Riemann Surfaces

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Introduction

Riemann Surfaces is the one-dimensional complex manifold. Also, it can be described as the one-dimensional complex algeraic curves. I first encounter the concept of Riemann Surfaces in complex analysis.Later I found that Riemann Surfaces is not only an interesting object to learn itself. Since it cna be described as algebraic curves, it also provides a path to the study of algebraic geometry. I want to learn algebraic geometry and Riemann Surfaces is a good place to start. OUr goal in this note is to obtain Riemann-Roch theorem and its application.

1 Riemann Surfaces and complex manifolds.

1.1 holomorphic functions in 1-variable

1.2 holomorphic functions in n-variables

1.3 Complex manifolds & Riemann Surfaces.

Definition 1.1. Let X be a **topological** space.

1. A n-dim **complex** chart on X is a **homeomorphism**

$$\phi : U \xrightarrow{\cong} V \subset \mathbb{C}^n \text{ open}$$

2. Two such charts are compatible if $U_1 \cap U_2 = \emptyset$ or $\phi_2 \circ \phi_1^{-1}\big|\phi_1(U_1 \cap U_2)$ is holomorphic
3. A n-dim complex atlas $\mathcal{A}$ is a collection of pairwise compatible charts on X.
4. Two such atlases on X are equivalent if $\mathcal{A} \cup \mathcal{B}$ is an atlas.
5. A n-dim $\mathbb{C}$ **manifold** is a topological space (is Hausdorff & 2^{nd} countable) with an equivalence class of n-dim $\mathbb{C}$ atlases.
6. A Riemann surface is a 1-dim $\mathbb{C}$ **manifold**.

Exercise :

1. Equivalence of atlases is an equivalence relation.
2. $\exists$ unique maximal $\mathbb{C}$ atlas.

Remark 1.1. (i) Refining an atlas doesn't change the complex structure.

(ii) If $\phi : U \to V$ is a chart on Riemann Surface X.

$$\alpha : V \xrightarrow{\wedge} W$$

then $\alpha \circ \phi : U \to W$ is a chart compatible with ϕ.

(iii) An n-dimensional **manifold** is a $2n$-dimensional real smooth **manifold**.

1.4 Examples of Riemann Surfaces.

Example 1.1. The first example is a **Non-Examples:**

1. $X = \mathbb{R}^2 \times U \to V = \mathbb{C}$ for $i = 1, 2$.

$$\phi_1(x, y) = x + iy$$
$$\phi_2(x, y) = \frac{x + iy}{1 + idx^2y^2}$$

ϕ_1 & ϕ_2 are not compatible. $\phi_2 \circ \phi_1^{-1}(z) = \frac{z}{1+|z|^2}$ not holomorphic.

2. The complex plane $\mathbb{C}$. $X = \mathbb{R}^2$. with $\phi_1 : \mathbb{R}^2 \to \mathbb{C}$.

$$(x, y) \mapsto x + iy$$

is a Riemann Surface.

3. The Riemann Sphere. $\mathbb{CP}^1$: $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. (the stereographic projection with some modifications)

