf{w} \rangle + b\langle \mathbf{v}_2, \mathbf{w} \rangle$ (linearity).

3. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0 \quad \forall \mathbf{v} \in V$. equality holds if and only if $\mathbf{v} = \mathbf{0}$.

Definition 3.7. The quadratic form corresponding to an inner product is a map $q : V \to \mathbb{R}$ defined by

$$\mathbf{v} \to \langle \mathbf{v}, \mathbf{v} \rangle.$$

Definition 3.8. The **First Fundamental Form** of S is given by the quadratic form $I_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle$ for $\mathbf{v} \in T_p S$.

In order to understand I_p better, we need to choose a basis of $T_p S$ and express I_p using matrices and vectors.

Let $\mathbf{x} : U \to$ VAS be a coordinate patch with $\mathbf{x}(p) = p$. Then $\mathcal{B} = \left\{ \mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}(p), \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}(p) \right\}$

$$\text{with } E_p = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$
$$F_p = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$
$$G_p = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

(We drop the subscript p if it does not cause confusion)

Then for all $\mathbf{v}$, $I_p(\mathbf{v}) = (a, b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.

More precisely, $\langle \mathbf{v}, \mathbf{w} \rangle_p = (a, b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$, where $\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{w} = c\mathbf{x}_u + d\mathbf{x}_v$.

Therefore, in a coordinate patch IFF refers to $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$.

Example 3.8.   1. **Cylinder:**
$$\mathbf{x}(u, v) = (\cos u, \sin u, v)$$
$$\mathbf{x}_u = (-\sin u, \cos u, 0)$$
$$\mathbf{x}_v = (0, 0, 1)$$

$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1.$ $F = 0.$ $G = 1.$

$$\Rightarrow I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. **Plane:**
$$\mathbf{x}(u, v) = \mathbf{x}_0 + u\mathbf{w}_1 + v\mathbf{w}_2$$

(Assuming $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthonormal, $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_{ij}$)

$$I_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.
$$\mathbf{x}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$
$$\mathbf{x}_u = \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}} \right), \quad \mathbf{x}_v = \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}} \right)$$
$$I_p = \begin{pmatrix} \frac{1-v^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & \frac{1-u^2}{1-u^2-v^2} \end{pmatrix}$$

(Note: This is the IFF for the upper hemisphere in Cartesian coordinates.) $\tilde{\mathbf{x}}_\varphi = (-\sin\phi\sin\varphi, \sin\phi\cos\varphi, 0)$.

$$I_p = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix} \quad \text{different expression of } I_p \text{ for the same surface } S$$

Question: Suppose we have 2 parameterizations
$$\mathbf{x} : U \to V \cap S, \quad \tilde{\mathbf{x}} : \tilde{U} \to V \cap S.$$

Note that
$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}, \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \right)^T \begin{pmatrix} \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \\ \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \end{pmatrix} = (d\tilde{\mathbf{x}})^T d\tilde{\mathbf{x}}.$$

Suppose $\Psi = \mathbf{x}^{-1} \circ \tilde{\mathbf{x}}$ is the transition function
$$\Psi(\tilde{u}, \tilde{v}) = (u(\tilde{u}, \tilde{v}), v(\tilde{u}, \tilde{v})).$$
$$\Rightarrow \tilde{\mathbf{x}} = \mathbf{x} \circ \Psi. \quad \text{so}$$
$$d\tilde{\mathbf{x}} = d\mathbf{x}\, d\Psi, \quad \text{and hence}$$
$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (d\mathbf{x}\, d\Psi)^T (d\mathbf{x}\, d\Psi)$$
$$= (d\Psi)^T (d\mathbf{x})^T (d\mathbf{x})(d\Psi)$$
$$= (d\Psi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} (d\Psi)$$

composition by **Jacobian matrix**.

Example 3.9. Back to the sphere example.

$$\Psi(\phi, \varphi) = (\sin\phi\cos\varphi, \sin\phi\sin\varphi)$$

$$d\Psi = \begin{pmatrix} \cos\phi\cos\varphi & -\sin\phi\sin\varphi \\ \cos\phi\sin\varphi & \sin\phi\cos\varphi \end{pmatrix} \text{ and}$$

$$(d\Psi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} d\Psi = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \sin^2\phi \end{pmatrix}$$

Corollary 3.4.

$$\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = \sqrt{EG - F^2} \, \det\left(\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}\right)$$

Remark 3.8. We are always working in the orientable cases.

With the first fundamental form, we can compute arc-length, area, etc.

1. Arc-length.

$$L(\gamma) = \int_I |\gamma'(t)| \, dt$$

$$\gamma'(t) = X_u(u,v)u'(t) + X_v(u,v)v'(t)$$

$$= \int_I \sqrt{|X_u u'(t) + X_v v'(t)|^2} \, dt$$

$$= \int_I \sqrt{E|u'(t)|^2 + 2Fu'(t)v'(t) + G|v'(t)|^2} \, dt.$$

(If we consider γ a curve on S then first fundamental form gives us **Advantage**: a way to determine the arc-length **intrinsically**.)

We just need to know $\alpha(t) = (x(t), y(t))$ to compute the arc-length of α.

2. Area of Surface

Definition 3.9. Let $Z : U \to S \cap V$ be a coordinate patch of S, and $R \in S$ be such that $Z(U) = R$ for $U \subset \mathbb{R}^2$. Then the area of R is given by

$$\int_A |X_u \times X_v| \, du \, dv.$$

Note that $|u \times w|^2 = |u|^2|w|^2 - (u \cdot w)^2$. That is $|X_u \times X_v| = \sqrt{EG - F^2}$.

What happens if we have $\tilde{Z} : \tilde{U} \to S \cap \tilde{V}$ and $Z(U) = \tilde{Z}(\tilde{U}) = R$? Note that

$$\Psi = Z^{-1} \circ \tilde{Z} : \tilde{U} \to U$$

$$\int_{\tilde{R}} \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} \, d\tilde{u} \, d\tilde{v} = \int_R \sqrt{EG - F^2} \left| \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} \right| d\tilde{u} \, d\tilde{v}$$

$$= \int_R \sqrt{EG - F^2} \, du \, dv.$$

$\implies$ Area of R is independent of the choice of coordinate patch.

Remark 3.9. **Recall**:

- First fundamental form (f.f.f): $I_p(w) = \langle w, w \rangle, w \in T_pS$.

- If $X : U \to V \cap S$, $w = aX_u + bX_v$, then $I_p(w) = (a \ b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.

4 Gauss Map and the Second Fundamental Form

Main Goal: Define curvature of a surface.

Recall that in the case of curves:

$$\vec{T}'(t) = \kappa(t)\vec{N}(t) \quad \text{in the case of plane curve.}$$

$$\vec{N}'(t) = -\kappa(t)\vec{T}(t)$$

So curvature of a plane curve can be measured by the changes of tangent/normal vector.

If we want to do this on surface, one would like to study the change of normal vector.

Difficulty 1:

The normal vector $N_p = \frac{X_u \times X_v}{\|X_u \times X_v\|}(q)$ is defined only up to a sign.

On each parametrization, one has a "smooth" choice of $N(p)$. So we have:

$$N : V \cap S \to S^2$$

$$p \mapsto N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(q)$$

Remark 4.1. **However** The map may not be able to be extended to whole S. For example, we have a problem defining $N : S \to S^2$ on a Möbius band. The problem is the **orientability of** S.

Definition 4.1. Let $f : S_1 \to S_2$ be a map between two regular surfaces. We say f is **smooth** if $Z_2^{-1} \circ f \circ Z_1$ is smooth. The surface S is **orientable** if there is a smooth map

$$N : S \to S^2$$

such that $N(p) \perp T_pS$ for all $p \in S$.

Example 4.1. 1. Suppose S is covered by one coordinate patch $Z : U \to S$. For example, (S is the graph of a function f). Then $N_{(p)} = \frac{X_u \times X_v}{\|X_u \times X_v\|}(q)$.

2. Suppose $S = f^{-1}(0)$ is defined by an equation, then we can take

$$N_{(p)} = \frac{Df(p)}{\|Df(p)\|} = \frac{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})}{|(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})|}(p)$$

Difficulty 2:

For curves, we have $n' = -\kappa t$. For surfaces, we have two degrees of freedom on S.

4.1 Gauss Map

Definition 4.2. The map N is called **Gauss Map** of S. N is well-defined up to a sign.

Definition 4.3. (Differential) Let $f : S_1 \to S_2$ be a map. The **differential** of f at p, df_p, is a map from T_pS_1 to $T_{f(p)}S_2$, defined by

$$df_p(v) := (f \circ \alpha)'(0),$$

where $\alpha : (-\varepsilon, \varepsilon) \to S_1$ such that $\alpha(0) = p$, $\alpha'(0) = v$.

Remark 4.2. One needs to check this definition is well-defined. **Hint**: Instead of using the definition that f is smooth if $\psi \circ f \circ \varphi^{-1}$ is smooth, one may use the definition that f is smooth on $S_1 \subset \mathbb{R}^3$, if $f = F|_{S_1}$, where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is smooth. It's much easier and intuitive in our cases.

Now we study $dN_p : T_pS_1 \to T_{N(p)}S^2 \simeq T_pS$. Geometrically, dN_p measures how $N(p)$ changes along any direction of T_pS.

Remark 4.3. One way to compute $dN_p : T_pS \to T_{N(p)}S^2$ is the following. Let $Z(u,v)$ coordinate patch of S.
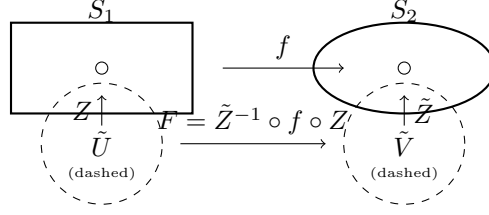
If $w = aX_u + bX_v$,

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$(df_p)(w) = c\tilde{Z}_{\tilde{u}} + d\tilde{Z}_{\tilde{v}}$$

Consider $N \circ Z : U \to S^2$, by abuse of notation. Write this map as $N(u,v)$. (No Z in that).

To compute dN_p, take the "principal curve". $\beta(t) = (t, v_0)$. $\alpha = Z \circ \beta$. Has $\alpha'(0) = X_u$. Then $dN_p(X_u)$ is given by

$$(N \circ \alpha)'(0) = (N \circ Z \circ \beta)'(0)$$

$$= \frac{d}{dt} N(t, v_0)|_{t=0} = \frac{\partial N}{\partial u} = N_u.$$

Similarly, $dN_p(X_v) = N_v = \frac{\partial N}{\partial v}$.

