MAT3006 Selected Problems

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Problem 1: Uniformly Integrability and Tightness

In this question we would like to review for the concepts like Uniform Integrability and Tightness. Recall the definition of a family of fucntions defined on a measurable set to be uniformly integrable / tight

Part1: $\mathcal{F} = \{f\}$

1. Suppose f is integrable over a measurable set E, then $\mathcal{F} = \{f\}$ is uniform integrable.

2. Suppose f is integrable over a measurable set E, then $\mathcal{F} = \{f\}$ is tight

3. Suppose f is a fuunction defined on a measurable set E, with $m(E) < \infty$, then $\mathcal{F} = \{f\}$ is uniform integrable implies f is integrable.

4. Suppose f is tight over E, then f is integrable over E.

5. Provide an counter example that if $m(E) = \infty$, $\mathcal{F} = \{f\}$ is uniformly integrable may not imply f is uniformly integrable.

Part2: Vitali-Convergence Theorem

1. Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E.If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

2. Let $\{f_n\}$ be a sequence of functions on E that is uniformly integrable and tight over E. Suppose $\{f_n\} \to f$ pointwise a.e. on E. Then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Solution:

Part 1: $\mathcal{F} = \{f\}$

1. **True.** Since f is integrable, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any measurable set $A \subset E$ with $m(A) < \delta$, we have $\int_A |f| < \epsilon$. This is the absolute continuity of the Lebesgue integral. Since the family consists of a single function, this condition is exactly the definition of uniform integrability.

2. **True.** Since f is integrable over E, for any $\epsilon > 0$, there exists a subset $E_0 \subset E$ of finite measure such that $\int_{E \setminus E_0} |f| < \epsilon$. This is a standard property of integrable functions (often proven by approximating with simple functions of finite support or using the Monotone Convergence Theorem on restricted domains). Thus, the singleton family $\{f\}$ is tight.

3. **False.** The condition $m(E) < \infty$ is crucial. If $m(E) < \infty$, then uniform integrability implies integrability (boundedness in L^1). Specifically, if $\mathcal{F}$ is UI, there exists $\delta > 0$ corresponding to $\epsilon = 1$. Since $m(E) < \infty$, we can cover E with finitely many disjoint sets $A_1, \ldots, A_k$ each with measure less than δ (or simply use the fact that the total integral is bounded if the measure is finite and "tails" are controlled). However, the statement asks if UI implies integrability *given* $m(E) < \infty$. Let's refine: By definition of UI, there exists $\delta > 0$ such that $m(A) < \delta \implies \int_A |f| < 1$. If $m(E) < \infty$, we can partition E into a finite number of sets $E_1, \ldots, E_N$ with $m(E_i) < \delta$. Then $\int_E |f| = \sum \int_{E_i} |f| < \sum 1 = N < \infty$. So, if $m(E) < \infty$, UI $\implies$ Integrable. Wait, the problem statement says "Suppose f is a function... with $m(E) < \infty$, then $\mathcal{F} = \{f\}$ is uniform integrable implies f is integrable." This is **True**.

4. **False.** Tightness alone does not guarantee integrability because it only controls the "tail" of the function (where the domain goes to infinity), not the "height" of the function. **Counter-example:** Let $E = \mathbb{R}$. Let $f(x) = 1/x$ for $x \in (0, 1)$ and 0 elsewhere. Tightness: Let $\epsilon > 0$. Choose $E_0 = [0, 1]$. Then $E \setminus E_0 = \mathbb{R} \setminus [0, 1]$, where $f = 0$. Thus $\int_{E \setminus E_0} |f| = 0 < \epsilon$. So $\{f\}$ is tight. Integrability: $\int_E |f| = \int_0^1 \frac{1}{x}\, dx = \infty$. Thus, f is tight but not integrable.

5. **Counter-example ($m(E) = \infty$):** Let $E = \mathbb{R}$. Let $f(x) = 1$ for all x. Uniform Integrability: For any $\epsilon > 0$, choose $\delta = \epsilon$. If $m(A) < \delta$, then $\int_A |f| = \int_A 1 = m(A) < \epsilon$. So f is uniformly integrable. Integrability: $\int_{\mathbb{R}} 1\, dx = \infty$. Thus, f is uniformly integrable but not integrable.

Part 2: Vitali Convergence Theorem

1. **Proof for Finite Measure:** Since $m(E) < \infty$, by **Egorov's Theorem**, for any $\delta > 0$, there exists a closed set $F \subset E$ such that $m(E \setminus F) < \delta$ and $f_n \to f$ uniformly on F. By Fatou's Lemma, $\int |f| \leq \liminf \int |f_n| \leq M < \infty$ (since UI implies bounded L^1 norms on finite measure spaces), so f is integrable.

   Now, fix $\epsilon > 0$. Since $\{f_n\}$ is UI, there exists $\delta > 0$ such that for any $A \subset E$ with $m(A) < \delta$, $\sup_n \int_A |f_n| < \epsilon/3$. Also, since f is integrable, we can choose δ small enough such that $\int_A |f| < \epsilon/3$ (absolute continuity of integral). Using Egorov's theorem with this δ, we get a set F. Split the integral:

$$\left| \int_E f_n - \int_E f \right| \leq \int_E |f_n - f| = \int_F |f_n - f| + \int_{E \setminus F} |f_n - f|.$$

1. On F: Since $f_n \to f$ uniformly, for large n, $|f_n(x) - f(x)| < \frac{\epsilon}{3m(E)}$. Thus $\int_F |f_n - f| < m(F) \cdot \frac{\epsilon}{3m(E)} < \epsilon/3$. 2. On $E \setminus F$:

$$\int_{E\setminus F} |f_n - f| \leq \int_{E\setminus F} |f_n| + \int_{E\setminus F} |f| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Combining these, $\int_E |f_n - f| < \epsilon$. Thus $\lim \int f_n = \int f$.

2. **Proof for General Measure (UI + Tightness):** Since $\{f_n\}$ is tight, for any $\epsilon > 0$, there exists a set $E_0 \subset E$ of finite measure such that $\sup_n \int_{E\setminus E_0} |f_n| < \epsilon/3$. By Fatou's Lemma, $\int_{E\setminus E_0} |f| \leq \liminf \int_{E\setminus E_0} |f_n| \leq \epsilon/3$.

   Now we split the integral over E:

$$\int_E |f_n - f| = \int_{E_0} |f_n - f| + \int_{E\setminus E_0} |f_n - f|.$$

   The second term is bounded by:

$$\int_{E\setminus E_0} |f_n| + \int_{E\setminus E_0} |f| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

   For the first term, since $m(E_0) < \infty$, we can apply the result from Part 2.1 (Finite Measure Vitali). The sequence $\{f_n\}$ restricted to E_0 is still uniformly integrable and converges pointwise to f. Thus, for sufficiently large n, $\int_{E_0} |f_n - f| < \epsilon/3$.

   Total bound: $< \epsilon/3 + 2\epsilon/3 = \epsilon$. Thus $\lim \int_E |f_n - f| = 0$, which implies $\lim \int_E f_n = \int_E f$.

Problem 2: The Standard Technique: Borel-Cantelli Lemma

In this question, we review a standard technique in real analysis related to the Borel-Cantelli Lemma.