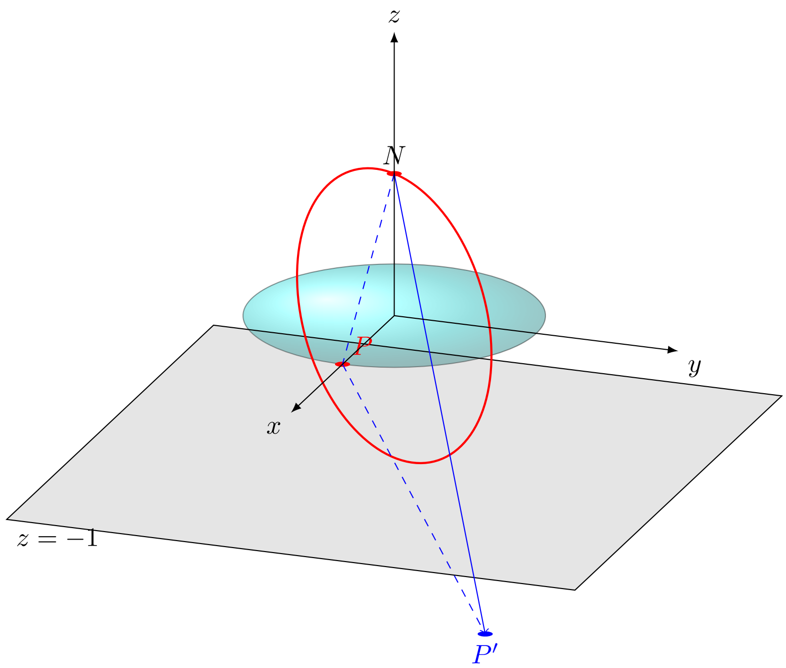


Figure 1: An illustration of the stereographic projection.

$S^2 \setminus \{(0,0,1)\}$

$$\begin{array}{c} \phi_0 : U_0 \longrightarrow V_0 = \mathbb{C} \\ \| \\ S^2 \setminus \{(0,0,1)\} \end{array}$$

$$(x,y,w) \longmapsto \frac{x+iy}{1-w} \quad \Rightarrow \quad \text{this is the stereographic projection}$$

$$\begin{array}{c} \phi_\infty : U_\infty \xrightarrow{\cong} V_\infty = \mathbb{C} \\ \| \\ S^2 \setminus \{(0,0,-1)\} \end{array}$$

$$(x,y,w) \longmapsto \frac{x-iy}{1+w}$$

Exercise : check these are charts.
ϕ_0 & ϕ_∞ are compatible. On $U_0 \cap U_\infty = S^2 \setminus \{(0,0,1),(0,0,-1)\}$, $\phi_0(U_0 \cap U_\infty) \subset \mathbb{C}^* \subset V_0$.

$$\frac{1}{\phi_0(x,y,w)} = \frac{1-w}{x+iy} = \frac{(1-w)(x-iy)}{x^2+y^2} = \frac{(1-w)(x-iy)}{1-w^2} = \frac{x-iy}{1+w} = \phi_\infty(x,y,w)$$

Thus,

$$\phi_\infty \circ \phi_0^{-1}(z) = \frac{1}{z} \text{ on } \mathbb{C}^* = \phi_0(U_0 \cap U_\infty) \subset \mathbb{C} = V_0.$$

holomorphic

Hence, $\{\phi_0, \phi_\infty\}$ are an atlas, and the corresponding Riemann Surface is called the **Riemann Sphere**.

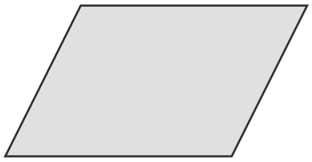
1. **Complex tori of dimension 1.** For $\omega_1, \omega_2 \in \mathbb{C}$ which are $\mathbb{R}$-linearly independent. consider the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\} \subset \mathbb{C}$. Let $X = \mathbb{C}/L$, with the quotient topology.

$$\pi : \mathbb{C} \longrightarrow X = \mathbb{C}/L.$$

$$z \longmapsto [z] = z + L.$$

Topologically X is a torus.

Every $z \in \mathbb{C}$ is equivalent to a unique point in the Fundamental domain.