Since $dN_p : T_pS \to T_{N(p)}S^2$ is a linear transformation

And $\{X_u, X_v\}$ forms a basis of T_pS, one can get from

$$\begin{cases} dN_p(X_u) = N_u \\ dN_p(X_v) = N_v \end{cases}$$

Moreover, $\langle N, N \rangle = 1$, so differentiating yields $\langle dN, N \rangle = 0$.

$$\implies \langle N_u, N \rangle = 0 \implies N_u \in T_pS.$$

$$\implies \langle N_v, N \rangle = 0 \implies N_v \in T_pS.$$

$$\implies dN_p : T_pS \to T_pS.$$

Example 4.2. Here are some concrete examples.

1. S is a plane, i.e. $Z(u,v) = X_0 + uW_1 + vW_2$.

$$X_u = W_1, \quad X_v = W_2$$

$$\implies N(u,v) = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \text{ constant.}$$

$$\implies N_u = N_v = 0. \quad \text{i.e. } dN_p \text{ is the zero map.}$$

2. Sphere S^2: $Z(u,v) = (u, v, \sqrt{1 - u^2 - v^2})$.

$$X_u = \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right)$$

$$X_v = \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\right)$$

$$N(u,v) = (u, v, \sqrt{1 - u^2 - v^2})$$

$$N_u = X_u, \quad N_v = X_v$$

$$\text{i.e. } dN_p(X_u) = X_u$$

$$\text{i.e. } dN_p(X_v) = X_v$$

Hence dN_p is the **identity map**.

3. $Z(u,v) = (u, v, u^2 - v^2)$.

$$X_u = (1, 0, 2u), \quad X_v = (0, 1, -2v).$$

$$N(u,v) = \left(\frac{-2u}{\sqrt{1 + 4u^2 + 4v^2}}, \frac{2v}{\sqrt{1 + 4u^2 + 4v^2}}, \frac{1}{\sqrt{1 + 4u^2 + 4v^2}} \right).$$

Let $p = (0,0,0)$. $(u = 0, v = 0)$.

$$X_u|_p = (1, 0, 0), \quad X_v|_p = (0, 1, 0).$$

$$N_u|_p = (-2, 0, 0), \quad N_v|_p = (0, 2, 0).$$

$$dN_p(X_u) = N_u|_p = -2X_u|_p, \quad dN_p(X_v) = N_v|_p = 2X_v|_p.$$

Proposition 4.1. *The linear operator dN_p is a **self-adjoint operator** with respect to the inner-product.*

Remark 4.4. In the finite-dimensional case, $A^* = A$. $\ker A = (\operatorname{Im} A^*)^\perp$. $\ker A^* = (\operatorname{Im} A)^\perp$.

Proof. We need to show for all $w_1, w_2 \in T_pS$,

$$\langle dN_p w_1, w_2 \rangle = \langle w_1, dN_p w_2 \rangle$$

Let $w_1 = \begin{pmatrix} a \\ b \end{pmatrix}_{\mathcal{B}} = aX_u + bX_v$

$$w_2 = \begin{pmatrix} c \\ d \end{pmatrix}_{\mathcal{B}} = cX_u + dX_v$$

$\square$

$$\text{LHS} = \langle aX_u + bX_v, cN_u + dN_v \rangle$$
$$\text{RHS} = \langle aN_u + bN_v, cX_u + dX_v \rangle$$

In fact, since we have

$$\langle X_u, N_v \rangle = \langle N_v, X_u \rangle$$
$$\langle X_v, N_u \rangle = \langle N_u, X_v \rangle$$

then we're done.

Definition 4.4. The self-adjoint operator

$$-\mathbf{d}N_p : T_pS \to T_pS \quad \text{is called the **Weingarten Operator**}$$

(**Shape Operator**)

Corollary 4.2. *For real vector spaces, a self-adjoint operator is equivalent to a real symmetric matrix. One knows symmetric matrices have an orthonormal basis of real eigenvectors. It's frequently used and the proof is simple and interesting. We would like to include the proof below.*

Proof. Let A be a real symmetric $n \times n$ matrix, so $A = A^T$. We must show that all eigenvalues are real and that there exists an orthonormal basis of eigenvectors for $\mathbb{R}^n$.

Step 1: All Eigenvalues are Real We temporarily consider A as an operator on $\mathbb{C}^n$. Let λ be an eigenvalue and $v \in \mathbb{C}^n$ be the corresponding eigenvector.

$$Av = \lambda v$$

Taking the conjugate transpose (†):

$$(Av)^\dagger = (\lambda v)^\dagger \implies v^\dagger A^\dagger = \bar{\lambda} v^\dagger$$

Since A is real symmetric, $A^\dagger = A^T = A$.

$$v^\dagger A = \bar{\lambda} v^\dagger$$

Now, consider the scalar quantity $v^\dagger A v$:

$$v^\dagger(Av) = v^\dagger(\lambda v) = \lambda(v^\dagger v) \quad \text{(Equation 1)}$$

$$(v^\dagger A)v = (\bar{\lambda} v^\dagger)v = \bar{\lambda}(v^\dagger v) \quad \text{(Equation 2)}$$

Since $v^\dagger A v$ is a scalar, its two expressions must be equal:

$$\lambda(v^\dagger v) = \bar{\lambda}(v^\dagger v)$$

Since v is an eigenvector, $v \neq 0$, so $v^\dagger v = \|v\|^2 > 0$. We can divide by $v^\dagger v$, which gives:

$$\lambda = \bar{\lambda}$$

Thus, λ is real.

Step 2: Eigenvectors corresponding to distinct eigenvalues are orthogonal Let $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, with $\lambda_1 \neq \lambda_2$. Consider $v_1^T A v_2$:

$$v_1^T(Av_2) = v_1^T(\lambda_2 v_2) = \lambda_2(v_1^T v_2) \quad \text{(Equation 3)}$$

Using the symmetry $A = A^T$:

$$v_1^T A v_2 = v_1^T A^T v_2 = (Av_1)^T v_2 = (\lambda_1 v_1)^T v_2 = \lambda_1(v_1^T v_2) \quad \text{(Equation 4)}$$

Equating (3) and (4):

$$\lambda_2(v_1^T v_2) = \lambda_1(v_1^T v_2)$$

$$(\lambda_2 - \lambda_1)(v_1^T v_2) = 0$$

Since $\lambda_1 \neq \lambda_2$ (by assumption), we must have $v_1^T v_2 = 0$. Thus, v_1 and v_2 are orthogonal.

Step 3: Existence of an Orthonormal Basis (Inductive Step) Since all roots of the characteristic polynomial are real (from Step 1), A has at least one real eigenvector v_1 with eigenvalue λ_1. Let $W = \text{span}\{v_1\}^\perp$. If $w \in W$, then $v_1^T w = 0$. We show that W is an invariant subspace under A. If $w \in W$, we check Aw:

$$v_1^T(Aw) = (v_1^T A)w = (A^T v_1)^T w = (Av_1)^T w$$

$$= (\lambda_1 v_1)^T w = \lambda_1(v_1^T w) = \lambda_1(0) = 0$$

Thus, $Aw \in W$. Since A restricted to W is still self-adjoint (symmetric), we can repeat the process on the $(n-1)$-dimensional space W. By induction, we obtain an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of eigenvectors for $\mathbb{R}^n$. $\square$

Therefore, so does $-dN_p$, has an orthonormal basis of real eigenvector.

Definition 4.5. The real eigenvalues $\lambda_1 \leq \lambda_2$ of $-dN_p$ are called the **principal curvatures** of $p \in S$. (If $\lambda_1 = \lambda_2 = \lambda$, i.e., $-dN_p = \lambda I$, then all directions are principal.)
The unit eigenvectors $e_1, e_2 \in T_p S$ corresponding to λ_1, λ_2 are called the **principal directions** of $p \in S$.

The **Mean Curvature** of $p \in S$ is $H = \frac{1}{2}(\lambda_1 + \lambda_2)$:

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\text{tr}(-dN_p)$$

Remark 4.5. One may observe that mean curvature depends on $-dN_p$. In fact, it depends on the way the surface is embedded in $\mathbb{R}^3$. We will introduce the **Gaussian Curvature** which we will show it is an **intrinsic curvature**, i.e., independent of the way we embed it. We will be more precise in the following discussion.

Definition 4.6. The **Gaussian Curvature**

$$K = \lambda_1 \lambda_2 = \det(-dN_p)$$

Example 4.3. Sphere of radius r. $\implies -dN_p = \begin{pmatrix} \frac{1}{r} & 1 & 0 \\ & 0 & 1 \end{pmatrix}$

$\implies$ principal directions are all directions on T_pS

Example 4.4. Cylinder

$$Z(u,v) = (a\cos u, a\sin u, v)$$
$$Z_u = (-a\sin u, a\cos u, 0)$$
$$Z_v = (0,0,1)$$

The unit normal vector is $N = (-\cos u, -\sin u, 0)$.

$$N_u = (\sin u, -\cos u, 0), \quad N_v = 0$$

The Weingarten Operator is $\mathbf{S}_p = -dN_p$. Using the basis $\{Z_u, Z_v\}$,

$$dN_p(Z_u) = N_u = \frac{1}{a}(-a\sin u, a\cos u, 0) = \frac{1}{a}Z_u$$

$$dN_p(Z_v) = N_v = 0Z_v$$

So, dN_p is represented by the matrix $\begin{pmatrix} 1/a & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, the Weingarten operator is represented by $\mathbf{S}_p = -dN_p = \begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix}$.

Principal curvatures are the eigenvalues of $\mathbf{S}_p$:

$$\lambda_1 = -\frac{1}{a}, \quad \lambda_2 = 0.$$

Principal directions are Z_u, Z_v.

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = -\frac{1}{2a}, \quad K = \lambda_1\lambda_2 = 0.$$

Saddle (Hyperbolic Paraboloid)

$$Z(u,v) = (u, v, v^2 - u^2)$$

Let $p = (0,0,0)$. (Here $Z_u|_p = (1,0,0)$ and $Z_v|_p = (0,1,0)$). $N_u|_p = (-2,0,0)$, $N_v|_p = (0,2,0)$ (from previous calculation).

$$dN_p(Z_u) = N_u|_p = -2Z_u|_p$$
$$dN_p(Z_v) = N_v|_p = 2Z_v|_p$$

The Weingarten Operator $\mathbf{S}_p = -dN_p$ is represented by the matrix:

$$\mathbf{S}_p = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

The matrix for dN_p is $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, so $-dN_p = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

Principal curvatures (eigenvalues of $\mathbf{S}_p$) are:

$$\lambda_1 = -2, \quad \lambda_2 = 2$$

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = 0, \quad K = \lambda_1\lambda_2 = -4.$$

Definition 4.7. A point $p \in S$ is called

1. **Elliptic** if $\det(-dN_p) > 0$.

2. **Hyperbolic** if $\det(-dN_p) < 0$.

3. **Parabolic** if $\det(-dN_p) = 0$. $dN_p \neq 0$.

4. **Planar** if $dN_p = 0$.

This tells the local behavior of $p \in S$.

4.2 Second Fundamental Form

Remark 4.6. Recall: $\langle -dN_p V, W \rangle = \langle V, -dN_p W \rangle$. Self-adjoint.

