Final Review: Real Analysis

Borel-Cantelli, Differentiation, L^p Spaces, and Modes of Convergence

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Outline

Topic 1: Overview

Key Concept: The Borel-Cantelli Lemma

It is the primary tool for translating conditions on integrals or measures into pointwise almost everywhere (a.e.) convergence results.

"The message is that the 'Borel-Cantelli' way of arguing is important!"

Problem 1: Rapid Convergence

Problem

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions on a measurable set E. Suppose that for every k:

$$\int_E |f_{k+1} - f_k| < \delta_k^2$$

where $\sum_{k=1}^{\infty} \delta_k < \infty$.
Prove: The sequence $\{f_k\}$ converges pointwise almost everywhere to some function f.

Solution to Problem 1

Strategy

To show f_k converges a.e., it suffices to show that $\sum |f_{k+1} - f_k| < \infty$ a.e.

- Let $A_k = \{x \in E \mid |f_{k+1}(x) - f_k(x)| > \delta_k\}$.
- By Chebyshev's Inequality:

$$m(A_k) \leq \frac{1}{\delta_k} \int_E |f_{k+1} - f_k| < \delta_k$$

- Since $\sum \delta_k < \infty$, by the **Borel-Cantelli Lemma**, almost every x belongs to only finitely many A_k.

Note: This is the key step in proving the Riesz-Fischer Theorem.

Problem 2: Beppo-Levi Argument

Problem

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions on a measurable set E. Suppose that for every k:

$$\int_E |f_{k+1} - f_k| < \delta_k$$

where $\sum_{k=1}^{\infty} \delta_k < \infty$.
Prove: The sequence $\{f_k\}$ converges pointwise almost everywhere to some function f.

Remark on Problem 2

Comparison: The proof from Problem 1 (using Chebyshev directly) fails here because the power of convergence is different.

Alternative Argument (Beppo-Levi)

Consider the function $g(x) = \sum_{k=1}^{\infty} |f_{k+1}(x) - f_k(x)|$.

- By the Monotone Convergence Theorem (MCT):

$$\int_E g = \sum \int_E |f_{k+1} - f_k| < \sum \delta_k < \infty$$

- Thus $g(x)$ is finite a.e., implying the series converges absolutely a.e.
- This implies $f_k \to f$ a.e.

Problem 3

Problem

Let $\{f_n\}_{n=1}^{\infty}$ be integrable on $\mathbb{R}$ such that:

$$\int_{\mathbb{R}} |f_n(x) - f(x)|\, dx \leq \frac{1}{n^{1+\epsilon}}$$

(a) If $\epsilon > 0$, prove that $f_n \to f$ pointwise almost everywhere.

(b) If $\epsilon = 0$, provide a counterexample.

Solution to Problem 3

Part (a) $\epsilon > 0$: Since $\sum \frac{1}{n^{1+\epsilon}} < \infty$, we can sum the integrals. By the Beppo-Levi argument (Problem 2), $\sum |f_n - f| < \infty$ a.e., implying convergence.

Part (b) $\epsilon = 0$:

Counter-Example: The Typewriter Sequence

Consider moving characteristic functions:

$$\chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/4]}, \chi_{[1/4,1/2]}, \cdots$$

The integral decays like $1/n$ (harmonic series diverges). The "bump" cycles through the interval infinitely often, so it diverges everywhere.

Problem 4: Cauchy in Measure

Definition & Problem

A sequence $\{f_n\}$ is **Cauchy in Measure** if $\forall \eta, \epsilon > 0$, $\exists N$ s.t. for $n, m \geq N$:

$$m(\{x \in E \mid |f_n(x) - f_m(x)| \geq \eta\}) < \epsilon$$

Prove: If $\{f_n\}$ is Cauchy in measure, then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise almost everywhere.

Remark: This is a standard result. The key strategy is utilizing the **Borel-Cantelli** argument to extract the subsequence efficiently.

Problem 5: Diophantine Approximation

Problem

(a) Order 2 is Universal: Use Pigeonhole Principle to prove Dirichlet's Approximation Theorem: $\forall$ irrational x, $\exists\infty$ rationals p/q s.t. $|x - p/q| < 1/q^2$.
(b) Order > 2 is Rare: Prove that the set of $x \in [0,1]$ such that $|x - p/q| < 1/q^3$ for infinitely many rationals p/q has Lebesgue measure zero.

Solution to Problem 5(b)

Proof

It suffices to show by Borel-Cantelli that the sum of measures is finite.

$$\sum_{q=1}^{\infty} m\left(\bigcup_{p=0}^{q}\left\{x\in[0,1]:\left|x-\frac{p}{q}\right|<\frac{1}{q^3}\right\}\right)$$

The measure at each level q is roughly $q\cdot\frac{2}{q^3}=\frac{2}{q^2}$. Since $\sum\frac{1}{q^2}<\infty$, the set of such x has measure 0.

Key Question: Under what conditions does the FTC hold?

$$\int_a^b f' = f(b) - f(a)$$

Classes of Functions:

- **Monotone:** Differentiable a.e.
- **Bounded Variation (BV):** Diff. a.e., but FTC may fail.
- **Absolute Continuity (AC):** The necessary and sufficient condition for FTC.

