

# Final Review for MAT2060: Honors Mathematical Analysis

## Metric Spaces and Multivariable Calculus

Cola

The Chinese University of HongKong, Shenzhen

December 10, 2025

# Outline

- 1 Topology of Metric Spaces
- 2 Tietze Extension Theorem
- 3 Stone-Weierstrass Theorem
- 4 Arzela-Ascoli Theorem
- 5 Baire Category Theorem
- 6 Multivariable Differentiation
- 7 Multivariable Integration



# Overview: Topology of Metric Spaces

### Key Concepts:

- **Compactness:** Recall that in a general metric space  $(X, d)$ :

Compactness  $\iff$  Sequential Compactness  $\iff$  Totally Bounded + Complete

In  $\mathbb{R}^n$ , we have the **Heine-Borel Theorem**: Compact  $\iff$  Closed and Bounded.

- **Connectedness:**

Path-Connected  $\implies$  Connected

The converse is not true (e.g., Topologist's Sine Curve).

# Problem 1: Compactness Equivalences

## Problem 1

Let  $(X, d)$  be a metric space and let  $K \subseteq X$ . Prove the equivalence of the following statements:

- ①  $K$  is compact (every open cover has a finite subcover).
- ②  $K$  is sequentially compact (every sequence has a convergent subsequence).
- ③  $K$  is totally bounded and complete.

Solution to Problem 1 (1/3): Seq. Compact  $\implies$  Complete & Totally Bounded

**Part A: Completeness** Let  $\{x_n\}$  be a Cauchy sequence in  $K$ . Since  $K$  is sequentially compact, there exists a subsequence  $\{x_{n_k}\}$  that converges to some  $x \in K$ . A Cauchy sequence with a convergent subsequence converges to the same limit. Thus  $x_n \rightarrow x$ , so  $K$  is complete.

**Part B: Totally Bounded** Suppose  $K$  is not totally bounded. Then  $\exists \varepsilon > 0$  such that  $K$  cannot be covered by finitely many  $\varepsilon$ -balls.

- Pick  $x_1 \in K$ . Then  $K \not\subseteq B(x_1, \varepsilon)$ , so choose  $x_2 \in K \setminus B(x_1, \varepsilon)$ .
- Inductively, choose  $x_n \in K \setminus \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$ .
- For  $n \neq m$ ,  $d(x_n, x_m) \geq \varepsilon$ .
- The sequence  $\{x_n\}$  has no Cauchy subsequence, hence no convergent subsequence. Contradiction.

## Solution to Problem 1 (2/3): Complete & Totally Bounded $\implies$ Compact

Suppose  $K$  is complete and totally bounded but not compact. Let  $\mathcal{U}$  be an open cover with no finite subcover.

- 1 Since  $K$  is totally bounded, cover it by finitely many balls of radius 1. At least one, say  $K \cap B_1$ , cannot be finitely covered by  $\mathcal{U}$ .
- 2 Cover  $K \cap B_1$  by finitely many balls of radius  $1/2$ . Pick  $B_2$  such that  $K \cap B_1 \cap B_2$  is not finitely covered.
- 3 Inductively, find nested sets with diameter  $\rightarrow 0$ . Pick  $x_n$  in the  $n$ -th intersection.
- 4 By completeness,  $x_n \rightarrow x \in K$ .
- 5 Since  $\mathcal{U}$  covers  $K$ ,  $x \in U$  for some  $U \in \mathcal{U}$ .  $\exists \delta > 0$  such that  $B(x, \delta) \subseteq U$ .
- 6 For large  $n$ , the chosen set lies inside  $B(x, \delta)$ , meaning it is covered by a single set  $U$ . Contradiction.

# Solution to Problem 1 (3/3): Compact $\implies$ Sequentially Compact

Let  $K$  be compact. We prove sequential compactness by contradiction.

- Suppose there is a sequence  $\{x_n\}$  with no convergent subsequence.
- Then the set of points  $S = \{x_n\}$  has no limit points.
- For any  $y \in K$ , there exists an open ball  $B(y, r_y)$  containing only finitely many terms of the sequence (if  $y \in S$ , it contains only  $y$ ; if not, it can be disjoint from the tail).
- The collection  $\{B(y, r_y)\}_{y \in K}$  is an open cover of  $K$ .
- By compactness, there is a finite subcover.
- The union of these finitely many balls contains only finitely many terms of the sequence, contradicting that  $\{x_n\}$  is an infinite sequence.



# Problem 2: Heine-Borel Theorem

## Problem 2

*Prove  $S \subseteq \mathbb{R}^n$  is compact if and only if  $S$  is closed and bounded.*

# Solution to Problem 2 (1/2): Compact $\implies$ Closed & Bounded

## Forward Direction.

**Boundedness:** Consider the open cover  $\mathcal{U} = \{B(0, n) \mid n \in \mathbb{N}\}$ . Since  $S \subseteq \bigcup B(0, n) = \mathbb{R}^n$ , this covers  $S$ . By compactness, there is a finite subcover  $\{B(0, n_1), \dots, B(0, n_k)\}$ . Let  $N = \max(n_i)$ . Then  $S \subseteq B(0, N)$ , so  $S$  is bounded.

