Final Review for MAT2060: Honors Mathematical Analysis

Metric Spaces and Multivariable Calculus

Cola

The Chinese University of HongKong, Shenzhen

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Outline

Topology of Metric Spaces

Overview: Topology of Metric Spaces

Key Concepts:

- **Compactness:** Recall that in a general metric space (X, d):

$$\text{Compactness} \iff \text{Sequential Compactness} \iff \text{Totally Bounded} + \text{Complete}$$

  In $\mathbb{R}^n$, we have the **Heine-Borel Theorem**: Compact $\iff$ Closed and Bounded.

- **Connectedness:**

$$\text{Path-Connected} \implies \text{Connected}$$

  The converse is not true (e.g., Topologist's Sine Curve).

Problem 1: Compactness Equivalences

Problem 1

Let (X, d) be a metric space and let $K \subseteq X$. Prove the equivalence of the following statements:

1. *K is compact (every open cover has a finite subcover).*
2. *K is sequentially compact (every sequence has a convergent subsequence).*
3. *K is totally bounded and complete.*

Solution to Problem 1 (1/3): Seq. Compact $\implies$ Complete & Totally Bounded

Part A: Completeness Let $\{x_n\}$ be a Cauchy sequence in K. Since K is sequentially compact, there exists a subsequence $\{x_{n_k}\}$ that converges to some $x \in K$. A Cauchy sequence with a convergent subsequence converges to the same limit. Thus $x_n \to x$, so K is complete.

Part B: Totally Bounded Suppose K is not totally bounded. Then $\exists \varepsilon > 0$ such that K cannot be covered by finitely many ε-balls.

- Pick $x_1 \in K$. Then $K \not\subseteq B(x_1, \varepsilon)$, so choose $x_2 \in K \setminus B(x_1, \varepsilon)$.
- Inductively, choose $x_n \in K \setminus \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$.
- For $n \neq m$, $d(x_n, x_m) \geq \varepsilon$.
- The sequence $\{x_n\}$ has no Cauchy subsequence, hence no convergent subsequence. Contradiction.

Solution to Problem 1 (2/3): Complete & Totally Bounded $\implies$ Compact

Suppose K is complete and totally bounded but not compact. Let $\mathcal{U}$ be an open cover with no finite subcover.

1. Since K is totally bounded, cover it by finitely many balls of radius 1. At least one, say $K \cap B_1$, cannot be finitely covered by $\mathcal{U}$.
2. Cover $K \cap B_1$ by finitely many balls of radius $1/2$. Pick B_2 such that $K \cap B_1 \cap B_2$ is not finitely covered.
3. Inductively, find nested sets with diameter $\to 0$. Pick x_n in the n-th intersection.
4. By completeness, $x_n \to x \in K$.
5. Since $\mathcal{U}$ covers K, $x \in U$ for some $U \in \mathcal{U}$. $\exists \delta > 0$ such that $B(x, \delta) \subseteq U$.
6. For large n, the chosen set lies inside $B(x, \delta)$, meaning it is covered by a single set U. Contradiction.

Solution to Problem 1 (3/3): Compact $\Longrightarrow$ Sequentially Compact

Let K be compact. We prove sequential compactness by contradiction.

- Suppose there is a sequence $\{x_n\}$ with no convergent subsequence.
- Then the set of points $S = \{x_n\}$ has no limit points.
- For any $y \in K$, there exists an open ball $B(y, r_y)$ containing only finitely many terms of the sequence (if $y \in S$, it contains only y; if not, it can be disjoint from the tail).
- The collection $\{B(y, r_y)\}_{y \in K}$ is an open cover of K.
- By compactness, there is a finite subcover.
- The union of these finitely many balls contains only finitely many terms of the sequence, contradicting that $\{x_n\}$ is an infinite sequence.

Problem 2: Heine-Borel Theorem

Problem 2

Prove $S \subseteq \mathbb{R}^n$ is compact if and only if S is closed and bounded.

Solution to Problem 2 (1/2): Compact $\implies$ Closed & Bounded

Forward Direction.

Boundedness: Consider the open cover $\mathcal{U} = \{B(0,n) \mid n \in \mathbb{N}\}$. Since $S \subseteq \bigcup B(0,n) = \mathbb{R}^n$, this covers S. By compactness, there is a finite subcover $\{B(0,n_1), \dots, B(0,n_k)\}$. Let $N = \max(n_i)$. Then $S \subseteq B(0,N)$, so S is bounded.

Closedness: We show S^c is open. Let $y \in S^c$. For each $x \in S$, let $r_x = \frac{1}{2}d(x,y)$. The collection $\{B(x,r_x)\}$ covers S. By compactness, take a finite subcover corresponding to $x_1, \dots, x_m$. Let $V = \bigcap_{i=1}^{m} B(y, r_{x_i})$. V is an open neighborhood of y disjoint from S. Thus S^c is open. □

Solution to Problem 2 (2/2): Closed & Bounded $\implies$ Compact

Converse Direction (Sequential Method).

Suppose S is closed and bounded. We show S is sequentially compact.

1. Let $\{\mathbf{x}_k\}$ be an arbitrary sequence in S.
2. Since S is bounded, the sequence $\{\mathbf{x}_k\}$ is bounded in $\mathbb{R}^n$.
3. By the **Bolzano-Weierstrass Theorem** for $\mathbb{R}^n$, every bounded sequence has a convergent subsequence.
4. Let $\{\mathbf{x}_{k_j}\}$ be a subsequence converging to some limit $\mathbf{x} \in \mathbb{R}^n$.
5. Since S is closed and $\{\mathbf{x}_{k_j}\} \subset S$, the limit must be in S (i.e., $\mathbf{x} \in S$).
6. Thus, every sequence in S has a subsequence converging to a point in S.
7. S is sequentially compact $\implies$ S is compact (by Problem 1).

