USTC Real Analysis Final Exam Problems

Collected Years: 2019 – 2025

Compiled Collection

November 30, 2025

Contents

1 Year 2025

1.1 Standard Class

1. Measure Theory Concepts

(a) Briefly state the definition of the Lebesgue outer measure m_*.

(b) Prove the countable subadditivity of the outer measure using the definition.

(c) Prove: For two sets E_1 and E_2 satisfying $d(E_1, E_2) > 0$, we have $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$.

(d) If the condition is changed to $E_1 \cap E_2 \neq \emptyset$, does the above conclusion still hold?

2. Convergence and Limits

(a) Let $\{f_n\}$ and f be defined on $[0,1]$. Prove that f_n converges to f in measure ($f_n \Rightarrow f$) if and only if:

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n - f|}{1 + |f_n - f|}\, dx = 0$$

(b) Calculate the limit (state the theorems used):

$$\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\, dx$$

3. Convergence Relationships

(a) Does there exist a nowhere continuous function that is equal to a continuous function almost everywhere?

(b) Does L^1 convergence imply the existence of a subsequence that converges almost everywhere?

(c) Does L^∞ convergence imply almost everywhere convergence?

4. Bounded Variation

(a) Let $f(x) = 3x - x^3$. Calculate the total variation $V_{-2}^{2}(f)$.

(b) Let $f \in BV[a,b]$. Prove that there exist $g \in AC[a,b]$ and $h \in BV[a,b]$ with $h'(x) = 0$ a.e. on $[a,b]$, such that $f = g - h$.

5. Absolute Continuity

(a) Briefly state the definition of $f \in AC[a,b]$.

(b) Let $\alpha \in \mathbb{R}$ and define:

$$f(x) = \begin{cases} x^\alpha, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Discuss for which α the function f is absolutely continuous.

(c) Use the definition of absolute continuity and the Vitali Covering Lemma to prove: If $f \in AC[a,b]$ and $f'(x) = 0$ a.e., then f is a constant.

6. Abstract Measure Theory

(a) Briefly state the definitions of an Algebra and a Pre-measure. Explain how to construct an outer measure from a pre-measure.

(b) Briefly explain how to construct a measure space from a pre-measure (Carathéodory extension).

1.2 Honors Class (H)

10.1. Prove the following are equivalent:

(i) $f \in \text{Lip}[a, b]$ with Lipschitz constant M.

(ii) $f \in AC[a, b]$ and $|f'(x)| \leq M$ a.e. $x \in [a, b]$.

10.2. Assume $f \in L^2(0, +\infty)$. Prove:

$$\lim_{n\to\infty} f(x+n) = 0, \quad \text{a.e. } x \in [0, +\infty).$$

10.3. Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$$f\left(\int_0^1 g(x)\,dx\right) \leq \int_0^1 f(g(x))\,dx, \quad \forall g \in L^\infty[0, 1].$$

Prove that f is a convex function.

10.4. Determine whether the following are correct (prove or provide a counterexample):

(1) Assuming $f, g \in AC[a, b]$, then $f(x)g(x) \in AC[a, b]$.

(2) Assuming $f, g \in AC[a, b]$ and $g : [a, b] \to [a, b]$, then $f(g(x)) \in AC[a, b]$.

10.5. Let ν and μ be finite measures such that $\nu \perp \mu$ and $\nu \ll \mu$. Prove that $\nu = 0$.

10.6. Let μ be the counting measure on the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ as:

$$f(x, y) = \begin{cases} 1, & \text{if } x = y \\ -1, & \text{if } x = y + 1 \\ 0, & \text{otherwise} \end{cases}$$

Prove that:

$$\int_{\mathbb{N}} \int_{\mathbb{N}} f(x, y)\,d\mu(x)\,d\mu(y) \neq \int_{\mathbb{N}} \int_{\mathbb{N}} f(x, y)\,d\mu(y)\,d\mu(x)$$

Explain why this result does not contradict the Fubini Theorem.

