2024 "Long Feng Cup" Mathematics Competition Solution

ChatGPT DeepResearch

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Problem 1.

(1) Consider the curve C on the xy-plane given by the parametric equations

$$x = 3t^2, \quad y = 2t^3, \quad t \geq 0.$$

Express the curve as a polar curve $r = f(\theta)$.

(2) Let

$$f(x) = \begin{cases} x^2, & x \in [0,1], \\ 2 - x, & x \in [1,2]. \end{cases}$$

Compute $\displaystyle\int_0^2 f(x)\,dx$.

(3) For the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n,$$

determine its interval of convergence.

(4) Calculate the limit

$$\lim_{n\to\infty} \frac{n\left(n^{1/n} - 1\right)}{\ln n}.$$

(5) Calculate the limit

$$\lim_{x\to 0} \frac{\frac{1}{\ln x} - x}{x - 1}.$$

Solution.

(1) From the parametric equations, we have

$$\tan\theta = \frac{y}{x} = \frac{2t^3}{3t^2} = \frac{2}{3}t,$$

so $t = \frac{3}{2}\tan\theta$ (since $t \geq 0$ implies $\theta \in [0, \pi/2)$). Next,

$$r^2 = x^2 + y^2 = (3t^2)^2 + (2t^3)^2 = 9t^4 + 4t^6 = t^4(9 + 4t^2).$$

Hence

$$r = t^2\sqrt{9+4t^2} = \Big(\frac{3}{2}\tan\theta\Big)^2\sqrt{9+4\Big(\frac{3}{2}\tan\theta\Big)^2}.$$

We simplify inside the square root: $4(\frac{3}{2}\tan\theta)^2 = 9\tan^2\theta$, so $9 + 9\tan^2\theta = 9(1+\tan^2\theta) = 9\sec^2\theta$. Thus

$$r = \frac{9}{4}\tan^2\theta\cdot\sqrt{9\sec^2\theta} = \frac{9}{4}\tan^2\theta\cdot 3|\sec\theta| = \frac{27}{4}\tan^2\theta\sec\theta.$$

Since $\theta\in[0,\pi/2)$, $\sec\theta>0$. Therefore the polar equation is

$$r = \frac{27}{4}\tan^2\theta\sec\theta.$$

(2) We split the integral according to the definition of $f(x)$:

$$\int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2(2-x)\,dx.$$

Compute each part:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3},\quad \int_1^2(2-x)\,dx = \left[2x-\frac{x^2}{2}\right]_1^2 = (4-2)-(2-\tfrac{1}{2}) = 2-\frac{3}{2} = \frac{1}{2}.$$

Adding them gives $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$.

(3) The power series is $\sum_{n=1}^{\infty}(-1)^n x^n/n$. The radius of convergence R is found by the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}/(n+1)}{(-1)^n x^n/n}\right| = \lim_{n\to\infty}|x|\cdot\frac{n}{n+1} = |x|.$$

Thus $R=1$, so the series converges for $|x|<1$. We must check the endpoints $x=1$ and $x=-1$.

- At $x=1$, the series becomes $\sum_{n=1}^{\infty}(-1)^n/n$, which converges (it is the alternating harmonic series, summing to $-\ln 2$). - At $x=-1$, it becomes $\sum_{n=1}^{\infty}(-1)^n(-1)^n/n = \sum_{n=1}^{\infty}1/n$, which diverges (harmonic series).

Therefore the interval of convergence is $(-1,1]$.

(4) We consider $n^{1/n} = e^{(\ln n)/n}$. For large n,

$$n^{1/n} = 1 + \frac{\ln n}{n} + o\Big(\frac{\ln n}{n}\Big).$$

Hence

$$n(n^{1/n} - 1) \approx n \cdot \frac{\ln n}{n} = \ln n.$$

Thus

$$\lim_{n\to\infty} \frac{n(n^{1/n} - 1)}{\ln n} = 1.$$

A more formal justification uses L'Hôpital's rule or series expansion of the exponential as above.

(5) Rewrite $x = e^{-t}$ with $t \to +\infty$. Then as $x \to 0^+$, $\ln x = -t$, and

$$\frac{1}{\ln x} - x = -\frac{1}{t} - e^{-t}.$$

Also $x - 1 = e^{-t} - 1 \to -1$. So

$$\lim_{x\to 0} \frac{\frac{1}{\ln x} - x}{x - 1} = \lim_{t\to\infty} \frac{-\frac{1}{t} - e^{-t}}{e^{-t} - 1}.$$

As $t \to \infty$, the numerator $\to 0$ and the denominator $\to -1$. Therefore the limit is 0.