$\square$

Problem 3: The Hilbert Cube

Problem 3

Let $H = [0,1]^{\mathbb{N}}$ *be the set of sequences* $x = (x_1, x_2, \dots)$ *with* $x_n \in [0,1]$. *Define a distance function:*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$$

1. *Show that* (H, d) *is a metric space.*
2. *Show that* H *is sequentially compact.*

Solution to Problem 3 (1/2): Metric Space

Part 1: Metric Space.

- **Well-defined:** Since $|x_n - y_n| \leq 1$, the series is dominated by $\sum 2^{-n} = 1$, so it converges.
- **Positivity:** $d(x, y) \geq 0$. If $d(x, y) = 0$, then $|x_n - y_n| = 0$ for all n, so $x = y$.
- **Symmetry:** $|x_n - y_n| = |y_n - x_n|$, so $d(x, y) = d(y, x)$.
- **Triangle Inequality:** For any $z \in H$:

$$|x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$$

Multiplying by 2^{-n} and summing gives $d(x, y) \leq d(x, z) + d(z, y)$.

□

Solution to Problem 3 (2/2): Sequential Compactness

Part 2: Diagonal Argument.

Let $\{x^{(k)}\}$ be a sequence in H.

- The first coordinates $x_1^{(k)}$ lie in $[0,1]$. By Bolzano-Weierstrass, there is a subsequence converging in the 1st slot.
- From this, extract a sub-subsequence converging in the 2nd slot, and so on.
- Let $x^{(k_j)}$ be the **diagonal sequence**. It converges pointwise to some $x \in H$: $\lim_{j\to\infty} |x_n^{(k_j)} - x_n| = 0$ for each n.
- **Convergence in metric d:** Given $\varepsilon > 0$, pick N such that $\sum_{N+1}^{\infty} 2^{-n} < \varepsilon/2$.
- Choose J large enough so for $j > J$, $\sum_{n=1}^{N} 2^{-n}|x_n^{(k_j)} - x_n| < \varepsilon/2$.
- Then $d(x^{(k_j)}, x) < \varepsilon$. Thus H is sequentially compact.

□

Problem 4: Connectedness Properties

Problem 4

1. *Show that the interval* $[0, 1]$ *is a connected set in* $\mathbb{R}$*.*
2. *Show that if a space* X *is path-connected, it is connected.*

Solution to Problem 4: Connectedness

Proof.

1. $[0,1]$ **is connected:** Suppose $[0,1] = A \cup B$ where A, B are disjoint, non-empty, closed sets in $[0,1]$. Assume $0 \in A$. Let $c = \sup A$.

- Since A is closed, $c \in A$. Thus $c < 1$ (otherwise B is empty).
- Since A is open in $[0,1]$ (complement of B), there is a neighborhood $[c, c+\varepsilon) \subseteq A$.
- This contradicts $c = \sup A$. Thus, no such separation exists.

2. Path-connected $\implies$ **Connected:** Suppose X is path-connected but disconnected, so $X = U \cup V$ disjoint open sets. Pick $u \in U, v \in V$. Let $\gamma : [0,1] \to X$ be a path from u to v.

- Consider sets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ in $[0,1]$.
- They are disjoint, non-empty, and open in $[0,1]$ (by continuity).
- Their union is $[0,1]$, contradicting that $[0,1]$ is connected.

□

Problem 5: Open Connected Sets

Problem 5

Show that any **open**, *connected subset of a Euclidean space (or normed vector space) is path-connected.*

Solution to Problem 5: Open Connected $\implies$ Path-Connected

Proof.

Let $U \subseteq \mathbb{R}^n$ be open and connected. Fix $x_0 \in U$. Define $A = \{x \in U \mid \exists \text{ path from } x_0 \text{ to } x \text{ in } U\}$.

1. **A is open:** Let $x \in A$. Since U is open, $\exists B(x, r) \subseteq U$. Balls are convex (hence path-connected). Any $y \in B(x, r)$ connects to x, then to x_0. So $B(x, r) \subseteq A$.
2. **$U \setminus A$ is open:** Let $y \in U \setminus A$. $\exists B(y, r) \subseteq U$. If any $z \in B(y, r)$ were in A, we could connect $x_0 \to z \to y$, implying $y \in A$ (contradiction). So $B(y, r) \subseteq U \setminus A$.
3. **Conclusion:** A is a non-empty $(x_0 \in A)$ open and closed subset of connected U. Thus $A = U$.

□

Problem 6: The Counterexample

Problem 6

Consider the "Topologist's Sine Curve":

$$S = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x \in (0,1]\} \cup \{(0,y) \mid y \in [-1,1]\}$$

Prove that S is connected but **not** *path-connected.*

Note: This illustrates that path-connectedness is strictly stronger than connectedness for general sets.

Solution to Problem 6: Topologist's Sine Curve

Proof.

Connected: Let $G = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$. G is the continuous image of the connected set $(0, 1]$, so G is connected. $S = \bar{G}$ (the closure adds the segment $\{0\} \times [-1, 1]$). The closure of a connected set is connected.

Not Path-Connected: Suppose there is a path $\gamma : [0, 1] \to S$ from $(0, 0)$ to $(1/\pi, 0)$. Let $\gamma(t) = (x(t), y(t))$.

