Final Review: Real Analysis

Borel-Cantelli, Differentiation, L^p Spaces, and Modes of Convergence

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Outline

Topic 1: Overview

Key Concept: The Borel-Cantelli Lemma.
It is the primary tool for translating conditions on integrals or measures into pointwise almost everywhere (a.e.) convergence results.

"The message is that 'Borel-Cantelli' way of arguing is important!"

Problem 1

Problem

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions on a measurable set E. Suppose that for every k:

$$\int_E |f_{k+1} - f_k| < \delta_k^2$$

where $\sum_{k=1}^{\infty} \delta_k < \infty$.
Prove: The sequence $\{f_k\}$ converges pointwise almost everywhere to some function f.

Solution to Problem 1

Strategy: In order to show f_k converges a.e., it suffices to show that $\sum |f_{k+1} - f_k| < \infty$ a.e.

- Let $A_k = \{x \in E \mid |f_{k+1}(x) - f_k(x)| > \delta_k\}$.
- By Chebyshev's Inequality: $m(A_k) \leq \frac{1}{\delta_k} \int_E |f_{k+1} - f_k| < \delta_k$.
- Since $\sum \delta_k < \infty$, by the Borel-Cantelli Lemma, almost every x belongs to only finitely many A_k.

Note: This is the key step in proving the Riesz-Fischer Theorem.

Problem 2

Problem

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions on a measurable set E. Suppose that for every k:

$$\int_E |f_{k+1} - f_k| < \delta_k$$

where $\sum_{k=1}^{\infty} \delta_k < \infty$.

Prove: The sequence $\{f_k\}$ converges pointwise almost everywhere to some function f.

Remark on Problem 2

Comparison: The proof from Problem 1 (using Chebyshev directly on the sets) works for Problem 1 but **not** for Problem 2 directly because the power of convergence is different.

Alternative Argument: Consider the function $g(x) = \sum_{k=1}^{\infty} |f_{k+1}(x) - f_k(x)|$.

- By the Monotone Convergence Theorem (MCT), $\int_E g = \sum \int_E |f_{k+1} - f_k| < \sum \delta_k < \infty$.
- Thus $g(x)$ is finite a.e., implying the series converges absolutely a.e.
- This implies $f_k \to f$ a.e. (The "Beppo-Levi" argument).

Problem 3

Problem

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on $\mathbb{R}$ such that:

$$\int_{\mathbb{R}} |f_n(x) - f(x)| \, dx \leq \frac{1}{n^{1+\epsilon}}$$

(a) If $\epsilon > 0$, prove that $f_n \to f$ pointwise almost everywhere.

(b) If $\epsilon = 0$, provide a counterexample to show that pointwise a.e. convergence may fail.

Solution to Problem 3

Part (a) $\epsilon > 0$**:** Since $\sum \frac{1}{n^{1+\epsilon}} < \infty$, we can sum the integrals. By the Beppo-Levi argument (similar to Problem 2), the series $\sum |f_n - f|$ converges a.e., which implies $|f_n - f| \to 0$ a.e.

Part (b) $\epsilon = 0$**:** Consider the "Typewriter Sequence" (moving characteristic functions):

$$\chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/4]}, \chi_{[1/4,1/2]}, \cdots$$

The integral decays like $1/n$ (harmonic series diverges), and the "bump" keeps cycling through the interval infinitely often, so it diverges everywhere.

Problem 4

Problem

A sequence $\{f_n\}$ of measurable functions is said to be **Cauchy in Measure** if for every $\eta > 0$ and $\epsilon > 0$, there exists an integer N such that for all $n, m \geq N$:

$$m(\{x \in E \mid |f_n(x) - f_m(x)| \geq \eta\}) < \epsilon$$

Prove: If $\{f_n\}$ is Cauchy in measure, then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise almost everywhere.

Remark on Problem 4

This is a standard homework result. The key takeaway is utilizing the **Borel-Cantelli** style of argument to extract the subsequence efficiently.