**Closedness:** We show  $S^c$  is open. Let  $y \in S^c$ . For each  $x \in S$ , let  $r_x = \frac{1}{2}d(x, y)$ . The collection  $\{B(x, r_x)\}$  covers  $S$ . By compactness, take a finite subcover corresponding to  $x_1, \dots, x_m$ . Let  $V = \bigcap_{i=1}^m B(y, r_{x_i})$ .  $V$  is an open neighborhood of  $y$  disjoint from  $S$ . Thus  $S^c$  is open. □

# Solution to Problem 2 (2/2): Closed & Bounded $\implies$ Compact

## Converse Direction (Sequential Method).

Suppose  $S$  is closed and bounded. We show  $S$  is sequentially compact.

- ① Let  $\{\mathbf{x}_k\}$  be an arbitrary sequence in  $S$ .
- ② Since  $S$  is bounded, the sequence  $\{\mathbf{x}_k\}$  is bounded in  $\mathbb{R}^n$ .
- ③ By the **Bolzano-Weierstrass Theorem** for  $\mathbb{R}^n$ , every bounded sequence has a convergent subsequence.
- ④ Let  $\{\mathbf{x}_{k_j}\}$  be a subsequence converging to some limit  $\mathbf{x} \in \mathbb{R}^n$ .
- ⑤ Since  $S$  is closed and  $\{\mathbf{x}_{k_j}\} \subset S$ , the limit must be in  $S$  (i.e.,  $\mathbf{x} \in S$ ).
- ⑥ Thus, every sequence in  $S$  has a subsequence converging to a point in  $S$ .
- ⑦  $S$  is sequentially compact  $\implies S$  is compact (by Problem 1).



# Problem 3: The Hilbert Cube

## Problem 3

Let  $H = [0, 1]^{\mathbb{N}}$  be the set of sequences  $x = (x_1, x_2, \dots)$  with  $x_n \in [0, 1]$ . Define a distance function:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$$

- 1 Show that  $(H, d)$  is a metric space.
- 2 Show that  $H$  is sequentially compact.

# Solution to Problem 3 (1/2): Metric Space

## Part 1: Metric Space.

- **Well-defined:** Since  $|x_n - y_n| \leq 1$ , the series is dominated by  $\sum 2^{-n} = 1$ , so it converges.
- **Positivity:**  $d(x, y) \geq 0$ . If  $d(x, y) = 0$ , then  $|x_n - y_n| = 0$  for all  $n$ , so  $x = y$ .
- **Symmetry:**  $|x_n - y_n| = |y_n - x_n|$ , so  $d(x, y) = d(y, x)$ .
- **Triangle Inequality:** For any  $z \in H$ :

$$|x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$$

Multiplying by  $2^{-n}$  and summing gives  $d(x, y) \leq d(x, z) + d(z, y)$ .



# Solution to Problem 3 (2/2): Sequential Compactness

## Part 2: Diagonal Argument.

Let  $\{x^{(k)}\}$  be a sequence in  $H$ .

- The first coordinates  $x_1^{(k)}$  lie in  $[0, 1]$ . By Bolzano-Weierstrass, there is a subsequence converging in the 1st slot.
- From this, extract a sub-subsequence converging in the 2nd slot, and so on.
- Let  $x^{(k_j)}$  be the **diagonal sequence**. It converges pointwise to some  $x \in H$ :  
 $\lim_{j \rightarrow \infty} |x_n^{(k_j)} - x_n| = 0$  for each  $n$ .
- **Convergence in metric  $d$ :** Given  $\varepsilon > 0$ , pick  $N$  such that  $\sum_{n=1}^{\infty} 2^{-n} < \varepsilon/2$ .
- Choose  $J$  large enough so for  $j > J$ ,  $\sum_{n=1}^N 2^{-n} |x_n^{(k_j)} - x_n| < \varepsilon/2$ .
- Then  $d(x^{(k_j)}, x) < \varepsilon$ . Thus  $H$  is sequentially compact.



# Problem 4: Connectedness Properties

## Problem 4

- ① Show that the interval  $[0, 1]$  is a connected set in  $\mathbb{R}$ .
- ② Show that if a space  $X$  is path-connected, it is connected.

# Solution to Problem 4: Connectedness

## Proof.

**1.  $[0, 1]$  is connected:** Suppose  $[0, 1] = A \cup B$  where  $A, B$  are disjoint, non-empty, closed sets in  $[0, 1]$ . Assume  $0 \in A$ . Let  $c = \sup A$ .

- Since  $A$  is closed,  $c \in A$ . Thus  $c < 1$  (otherwise  $B$  is empty).
- Since  $A$  is open in  $[0, 1]$  (complement of  $B$ ), there is a neighborhood  $[c, c + \varepsilon) \subseteq A$ .
- This contradicts  $c = \sup A$ . Thus, no such separation exists.