- Since $x(t)$ is continuous and $x(0) = 0, x(1) > 0$, there are points arbitrarily close to $t = 0$ with $x(t) > 0$.
- As $t \to 0$, $x(t) \to 0$, so $1/x(t) \to \infty$.
- $\sin(1/x(t))$ oscillates between -1 and 1 infinitely often.
- This prevents $y(t)$ from converging to any specific value if we approach along the curve, contradicting the continuity of γ at 0.

□

Tietze Extension Theorem

Overview: Tietze Extension Theorem

The Theorem: Let X be a metric space (or normal topological space) and $F \subseteq X$ be a closed set. If $f : F \to \mathbb{R}$ is continuous, there exists a continuous extension $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_F = f$.

Key Tool (Urysohn's Lemma): The proof relies heavily on the ability to separate disjoint closed sets. If $A, B \subseteq X$ are disjoint and closed, there exists a continuous function $\varphi : X \to [0, 1]$ such that:

$$\varphi(A) = \{0\} \quad \text{and} \quad \varphi(B) = \{1\}.$$

Problem 7: Urysohn's Lemma

Problem 7

Let (X, d) *be a metric space. Let* A, B *be disjoint closed subsets of* X*. Construct a continuous function* $\varphi : X \to [0, 1]$ *satisfying Urysohn's property.*

Solution to Problem 7

Proof.

Using the distance function to a set, $d(x, E) = \inf\{d(x, y) \mid y \in E\}$, which is continuous. Define:

$$\varphi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

- Since A, B are closed and disjoint, $d(x, A) + d(x, B) > 0$ for all x, so φ is well-defined and continuous.
- If $x \in A$, $d(x, A) = 0 \implies \varphi(x) = 0$.
- If $x \in B$, $d(x, B) = 0 \implies \varphi(x) = 1$.
- Clearly $0 \leq \varphi(x) \leq 1$.

$\square$

Problem 8: Proof Strategy of Tietze Extension

Problem 8

Describe the strategy to prove the Tietze Extension Theorem using Urysohn's Lemma.

Solution Sketch: The proof uses an iterative approximation: 1. Assume $|f(x)| \leq M$. 2. Define sets $A = \{x \mid f(x) \leq -M/3\}$ and $B = \{x \mid f(x) \geq M/3\}$. 3. Use Urysohn to find g_1 approximating f on these sets. 4. Consider the error $f - g_1$, which is bounded by $2M/3$. 5. Repeat inductively to get a series of functions $\sum g_n$ that converges uniformly to an extension of f.

Problem 9 & 10: Extensions on $\mathbb{R}$

Problem 9

If $F \subseteq \mathbb{R}$ is closed, show explicitly that any continuous $f : F \to \mathbb{R}$ can be extended to $\mathbb{R}$ (e.g., by linear interpolation on the gaps).

Problem 10

Prove or Disprove: *Any continuous function $f : (0, 1] \to \mathbb{R}$ can be extended to a continuous function on $\mathbb{R}$.*

Answer to 10: False. Consider $f(x) = 1/x$. It is continuous on $(0, 1]$ but cannot be extended to $x = 0$ continuously. (Extension requires uniform continuity or boundedness if the domain is not closed).

Stone-Weierstrass Theorem

Overview: Stone-Weierstrass

Theorem: Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{R})$ such that:

1. $\mathcal{A}$ separates points ($x \neq y \implies \exists f \in \mathcal{A}, f(x) \neq f(y)$).
2. $\mathcal{A}$ vanishes at no point (or contains constants).

Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ in the uniform norm.

Corollary: Polynomials are dense in $C[a, b]$.

Problem 11: The Moment Problem

Problem 11

Let $f \in C[0,1]$. *Suppose that for all* $n = 0, 1, 2, \dots,$

$$\int_0^1 x^n f(x)\, dx = 0.$$

Prove that $f(x) \equiv 0$.

Solution to Problem 11

Proof.

1. By the Stone-Weierstrass theorem, the set of polynomials is dense in $C[0,1]$.
2. Therefore, there exists a sequence of polynomials $P_k(x)$ such that $P_k \to f$ uniformly.
3. By linearity of the integral and the hypothesis, $\int_0^1 P_k(x)f(x)\,dx = 0$ for any polynomial P_k.
4. Taking the limit as $k \to \infty$:

$$\int_0^1 (f(x))^2\,dx = \lim_{k\to\infty} \int_0^1 P_k(x)f(x)\,dx = 0.$$

5. Since $f^2 \geq 0$ and is continuous, $\int f^2 = 0 \implies f \equiv 0$.

$\square$

Problem 12 & 13: Separability

Problem 12

Prove that $C[0,1]$ is a **separable** *metric space (has a countable dense subset).*

Hint: Use polynomials with rational coefficients $\mathbb{Q}[x]$.

Problem 13

Prove that a compact metric space is separable.

Solution to Problem 12: Separability of $C[0,1]$

Proof.

We show that the set of polynomials with rational coefficients, denoted $\mathbb{Q}[x]$, is a countable dense subset of $C[0,1]$.

1. **Countability:** $\mathbb{Q}[x] = \bigcup_{n=0}^{\infty} \{a_n x^n + \cdots + a_0 \mid a_i \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, each set of degree n polynomials is countable. A countable union of countable sets is countable.
2. **Density:** Let $f \in C[0,1]$ and $\varepsilon > 0$. By the **Weierstrass Approximation Theorem**, there exists a polynomial $P(x)$ (with real coefficients) such that $\|f - P\|_\infty < \varepsilon/2$.
3. Let $P(x) = \sum_{k=0}^{n} c_k x^k$. For each c_k, choose $q_k \in \mathbb{Q}$ such that $|c_k - q_k| < \frac{\varepsilon}{2(n+1)}$. Let $Q(x) = \sum q_k x^k$.
4. Then $\|P - Q\|_\infty \le \sum |c_k - q_k| < \varepsilon/2$.
5. By Triangle Inequality: $\|f - Q\|_\infty \le \|f - P\| + \|P - Q\| < \varepsilon$.