10.7. Assume $f \in AC[a, b]$. Prove that f maps sets of measure zero to sets of measure zero (the Luzin N-property).

10.8. Assume $[0, 1] \cap \mathbb{Q} = \{r_1, r_2, \dots\}$. Let $I_n = (r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n})$. Prove:

$$\sum_{n=1}^{\infty} \chi_{I_n}(x) \in L^2[0, 1] \setminus \mathcal{R}[0, 1]$$

(Note: $\mathcal{R}[0, 1]$ denotes the class of Riemann integrable functions).

10.9. Assume $f : [0, 1] \to [0, 1]$ is the Cantor function.

(a) Determine whether the following are true: (i) $f \in AC[0, 1]$; (ii) $f \in BV[0, 1]$; (iii) $f \in \text{Lip}[0, 1]$; (iv) f' exists a.e. and $f' \in L^1[0, 1]$.

(b) Calculate $\int_0^1 f'(x)\,dx - f(1) + f(0)$.

(c) Calculate: (i) $\operatorname{ess\,sup}_{x\in[0,1]} f(x)$; (ii) $\operatorname{ess\,sup}_{x\in[0,1]} f'(x)$.

(d) Calculate $V_0^1(f)$ (the total variation of f on $[0, 1]$).

2 Year 2024

2.1 Standard Class

1. **True or False** (Prove or give a counterexample) (30 points)

   (a) L^1 convergence implies almost everywhere convergence.

   (b) Almost everywhere convergence implies convergence in measure.

   (c) L^1 convergence implies convergence in measure.

2. **Integration Theorems** (20 points)

   (a) State and prove the Dominated Convergence Theorem.

   (b) Use the Dominated Convergence Theorem to calculate the limit:

$$\lim_{n\to\infty}\int_0^{\infty}\frac{1}{1+x^{\frac{\sqrt{n}}{\log(n+2024)}}}\,dx$$

3. **Bounded Variation and Hölder Continuity** (15 points) Let $a, b > 0$ and define the function:

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

   Prove: $f \in BV[0,1]$ if and only if $a > b$. Furthermore, if $a = b$, for any given $\alpha \in (0,1)$, construct a function that satisfies the Hölder continuity condition of order α (i.e., $|f(x) - f(y)| \le A|x-y|^{\alpha}$) but is not of bounded variation.

4. **Measure Theory** (15 points) Let μ^* be an outer measure defined on a set X.

   (a) State the definition of a μ^*-measurable set.

   (b) Let $\mathcal{M} = \{E \subset X : E \text{ is } \mu^*\text{-measurable}\}$. Prove that $\mathcal{M}$ is a σ-algebra on X, and $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure on $(X, \mathcal{M})$.

5. **Absolute Continuity** (10 points) Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is Lipschitz continuous (exists $L > 0$ such that $|f(x) - f(y)| \le L|x-y|$) if and only if f is absolutely continuous and $|f'(x)| \le L$ for a.e. $x \in \mathbb{R}$.

6. **Integral Kernels** (10 points) Let $f \in L^1(\mathbb{R})$ satisfy:

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|f(x)f(y)|}{(x-y)^2+\epsilon^2}\,dx\,dy < \infty$$

   Prove that $f(x) = 0$ a.e. on $\mathbb{R}$.

3 Year 2023

3.1 Standard Class

1. **True or False** (20 points)

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then its derivative function f' is measurable.

   (b) For any monotonically increasing sequence of measurable sets $\{A_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$, we have $m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

2. **Monotonicity** (15 points) If a function f is absolutely continuous on $[a,b]$ and $f'(x) \geq 0$ a.e., prove that f is an increasing function.