**2. Path-connected  $\implies$  Connected:** Suppose  $X$  is path-connected but disconnected, so  $X = U \cup V$  disjoint open sets. Pick  $u \in U, v \in V$ . Let  $\gamma : [0, 1] \rightarrow X$  be a path from  $u$  to  $v$ .

- Consider sets  $\gamma^{-1}(U)$  and  $\gamma^{-1}(V)$  in  $[0, 1]$ .
- They are disjoint, non-empty, and open in  $[0, 1]$  (by continuity).
- Their union is  $[0, 1]$ , contradicting that  $[0, 1]$  is connected.





# Problem 5: Open Connected Sets

## Problem 5

Show that any **open**, connected subset of a Euclidean space (or normed vector space) is path-connected.

# Solution to Problem 5: Open Connected $\implies$ Path-Connected

## Proof.

Let  $U \subseteq \mathbb{R}^n$  be open and connected. Fix  $x_0 \in U$ . Define  $A = \{x \in U \mid \exists \text{ path from } x_0 \text{ to } x \text{ in } U\}$ .

- ①  **$A$  is open:** Let  $x \in A$ . Since  $U$  is open,  $\exists B(x, r) \subseteq U$ . Balls are convex (hence path-connected). Any  $y \in B(x, r)$  connects to  $x$ , then to  $x_0$ . So  $B(x, r) \subseteq A$ .
- ②  **$U \setminus A$  is open:** Let  $y \in U \setminus A$ .  $\exists B(y, r) \subseteq U$ . If any  $z \in B(y, r)$  were in  $A$ , we could connect  $x_0 \rightarrow z \rightarrow y$ , implying  $y \in A$  (contradiction). So  $B(y, r) \subseteq U \setminus A$ .
- ③ **Conclusion:**  $A$  is a non-empty ( $x_0 \in A$ ) open and closed subset of connected  $U$ . Thus  $A = U$ .



## Problem 6: The Counterexample

### Problem 6

Consider the "Topologist's Sine Curve":

$$S = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x \in (0, 1]\} \cup \{(0, y) \mid y \in [-1, 1]\}$$

Prove that  $S$  is connected but **not** path-connected.

*Note: This illustrates that path-connectedness is strictly stronger than connectedness for general sets.*

# Solution to Problem 6: Topologist's Sine Curve

## Proof.

**Connected:** Let  $G = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$ .  $G$  is the continuous image of the connected set  $(0, 1]$ , so  $G$  is connected.  $S = \bar{G}$  (the closure adds the segment  $\{0\} \times [-1, 1]$ ). The closure of a connected set is connected.

**Not Path-Connected:** Suppose there is a path  $\gamma : [0, 1] \rightarrow S$  from  $(0, 0)$  to  $(1/\pi, 0)$ . Let  $\gamma(t) = (x(t), y(t))$ .

- Since  $x(t)$  is continuous and  $x(0) = 0, x(1) > 0$ , there are points arbitrarily close to  $t = 0$  with  $x(t) > 0$ .
- As  $t \rightarrow 0$ ,  $x(t) \rightarrow 0$ , so  $1/x(t) \rightarrow \infty$ .
- $\sin(1/x(t))$  oscillates between -1 and 1 infinitely often.
- This prevents  $y(t)$  from converging to any specific value if we approach along the curve, contradicting the continuity of  $\gamma$  at 0.



# Tietze Extension Theorem

# Overview: Tietze Extension Theorem

**The Theorem:** Let  $X$  be a metric space (or normal topological space) and  $F \subseteq X$  be a closed set. If  $f : F \rightarrow \mathbb{R}$  is continuous, there exists a continuous extension  $\tilde{f} : X \rightarrow \mathbb{R}$  such that  $\tilde{f}|_F = f$ .

**Key Tool (Urysohn's Lemma):** The proof relies heavily on the ability to separate disjoint closed sets. If  $A, B \subseteq X$  are disjoint and closed, there exists a continuous function  $\varphi : X \rightarrow [0, 1]$  such that:

$$\varphi(A) = \{0\} \quad \text{and} \quad \varphi(B) = \{1\}.$$

# Problem 7: Urysohn's Lemma

## Problem 7

*Let  $(X, d)$  be a metric space. Let  $A, B$  be disjoint closed subsets of  $X$ . Construct a continuous function  $\varphi : X \rightarrow [0, 1]$  satisfying Urysohn's property.*

# Solution to Problem 7

## Proof.

Using the distance function to a set,  $d(x, E) = \inf\{d(x, y) \mid y \in E\}$ , which is continuous. Define:

$$\varphi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

- Since  $A, B$  are closed and disjoint,  $d(x, A) + d(x, B) > 0$  for all  $x$ , so  $\varphi$  is well-defined and continuous.
- If  $x \in A$ ,  $d(x, A) = 0 \implies \varphi(x) = 0$ .
- If  $x \in B$ ,  $d(x, B) = 0 \implies \varphi(x) = 1$ .
- Clearly  $0 \leq \varphi(x) \leq 1$ .