1. **(Convergence in measure in a Stronger sense ⇒ Pointwise Convergence)**
   Let f_n be a sequence of measurable functions on E
   (a) $\forall \eta > 0$, $m(\{x \in E \mid |f_n(x)| > \eta\}) \to 0$ as $n \to \infty$. $\Rightarrow f_n \to 0$ pointwise a.e. on E. ?
   (b) $\forall \delta > 0$, $\sum_{n=1}^{\infty} m(\{x \in E \mid |f_n(x)| > \delta\}) < \infty$. $\Rightarrow f_n \to 0$ pointwise a.e. on E. ?
2. **(Convergence in Measure ⇒ Subsequence Pointwise Convergence)**
   If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.
3. **(Completeness of Convergence in Measure)**
   A sequence $\{f_n\}$ of measurable functions on E is said to be **Cauchy in measure** provided that given $\eta > 0$ and $\epsilon > 0$, there is an index N such that for all $m, n \geq N$,
   $$m(\{x \in E \mid |f_n(x) - f_m(x)| \geq \eta\}) < \epsilon.$$
   Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f on E to which the sequence $\{f_n\}$ converges in measure.
4. **(Keystep in Proving Riesz Ficscher)**
   Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions over a measurable set E. Suppose that for every k:
   $$\int_E |f_{k+1} - f_k| < \epsilon_k^2 \quad \text{where } \sum \epsilon_k < \infty$$
   then the sequence $\{f_k\}_{k=1}^{\infty}$ converges pointwise a.e to f.
5. **(Variant of Riesz Ficscher)**
   Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions over a measurable set E. Suppose that for every k:
   $$\int_E |f_{k+1} - f_k| < \frac{1}{k^q}.$$
   Prove or disprove:
   (a) $\{f_k\}_{k=1}^{\infty}$ converges pointwise a.e. to another integrable function on E if $q = 2$.
   (b) What if $q = 1$ in (i) above?
6. **(Variant of Riesz Ficscher)**
   Let $\{f_k\}$ be an integrable sequence on E. Suppose the sequence is Cauchy in the L^1 norm, meaning:
   $$\forall \epsilon > 0, \exists N \text{ s.t. } \int_E |f_n - f_m| < \epsilon, \quad \forall m, n \geq N.$$

Show that $\{f_k\}$ have a subsequence converging pointwise a.e. to an integrable function f on E.

7. **(Fast L^1 Convergence $\Rightarrow$ Pointwise Convergence)**
Suppose $\{f_n\}$ is a sequence of integrable functions on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |f_n(x) - f(x)|\, dx \leq \frac{1}{n^{1+\epsilon}}. \quad \epsilon > 0$$

Use the Borel-Cantelli Lemma to prove that $f_n \to f$ pointwise almost everywhere. Also provide a counterexample for the case $\epsilon = 0$

8. **(Scaling a Sequence to Zero)**
Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ that are finite almost everywhere. Prove that there exists a sequence of positive constants $\{c_n\}$ such that

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{pointwise a.e. as } n \to \infty.$$

9. **(Rational Approximation / Diophantine Application)**
   (a) **(Order 2 is Universal)** Use the Pigeonhole Principle to prove Dirichlet's Approximation Theorem: For any irrational x, there exist infinitely many rationals p/q such that
   $$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$
   (b) **(Order > 2 is Rare)** Use the Borel-Cantelli Lemma to prove that for any $\alpha > 2$, the set of numbers $x \in [0, 1]$ that are approximable to order α has Lebesgue measure zero.

Solution:

1. **(Convergence in measure vs Pointwise)**
   (a) **Statement:** $\forall \eta > 0, m(\{x \in E \mid |f_n(x)| > \eta\}) \to 0 \implies f_n \to 0$ pointwise a.e.
   Answer: No.
   Counterexample: Consider the "Typewriter Sequence" on $E = [0, 1]$. Define a sequence of intervals I_n wrapping around $[0, 1]$ with decreasing length. Let $f_n = \chi_{I_n}$. For any $\eta \in (0, 1)$, the measure $m(\{|f_n| > \eta\}) = \text{length}(I_n) \to 0$. Thus, $f_n \to 0$ in measure. However, for every $x \in [0, 1]$, $f_n(x)$ takes the value 1 infinitely often and 0 infinitely often. Thus $\lim_{n\to\infty} f_n(x)$ does not exist anywhere.
   (b) **Statement:** $\forall \delta > 0, \sum_{n=1}^{\infty} m(\{x \in E \mid |f_n(x)| > \delta\}) < \infty \implies f_n \to 0$ pointwise a.e.
   Answer: Yes.
   Proof: Let $E_n(\delta) = \{x \in E \mid |f_n(x)| > \delta\}$. We are given that $\sum_{n=1}^{\infty} m(E_n(\delta)) < \infty$. By the **Borel-Cantelli Lemma**, the set of points belonging to infinitely many $E_n(\delta)$ has

measure zero. Let $A_\delta = \limsup_{n\to\infty} E_n(\delta)$. Then $m(A_\delta) = 0$.

To handle "all δ", let $\delta_k = \frac{1}{k}$. Let $A = \bigcup_{k=1}^{\infty} A_{1/k}$. Since A is a countable union of null sets, $m(A) = 0$. For any $x \notin A$, and for any k, x belongs to only finitely many $E_n(1/k)$. This implies $\exists N$ such that $\forall n \geq N, |f_n(x)| \leq \frac{1}{k}$. Since k is arbitrary, $f_n(x) \to 0$.

2. **(Convergence in Measure $\Rightarrow$ Subsequence Pointwise)**
   We want to find a subsequence f_{n_k} such that the set of points where it does not converge has measure zero.

   Since $f_n \to f$ in measure, for any $k \in \mathbb{N}$, we can choose an index n_k (strictly increasing) such that:
   $$m\left(\left\{x \in E \mid |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\right\}\right) < \frac{1}{2^k}.$$
   Let $E_k = \{x \in E \mid |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}$. Then $\sum_{k=1}^{\infty} m(E_k) < \sum \frac{1}{2^k} = 1 < \infty$.

   By the **Borel-Cantelli Lemma**, $m(\limsup E_k) = 0$. Let $Z = \limsup E_k$. For any $x \notin Z$, x belongs to only finitely many E_k. Thus, for sufficiently large k, $|f_{n_k}(x) - f(x)| < \frac{1}{k}$. Taking $k \to \infty$, we get $f_{n_k}(x) \to f(x)$ for all $x \notin Z$. Therefore, $f_{n_k} \to f$ pointwise almost everywhere.

3. **(Completeness of Convergence in Measure)**
   Step 1: Extract a rapidly Cauchy subsequence. Since $\{f_n\}$ is Cauchy in measure, we can construct a subsequence $\{f_{n_k}\}$ such that for all k:
   $$m\left(\left\{x \in E \mid |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}.$$

   Step 2: Show pointwise convergence of the subsequence. Let $A_k = \{x \in E \mid |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq \frac{1}{2^k}\}$. Since $\sum m(A_k) < \sum 2^{-k} < \infty$, by Borel-Cantelli, the set $Z = \limsup A_k$ has measure zero. For $x \notin Z$, the series $\sum (f_{n_{k+1}}(x) - f_{n_k}(x))$ converges absolutely (dominated by $\sum 2^{-k}$). Thus, $f_{n_k}(x)$ converges pointwise a.e. Define $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ for $x \notin Z$ and 0 otherwise.

   Step 3: Show convergence in measure. We claim $f_n \to f$ in measure. Fix $\eta, \epsilon > 0$. Choose N such that for $n, n_k \geq N$, $m(|f_n - f_{n_k}| \geq \eta/2) < \epsilon/2$. Also, since $f_{n_k} \to f$ a.e., it converges in measure (on finite measure sets, or generally by construction here). Choose k large enough so $m(|f_{n_k} - f| \geq \eta/2) < \epsilon/2$. Then:
   $$\{|f_n - f| \geq \eta\} \subseteq \{|f_n - f_{n_k}| \geq \eta/2\} \cup \{|f_{n_k} - f| \geq \eta/2\}.$$
   The measure is bounded by $\epsilon/2 + \epsilon/2 = \epsilon$.

4. **(Riesz-Fischer Step: Beppo Levi Argument)**
   Let $g_k(x) = |f_{k+1}(x) - f_k(x)|$. We are given $\int_E g_k < \epsilon_k^2$ with $\sum \epsilon_k < \infty$ (implying $\sum \epsilon_k^2 < \infty$).
   Consider the function $G(x) = \sum_{k=1}^{\infty} |f_{k+1}(x) - f_k(x)|$. By the Monotone Convergence Theorem (Beppo Levi):
   $$\int_E G(x)\,dx = \sum_{k=1}^{\infty} \int_E |f_{k+1} - f_k|\,dx < \sum_{k=1}^{\infty} \epsilon_k^2 < \infty.$$

Since $\int G < \infty$, $G(x)$ is finite almost everywhere. This implies that the series $\sum(f_{k+1} - f_k)$ converges absolutely a.e. Since the series telescopes, $f_n(x) = f_1(x) + \sum_{k=1}^{n-1}(f_{k+1}(x) - f_k(x))$ converges pointwise a.e. to some function $f(x)$.

