Riemann Surfaces

Zihan Ke

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Introduction

Riemann Surfaces is the one-dimensional complex manifold. Also, it can be described as the one-dimensional complex algeraic curves. I first encounter the concept of Riemann Surfaces in complex analysis.Later I found that Riemann Surfaces is not only an interesting object to learn itself. Since it cna be described as algebraic curves, it also provides a path to the study of algebraic geometry. I want to learn algebraic geometry and Riemann Surfaces is a good place to start. OUr goal in this note is to obtain Riemann-Roch theorem and its application.

1 Riemann Surfaces and complex manifolds.

1.1 holomorphic functions in 1-variable

1.2 holomorphic functions in n-variables

1.3 Complex manifolds & Riemann Surfaces.

Definition 1.1. Let X be a **topological** space.

1. A n-dim **complex** chart on X is a **homeomorphism**

$$\phi : U \xrightarrow{\cong} V \subset \mathbb{C}^n \text{ open}$$

2. Two such charts are compatible if $U_1 \cap U_2 = \emptyset$ or $\phi_2 \circ \phi_1^{-1}\big|\phi_1(U_1 \cap U_2)$ is holomorphic
3. A n-dim complex atlas $\mathcal{A}$ is a collection of pairwise compatible charts on X.
4. Two such atlases on X are equivalent if $\mathcal{A} \cup \mathcal{B}$ is an atlas.
5. A n-dim $\mathbb{C}$ **manifold** is a topological space (is Hausdorff & 2^{nd} countable) with an equivalence class of n-dim $\mathbb{C}$ atlases.
6. A Riemann surface is a 1-dim $\mathbb{C}$ **manifold**.

Exercise :

1. Equivalence of atlases is an equivalence relation.
2. $\exists$ unique maximal $\mathbb{C}$ atlas.

Remark 1.1. (i) Refining an atlas doesn't change the complex structure.

(ii) If $\phi : U \to V$ is a chart on Riemann Surface X.

$$\alpha : V \xrightarrow{\wedge} W$$

then $\alpha \circ \phi : U \to W$ is a chart compatible with ϕ.

(iii) An n-dimensional **manifold** is a $2n$-dimensional real smooth **manifold**.

1.4 Examples of Riemann Surfaces.

Example 1.1. The first example is a **Non-Examples:**

1. $X = \mathbb{R}^2 \times U \to V = \mathbb{C}$ for $i = 1, 2$.

$$\phi_1(x, y) = x + iy$$
$$\phi_2(x, y) = \frac{x + iy}{1 + idx^2y^2}$$

ϕ_1 & ϕ_2 are not compatible. $\phi_2 \circ \phi_1^{-1}(z) = \frac{z}{1+|z|^2}$ not holomorphic.

2. The complex plane $\mathbb{C}$. $X = \mathbb{R}^2$. with $\phi_1 : \mathbb{R}^2 \to \mathbb{C}$.

$$(x, y) \mapsto x + iy$$

is a Riemann Surface.

3. The Riemann Sphere. $\mathbb{CP}^1$: $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. (the stereographic projection with some modifications)

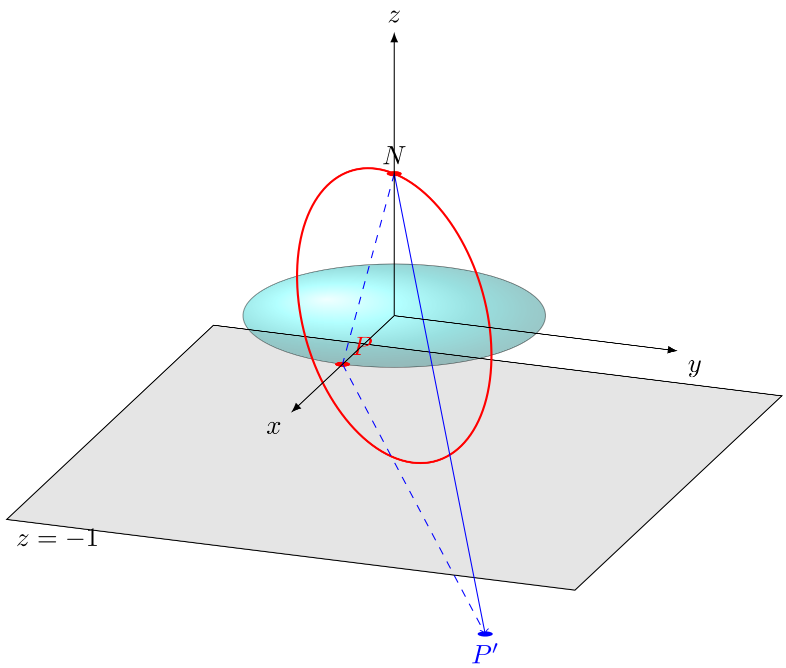


Figure 1: An illustration of the stereographic projection.