Definition 4.8. Let $\langle\langle \cdot, \cdot \rangle\rangle : T_p S \times T_p S \to \mathbb{R}$ be given by

$$\langle\langle V, W \rangle\rangle = \langle -dN_p V, W \rangle$$

Then $\langle\langle \cdot, \cdot \rangle\rangle$ is a **bilinear form** i.e.

1. $\langle\langle V, W \rangle\rangle = \langle\langle W, V \rangle\rangle$.

2. **Linearity**.

But $\langle\langle V, W \rangle\rangle$ can be ≤ 0 for non-zero V, W. i.e. it may not be an inner product.

Definition 4.9 (The Second Fundamental Form)**.**

$$\text{II}_p : T_p S \times T_p S \to \mathbb{R} \text{ is defined as}$$

$$\text{II}_p(W) = \langle\langle W, W \rangle\rangle = \langle -dN_p W, W \rangle$$

Question: why II_p? How do we understand $\text{II}_p(w)$?

Remark 4.7. Suppose $\alpha : (-\varepsilon, \varepsilon) \to S$ (p.a.l.) with $\alpha(0) = p$, $\alpha'(0) = w = t(0)$. Then we have $\alpha''(0) = \kappa \mathbf{n}(0)$ (assuming $\mathbf{n}(0)$ normal on α; $N(\alpha(t))$ normal on S)
    Since $\langle t'(0), t(0) \rangle = 0$, we have

$$\kappa \mathbf{n} = \alpha''(0) = t'(0) \in \text{span}\{N(\alpha(0)), N(\alpha(0)) \times t(0)\}$$

$$\kappa \mathbf{n} = k_n N + k_g T$$

$$k_n = \langle \kappa \mathbf{n}, N \rangle = \langle t'(0), N \rangle = \langle \alpha''(0), N \rangle$$

$$k_g = \langle \kappa \mathbf{n}, T \rangle = \langle t'(0), T \rangle = \langle \alpha''(0), T \rangle$$

where we denote N, T as $\{N(\alpha(0)), N(\alpha(0)) \times t(0)\}$

Definition 4.10.
- k_n is called the **normal curvature**.
- k_g is called the **geodesic curvature**.

Example 4.5. Let $w \in T_p S$ of unit length, define the **normal section** of w as the plane spanned by N_p and w. Then its intersection with S gives a regular curve β p.a,l with $\beta(0) = p$, $\beta'(0) = w$. (β is called sectional curve of w).

Remark 4.8.

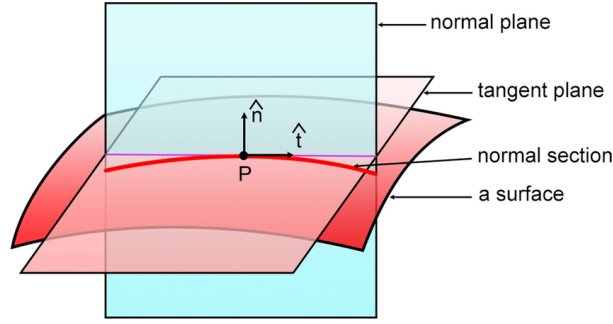$$\beta''(0) = t'(0) = \kappa \mathbf{n} = k_n N + k_g T.$$

Figure 3: The normal section β formed by the intersection of the surface S and the plane spanned by the normal vector N and the tangent vector w.

But β obviously must lie on Σ, so $k_g = 0$.

$$\Rightarrow k_g = \begin{cases} +\kappa & \text{if } \mathbf{n} \text{ is along the direction of } N. \\ -\kappa & \text{if } \mathbf{n} \text{ is opposite the direction of } N. \end{cases}$$

Proposition 4.3. *Let $\alpha : (-\varepsilon, \varepsilon) \to S$ piecewise unit speed curve (p.u.c.) with $\alpha(0) = p$ and $\alpha'(0) = w$. Then the normal curvature of α on S is equal to $\mathrm{II}_p(w)$.*

Proof.
$$\mathrm{II}_p(w) = \langle -dN_p w, w \rangle = \langle -dN_p(\alpha'(0)), \alpha'(0) \rangle$$
$$= -\langle (N \circ \alpha)'(0), \alpha'(0) \rangle = k_n.$$

$\square$

Corollary 4.4.   *1. The normal curvature of any $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha'(0) = w$ are the same $\mathrm{II}_p(w)$.*

2. $\mathrm{II}_p(w)$ measures the (normal) curvature of the sectional curve of w.
This gives a geometric interpretation of $\mathrm{II}_p(w)$.

Corollary 4.5. *Let $w \in T_p S$ be of norm 1. Then the curvature of the sectional curve of w must lie within λ_1 and λ_2. i.e:*

$$\lambda_1 \leq \mathrm{II}_p(w) \leq \lambda_2 \quad \forall w \in T_p S, \quad \|w\| = 1.$$

Proof. Without loss of generality (WLOG), write $w = \cos\theta e_1 + \sin\theta e_2$, where e_1, e_2 principal directions of $-dN_p$ and orthogonal basis of $T_p S$. Then:

$$\mathrm{II}_p(w) = \langle \langle w, w \rangle \rangle = \langle -dN_p w, w \rangle$$
$$= \langle -dN_p(\cos\theta e_1 + \sin\theta e_2), \cos\theta e_1 + \sin\theta e_2 \rangle$$
$$= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$$

Then we're done. $\square$

Remark 4.9. Finally we can give a better geometric picture of understanding II_p.

Example 4.6. Local behavior for parabolic point $p \in S$.

$$-\lambda \leq \mathrm{II}_p(w) \leq 0 \quad \text{or} \quad 0 \leq \mathrm{II}_p(w) \leq \lambda$$

Remark 4.10. **Elliptic pt:**
$$\lambda_1 \leq \mathrm{II}_p(w) \leq \lambda_2. \quad \lambda_1 \lambda_2 > 0.$$

Hyperbolic pt:
$$\lambda_1 \leq \mathrm{II}_p(w) \leq \lambda_2. \quad \lambda_1 < 0 < \lambda_2.$$

4.3   Second Fundamental Form in Local Coordinates

Remark 4.11. **Our goal:** Find an easy way to compute II_p and dN_p.

Definition 4.11. Let $N : U \to \mathbb{S}^2$ be the **normal map**. (Recall this is actually $N \circ X$).

Definition 4.12.

$$e = \langle -N_u, X_u \rangle.$$
$$f = \langle -N_u, X_v \rangle = \langle -N_v, X_u \rangle.$$
$$g = \langle -N_v, X_v \rangle.$$

Proposition 4.6. *If*

$$w_1 = aX_u + bX_v$$
$$w_2 = cX_u + dX_v$$

then

$$\langle\langle w_1, w_2 \rangle\rangle = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

Remark 4.12. As before, we call

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

the **second fundamental form** (matrix).

Proof. By straightforward calculations. $\qquad\square$

Remark 4.13. N_u, N_v are hard to compute, but we have $\langle N, X_u \rangle = 0$.

$$\Rightarrow \langle N_u, X_u \rangle + \langle N, X_{uu} \rangle = 0.$$

$$\Rightarrow \begin{cases} e = \langle N, X_{uu} \rangle \\ f = \langle N, X_{uv} \rangle \\ g = \langle N, X_{vv} \rangle \end{cases}$$

Example 4.7.

$$X(u,v) = (v \cos u, v \sin u, u) \quad \text{Helicoid}$$

$$\Rightarrow \text{f.f.f} = \mathrm{I}_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} v^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

Then we compute X_{uu}, X_{uv}, X_{vv}. Then

$$e = \langle N, X_{uu} \rangle$$
$$f = \langle N, X_{uv} \rangle$$
$$g = \langle N, X_{vv} \rangle$$

$$\Rightarrow \text{s.f.f.} = \mathrm{II}_p = \begin{pmatrix} 0 & \frac{1}{\sqrt{v^2+1}} \\ \frac{1}{\sqrt{v^2+1}} & 0 \end{pmatrix}$$

Remark 4.14. After understanding the second fundamental form, one can express:

$$-dN_p : T_pS \to T_pS$$

in coordinates with $\mathcal{B} = \{X_u, X_v\}$.

Suppose $(-dN_p)_{\mathcal{B}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, i.e.:

$$-dN_p(X_u) = a_{11}X_u + a_{21}X_v.$$
$$\Rightarrow -N_u = a_{11}X_u + a_{21}X_v.$$
$$-dN_p(X_v) = a_{12}X_u + a_{22}X_v = -N_v.$$

$$\Rightarrow \langle -N_u, X_u \rangle = e = a_{11}E + a_{21}F.$$
$$\Rightarrow \langle -N_u, X_v \rangle = f = a_{11}F + a_{21}G.$$
$$\langle -N_v, X_u \rangle = f = a_{12}E + a_{22}F.$$
$$\langle -N_v, X_v \rangle = g = a_{12}F + a_{22}G.$$

$$\Rightarrow \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\Rightarrow (-dN_p)_{\mathcal{B}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

$$= \mathrm{I}_p^{-1} \mathrm{II}_p.$$

Corollary 4.7. *The Gaussian curvature $K = \frac{\det(\mathrm{II}_p)}{\det(\mathrm{I}_p)} = \frac{eg - f^2}{EG - F^2}$. Which gives a neat formula for calculating the Gaussian curvature.*

Example 4.8. Back to the previous Helicoid example. We have $K = \det(-dN_p) = (1 + v^2)^{-2}$. $H = 0 = \frac{1}{2}\operatorname{tr}(-dN_p)$. (Example of **minimum surface**). From here, one can easily compute the principal directions and principal curvature.

Example 4.9. Compute $(-dN_p)_{\mathcal{B}}$, K, H. See that $H = 0$.

Example 4.10 (Torus)**.**

$$X(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$$

I_p. II_p. K.

$$\Rightarrow K > 0 \quad \text{when} \quad -\frac{\pi}{2} < u < \frac{\pi}{2}.$$
$$K = 0 \quad \text{when} \quad u = \frac{\pi}{2}, \frac{3\pi}{2}.$$
$$K < 0 \quad \text{when} \quad \frac{\pi}{2} < u < \frac{3\pi}{2}.$$

Remark 4.15.

$$\int K\, dA = 0.$$

We can check this by direct calculation.

Exercise. *Check $H = \frac{1}{2} \cdot \frac{eG - 2fF + gE}{EG - F^2}$.*

4.4 Gaussian Curvature

$K = \det(II)/\det(I)$. Since λ_1, λ_2 principal curvature are roots of characteristic poly of $-dN_p$.

$$\Rightarrow \lambda^2 - 2\lambda H + K = 0.$$
$$\Rightarrow \lambda = H \pm \sqrt{H^2 - K}$$

A differentiable fcn of local charts. except possible at pt where $H^2 = K$ we define these as "umbilical pts"

Exercise (HW). *prove A planar or sphere is the only totally "umbilical pts" surface Here "totally umbilical" means every pt is umbilical in $\mathbb{R}^3$.*

Also we have another characterization of gaussian curvature. In fact this is the orginal definition of Gauss.