# Problem 8: Proof Strategy of Tietze Extension

## Problem 8

*Describe the strategy to prove the Tietze Extension Theorem using Urysohn's Lemma.*

**Solution Sketch:** The proof uses an iterative approximation: 1. Assume  $|f(x)| \leq M$ . 2. Define sets  $A = \{x \mid f(x) \leq -M/3\}$  and  $B = \{x \mid f(x) \geq M/3\}$ . 3. Use Urysohn to find  $g_1$  approximating  $f$  on these sets. 4. Consider the error  $f - g_1$ , which is bounded by  $2M/3$ . 5. Repeat inductively to get a series of functions  $\sum g_n$  that converges uniformly to an extension of  $f$ .

# Problem 9 & 10: Extensions on $\mathbb{R}$

## Problem 9

*If  $F \subseteq \mathbb{R}$  is closed, show explicitly that any continuous  $f : F \rightarrow \mathbb{R}$  can be extended to  $\mathbb{R}$  (e.g., by linear interpolation on the gaps).*

## Problem 10

**Prove or Disprove:** *Any continuous function  $f : (0, 1] \rightarrow \mathbb{R}$  can be extended to a continuous function on  $\mathbb{R}$ .*

**Answer to 10:** False. Consider  $f(x) = 1/x$ . It is continuous on  $(0, 1]$  but cannot be extended to  $x = 0$  continuously. (Extension requires uniform continuity or boundedness if the domain is not closed).

# Stone-Weierstrass Theorem

# Overview: Stone-Weierstrass

**Theorem:** Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a subalgebra of  $C(X, \mathbb{R})$  such that:

- 1  $\mathcal{A}$  separates points ( $x \neq y \implies \exists f \in \mathcal{A}, f(x) \neq f(y)$ ).
- 2  $\mathcal{A}$  vanishes at no point (or contains constants).

Then  $\mathcal{A}$  is dense in  $C(X, \mathbb{R})$  in the uniform norm.

**Corollary:** Polynomials are dense in  $C[a, b]$ .

# Problem 11: The Moment Problem

## Problem 11

Let  $f \in C[0, 1]$ . Suppose that for all  $n = 0, 1, 2, \dots$ ,

$$\int_0^1 x^n f(x) dx = 0.$$

Prove that  $f(x) \equiv 0$ .

# Solution to Problem 11

## Proof.

- ① By the Stone-Weierstrass theorem, the set of polynomials is dense in  $C[0, 1]$ .
- ② Therefore, there exists a sequence of polynomials  $P_k(x)$  such that  $P_k \rightarrow f$  uniformly.
- ③ By linearity of the integral and the hypothesis,  $\int_0^1 P_k(x)f(x) dx = 0$  for any polynomial  $P_k$ .
- ④ Taking the limit as  $k \rightarrow \infty$ :

$$\int_0^1 (f(x))^2 dx = \lim_{k \rightarrow \infty} \int_0^1 P_k(x)f(x) dx = 0.$$

- ⑤ Since  $f^2 \geq 0$  and is continuous,  $\int f^2 = 0 \implies f \equiv 0$ .



# Problem 12 & 13: Separability

## Problem 12

Prove that  $C[0, 1]$  is a **separable** metric space (has a countable dense subset).

**Hint:** Use polynomials with rational coefficients  $\mathbb{Q}[x]$ .

## Problem 13

Prove that a compact metric space is separable.

# Solution to Problem 12: Separability of $C[0, 1]$

## Proof.

We show that the set of polynomials with rational coefficients, denoted  $\mathbb{Q}[x]$ , is a countable dense subset of  $C[0, 1]$ .

- ① **Countability:**  $\mathbb{Q}[x] = \bigcup_{n=0}^{\infty} \{a_n x^n + \cdots + a_0 \mid a_i \in \mathbb{Q}\}$ . Since  $\mathbb{Q}$  is countable, each set of degree  $n$  polynomials is countable. A countable union of countable sets is countable.
- ② **Density:** Let  $f \in C[0, 1]$  and  $\varepsilon > 0$ . By the **Weierstrass Approximation Theorem**, there exists a polynomial  $P(x)$  (with real coefficients) such that  $\|f - P\|_{\infty} < \varepsilon/2$ .
- ③ Let  $P(x) = \sum_{k=0}^n c_k x^k$ . For each  $c_k$ , choose  $q_k \in \mathbb{Q}$  such that  $|c_k - q_k| < \frac{\varepsilon}{2(n+1)}$ . Let  $Q(x) = \sum q_k x^k$ .
- ④ Then  $\|P - Q\|_{\infty} \leq \sum |c_k - q_k| < \varepsilon/2$ .
- ⑤ By Triangle Inequality:  $\|f - Q\|_{\infty} \leq \|f - P\| + \|P - Q\| < \varepsilon$ .





# Solution to Problem 13: Separability of Compact Spaces

## Proof.

Let  $(X, d)$  be a compact metric space.