$S^2 \setminus \{(0,0,1)\}$

$$\begin{aligned} \phi_0 : U_0 &\longrightarrow V_0 = \mathbb{C} \\ \| \\ S^2 \setminus \{(0,0,1)\} \\ (x,y,w) &\longmapsto \frac{x+iy}{1-w} \quad \Rightarrow \quad \text{this is the stereographic projection} \end{aligned}$$

$$\begin{aligned} \phi_\infty : U_\infty &\xrightarrow{\cong} V_\infty = \mathbb{C} \\ \| \\ S^2 \setminus \{(0,0,-1)\} \\ (x,y,w) &\longmapsto \frac{x-iy}{1+w} \end{aligned}$$

Exercise : check these are charts.
ϕ_0 & ϕ_∞ are compatible. On $U_0 \cap U_\infty = S^2 \setminus \{(0,0,1),(0,0,-1)\}$, $\phi_0(U_0 \cap U_\infty) \subset \mathbb{C}^* \subset V_0$.

$$\frac{1}{\phi_0(x,y,w)} = \frac{1-w}{x+iy} = \frac{(1-w)(x-iy)}{x^2+y^2} = \frac{(1-w)(x-iy)}{1-w^2} = \frac{x-iy}{1+w} = \phi_\infty(x,y,w)$$

Thus,

$$\phi_\infty \circ \phi_0^{-1}(z) = \frac{1}{z} \text{ on } \mathbb{C}^* = \phi_0(U_0 \cap U_\infty) \subset \mathbb{C} = V_0.$$

holomorphic

Hence, $\{\phi_0, \phi_\infty\}$ are an atlas, and the corresponding Riemann Surface is called the **Riemann Sphere**.

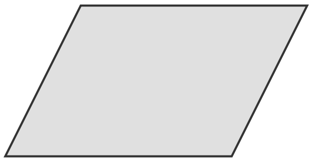
1. **Complex tori of dimension 1.** For $\omega_1, \omega_2 \in \mathbb{C}$ which are $\mathbb{R}$-linearly independent. consider the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\} \subset \mathbb{C}$. Let $X = \mathbb{C}/L$, with the quotient topology.

$$\pi : \mathbb{C} \longrightarrow X = \mathbb{C}/L.$$

$$z \longmapsto [z] = z + L.$$

Topologically X is a torus.

Every $z \in \mathbb{C}$ is equivalent to a unique point in the Fundamental domain.



Given $X = \mathbb{C}/L$, we construct an atlas using $\pi : \mathbb{C} \to \mathbb{C}/L$.
Pick $\varepsilon > 0$ s.t. $\forall p \in \mathbb{C}$, $B_\varepsilon(p)$ intersects each $[z]$ in at most one point.
Thus gives a homeomorphism

$$\pi : B_\varepsilon(p) \xrightarrow{\cong} \pi(B_\varepsilon(p))$$

$$\text{with } \phi_p = \pi\big|_{B_\varepsilon(p)} : U_p \subset X$$

where $U_p = \pi(B_\varepsilon(p))$.
Claim :

$$\mathcal{A} = \{\phi_p : U_p \to V_p\}_{p \in \mathbb{C}} \text{ is an atlas.}$$

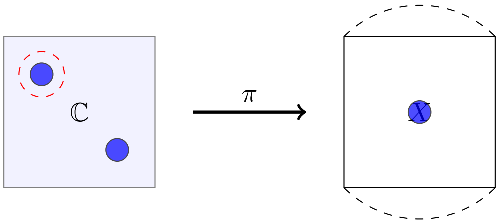
Compatibility of ϕ_p & ϕ_q: Assume $U_{p,q} = U_p \cap U_q \neq \emptyset$.
The transition map is:

$$T = \phi_q \circ \phi_p^{-1} : \phi_p(U_{p,q}) \longrightarrow \phi_q(U_{p,q})$$

T satisfies $\pi(T([z])) = \phi_p^{-1}([z]) = \pi([z])$ i.e., $T([z]) - z \in L = \ker(\pi)$, which is constant.

$$\Rightarrow T - \text{id is locally constant: locally } T - \text{id} = w \in L.$$

$$T(z) = z + w \text{ is holomorphic}$$



1.5 Examples of complex manifolds.

Example 1.2 (Complex Projective Plane)**.**

$$\mathbb{CP}^n = \{\text{1-dimensional complex vector subspace in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*.$$

$$\cong S^{2n+1}/S^1 \quad \text{quotient topology.}$$

Give $\mathbb{CP}^n$ the quotient topology.

