Real Analysis Midterm-ustc-2015

October 13, 2025

Problem 1.

(15 points) Let N be the non-measurable set on the interval $[0, 1]$ constructed in the textbook. If E is a measurable subset of N, prove that $m(E) = 0$.

Problem 2.

(15 points) Let f be an integrable function on $\mathbb{R}$ that is also uniformly continuous. Prove that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Problem 3.

(15 points) Let $\{f_n\}$ be a sequence of integrable functions such that $0 \leq f_1 \leq f_2 \leq \dots$, a.e. Prove that $\lim_{n\to\infty} \int f_n = 0$ if and only if $\lim_{n\to\infty} f_n(x) = 0$, a.e.

Problem 4.

(15 points) Let $\{f_n\}$ be a sequence of integrable functions on $\mathbb{R}$, and f be an

integrable function on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} |f_n(x) - f(x)| \, dx \leq \frac{1}{n^2} \quad (n = 1, 2, \dots).$$

Prove that $f_n(x) \to f(x)$, a.e.

Problem 5.

(15 points) Let $E \subset \mathbb{R}^d$ be a bounded measurable set. Prove that

$$\lim_{h \to 0} m(E \cap (E + h)) = m(E).$$

Problem 6.

(15 points) Let $m(E) < \infty$, $\{f_n\}$ be a sequence of measurable functions on E, and f be a measurable function on E. Prove that f_n converges to f in measure on E if and only if every subsequence f_{n_k} of f_n has a further subsequence $f_{n_{k_i}}$ that converges to f almost everywhere on E.

Problem 7.

(10 points) Define the function $f(x) = x^3$ on $\mathbb{R}$. Consider $f : \mathbb{R} \to \mathbb{R}$ as a mapping. Prove:

(a) (5 points) f maps null sets to null sets.

(b) (5 points) f maps measurable sets to measurable sets.