Lecture Notes: Differential Forms

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This note is a quick introduction to differential forms. The aim is to introduce the subject to unify the formula: Generalized Stokes formula.

$$\int_{\partial D} \vec{V} \cdot d\vec{\varphi} := \int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{1}$$

$$\int_{\partial D} \vec{V} \cdot d\vec{u} := \int_{\partial D} -Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dxdy \tag{2}$$

$$\int_{\partial D} \vec{V} \cdot d\vec{n} := \iint_{\partial D} Pdydz + Qdxdz + Rdxdy = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz \tag{3}$$

As $\int_{\partial D} \omega = \int_D d\omega$.

From now on, we restrict ourselves in $\mathbb{R}^n$ with coordinates $x^1, x^2, \ldots, x^n$.

0-forms:

All smooth functions $f : \mathbb{R}^n \to \mathbb{R}$, $C^\infty(\mathbb{R}^n, \mathbb{R})$. e.g.: $f(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$

1-forms:

A n-dimensional vector space over $C^\infty(\mathbb{R}^n, \mathbb{R})$. $\mathrm{Span}_{C^\infty(\mathbb{R}^n,\mathbb{R})}\{dx^1, dx^2, \ldots, dx^n\}$, where $dx^1, dx^2, \ldots, dx^n$ are canonical differential 1-forms. Here we just treat it as a "pure symbol". e.g., $\omega = (x^1)^2 dx^1 + (x^2) dx^2$.

2-forms:

A $\binom{n}{2}$ dimensional vector space over $C^\infty(\mathbb{R}^n, \mathbb{R})$. $\mathrm{Span}_{C^\infty(\mathbb{R}^n,\mathbb{R})}\{dx^i \wedge dx^j\}$. Here we require $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$. The dimension is $\binom{n}{2}$.

e.g.: $\omega = f(x^1, \ldots, x^n) dx^1 \wedge dx^2 + g(x^1, \ldots, x^n) dx^2 \wedge dx^3$.

k-forms ($1 \le k \le n$):

A $\binom{n}{k}$ dimensional vector space over $C^\infty(\mathbb{R}^n, \mathbb{R})$. $\mathrm{Span}_{C^\infty(\mathbb{R}^n,\mathbb{R})}\{dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}\}$, where we require: $dx^i \wedge dx^j \wedge dx^k = -dx^j \wedge dx^i \wedge dx^k$, $dx^i \wedge dx^i \wedge dx^k = 0$ (the so-called anti-symmetric property).

If $k > n$, then it must be 0, since $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \wedge dx^1 = 0$ (since two dx^1 appear).

Exterior derivatives d

We want to define a map that maps k-forms to $(k+1)$-forms: $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, where $\omega \mapsto d\omega$.

0-forms $\to$ 1-forms

$f \to df = \sum_i \frac{\partial f}{\partial x^i} dx^i$.

k-forms $\to$ $(k+1)$-forms

$\omega = \sum a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \implies d\omega = \sum da_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

Examples:

$n = 3$: 2-forms $\to$ 3-forms: $\omega = Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy$. $d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$.

$n = 2$: 1-forms $\to$ 2-forms: $\omega = Pdx + Qdy \implies d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$.

Integration of k-forms

Suppose $D \subseteq \mathbb{R}^n$ is a k-dimensional region, $\omega \in \Omega^k(\mathbb{R}^n)$, $\omega = \sum a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$. Define $\int_D \omega := \sum \int_D a_{i_1 \dots i_k} dx^{i_1} \dots dx^{i_k}$.

Example:

$n = 3$. $\omega = Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy$. $S \subseteq \mathbb{R}^3$ a two-dimensional surface. $\int_S \omega := \int_S Pdydz + \int_S Qdxdz + \int_S Rdxdy$.

Stokes's Theorem

Suppose D is a $(k+1)$-dimensional region, and ∂D is its k-dimensional boundary region. If ω is a k-form, then $d\omega$ is a $(k+1)$-form. We have two integrals defined: $\int_D d\omega$ and $\int_{\partial D} \omega$. Stokes's theorem claims that:

$$\int_D d\omega = \int_{\partial D} \omega \tag{4}$$

Examples

1. $n = 3$: $\omega = Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy$. $d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$. $\int_D d\omega = \int_{\partial D} \omega$ is just the divergence theorem.

2. $n = 2$: $\omega = Pdx + Qdy$. $d\omega = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dxdy$. $\int_D d\omega = \int_{\partial D} \omega$ is Green's formula.