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$$

$$(z_0, \ldots, z_n) \longmapsto \pi(z_0, \ldots, z_n) = [z_0 : z_1 : \cdots : z_n].$$

homogeneous coordinates.
Atlases: Let

$$U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\} \subset \mathbb{CP}^n \quad \text{open}$$

The chart ϕ_i is given by:

$$\phi_i : U_i \longrightarrow V_i = \mathbb{C}^n$$

$$[z_0 : \cdots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$$

where $\widehat{\frac{z_i}{z_i}}$ denotes the omission of the i-th coordinate.

2 Morphisms of complex manifolds & meromorphic functions.

2.1 Morphisms of manifolds.

Definition 2.1. Let X & Y be complex **manifolds** of dimensions n & m respectively. Let $W \subset X$ & $W' \subset Y$ be open sets.

1. A continuous map $F : W \to W'$ is holomorphic at $p \in W$ if $\exists$ charts $\phi : U \to V$ & $\psi : W' \to V'$ s.t. $p \in U$ & $F(p) \in W'$, s.t. $\psi \circ F \circ \phi^{-1}$ is holo at $\phi(p)$.
2. Biholomorphism.

Example 2.1. (Examples of Morphisms)

1. A chart on a Riemann Surface $\phi : U \to V \subset \mathbb{C}$ on a Riemann Surface is a holomorphic function.
2. Let $U \subset X$ for a Riemann Surface X. Then U has a unique Riemann Surface structure s.t. the inclusion map $U \hookrightarrow X$ is holomorphic.
3. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \cong S^2 = U_0 \cup U_\infty$. Let f be a holomorphic function on $\mathbb{C}$. Let $f_0 := f \circ \phi_0^{-1} : \mathbb{C} \to \mathbb{C}$. Let $f_\infty := f \circ \phi_\infty^{-1} : \mathbb{C} \to \mathbb{C}$. On $\mathbb{C}^*$:
$$f_\infty(w) = f \circ \phi_\infty^{-1}(w) = f \circ \phi_0^{-1} \circ \phi_0 \circ \phi_\infty^{-1}(w) = f_0\left(\frac{1}{w}\right)$$
This f is holomorphic at $w \in \mathbb{C}_\infty$
$$\iff f\left(\frac{1}{z}\right) \text{ is holomorphic at } 0$$
$$\Updownarrow \text{ Def}$$
$$f_\infty(w) \text{ is holomorphic at } 0 \iff f_0\left(\frac{1}{w}\right) \text{ is holomorphic at } \infty$$
4. the quotient map $\pi : \mathbb{C}^n \to \mathbb{C}^n/L$ for a complex torus is holomorphic.

subsection*§ 2.2. Properties of holomorphic maps of Riemann Surfaces.

Theorem 2.1 (The identity theorem)**.** *Let $F, G : X \to Y$ be holomorphic maps of Riemann Surfaces s.t. F & G agree on a subset of X with an accumulation point. Then $F = G$.*

Theorem 2.2 (Local form of holomorphic maps)**.** *Let $F : X \to Y$ be a non-constant holomorphic map of Riemann Surfaces. For $p \in X$ & $q = F(p)$, $\exists$ unique $k \in \mathbb{Z}_{>0}$ and local charts $\phi : U \to V$ & $\psi : U' \to V'$ s.t. F has local form*
$$\psi \circ F \circ \phi^{-1} : V \to V'$$
$$z \mapsto z^k.$$

Proof. take any charts $\phi : U \to V \subset \mathbb{C}$ & $\psi : U' \to V' \subset \mathbb{C}$.
$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow \phi & & \downarrow \psi \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \end{array} \qquad \begin{array}{l} p \mapsto 0. \\ q \mapsto 0. \end{array}$$

Shrink V so $F(U) \subset U'$. Then $f = \psi \circ F \circ \phi^{-1} : V \to V'$ is holomorphic and $0 \mapsto 0$.