- ① Since  $X$  is compact, it is totally bounded.
- ② For each  $n \in \mathbb{N}$ , there exists a finite set of points  $A_n = \{x_{n,1}, \dots, x_{n,k_n}\}$  such that the balls of radius  $1/n$  centered at  $A_n$  cover  $X$ .
- ③ Define  $D = \bigcup_{n=1}^{\infty} A_n$ . As a countable union of finite sets,  $D$  is countable.
- ④ **Density:** Let  $x \in X$  and  $\varepsilon > 0$ . Choose  $n$  such that  $1/n < \varepsilon$ .
- ⑤ Since  $A_n$  centers cover  $X$ , there exists  $y \in A_n \subseteq D$  such that  $d(x, y) < 1/n < \varepsilon$ .
- ⑥ Thus  $D$  is dense in  $X$ .



# Arzela-Ascoli Theorem

# Overview: Arzela-Ascoli Theorem

**Theorem:** Let  $X$  be a compact metric space. A subset  $K \subseteq C(X, \mathbb{R})$  is compact in the uniform topology if and only if  $K$  is:

- ① **Closed**,
- ② **Bounded** (pointwise bounded, which implies uniformly bounded on compact  $X$ ), and
- ③ **Equicontinuous**.

**Interpretation:** This theorem characterizes the compact sets in the infinite-dimensional space  $C(X, \mathbb{R})$ . It is the function space analog of the Heine-Borel theorem, adding the condition of "equicontinuity" to replace the loss of finite dimensionality.

# Problem 14: Equicontinuity Definitions

## Problem 14

Let  $\mathcal{F}$  be a family of functions on a compact metric space  $(X, d)$ . Show that the following definitions are equivalent:

- ① **Pointwise Equicontinuity:**  $\forall x \in X, \forall \varepsilon > 0, \exists \delta(x, \varepsilon) > 0$  such that  
 $d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon, \forall f \in \mathcal{F}.$
- ② **Uniform Equicontinuity:**  $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$  such that  
 $d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon, \forall f \in \mathcal{F}, \forall x, y \in X.$

# Solution to Problem 14: Equicontinuity

## Proof.

(2  $\implies$  1) is trivial. We prove (1  $\implies$  2) using compactness.

- ① Let  $\varepsilon > 0$ . By (1), for each  $x \in X$ , there exists  $\delta_x > 0$  such that  $f(B(x, \delta_x)) \subseteq B(f(x), \varepsilon/2)$  for all  $f \in \mathcal{F}$ .
- ② The balls  $\{B(x, \delta_x/2)\}_{x \in X}$  form an open cover of  $X$ .
- ③ By compactness, there is a finite subcover centered at  $x_1, \dots, x_k$ .
- ④ Let  $\delta = \min\{\delta_{x_1}/2, \dots, \delta_{x_k}/2\} > 0$ .
- ⑤ Let  $y, z \in X$  with  $d(y, z) < \delta$ . Then  $y \in B(x_i, \delta_{x_i}/2)$  for some  $i$ .
- ⑥ By triangle inequality:  $d(z, x_i) \leq d(z, y) + d(y, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}$ .
- ⑦ So both  $y, z \in B(x_i, \delta_{x_i})$ . For any  $f \in \mathcal{F}$ :

$$|f(y) - f(z)| \leq |f(y) - f(x_i)| + |f(x_i) - f(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$



# Problem 15: Necessity of Compactness

## Problem 15

*Show that the assumption that  $X$  is compact is necessary for the Arzela-Ascoli theorem. Specifically, find a sequence of functions  $f_n$  on a non-compact space  $X$  that is uniformly bounded and equicontinuous, but admits no uniformly convergent subsequence.*

# Solution to Problem 15

Let  $X = \mathbb{R}$  (which is not compact). Consider the "sliding bump" functions:

$$f_n(x) = \max(0, 1 - |x - n|)$$

- **Bounded:**  $|f_n(x)| \leq 1$  for all  $x, n$ .
- **Equicontinuous:** Since  $|f'_n(x)| \leq 1$  wherever defined, they are all Lipschitz continuous with constant 1. Thus, they are equicontinuous.
- **No Convergent Subsequence:** For any  $n \neq m$ ,  $\|f_n - f_m\|_\infty = 1$  (since the supports  $[n - 1, n + 1]$  and  $[m - 1, m + 1]$  are disjoint for large enough difference). Since the distance between any distinct terms is 1, no subsequence is Cauchy, so no subsequence converges uniformly.

# Problem 16: Dini's Theorem

## Problem 16

Let  $f_n$  be a monotone sequence of continuous functions on a compact metric space  $S$ . Suppose that  $f_n$  converges pointwise to a continuous function  $f$  on  $S$ . Prove or disprove that the convergence is uniform.

*This is known as Dini's Theorem.*



# Solution to Problem 16: Dini's Theorem

## Proof.

**Prove:** The convergence is uniform. Let  $g_n = |f_n - f|$ . Since  $f_n$  is monotone and converges to  $f$ ,  $g_n$  is a monotonic sequence decreasing to 0 pointwise (assume  $f_n \downarrow f$ ; if  $f_n \uparrow f$ , take  $f - f_n$ ). Since  $f_n, f$  are continuous,  $g_n$  is continuous.