Problem 5 (Diophantine Approximation)

Problem

(a) Order 2 is Universal: Use the Pigeonhole Principle to prove Dirichlet's Approximation Theorem: For any irrational x, there exist infinitely many rational numbers p/q such that $|x - p/q| < 1/q^2$.

(b) Order > 2 is Rare: Prove that the set of $x \in [0, 1]$ such that $|x - p/q| < 1/q^3$ for infinitely many rationals p/q has Lebesgue measure zero.

Solution to Problem 5(b)

Proof: It suffices to show by the Borel-Cantelli Lemma that the sum of the measures of the target intervals is finite.

$$\sum_{q=1}^{\infty} m\left(\bigcup_{p=0}^{q}\left\{x \in [0,1] : \left|x - \frac{p}{q}\right| < \frac{1}{q^3}\right\}\right)$$

The measure at each level q is roughly $q \cdot \frac{2}{q^3} = \frac{2}{q^2}$. Since $\sum \frac{1}{q^2} < \infty$, the set of such x has measure 0.

Topic 2: Overview

Key Questions:

1. Which classes of functions are differentiable almost everywhere?
2. Under what conditions does the Fundamental Theorem of Calculus (FTC) hold? i.e., $\int_a^b f' = f(b) - f(a), \quad \frac{d}{dx}\int_a^x f = f.$

Summary:

- **Monotone Functions:** Differentiable a.e. (Proof uses Vitali Covering Lemma).
- **Bounded Variation (BV):** Differentiable a.e., but FTC may fail (e.g., Cantor-Lebesgue function).
- **Absolute Continuity (AC):** The necessary and sufficient condition for FTC.

Properties of Absolutely Continuous Functions

- **Equivalence with Stronger Definition:**
  The standard definition of absolute continuity (requiring $\sum |f(b_k) - f(a_k)| < \epsilon$) is equivalent to the stronger condition:

  $$\sum_{k=1}^{n} TV(f|_{[a_k,b_k]}) < \epsilon$$

  whenever $\sum (b_k - a_k) < \delta$.
- **Indefinite Integral Characterization:**
  A function f is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral:

  $$f(x) = f(a) + \int_a^x g(t)\,dt$$

  for some integrable function g (specifically, $g = f'$ a.e.).

Properties of Absolutely Continuous Functions

- **FTC for Increasing Functions:**
  An increasing continuous function f is absolutely continuous if and only if it satisfies the Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Problem 6

Problem

Let $\alpha, \beta > 0$. Define $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} x^{\alpha} \sin(\frac{1}{x^{\beta}}) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

(i) Prove that if $\alpha > \beta$, then f is of Bounded Variation (BV) on $[0,1]$.

(ii) Prove that if $\alpha \leq \beta$, then f is **not** of Bounded Variation on $[0,1]$.

Solution to Problem 6

(i) Case $\alpha > \beta$: f is continuously differentiable on $(0, 1]$. We calculate the derivative:

$$|f'(x)| \leq \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1}$$

Since $\alpha > \beta$, the power $\alpha - \beta - 1 > -1$. Thus $|f'|$ is integrable near 0. Therefore, $TV(f) = \int_0^1 |f'| < \infty$.

(ii) Case $\alpha \leq \beta$: We construct a partition where the variation diverges. Choose points x_n where $\sin(1/x^\beta) = \pm 1$. The sum of oscillations behaves like the harmonic series $\sum \frac{1}{n}$ (or worse), which diverges to infinity.

Problem 7

Problem

Let Lip, AC, and BV denote the spaces of Lipschitz, Absolutely Continuous, and Bounded Variation functions on $[a, b]$, respectively.
Prove the strict inclusions:

$$\mathrm{Lip} \subsetneq AC \subsetneq BV$$

Solution to Problem 7

- Lip $\subsetneq AC$: Consider $f(x) = \sqrt{x}$ on $[0, 1]$. It is AC (integral of $1/2\sqrt{x}$) but not Lipschitz (derivative is unbounded at 0).
- $AC \subsetneq BV$: Consider the Cantor-Lebesgue function. It is BV (since it is monotone) but not AC (it maps a set of measure 0 to a set of measure 1).