3. **Variation Inequality** (15 points) Let $f \in BV[a,b]$ and $V(f)$ be its total variation. Prove:

$$\int_a^b |f'(t)|\, dt \leq V_a^b(f).$$

4. **Indefinite Integral** (15 points) Let $f \in L^1[0,1]$ and define $F(x) := \int_0^x f(t)\, dt$. Prove:
   (a) $F \in L^1[0,1]$.
   (b) $\lim_{x\to 0^+} xF(x) = 0$.
   (c) $\int_0^1 F(x)\, dx = \int_0^1 (1-x)f(x)\, dx$. (Note: Standard identity, user text slightly cut off but implies integration by parts formula).

5. **Convergence in Measure** (15 points) Let $m(E) < \infty$. If a sequence of measurable functions $\{f_n\}$ converges in measure to f on E, and a sequence of measurable functions $\{g_n\}$ converges in measure to g on E, prove that $\{f_n g_n\}$ converges in measure to fg on E.

6. **Brezis-Lieb Lemma Variant** (10 points) Let $\{f_k\} \subset L^1(\mathbb{R})$ satisfy $f_k \to f$ almost everywhere and $\lim_{k\to\infty} \|f_k\|_1 = \|f\|_1 < \infty$. Prove:

$$\lim_{k\to\infty} \|f_k - f\|_1 = 0.$$

7. **Topology of Measure** (10 points) Prove: There exists a set $E \subset [0,1]$ with positive measure such that for any open interval $I \subset [0,1]$,

$$0 < m(E \cap I) < m(I).$$

3.2 Honors Class (H)

1. Give examples to show that the following inclusion relationships are strict:

$$\mathrm{Lip}[0,1] \subset AC[0,1] \subset BV[0,1].$$

2. Known: Two sequences of non-negative integrable functions $\{f_j\}$, $\{g_j\}$ converge a.e. to f, g respectively. Also $|f_j(x)| \leq g_j(x)$. If $\{g_j\}$ converges in L^1 norm (to g), prove that $\{f_j\}$ also converges in L^1 norm (Generalized Dominated Convergence Theorem).

3. Known: Two sequences of non-negative integrable functions $\{f_j\}, \{g_j\}$ converge in measure to f, g respectively. Determine if $\{f_j g_j\}$ converges in measure to fg. Explain your reason.

4. Give an example to illustrate that Fubini's Theorem may not hold for general measurable functions (if conditions like σ-finiteness or integrability are violated).

5. Prove that there exist countable disjoint closed balls $B_j \subset [0,1]^3$ in $\mathbb{R}^3$ such that:

$$m\left([0,1]^3 \setminus \left(\bigcup_j B_j\right)\right) = 0.$$

6. Let $f(x) \in L^1(E)$ be an integrable function. Write the definition of the distribution function $f_*(t)$ (or $m(\{x : |f(x)| > t\})$) and prove:

$$\int_E |f(x)|\, dx = \int_0^\infty f_*(t)\, dt.$$

7. True or False (Give reasons):

(a) For a monotonically increasing function $f : \mathbb{R} \to \mathbb{R}$, the derivative exists almost everywhere, the derivative is measurable, and $f' \in L^1_{loc}(\mathbb{R})$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a monotonic function, then $\int_0^1 f'(x) = f(1) - f(0)$.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is Generalized Riemann integrable and Lebesgue integrable, then the two integral values are the same.

8. Prove that the space $L^\infty(E)$ is a Banach space.

9. Let $f : [0, 1] \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} x^{3/2} \sin(1/x), & x > 0 \\ 0, & x = 0 \end{cases}$$

Prove that the total variation $V_0^1(f) \leq 3$.

10. Let $f \in L^1[a, b]$. Prove that:

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0$$

holds almost everywhere (Lebesgue Differentiation Theorem).

4 Year 2022

4.1 Standard Class

1. Concepts (20 points)

(a) State Lusin's Theorem (Hint: The theorem concerning measurable functions and continuity).

(b) State the Dominated Convergence Theorem.

(c) Let f be a finite-valued measurable function defined on $\mathbb{R}^d$. State the definition of the Hardy-Littlewood maximal function of f.

(d) State the definition of an absolutely continuous function defined on $[a, b]$.