Theorem 4.8. *The Gaussian Curvature K of a surface S at a point $p \in S$ is given by the limit of the ratio of the area of the spherical image of a small neighborhood of p to the area of the neighborhood itself.*
$$K(p) = \lim_{n \to \infty} \frac{signed\ area\ (N(B_n))}{area\ (B_n)}$$
where $B_1 \supset B_2 \supset \cdots \supset B_n \supset \ldots$ is a sequence of nested neighborhoods (e.g., small geodesic balls) around p such that $\lim_{n \to \infty} area(B_n) = 0$.
This can be written infinitesimally as $d(area_N) = K \cdot d(area_S)$, where $d(area_N)$ is the area element on the sphere S^2 via the Gauss map N.

Remark 4.16. This was Gauss's original definition of the curvature. Let $X : R \to B_n$ be a parametrization of the neighborhood B_n.

$$\text{area}(B_n) = \iint_R |X_u \times X_v| du dv = \iint_R \sqrt{EG - F^2} du dv$$

$$\text{area}(N(B_n)) = \iint_R |N_u \times N_v| du dv$$

Proof (Sketch). We know that the differential of the Gauss map $dN_p : T_pS \to T_pS$ (identifying $T_{N(p)}S^2$ with T_pS) relates the tangent vectors. The relationship between the area elements is given by the determinant of this map.
$$|N_u \times N_v| = |\det(dN_p)||X_u \times X_v|$$
The Gaussian curvature is $K = \det(dN_p)$ (or $\det(-dN_p)$ depending on convention, but for area ratios, we use $\det(dN_p)$).
$$|N_u \times N_v| = |K||X_u \times X_v|$$
Therefore,
$$\frac{\text{area}(N(B_n))}{\text{area}(B_n)} = \frac{\iint_{R_n} |N_u \times N_v| du dv}{\iint_{R_n} |X_u \times X_v| du dv} = \frac{\iint_{R_n} |K(u,v)|\sqrt{EG - F^2} du dv}{\iint_{R_n} \sqrt{EG - F^2} du dv}$$
By the Mean Value Theorem for integrals, as the region R_n shrinks to the point p, this ratio limits to $|K(p)|$. Taking into account the orientation (sign), the limit is $K(p)$.
$$\lim_{n \to \infty} \frac{\text{signed area } (N(B_n))}{\text{area } (B_n)} = K(p)$$
$\square$

Third Fundamental Form

Definition 4.13.
$$III_p(w_1, w_2) = \langle (-dN_p)^2 w_1, w_2 \rangle$$

also we can define the N-fundamental form as

$$IV_p(w_1, w_2) := \langle -(dN_p)^{(n-1)} w_1, w_2 \rangle.$$

this is a bilinear form symmetric form.

Exercise.
$$III_p = 2H II_p - K I_p.$$

Remark 4.17 (Rank). By the above computation we see f.f.f & s.f.f. indeed gives all the information.

4.5 Minimal Surfaces

Idea

Given a coordinate patch $z : U \to S \cap V$ take $\overline{D} \subset U$ closed. if we "perturb" the surface a bit will the area go up or down?

Definition 4.14 (Minimal surface "="). Surface that has minimal area If we "perturb" the surface a little bit, its area will always go up !

Now we are making the concept of minimal surface rigorously. Let $\phi : \overline{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$

$$\phi(u, v, t) = z(u, v) + th(u, v)N(u, v)$$

for some $h : \overline{D} \to \mathbb{R}$ Normal to surfaces. with $h|_{\partial \overline{D}} = 0$ (this is the boundary condition, note that there is always a boundary condition when we do Calculus of variation) and $X^t(u, v) = \phi(u, v, t)$. a family of parametrization

$$\Rightarrow X_u^t = X_u + thN_u + th_u N$$
$$X_v^t = X_v + thN_v + th_v N$$

$$E^t = E + 2th\langle X_u, N_u \rangle + t^2 h^2 \langle N_u, N_u \rangle + t^2 h_u^2$$
$$F^t = F + 2th\langle X_u, N_v \rangle + t^2 h^2 \langle N_u, N_v \rangle + t^2 h_u h_v$$
$$G^t = G + 2th\langle X_v, N_v \rangle + t^2 h^2 \langle N_v, N_v \rangle + t^2 h_v^2$$

$\Rightarrow E^t G^t - F^{t^2}$ we expand it and pick the order 1-term

$$= EG - F^2 - 2th(eG - 2fF + gE) + O(t^2)$$

with

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

$$= (EG - F^2)(1 - 4thH) + O(t^2)$$

$$\Rightarrow A(t) = \int_{\overline{D}} \sqrt{E^t G^t - F^{t^2}} \, du dv.$$

we have.

$$= \int_{\overline{D}} \sqrt{EG - F^2} \sqrt{1 - 4thH + O(t^2)} \, du dv$$

then $A'(0) = -\int_{\overline{D}} 2hH\sqrt{EG - F^2} \, du dv$.

Proposition 4.9. $H = 0 \Leftrightarrow A'(0) = 0$ *for all h.*

- *"$\Leftarrow$" is obvious.*

- *"$\Rightarrow$" Suppose $H(q) \neq 0$ for some $q \in \overline{D}$. Then one can choose h such that $h(q) = H(q)$ & hence $A'(0) < 0$.*

Definition 4.15. A regular parametrized surface is a minimal surface if $H \equiv 0$. a minimal regular surface. by the above discussion. hits the critical value of the area functional, yet one may not know whether this is the minimal area: local min? global min?

Proposition 4.10. *Let S be a minimal surface. and a graph.*

$$X(u, v) = (u, v, f(u, v)).$$

then f satisfies the PDE.

$$(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0.$$

This is a Elliptic PDE.

Proof. If $h : \overline{D} \to \mathbb{R}$ smooth & $h|_{\partial D} = 0$ consider family of graphs

$$F(t) = f + th$$

$$X^t(u, v) = (u, v, F(t)) \text{ All above share boundary}$$

$$\nabla F = \nabla f + t\nabla h$$

$$\Rightarrow \frac{d}{dt}(\text{area } X^t)\Big|_{t=0} = \int_D \frac{d}{dt}\sqrt{1 + |\nabla f + t\nabla h|^2}\Big|_{t=0} dudv.$$

$$= \int_D \frac{\langle \nabla f, \nabla h \rangle}{\sqrt{1 + |\nabla f|^2}} dudv$$

$$\overset{HW}{=} \int_D -h \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) dudv = 0$$

$\Rightarrow \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$

then we can verify this is our PDE. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 4.11. Suppose a solution of form

$$f = \alpha(u) + \beta(v)$$

The minimal surface equation is $(1+f_u^2)f_{vv}+(1+f_v^2)f_{uu}-2f_u f_v f_{uv} = 0$. Since $f_{uv} = 0$, $f_{uu} = \alpha_{uu}$, $f_{vv} = \beta_{vv}$, $f_u = \alpha_u$, $f_v = \beta_v$, this becomes:

$$(1 + \alpha_u^2)\beta_{vv} + (1 + \beta_v^2)\alpha_{uu} = 0$$

$$\Rightarrow -\frac{1 + \alpha_u^2}{\alpha_{uu}} = \frac{1 + \beta_v^2}{\beta_{vv}}$$

Since the left side depends only on u and the right side only on v, both must be equal to a constant λ.

$$\begin{cases} -(1 + \alpha_u^2) = \lambda\alpha_{uu} \\ (1 + \beta_v^2) = \lambda\beta_{vv} \end{cases}$$

A solution to this system is $\alpha(u) = \lambda\ln(\cos(u/\lambda))$ and $\beta(v) = -\lambda\ln(\cos(v/\lambda))$. This gives the function:

$$f(u, v) = \alpha(u) + \beta(v) = \lambda\ln\left(\frac{\cos(\frac{u}{\lambda})}{\cos(\frac{v}{\lambda})}\right)$$

The surface given by $(u, v, f(u, v))$ is a minimal surface. This is "Scherk's" first surface.

Definition 4.16 (Isothermal Coordinates). A coordinate patch $X(u, v)$ provides **isothermal coordinates** if the first fundamental form satisfies $E = G$ and $F = 0$.

$$I_p = E(u, v)(du^2 + dv^2)$$

(Sometimes written $I_p = e^\phi(du^2 + dv^2)$ where $E = e^\phi$.)

Theorem 4.11. *Isothermal coordinates exist locally for any minimal surface. In fact, they exist locally on any smooth (real analytic) surface. (A special theorem for $S \subseteq \mathbb{R}^3$)*

Proof. (Sketch for minimal part) Let S be a minimal surface. We can locally represent it as a graph $X(u,v) = (u, v, f(u,v))$. Since S is minimal, $H = 0$, and f satisfies the minimal surface PDE:

$$(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0$$

Let $F = (1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu}$. One can verify the following identities:

$$\frac{\partial}{\partial v}\left(\frac{1 + f_u^2}{\sqrt{1 + f_u^2 + f_v^2}}\right) - \frac{\partial}{\partial u}\left(\frac{f_u f_v}{\sqrt{1 + f_u^2 + f_v^2}}\right) = \frac{-f_v}{\sqrt{1 + f_u^2 + f_v^2}}F$$

$$\frac{\partial}{\partial u}\left(\frac{1 + f_v^2}{\sqrt{1 + f_u^2 + f_v^2}}\right) - \frac{\partial}{\partial v}\left(\frac{f_u f_v}{\sqrt{1 + f_u^2 + f_v^2}}\right) = -\frac{f_u}{\sqrt{1 + f_u^2 + f_v^2}}F$$

Since $F = 0$ (minimal surface), both right-hand sides are zero. Let $a = \sqrt{1 + f_u^2 + f_v^2}$ and define:

$$P = \frac{f_u f_v}{a}, \quad Q = \frac{1 + f_u^2}{a}, \quad R = \frac{1 + f_v^2}{a}$$

The equations become $\frac{\partial P}{\partial u} - \frac{\partial Q}{\partial v} = 0$ and $\frac{\partial R}{\partial u} - \frac{\partial P}{\partial v} = 0$. These are the conditions for the vector fields $V = (Q, P)$ and $W = (P, R)$ to be conservative (on a simply connected domain D). By Green's Theorem (Gauss's theorem in 2D):

$$\iint_D \left(\frac{\partial P}{\partial u} - \frac{\partial Q}{\partial v}\right) du\, dv = \oint_{\partial D} Q\, du + P\, dv = 0$$

$$\iint_D \left(\frac{\partial R}{\partial u} - \frac{\partial P}{\partial v}\right) du\, dv = \oint_{\partial D} P\, du + R\, dv = 0$$

This implies V and W have potentials, i.e., there exist functions θ, w such that $\nabla\theta = V$ and $\nabla w = W$.