$\square$

Solution to Problem 13: Separability of Compact Spaces

Proof.

Let (X, d) be a compact metric space.

1. Since X is compact, it is totally bounded.
2. For each $n \in \mathbb{N}$, there exists a finite set of points $A_n = \{x_{n,1}, \ldots, x_{n,k_n}\}$ such that the balls of radius $1/n$ centered at A_n cover X.
3. Define $D = \bigcup_{n=1}^{\infty} A_n$. As a countable union of finite sets, D is countable.
4. **Density:** Let $x \in X$ and $\varepsilon > 0$. Choose n such that $1/n < \varepsilon$.
5. Since A_n centers cover X, there exists $y \in A_n \subseteq D$ such that $d(x, y) < 1/n < \varepsilon$.
6. Thus D is dense in X.

$\square$

Arzela-Ascoli Theorem

Overview: Arzela-Ascoli Theorem

Theorem: Let X be a compact metric space. A subset $K \subseteq C(X, \mathbb{R})$ is compact in the uniform topology if and only if K is:

1. **Closed**,
2. **Bounded** (pointwise bounded, which implies uniformly bounded on compact X), and
3. **Equicontinuous**.

Interpretation: This theorem characterizes the compact sets in the infinite-dimensional space $C(X, \mathbb{R})$. It is the function space analog of the Heine-Borel theorem, adding the condition of "equicontinuity" to replace the loss of finite dimensionality.

Problem 14: Equicontinuity Definitions

Problem 14

Let $\mathcal{F}$ be a family of functions on a compact metric space (X, d). Show that the following definitions are equivalent:

1. **Pointwise Equicontinuity:** $\forall x \in X, \forall \varepsilon > 0, \exists \delta(x, \varepsilon) > 0$ *such that* $d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon, \forall f \in \mathcal{F}$.
2. **Uniform Equicontinuity:** $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ *such that* $d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon, \forall f \in \mathcal{F}, \forall x, y \in X$.

Solution to Problem 14: Equicontinuity

Proof.

$(2 \implies 1)$ is trivial. We prove $(1 \implies 2)$ using compactness.

1. Let $\varepsilon > 0$. By (1), for each $x \in X$, there exists $\delta_x > 0$ such that $f(B(x, \delta_x)) \subseteq B(f(x), \varepsilon/2)$ for all $f \in \mathcal{F}$.
2. The balls $\{B(x, \delta_x/2)\}_{x \in X}$ form an open cover of X.
3. By compactness, there is a finite subcover centered at $x_1, \ldots, x_k$.
4. Let $\delta = \min\{\delta_{x_1}/2, \ldots, \delta_{x_k}/2\} > 0$.
5. Let $y, z \in X$ with $d(y, z) < \delta$. Then $y \in B(x_i, \delta_{x_i}/2)$ for some i.
6. By triangle inequality: $d(z, x_i) \leq d(z, y) + d(y, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}$.
7. So both $y, z \in B(x_i, \delta_{x_i})$. For any $f \in \mathcal{F}$:

$$|f(y) - f(z)| \leq |f(y) - f(x_i)| + |f(x_i) - f(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

Problem 15: Necessity of Compactness

Problem 15

Show that the assumption that X is compact is necessary for the Arzela-Ascoli theorem. Specifically, find a sequence of functions f_n on a non-compact space X that is uniformly bounded and equicontinuous, but admits no uniformly convergent subsequence.

Solution to Problem 15

Let $X = \mathbb{R}$ (which is not compact). Consider the "sliding bump" functions:

$$f_n(x) = \max(0, 1 - |x - n|)$$

- **Bounded:** $|f_n(x)| \leq 1$ for all x, n.
- **Equicontinuous:** Since $|f_n'(x)| \leq 1$ wherever defined, they are all Lipschitz continuous with constant 1. Thus, they are equicontinuous.
- **No Convergent Subsequence:** For any $n \neq m$, $\|f_n - f_m\|_\infty = 1$ (since the supports $[n-1, n+1]$ and $[m-1, m+1]$ are disjoint for large enough difference). Since the distance between any distinct terms is 1, no subsequence is Cauchy, so no subsequence converges uniformly.

Problem 16: Dini's Theorem

Problem 16

Let f_n be a monotone sequence of continuous functions on a compact metric space S. Suppose that f_n converges pointwise to a continuous function f on S. Prove or disprove that the convergence is uniform.

This is known as Dini's Theorem.

Solution to Problem 16: Dini's Theorem

Proof.

Prove: The convergence is uniform. Let $g_n = |f_n - f|$. Since f_n is monotone and converges to f, g_n is a monotonic sequence decreasing to 0 pointwise (assume $f_n \downarrow f$; if $f_n \uparrow f$, take $f - f_n$). Since f_n, f are continuous, g_n is continuous.

Let $\varepsilon > 0$. For each $x \in S$, $g_n(x) \downarrow 0$, so $\exists N_x$ s.t. $g_{N_x}(x) < \varepsilon$. By continuity, there is an open neighborhood U_x such that $g_{N_x}(y) < \varepsilon$ for all $y \in U_x$.