Let  $\varepsilon > 0$ . For each  $x \in S$ ,  $g_n(x) \downarrow 0$ , so  $\exists N_x$  s.t.  $g_{N_x}(x) < \varepsilon$ . By continuity, there is an open neighborhood  $U_x$  such that  $g_{N_x}(y) < \varepsilon$  for all  $y \in U_x$ .

$\{U_x\}_{x \in S}$  covers  $S$ . By compactness, there is a finite subcover  $U_{x_1}, \dots, U_{x_k}$ . Let  $N = \max(N_{x_1}, \dots, N_{x_k})$ . For any  $y \in S$ ,  $y \in U_{x_i}$  for some  $i$ . Since  $g_n$  is decreasing, for all  $n \geq N \geq N_{x_i}$ :

$$0 \leq g_n(y) \leq g_{N_{x_i}}(y) < \varepsilon.$$

Thus convergence is uniform. □

# Baire Category Theorem



# Problem 17: Removing Lines from $\mathbb{R}^2$

## Problem 17

Let  $\{l_i\}_{i=1}^{\infty}$  be a countable collection of straight lines in  $\mathbb{R}^2$ . Show that  $\mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} l_i$  is dense in  $\mathbb{R}^2$ .

We will provide a direct proof using nested balls (simulating the proof of BCT) as suggested in the notes.

Proof.

**Conclusion:** By completeness of  $\mathbb{R}^2$ ,  $\bigcap_{n=1}^{\infty} \overline{B}_n = \{x\}$ .  $x \in B_0$ , and for all  $n$ ,  $x \notin l_n$ . Thus  $x \in B_0 \setminus \bigcup l_i$ .

## Problem 18: Generic Nowhere Differentiability

### Problem 18

Let  $C[0, 1]$  be the complete metric space of continuous functions equipped with the sup-norm. Show that the set of functions which are nowhere differentiable is of the **Second Category** (i.e., "most" continuous functions are nowhere differentiable).

*This is a classic application of BCT, credited to Banach and Mazurkiewicz.*

# Solution to Problem 18 (1/4): Setup

**Goal:** Show that the set of functions differentiable at even one point is of the **First Category**.

Define the set  $E_n$  for each  $n \in \mathbb{N}$ :

$$E_n = \{f \in C[0, 1] \mid \exists x_0 \in [0, 1] \text{ s.t. } \forall y \in [0, 1], |f(y) - f(x_0)| \leq n|y - x_0|\}$$

**Connection to Differentiability:** If  $f$  is differentiable at  $x_0$ , then  $f'(x_0)$  exists, so the difference quotient is bounded near  $x_0$ . Since  $f$  is bounded on  $[0, 1]$ , the difference quotient is bounded globally by some integer  $n$ . Thus,  $\{f \in C[0, 1] \mid f \text{ is diff. at some point}\} \subseteq \bigcup_{n=1}^{\infty} E_n$ .

We must show each  $E_n$  is **nowhere dense**.

# Solution to Problem 18 (2/4): Closedness

**Step 1: Show  $E_n$  is closed.** Let  $\{f_k\} \subset E_n$  be a sequence converging uniformly to  $f$ .

- For each  $k$ , there exists  $x_k \in [0, 1]$  such that  $|f_k(y) - f_k(x_k)| \leq n|y - x_k|$  for all  $y$ .
- By Bolzano-Weierstrass,  $\{x_k\}$  has a subsequence converging to some  $x \in [0, 1]$ . Assume w.l.o.g.  $x_k \rightarrow x$ .
- Fix  $y$ . By uniform convergence  $f_k \rightarrow f$  and continuity:

$$|f(y) - f(x)| = \lim_{k \rightarrow \infty} |f_k(y) - f_k(x_k)| \leq \lim_{k \rightarrow \infty} n|y - x_k| = n|y - x|.$$

- Thus  $f \in E_n$ . So  $E_n$  is closed.



# Solution to Problem 18 (3/4): Nowhere Dense

**Step 2: Show  $E_n$  has empty interior.** It suffices to show that for any  $f \in C[0, 1]$  and  $\varepsilon > 0$ , there exists  $g \in B(f, \varepsilon)$  such that  $g \notin E_n$ .

- Approximate  $f$  by a piecewise linear function  $p$  (polygonal path) such that  $\|f - p\| < \varepsilon/2$ .
- Let  $M$  be the maximum slope of  $p$ .
- Construct a "sawtooth" function  $\phi(x)$  that oscillates very rapidly with slope  $K > n + M$  and amplitude bounded by  $\varepsilon/2$ .
- Define  $g(x) = p(x) + \phi(x)$ . Then  $\|g - f\| \leq \|g - p\| + \|p - f\| < \varepsilon$ .
- At any point  $x$ , the slope of  $g$  is roughly  $\text{slope}(p) + \text{slope}(\phi)$ . Since  $\text{slope}(\phi)$  dominates,  $|g(y) - g(x)|/|y - x|$  will exceed  $n$  locally.
- Thus  $g \notin E_n$ .  $E_n$  contains no open ball.