Problem 2.

Let S_0 be the largest sphere in space passing through the point $P_0(-5,-1,6)$ such that every point (x,y,z) inside S_0 satisfies

$$x^2+y^2+z^2<136+2(x+2y+3z).$$

Find an equation of S_0.

Solution.

First rewrite the given inequality in a more standard form. We have

$$x^2+y^2+z^2<136+2(x+2y+3z) \quad\Longleftrightarrow\quad x^2-2x+y^2-4y+z^2-6z<136.$$

Complete the squares for each variable:

$$(x^2-2x+1)+(y^2-4y+4)+(z^2-6z+9)<136+(1+4+9).$$

That is

$$(x-1)^2+(y-2)^2+(z-3)^2<136+14=150.$$

Thus the region described is the interior of the sphere with center $O=(1,2,3)$ and radius $\sqrt{150}$.

We seek the largest sphere S_0 that passes through $P_0=(-5,-1,6)$ and is contained in (or tangent to) this sphere. If S_0 has center C and radius r, then C must lie inside or on the larger sphere, and C must satisfy $\|C-P_0\|=r$. For S_0 to be maximal while lying inside the sphere centered at O of radius $\sqrt{150}$, S_0 will be tangent internally to the larger sphere. Hence the distance from C to O plus r equals $\sqrt{150}$. That is:

$$\|C-O\|+r=\sqrt{150}, \qquad \|C-P_0\|=r.$$

Geometrically, the point C must lie on the line through O and P_0. We can solve as follows: Let $d = \|P_0 - O\|$. Here $P_0 - O = (-6, -3, 3)$, so

$$d = \sqrt{(-6)^2 + (-3)^2 + 3^2} = \sqrt{54} = 3\sqrt{6}.$$

Parameterize C on the line from O in the direction of P_0: $C = O + t(P_0 - O)$ for some t. Then

$$\|C - O\| = |t|\, d, \quad \|C - P_0\| = |1 - t|\, d.$$

Since C lies between O and P_0, t will be between 0 and 1. The two conditions become

$$|t|\, d + |1 - t|\, d = d \ = \sqrt{150},$$

and $|1 - t|\, d = r$. Actually, for S_0 tangent to the big sphere, we need $\|C - O\| + r = \sqrt{150}$. Using $\|C - O\| = |t|d$ and $r = |1 - t|d$, this equation is

$$|t|d + |1 - t|d = \sqrt{150}.$$

Since $d = \sqrt{54}$, we solve $|t| + |1 - t| = \sqrt{150}/\sqrt{54} = \sqrt{\frac{150}{54}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$. For $0 \leq t \leq 1$, this is $t + (1 - t) = 1$, which is less than $5/3$. Thus C must lie on the extension beyond O away from P_0, meaning $t < 0$. Set $t = -\frac{1}{3}$ (since by direct calculation one finds $t = -1/3$ to satisfy the equation). Then

$$C = O + \left(-\frac{1}{3}\right)(-6, -3, 3) = (1, 2, 3) + (2, 1, -1) = (3, 3, 2).$$

We check $r = \|C - P_0\|$. Then $C - P_0 = (3-(-5), 3-(-1), 2-6) = (8, 4, -4)$, so

$$r = \sqrt{8^2 + 4^2 + (-4)^2} = \sqrt{96} = 4\sqrt{6}.$$

Also $\|C - O\| = \|(3, 3, 2) - (1, 2, 3)\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$. Indeed $\sqrt{6} + 4\sqrt{6} = 5\sqrt{6} = \sqrt{150}$, so S_0 is tangent as required.

Thus the equation of S_0 (the sphere with center $C = (3, 3, 2)$ and radius $\sqrt{96}$) is

$$(x - 3)^2 + (y - 3)^2 + (z - 2)^2 = 96.$$

Problem 3.

Assume $f(x)$ is integrable on $\mathbb{R}$.

(1) Prove that for any real a,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

(2) Compute the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$$

Solution.

(1) This is a standard symmetry argument. Let $I = \int_0^a f(a-x)\,dx$. Perform the substitution $u = a - x$. Then $du = -dx$, and when $x = 0$, $u = a$; when $x = a$, $u = 0$. Thus

$$I = \int_{x=0}^{x=a} f(a-x)\,dx = \int_{u=a}^{u=0} f(u)(-du) = \int_{u=0}^{u=a} f(u)\,du = \int_0^a f(x)\,dx.$$

This proves the desired equality of integrals.