Problem 8

Problem

Prove that if f is an absolutely continuous function, then f maps measurable sets to measurable sets.

Solution to Problem 8

Step 1: Show f satisfies the condition (N): it maps sets of measure 0 to sets of measure 0. For any ϵ, cover the zero-measure set Z with intervals of small total length. By absolute continuity, the image of these intervals has small total length.
Step 2: General Measurable Set E. Write $E = F \cup Z$ where F is an F_σ set (countable union of closed sets) and $m(Z) = 0$.

- $f(F)$ is measurable (continuous image of an F_σ set is F_σ, hence Borel).
- $f(Z)$ has measure 0 by Step 1.

Thus $f(E) = f(F) \cup f(Z)$ is measurable.

Problem 9

Problem

Let f be a function of bounded variation on $[a, b]$, and define the total variation function $V(x) = TV(f|_{[a,x]})$.

(i) Show that $|f'(x)| \leq V'(x)$ almost everywhere, and $\int_a^b |f'(x)|\, dx \leq TV(f)$.

(ii) Prove that $V'(x) = |f'(x)|$ a.e. if and only if f is absolutely continuous.

Problem 10

Problem

Let $f : [a, b] \to \mathbb{R}$ be an increasing function.
Prove: f is singular (i.e., $f' = 0$ a.e.) if and only if: For every $\epsilon > 0$, there exists a finite disjoint collection of intervals $\{(a_k, b_k)\}$ such that: 1. $\sum(b_k - a_k) < \epsilon$ (Small total length in domain) 2. $\sum(f(b_k) - f(a_k)) > f(b) - f(a) - \epsilon$ (Captures most of the growth in range).

Remark on Problem 10

Intuition: If f is AC, it increases "gradually" everywhere. If f is singular and increasing, it must accomplish a "big increase" within a "very small region" (effectively on a set of measure zero).

Topic 3: Overview

Density:

- Simple functions, Step functions, and Continuous functions with compact support (C_c) are dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Separability:

- L^p is separable for $1 \leq p < \infty$.
- L^∞ is **not** separable.
- $C[a, b]$ is separable (by the Stone-Weierstrass Theorem).

Problem 11

Problem (Weak Compactness)

Let $1 < p < \infty$. Prove that if $\{f_n\}$ is a bounded sequence in $L^p(E)$, then $\{f_n\}$ possesses a subsequence that converges in the **weak sense**.

Problem 12

Problem

Let $1 < p < \infty$. Prove that a bounded sequence $\{f_n\}$ in $L^p(E)$ converges weakly to f if and only if for every measurable subset $A \subseteq E$:

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Problem 13

Problem (Riemann-Lebesgue Lemma)

Let $f \in L^1([a,b])$. Prove that:

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)\,dx = 0$$

Problem 14

Problem

Provide an explicit example of a sequence that converges weakly to 0 but does not converge strongly (in norm) to 0.

Solution to Problem 14

Example: The Rademacher sequence $R_n(x) = \mathrm{sgn}(\sin(2^n \pi x))$ on $[0, 1]$. Or simply $f_n(x) = \sin(nx)$.

- $f_n \rightharpoonup 0$ (weakly) by the Riemann-Lebesgue Lemma.
- However, $\|f_n\|_p$ does not go to 0 (it stays constant for Rademacher, or constant average for sin), so it does not converge strongly.