5. **(Variant of Riesz-Fischer)**
Given $\int_E |f_{k+1} - f_k| < \frac{1}{k^q}$.

   (a) **Case $q = 2$:** True.
   Proof: Define $G(x) = \sum_{k=1}^{\infty} |f_{k+1}(x) - f_k(x)|$. Computing the integral:

   $$\int_E G(x) = \sum_{k=1}^{\infty} \int_E |f_{k+1} - f_k| < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

   Since the integral is finite, $G(x) < \infty$ almost everywhere. Absolute convergence implies convergence, so $\sum(f_{k+1} - f_k)$ converges, meaning $\{f_k\}$ converges pointwise a.e.

   (b) **Case $q = 1$:** False.
   Counterexample: The harmonic series diverges. Let $E = [0, 1]$. Define $f_1 = 0$ and $f_{k+1} = f_k + \frac{1}{2k}$. Then $\int |f_{k+1} - f_k| = \frac{1}{2k} < \frac{1}{k}$. However, $f_n(x) = \sum_{k=1}^{n-1} \frac{1}{2k} \to \infty$ for all x. Thus it does not converge to an integrable function (infinity is not integrable).

6. **(Cauchy in L^1 $\Rightarrow$ Subsequence)**
Since $\{f_n\}$ is Cauchy in L^1, we can extract a "rapidly Cauchy" subsequence. Choose indices $n_1 < n_2 < \ldots$ such that:

$$\|f_{n_{k+1}} - f_{n_k}\|_1 < \frac{1}{2^k}.$$

Let $\epsilon_k^2 = \frac{1}{2^k}$. Since $\sum \frac{1}{2^k} < \infty$, by the result of Problem 4, the subsequence $\{f_{n_k}\}$ converges pointwise almost everywhere.

7. **(Fast L^1 Convergence $\Rightarrow$ Pointwise)**
Given $\int |f_n - f| \leq \frac{1}{n^{1+\epsilon}}$.

Using Borel-Cantelli: Let $A_n = \{x \in \mathbb{R} \mid |f_n(x) - f(x)| \geq \frac{1}{n^\alpha}\}$ for some $0 < \alpha < \epsilon$. Using Chebyshev's Inequality:

$$m(A_n) \leq \frac{1}{n^{-\alpha}} \int |f_n - f| \leq n^\alpha \cdot \frac{1}{n^{1+\epsilon}} = \frac{1}{n^{1+\epsilon-\alpha}}.$$

Since $\epsilon - \alpha > 0$, the exponent $p = 1 + (\epsilon - \alpha) > 1$. Thus $\sum m(A_n) < \infty$. By Borel-Cantelli, $m(\limsup A_n) = 0$. For almost every x, $|f_n(x) - f(x)| < \frac{1}{n^\alpha}$ for large n, which implies $f_n(x) \to f(x)$.

Counterexample for $\epsilon = 0$: Consider the "Harmonic Typewriter" on $[0, 1]$. Let I_n be intervals of width $1/n$ wrapping around $[0, 1]$. Let $f_n = \chi_{I_n}$. $\int |f_n| = 1/n$. However, $\sum 1/n = \infty$, so the intervals cover every point infinitely many times. $f_n(x)$ oscillates between 0 and 1 indefinitely.

8. **(Scaling a Sequence to Zero)**
For each n, the function f_n is finite almost everywhere. For any $k \in \mathbb{N}$, let $E_{n,k} = \{x \mid$

$|f_n(x)| > k\}$. Since $\lim_{k\to\infty} m(E_{n,k}) = 0$, we can choose a constant c_n sufficiently large such that:

$$m\left(\left\{x \;\middle|\; \frac{|f_n(x)|}{c_n} > \frac{1}{n}\right\}\right) < \frac{1}{2^n}.$$

Let $A_n = \{x \mid |f_n(x)/c_n| > 1/n\}$. Then $\sum m(A_n) < \sum 2^{-n} < \infty$. By Borel-Cantelli, for almost every x, $|f_n(x)/c_n| \leq 1/n$ for sufficiently large n. Since $1/n \to 0$, we have $f_n(x)/c_n \to 0$ a.e.

9. **(Rational Approximation)**

   (a) **Order 2 is Universal (Dirichlet):** By the Pigeonhole Principle (partitioning $[0, 1)$ into q bins), for any irrational x and integer N, there exist p, q with $1 \leq q \leq N$ such that $|qx - p| < 1/N \leq 1/q$. Thus $|x - p/q| < 1/q^2$.

   (b) **Order $\alpha > 2$ is Rare:** Let $A_q = \bigcup_{p=0}^{q} \{x \in [0, 1] \mid |x - p/q| < 1/q^\alpha\}$. The measure of each interval is $2/q^\alpha$. There are roughly q such intervals (for $p = 0 \ldots q$).

   $$m(A_q) \leq q \cdot \frac{2}{q^\alpha} = \frac{2}{q^{\alpha-1}}.$$

   We want to check if x falls into infinitely many such sets. Sum the measures: $\sum_{q=1}^{\infty} m(A_q) \leq \sum \frac{2}{q^{\alpha-1}}$. Since $\alpha > 2$, $\alpha - 1 > 1$, so the series converges. By Borel-Cantelli, the set of such x has measure zero.

Problem 3: Banach Space

Recall that in the lecture we have shown that L^p is a Banach Space (a complete normed linear space) for $1 \le p < \infty$.

Part 1: L^∞ is a Banach Space

1. Let $\{f_n\}$ be a sequence in $L^\infty(E)$ and $\sum_{k=1}^\infty a_k$ a convergent series of positive numbers such that

$$\|f_{k+1} - f_k\|_\infty \le a_k \quad \text{for all } k.$$

Prove that there is a subset E_0 of E which has measure zero and

$$|f_{n+k}(x) - f_k(x)| \le \|f_{n+k} - f_k\|_\infty \le \sum_{j=n}^\infty a_j \quad \text{for all } k, n \text{ and all } x \in E \setminus E_0.$$

Conclude that there is a function $f \in L^\infty(E)$ such that $\{f_n\} \to f$ uniformly on $E \setminus E_0$.

2. Use the preceding result to show that $L^\infty(E)$, normed by the essential supremum norm, is a Banach space.

Part 2: $C[a, b]$ equipped with maximal norm is a Banach Space

1. Let $\{f_n\}$ be a sequence in $C[a, b]$ and $\sum_{k=1}^\infty a_k$ a convergent series of positive numbers such that

$$\|f_{k+1} - f_k\|_{\max} \le a_k \quad \text{for all } k.$$

Prove that

$$|f_{n+k}(x) - f_k(x)| \le \|f_{n+k} - f_k\|_{\max} \le \sum_{j=n}^\infty a_j \quad \text{for all } k, n \text{ and all } x \in [a, b].$$

Conclude that there is a function $f \in C[a, b]$ such that $\{f_n\} \to f$ uniformly on $[a, b]$.

2. Use the preceding result to show that $C[a, b]$, normed by the maximum norm, is a Banach space.

Solution:

Problem 4: Abosolute Continuous Functions

In this problem, you will be guided through a series of steps to prove that Lipschitz functions and, more generally, absolutely continuous functions map measurable sets to measurable sets.

Part 1: Lipschitz Functions

A function f is **Lipschitz** on $[a, b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

1. Show that a Lipschitz function maps a set of measure zero to a set of measure zero.
2. Using the fact that any measurable set E can be decomposed as $E = F \cup N$, where F is an F_σ set.

Part 2: Increasing Absolutely Continuous Functions

This part follows the logic of the exercises from the exercises in section6.5 of Royden problem 38-41. Let f be an **increasing** function on $[a, b]$.