2. Counterexamples (20 points) Construct specific sequences of real-valued functions defined on $\mathbb{R}$ to illustrate the following propositions:

(a) L^1 convergence does not guarantee almost everywhere convergence.

(b) L^3 convergence does not guarantee L^2 convergence.

(c) Convergence in measure does not guarantee L^1 convergence.

(d) Convergence in measure does not guarantee almost everywhere convergence.

3. Differentiability and Variation (10 points) Let $f : [0, 1] \to \mathbb{R}$ have a continuous derivative. Is f a function of bounded variation? Explain why.

4. Bounded Convergence on Balls (15 points) Let B be the unit ball in $\mathbb{R}^d$. Let $f_n : B \to \mathbb{R}$ be a sequence of measurable functions satisfying:

(a) f_n converges to f almost everywhere.

(b) $\|f_n\|_{L^\infty(B)} \leq 1$ for all n.

Prove:

$$\lim_{n\to\infty}\int_B f_n = \int_B f.$$

5. **Series of AC Functions** (15 points) Let $\{f_n\}$ be a sequence of monotonically increasing absolutely continuous functions defined on $[a,b]$. If the series $\sum f_n$ converges pointwise to f on $[a,b]$, prove that f is also absolutely continuous.

6. **Approximation of Identity** (15 points) Let $\phi \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\phi = 1$. For any $t > 0$, define $\phi_t(x) = t^{-d}\phi(x/t)$.
   (a) If f is a continuous function with compact support, prove that $f * \phi_t$ converges uniformly to f as $t \to 0$.
   (b) If $f \in L^1(\mathbb{R}^d)$, prove that $\lim_{t\to 0}\|f * \phi_t - f\|_{L^1} = 0$.

7. **Singular Behavior** (5 points) Let E be a specified set of measure zero in $\mathbb{R}$. Prove that there exists a monotonic function f such that f is not differentiable on the set E.

5 Year 2021

5.1 Standard Class

1. **True or False** (10 points) Judge whether the following are true or false and provide reasons:
   (a) Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of uniformly bounded integrable functions on $\mathbb{R}^n$. If this sequence converges almost everywhere to f, then there exists a subsequence that converges to f in measure.
   (b) Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of uniformly bounded non-negative integrable functions on $\mathbb{R}^n$. If the sequence converges uniformly to a non-negative integrable function f, then there exists a subsequence $\{f_{i_k}\}$ such that $\lim_{k\to\infty}\int f_{i_k} = \int f$.

2. **Integrability Condition** (15 points) Let $E \subset \mathbb{R}$ with $0 < m(E) < \infty$, and let $f(x)$ be non-negative and measurable on $\mathbb{R}$. Prove that $f \in L^1(\mathbb{R})$ if and only if $g(x) = \int_E f(x-t)\,dt$ is integrable on $\mathbb{R}$.

3. **Properties of AC Functions** (15 points) Assume $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous.
   (a) Prove that f maps sets of measure zero to sets of measure zero.
   (b) Prove that f maps measurable sets to measurable sets.

4. **Maximal Function Integrability** (15 points) If $f \in L^1(\mathbb{R}^d)$ and f is not identically zero, prove that the Hardy-Littlewood maximal function f^* is not in $L^1(\mathbb{R}^d)$.

5. **Bounded Variation Construction** (15 points) Let H be a function with period 2 on $\mathbb{R}$ given by:

$$H(x) = \begin{cases} 0, & 2k-1 < x \le 2k \\ 1, & 2k < x \le 2k+1 \end{cases}$$

   where $k \in \mathbb{Z}$. Prove that the function

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}H(2^n x)$$

   is not a function of bounded variation on $[0,1]$.

6. Nowhere Monotonic Continuous Function (15 points) We wish to construct a continuous function on $[0, 1]$ that is not monotonic on any subinterval.