Now define a new coordinate transformation $T : D \to \mathbb{R}^2$ by

$$(u, v) \mapsto (x, y) = (u + \theta, v + w)$$

The Jacobian matrix is $J(T) = \begin{pmatrix} 1 + \theta_u & \theta_v \\ w_u & 1 + w_v \end{pmatrix} = \begin{pmatrix} 1 + Q & P \\ P & 1 + R \end{pmatrix}$. One can compute $\det(J(T)) = (1 + Q)(1 + R) - P^2 = \cdots = \frac{(1 + f_u^2 + f_v^2 + 1)^2}{1 + f_u^2 + f_v^2} > 0$. By the Inverse Function Theorem, this map is a local diffeomorphism. We can get $T^{-1}(x, y) = (u(x, y), v(x, y))$.

The final step is to consider the new parametrization $\tilde{X}(x, y) = (u(x, y), v(x, y), f(u(x, y), v(x, y)))$. One computes the new first fundamental form coefficients $\tilde{E} = \langle \tilde{X}_x, \tilde{X}_x \rangle$ and $\tilde{G} = \langle \tilde{X}_y, \tilde{X}_y \rangle$ and $\tilde{F} = \langle \tilde{X}_x, \tilde{X}_y \rangle$. After a long calculation using the chain rule and the inverse Jacobian, one finds:

$$\tilde{E} = \tilde{G} \quad \text{and} \quad \tilde{F} = 0$$

Thus, (x, y) are isothermal coordinates. $\qquad \square$

> **Theorem 4.12.** *For any isothermal parametrization $X(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$, the Laplacian of X is related to the mean curvature H by:*
>
> $$\Delta X = 2EHN$$
>
> *where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$, E is the E in the first fundamental form, and N is the unit normal vector.*

Proof. We are given X is isothermal, so: $\langle X_u, X_u \rangle = \langle X_v, X_v \rangle = E(u, v)$ and $\langle X_u, X_v \rangle = 0$.

Differentiating these relations:

1. $\frac{\partial}{\partial u}\langle X_u, X_v \rangle = 0 \Rightarrow \langle X_{uu}, X_v \rangle + \langle X_u, X_{uv} \rangle = 0$

2. $\frac{\partial}{\partial v}\langle X_u, X_v \rangle = 0 \Rightarrow \langle X_{uv}, X_v \rangle + \langle X_u, X_{vv} \rangle = 0$

From $\langle X_u, X_u \rangle = \langle X_v, X_v \rangle = E$:

1. $\frac{\partial}{\partial u}\langle X_u, X_u\rangle = 2\langle X_{uu}, X_u\rangle = \frac{\partial E}{\partial u}$

2. $\frac{\partial}{\partial u}\langle X_v, X_v\rangle = 2\langle X_{uv}, X_v\rangle = \frac{\partial E}{\partial u}$

3. $\frac{\partial}{\partial v}\langle X_u, X_u\rangle = 2\langle X_{uv}, X_u\rangle = \frac{\partial E}{\partial v}$

4. $\frac{\partial}{\partial v}\langle X_v, X_v\rangle = 2\langle X_{vv}, X_v\rangle = \frac{\partial E}{\partial v}$

From this, we see $\langle X_{uu}, X_u\rangle = \langle X_{uv}, X_v\rangle$ and $\langle X_{uv}, X_u\rangle = \langle X_{vv}, X_v\rangle$.

Now consider the Laplacian $\Delta X = X_{uu} + X_{vv}$. We check its components in the tangent plane $T_p S = \text{span}\{X_u, X_v\}$:

$$\langle \Delta X, X_u\rangle = \langle X_{uu} + X_{vv}, X_u\rangle = \langle X_{uu}, X_u\rangle + \langle X_{vv}, X_u\rangle$$
$$= \langle X_{uv}, X_v\rangle - \langle X_v, X_{uv}\rangle = 0 \quad \text{(using 2 and a=c)}$$

$$\langle \Delta X, X_v\rangle = \langle X_{uu} + X_{vv}, X_v\rangle = \langle X_{uu}, X_v\rangle + \langle X_{vv}, X_v\rangle$$
$$= -\langle X_u, X_{uv}\rangle + \langle X_{uv}, X_u\rangle = 0 \quad \text{(using 1 and b=d)}$$

Since ΔX is orthogonal to both X_u and X_v, it must be parallel to the normal vector N.

$$\Delta X = \lambda N \quad \text{for some scalar } \lambda$$

To find λ, we take the dot product with N:

$$\lambda = \langle \Delta X, N\rangle = \langle X_{uu} + X_{vv}, N\rangle = \langle X_{uu}, N\rangle + \langle X_{vv}, N\rangle$$

By definition, $e = \langle X_{uu}, N\rangle$ and $g = \langle X_{vv}, N\rangle$. So, $\lambda = e + g$.

For an isothermal surface ($F = 0, E = G$), the mean curvature is:

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{eE + gE}{2(E^2)} = \frac{E(e+g)}{2E^2} = \frac{e+g}{2E}$$

Therefore, $e + g = 2EH$. Substituting this back, we get $\lambda = 2EH$.

$$\Delta X = 2EHN$$

$\square$

Corollary 4.13. *Let S be a minimal surface ($H = 0$) with an isothermal parametrization $X(u,v)$. Then*
$$\Delta X = 0$$
*This means the coordinate functions $x^1(u,v)$, $x^2(u,v)$, and $x^3(u,v)$ are **harmonic functions**.*

Remark 4.18 (Complex Analysis Connection). Recall from complex analysis:

- Let $z = u + iv$, where $i = \sqrt{-1}$. Then $u = \frac{z+\bar{z}}{2}$ and $v = \frac{z-\bar{z}}{2i}$.

- A function $\phi(z, \bar{z})$ is **holomorphic** if it satisfies the Cauchy-Riemann equations, which is equivalent to $\frac{\partial \phi}{\partial \bar{z}} = 0$.

- A function is **harmonic** if $\Delta \phi = 0$.

- The Wirtinger derivatives are:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

- The Laplacian is $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

Definition 4.17. Let $X(u,v) = X(z,\overline{z}) = (X^1, X^2, X^3)$ be a surface parametrization. Define the complex vector $\phi(z) = (\phi^1, \phi^2, \phi^3)$ where

$$\phi^j(z) = \frac{\partial X^j}{\partial z} \quad \text{for } j = 1,2,3$$

Also define the complex scalar Φ^2 :

$$\Phi^2 := \sum_{j=1}^{3} (\phi^j)^2 = \sum_{j=1}^{3} \left(\frac{\partial X^j}{\partial z} \right)^2$$

Proposition 4.14. *Let $X(u,v) = (X^1, X^2, X^3)$ be a parametrization of a surface. Then:*

1. *The components ϕ^j are **holomorphic** $\iff$ the coordinate functions X^j are **harmonic**.*

2. *The parametrization X is **isothermal** $\iff$ $\Phi^2 = 0$.*

3. *If X is isothermal, then S is **regular** $\iff \sum_{j=1}^{3} |\phi^j|^2 \neq 0$.*

Proof. 1. We check the Cauchy-Riemann condition for ϕ^j:

$$\frac{\partial \phi^j}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \left(\frac{\partial X^j}{\partial z} \right) = \frac{1}{4} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \left(\frac{\partial X^j}{\partial u} - i \frac{\partial X^j}{\partial v} \right)$$

$$= \frac{1}{4} \left((X_{uu}^j - i X_{uv}^j) + (i X_{vu}^j + X_{vv}^j) \right) = \frac{1}{4} (X_{uu}^j + X_{vv}^j) = \frac{1}{4} \Delta X^j$$

Thus, $\frac{\partial \phi^j}{\partial \overline{z}} = 0$ (i.e., ϕ^j is holomorphic) if and only if $\Delta X^j = 0$ (i.e., X^j is harmonic).

2. We compute Φ^2:

$$\Phi^2 = \sum_{j=1}^{3} \left(\frac{\partial X^j}{\partial z} \right)^2 = \sum_{j=1}^{3} \left(\frac{1}{2} \left(\frac{\partial X^j}{\partial u} - i \frac{\partial X^j}{\partial v} \right) \right)^2$$

$$= \frac{1}{4} \sum_{j=1}^{3} \left((X_u^j)^2 - (X_v^j)^2 - 2i X_u^j X_v^j \right)$$

$$= \frac{1}{4} \left(\sum_{j=1}^{3} (X_u^j)^2 - \sum_{j=1}^{3} (X_v^j)^2 - 2i \sum_{j=1}^{3} X_u^j X_v^j \right)$$

$$= \frac{1}{4} \left(\langle X_u, X_u \rangle - \langle X_v, X_v \rangle - 2i \langle X_u, X_v \rangle \right)$$

$$= \frac{1}{4} (E - G - 2iF)$$

Therefore, $\Phi^2 = 0$ if and only if $E = G$ and $F = 0$, which is the definition of an isothermal parametrization.

3. (Homework) Regularity means $X_u \times X_v \neq 0$, which is equivalent to $EG - F^2 \neq 0$. For an isothermal patch, this is $E^2 \neq 0$ or $E \neq 0$. $E = \langle X_u, X_u \rangle$. One must relate this to $\sum |\phi^j|^2$. $\qquad \square$

Exercise. *Show the third proposition above.*

Theorem 4.15 (Weierstrass-Enneper Integral Representation). *Let D be a simply-connected domain in $\mathbb{C}$. Let $g(z)$ be a meromorphic function and $f(z)$ be a holomorphic function on D such that at any pole z_0 of g, the order of the pole is m, and f has a zero of order at least $2m$. (This ensures $f(1 - g^2)$, $f(1 + g^2)$, and fg are all holomorphic).*

Define the holomorphic map $\phi(z) = (\phi^1, \phi^2, \phi^3)$ *by*

$$\begin{cases} \phi^1 = \frac{1}{2}f(z)(1 - g(z)^2) \\ \phi^2 = \frac{i}{2}f(z)(1 + g(z)^2) \\ \phi^3 = f(z)g(z) \end{cases}$$

Then the surface S *defined by the parametrization*

$$X(z, \bar{z}) = (X^1, X^2, X^3) \quad with \quad X^j(z, \bar{z}) = c^j + \Re\left(\int_{z_0}^z \phi^j(\zeta)d\zeta\right)$$

is a minimal surface with isothermal coordinates.
Conversely, any non-planar minimal surface can be locally represented in this way.