Define $k = \mathrm{ord}_0(f) \in \mathbb{Z}_{\geq 0}$. $\mathrm{ord}_0(f)$ is the order of vanishing of f at $0 = \min\{n \mid c_n \neq 0\}$ where $f(z) = \sum_{n \geq 0} c_n z^n$ is the Taylor expansion. $f(z) = z^k g(z)$, where $g(z)$ is non-zero.

Shrink V so $g : V \to \mathbb{C}$ is non-zero and holomorphic. Thus, $\exists$ holomorphic k^{th} root $h : V \to \mathbb{C}$ of g i.e., $(h(z))^k = g(z)$.

Thus $f(z) = z^k g(z) = (zh(z))^k$.

Let $\alpha : V \xrightarrow{\cong} \alpha(V)$ be biholomorphic

$$z \mapsto zh(z) = w.$$

Note $\alpha(0) = 0$, $\alpha'(0) = h(0) \neq 0$, then $\operatorname{ord}_0(\alpha) = 1$.

Now replace ϕ by $\alpha \circ \phi$.

$$\psi \circ F \circ (\alpha \circ \phi)^{-1}(w) = \psi \circ F \circ \phi^{-1} \circ \alpha^{-1}(w)$$

$$= f(\alpha^{-1}(w)) = (\alpha^{-1}(w)h(\alpha^{-1}(w)))^k = w^k.$$

Local form is $w \mapsto w^k$. $\square$

Exercise: Show k is independent of the choice of chart.

Definition 2.2. The **multiplicity** of a non-constant holomorphic map $F : X \to Y$ of Riemann Surfaces at $p \in X$ is the unique positive integer k given by Theorem 2.2.

We say p is an unramified point of F if $\operatorname{mult}_p(F) = 1$. p is a ramified point of F if $\operatorname{mult}_p(F) > 1$.

$$R(F) = \{p \in X : \operatorname{mult}_p(F) > 1\} \text{ ramification locus.}$$

$$B(F) = F(R(F)) \subset Y \text{ branch locus.}$$

Theorem 2.3. *(Open mapping theorem) A non-constant holomorphic map of Riemann Surfaces is an open mapping.*

Proof: The local form $z \mapsto z^k$ is open ("maps circle to circle").

Theorem 2.4. *(Biholomorphic maps are biholomorphisms) The inverse of a bijective holomorphic map of Riemann Surfaces is holomorphic (since F is an open mapping).*

Theorem 2.5. *(Discreteness of preimages) Let $F : X \to Y$ non constant map of RS. Then $\forall q \in Y$. the preimage $F^{-1}(q)$ is discrete in X. (If X is compact then $F^{-1}(q)$ is finite)*

Proof. F follows as a holomorphic map of the complex plane are discrete. $\square$

Theorem 2.6. *(Surjectivity of non-constant holomorphic map from compact RS) Let $F : X \to Y$ a non-constant holomorphic map of RS with X compact. Then F is surjective and Y is compact.*

Proof. Open mapping theorem $\implies$ $F(X) \subset Y$ is open F continuous and X compact $\implies$ $F(X)$ compact hence closed (and Y hausdorff) $\square$. $\square$

Corollary 2.7. *Every holomorphic function on a compact Riemann Surface (**RS**) is constant.*

Proof: Let $f : X \to \mathbb{C}$ be a non-constant holomorphic function. Then f is constant. $\square$

Theorem 2.8 (Riemann's extension theorem)**.** *Let X be a **RS**, $p \in U \subset_{open} X$. If $f : U \setminus \{p\} \to \mathbb{C}$ is holomorphic and bounded in a punctured neighborhood of p, then f extends to a holomorphic function on U.*

Proof: Follows from complex analysis using charts. $\square$

Theorem 2.9 (Maximum principle)**.** *Let $f : X \to \mathbb{C}$ be a non-constant holomorphic function on a* **RS** *X. Then $|f|$ has no local maximum.*

2.2 Meromorphic functions on RS

Definition 2.3. Let X be a **RS** and $p \in W \subset_{\text{open}} X$. If $f : W \setminus \{p\} \to \mathbb{C}$ is holomorphic, we say that p is a

$$\left\{\begin{array}{l}\text{removable singularity}\\ \text{pole}\\ \text{essential singularity}\end{array}\right\} \text{ if } \exists \text{ chart } \phi : U \to V, \text{ s.t. } \phi(p) \text{ is a } \left\{\begin{array}{l}\text{removable singularity}\\ \text{pole}\\ \text{essential singularity}\end{array}\right\}$$

We say f is **meromorphic** at p if p is a **non-essential singularity**.
If $S \subset W \subset_{\text{open}} X$ and $f : W \setminus S \to \mathbb{C}$ is holomorphic, we say f is **meromorphic** on W if it is **meromorphic** at each $p \in S$.