Given $X = \mathbb{C}/L$, we construct an atlas using $\pi : \mathbb{C} \to \mathbb{C}/L$.
Pick $\varepsilon > 0$ s.t. $\forall p \in \mathbb{C}$, $B_\varepsilon(p)$ intersects each $[z]$ in at most one point.
Thus gives a homeomorphism

$$\pi : B_\varepsilon(p) \xrightarrow{\cong} \pi(B_\varepsilon(p))$$

$$\text{with } \phi_p = \pi\big|_{B_\varepsilon(p)} : U_p \subset X$$

where $U_p = \pi(B_\varepsilon(p))$.
Claim :

$$\mathcal{A} = \{\phi_p : U_p \to V_p\}_{p\in\mathbb{C}} \text{ is an atlas.}$$

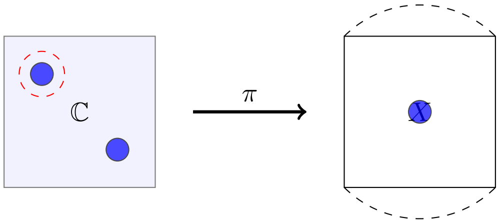
Compatibility of ϕ_p & ϕ_q: Assume $U_{p,q} = U_p \cap U_q \neq \emptyset$.
The transition map is:

$$T = \phi_q \circ \phi_p^{-1} : \phi_p(U_{p,q}) \longrightarrow \phi_q(U_{p,q})$$

T satisfies $\pi(T([z])) = \phi_p^{-1}([z]) = \pi([z])$ i.e., $T([z]) - z \in L = \ker(\pi)$, which is constant.

$$\Rightarrow T - \mathrm{id} \text{ is locally constant: locally } T - \mathrm{id} = w \in L.$$

$$T(z) = z + w \text{ is holomorphic}$$



1.5 Examples of complex manifolds.

Example 1.2 (Complex Projective Plane)**.**

$$\mathbb{CP}^n = \{\text{1-dimensional complex vector subspace in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*.$$

$$\cong S^{2n+1}/S^1 \quad \text{quotient topology.}$$

Give $\mathbb{CP}^n$ the quotient topology.

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$$

$$(z_0, \ldots, z_n) \longmapsto \pi(z_0, \ldots, z_n) = [z_0 : z_1 : \cdots : z_n].$$

homogeneous coordinates.
Atlases: Let

$$U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\} \subset \mathbb{CP}^n \quad \text{open}$$

The chart ϕ_i is given by:

$$\phi_i : U_i \longrightarrow V_i = \mathbb{C}^n$$

$$[z_0 : \cdots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$$

where $\widehat{\frac{z_i}{z_i}}$ denotes the omission of the i-th coordinate.

2 Morphisms of complex manifolds & meromorphic functions.

2.1 Morphisms of manifolds.

Definition 2.1. Let X & Y be complex **manifolds** of dimensions n & m respectively. Let $W \subset X$ & $W' \subset Y$ be open sets.

1. A continuous map $F : W \to W'$ is holomorphic at $p \in W$ if $\exists$ charts $\phi : U \to V$ & $\psi : W' \to V'$ s.t. $p \in U$ & $F(p) \in W'$, s.t. $\psi \circ F \circ \phi^{-1}$ is holo at $\phi(p)$.
2. Biholomorphism.

Example 2.1. (Examples of Morphisms)

1. A chart on a Riemann Surface $\phi : U \to V \subset \mathbb{C}$ on a Riemann Surface is a holomorphic function.
2. Let $U \subset X$ for a Riemann Surface X. Then U has a unique Riemann Surface structure s.t. the inclusion map $U \hookrightarrow X$ is holomorphic.
3. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \cong S^2 = U_0 \cup U_\infty$. Let f be a holomorphic function on $\mathbb{C}$. Let $f_0 := f \circ \phi_0^{-1} : \mathbb{C} \to \mathbb{C}$. Let $f_\infty := f \circ \phi_\infty^{-1} : \mathbb{C} \to \mathbb{C}$. On $\mathbb{C}^*$:
$$f_\infty(w) = f \circ \phi_\infty^{-1}(w) = f \circ \phi_0^{-1} \circ \phi_0 \circ \phi_\infty^{-1}(w) = f_0\left(\frac{1}{w}\right)$$
This f is holomorphic at $w \in \mathbb{C}_\infty$
$$\iff f\left(\frac{1}{z}\right) \text{ is holomorphic at } 0$$
$$\Updownarrow \text{ Def}$$
$$f_\infty(w) \text{ is holomorphic at } 0 \iff f_0\left(\frac{1}{w}\right) \text{ is holomorphic at } \infty$$
4. the quotient map $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ for a complex torus is holomorphic.

subsection*§ 2.2. Properties of holomorphic maps of Riemann Surfaces.