# Solution to Problem 18 (4/4): Conclusion

## Final Argument.

- ① We showed each  $E_n$  is closed and has empty interior, i.e.,  $E_n$  is **nowhere dense**.
- ② The set of functions differentiable at at least one point is contained in the countable union  $\bigcup_{n=1}^{\infty} E_n$ .
- ③ Therefore, the set of differentiable functions is of the **First Category** (meager).
- ④ By the Baire Category Theorem,  $C[0, 1]$  is of Second Category.
- ⑤ The complement—functions that are **nowhere differentiable**—must be of the **Second Category** and is therefore dense in  $C[0, 1]$ .



# Problem 19: Generic Nowhere Monotonicity

## Problem 19

Let  $C[0, 1]$  be the space of continuous functions on  $[0, 1]$ . Show that the set of functions which are nowhere monotone (i.e., not monotone on any sub-interval) is of the **Second Category** (generic).

*This result implies that "most" continuous functions wiggle infinitely often at all scales.*

# Solution to Problem 19 (1/2): Setup & Closedness

Let  $M \subset C[0, 1]$  be the set of functions monotone on some open sub-interval. We show  $M$  is of the First Category.

Let  $\{I_n\}_{n=1}^{\infty}$  enumerate all open intervals in  $[0, 1]$  with rational endpoints. Let  $M_n^+$  be the set of functions non-decreasing on  $I_n$ , and  $M_n^-$  be non-increasing on  $I_n$ . Then  $M = \bigcup_{n=1}^{\infty} (M_n^+ \cup M_n^-)$ .

**Step 1: Closedness.** If sequence  $\{f_k\} \subset M_n^+$  converges uniformly to  $f$ , then monotonicity is preserved. For any  $x, y \in I_n$  with  $x < y$ :  $f(y) - f(x) = \lim(f_k(y) - f_k(x)) \geq 0$ . Thus  $f \in M_n^+$ . Similarly,  $M_n^-$  is closed.

## Solution to Problem 19 (2/2): Nowhere Dense & Conclusion

**Step 2: Nowhere Dense.** We show  $M_n^+$  has empty interior. Let  $f \in M_n^+$  and  $\varepsilon > 0$ .

- Add a high-frequency "zig-zag" function  $\phi(x)$  (small amplitude  $< \varepsilon$ ) to  $f$ .
- The zig-zags will break the monotonicity of  $f$  within  $I_n$ .
- Thus, every ball around  $f$  contains a function not in  $M_n^+$ .

Since  $M_n^+$  is closed and has empty interior, it is nowhere dense (same for  $M_n^-$ ).

**Conclusion:**  $M$  is a countable union of nowhere dense sets  $\implies$  First Category. The complement (nowhere monotone functions) is Second Category.

# Multivariable Differentiation

# Overview: Differentiation

## Topics:

- **Taylor Series** ( $n = 2$ ): Using differential operators:

$$f(x + h, y + k) \approx \sum_{j=0}^2 \frac{1}{j!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^j f(x, y)$$

- **Implicit Function Theorem (IFT)**: Conditions under which non-linear equations  $F(\mathbf{x}, \mathbf{y}) = 0$  can be locally solved for  $\mathbf{y}$  as a function of  $\mathbf{x}$ . Crucially depends on the invertibility of the derivative with respect to  $\mathbf{y}$ .
- **Inverse Function Theorem**: Regularity of the Jacobian determinant implies local invertibility.
- **Bump Functions**: Smooth functions with compact support, used in partitions of unity.

# Problem 20: Taylor Series

## Problem 20

Compute the second-order Taylor expansion of  $f(x, y) = e^x \cos y$  at the point  $(0, 0)$ . Use the operator notation  $h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}$  in your solution.





# Problem 21: IFT in $\mathbb{R}^2 \times \mathbb{R}^2$

## Problem 21

Consider a  $C^1$  mapping  $F : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ , where we write points as  $(x, y, u, v)$  and  $F(x, y, u, v) = 0$ .

- ① State the Implicit Function Theorem conditions for solving for  $u, v$  in terms of  $x, y$  locally around a point  $P_0$ .
- ② State the formula for the Jacobian matrix of the implicit function  $G(x, y) = (u, v)$ .
- ③ Prove this derivative formula in detail.

## Solution to Problem 21 (1/2): Statement

**Theorem:** Let  $F = (F_1, F_2)$  be a  $C^1$  mapping near  $P_0 = (x_0, y_0, u_0, v_0)$  such that  $F(P_0) = 0$ . If the  $2 \times 2$  matrix of partial derivatives with respect to the dependent variables  $u, v$  is invertible at  $P_0$ :

$$\det \frac{\partial(F_1, F_2)}{\partial(u, v)} = \det \begin{pmatrix} \partial_u F_1 & \partial_v F_1 \\ \partial_u F_2 & \partial_v F_2 \end{pmatrix} \neq 0,$$

then there exists a neighborhood  $U$  of  $(x_0, y_0)$  and a unique  $C^1$  function  $G : U \rightarrow \mathbb{R}^2$ ,  $G(x, y) = (u(x, y), v(x, y))$ , such that  $F(x, y, u(x, y), v(x, y)) = 0$  for all  $(x, y) \in U$ .