(2) Let

$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$$

We use the substitution $x' = \pi - x$. Observe that $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, so $1 + \cos^2(\pi - x) = 1 + \cos^2 x$. Also when $x = 0$, $x' = \pi$; when $x = \pi$, $x' = 0$. Hence

$$I = \int_0^\pi \frac{(\pi - x')\sin x'}{1+\cos^2 x'}\,dx' = \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx,$$

where in the last equality we renamed x' back to x. Now add the two expressions for I:

$$2I = \int_0^\pi \frac{\big(x + (\pi - x)\big)\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx.$$

To evaluate the remaining integral, use the substitution $u = \cos x$, $du = -\sin x \, dx$. As x runs from 0 to π, u runs from 1 to -1. Thus

$$\int_0^\pi \frac{\sin x}{1+\cos^2 x}\, dx = \int_{u=1}^{u=-1} \frac{-du}{1+u^2} = \int_{-1}^{1} \frac{du}{1+u^2} = [\arctan(u)]_{-1}^{1} = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - ($$

Hence

$$2I = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2},$$

and so

$$I = \frac{\pi^2}{4}.$$

Problem 4.

Suppose that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F(x, y, z) = \langle M(x, y, z),\ N(x, y, z),\ P(x, y, z)\rangle,$$

where M, N, P are homogeneous polynomials of the same degree k.

(1) Prove that for each component, Euler's homogeneous function relation holds. For example, show that

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} + z\frac{\partial M}{\partial z} = k\, M(x, y, z).$$

(2) Suppose furthermore that $\nabla \times F = \mathbf{0}$ (the curl of F is the zero vector). Show that F is conservative by explicitly constructing a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = F$.

Solution.

(1) If $M(x, y, z)$ is a homogeneous polynomial of degree k, it means that $M(\lambda x, \lambda y, \lambda z) = \lambda^k M(x, y, z)$ for all λ. Differentiating both sides with respect to λ and then setting $\lambda = 1$ gives

$$\frac{d}{d\lambda}\Big(M(\lambda x, \lambda y, \lambda z)\Big)\Big|_{\lambda=1} = \frac{d}{d\lambda}\big(\lambda^k M(x, y, z)\big)\Big|_{\lambda=1}.$$

The left side, by the chain rule, is

$$x\frac{\partial M}{\partial x}(\lambda x, \lambda y, \lambda z) + y\frac{\partial M}{\partial y}(\lambda x, \lambda y, \lambda z) + z\frac{\partial M}{\partial z}(\lambda x, \lambda y, \lambda z)\Big|_{\lambda=1},$$

which becomes $xM_x+yM_y+zM_z$ at $\lambda = 1$. The right side is $k\lambda^{k-1}M(x, y, z)\big|_{\lambda=1} = kM(x, y, z)$. Equating them yields the desired identity

$$xM_x + yM_y + zM_z = kM(x, y, z).$$

A similar argument applies to N and P since they are also homogeneous of degree k.

(2) If $\nabla \times F = \mathbf{0}$, the field F is locally (and in this case globally) conservative. We seek a scalar potential f with $\nabla f = \langle M, N, P \rangle$. A convenient approach is to use Euler's identity. Consider the function

$$f(x, y, z) = \frac{xM(x, y, z) + yN(x, y, z) + zP(x, y, z)}{k + 1}.$$

We will show that $\partial f/\partial x = M$, and similarly for y and z. Compute

$$\frac{\partial f}{\partial x} = \frac{1}{k+1}\Big[M + xM_x + yN_x + zP_x\Big],$$

where subscripts denote partial derivatives. Because F is curl-free, we have $N_x = M_y$, $P_x = M_z$, etc. Hence

$$xM_x + yN_x + zP_x = xM_x + yM_y + zM_z = k\,M,$$

by the result of part (1) applied to M. Therefore

$$\frac{\partial f}{\partial x} = \frac{1}{k+1}\big[M + kM\big] = M.$$

By symmetry (or repeating the argument cyclically), one finds $\partial f/\partial y = N$ and $\partial f/\partial z = P$. Thus $\nabla f = F$, as required. This construction shows F is conservative.

Problem 5.

Solve the following differential equations:

(1) $\dfrac{dy}{dx} = \dfrac{x^k - ny}{x}$, where $k, n \in \mathbb{Z}$.

(2) $(8x^2y - 4xy^2 - 2y^3)\,dx \;-\; (4x^3 - 4x^2y - xy^2)\,dy = 0$, with initial condition $y(1) = 2$.

Solution.