(a) Prove: There exists a measurable subset A of $[0, 1]$ such that for any subinterval $I \subset [0, 1]$, we have $0 < m(A \cap I) < m(I)$. (Hint: Use a construction similar to the Cantor set).

(b) Construct a continuous function on $[0, 1]$ such that it is not monotonic on any subinterval of $[0, 1]$. (Hint: Use the conclusion from part (1) and the Fundamental Theorem of Calculus).

5.2 Honors Class

1. Is a Generalized Riemann integrable function on $\mathbb{R}$ necessarily Lebesgue integrable? Does the converse hold? (Give examples).

2. Let $f \in BV[0, 1]$. Is the set $E = \{x \in [0, 1] \mid f'(x) = \infty\}$ Lebesgue measurable?

3. Let $f, g \in AC[0, 1]$ and $g([0, 1]) \subset [0, 1]$. Is it necessarily true that $f \circ g \in AC[0, 1]$?

4. Let $f \in L^p(E)$. Prove:

$$\int_E |f(x)|^p \, dx = \int_0^\infty p\lambda^{p-1} m(\{x \in E \mid |f(x)| > \lambda\}) \, d\lambda$$

5. Let μ be an abstract positive measure. Define $\|f\|_{L^\infty} = \inf\{M > 0 \mid \mu(\{x : |f(x)| > M\}) = 0\}$.

(a) For $f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ and Lebesgue measure m, prove $\|f\|_{L^\infty} = \sup_{x\in\mathbb{R}} |f(x)|$.

(b) For the Dirac measure δ_0, give an example showing (1) is false.

(c) If $f \in L^\infty(\mathbb{R})$, does there necessarily exist $g \in C(\mathbb{R})$ such that $f = g$ a.e.?

6. Let $\{f_k\}$ be a countable sequence of measurable functions on E satisfying $\int_E f_k^2 \, dx \le C$ and $\int_E f_k f_j \, dx = 0$ for $k \neq j$. Prove that $\lim_{n\to\infty} \sum_{k=1}^{n^2} (n^{-\beta} f_k)^2 = 0$ a.e. (for appropriate β derived from the inequality provided).

7. Let $g \in \mathcal{L}^+(E)$ satisfying the weak-type inequality $m\{x \in E \mid g(x) > t\} \le \frac{1}{t} \int_E f(x) \, dx$. For $p \in (1, \infty)$, prove:

$$\left(\int_E (g(x))^p \, dx\right)^{1/p} \le \frac{p}{p-1} \left(\int_E (f(x))^p \, dx\right)^{1/p}$$

8. Let E be a null set. Let $\{\mathcal{O}_k\}$ be a sequence of open sets containing E with $m(\mathcal{O}_k) < 2^{-k}$. Define $\Phi(x) = \int_0^x \sum_{k=1}^\infty \chi_{\mathcal{O}_k}(t) \, dt$. Prove $\Phi \in C(\mathbb{R})$ and discuss the behavior of $\Phi'(x)$ for $x \in E$.

9. Let f be a measurable function on $\mathbb{R}^2$ with $\|f\|_{L^1} = 1$. By estimating $\int_{|x|\le 1} \int_{\mathbb{R}^2} \frac{f(y)}{|y-x|} dy dx$, prove there exists $|x_0| \le 1$ such that $\int_{\mathbb{R}^2} \frac{f(y)}{|y-x_0|} dy < 2021$.

10. Let $f(x)$ be differentiable everywhere on $\mathbb{R}$, with $f \in L^2(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$. Prove:

(a) For any closed interval $[a, b]$, $f \in AC[a, b]$.

(b) $\lim_{x\to\infty} f(x) = 0$.

6 Year 2020

6.1 Standard Class

1. **Positivity from Integral** (20 points) If f is a real-valued integrable function on $\mathbb{R}$, and for any measurable set E, $\int_E f(x)\,dx \geq 0$, prove that $f \geq 0$ almost everywhere.