Warning

The Cantor-Lebesgue function is BV, Continuous, and Monotone, but **not** AC.

Properties of AC Functions

Equivalence with Stronger Definition

The standard definition ($\sum |f(b_k) - f(a_k)| < \epsilon$) implies:

$$\sum_{k=1}^{n} TV(f|_{[a_k,b_k]}) < \epsilon \quad \text{whenever} \quad \sum (b_k - a_k) < \delta$$

Indefinite Integral Characterization

f is AC on $[a, b]$ iff $f(x) = f(a) + \int_a^x g(t)\,dt$ for some integrable g (where $g = f'$ a.e.).

Problem 6

Problem

Let $\alpha, \beta > 0$. Define $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} x^\alpha \sin(\frac{1}{x^\beta}) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

(i) Prove that if $\alpha > \beta$, then $f \in BV[0,1]$.

(ii) Prove that if $\alpha \leq \beta$, then $f \notin BV[0,1]$.

Solution to Problem 6

(i) Case $\alpha > \beta$: We calculate the derivative: $|f'(x)| \leq \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1}$. Since $\alpha > \beta$, the power is > -1. Thus $|f'|$ is integrable near 0, so $TV(f) = \int |f'| < \infty$.

(ii) Case $\alpha \leq \beta$:

Divergence

We construct a partition where points x_n satisfy $\sin(1/x^{\beta}) = \pm 1$. The sum of oscillations behaves like the harmonic series $\sum \frac{1}{n}$, which diverges.

Problem 7: Inclusions

Problem

Prove the strict inclusions: Lip $\subsetneq AC \subsetneq BV$.

Counter-Examples

- Lip $\subsetneq AC$: $f(x) = \sqrt{x}$ on $[0,1]$. It is AC (integral of $1/2\sqrt{x}$) but not Lipschitz (unbounded derivative at 0).
- $AC \subsetneq BV$: The **Cantor-Lebesgue function**. It is BV (monotone) but not AC (maps measure 0 set to measure 1 set).

Problem 8: Mapping Properties

Problem

Prove that if f is absolutely continuous, then f maps measurable sets to measurable sets.

Sketch of Proof:

1. **Condition (N):** Show f maps sets of measure 0 to sets of measure 0 (using the $\delta - \epsilon$ definition of AC).
2. Write measurable set $E = F \cup Z$ (F is F_σ, $m(Z) = 0$).
3. $f(F)$ is F_σ (hence Borel/Measurable) by continuity.
4. $f(Z)$ is measure 0 by step 1.

Topic 3: Overview

Density:

- Simple functions
- Step functions
- C_c (Continuous with compact support)

are dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Separability:

- L^p is separable for $1 \leq p < \infty$.
- L^∞ is NOT separable.
- $C[a, b]$ is separable (Stone-Weierstrass).

Problem 11: Weak Compactness

Theorem (Reflexivity)

Let $1 < p < \infty$. If $\{f_n\}$ is a bounded sequence in $L^p(E)$, then $\{f_n\}$ possesses a subsequence that converges in the **weak sense**.

Note: This fails for L^1 (not reflexive).

Problem 14: Weak vs Strong

Problem

Example of a sequence converging weakly to 0 but not strongly.

Example: Rademacher or Sine

$f_n(x) = \sin(nx)$ on $[0, 2\pi]$.

- **Weakly:** $\int f_n g \to 0$ by Riemann-Lebesgue Lemma.
- **Strongly:** $\|f_n\|_p$ is constant non-zero, so it cannot converge to 0 in norm.

Convergence Relationships

Does Property A imply a Subsequence has Property B? (Assume $m(E) < \infty$ and bounded in L^p)				
A $\downarrow$ \ B $\rightarrow$	**Strong (L^p)**	**Weak (L^p)**	**Ptwise a.e.**	**Measure**
Case 1: Reflexive Range ($1 < p < \infty$)				
Strong	YES	YES	YES	YES
Weak	NO	YES	NO	NO
Pointwise a.e.	NO	**YES**	YES	YES
Measure	NO	**YES**	YES	YES
Case 2: Non-Reflexive ($p = 1$)				
Pointwise a.e.	NO	**NO**	YES	YES

Problem 16: Radon-Riesz Property

Problem

Let $1 \le p < \infty$. Suppose $f_n \to f$ a.e. and $f_n, f \in L^p$. **Prove:** $f_n \to f$ in L^p (Strongly) iff $\|f_n\|_p \to \|f\|_p$.

Proof Hint

Apply **Fatou's Lemma** to the sequence:

$$g_n = 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p$$

Since $g_n \ge 0$, Fatou yields the result.

Standard Counter-Example Library

- **Cantor-Lebesgue Function**
  - Continuous, Monotone, $f' = 0$ a.e.
  - Not Absolutely Continuous.
- **Typewriter Sequence**
  - Converges in measure.
  - Diverges pointwise everywhere.
- **The Spikes ($n\chi_{[0,1/n]}$)**
  - Converges pointwise to 0.
  - Integral stays 1 (No L^1 convergence).
- **Rademacher Functions / $\sin(nx)$**
  - Weak convergence to 0.
  - No strong convergence.

Good Luck!