Theorem 2.1 (The identity theorem)**.** *Let $F, G : X \to Y$ be holomorphic maps of Riemann Surfaces s.t. F & G agree on a subset of X with an accumulation point. Then $F = G$.*

Theorem 2.2 (Local form of holomorphic maps)**.** *Let $F : X \to Y$ be a non-constant holomorphic map of Riemann Surfaces. For $p \in X$ & $q = F(p)$, $\exists$ unique $k \in \mathbb{Z}_{>0}$ and local charts $\phi : U \to V$ & $\psi : U' \to V'$ s.t. F has local form*
$$\psi \circ F \circ \phi^{-1} : V \to V'$$
$$z \mapsto z^k.$$

Proof. take any charts $\phi : U \to V \subset \mathbb{C}$ & $\psi : U' \to V' \subset \mathbb{C}$.
$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow \phi & & \downarrow \psi \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \end{array} \qquad \begin{array}{l} p \mapsto 0. \\ q \mapsto 0. \end{array}$$

Shrink V so $F(U) \subset U'$. Then $f = \psi \circ F \circ \phi^{-1} : V \to V'$ is holomorphic and $0 \mapsto 0$.

Define $k = \mathrm{ord}_0(f) \in \mathbb{Z}_{\geq 0}$. $\mathrm{ord}_0(f)$ is the order of vanishing of f at $0 = \min\{n \mid c_n \neq 0\}$ where $f(z) = \sum_{n \geq 0} c_n z^n$ is the Taylor expansion. $f(z) = z^k g(z)$, where $g(z)$ is non-zero.

Shrink V so $g : V \to \mathbb{C}$ is non-zero and holomorphic. Thus, $\exists$ holomorphic k^{th} root $h : V \to \mathbb{C}$ of g i.e., $(h(z))^k = g(z)$.

Thus $f(z) = z^k g(z) = (zh(z))^k$.

Let $\alpha : V \xrightarrow{\cong} \alpha(V)$ be biholomorphic

$$z \mapsto zh(z) = w.$$

Note $\alpha(0) = 0$, $\alpha'(0) = h(0) \neq 0$, then $\operatorname{ord}_0(\alpha) = 1$.

Now replace ϕ by $\alpha \circ \phi$.

$$\psi \circ F \circ (\alpha \circ \phi)^{-1}(w) = \psi \circ F \circ \phi^{-1} \circ \alpha^{-1}(w)$$

$$= f(\alpha^{-1}(w)) = (\alpha^{-1}(w)h(\alpha^{-1}(w)))^k = w^k.$$

Local form is $w \mapsto w^k$. $\square$

Exercise: Show k is independent of the choice of chart.

Definition 2.2. The **multiplicity** of a non-constant holomorphic map $F : X \to Y$ of Riemann Surfaces at $p \in X$ is the unique positive integer k given by Theorem 2.2.

We say p is an unramified point of F if $\operatorname{mult}_p(F) = 1$. p is a ramified point of F if $\operatorname{mult}_p(F) > 1$.

$$R(F) = \{p \in X : \operatorname{mult}_p(F) > 1\} \text{ ramification locus.}$$
$$B(F) = F(R(F)) \subset Y \text{ branch locus.}$$

Theorem 2.3 (Open mapping theorem)**.** *A non-constant holomorphic map of Riemann Surfaces is an open mapping.*

Proof: The local form $z \mapsto z^k$ is open ("maps circle to circle").

Theorem 2.4 (Biholomorphic maps are biholomorphisms)**.** *The inverse of a bijective holomorphic map of Riemann Surfaces is holomorphic (since F is an open mapping).*