2. **Measurability of Slices** (20 points) Determine if correct (proof not required) or false (give counterexample):

   (a) If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then for almost every $y \in \mathbb{R}^{d_2}$, the slice $E^y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\}$ is measurable in $\mathbb{R}^{d_1}$.

   (b) If for almost every $y \in \mathbb{R}^{d_2}$, E^y is measurable in $\mathbb{R}^{d_1}$, then E is measurable in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

3. **Signed Measures** (20 points) Let ν, ν_1, ν_2 be signed measures on $(X, \mathcal{M})$ and μ be a positive measure. Prove:

   (a) If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

   (b) If $\nu_1 \perp \nu_2$, then $|\nu_1| \perp |\nu_2|$.

4. **Maximal Function Lower Bound** (20 points) Let f be an integrable function on $\mathbb{R}^d$ that is not identically zero. Prove there exists a constant $c > 0$ such that for all $|x| \geq 1$:

$$f^*(x) \geq \frac{c}{|x|^d}$$

   where f^* is the Hardy-Littlewood maximal function.

5. **Vitali Convergence Variant** (15 points) Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ satisfying:

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{a.e. } x \in [0, 1]$$

   and $\sup_n \|f_n\|_{L^2([0,1])} \leq 1$. Prove that $\lim_{n\to\infty} \|f_n\|_{L^1([0,1])} = 0$.

6. **Density Topology** (15 points) Let m denote the Lebesgue measure on $\mathbb{R}$ and $A \subset \mathbb{R}$ be a Lebesgue measurable set. Assume that for all real numbers $a < b$:

$$m(A \cap [a, b]) < \frac{b - a}{2}.$$

   Prove that $m(A) = 0$.

7. **Differentiation in L^1** (15 points) Let f be absolutely continuous on $\mathbb{R}$ and $f \in L^1(\mathbb{R})$. If

$$\lim_{t\to 0} \int_{\mathbb{R}} \left| \frac{f(x + t) - f(x)}{t} \right| dx = 0,$$

   prove that $f \equiv 0$.

6.2 Honors Class (Midterm)

1. Let $\{f_i\}_{i\in I}$ be a family of measurable functions. Let $g = \sup_{i\in I} f_i$. If I is countable, is g measurable? If I is uncountable, is g measurable? Prove your conclusion.

2. Prove: A non-empty perfect set is uncountable.

3. Let X be an infinite set. Prove that X has the same cardinality as $X \times X$.

4. Let $\{f_n\}$ and f be integrable functions satisfying $\int_0^1 |f_n(x) - f(x)| \le \frac{1}{n^2}$ for all n. Prove $f_n \to f$ almost everywhere.

5. Prove: There exists $f : [0,1] \to [0,1]$ such that f' exists and is integrable, satisfying the strict inequality $\int_0^1 f'(x)\,dx < f(1) - f(0)$.

6. Prove: There exists a non-measurable set W such that every measurable subset of W is a null set.

7. Calculate the limits:
$$\lim_{n\to\infty} \int_0^1 \frac{\sqrt{nx}}{1+nx}\,dx \quad \text{and} \quad \lim_{n\to\infty} \int_0^2 (1+x^{2n})^{1/n}\,dx.$$

8. Let $f : [0,1] \to [-1,1]$ satisfy: For all n and all $x_1, \ldots, x_n \in [0,1]$, $|\sum_{k=1}^n f(x_k)| \le 1$. Prove $f = 0$.

9. Let $m(E) < \infty$ and f be a measurable function on E. Prove: For any $\epsilon > 0$, there exists a bounded measurable function g such that $m(\{x \in E : f(x) \ne g(x)\}) < \epsilon$.

10. Let C be the standard Cantor set. Prove that C has no interior points, and $\{(x,y) : e^x y \in C\}$ is a measurable set in $\mathbb{R}^2$. Further prove there exists a set of positive measure with no interior points.

7 Year 2019

7.1 Standard Class

1. **Measure Theory Definitions** (15 points) Write down the definitions of a measure and a pre-measure. Describe the steps to construct a measure from a pre-measure.