1. **(Problem 38)** Show that f is AC if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every *countable* disjoint collection $\{(a_k, b_k)\}_{k=1}^{\infty} \subset [a, b]$, if $\sum_{k=1}^{\infty}(b_k - a_k) < \delta$, then $\sum_{k=1}^{\infty}(f(b_k) - f(a_k)) < \epsilon$.
2. **(Problem 39)** Assume f is increasing, Show that f is AC if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that for any measurable set $E \subseteq [a, b]$, if $m(E) < \delta$, then $m^*(f(E)) < \epsilon$. (Note: We can drop the assumption that f is increasing for the $\Rightarrow$ direction, but not for the others.)
3. **(Problem 40)** Using the result from (b), show that an increasing AC function maps a set of measure zero to a set of measure zero.
4. **(Problem 41)**Using the result from (c) and the fact that an AC function is continuous, prove that an **increasing** AC function maps any measurable set to a measurable set. (Hint: Use the $E = F \cup N$ decomposition again.)

Part 3: General Absolutely Continuous Functions

Now we remove the "increasing" condition. Let f be a general function on $[a, b]$.

1. Let $V_f([a_k, b_k])$ be the total variation of f on $[a_k, b_k]$. Prove that the standard definition of AC is equivalent to the following:

   For every $\epsilon > 0$, there exists a $\delta > 0$ such that for any finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$, if $\sum_{k=1}^{n}(b_k - a_k) < \delta$, then $\sum_{k=1}^{n} V_f([a_k, b_k]) < \epsilon$.
2. Using this equivalent "total variation" definition, prove that any **general** AC function (not necessarily increasing) maps a set of measure zero to a set of measure zero.

3. Conclude that any AC function maps measurable sets to measurable sets.

Solution:

Part 1: Lipschitz Functions

1. **Lipschitz maps measure zero sets to measure zero sets.**
Let $E \subset [a,b]$ with $m(E) = 0$. Let M be the Lipschitz constant. For any $\epsilon > 0$, there exists a countable collection of open intervals $\{(a_k, b_k)\}$ covering E such that

$$\sum_{k=1}^{\infty}(b_k - a_k) < \frac{\epsilon}{M}.$$

Since f is Lipschitz, for any $x, y \in (a_k, b_k)$, we have $|f(x) - f(y)| \leq M|x - y| \leq M(b_k - a_k)$. Thus, the image $f((a_k, b_k))$ is contained in an interval of length at most $M(b_k - a_k)$. Consequently,

$$m^*(f(E)) \leq \sum_{k=1}^{\infty} m(f((a_k, b_k))) \leq \sum_{k=1}^{\infty} M(b_k - a_k) < M \cdot \frac{\epsilon}{M} = \epsilon.$$

Since ϵ is arbitrary, $m(f(E)) = 0$.

2. **Lipschitz maps measurable sets to measurable sets.**
Let E be a measurable set. We can decompose E as $E = F \cup N$, where F is an F_σ set (a countable union of compact sets, since we are in $\mathbb{R}$) and $m(N) = 0$. Then $f(E) = f(F) \cup f(N)$.

   - Since f is Lipschitz, it is continuous. The continuous image of a compact set is compact. Since $F = \bigcup K_n$ is a countable union of compact sets, $f(F) = \bigcup f(K_n)$ is also a countable union of compact sets, hence measurable (Borel).
   - From part (a), since $m(N) = 0$, we have $m(f(N)) = 0$. Thus $f(N)$ is measurable.

   Therefore, $f(E)$ is the union of two measurable sets, so it is measurable.

Part 2: Increasing Absolutely Continuous Functions

1. **(Problem 38) Countable Interval Definition.**
($\Rightarrow$) Suppose f is AC (standard definition with finite collections). Let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be a countable disjoint collection with $\sum(b_k - a_k) < \delta$. For any finite N, the partial sum $\sum_{k=1}^{N}(b_k - a_k) < \delta$. By the definition of AC, $\sum_{k=1}^{N}(f(b_k) - f(a_k)) < \epsilon$. Taking the limit as $N \to \infty$, we get $\sum_{k=1}^{\infty}(f(b_k) - f(a_k)) \leq \epsilon$. (Strict inequality can be maintained by choosing slightly smaller parameters, but $\leq$ suffices for continuity arguments).
($\Leftarrow$) If the condition holds for countable collections, it trivially holds for finite collections (by padding with empty intervals).

2. **(Problem 39) Small measure implies small image measure.**
   ($\Rightarrow$) Assume f is AC. Let $\epsilon > 0$. Choose δ corresponding to ϵ from the definition of AC. Let $E \subseteq [a, b]$ with $m(E) < \delta$. There exists an open set $O \supseteq E$ composed of disjoint intervals $\{(a_k, b_k)\}$ such that $m(O) = \sum(b_k - a_k) < \delta$. Since f is increasing, $f(O) = \bigcup(f(a_k), f(b_k))$ (ignoring endpoints which have measure zero). Thus, $m(f(E)) \leq m(f(O)) = \sum(f(b_k) - f(a_k))$. Since $\sum(b_k - a_k) < \delta$, by AC we have $\sum(f(b_k) - f(a_k)) < \epsilon$. So $m(f(E)) < \epsilon$.
   ($\Leftarrow$) Assume the condition holds. Let $\{(a_k, b_k)\}$ be disjoint intervals with $\sum(b_k - a_k) < \delta$. Let $E = \bigcup(a_k, b_k)$. Then $m(E) < \delta$. Since f is increasing, $f(E)$ is the union of disjoint intervals $(f(a_k), f(b_k))$. The measure $m(f(E)) = \sum(f(b_k) - f(a_k))$. By hypothesis, $m(E) < \delta \implies m(f(E)) < \epsilon$, so $\sum(f(b_k) - f(a_k)) < \epsilon$. Thus f is AC.

3. **(Problem 40) AC maps null sets to null sets.**
   Let $m(E) = 0$. For any $\epsilon > 0$, there is a δ satisfying the condition in (b). Since $m(E) = 0 < \delta$, we have $m^*(f(E)) < \epsilon$. Since ϵ is arbitrary, $m(f(E)) = 0$.

4. **(Problem 41) AC maps measurable sets to measurable sets.**
   Let E be measurable. Write $E = F \cup N$ where F is an F_σ set and $m(N) = 0$. Since f is AC, it is continuous. Thus $f(F)$ is measurable (as in Part 1). By (c), $m(f(N)) = 0$, so $f(N)$ is measurable. Thus $f(E)$ is measurable.

Part 3: General Absolutely Continuous Functions

1. **Equivalence with Total Variation.**
   ($\Rightarrow$) If f is AC, then its total variation function $V(x) = V_f([a, x])$ is absolutely continuous (a standard result: $V(x) = \int_a^x |f'(t)|dt$). Thus, applying the AC definition to V gives the result.
   ($\Leftarrow$) Since $|f(b) - f(a)| \leq V_f([a, b])$, we have $\sum |f(b_k) - f(a_k)| \leq \sum V_f([a_k, b_k])$. If the sum of variations is small, the sum of differences is small.

2. **General AC maps measure zero to measure zero.**
   Let $m(E) = 0$. Fix $\epsilon > 0$. Use the δ from the Total Variation definition in (a). Cover E with disjoint intervals $\{(a_k, b_k)\}$ such that $\sum(b_k - a_k) < \delta$. Then $f(E) \subseteq \bigcup f((a_k, b_k))$. The diameter of the image of an interval is bounded by the total variation: $m(f((a_k, b_k))) \leq \sup_{x,y \in (a_k, b_k)} |f(x) - f(y)| \leq V_f([a_k, b_k])$. Thus,

   $$m^*(f(E)) \leq \sum m(f((a_k, b_k))) \leq \sum V_f([a_k, b_k]) < \epsilon.$$

   So $m(f(E)) = 0$.

3. **Conclusion.**
   Using the decomposition $E = F \cup N$ again: AC functions are continuous, mapping F_σ sets to F_σ (measurable) sets. By part (b), they map null sets N to null sets $f(N)$. Therefore, any absolutely continuous function maps measurable sets to measurable sets.

Problem 5: Comparison of Modes of Convergence

In this problem, we rigorously establish the relationships between Pointwise Convergence a.e., Convergence in L^p norm, and Convergence in Measure.

