Real Analysis midterm-ustc-2022

Translated by Gemini DeepResearch

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Problem 1.

(20 points)

(a) State the definition of the **outer measure** m_*.

(b) State the definition of a **measurable set**.

(c) Prove that E is a measurable set if and only if for every $\epsilon > 0$, there exists an open set $O \supset E$ and a closed set $F \subset E$ such that $m_*(O - F) < \epsilon$.

Problem 2.

(15 points) Let E be a measurable set in $\mathbb{R}^d$ satisfying $m(E) < \infty$. State **Egorov's Theorem** on E, and provide an example to show that the condition $m(E) < \infty$ is necessary.

Problem 3.

(15 points) Let $A \subset [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be a measurable set with **positive**

measure. Prove that A must contain at least one **non-measurable subset**.

Problem 4.

(15 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a **continuous function**.

(a) Prove that the graph of f, $\Gamma = \{(x, y)|y = f(x)\}$, is a **closed set** in $\mathbb{R}^2$, and is therefore a measurable set.

(b) Prove that Γ is a **null set** (zero measure set) in $\mathbb{R}^2$.

Problem 5.

(15 points) If f is a **continuous function with compact support** on $\mathbb{R}$ (where f having compact support means the closure of the set $\{x \in \mathbb{R}|f(x) \neq 0\}$ is compact), prove:

(a) $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\sqrt{-1}\pi\xi x}dx$ is a **continuous function** of ξ on $\mathbb{R}$.

(b) $\lim_{\xi\to\infty} \hat{f}(\xi) = 0$.

Problem 6.

(10 points, Borel-Cantelli Lemma) Assume $\{E_k\}_{k=1}^{\infty}$ is a sequence of measurable sets that satisfies $\sum_{k=1}^{\infty} m(E_k) < \infty$. Let $E = \{x \in \mathbb{R}^d | x \in E_k \text{ for infinitely many } k\}$. Prove that $m(E) = 0$.

Problem 7.

(10 points) Use the **Borel-Cantelli Lemma** to prove the following conclu-

sion. Let

$$E = \left\{ x \in \mathbb{R} \,\middle|\, \text{there exist infinitely many rational numbers } \frac{p}{q}, \text{ with } p \text{ and } q \text{ coprime, such that } \right|$$

Prove that $m(E) = 0$.