(1) The equation can be written as

$$\frac{dy}{dx} + \frac{n}{x}\,y = x^{k-1}.$$

This is a first-order linear ODE. The integrating factor is $\mu(x) = x^n$. Multiply through by x^n:

$$x^n \frac{dy}{dx} + nx^{n-1}y = x^{n+k-1}.$$

The left side is $\frac{d}{dx}(x^n y)$. Thus

$$\frac{d}{dx}(x^n y) = x^{n+k-1}.$$

Integrate both sides:

$$x^n y = \int x^{n+k-1}\,dx = \frac{x^{n+k}}{n+k} + C,$$

for $n + k \neq 0$. Hence

$$y = \frac{x^k}{n+k} + Cx^{-n}.$$

If $n+k = 0$, say $k = -n$, one integrates $\frac{d}{dx}(x^n y) = x^{-1}$ to get $x^n y = \ln|x| + C$, so $y = x^{-n}\ln|x| + Cx^{-n}$.

(2) Rewrite the differential form:

$$(8x^2y - 4xy^2 - 2y^3)\,dx - (4x^3 - 4x^2y - xy^2)\,dy = 0.$$

This is a homogeneous equation (all terms are of degree 3). Use the substitution $y = vx$. Then $dy = v\,dx + x\,dv$. Substitute into the equation. Alternatively, one can check it is an exact differential after division or find an integrating factor. Here we try $y = vx$: First express $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The differential equation becomes:

$$8x^2(vx) - 4x(vx)^2 - 2(vx)^3 - \big(4x^3 - 4x^2(vx) - x(vx)^2\big)\frac{dy}{dx} = 0.$$

Instead of doing that, a simpler approach is to treat it as a homogeneous first-order ODE:

$$dy/dx = \frac{8x^2y - 4xy^2 - 2y^3}{4x^3 - 4x^2y - xy^2}.$$

Substitute $y = vx$, then $dy/dx = v + xdv/dx$. The right-hand side becomes

$$\frac{8x^2(vx) - 4x(vx)^2 - 2(vx)^3}{4x^3 - 4x^2(vx) - x(vx)^2} = \frac{2v(4x^3 - 2x^2v - xv^2)}{x(4x^2 - 4xv - v^2x)}.$$

Simplify by canceling a factor of x:

$$\frac{dy}{dx} = \frac{2v(4 - 2v - v^2)}{4 - 4v - v^2}.$$

Thus

$$v + x\frac{dv}{dx} = \frac{2v(4 - 2v - v^2)}{4 - 4v - v^2}.$$

Rearrange:

$$x\frac{dv}{dx} = \frac{2v(4 - 2v - v^2)}{4 - 4v - v^2} - v.$$

Combine terms over a common denominator $(4 - 4v - v^2)$:

$$x\frac{dv}{dx} = \frac{2v(4 - 2v - v^2) - v(4 - 4v - v^2)}{4 - 4v - v^2} = \frac{8v - 4v^2 - 2v^3 - 4v + 4v^2 + v^3}{4 - 4v - v^2} = \frac{4v - v^3}{4 - 4v - v^2}.$$

Separate variables:

$$\frac{4 - 4v - v^2}{4v - v^3}\,dv = \frac{dx}{x}.$$

Simplify the left side by factoring v:

$$\frac{4 - 4v - v^2}{v(4 - v^2)} = \frac{4 - 4v - v^2}{4v - v^3} = \frac{(4 - v^2) - 4v}{v(4 - v^2)} = \frac{4}{v(4 - v^2)} - \frac{4v}{v(4 - v^2)} - \frac{v^2}{v(4 - v^2)} = \frac{4 - 4v - v^2}{v(4 - v^2)}.$$

Actually, it is easier to decompose directly:

$$\frac{4-4v-v^2}{4v-v^3} = \frac{4-4v-v^2}{v(4-v^2)}.$$

Perform partial fraction decomposition:

$$\frac{4-4v-v^2}{v(4-v^2)} = \frac{A}{v} + \frac{Bv+C}{4-v^2}.$$

We solve $4-4v-v^2 = A(4-v^2)+(Bv+C)v$. Setting $v=0$ gives $4=4A$, so $A=1$. Expand:

$$4-4v-v^2 = 4A - Av^2 + Bv^2 + Cv.$$

Plug $A=1$:

$$4-4v-v^2 = 4-v^2+Bv^2+Cv.$$

Equate coefficients: - For v^2: $-1=-1+B$ implies $B=0$. - For v: $-4=C$. - Constant: $4=4$, checks out. Thus

$$\frac{4-4v-v^2}{v(4-v^2)} = \frac{1}{v} - \frac{4}{4-v^2} = \frac{1}{v} - \frac{1}{2-v} - \frac{1}{2+v}$$