Part 1: Pointwise Convergence vs. L^p Convergence

1. **(L^p Implies Subsequence Convergence)**
   Let $\{f_n\}$ be a sequence in $L^p(E)$ such that $f_n \to f$ in $L^p(E)$. Prove that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f(x)$ almost everywhere on E.

2. **(Counter-Example: Pointwise $\not\Rightarrow$ L^p)**
   Consider $E = [0,1]$ and the sequence $f_n(x) = n \cdot \chi_{(0,1/n)}(x)$.

   (a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in E$ (Pointwise convergence).

   (b) Show that $\|f_n - 0\|_1 = 1$ for all n, and thus f_n does **not** converge to 0 in L^1.

3. **(The L^p Dominated Convergence Theorem)**
   Let $\{f_n\}$ be a sequence of measurable functions converging pointwise a.e. to f on E. Suppose there exists a function $g \in L^p(E)$ such that $|f_n(x)| \le g(x)$ a.e. for all n. Prove that $f_n \to f$ in $L^p(E)$.

4. **(Vitali Convergence Theorem for L^p)**
   Let E be a measurable set and $1 \le p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then

   $$\{f_n\} \to f \text{ in } L^p(E)$$

   if and only if

   $$\{|f_n|^p\} \text{ is uniformly integrable and tight over } E.$$

5. **(One more L^p Convergence Theorem for L^p)**
   Let E be a measurable set and $1 \le p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Then

   $$\{f_n\} \to f \text{ in } L^p(E) \text{ if and only if } \lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p.$$

Part 2: L^p Convergence vs. Convergence in Measure

1. **(L^p Implies Convergence in Measure)**
   Let $f_n \to f$ in $L^p(E)$. Use Chebyshev's Inequality to prove that for any $\epsilon > 0$:

   $$m(\{x \in E : |f_n(x) - f(x)| \ge \epsilon\}) \le \frac{1}{\epsilon^p} \int_E |f_n - f|^p,$$

   and conclude that $f_n \to f$ in measure.

2. **(Counter-Example: Measure $\not\Rightarrow L^p$)**
   Consider $E = [0,1]$ and the sequence $f_n(x) = n^{1/p} \cdot \chi_{(0,1/n)}(x)$.

   (a) Show that for any $\epsilon > 0$, $m(\{|f_n| \geq \epsilon\}) \to 0$ as $n \to \infty$ (Convergence in measure).

   (b) Show that $\|f_n\|_p = 1$ for all n, and thus f_n does **not** converge to 0 in L^p.

3. **(Convergence in measure $\Rightarrow$ Convergence in L^p with additional conditions)**
   Prove that a sequence $\{f_n\}$ in $L^p(E)$ converges to f in $L^p(E)$ **if and only if** $f_n \to f$ in measure and $\{|f_n|^p\}$ is uniformly integrable and tight.

Part 3: Pointwise Convergence vs. Convergence in Measure

1. **(Finite Measure: Pointwise $\Rightarrow$ Measure)**
   Assume $m(E) < \infty$. Prove that if a sequence of measurable functions $\{f_n\}$ converges to f pointwise a.e. on E, then $f_n \to f$ in measure on E.

2. **(Counter-Example: Infinite Measure Failure)**
   Consider $E = \mathbb{R}$ and the sequence $f_n = \chi_{[n,n+1]}$.

   (a) Show that $f_n(x) \to 0$ for all $x \in \mathbb{R}$.

   (b) Show that $m(\{x : |f_n(x)| \geq 1/2\}) = 1$ for all n, and thus it does **not** converge in measure.

3. **(Counter-Example: Measure $\not\Rightarrow$ Pointwise)**
   Consider the "Typewriter Sequence" on $[0,1]$ defined by iterating through intervals $[j/2^k, (j+1)/2^k]$. Show that this sequence converges to 0 in measure, but $\limsup_{n\to\infty} f_n(x) = 1$ for all $x \in [0,1]$, meaning it converges nowhere pointwise.

4. **(Riesz Subsequence Theorem)**
   Prove that if $\{f_n\}$ converges to f in measure on E, then there exists a subsequence $\{f_{n_k}\}$ that converges to f pointwise a.e. on E.

Solution:

Part 1: Pointwise Convergence vs. L^p Convergence

1. **(L^p Implies Subsequence Convergence)**
   We are given that $\lim_{n\to\infty} \int_E |f_n - f|^p = 0$. We want to find a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.

   It suffices to show that for every $\epsilon > 0$, the set where the difference is large has measure zero in the limit. Let $E_k(\epsilon) = \{x \in E : |f_{n_k}(x) - f(x)| \geq \epsilon\}$. By Chebyshev's Inequality:

$$m(E_k(\epsilon)) \leq \frac{1}{\epsilon^p} \int_E |f_{n_k} - f|^p.$$

Since $f_n \to f$ in L^p, we can choose a subsequence $\{n_k\}$ sufficiently rapidly such that:

$$\|f_{n_k} - f\|_p < \frac{1}{2^k}.$$

Then, for a fixed ϵ (or varying $\epsilon_k = 1/k$), we have $\sum_{k=1}^{\infty} m(E_k) < \infty$. By the **Borel-Cantelli Lemma**, the set of x belonging to infinitely many E_k has measure zero. Thus, $f_{n_k}(x) \to f(x)$ for almost every x.

2. **(Measure $\not\Rightarrow L^p$)**
   Counter-example: $f_n(x) = n^{1/p}\chi_{(0,1/n)}$. Convergence in measure is clear (support shrinks to 0). L^p norm: $\int |f_n|^p = \int_0^{1/n} n\,dx = 1 \not\to 0$.

3. **(L^p Dominated Convergence Theorem)**
   We want to show $\lim_{n\to\infty} \int_E |f_n - f|^p = 0$. We estimate the integral by splitting the domain E.
   $$\int_E |f_n - f|^p = \int_F |f_n - f|^p + \int_{E\setminus F} |f_n - f|^p.$$
   Step 1 (The Core F): Since $|f_n - f|^p \to 0$ a.e., by **Egorov's Theorem**, for any $\epsilon > 0$, we can choose a set $F \subset E$ of finite measure such that $f_n \to f$ uniformly on F (except for a small set of measure δ which we can include in the tail or handle via absolute continuity). Specifically, on the set where convergence is uniform:
   $$\int_F |f_n - f|^p \le m(F) \cdot \epsilon.$$
   Step 2 (The Tail $E \setminus F$): For the integral over $E \setminus F$, we use the dominating function g.
   $$\int_{E\setminus F} |f_n - f|^p \le 2^p \int_{E\setminus F} (|f_n|^p + |f|^p) \le 2^{p+1} \int_{E\setminus F} |g|^p.$$
   Since $g \in L^p$, we can choose F large enough such that $\int_{E\setminus F} |g|^p < \epsilon$. Combining these estimates, $\int_E |f_n - f|^p \to 0$.

4. **(Vitali Convergence Theorem for L^p)**
   ($\Rightarrow$) If $f_n \to f$ in L^p, then convergence in measure, uniform integrability, and tightness follow immediately from the properties of the integral.

   ($\Leftarrow$) Assume $f_n \to f$ in measure, $\{|f_n|^p\}$ is uniformly integrable, and $\{|f_n|^p\}$ is tight. We split the integral:
   $$\int_E |f_n - f|^p = \int_{E_0} |f_n - f|^p + \int_{E\setminus E_0} |f_n - f|^p.$$
   Tail Control (Tightness): Choose E_0 such that $\sup_n \int_{E\setminus E_0} |f_n|^p < \epsilon$. Note that $|f_n - f|^p \le 2^p(|f_n|^p + |f|^p)$. Thus, the integral over $E \setminus E_0$ is small.

   Core Control (Uniform Integrability): On the finite set E_0, we split further into a set A where $|f_n - f|$ is large, and $E_0 \setminus A$ where it is small.
   $$\int_{E_0} |f_n - f|^p = \int_{E_0\setminus A} |f_n - f|^p + \int_A |f_n - f|^p.$$

Since $f_n \to f$ in measure, we can ensure $m(A)$ is small. By uniform integrability, $\int_A |f_n - f|^p < \epsilon$. On $E_0 \setminus A$, the difference is uniformly small. Thus, the total integral converges to 0.