2. **True or False** (20 points) Determine whether the following are true or false. Give a counterexample if false, or a brief proof if true.

   (a) Let f be monotonically increasing and almost everywhere differentiable on $[a,b]$. Then:
   $$\int_a^b f'(x)\,dx = f(b) - f(a).$$

   (b) Let E be a measurable set in $\mathbb{R}^d$. Then for almost every $x \in E$:
   $$\lim_{m(B)\to 0} \frac{m(B \cap E)}{m(B)} = 1$$
   where B denotes a ball containing x.

3. **Limit of Integral** (10 points) Let $f \in L^1(\mathbb{R})$. Calculate:
$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x-n)\frac{x}{1+|x|}\,dx.$$

4. **Distribution Functions** (15 points) Let f and g be non-negative real-valued measurable functions on $(0,1)$ satisfying:
$$m(\{x \in (0,1) : f(x) > \alpha\}) = m(\{x \in (0,1) : g(x) > \alpha\})$$
for all $\alpha > 0$. Prove that:
$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

5. **Bounded Variation** (15 points) Prove that the function defined by:

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x^2 \cos(1/x^2), & \text{if } 0 < x \le 1 \end{cases}$$

is not of bounded variation on $[0, 1]$.

6. **Luzin N-Property** (15 points) Let f be a real-valued function on $\mathbb{R}$ satisfying a Lipschitz-type condition (e.g., $|f(x) - f(y)| \le C|x - y|$ for all $x, y \in \mathbb{R}$). Prove that f maps every set of measure zero to a set of measure zero.

7. **Continuity of Convolution** (10 points) Let $E \subset \mathbb{R}$ be measurable with $m(E) > 0$. Let

$$f(x) := \int_{\mathbb{R}} \chi_E(tx)\chi_E(t)\, dt.$$

Prove that f is continuous at $x = 1$.

7.2 Honors Class

1. **Strictly Increasing Singular Function** (15 points) Is the following statement true? (Prove or give a counterexample): There exists a strictly monotonically increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} f'(x)\, dx = 0.$$

2. **Tonelli's Theorem Proof** (15 points) Provide a detailed proof of Tonelli's Theorem for the following special case: Assume K is a compact subset of $\mathbb{R}^2$ and χ_K is the characteristic function on K. Prove:

$$\int_{\mathbb{R}^2} \chi_K(x, y)\, dx\, dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_K(x, y)\, dy \right) dx.$$

3. **Orthogonal Basis and Limit** (15 points) Let $L^2[0, 2\pi]$ be the set of real-valued square-integrable functions on $[0, 2\pi]$, with the inner product defined as $\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)\, dx$.

   (a) Write down an orthonormal basis for $L^2[0, 2\pi]$.

   (b) Prove: $\lim_{n\to\infty}\langle f(x), \sin(nx)\rangle = 0$.

4. **Vitali Covering Application** (15 points) Assume $f : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing function, and for all $x \in \mathbb{R}$, the limit

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

exists (values in $\mathbb{R}$ or ∞). Use the Vitali Covering Lemma to prove:

$$m(\{x \in \mathbb{R} : f'(x) = \infty\}) = 0.$$

5. **Generalized Dominated Convergence** (15 points) Assume that for all $n \in \mathbb{N}$, f_n and f are non-negative Lebesgue integrable functions on $\mathbb{R}$ satisfying:

   (a) $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.

   (b) $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} f(x)\, dx$.

   Prove that:

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R})} = 0.$$

6. **BV implies AC Condition** (15 points) Assume α, β are positive constants. Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying:

$$f(x) = x^\alpha \sin(x^{-\beta}) \quad \text{for } x \in (0,1].$$

If $f \in BV[0,1]$, prove that $f \in AC[0,1]$. (Provide a detailed proof).

7. **Total Variation Calculation** (15 points) Assume $f(x) = x - \frac{1}{2}x^2$. Calculate the total variation $V_0^1(f)$.