# Real Analysis

# midterm-ustc-2023-version1

October 13, 2025

**Problem 1.**

(20 points) Determine whether the following statements are true or false (provide a proof or a counterexample to explain your conclusion):

(a) Continuous functions are measurable.

(b) Integrable functions are almost everywhere finite.

**Problem 2.**

(15 points) State the **Fatou's Lemma** and give an example where the inequality in the lemma is strict.

**Problem 3.**

(20 points) Calculate:

(a)

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx$$

(b)

$$\lim_{n\to\infty}\int_{-1}^{1}\frac{1+nx^2}{(1+x^2)^n}\,dx$$

**Problem 4.**

(15 points) Let $f$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and assume that it preserves the outer measure of Borel sets. Prove: For any measurable set $E \subset \mathbb{R}^n$, $f(E)$ is measurable.

**Problem 5.**

(10 points) State and prove the **Borel-Cantelli Lemma**.

**Problem 6.**

(10 points) Let $f, f_n \in L^1$, $n = 1, 2, \ldots$ satisfying $f_n \to f$ a.e., and

$$\lim_{n\to\infty}\int_{\mathbb{R}^n}|f_n|\,dm = \int_{\mathbb{R}^n}|f|\,dm.$$

Prove: For any measurable set $E \subset \mathbb{R}^n$,

$$\lim_{n\to\infty}\int_{E}|f_n|\,dm = \int_{E}|f|\,dm.$$

**Problem 7.**

(10 points) Let $E \subset \mathbb{R}^n$ be measurable, $m(E) < +\infty$. Let $f$ be a non-negative measurable function on $E$. For $\epsilon > 0$, define

$$E_k(\epsilon) := \{x \in E : k\epsilon \le f(x) < (k+1)\epsilon\}, \quad k = 0, 1, 2, \ldots$$

and

$$A(\epsilon) := \sum_{k=0}^{\infty} k\epsilon m(E_k(\epsilon)).$$

Prove:

$$\int_E f\, dm = \lim_{\epsilon \to 0} A(\epsilon).$$