$\{U_x\}_{x \in S}$ covers S. By compactness, there is a finite subcover $U_{x_1}, \dots, U_{x_k}$. Let $N = \max(N_{x_1}, \dots, N_{x_k})$. For any $y \in S$, $y \in U_{x_i}$ for some i. Since g_n is decreasing, for all $n \geq N \geq N_{x_i}$:

$$0 \leq g_n(y) \leq g_{N_{x_i}}(y) < \varepsilon.$$

Thus convergence is uniform. $\square$

Problem 17: Diagonal Argument Lemma

Problem 17

Let X be a metric space with a countable dense subset $S = \{x_1, x_2, \dots\}$. Let $\{f_n\}$ be a sequence of functions in $C(X, \mathbb{R})$ that is uniformly bounded. Show that there exists a subsequence $\{f_{n_k}\}$ that converges pointwise on S.

This is the first step in the proof of the Arzela-Ascoli Theorem.

Solution to Problem 17: Diagonal Argument

Proof.

1. Consider the sequence evaluated at x_1: $\{f_n(x_1)\}$. Since $\{f_n\}$ is uniformly bounded, this scalar sequence is bounded. By Bolzano-Weierstrass, there is a subsequence $\{f_{1,k}\}$ converging at x_1.
2. From $\{f_{1,k}\}$, extract a subsequence $\{f_{2,k}\}$ converging at x_2. Note it still converges at x_1.
3. Inductively, construct subsequence $\{f_{m,k}\}$ converging at $x_1, \ldots, x_m$.
4. **Diagonal Sequence:** Let $g_k = f_{k,k}$.
5. For any fixed $x_m \in S$, the sequence $\{g_k\}$ eventually becomes a subsequence of $\{f_{m,k}\}$ (for $k \geq m$), so it converges at x_m.
6. Thus, $\{g_k\}$ converges pointwise on S.

□

Baire Category Theorem

Overview: Baire Category Theorem

Theorem (Baire Category Theorem): Let (X, d) be a complete metric space. If $\{U_n\}_{n=1}^{\infty}$ is a countable collection of open dense subsets of X, then their intersection $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

Corollary: A complete metric space X is not a countable union of nowhere dense sets.
(i.e., X is of **Second Category**).

Connection: If $X = \bigcup A_n$ with A_n nowhere dense, then $\bar{A}_n$ has empty interior, so $U_n = (\bar{A}_n)^c$ is open and dense. BCT implies $\bigcap U_n \neq \emptyset$, so $X \neq \bigcup \bar{A}_n \supseteq \bigcup A_n$.

Problem 18: Removing Lines from $\mathbb{R}^2$

Problem 18

Let $\{l_i\}_{i=1}^{\infty}$ *be a countable collection of straight lines in* $\mathbb{R}^2$. *Show that* $\mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} l_i$ *is dense in* $\mathbb{R}^2$.

We will provide a direct proof using nested balls (simulating the proof of BCT) as suggested in the notes.

Solution to Problem 18 (Nested Balls)

Proof.

Let B_0 be an arbitrary open ball in $\mathbb{R}^2$. We want to show $B_0 \cap (\mathbb{R}^2 \setminus \bigcup l_i) \neq \emptyset$.
Step 1: Since l_1 has empty interior, $B_0 \not\subseteq l_1$. We can find a closed ball $\overline{B}_1 \subset B_0$ such that $\overline{B}_1 \cap l_1 = \emptyset$. We can ensure radius $r_1 \leq r_0/2$.
Step 2: Similarly, inside B_1, pick $\overline{B}_2$ such that $\overline{B}_2 \cap l_2 = \emptyset$ and $r_2 \leq r_1/2$.
Induction: We obtain a sequence of nested closed balls $\overline{B}_1 \supset \overline{B}_2 \supset \dots$ such that $\overline{B}_n \cap l_n = \emptyset$ and $r_n \to 0$.
Conclusion: By completeness of $\mathbb{R}^2$, $\bigcap_{n=1}^{\infty} \overline{B}_n = \{x\}$. $x \in B_0$, and for all n, $x \notin l_n$. Thus $x \in B_0 \setminus \bigcup l_i$. □

Problem 19: Generic Nowhere Differentiability

Problem 19

Let $C[0,1]$ be the complete metric space of continuous functions equipped with the sup-norm. Show that the set of functions which are nowhere differentiable is of the **Second Category** *(i.e., "most" continuous functions are nowhere differentiable).*

This is a classic application of BCT, credited to Banach and Mazurkiewicz.

Solution to Problem 19 (1/4): Setup

Goal: Show that the set of functions differentiable at even one point is of the **First Category**.
Define the set E_n for each $n \in \mathbb{N}$:

$$E_n = \{f \in C[0,1] \mid \exists x_0 \in [0,1] \text{ s.t. } \forall y \in [0,1], |f(y) - f(x_0)| \le n|y - x_0|\}$$

Connection to Differentiability: If f is differentiable at x_0, then $f'(x_0)$ exists, so the difference quotient is bounded near x_0. Since f is bounded on $[0,1]$, the difference quotient is bounded globally by some integer n. Thus, $\{f \in C[0,1] \mid f \text{ is diff. at some point}\} \subseteq \bigcup_{n=1}^{\infty} E_n$.
We must show each E_n is **nowhere dense**.

Solution to Problem 19 (2/4): Closedness

Step 1: Show E_n is closed. Let $\{f_k\} \subset E_n$ be a sequence converging uniformly to f.