Topic 4: Overview

We study the relationships between:

1. Convergence in L^p (Norm)
2. Convergence in Measure
3. Pointwise Convergence a.e.
4. Weak Convergence

Does Property A imply a Subsequence has Property B? (Assume $m(E) < \infty$ and sequence is bounded in L^p)				
Prop A $\downarrow$ \ Prop B $\rightarrow$	**Strong (L^p)**	**Weak (L^p)**	**Ptwise a.e.**	**Measure**
Case 1: Reflexive Range $(1 < p < \infty)$				
Strong	YES	YES	YES	YES
Weak	NO	YES	NO	NO
Pointwise a.e.	NO	**YES**	YES	YES
Measure	NO	**YES**	YES	YES
Case 2: Non-Reflexive $(p = 1)$				
Pointwise a.e.	NO	**NO**	YES	YES
Measure	NO	**NO**	YES	YES

Problem 15

Problem

Let $\{f_n\}$ be a bounded sequence in $L^p(E)$ such that $f_n \to f$ pointwise almost everywhere.

(i) If $p > 1$, prove that $f_n \rightharpoonup f$ (converges weakly).

(ii) If $p = 1$, prove or disprove that $f_n \rightharpoonup f$.

Solution to Problem 15

(i) Case $p > 1$: YES. By weak compactness (reflexivity of L^p), any subsequence contains a further weakly convergent subsequence. Since the pointwise limit is f, the weak limit must be unique and equal to f. Thus the entire sequence converges weakly to f.

(ii) Case $p = 1$: NO. Counter-example: $f_n = n\chi_{[0,1/n]}$.

- Bounded in L^1 ($\|f_n\|_1 = 1$).
- Converges pointwise a.e. to 0.
- Test against $g \equiv 1 \in L^\infty$: $\int f_n g = 1 \not\to 0$. Thus it does not converge weakly to 0.

Problem 16

Problem (Radon-Riesz Property)

Let $1 \leq p < \infty$. Suppose $\{f_n\} \subset L^p(E)$ converges pointwise almost everywhere to $f \in L^p(E)$.

Prove: $f_n \to f$ in L^p (Strongly) if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Remark on Problem 16

The "$\Rightarrow$" direction is trivial by continuity of the norm. The "$\Leftarrow$" direction is non-trivial. It usually involves applying Fatou's Lemma to the sequence:

$$g_n = 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p$$

Since $g_n \geq 0$, applying Fatou yields the result.

Topic 5: Overview (Standard Library)

It is hard to remember every counter-example, but useful to keep a "Standard Library" to test hypotheses:

1. **Cantor-Lebesgue Function:** (Continuous, Monotone, derivative 0 a.e., not AC).
2. **Typewriter Sequence:** (Converges in measure, but diverges pointwise everywhere).
3. **The Spikes:** $n\chi_{[0,1/n]}$ (Converges pointwise to 0, but Integral stays 1).
4. **Rademacher Functions:** (Weak convergence to 0, no strong convergence).

Problem 17

Problem

Construct a sequence of Riemann integrable functions $f_n : [a, b] \to \mathbb{R}$ such that $\{f_n\}$ converges pointwise to a function f, but f is **not** Riemann Integrable.

Remark on Problem 17

Example: Enumerate rationals in $[0, 1]$ as $\{r_1, r_2, \dots\}$. Let f_n be the characteristic function of $\{r_1, \dots, r_n\}$. Then $f_n \to \chi_{\mathbb{Q}}$, which is not Riemann integrable (Dirichlet function).
Note: If the sequence is uniformly bounded, the limit is always **Lebesgue Integrable** by the Dominated Convergence Theorem.

Problem 18

Problem

Construct an absolutely continuous, strictly increasing function $f : [0, 1] \to \mathbb{R}$ for which $f'(x) = 0$ on a set of positive measure.

Solution to Problem 18

This requires a ”Fat Cantor Set” construction (a Cantor-like set with positive measure). We construct a homeomorphism that maps the Fat Cantor set (measure > 0) to a regular Cantor set (measure 0) or vice versa, ensuring strict monotonicity while keeping the derivative zero on the ”gaps”.