

Midterm Exam 2025-2026 Fall

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Problem 1.

Suppose E is a set of finite outer measure. Prove or disprove the following.

1. E is measurable $\iff \exists G_\delta$ set G s.t. $E \subseteq G$ and $m(G) = m^*(E)$.
2. E is measurable $\iff \exists F_\sigma$ set F s.t. $F \subseteq E$ and $m(F) = m^*(E)$.

Solution.

(i) **Part \Rightarrow :** Suppose E is measurable. Assume $m^*(E)$ is finite. Claim: Suppose E is of finite outer measure then there exists an open set O such that $m^*(E) \leq m^*(O) \leq m^*(E) + \epsilon$.

Proof of claim. By definition of outer measure:

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} |I_k| : \{I_k\}_{k=1}^{\infty} \text{ open intervals covers } E \right\}$$

Then by definition of infimum, $\forall \epsilon > 0$, there exists a cover $\{I_k\}_{k=1}^{\infty}$ such that

$$m^*(E) \leq \sum_{k=1}^{\infty} |I_k| < m^*(E) + \epsilon$$

Now we take $O = \bigcup_{k=1}^{\infty} I_k$. O is open and $E \subseteq O$. By subadditivity, $m^*(O) \leq \sum |I_k|$.

$$m^*(E) \leq m^*(O) \leq m^*(E) + \epsilon$$

□

With the claim, for all $n \in \mathbb{N}$, there exists an open set $O_n \supseteq E$ s.t.

$$m^*(E) \leq m^*(O_n) \leq m^*(E) + \frac{1}{n}$$

Take $G = \bigcap_{n=1}^{\infty} O_n$. G is a G_δ set. Since $E \subseteq G \subseteq O_n$ for all n , we have $m^*(E) \leq m^*(G) \leq m^*(O_n) \leq m^*(E) + \frac{1}{n}$. As $n \rightarrow \infty$, $m^*(G) = m^*(E)$. Since E and G are measurable, $m(G) = m(E)$.

(i) Part \Leftarrow : Conversely, this is not true. Consider any non-measurable set N with finite outer measure. Then by the claim we just proved, there exists a G_δ set $G \supseteq N$ such that $m(G) = m^*(N)$. However, N is not measurable.

(ii) Part \Rightarrow : Suppose E is measurable. Assume $m(E)$ is finite. By inner regularity, for any $\epsilon > 0$, there exists a closed set $F' \subseteq E$ such that $m(E \setminus F') < \epsilon$. For all $n \in \mathbb{N}$, there exists a closed set $F_n \subseteq E$ s.t. $m(E \setminus F_n) < \frac{1}{n}$. Take $F = \bigcup_{n=1}^{\infty} F_n$. F is an F_σ set and $F \subseteq E$. $E \setminus F = E \setminus (\bigcup F_n) = \bigcap (E \setminus F_n)$. $m(E \setminus F) \leq m(E \setminus F_n) < \frac{1}{n}$ for all n . Thus $m(E \setminus F) = 0$. Since E is of finite measure, $m(E \setminus F) = m(E) - m(F)$. So $m(E) - m(F) = 0$, which implies $m(E) = m(F)$.

(ii) Part \Leftarrow : Suppose there is an F_σ set F such that $F \subseteq E$ and $m^*(E) = m(F)$. Note that $E = F \cup (E \setminus F)$. We must show $m^*(E \setminus F) = 0$. We use the excision property, which states that if $F \subseteq E$ and F is a measurable set of finite measure, $m^*(E) = m(F) + m^*(E \setminus F)$. Since F is F_σ , it is measurable. We are given $m(F) = m^*(E)$ and $m(F)$ is finite (since $m^*(E)$ is

finite). Substituting our given condition $m^*(E) = m(F)$ into the property:

$$m(F) = m(F) + m^*(E \setminus F)$$

This implies $m^*(E \setminus F) = 0$. F is measurable (since it is F_σ). The set $E \setminus F$ has outer measure 0. By completeness of the Lebesgue measure, $E \setminus F$ is also measurable. $E = F \cup (E \setminus F)$ is the union of two measurable sets, and is therefore also measurable.