- For each k, there exists $x_k \in [0, 1]$ such that $|f_k(y) - f_k(x_k)| \leq n|y - x_k|$ for all y.
- By Bolzano-Weierstrass, $\{x_k\}$ has a subsequence converging to some $x \in [0, 1]$. Assume w.l.o.g. $x_k \to x$.
- Fix y. By uniform convergence $f_k \to f$ and continuity:

$$|f(y) - f(x)| = \lim_{k\to\infty} |f_k(y) - f_k(x_k)| \leq \lim_{k\to\infty} n|y - x_k| = n|y - x|.$$

- Thus $f \in E_n$. So E_n is closed.

Solution to Problem 19 (3/4): Nowhere Dense

Step 2: Show E_n has empty interior. It suffices to show that for any $f \in C[0,1]$ and $\varepsilon > 0$, there exists $g \in B(f, \varepsilon)$ such that $g \notin E_n$.

- Approximate f by a piecewise linear function p (polygonal path) such that $\|f - p\| < \varepsilon/2$.
- Let M be the maximum slope of p.
- Construct a "sawtooth" function $\phi(x)$ that oscillates very rapidly with slope $K > n + M$ and amplitude bounded by $\varepsilon/2$.
- Define $g(x) = p(x) + \phi(x)$. Then $\|g - f\| \leq \|g - p\| + \|p - f\| < \varepsilon$.
- At any point x, the slope of g is roughly slope(p) + slope(ϕ). Since slope(ϕ) dominates, $|g(y) - g(x)|/|y - x|$ will exceed n locally.
- Thus $g \notin E_n$. E_n contains no open ball.

Solution to Problem 19 (4/4): Conclusion

Final Argument.

1. We showed each E_n is closed and has empty interior, i.e., E_n is **nowhere dense**.
2. The set of functions differentiable at at least one point is contained in the countable union $\bigcup_{n=1}^{\infty} E_n$.
3. Therefore, the set of differentiable functions is of the **First Category** (meager).
4. By the Baire Category Theorem, $C[0,1]$ is of Second Category.
5. The complement—functions that are **nowhere differentiable**—must be of the **Second Category** and is therefore dense in $C[0,1]$.

□

Problem 20: Generic Nowhere Monotonicity

Problem 20

Let $C[0,1]$ be the space of continuous functions on $[0,1]$. Show that the set of functions which are nowhere monotone (i.e., not monotone on any sub-interval) is of the **Second Category** *(generic).*

This result implies that "most" continuous functions wiggle infinitely often at all scales.

Solution to Problem 20 (1/2): Setup & Closedness

Let $M \subset C[0,1]$ be the set of functions monotone on some open sub-interval. We show M is of the First Category.
Let $\{I_n\}_{n=1}^{\infty}$ enumerate all open intervals in $[0,1]$ with rational endpoints. Let M_n^+ be the set of functions non-decreasing on I_n, and M_n^- be non-increasing on I_n. Then $M = \bigcup_{n=1}^{\infty}(M_n^+ \cup M_n^-)$.
Step 1: Closedness. If sequence $\{f_k\} \subset M_n^+$ converges uniformly to f, then monotonicity is preserved. For any $x, y \in I_n$ with $x < y$: $f(y) - f(x) = \lim(f_k(y) - f_k(x)) \geq 0$. Thus $f \in M_n^+$.
Similarly, M_n^- is closed.

Solution to Problem 20 (2/2): Nowhere Dense & Conclusion

Step 2: Nowhere Dense. We show M_n^+ has empty interior. Let $f \in M_n^+$ and $\varepsilon > 0$.

- Add a high-frequency "zig-zag" function $\phi(x)$ (small amplitude $< \varepsilon$) to f.
- The zig-zags will break the monotonicity of f within I_n.
- Thus, every ball around f contains a function not in M_n^+.

Since M_n^+ is closed and has empty interior, it is nowhere dense (same for M_n^-).
Conclusion: M is a countable union of nowhere dense sets $\implies$ First Category. The complement (nowhere monotone functions) is Second Category.

Problem 21: Continuity on Dense Subsets

Problem 21

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on a dense subset $D \subseteq \mathbb{R}$. Show that the set of all discontinuity points of f must be of the **First Category***.*

Note: As a consequence of BCT, we cannot have a function continuous precisely on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$, because $\mathbb{R} \setminus \mathbb{Q}$ is of Second Category.

Solution to Problem 21

Proof.

Let $C(f)$ be the set of continuity points and $D(f)$ be the set of discontinuity points. Recall that $D(f)$ is always an F_σ set (countable union of closed sets).

$$D(f) = \bigcup_{n=1}^{\infty} \{x \mid \omega_f(x) \geq 1/n\}$$

where $\omega_f(x)$ is the oscillation of f at x. The sets $A_n = \{x \mid \omega_f(x) \geq 1/n\}$ are closed.
We are given that $D \subseteq C(f)$ is dense in $\mathbb{R}$. Since $C(f)$ is dense, $D(f)$ has empty interior (if it contained an interval, that interval would contain points of D, a contradiction). Thus, each closed set $A_n \subseteq D(f)$ has empty interior, making A_n nowhere dense. Therefore, $D(f) = \bigcup A_n$ is a countable union of nowhere dense sets (First Category). $\square$

Problem 22: Pointwise Limits (Baire Class 1)

Problem 22

Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions, and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x. Show that the set of discontinuity points of f is of the **First Category***.*

This means f is continuous on a dense, Second Category set (almost continuous).

Solution to Problem 22

Proof.