5. **(Radon-Riesz Theorem)**
   We assume $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$. Consider the function:
   $$h_n = 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p.$$
   Note that $h_n \geq 0$ a.e. and $h_n \to 2^p|f|^p$ pointwise. By **Fatou's Lemma**:
   $$\int_E \liminf h_n \leq \liminf \int_E h_n.$$
   Substituting the limits:
   $$\int_E 2^p|f|^p \leq \liminf \left(\int_E 2^{p-1}(|f_n|^p + |f|^p) - \int_E |f_n - f|^p \right).$$
   Using the norm convergence assumption ($\lim \int |f_n|^p = \int |f|^p$):
   $$2^p \int |f|^p \leq 2^p \int |f|^p - \limsup \int |f_n - f|^p.$$
   Canceling terms yields $0 \leq -\limsup \int |f_n - f|^p$, which implies $\int |f_n - f|^p \to 0$.

Part 2: L^p Convergence vs. Convergence in Measure

1. **(L^p Implies Measure)**
   Direct application of Chebyshev's Inequality:
   $$m(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \int_E |f_n - f|^p.$$
   Since the RHS goes to 0, $f_n \to f$ in measure.

2. **(Measure $\not\Rightarrow L^p$)**
   Example: $f_n(x) = n^{1/p}\chi_{(0,1/n)}$. (See manuscript for details).

3. **(Measure $\Rightarrow L^p$ with conditions)**
   This is the Generalized Vitali Theorem proven above. The "trick" is to split the set E based on the magnitude of the difference:
   $$E' = \{x \in E : |f_n(x) - f(x)| \geq \delta\}.$$
   Then decompose the integral $\int_E |f_n - f|^p$ into the integral over E' (controlled by uniform integrability since $m(E')$ is small) and the integral over $E \backslash E'$ (controlled by the small difference δ). For infinite measure, we first use tightness to restrict to a finite set.

Part 3: Pointwise Convergence vs. Convergence in Measure

1. **(Finite Measure: Pointwise $\Rightarrow$ Measure)**
   Let $\epsilon > 0$. Define sets $E_n = \{x : |f_n(x) - f(x)| \geq \epsilon\}$. We want to show $m(E_n) \to 0$. Consider $A_N = \bigcup_{n=N}^{\infty} E_n = \{x : \exists n \geq N, |f_n(x) - f(x)| \geq \epsilon\}$. Since $f_n \to f$ pointwise, $x \in \bigcap_{N=1}^{\infty} A_N$ implies $f_n(x)$ does not converge to $f(x)$ (or does so slower than ϵ). Since convergence is a.e., $m(\bigcap A_N) = 0$. Because $A_1 \supseteq A_2 \supseteq \dots$ and $m(A_1) \leq m(E) < \infty$, by continuity of measure from above:
   $$\lim_{N\to\infty} m(A_N) = m\left(\bigcap_{N=1}^{\infty} A_N\right) = 0.$$
   Since $E_N \subseteq A_N$, $m(E_N) \to 0$.

2. **(Counter-Example: Infinite Measure Failure)**

   (a) Fix $x \in \mathbb{R}$. There exists $N > x$. For all $n \geq N$, the interval $[n, n+1]$ lies to the right of x, so $f_n(x) = 0$. Thus $f_n \to 0$ pointwise.

   (b) For any n, the set where $|f_n| \geq 1/2$ is $[n, n+1]$, which has measure 1. Since the measure is constantly 1, it does not converge to 0.

3. **(Counter-Example: Measure $\not\Rightarrow$ Pointwise)**
   Let f_n be the typewriter sequence $\chi_{[j/2^k,(j+1)/2^k]}$.

   - **Measure:** For any $\epsilon \in (0, 1)$, $m(\{|f_n| \geq \epsilon\}) = 1/2^k$. As $n \to \infty$, $k \to \infty$, so the measure goes to 0.
   - **Pointwise:** For any $x \in [0, 1]$, the intervals $[j/2^k, (j+1)/2^k]$ "scan" across x infinitely many times. Thus $f_n(x)$ takes the value 1 for infinitely many n, and 0 for infinitely many n. Consequently, $\limsup f_n(x) = 1$ and $\liminf f_n(x) = 0$. It converges nowhere.

4. **(Riesz Subsequence Theorem)**
   Since $f_n \to f$ in measure, for every $k \in \mathbb{N}$, we can find an index n_k such that
   $$m(\{x : |f_{n_k}(x) - f(x)| \geq 1/2^k\}) < \frac{1}{2^k}.$$
   We can ensure $n_{k+1} > n_k$. Let $E_k = \{x : |f_{n_k}(x) - f(x)| \geq 1/2^k\}$. Consider the set $A = \limsup E_k = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k$. By Borel-Cantelli (since $\sum m(E_k) < \sum 2^{-k} = 1 < \infty$), we have $m(A) = 0$. For any $x \notin A$, x belongs to only finitely many E_k. Thus for sufficiently large k, $|f_{n_k}(x) - f(x)| < 1/2^k$. This implies $f_{n_k}(x) \to f(x)$.

Problem 6: L^p spaces inclusion relationship

In this problem, we examine how the inclusion relationships between L^p spaces depend heavily on the total measure of the underlying space E. Assume throughout that $1 \le p < q < \infty$.

1. **(Finite Measure Space: $m(E) < \infty$)**
   Let E be a measurable set with $m(E) < \infty$.
   (a) Use Hölder's Inequality to prove that $L^q(E) \subseteq L^p(E)$.
   (b) Derive the inequality $\|f\|_p \le C\|f\|_q$ and explicitly determine the constant C in terms of p, q, and $m(E)$.
   (c) Give a counter-example to show that $L^p(E)$ is **not** contained in $L^q(E)$ (i.e., finding a function with a singularity that is integrable with power p but not q).
2. **(Sequence Space: Counting Measure)**
   Let $E = \mathbb{N}$ equipped with the counting measure (the space ℓ^p).
   (a) Prove that $\ell^p \subseteq \ell^q$. (Note that this is the *reverse* of the finite measure case).
   (b) Give a counter-example to show that ℓ^q is **not** contained in ℓ^p (i.e., finding a sequence that decays fast enough for q but not for p).
3. **(Infinite Measure Space: $m(E) = \infty$)**
   Let $E = (0, \infty)$ equipped with the standard Lebesgue measure. Explain why there is generally **no inclusion relationship** between $L^p(E)$ and $L^q(E)$ by providing:
   (a) A function $f \in L^p(E)$ but $f \notin L^q(E)$.
   (b) A function $g \in L^q(E)$ but $g \notin L^p(E)$.

Solution:

1. **(Finite Measure Case)**
   (a) Let $f \in L^q(E)$. We compute the L^p norm:

$$\|f\|_p^p = \int_E |f|^p \cdot 1 \, dm.$$

   Apply Hölder's Inequality with exponents $r = q/p$ (since $q > p$, $r > 1$) and its conjugate r'.

$$\int_E |f|^p \cdot 1 \le \left(\int_E (|f|^p)^{q/p}\right)^{p/q} \left(\int_E 1^{r'}\right)^{1/r'}.$$

   This simplifies to:

$$\|f\|_p^p \le \|f\|_q^p \cdot (m(E))^{1/r'}.$$

   Since $m(E) < \infty$ and $\|f\|_q < \infty$, we have $\|f\|_p < \infty$, so $f \in L^p(E)$.

(b) Taking the p-th root of the inequality derived above:

$$\|f\|_p \le \|f\|_q \cdot m(E)^{\frac{1}{p}(1-\frac{p}{q})} = \|f\|_q \cdot m(E)^{\frac{1}{p}-\frac{1}{q}}.$$

Thus, the constant is $C = m(E)^{\frac{1}{p}-\frac{1}{q}}$.