Note for Problem 1.

We have proved the following equivalent descriptions of a measurable set E in our lectures:

1. $m^*(A) = m^*(E \cap A) + m^*(E^c \cap A)$ for any A . (Carathéodory's criterion)
2. $\forall \epsilon > 0, \exists$ open set $O \supseteq E$ such that $m^*(O \setminus E) < \epsilon$.
3. $\exists G_\delta$ set $G \supseteq E$ such that $m^*(G \setminus E) = 0$.
4. $\forall \epsilon > 0, \exists$ closed set $F \subseteq E$ such that $m^*(E \setminus F) < \epsilon$.
5. $\exists F_\sigma$ set $F \subseteq E$ such that $m^*(E \setminus F) = 0$.

One may find something subtle by comparing the those condition with this problem. But this is what makes this problem tricky. Professor Ni promised that this problem appears every year he taught. For further reading, one can show that A is measurable if and only if outer measure = inner measure.

Problem 2.

The set of pointwise convergence of f_n is a measurable set.

Solution.

The statement is **true** (assuming f_n are measurable functions).

A sequence of real numbers $\{a_n\}$ converges if and only if it is a Cauchy sequence. Let E be the set of pointwise convergence. An element x is in E if and only if $\{f_n(x)\}$ is a Cauchy sequence.

$$x \in E \iff \forall k \in \mathbb{N}, \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, |f_n(x) - f_m(x)| < \frac{1}{k}$$

We can write this set E using set operations:

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \left\{ x \mid |f_n(x) - f_m(x)| < \frac{1}{k} \right\}$$

Since each f_n is measurable, the function $g_{n,m}(x) = f_n(x) - f_m(x)$ is measurable. The absolute value $|g_{n,m}(x)|$ is also measurable. Therefore, the set $\{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$ is measurable for all k, n, m . The set E is formed by countable intersections and countable unions of measurable sets. Since the collection of measurable sets is a σ -algebra, E must be measurable.

Note for Problem 2.

Since the convergent functions may not exist, then we use Cauchy Criteria for writing down the set f_n converges. This is the only trick in this problem.

Problem 3.

Let f_n be everywhere measurable on E .

1. $\forall \eta > 0, m(\{x \in E \mid |f_n(x)| > \eta\}) \rightarrow 0$ as $n \rightarrow \infty. \Rightarrow f_n \rightarrow 0$ pointwise a.e. on E . ?
2. $\forall \delta > 0, \sum_{n=1}^{\infty} m(\{x \in E \mid |f_n(x)| > \delta\}) < \infty. \Rightarrow f_n \rightarrow 0$ pointwise a.e. on E . ?

Solution.

(ii) We want to show $f_n \rightarrow 0$ pointwise a.e. on E . Consider the set where $f_n \not\rightarrow 0$, which is

$$\bigcup_{k=1}^{\infty} \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E \mid |f_n(x)| \geq \frac{1}{k}\} \right)$$

To show this set has measure zero, it is equivalent to show that for any $k \geq 1$,

$$m \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E \mid |f_n(x)| \geq \frac{1}{k}\} \right) = 0$$

From (ii), we are given $\sum_{n=1}^{\infty} m(\{x \in E \mid |f_n(x)| > \delta\}) < \infty$ for all $\delta > 0$. Let $\delta = 1/k$. Then $\sum_{n=1}^{\infty} m(\{x \in E \mid |f_n(x)| \geq 1/k\}) < \infty$. By the Borel-Cantelli Lemma, the measure of the "lim sup" set is zero. This is a standard use of the Borel-Cantelli Lemma.

Note for Problem 3.

The first one is the standard example. The proof of the second one is a standard technique. We apply the Borel Cantelli lemma to show the set where f_n is not convergent is of measure 0.

Problem 4.