This is the **Baire Classification Theorem**. A pointwise limit of continuous functions is a Baire Class 1 function. For such functions, the set of points of discontinuity $D(f)$ is always of the First Category.
Sketch: Define the oscillation $\omega_f(x)$. We check sets $F_k = \{x \mid \omega_f(x) \geq 1/k\}$. We can express these sets using the continuity of f_n. It can be shown that $D(f)$ is an F_σ set of first category. Specifically, Baire proved that for Baire-1 functions, the set of continuity points $C(f)$ is a dense G_δ set. Since $\mathbb{R}$ is complete, a dense G_δ set is Second Category (comeager), so $D(f)$ is First Category (meager). □

Multivariable Differentiation

Overview: Differentiation

Topics:

- **Taylor Series ($n = 2$):** Using differential operators:

$$f(x+h, y+k) \approx \sum_{j=0}^{2} \frac{1}{j!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^j f(x,y)$$

- **Implicit Function Theorem (IFT):** Conditions under which non-linear equations $F(\mathbf{x}, \mathbf{y}) = 0$ can be locally solved for $\mathbf{y}$ as a function of $\mathbf{x}$. Crucially depends on the invertibility of the derivative with respect to $\mathbf{y}$.
- **Inverse Function Theorem:** Regularity of the Jacobian determinant implies local invertibility.
- **Bump Functions:** Smooth functions with compact support, used in partitions of unity.

Problem 23: Taylor Series

Problem 23

Compute the second-order Taylor expansion of $f(x, y) = e^x \cos y$ *at the point* $(0, 0)$*. Use the operator notation* $h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$ *in your solution.*

Solution to Problem 23

Let $\mathbf{h} = (h, k) = (x, y)$ since we expand at $(0, 0)$. The operator is $D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

$$f(x, y) \approx \sum_{j=0}^{2} \frac{1}{j!} D^j f(0, 0) = f(0, 0) + Df(0, 0) + \frac{1}{2} D^2 f(0, 0)$$

- $j = 0$: $f(0, 0) = e^0 \cos 0 = 1$.
- $j = 1$: $Df = x(e^x \cos y) + y(-e^x \sin y)$. At $(0, 0)$, $Df = x(1) + y(0) = x$.
- $j = 2$: $D^2 f = (x\partial_x + y\partial_y)(xe^x \cos y - ye^x \sin y)$

$$= x(xe^x \cos y - ye^x \sin y) + y(-xe^x \sin y - ye^x \cos y)$$

At $(0, 0)$, $D^2 f = x(x) + y(-y) = x^2 - y^2$.

$$f(x, y) \approx 1 + x + \frac{1}{2}(x^2 - y^2)$$

Problem 24: IFT in $\mathbb{R}^2 \times \mathbb{R}^2$

Problem 24

Consider a C^1 mapping $F : \mathbb{R}^4 \to \mathbb{R}^2$, where we write points as (x, y, u, v) and $F(x, y, u, v) = 0$.

1. *State the Implicit Function Theorem conditions for solving for u, v in terms of x, y locally around a point P_0.*
2. *State the formula for the Jacobian matrix of the implicit function $G(x, y) = (u, v)$.*
3. *Prove this derivative formula in detail.*

Solution to Problem 24 (1/2): Statement

Theorem: Let $F = (F_1, F_2)$ be a C^1 mapping near $P_0 = (x_0, y_0, u_0, v_0)$ such that $F(P_0) = 0$. If the 2×2 matrix of partial derivatives with respect to the dependent variables u, v is invertible at P_0:

$$\det \frac{\partial(F_1, F_2)}{\partial(u, v)} = \det \begin{pmatrix} \partial_u F_1 & \partial_v F_1 \\ \partial_u F_2 & \partial_v F_2 \end{pmatrix} \neq 0,$$

then there exists a neighborhood U of (x_0, y_0) and a unique C^1 function $G : U \to \mathbb{R}^2$, $G(x, y) = (u(x, y), v(x, y))$, such that $F(x, y, u(x, y), v(x, y)) = 0$ for all $(x, y) \in U$.

Solution to Problem 24 (2/2): Derivative Formula Proof

Proof: We differentiate the identity $F(x, y, u(x, y), v(x, y)) = 0$ with respect to the independent variables. By the Chain Rule, for any variable $\xi \in \{x, y\}$:

$$\frac{\partial F}{\partial \xi} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial \xi} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial \xi} = 0$$

Writing this in matrix form for the Jacobian of the implicit function $DG = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$:

$$[D_{(x,y)}F] + [D_{(u,v)}F] \cdot [DG] = 0$$

Since $[D_{(u,v)}F]$ is invertible by hypothesis:

$$DG = -[D_{(u,v)}F]^{-1}[D_{(x,y)}F]$$

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_{1u} & F_{1v} \\ F_{2u} & F_{2v} \end{pmatrix}^{-1} \begin{pmatrix} F_{1x} & F_{1y} \\ F_{2x} & F_{2y} \end{pmatrix}$$

Problem 25: IFT Calculation

Problem 25

Consider the system:

$$\begin{cases} xu + yv + uv = 1 \\ xu^3 + yv^3 = 1 \end{cases}$$

1. *Verify that $P = (x, y, u, v) = (1, 1, 1, 0)$ is a solution.*
2. *Can we solve for u, v in terms of x, y near this point?*
3. *Compute the derivative matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ at $(x, y) = (1, 1)$.*

This applies the IFT conditions and formula from Problem 20.

Solution to Problem 25: Calculation

Let $F_1 = xu + yv + uv - 1$ and $F_2 = xu^3 + yv^3 - 1$.