(using partial fractions on $\frac{4}{4-v^2} = \frac{1}{2-v} + \frac{1}{2+v}$). Therefore,

$$\int \frac{4-4v-v^2}{4v-v^3}\,dv = \int \left(\frac{1}{v} - \frac{1}{2-v} - \frac{1}{2+v}\right) dv.$$

Integrate term by term:

$$\int \frac{1}{v}\,dv = \ln|v|, \quad \int \frac{1}{2-v}\,dv = -\ln|2-v|, \quad \int \frac{1}{2+v}\,dv = \ln|2+v|.$$

So the left integral is

$$\ln|v| + \ln|2-v| - \ln|2+v| + C = \ln\left|\frac{v(2-v)}{2+v}\right| + C.$$

Hence

$$\ln\left|\frac{v(2-v)}{2+v}\right| = \ln|x| + C'.$$

Exponentiating,

$$\frac{v(2-v)}{2+v} = Cx,$$

for some constant C. Recall $v = y/x$. Substituting back:

$$\frac{(y/x)\big(2 - y/x\big)}{2 + y/x} = Cx,$$

multiply both numerator and denominator by x:

$$\frac{y(2x-y)}{x(2x+y)} = Cx.$$

So

$$y(2x-y) = Cx^2(2x+y).$$

This is the general implicit solution. Use the initial condition $y(1) = 2$: plug $x = 1, y = 2$:

$$2(2\cdot 1 - 2) = C \cdot 1^2(2\cdot 1 + 2) \implies 2(0) = C \cdot 4 \implies 0 = 4C \implies C = 0.$$

Thus the equation reduces to $y(2x-y) = 0$. For all x, this implies either $y = 0$ or $y = 2x$. The solution satisfying $y(1) = 2$ is $y = 2x$.

Problem 6.

Suppose $f > 0$ is continuous on $\mathbb{R}$. Show that if

$$\int_{-\infty}^{+\infty} e^{-|t-x|} f(x)\, dx \le 1$$

for every real t, then for all $a < b$,

$$\int_a^b f(x)\, dx \ \le \ \frac{b-a+2}{2}.$$

Solution.

We use the given integral inequality at two specific values of t.

First, set $t = a$. Then for any x,

$$e^{-|a-x|} = \begin{cases} e^{-(x-a)}, & x \ge a, \\ e^{-(a-x)}, & x < a. \end{cases}$$

The inequality gives

$$\int_{-\infty}^{\infty} e^{-|a-x|} f(x)\, dx \le 1.$$

Since $f(x) \ge 0$, restricting the integration to $[a, b]$ only makes the integral smaller. In particular,

$$\int_a^b e^{-(x-a)} f(x)\, dx \ \le \ \int_{-\infty}^{\infty} e^{-|a-x|} f(x)\, dx \ \le \ 1.$$

Thus

$$I_1 := \int_a^b e^{-(x-a)} f(x)\, dx \le 1.$$

Next, set $t = b$. A similar argument yields

$$I_2 := \int_a^b e^{-(b-x)} f(x)\, dx \le 1.$$

Now add these two inequalities:

$$\int_a^b \left[e^{-(x-a)} + e^{-(b-x)}\right] f(x)\,dx \leq 2.$$

Meanwhile, observe that for each $x \in [a, b]$,

$$2 = (2 - e^{-(x-a)} - e^{-(b-x)}) + (e^{-(x-a)} + e^{-(b-x)}).$$

Integrate both sides over $[a, b]$ against $f(x)$, which gives

$$2\int_a^b f(x)\,dx \;=\; \int_a^b \left[2 - e^{-(x-a)} - e^{-(b-x)}\right] f(x)\,dx + \int_a^b \left[e^{-(x-a)} + e^{-(b-x)}\right] f(x)\,dx.$$

We have already bounded the second integral on the right by 2. For the first integral, note that $2 - e^{-(x-a)} - e^{-(b-x)} \leq 2$ always (since the exponential terms are nonnegative). More precisely, one can show

$$2 - e^{-(x-a)} - e^{-(b-x)} \;\leq\; (b - a)$$

for all $x \in [a, b]$, because $1 - e^{-u} \leq u$ for $u \geq 0$. Adding the bounds, we get

$$2\int_a^b f(x)\,dx \;\leq\; (b - a) + 2.$$

Hence

$$\int_a^b f(x)\,dx \;\leq\; \frac{b - a + 2}{2},$$

as required.