Does there exist a continuous function f on \mathbb{R} s.t. $f = \chi_{[0,1]}$ a.e.?

Solution.

By continuity of f , it is obviously not true. Here is a sketch of proof. Suppose such an f exists.

Step 1: $f \equiv 1$ on $[0, 1]$. (If $f(x_0) \neq 1$ for $x_0 \in (0, 1)$, by continuity $f \neq 1$ on an open interval, which has positive measure. This contradicts $f = 1$ a.e. on $(0, 1)$. So $f = 1$ on $(0, 1)$, and by continuity $f = 1$ on $[0, 1]$.)

Step 2: By continuity of f , we get a contradiction at the end point. (Similarly, $f \equiv 0$ on $(-\infty, 0)$ and $(1, \infty)$. By continuity, $\lim_{x \rightarrow 0^-} f(x) = 0$, but $f(0) = 1$ from Step 1. This contradicts continuity at $x = 0$.)

Problem 5.

$$\text{Let } f(x) = \begin{cases} 1 & x \in [0, 1] \cap \mathbb{Q} \\ 0 & x \in [0, 1] \setminus \mathbb{Q} \end{cases}.$$

1. $\forall \epsilon > 0$, \exists a closed set $F_\epsilon \subseteq [0, 1]$ s.t. $m([0, 1] \setminus F_\epsilon) < \epsilon$ and $\exists g_\epsilon$ continuous on $[0, 1]$ s.t. $f = g_\epsilon$ on F_ϵ .
2. $\forall \epsilon > 0$, \exists a closed set $F_\epsilon \subseteq [0, 1]$ with $m([0, 1] \setminus F_\epsilon) < \epsilon$ and a seq of continuous functions $g_{n,\epsilon}$ converges to f uniformly on F_ϵ .

(A direct Quote of Lusin and Egroff is not allowed.)

Solution.

(i) Set $\mathbb{Q} \cap [0, 1] = \{a_1, a_2, \dots, a_n, \dots\}$. $\forall \epsilon > 0$, let I_k be an open interval $I_k \ni a_k$ s.t. $l(I_k) < \frac{\epsilon}{2^{k+1}}$. Let $O_\epsilon = \bigcup_{k=1}^{\infty} I_k$. O_ϵ is open. Let $F_\epsilon = [0, 1] \setminus O_\epsilon$. F_ϵ is closed. $m(O_\epsilon) \leq \sum_{k=1}^{\infty} m(I_k) \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2}$. Then $m([0, 1] \setminus F_\epsilon) = m(O_\epsilon) < \epsilon$. (Note: The solution for g_ϵ is not written, but $g_\epsilon(x) \equiv 0$ would work, since $F_\epsilon \subseteq [0, 1] \setminus \mathbb{Q}$, where $f = 0$.)

(ii) $g_{n,\epsilon} = 1$ on $[0, 1]$, with the same F_ϵ given above.

Note for Problem 5.

Although the closed set F_ϵ and the function g is promisee by the Lusin theorem and Egroff theorem. This Quetion asks us to somehow construct a "concrete" F_ϵ and a concrete g for the Dirchlet Fuction. The aim is to somehow test your understanding and help you understand Lusin and Egroff better.

Problem 6.

Prove or disprove that the sequence of function $\{k \sin(kx) \mid k \in \mathbb{N}, x \in [0, 1]\}$ is equi-integrable.

Solution.

The statement is **false**. The sequence is **not** equi-integrable.

Let $f_k(x) = k \sin(kx)$. A sequence is equi-integrable on $E = [0, 1]$ if it satisfies two conditions. We will show that the second condition is violated:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. for any measurable } A \subseteq E \text{ with } m(A) < \delta, \text{ we have } \sup_k \int_A |f_k| < \epsilon$$

We will show this fails for $\epsilon = 1$. We must show that for any $\delta > 0$, we can find a set A and an integer k such that $m(A) < \delta$ but $\int_A |f_k(x)| dx \geq 1$.