Proof. We have a map $\phi = (\phi^1, \phi^2, \phi^3)$ whose components are holomorphic by construction. By Proposition 1, the resulting coordinate functions $X^j = \Re \int \phi^j dz$ are harmonic ($\Delta X^j = 0$). By the Corollary to the previous theorem, if X is isothermal, $H = 0$, so S is minimal. We only need to check that X is isothermal. By Proposition 2, this is equivalent to checking if $\Phi^2 = 0$.

$$\Phi^2 = (\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2$$

$$= \left(\frac{1}{2}f(1 - g^2)\right)^2 + \left(\frac{i}{2}f(1 + g^2)\right)^2 + (fg)^2$$

$$= \frac{1}{4}f^2(1 - 2g^2 + g^4) - \frac{1}{4}f^2(1 + 2g^2 + g^4) + f^2g^2$$

$$= \frac{f^2}{4}(1 - 2g^2 + g^4 - (1 + 2g^2 + g^4)) + f^2g^2$$

$$= \frac{f^2}{4}(-4g^2) + f^2g^2 = -f^2g^2 + f^2g^2 = 0$$

Since $\Phi^2 = 0$, the parametrization is isothermal. Since ϕ^j are holomorphic, the X^j are harmonic ($\Delta X = 0$). Since X is isothermal and $\Delta X = 0$, we have $2EHN = 0$. Assuming the surface is regular ($E \neq 0$), this implies $H = 0$. Thus, the surface is minimal. $\square$

Example 4.12 (Enneper's Surface). Let $f(z) = c$ (e.g., $f(z) = 2$) and $g(z) = z$.

$$\phi^1 = \frac{1}{2}(2)(1 - z^2) = 1 - z^2$$

$$\phi^2 = \frac{i}{2}(2)(1 + z^2) = i(1 + z^2)$$

$$\phi^3 = 2z$$

Integrating to find X ($z = u + iv$):

$$X^1 = \Re \int (1 - z^2)dz = \Re(z - \frac{z^3}{3}) = u - \frac{u^3}{3} + uv^2$$

$$X^2 = \Re \int i(1 + z^2)dz = \Re(i(z + \frac{z^3}{3})) = -v - \frac{v^3}{3} + u^2v$$

$$X^3 = \Re \int 2zdz = \Re(z^2) = u^2 - v^2$$

One can check that this is a regular, isothermal parametrization.

$$\langle X_u, X_v \rangle = 0 \Rightarrow F = 0$$

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle = (1 + u^2 + v^2)^2 \Rightarrow E = G$$

Since it is isothermal and its coordinate functions are harmonic ($\Delta X = 0$), it is a minimal surface.

Example 4.13 (Catenoid-Helicoid Family). Let $f(z) = e^{-i\theta}e^z$ and $g(z) = e^{-z}$.

$$\phi^1 = \frac{1}{2}f(1-g^2) = \frac{1}{2}e^{-i\theta}e^z(1-e^{-2z}) = \frac{1}{2}e^{-i\theta}(e^z - e^{-z}) = e^{-i\theta}\sinh z$$

$$\phi^2 = \frac{i}{2}f(1+g^2) = \frac{i}{2}e^{-i\theta}e^z(1+e^{-2z}) = \frac{i}{2}e^{-i\theta}(e^z + e^{-z}) = ie^{-i\theta}\cosh z$$

$$\phi^3 = fg = (e^{-i\theta}e^z)(e^{-z}) = e^{-i\theta}$$

So, $\phi(z) = (e^{-i\theta}\sinh z, ie^{-i\theta}\cosh z, e^{-i\theta})$.

- If $\theta = 0$, $\phi = (\sinh z, i\cosh z, 1)$. Integrating gives the **Catenoid**.

- If $\theta = \frac{\pi}{2}$, $\phi = (-i\sinh z, \cosh z, -i)$. Integrating gives the **Helicoid**.

Remark 4.19. This provides a continuous deformation from the Catenoid to the Helicoid by varying the parameter θ.

4.6   Geodesics

Recall:
$$\alpha''(t) = k_n(t)N(\alpha(t)) + k_g(t)T(\alpha(t))$$
where k_g is the **geodesic curvature**.
Question: How shall we understand $k_g(t)$?

Example 4.14. Suppose S = plane and $Z(u,v) = (u,v,0)$. Then for any $\alpha = (x(t), y(t), 0)$, $\alpha'' = (x'', y'', 0) = 0N + k_gT$. i.e:
$$k = k_g$$

If S is plane, k_g means the curvature of the curve.

Definition 4.18. A **geodesic** on a surface S is a curve $\alpha : I \to S$ such that $k_g(t) = 0, \forall t \in I$.

Example 4.15. If S is a plane, then we have just seen that the path with shortest distance between 2 pts, i.e: Straight lines are geodesics.

In fact, we'll show path α joining a, b with shortest distance is a geodesic.

Theorem 4.16 (Gauss). *Let $\alpha : I \to S$ p.a.l (parameterized by arc length) be the shortest curve between $\alpha(t_1)$ and $\alpha(t_2)$. Then α is a geodesic.*

Proof. As in the minimal surface proof, this requires the technique of variations: Focus on a small subinterval of $I' \subseteq I$ such that $\alpha(I')$ lies on a single coord patch. with $\alpha = Z \circ \gamma$. $h : I' \to \mathbb{R}^2$ with $h(a_0) = h(a_1) = 0$. Consider
$$\gamma^t(s) = \gamma(s) + th(s) : I' \to U$$
Such that $\gamma^t(a_0) = \gamma(a_0)$ and $\gamma^t(a_1) = \gamma(a_1)$. Let $\alpha(t; s) = \alpha^t(s) = Z \circ \gamma^t(s) = Z(\gamma_1(s) + th_1(s), \gamma_2(s) + th_2(s))$.

$$L(t) = \int_{a_0}^{a_1} \langle \alpha'(t; s), \alpha'(t; s) \rangle^{1/2} ds$$

then

$$L'(t) = \int_{a_0}^{a_1} \frac{\langle \alpha_{ts}, \alpha_s \rangle}{\langle \alpha_s, \alpha_s \rangle^{1/2}} ds$$

Assuming α is p.a.l (so $\langle \alpha_s, \alpha_s \rangle = 1$ at $t = 0$),

$$L'(0) = \int_{a_0}^{a_1} \langle \alpha_{ts}, \alpha_s \rangle|_{t=0} ds$$

Consider

$$\frac{\partial}{\partial s}\langle \alpha_t, \alpha_s \rangle|_{t=0} = \langle \alpha_{ts}, \alpha_s \rangle|_{t=0} + \langle \alpha_t, \alpha_{ss} \rangle|_{t=0}$$

$$\Rightarrow L'(0) = \int_{a_0}^{a_1} \left(\frac{\partial}{\partial s}\langle \alpha_t, \alpha_s \rangle|_{t=0} - \langle \alpha_t, \alpha_{ss} \rangle|_{t=0} \right) ds$$

Note that $\alpha(t,s) = Z(\gamma_1(s) + th_1(s), \gamma_2(s) + th_2(s))$ $\partial_t \alpha(t,s)|_{t=0} = Z_u h_1 + Z_v h_2$ $\partial_{ss}\alpha(t,s)|_{t=0} = \alpha''(s)$
Since $h(a_0) = h(a_1) = 0$, the boundary term $[\langle \alpha_t, \alpha_s \rangle]_{a_0}^{a_1}$ vanishes.

$$\Rightarrow L'(0) = -\int_{a_0}^{a_1} \langle \alpha_t, \alpha_{ss} \rangle|_{t=0} ds$$

$$= -\int_{a_0}^{a_1} \langle h_1 Z_u + h_2 Z_v, \alpha''(s) \rangle ds$$

$$= -\int_{a_0}^{a_1} k_g \langle h_1 Z_u + h_2 Z_v, T \rangle ds$$

Since the curve α attains its minimum length when $t = 0$, we have $L'(0) = 0$ for any choice of h. This implies $k_g = 0$. $\qquad\square$

Remark 4.20. Geodesics may not be global minimum, but it is local minimum.

Question: Given a p.a.l curve $\alpha : I \to S$, how can we check if it is a geodesic?

Lemma 4.17. $k_g(s) = \det(N(\alpha(s)), \alpha', \alpha'')$

Proof. $t' = \alpha'' = k_n N + k_g T$ where $T = N \times t = N \times \alpha'$. $k_g = \langle t', T \rangle = \langle \alpha'', N \times \alpha' \rangle = \det(N, \alpha', \alpha'')$. Therefore, $\alpha(s)$ is a geodesic $\iff \det(N, \alpha', \alpha'') = 0$. $\qquad\square$

Example 4.16. $S = S^2$. For any two pts on S^2, one can perform rigid motion such that these two pts lie on the equator. Now assume these two pts lie on the equator. We claim the great circle, i.e. the equator lines joining the two pts $\alpha(s) = (\cos(s), \sin(s), 0)$ is a geodesic. Indeed: One can check $\det(N, \alpha', \alpha'') = 0 \Rightarrow k_g = 0$.

Remark 4.21. It's obvious that the great circle may not be the global minimum, but it's indeed a local minimum. The long path is a geodesic but not global minimum.

Next Question is study which curve $\gamma : I \to U$ such that $Z \circ \gamma$ is a geodesic?

Definition 4.19 (The Christoffel Symbols). For a coord patch X is given by: $\Gamma_{ij}^k : U \to \mathbb{R}$ Satisfies (where we denote $X_u = X_1, X_v = X_2$)

$$X_{uu} = \Gamma_{11}^1 X_1 + \Gamma_{11}^2 X_2 + eN$$
$$X_{uv} = \Gamma_{12}^1 X_1 + \Gamma_{12}^2 X_2 + fN$$
$$X_{vv} = \Gamma_{22}^1 X_1 + \Gamma_{22}^2 X_2 + gN$$

Remark 4.22. Γ_{ij}^k gives the "Levi-Civita Connection".

Remark 4.23. For short hand, we may write: $Z_u = Z_1, Z_v = Z_2$. $\Gamma_{12}^k = \Gamma_{21}^k$ for $k = 1, 2$. Since $Z_{uv} = Z_{vu}$.

Proposition 4.18. $\alpha(s) = Z \circ \beta(s) = Z(u(s), v(s))$ *is a geodesic* $\iff$

$$u'' + \Gamma_{11}^1(u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1(v')^2 = 0$$
$$v'' + \Gamma_{11}^2(u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2(v')^2 = 0$$

Proof. $\alpha' = X_u u' + X_v v'$ $\alpha'' = X_u u'' + X_{uu}(u')^2 + X_{uv}u'v' + X_v v'' + X_{vu}u'v' + X_{vv}(v')^2$
In order $k_g = 0$. Then $\alpha'' = k_n N \iff \alpha''||N$. So α'' has no X_u, X_v component. Substitute the definitions of X_{uu}, X_{uv}, X_{vv} using Christoffel symbols. X_u component and X_v component should be 0. (Collecting X_u terms): $u'' + \Gamma_{11}^1(u')^2 + \Gamma_{12}^1 u'v' + \Gamma_{21}^1 v'u' + \Gamma_{22}^1(v')^2 = 0$ (Collecting X_v terms): $v'' + \Gamma_{11}^2(u')^2 + \Gamma_{12}^2 u'v' + \Gamma_{21}^2 v'u' + \Gamma_{22}^2(v')^2 = 0$ then we're done. $\qquad\square$

Definition 4.20. The equations above are called **geodesic equations**.