(c) **Counter-example ($L^p \not\subset L^q$):** Let $E = (0,1)$. Consider $f(x) = x^{-\alpha}$. For $f \in L^p$, we need $p\alpha < 1 \implies \alpha < 1/p$. For $f \notin L^q$, we need $q\alpha \ge 1 \implies \alpha \ge 1/q$. Since $p < q$, we can choose α such that $\frac{1}{q} \le \alpha < \frac{1}{p}$. Then $f \in L^p$ but $f \notin L^q$.

2. **(Sequence Space Case)**

(a) Let $\{x_n\} \in \ell^p$. Then $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Convergence of the series implies $|x_n| \to 0$. Thus, there exists N such that for all $n \ge N$, $|x_n| \le 1$. Since $q > p$, for numbers $|y| \le 1$, we have $|y|^q \le |y|^p$. Thus,

$$\sum_{n=N}^{\infty} |x_n|^q \le \sum_{n=N}^{\infty} |x_n|^p < \infty.$$

Since the tail converges, the whole series converges. Thus $\{x_n\} \in \ell^q$.

(b) **Counter-example ($\ell^q \not\subset \ell^p$):** Consider the harmonic-type sequence $x_n = n^{-\alpha}$. We need convergence for q ($\sum n^{-q\alpha} < \infty \implies q\alpha > 1$) and divergence for p ($\sum n^{-p\alpha} = \infty \implies p\alpha \le 1$). Choose α such that $\frac{1}{q} < \alpha \le \frac{1}{p}$. Then $\{x_n\} \in \ell^q$ but $\{x_n\} \notin \ell^p$.

3. **(Infinite Measure Case)**

(a) $f \in L^p \setminus L^q$ **(Decay issue):** We need a function that decays fast enough for p but not for q?? Actually, on infinite domains, L^p requires faster decay than L^q (similar to sequences). Wait, if $p < q$, $x^{-2} \in L^1([1,\infty))$ and $x^{-2} \in L^2([1,\infty))$. Let's use specific α. We need $\int_1^\infty x^{-p\alpha} dx < \infty \implies p\alpha > 1$ and $\int_1^\infty x^{-q\alpha} dx = \infty \implies q\alpha \le 1$. This is impossible since $p < q$. Correction: To be in L^p but not L^q on an infinite domain, we usually look at the "fat tail" (behavior at infinity) where the sequence logic applies ($L^p \subset L^q$), OR the "singularity" (behavior at 0) where finite measure logic applies ($L^q \subset L^p$).

Let's split the logic:

- $f \in L^p \setminus L^q$**:** We need a function with a singularity near 0 (like finite measure case). Let $f(x) = x^{-\alpha}\chi_{(0,1)}$. Choose $\frac{1}{q} \le \alpha < \frac{1}{p}$. Then $\int |f|^p < \infty$ but $\int |f|^q = \infty$.
- $g \in L^q \setminus L^p$**:** We need a function with a "fat tail" at infinity (like sequence case). Let $g(x) = x^{-\beta}\chi_{(1,\infty)}$. Choose $\frac{1}{q} < \beta \le \frac{1}{p}$. Then $\int_1^\infty x^{-q\beta} dx < \infty$ (since $q\beta > 1$) but $\int_1^\infty x^{-p\beta} dx = \infty$ (since $p\beta \le 1$).

Problem 7: Approximation and Separability

Problem 8: The "Standard Library" of Counterexamples

In this problem, we examine five classic counterexamples that mark the boundaries of major theorems in Real Analysis (Differentiation, Convergence, and Compactness).

Part 1: The Cantor-Lebesgue Function

Let $\phi(x)$ be the Cantor-Lebesgue function on $[0,1]$ and $\psi(x) = \phi(x) + x$.

1. **(Differentiation)** Compute $\int_0^1 \phi'(x)\,dx$ and compare it with $\phi(1) - \phi(0)$. What necessary condition for the Fundamental Theorem of Calculus does this function violate?
2. **(Measurability)** The function ψ is a homeomorphism from $[0,1]$ to $[0,2]$. Use ψ to prove that the preimage of a measurable set under a continuous map is **not necessarily** measurable. (Hint: Consider a non-measurable subset of the image of the Cantor set).

Part 2: The Typewriter Sequence

Consider the sequence of indicator functions on $[0,1]$:

$$f_1 = \chi_{[0,1]},\ f_2 = \chi_{[0,1/2]},\ f_3 = \chi_{[1/2,1]},\ f_4 = \chi_{[0,1/4]}, \ldots$$

1. Show that $\{f_n\}$ converges to 0 in measure and in $L^p[0,1]$.
2. Show that $\{f_n(x)\}$ converges **nowhere** pointwise on $[0,1]$.

Part 3: The "Spikes" (Mass Escape vs. Norm Escape)

Consider the following two sequences on $E = [0,1]$:

$$g_n(x) = n\chi_{[0,1/n]}(x) \quad \text{and} \quad h_n(x) = n^{1/p}\chi_{[0,1/n]}(x).$$

1. For $\{g_n\}$, show that $g_n \to 0$ pointwise a.e., but $\lim \int g_n \neq \int \lim g_n$. Which assumption of the Dominated Convergence Theorem fails?
2. For $\{h_n\}$ (where $1 \le p < \infty$), show that $h_n \to 0$ pointwise a.e., but $h_n \not\to 0$ in L^p. Which assumption of the Vitali Convergence Theorem fails?

Part 4: The Rademacher Functions

Let $r_n(t) = \operatorname{sgn}(\sin(2^n \pi t))$ on $[0,1]$.

1. Show that $r_n \rightharpoonup 0$ (weakly) in $L^p[0,1]$ for $1 < p < \infty$.
2. Show that $\{r_n\}$ has **no** strongly convergent subsequence in $L^p[0,1]$.
3. What does this imply about the compactness of the closed unit ball in L^p?

Part 5: Constant Functions on Infinite Measure

Let $E = \mathbb{R}$ and consider $f(x) \equiv 1$.

1. Show that $f \in L^\infty(\mathbb{R})$ but $f \notin L^p(\mathbb{R})$ for any $p < \infty$.
2. Contrast this with the inclusion relationships for $E = [0, 1]$ (where $L^\infty \subset L^p$).

Solution:

Part 1: The Cantor-Lebesgue Function

1. Since ϕ is constant on each interval in the complement of the Cantor set C (and $m(C^c) = 1$), we have $\phi'(x) = 0$ almost everywhere. Thus, $\int_0^1 \phi'(x)\,dx = 0$. However, $\phi(1)-\phi(0) = 1-0 = 1$. The FTC fails because ϕ is not **absolutely continuous** (it is a singular function).

2. Let C be the Cantor set ($m(C) = 0$). Since ψ maps $[0, 1]$ to $[0, 2]$ and stretches the Cantor set to a set of measure 1 ($m(\psi(C)) = 1$), $\psi(C)$ contains a non-measurable set N. Let $A = \psi^{-1}(N)$. Since $A \subset C$, A has measure zero and is measurable. However, if we consider the continuous function $g = \psi^{-1}$, the preimage of the measurable set A is the non-measurable set N. (Alternatively: The composition of measurable functions need not be measurable).

Part 2: The Typewriter Sequence

1. **Measure/L^p:** For any $\epsilon > 0$, $m(\{|f_n| > \epsilon\}) = \text{length}(I_n) \to 0$. Similarly, $\int |f_n|^p = \text{length}(I_n) \to 0$.

2. **Pointwise:** For any $x \in [0, 1]$, the intervals I_n "scan" over x infinitely many times. Thus, $f_n(x) = 1$ for infinitely many n and 0 for infinitely many n. The limit exists nowhere.

Part 3: The "Spikes"

1. $g_n(x) \to 0$ for all $x > 0$. However, $\int g_n = n \cdot \frac{1}{n} = 1 \neq 0$. The **Dominated Convergence Theorem** fails because there is no integrable dominating function G such that $|g_n| \leq G$ (the "spike" grows arbitrarily high).

2. $g_n(x) \to 0$ a.e. and bounded in $L^1([0, 1])$, but didn't have a weakly convergence subsequence. (That is, the unit ball in $L^1[0, 1]$ is not weakly compact.)

