Real Analysis
midterm-ustc-2025-version1

October 13, 2025

Problem 1.

(1) State the definition of a **measurable function** f.

(2) Prove: f is measurable if and only if the set $\{f > 0\}$ is measurable.

Problem 2.

(1) A measurable subset of the non-measurable set N constructed in the textbook must be a null set. (Prove)

(2) $m(E) = 0 \Leftrightarrow$ every measurable subset of E is a null set. (Prove)

Problem 3.

Let $B = \{\|x\| < 1\}$ be the open ball in $\mathbb{R}^d$. Let f be a non-negative measurable function such that $\int_B f \, dx = 1$. Prove:

$$\int_B f(x)\|x\| \, dx < 1$$

Problem 4.

Evaluate:

$$\lim_{k\to\infty}\int_0^{+\infty}\frac{x+\sin^k x}{1-e^{-kx}+x^k}\,dx$$

Problem 5.

Let $f, f_1, f_2, \ldots$ all be measurable on $[0,1]$.

(1) If $f_n \xrightarrow{\text{a.e.}} f$, can we conclude $f_n \xrightarrow{L^1} f$? Prove or give a counterexample.

(2) If $f_n \xrightarrow{L^1} f$, can we conclude $f_n \xrightarrow{\text{a.e.}} f$? Prove or give a counterexample.

(3) If $f_n \xrightarrow{m} f$ (converges in measure), can we conclude $\lim_{n\to\infty} m(\{|f_n - f| > \epsilon\}) = 0$? Prove or give a counterexample.

Problem 6.

Let $E_k = \{|f| \geq k\}$ and f be integrable. Prove:

(1) $\lim_{k\to\infty} m(E_k) = 0$

(2) $\sum_{k=1}^{\infty} m(E_k) < +\infty$

Problem 7.

Let g be a **1-periodic smooth function** such that $\int_0^1 g(x)\,dx = 0$. Prove that the following hold:

(1) For any finite interval $[a, b]$,

$$\lim_{n\to\infty} \int_a^b g(nx)\, dx = 0$$

(2) For any integrable function f,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) g(nx)\, dx = 0$$