For any (p_0, v_0) and $w \in \mathbb{R}^2$, $\exists!$ solution of the geodesics. The Christoffel symbol can be given by the first fundamental form (i.e. the metric).

Proposition 4.19 (Gauss Equation).

$$\Gamma_{ij}^k = \frac{1}{2}g^{mk}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})$$

where $g_{mk} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, g^{mk} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$

Computation of Christoffel Symbols. We start with the definition of the metric tensor components $g_{ij} = \langle X_i, X_j \rangle$, where $X_1 = X_u$ and $X_2 = X_v$. We assume the connection is torsion-free (i.e., $\Gamma_{ij}^k = \Gamma_{ji}^k$), which is true for the Levi-Civita connection on a surface derived from the embedding, as $X_{uv} = X_{vu}$.

We take the partial derivatives of the metric tensor:

$$\partial_k g_{ij} = \frac{\partial}{\partial x^k}\langle X_i, X_j \rangle = \langle X_{ik}, X_j \rangle + \langle X_i, X_{kj} \rangle$$

Using the definition of the Christoffel symbols $X_{ij} = \Gamma_{ij}^l X_l + $ (normal component), and noting that the inner product of a tangential vector (X_l) with the normal component is zero, we get:

$$\partial_k g_{ij} = \langle \Gamma_{ik}^l X_l, X_j \rangle + \langle X_i, \Gamma_{kj}^l X_l \rangle$$
$$= \Gamma_{ik}^l \langle X_l, X_j \rangle + \Gamma_{kj}^l \langle X_i, X_l \rangle$$
$$\partial_k g_{ij} = \Gamma_{ik}^l g_{lj} + \Gamma_{kj}^l g_{il}$$

This equation relates the derivative of the metric to the Christoffel symbols. To solve for Γ, we write this equation three times with permuted indices (this is sometimes called the Koszul formula):

(1) $\partial_i g_{mj} = \Gamma_{im}^l g_{lj} + \Gamma_{ij}^l g_{ml}$

(2) $\partial_j g_{mi} = \Gamma_{jm}^l g_{li} + \Gamma_{ji}^l g_{ml}$

(3) $\partial_m g_{ij} = \Gamma_{mi}^l g_{lj} + \Gamma_{mj}^l g_{il}$

Using the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$, we compute $(1) + (2) - (3)$:

$$\text{LHS} = \partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}$$
$$\text{RHS} = (\Gamma_{im}^l g_{lj} + \Gamma_{ij}^l g_{ml}) + (\Gamma_{jm}^l g_{li} + \Gamma_{ij}^l g_{ml}) - (\Gamma_{im}^l g_{lj} + \Gamma_{jm}^l g_{il})$$
$$= 2\Gamma_{ij}^l g_{ml}$$

This gives the relation:

$$2\Gamma_{ij}^l g_{ml} = \partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}$$

To isolate Γ_{ij}^k, we contract with the inverse metric g^{mk}.

$$2(\Gamma_{ij}^l g_{ml})g^{mk} = g^{mk}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})$$

$$2\Gamma_{ij}^l(\delta_l^k) = g^{mk}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})$$

$$2\Gamma_{ij}^k = g^{mk}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})$$

This yields the final formula:

$$\Gamma_{ij}^k = \frac{1}{2}g^{mk}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})$$

$\square$

Remark 4.24. It's useful to memorize the formula, and also useful to perform the calculation in this two-dimensional surface case.

Example 4.17 (Geodesics of surface of Revolution). $X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ $X_u = (-f\sin u, f\cos u, 0)$ $X_v = (f'\cos u, f'\sin u, g') \Rightarrow E = f^2, F = 0, G = f'^2 + g'^2$
Then $\Gamma_{11}^1 = 0$, $\Gamma_{12}^1 = \frac{f'}{f}$ $\Gamma_{11}^2 = \frac{-ff'}{f'^2+g'^2}$ $\Gamma_{22}^1 = 0$ $\Gamma_{12}^2 = 0$ $\Gamma_{22}^2 = \frac{f'f''+g'g''}{f'^2+g'^2}$
Geodesics Equation:

$$u'' + 2\left(\frac{f'}{f}\right)u'v' = 0$$

$$v'' + \left(\frac{-ff'}{f'^2+g'^2}\right)(u')^2 + \left(\frac{f'f''+g'g''}{f'^2+g'^2}\right)(v')^2 = 0$$

- Any **meridian curve** ($u =$ const) parametrized by constant speed is geodesic. If $u =$ const, $u' = 0, u'' = 0$. The first geodesic equation $0 = 0$ is satisfied. The second equation becomes $v'' + \left(\frac{f'f''+g'g''}{f'^2+g'^2}\right)(v')^2 = 0$.

- $v =$ const (**parallel**). Then $v' = 0, v'' = 0$. The second equation becomes $\left(\frac{-ff'}{f'^2+g'^2}\right)(u')^2 = 0$. This holds if $f' = 0$ (e.g., a cylinder) or $u' = 0$ (a point). So, a parallel is a geodesic iff $f'(v) = 0$.

For $\alpha = (u(t), v(t))$ constant speed, we have:
Clairaut's Relation: From the first geodesic equation: $u'' + 2(\frac{f'}{f})u'v' = 0$. This can be rewritten as $(f^2 u')' = 0$.

$$\Rightarrow f^2 u' = \text{const}$$

Note $\langle \alpha', X_u \rangle = \langle X_u u' + X_v v', X_u \rangle = Eu' + Fv' = f^2 u'$. Let θ be the angle between α' and the parallel (spanned by X_u). Then $f^2 u' = \text{const}$ is the same as $f\cos\theta = \text{const}$.

Remark 4.25. One can refer this to Do Carmo Chap 3. Exercise.

Riemannian Geometry

4.7 Fundamental Theorem of Surfaces

Let $\{X_u, X_v, N\}$ be the moving frame. $\Phi = (X_u, X_v, N) : S \to GL(\mathbb{R}^3)$

$$\Phi_u = \Phi\Lambda, \quad \Phi_v = \Phi\Theta$$

where

$$\Lambda = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -a_{11} \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -a_{21} \\ e & f & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{22}^1 & -a_{12} \\ \Gamma_{12}^2 & \Gamma_{22}^2 & -a_{22} \\ f & g & 0 \end{pmatrix}$$

where a_{ij} are components of $(-dN_p)$.

Computation of Gauss-Weingarten Equations. We want to verify the matrix equations $\Phi_u = \Phi\Lambda$ and $\Phi_v = \Phi\Theta$, where $\Phi = (X_u, X_v, N)$. We denote $X_1 = X_u$ and $X_2 = X_v$.
1. Check $\Phi_u = \Phi\Lambda$:
The left side is $\Phi_u = \partial_u \Phi = (\partial_u X_u, \partial_u X_v, \partial_u N) = (X_{uu}, X_{uv}, N_u)$. The right side is $\Phi\Lambda = (X_u, X_v, N)\Lambda$. Using the matrix Λ from our notes:

$$\Phi\Lambda = (X_u, X_v, N)\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -a_{11} \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -a_{21} \\ e & f & 0 \end{pmatrix}$$

We check this column by column:

- **Column 1:** $(\Phi\Lambda)_1 = X_u\Gamma_{11}^1 + X_v\Gamma_{11}^2 + Ne$. By definition of the Christoffel symbols, this is exactly X_{uu}. This matches.

- **Column 2:** $(\Phi\Lambda)_2 = X_u\Gamma_{12}^1 + X_v\Gamma_{12}^2 + Nf$. By definition, this is exactly X_{uv}. This matches.

- **Column 3:** $(\Phi\Lambda)_3 = X_u(-a_{11}) + X_v(-a_{21}) + N(0) = -a_{11}X_u - a_{21}X_v$. This must equal $N_u = \partial_u N$. By definition of the Weingarten map $S = -dN_p$, we have $S(X_u) = -dN_p(X_u) = -N_u$. The matrix (a_{ij}) of S is defined by $S(X_i) = \sum a_{ij} X_j$. So, $S(X_u) = a_{11}X_u + a_{21}X_v$. This means $-N_u = a_{11}X_u + a_{21}X_v$, or $N_u = -a_{11}X_u - a_{21}X_v$. This matches.

Thus, the equation $\Phi_u = \Phi\Lambda$ holds.

2. Check $\Phi_v = \Phi\Theta$:

The left side is $\Phi_v = \partial_v \Phi = (\partial_v X_u, \partial_v X_v, \partial_v N) = (X_{vu}, X_{vv}, N_v)$. The right side is $\Phi\Theta = (X_u, X_v, N)\Theta$. Using the matrix Θ from your notes (and $\Gamma^k_{21} = \Gamma^k_{12}$):

$$\Phi\Theta = (X_u, X_v, N) \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} & -a_{12} \\ \Gamma^2_{12} & \Gamma^2_{22} & -a_{22} \\ f & g & 0 \end{pmatrix}$$

We check this column by column:

- **Column 1:** $(\Phi\Theta)_1 = X_u\Gamma^1_{12} + X_v\Gamma^2_{12} + Nf$. By definition, this is X_{uv}. Since $X_{uv} = X_{vu}$, this matches.

- **Column 2:** $(\Phi\Theta)_2 = X_u\Gamma^1_{22} + X_v\Gamma^2_{22} + Ng$. By definition, this is X_{vv}. This matches.

- **Column 3:** $(\Phi\Theta)_3 = X_u(-a_{12}) + X_v(-a_{22}) + N(0) = -a_{12}X_u - a_{22}X_v$. This must equal $N_v = \partial_v N$. From the Weingarten map, $S(X_v) = -dN_p(X_v) = -N_v$. Using the matrix definition, $S(X_v) = a_{12}X_u + a_{22}X_v$. This means $-N_v = a_{12}X_u + a_{22}X_v$, or $N_v = -a_{12}X_u - a_{22}X_v$. This matches.

Thus, the equation $\Phi_v = \Phi\Theta$ also holds. $\qquad\square$

The above formula is called **Gauss-Weingarten Equation**.

Proposition 4.20. *The matrix Λ, Θ satisfy:*

$$\frac{\partial \Lambda}{\partial v} - \frac{\partial \Theta}{\partial u} = \Lambda\Theta - \Theta\Lambda$$

(Compatibility Condition)

Proof. By comparing Φ_{uv} and Φ_{vu}. $\qquad\square$

A Generalizing the Covariant Derivative to Tensor Fields

A.1 From Vectors to Tensors: The Guiding Principles

We have defined the covariant derivative $\nabla_X Y$ of a vector field Y along X. Our goal is to extend this definition to an arbitrary tensor field T of type (r, s). We do this by enforcing two fundamental rules:

1. **Action on scalars:** For any scalar function f, $\nabla_X f = X(f)$, which is the standard directional derivative.

2. **Leibniz Rule (Product Rule):** The covariant derivative must obey the product rule for all tensor products. For example:

$$\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S)$$

From these rules, we can derive the action on all tensor types.