- **1. Verify Solution:** $1(1) + 1(0) + 1(0) = 1$ and $1(1)^3 + 1(0)^3 = 1$. ✓
- **2. Check IFT Condition:** Calculate $J_{u,v} = \frac{\partial(F_1,F_2)}{\partial(u,v)}$ at $P(1,1,1,0)$.

$$J_{u,v} = \begin{pmatrix} x+v & y+u \\ 3xu^2 & 3yv^2 \end{pmatrix}_P = \begin{pmatrix} 1+0 & 1+1 \\ 3(1)(1)^2 & 3(1)(0)^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$\det(J_{u,v}) = -6 \neq 0$. Thus, implicit functions $u(x,y), v(x,y)$ **exist**.

- **3. Compute Derivatives:** $J_{x,y} = \begin{pmatrix} u & v \\ u^3 & v^3 \end{pmatrix}_P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = -\frac{1}{-6}\begin{pmatrix} 0 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -2 & 0 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -1/3 & 0 \\ -1/3 & 0 \end{pmatrix}$$

Further Learning: Alternative Proof of IFT

Alternative Proof: Fixed Point Method

For those interested in a proof of the **Implicit/Inverse Function Theorem** using the **Contraction Mapping Principle** (Fixed Point Theorem), please check **Liu Siqi's video series** on Mathematical Analysis.

Multivariable Integration

Overview: Multivariable Integration

Key Concepts:

1. **Riemann Integrability:** Covered in the course, but note that the modern standard is the **Lebesgue Integration** theory (MAT3010).
2. **Green's Formula & Divergence Theorem:** Tools to convert between domain integrals and boundary integrals.
   - These are special cases of the generalized **Stokes' Theorem** on manifolds.
   - **Recommendation:** *Differential Forms and Applications* by Do Carmo.
3. **Leibniz Rule (Feynman's Trick):** Differentiation under the integral sign:

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

Valid for both proper and convergent improper integrals (under specific uniform convergence conditions).

Problem 26: Feynman's Trick Practice

Problem 26

Find the following integrals:

(a) $$\int_0^1 \frac{x^b - x^a}{\ln x}\,dx, \quad b > a > 0.$$

(b) $$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx.$$

Solution to Problem 26 (1/2): Part (a)

Technique: Parameter Integration Observe that $\frac{x^b - x^a}{\ln x} = \int_a^b x^y \, dy$.

$$I = \int_0^1 \left(\int_a^b x^y \, dy \right) dx$$

By Fubini's Theorem (justification required for improper integral, but valid here since integrand is positive and measurable):

$$I = \int_a^b \left(\int_0^1 x^y \, dx \right) dy$$

$$I = \int_a^b \left[\frac{x^{y+1}}{y+1} \right]_0^1 dy = \int_a^b \frac{1}{y+1} \, dy$$

$$I = \left[\ln(y+1) \right]_a^b = \ln(b+1) - \ln(a+1) = \ln \left(\frac{b+1}{a+1} \right)$$

Solution to Problem 26 (2/2): Part (b)

Technique: Variable Substitution Let $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$. Limits: $0 \to 0, 1 \to \pi/4$.

$$I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

Use the symmetry property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$$I = \int_0^{\pi/4} \ln\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta$$

Using $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$:

$$1 + \tan\left(\frac{\pi}{4} - \theta\right) = 1 + \frac{1-\tan\theta}{1+\tan\theta} = \frac{2}{1+\tan\theta}$$

$$I = \int_0^{\pi/4} (\ln 2 - \ln(1+\tan\theta))\, d\theta = \frac{\pi}{4}\ln 2 - I \implies I = \frac{\pi}{8}\ln 2$$

Problem 27: Improper Integrals with Parameters

Problem 27

Compute:

(a) $J = \displaystyle\int_0^\infty e^{-px} \frac{\sin bx - \sin ax}{x}\, dx, \quad p > 0, b > a.$

(b) $\displaystyle\int_0^\infty \frac{\sin ax}{x}\, dx, \quad a \neq 0.$

Solution to Problem 27 (1/2): Part (a)

Let $I(y) = \int_0^\infty e^{-px} \frac{\sin yx}{x}\, dx$. Then $J = I(b) - I(a)$. Differentiate w.r.t. parameter y:

$$I'(y) = \int_0^\infty e^{-px} \frac{\partial}{\partial y}\left(\frac{\sin yx}{x}\right) dx = \int_0^\infty e^{-px} \cos(yx)\, dx$$

This is a standard Laplace transform or integration by parts:

$$I'(y) = \frac{p}{p^2 + y^2}$$

Integrate w.r.t. y:

$$I(y) = \int \frac{p}{p^2 + y^2} dy = \arctan\left(\frac{y}{p}\right) + C$$

Since $I(0) = 0 \implies C = 0$, we have $I(y) = \arctan(y/p)$.

$$J = \arctan\left(\frac{b}{p}\right) - \arctan\left(\frac{a}{p}\right)$$

Solution to Problem 27 (2/2): Part (b)

Dirichlet Integral: We can view this as the limit of part (a) as $p \to 0^+$ (with $b = a$ and the second term 0, effectively). Alternatively, define $K(p) = \int_0^\infty e^{-px} \frac{\sin ax}{x} dx$ for $p \geq 0$.

$$K'(p) = -\int_0^\infty e^{-px} \sin ax \, dx = -\frac{a}{p^2 + a^2}$$

$$K(p) = -\arctan\left(\frac{p}{a}\right) + C$$

As $p \to \infty$, $K(p) \to 0$ (Riemann-Lebesgue type decay), so $0 = -\frac{\pi}{2} + C \implies C = \frac{\pi}{2}$ (assuming $a > 0$).
Thus $K(0) = \frac{\pi}{2}$.
If $a < 0$, the sign flips. Thus:

$$\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \operatorname{sgn}(a)$$

Good luck with the Final!