Real Analysis
midtern-ustc-2023-version2

October 13, 2025

Problem 1.

(20 points) Determine whether the following statements are true or false (provide a proof or a counterexample to explain your conclusion):

(i) Every monotonic function on an interval is measurable.

(ii) Every closed set in $\mathbb{R}^n$ is a G_δ set.

Problem 2.

(15 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. Prove that the following two statements are equivalent:

(i) For all $a \in \mathbb{R}$, $f^{-1}((a, +\infty))$ is Lebesgue measurable.

(ii) For any open set $G \subset \mathbb{R}$, $f^{-1}(G)$ is Lebesgue measurable.

Problem 3.

(10 points) Calculate:

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$$

Problem 4.

(15 points) Let $f, f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ be non-negative and $f_n \to f$ a.e. on $[a,b]$. Prove:

$$\lim_{n\to\infty} \int_{[a,b]} f_n e^{-f_n}\,dm = \int_{[a,b]} f e^{-f}\,dm$$

Problem 5.

(15 points) Let $E \subset \mathbb{R}^n$. Prove that there exists a G_δ set H containing E such that $m(H) = m_*(E)$.

Problem 6.

(15 points) Let $E_1, E_2, \ldots, E_n$ be measurable subsets of $[0,1]$. If every point of $[0,1]$ appears in at least k of these sets, prove that there exists an index i_0 such that $m(E_{i_0}) \geq k/n$.

Problem 7.

(10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Let $V \subset \mathbb{R}$ be an open set, $0 \in V$. Prove that there exists a measurable set E, with $m(E) > 0$, such that for all $x, y \in E$, $f(x) - f(y) \in V$.