A.2 Step 1: Deriving the Action on 1-forms (Covectors)

Let ω be a 1-form (a $(0, 1)$ tensor) and Y be a vector field (a $(1, 0)$ tensor). Their contraction $\omega(Y)$ is a scalar.

Let's apply our rules to this scalar:

*   By **Rule 1**: $\nabla_X(\omega(Y)) = X(\omega(Y))$.

*   By **Rule 2**: $\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$.

Equating these two expressions gives us a definition for $(\nabla_X \omega)$:

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

This defines how the covariant derivative acts on 1-forms.

A.3 Step 2: The General Rule for an (r, s) Tensor

We can generalize this idea. An (r, s) tensor θ is a multilinear map that takes r covectors $(\Omega^1, \ldots, \Omega^r)$ and s vectors $(Y_1, \ldots, Y_s)$ to produce a scalar:

$$f = \theta(\Omega^1, \ldots, \Omega^r, Y_1, \ldots, Y_s)$$

Applying the Leibniz rule repeatedly to this scalar defines the action of ∇_X on θ. The result is:

$$(\nabla_X \theta)(\Omega^1, \ldots, Y_s) = X(\theta(\Omega^1, \ldots, Y_s)) \quad \text{(Derivative of the scalar output)}$$

$$- \sum_{p=1}^{r} \theta(\Omega^1, \ldots, \nabla_X \Omega^p, \ldots, Y_s) \quad \text{(Terms for covector inputs)}$$

$$- \sum_{q=1}^{s} \theta(\Omega^1, \ldots, Y_q, \ldots, \nabla_X Y_q, \ldots) \quad \text{(Terms for vector inputs)}$$

A.4 Step 3: Component Computation

Now we find the components of $\nabla \theta$. An (r, s) tensor θ can be written in a basis as:

$$\theta = \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

The covariant derivative $\nabla \theta$ is an $(r, s+1)$ tensor. We denote its components by $\nabla_k \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s}$. These components are found by applying $\nabla_{\frac{\partial}{\partial x^k}}$ to θ and evaluating the result on the basis covectors and vectors:

$$\nabla_k \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \left(\nabla_{\frac{\partial}{\partial x^k}} \theta \right) \left(dx^{i_1}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}} \right)$$

Using our general rule from Step 2 (with $X = \frac{\partial}{\partial x^k}$):

$$\nabla_k \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial}{\partial x^k} \left(\theta \left(dx^{i_1}, \ldots, \frac{\partial}{\partial x^{j_s}} \right) \right)$$

$$- \sum_{p=1}^{r} \theta \left(dx^{i_1}, \ldots, \nabla_k dx^{i_p}, \ldots, \frac{\partial}{\partial x^{j_s}} \right)$$

$$- \sum_{q=1}^{s} \theta \left(dx^{i_1}, \ldots, \nabla_k \frac{\partial}{\partial x^{j_q}}, \ldots, \frac{\partial}{\partial x^{j_s}} \right)$$

Let's evaluate these three terms (using h as a dummy summation index):

1. **First Term:** $\theta \left(dx^{i_1}, \ldots, \frac{\partial}{\partial x^{j_s}} \right)$ is just the component $\theta^{i_1 \ldots i_r}_{j_1 \ldots j_s}$. So the first term is the partial derivative:

$$\frac{\partial \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^k}$$

2. **Contravariant terms (p sum):** We need the components of $\nabla_k dx^{i_p}$. From Step 1:

$$(\nabla_k dx^{i_p}) \left(\frac{\partial}{\partial x^h} \right) = \frac{\partial}{\partial x^k} \left(dx^{i_p} \left(\frac{\partial}{\partial x^h} \right) \right) - dx^{i_p} \left(\nabla_k \frac{\partial}{\partial x^h} \right)$$

$$= \frac{\partial}{\partial x^k} (\delta^{i_p}_h) - dx^{i_p} \left(\Gamma^m_{kh} \frac{\partial}{\partial x^m} \right) = 0 - \Gamma^m_{kh} \delta^{i_p}_m = -\Gamma^{i_p}_{kh}$$

So, $\nabla_k dx^{i_p} = -\Gamma^{i_p}_{kh} dx^h$. Substituting this into the p-th term:

$$\theta \left(\ldots, \nabla_k dx^{i_p}, \ldots \right) = \theta \left(\ldots, -\Gamma^{i_p}_{kh} dx^h, \ldots \right) = -\Gamma^{i_p}_{kh} \theta^{i_1 \ldots h \ldots i_r}_{j_1 \ldots j_s}$$

(where h is in the p-th slot). The full sum becomes:

$$- \sum_{p=1}^{r} \left(-\Gamma^{i_p}_{kh} \theta^{i_1 \ldots h \ldots i_r}_{j_1 \ldots j_s} \right) = + \sum_{p=1}^{r} \Gamma^{i_p}_{kh} \theta^{i_1 \ldots h \ldots i_r}_{j_1 \ldots j_s}$$

3. **Covariant terms (q sum):** We need $\nabla_k \frac{\partial}{\partial x^{j_q}}$. This is given by the definition of the connection coefficients:

$$\nabla_k \frac{\partial}{\partial x^{j_q}} = \Gamma^h_{kj_q} \frac{\partial}{\partial x^h}$$

Substituting this into the q-th term:

$$\theta \left(\ldots, \nabla_k \frac{\partial}{\partial x^{j_q}}, \ldots \right) = \theta \left(\ldots, \Gamma^h_{kj_q} \frac{\partial}{\partial x^h}, \ldots \right) = \Gamma^h_{kj_q} \theta^{i_1 \ldots i_r}_{j_1 \ldots h \ldots j_s}$$

(where h is in the q-th slot). The full sum becomes:

$$- \sum_{q=1}^{s} \Gamma^h_{kj_q} \theta^{i_1 \ldots i_r}_{j_1 \ldots h \ldots j_s}$$

A.4.1   Final Component Formula

Combining all three terms gives the complete formula for the components of the covariant derivative of a tensor θ:

$$\nabla_k \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial \theta^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^k} + \sum_{p=1}^{r} \Gamma^{i_p}_{kh} \theta^{i_1 \ldots h \ldots i_r}_{j_1 \ldots j_s} - \sum_{q=1}^{s} \Gamma^h_{kj_q} \theta^{i_1 \ldots i_r}_{j_1 \ldots h \ldots j_s}$$

(Note: We assume a torsion-free connection, so $\Gamma^i_{kj} = \Gamma^i_{jk}$).

Now that we have this formula, we have successfully generalized the ∇ operator from acting only on a vector field to acting on any tensor field.

B The Riemann Curvature Tensor

B.1 Motivation: The Commutator of Derivatives

In basic multivariable calculus, Schwarz's theorem guarantees that for a smooth function f, partial derivatives commute: $\partial_i \partial_j f = \partial_j \partial_i f$. This "flatness" of Euclidean space means the order of differentiation doesn't matter.

On a general manifold with a covariant connection ∇, we must ask the same question: Does the order of covariant differentiation matter? That is, for two vector fields X, Y and a third vector field Z, is $\nabla_X(\nabla_Y Z)$ the same as $\nabla_Y(\nabla_X Z)$?

It turns out that in general, they are not equal. Their failure to commute is precisely what defines the curvature of the manifold.

B.2 Coordinate-Free Definition

The operator $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ measures this failure. The $\nabla_{[X,Y]}$ term is crucial to ensure that the operator is a **tensor**—that is, its value at a point p depends only on the vectors X_p, Y_p, Z_p at that point, not on how they are extended to vector fields.

> **Definition B.1** (Riemann Curvature Tensor). The **Riemann Curvature Tensor** R is a $(1,3)$ tensor field that is defined by its action on three vector fields X, Y, Z:
>
> $$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The curvature tensor R is the fundamental measure of a manifold's intrinsic curvature. A manifold is called **flat** (locally indistinguishable from Euclidean space) if and only if its Riemann tensor is zero everywhere.

B.3 Component Expression

We can find the components of R by evaluating it on a local coordinate basis $\{\partial_j = \frac{\partial}{\partial x^j}\}$.

First, recall that for a coordinate basis, the Lie bracket is zero: $[\partial_k, \partial_j] = 0$. This simplifies the definition significantly:

$$R(\partial_k, \partial_j)\partial_l = \nabla_k \nabla_j \partial_l - \nabla_j \nabla_k \partial_l$$

The components R^i_{lkj} are defined as the components of the resulting vector:

$$R(\partial_k, \partial_j)\partial_l = R^i_{lkj} \partial_i$$

Let's apply the commutator to an arbitrary vector field $V = V^l \partial_l$ in this basis:

$$R(\partial_k, \partial_j)V = (\nabla_k \nabla_j - \nabla_j \nabla_k)V$$

Expanding the left side:

$$R(\partial_k, \partial_j)(V^l \partial_l) = V^l R(\partial_k, \partial_j)\partial_l = V^l (R^i_{lkj} \partial_i) = R^i_{lkj} V^l \partial_i$$

Expanding the right side (as was done in the previous version) yields:

$$(\nabla_k \nabla_j - \nabla_j \nabla_k)V = (\nabla_k \nabla_j - \nabla_j \nabla_k)(V^i \partial_i) = \left(\nabla_k(\nabla_j V^i) - \nabla_j(\nabla_k V^i)\right)\partial_i$$

A direct calculation (involving the component formula for ∇ on a $(1,1)$ tensor) shows that this simplifies to:

$$(\nabla_k \nabla_j - \nabla_j \nabla_k)V = \left(\partial_k \Gamma^i_{lj} - \partial_j \Gamma^i_{lk} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{jm}\Gamma^m_{lk}\right) V^l \partial_i$$

By comparing the components of ∂_i in our two results, we arrive at the component formula for the Riemann tensor:

$$R^i_{lkj} = \partial_k \Gamma^i_{lj} - \partial_j \Gamma^i_{lk} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{jm}\Gamma^m_{lk}$$

B.4 Properties and Symmetries

The Riemann tensor possesses several fundamental symmetries, which can be expressed in both coordinate-free and component forms.

1. **Antisymmetry in X and Y:**

   - $R(X, Y)Z = -R(Y, X)Z$
   - $R^i_{lkj} = -R^i_{ljk}$

2. **First Bianchi Identity (Cyclic sum):**

   - $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
   - $R^i_{lkj} + R^i_{jlk} + R^i_{kjl} = 0$

If we lower the first index using the metric g_{il} to create the $(0, 4)$ tensor $R_{ilkj} = g_{im}R^m_{lkj}$, we find two additional symmetries:

1. **Antisymmetry in the first two indices:**

$$R_{ilkj} = -R_{likj}$$

2. **Block (Pair) Symmetry:**

$$R_{ilkj} = R_{kjil}$$

These symmetries drastically reduce the number of independent components of the curvature tensor. In an n-dimensional manifold, there are only $\frac{n^2(n^2-1)}{12}$ independent components.