# Solution to Problem 21 (2/2): Derivative Formula Proof

**Proof:** We differentiate the identity  $F(x, y, u(x, y), v(x, y)) = 0$  with respect to the independent variables. By the Chain Rule, for any variable  $\xi \in \{x, y\}$ :

$$\frac{\partial F}{\partial \xi} + \frac{\partial F}{\partial u} \frac{\partial u}{\partial \xi} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial \xi} = 0$$

Writing this in matrix form for the Jacobian of the implicit function  $DG = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ :

$$[D_{(x,y)}F] + [D_{(u,v)}F] \cdot [DG] = 0$$

Since  $[D_{(u,v)}F]$  is invertible by hypothesis:

$$DG = -[D_{(u,v)}F]^{-1}[D_{(x,y)}F]$$

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = - \begin{pmatrix} F_{1u} & F_{1v} \\ F_{2u} & F_{2v} \end{pmatrix}^{-1} \begin{pmatrix} F_{1x} & F_{1y} \\ F_{2x} & F_{2y} \end{pmatrix}$$



# Solution to Problem 22: Calculation

Let  $F_1 = xu + yv + uv - 1$  and  $F_2 = xu^3 + yv^3 - 1$ .

- **1. Verify Solution:**  $1(1) + 1(0) + 1(0) = 1$  and  $1(1)^3 + 1(0)^3 = 1$ . ✓
- **2. Check IFT Condition:** Calculate  $J_{u,v} = \frac{\partial(F_1, F_2)}{\partial(u, v)}$  at  $P(1, 1, 1, 0)$ .

$$J_{u,v} = \begin{pmatrix} x + v & y + u \\ 3xu^2 & 3yv^2 \end{pmatrix}_P = \begin{pmatrix} 1 + 0 & 1 + 1 \\ 3(1)(1)^2 & 3(1)(0)^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$\det(J_{u,v}) = -6 \neq 0$ . Thus, implicit functions  $u(x, y), v(x, y)$  **exist**.

- **3. Compute Derivatives:**  $J_{x,y} = \begin{pmatrix} u & v \\ u^3 & v^3 \end{pmatrix}_P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ .

$$\begin{aligned} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} &= - \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = -\frac{1}{-6} \begin{pmatrix} 0 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} -2 & 0 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -1/3 & 0 \\ -1/3 & 0 \end{pmatrix} \end{aligned}$$

# Further Learning: Alternative Proof of IFT

## Alternative Proof: Fixed Point Method

For those interested in a proof of the **Implicit/Inverse Function Theorem** using the **Contraction Mapping Principle** (Fixed Point Theorem), please check **Liu Siqi's video series** on Mathematical Analysis.

# Multivariable Integration



**Kiehlmann T**  
**Lebensweg**

- $$\frac{d}{dy} \int_a^b f(x, y) dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx$$

## Problem 23: Feynman's Trick Practice

### Problem 23

Find the following integrals:

(a)  $\int_0^1 \frac{x^b - x^a}{\ln x} dx, \quad b > a > 0.$

(b)  $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx.$

# Solution to Problem 23 (1/2): Part (a)

**Technique: Parameter Integration** Observe that  $\frac{x^b - x^a}{\ln x} = \int_a^b x^y dy$ .

$$I = \int_0^1 \left( \int_a^b x^y dy \right) dx$$

By Fubini's Theorem (justification required for improper integral, but valid here since integrand is positive and measurable):

$$I = \int_a^b \left( \int_0^1 x^y dx \right) dy$$

$$I = \int_a^b \left[ \frac{x^{y+1}}{y+1} \right]_0^1 dy = \int_a^b \frac{1}{y+1} dy$$

$$I = [\ln(y+1)]_a^b = \ln(b+1) - \ln(a+1) = \ln \left( \frac{b+1}{a+1} \right)$$



## Problem 24: Improper Integrals with Parameters

### Problem 24

Compute:

(a)  $J = \int_0^{\infty} e^{-px} \frac{\sin bx - \sin ax}{x} dx, \quad p > 0, b > a.$

(b)  $\int_0^{\infty} \frac{\sin ax}{x} dx, \quad a \neq 0.$



## Solution to Problem 24 (2/2): Part (b)

**Dirichlet Integral:** We can view this as the limit of part (a) as  $p \rightarrow 0^+$  (with  $b = a$  and the second term 0, effectively). Alternatively, define  $K(p) = \int_0^\infty e^{-px} \frac{\sin ax}{x} dx$  for  $p \geq 0$ .

$$K'(p) = - \int_0^\infty e^{-px} \sin ax \, dx = - \frac{a}{p^2 + a^2}$$

$$K(p) = - \arctan \left( \frac{p}{a} \right) + C$$

As  $p \rightarrow \infty$ ,  $K(p) \rightarrow 0$  (Riemann-Lebesgue type decay), so  $0 = -\frac{\pi}{2} + C \implies C = \frac{\pi}{2}$  (assuming  $a > 0$ ). Thus  $K(0) = \frac{\pi}{2}$ .

If  $a < 0$ , the sign flips. Thus:

$$\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \operatorname{sgn}(a)$$