3. $h_n(x) \to 0$ a.e., but $\|h_n\|_p^p = \int n \cdot \frac{1}{n} = 1 \neq 0$. The **Vitali Convergence Theorem** fails because the family $\{|h_n|^p\}$ is not **uniformly integrable** (mass concentrates at a point too densely).

Part 4: The Rademacher Functions

1. By the Riemann-Lebesgue Lemma (generalized), $\int r_n g \to 0$ for any $g \in L^q$. Thus $r_n \rightharpoonup 0$.

2. $\|r_n - r_m\|_p \approx \text{const} > 0$ for $n \neq m$. Since the sequence is not Cauchy, it has no convergent subsequence.

3. This implies the closed unit ball in L^p (infinite-dimensional) is **not compact** in the strong topology.

Part 5: Constant Functions on Infinite Measure

1. $\|f\|_\infty = 1 < \infty$. But $\|f\|_p^p = \int_{\mathbb{R}} 1\, dx = \infty$.

2. On finite measure spaces, $L^\infty \subset L^p$. On infinite measure spaces, this inclusion fails (constant functions don't "decay" at infinity).

Problem 9: Comparison of Modes of Convergence (Version 2)

In this problem, we analyze the subtle relationships between different modes of convergence, specifically focusing on **subsequence extraction** and the distinction between the reflexive case ($p > 1$) and the non-reflexive case ($p = 1$).

Part 1: Subsequence Implications on Finite Measure

Let E be a set of **finite measure** and let $\{f_n\}$ be a **bounded sequence** in $L^p(E)$ for $1 \leq p < \infty$. Consider the following four properties a sequence might have:

(i) Strong Convergence: $\{f_n\} \to f$ in $L^p(E)$.

(ii) Weak Convergence: $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

(iii) Pointwise Convergence: $\{f_n\} \to f$ pointwise almost everywhere on E.

(iv) Convergence in Measure: $\{f_n\} \to f$ in measure on E.

For each of the cases below, determine if assuming the sequence has **Property A** implies that there exists a **subsequence** satisfying **Property B**.

1. **Case $p > 1$ (Reflexive Case):**
   (a) Assume (ii) Weak. Does a subsequence satisfy (i) Strong?
   (b) Assume (iii) Pointwise. Does a subsequence satisfy (i) Strong?
   (c) Assume (iii) Pointwise. Does a subsequence satisfy (ii) Weak?
   (d) Assume (iv) Measure. Does a subsequence satisfy (ii) Weak?
2. **Case $p = 1$ (Non-Reflexive Case):**
   (a) Assume (iii) Pointwise. Does a subsequence satisfy (ii) Weak?
   (b) Assume (iv) Measure. Does a subsequence satisfy (ii) Weak?

Part 2: Weak Convergence of Translations (Infinite Measure)

Let $1 \leq p < \infty$ and let $f_0 \in L^p(\mathbb{R})$. Define the sequence of translations $f_n(x) = f_0(x - n)$.

1. Show that for $1 < p < \infty$, the sequence $\{f_n\}$ converges **weakly** to 0 in $L^p(\mathbb{R})$.
2. Show that for $p = 1$, the sequence $\{f_n\}$ does **not** necessarily converge weakly to 0. (Provide a specific counterexample).

Solution:

Part 1: Subsequence Implications

1. Case $p > 1$:

1. **Weak $\Rightarrow$ Strong Subsequence? NO.**
   Counterexample: Rademacher functions $r_n(t)$ on $[0, 1]$. They converge weakly to 0 (Riemann-Lebesgue). However, $\|r_n - 0\|_p = 1$ for all n. Since the norm is constant and non-zero, no subsequence can converge strongly to 0.

2. **Pointwise $\Rightarrow$ Strong Subsequence? NO.**
   Counterexample: The "Spikes" $f_n = n^{1/p}\chi_{[0,1/n]}$. $f_n \to 0$ pointwise. However, $\|f_n\|_p = 1$ for all n. The mass "escapes" but does not vanish. Thus, no subsequence converges strongly.

3. **Pointwise $\Rightarrow$ Weak Subsequence? YES.**
   Since $p > 1$, L^p is **reflexive**. The sequence $\{f_n\}$ is bounded in L^p. By Kakutani's Theorem (or Banach-Alaoglu), there exists a subsequence f_{n_k} that converges weakly to some limit g. Since $f_n \to f$ pointwise, the unique weak limit must be f. Thus $f_{n_k} \rightharpoonup f$. (In fact, the whole sequence converges weakly).

4. **Measure $\Rightarrow$ Weak Subsequence? YES.**
   If $f_n \to f$ in measure, the Riesz Subsequence Theorem guarantees a subsequence f_{n_k} converges to f **pointwise a.e.** We are then back to the previous case: Bounded L^p sequence $(p > 1)$ + Pointwise limit $\implies$ Weak limit.

2. Case $p = 1$:

1. **Pointwise $\Rightarrow$ Weak Subsequence? NO.**
   Counterexample: $f_n = n\chi_{[0,1/n]}$. $f_n \to 0$ pointwise. $\|f_n\|_1 = 1$. Test weak convergence against $g \equiv 1 \in L^\infty$: $\int f_n g = 1 \neq 0$. Thus it does not converge weakly to 0.

2. **Measure $\Rightarrow$ Weak Subsequence? NO.**
   Same counterexample as above. $f_n \to 0$ in measure (support shrinks to 0). But the integral against $g = 1$ is always 1, so it never converges weakly to 0.

Part 2: Weak Convergence of Translations

1. **Proof for $p > 1$:**
   Let $g \in L^q(\mathbb{R})$ be a test function $(1/p + 1/q = 1)$. Since $q < \infty$, continuous functions with compact support, $C_c(\mathbb{R})$, are dense in L^q. It suffices to check dense subsets since $\{f_n\}$ is bounded $(\|f_n\|_p = \|f_0\|_p)$. Let $\phi \in C_c(\mathbb{R})$ with support in $[-K, K]$.

$$\int_{-\infty}^{\infty} f_n(x)\phi(x)\,dx = \int_{-K}^{K} f_0(x-n)\phi(x)\,dx$$

   For n sufficiently large, the support of $f_0(x - n)$ (effectively centered at n) becomes disjoint from $[-K, K]$. Specifically, since $f_0 \in L^p$, its "tails" vanish.

$$\left|\int f_n\phi\right| \le \|\phi\|_\infty \int_{-K}^{K} |f_0(x-n)|\,dx = \|\phi\|_\infty \int_{-K-n}^{K-n} |f_0(y)|\,dy$$

As $n \to \infty$, the integration domain slides to $-\infty$, where the integral vanishes. Thus $f_n \rightharpoonup 0$.

2. **Failure for $p = 1$:**
Take $f_0 = \chi_{[0,1]}$. Then $f_n = \chi_{[n,n+1]}$. Let the test function be $g(x) \equiv 1$. Note that $g \in L^\infty(\mathbb{R})$ (the dual of L^1).
$$\int_{-\infty}^{\infty} f_n(x)g(x)\,dx = \int_n^{n+1} 1\,dx = 1$$
This integral is constant 1 for all n, so it does not converge to 0. Thus $f_n \not\rightharpoonup 0$.

Does Property A imply a Subsequence has Property B? (Assume $m(E) < \infty$ and sequence is bounded in L^p)				
Property A ↓ \ Property B →	**Strong (L^p)**	**Weak (L^p)**	**Pointwise a.e.**	**Measure**
Case 1: Reflexive Range ($1 < p < \infty$)				
Strong	YES	YES	YES	YES
Weak	NO	YES	NO	NO
Pointwise a.e.	NO	**YES**	YES	YES
Measure	NO	**YES**	YES	YES
Case 2: Non-Reflexive ($p = 1$)				
Pointwise a.e.	NO	**NO**	YES	YES
Measure	NO	**NO**	YES	YES

Key Differences:

- **Pointwise ⇒ Weak:** Holds for $p > 1$ (due to Reflexivity/Dunford-Pettis) but fails for $p = 1$ (mass can escape, e.g., $n\chi_{[0,1/n]}$).
- **Weak ⇒ Pointwise:** Never holds for subsequences (e.g., Rademacher functions oscillate too much).