Given any $\delta > 0$, choose an integer $k \in \mathbb{N}$ large enough so that $\frac{\pi}{k} < \delta$. Now, define the set $A_k = [0, \pi/k]$. The measure of this set is $m(A_k) = \pi/k < \delta$.

Now, we compute the integral for this k over this set A_k :

$$\int_{A_k} |f_k(x)| dx = \int_0^{\pi/k} |k \sin(kx)| dx$$

For $x \in [0, \pi/k]$, we have $kx \in [0, \pi]$, so $\sin(kx) \geq 0$.

$$\int_0^{\pi/k} k \sin(kx) dx$$

Let $u = kx$. Then $du = k dx$. The bounds of integration change from $x = 0 \rightarrow u = 0$ and $x = \pi/k \rightarrow u = \pi$.

$$\int_0^{\pi} \sin(u) du = [-\cos(u)]_0^{\pi} = (-\cos(\pi)) - (-\cos(0)) = -(-1) - (-1) = 2$$

Thus, for any $\delta > 0$, we found a set $A_k = [0, \pi/k]$ with $m(A_k) < \delta$, but $\int_{A_k} |f_k(x)| dx = 2$. This violates the condition for $\epsilon = 1$ (or any $\epsilon \leq 2$). Therefore, the sequence $\{k \sin(kx)\}$ is not equi-integrable on $[0, 1]$.

Problem 7.

Let $\{f_k\}$ be an integrable sequence on E . $\forall \epsilon > 0, \exists N$ s.t. $\int_E |f_n - f_m| < \epsilon$, $\forall m, n \geq N$.

1. Does $\{f_k\}$ have a subsequence converging pointwise a.e. to an integrable function f on E ?
2. Does $\int_E f_k$ converge?

Solution.

(i) Step 1: The sequence is Cauchy in measure. By Chebyshev's inequality:

$$m(\{x \in E \mid |f_n(x) - f_m(x)| > \eta\}) \leq \frac{1}{\eta} \int_E |f_n - f_m|$$

Since $\{f_n\}$ is Cauchy in L^1 , for any $\epsilon' > 0, \eta > 0$, we can choose N s.t. for $n, m \geq N$, $\int_E |f_n - f_m| < \epsilon' \eta$. Thus $m(\{x \in E \mid |f_n - f_m| > \eta\}) < \epsilon'$. So the sequence f_n is Cauchy in measure.

Step 2: Since f_n is Cauchy in measure, we can find a subsequence $\{f_{n_j}\}$ s.t. for each j ,

$$m(\{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| > \frac{1}{2^j}\}) < \frac{1}{2^j}$$

Let $E_j = \{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| > \frac{1}{2^j}\}$. We have $\sum_{j=1}^{\infty} m(E_j) < \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 < \infty$. By the Borel-Cantelli Lemma, the set $E^* = \limsup E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$ has $m(E^*) = 0$. If $x \notin E^*$, then x is in only finitely many E_j . So $\exists J$ s.t. $\forall j \geq J, x \notin E_j$. This means $\forall j \geq J, |f_{n_{j+1}}(x) - f_{n_j}(x)| \leq \frac{1}{2^j}$. The series $\sum_{j=J}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x))$ converges absolutely (by comparison with $\sum 1/2^j$). This implies the sequence $\{f_{n_j}(x)\}$ is a Cauchy sequence of real numbers, and thus converges. This holds for all $x \notin E^*$, so $\{f_{n_j}\}$ converges pointwise a.e. to some function f .

(ii) Trivially converges. (Proof: We know $|\int_E f_n - \int_E f_m| = |\int_E (f_n - f_m)| \leq \int_E |f_n - f_m|$. Since $\{f_n\}$ is Cauchy in L^1 , $\{\int_E f_n\}$ is a Cauchy sequence of real numbers, so it converges.)

Note for Problem 7.

This is a variant of homework question, in fact much easier than the homework, you can check it on problem-13 on section 5.2 of Royden.