Real Analysis
midterm-ustc-2024-ren's version

October 13, 2025

Problem 1.

(10 points) State the definition of a **Lebesgue measurable function** on $[a, b]$, and explain why the concept of measurable functions must be introduced in Lebesgue integration theory.

Problem 2.

(10 points) Determine whether the following statement is correct, and provide a proof or a counterexample: Suppose $f : [a, b] \to [a, b]$ is a **monotonically increasing continuous function** that is both injective and surjective. Then the preimage of any Lebesgue measurable subset of $[a, b]$ under the mapping f must be a Lebesgue measurable set.

Problem 3.

(10 points) Suppose $k \in \mathbb{N}$, $a_i^k \in \mathbb{R}$, $b_k \in \mathbb{R}$, and

$$(a_1^k)^2 + \cdots + (a_n^k)^2 = 1.$$

We define

$$E_k = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_1^k x_1 + \cdots + a_n^k x_n = b_k\}.$$

Prove:

$$\bigcup_{k=1}^{\infty} E_k \neq \mathbb{R}^n.$$

Problem 4.

(10 points) Suppose $f : \mathbb{R} \to \overline{\mathbb{R}}$ is defined as

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \in \mathbb{R} \setminus \{0\} \\ +\infty, & x = 0. \end{cases}$$

Starting from the definition of a **Lebesgue integrable function**, prove that this function is **not** Lebesgue integrable on $\mathbb{R}$. (Using other methods will result in zero points.)

Problem 5.

(10 points) Suppose $f : [a, b] \to \overline{\mathbb{R}}$ is a **non-negative bounded measurable function**. Prove:

$$\int_{[a,b]} f(x)\, dx = \inf_{f \le \psi} \int_{[a,b]} \psi(x)\, dx$$

where $\psi : [a, b] \to \mathbb{R}$ is a non-negative simple measurable function.

Problem 6.

(10 points) Consider the sequence of functions:

$$\{f_{1,1}, f_{2,1}, f_{2,2}, \ldots, f_{n,1}, f_{n,2}, \ldots, f_{n,n}, \ldots\}.$$

where $f_{n,j} : [0,1] \to \mathbb{R}$ is defined as:

$$f_{n,j}(x) = \chi_{\left[\frac{j-1}{n}, \frac{j}{n}\right)}(x), \quad j = 1, \ldots, n;\ n \in \mathbb{N}.$$

Explain whether this sequence of functions converges in the following senses: **Convergence in measure, pointwise convergence, almost everywhere convergence, almost uniform convergence, L^p convergence.**

Problem 7.

(10 points) Determine whether the following statement is correct, and provide a proof or a counterexample: Suppose $E \subset \mathbb{R}$ is a **Lebesgue measurable set** and E is a **closed set**, with $m(E) = 1$. Then E must have an **interior point**.

Problem 8.

(10 points) Suppose $E \subset \mathbb{R}$ is a measurable set with finite measure. The function $f : \mathbb{R} \to \mathbb{R}$ is defined as:

$$f(x) = \int_E \chi_{x+E}(y)\, dy.$$

Prove that the function f is **continuous at** $0 \in \mathbb{R}$.

Problem 9.

(10 points) Suppose $g : [0,1] \to [0,1]$ is a **Lebesgue measurable function**, and $f : [0,1] \to \mathbb{R}$ is a **continuous function**, with $f(0) \leq f(1)$. Prove that the following limit exists and belongs to the interval $[f(0), f(1)]$:

$$\lim_{n\to\infty} \int_0^1 f((g(x))^n)\, dx.$$

Problem 10.

(10 points) Suppose $a > 0$. The function $G : (0, +\infty) \to \mathbb{R}$ is defined as:

$$G(x) = \int_0^{+\infty} e^{-x(t+t^{-1})}(t^{1+\alpha} + t^{1-\alpha})\, dt.$$

Prove that G is **well-defined**, and $G \in C^\infty(0, +\infty)$.