Notation

Let $U \subset_{\text{open}} X$.

- $\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$
- $\mathcal{M}_X(U) = \{f : U \setminus S \to \mathbb{C} \mid f \text{ is meromorphic on } U\}$

Lemma 2.10. *i) The above definition is independent of the choice of chart.*

ii) $\mathcal{O}(X) = \mathcal{O}_X(X)$ *is a* $\mathbb{C}$*-algebra.*

iii) $\mathcal{M}(X) = \mathcal{M}_X(X)$ *is a field, called the **function field of X**.*

iv) $\mathcal{M}(X) = \mathit{Frac}(\mathcal{O}(X))$.

v) p has a

$$\left\{\begin{array}{l}\textit{removable singularity}\\ \textit{pole}\\ \textit{essential singularity}\end{array}\right\} \textit{ at } p \textit{ if } \left\{\begin{array}{l}|f| \textit{ is bounded in a neighborhood of } p\\ \lim_{z\to p}|f(z)| = \infty\\ \lim_{z\to p} f(z) \textit{ doesn't exist}\end{array}\right\}$$

2.3 Laurent Expansions & Orders of Singularities

Let $f : W \setminus \{p\} \to \mathbb{C}$ be a holomorphic function with $p \in W \subset_{\text{open}} X$, where X is a **Riemann Surface**. Let $\phi : U \to V$ be a chart on X, such that $\phi(p) = 0$. Then $f \circ \phi^{-1} : V \setminus \{0\} \to \mathbb{C}$ is holomorphic and has a Laurent expansion at 0.

$$f \circ \phi^{-1}(z) = \sum_{n=-\omega}^{\infty} c_n z^n \quad (\textbf{Note}\text{: This depends on the choice of chart.})$$

Definition 2.4. The **order of f** is defined as:

$$\text{ord}_p(f) := \text{ord}_0(f \circ \phi^{-1}) = \min\{n \in \mathbb{Z} \mid c_n \neq 0\}$$

If $f(z) = 0$ for all z, then $\text{ord}_p(0) = \infty$.

Lemma 2.11. *i)* $\mathit{ord}_p(f)$ *is independent of the chart ϕ centered at p.*

ii) X is a

$$\left\{\begin{array}{l}\textit{removable singularity}\\ \textit{pole}\\ \textit{essential singularity}\end{array}\right\} \textit{ of } f \textit{ if } ord_p(f) = \left\{\begin{array}{l}\geq 0\\ -m, \quad m>0 \quad (\textit{i.e.}, \ <0)\\ -\infty\end{array}\right\}$$

iii) $ord_p(f^{-1}) = -ord_p(f)$.

iv) $ord_p(fg) = ord_p(f) + ord_p(g)$.

v) $ord_p(f+g) \geq \min\{ord_p(f), ord_p(g)\}$.

Example 2.2. The map $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic on $\mathbb{C}$. Is it holomorphic or meromorphic on $\mathbb{C}_\infty$? Consider $w = 1/z$. For $f : \mathbb{C}_\infty \setminus \{0\} \to \mathbb{C}$ we have $\mathrm{ord}_\infty(f) = \mathrm{ord}_0(f(1/z))$. Thus, $\exp(w)$ has an essential singularity at $w = 0$, so exp is not meromorphic on $\mathbb{C}_\infty$.

Example 2.3 (Meromorphic functions on $\mathbb{C}_\infty$)**.** Let $z = \phi_0 : U_0 \to V_0 = \mathbb{C}$. $\mathrm{ord}_0(z) = -1$. Let $P, Q \in \mathbb{C}[z]$ with $Q \not\equiv 0$. We claim $f(z) = P(z)/Q(z) \in \mathcal{M}(\mathbb{C}_\infty)$. We know $f \in \mathcal{M}(\mathbb{C})$: what about at ∞? Let $f(z) = \lambda \prod_i (z - a_i)^{k_i}$, where $a_i \in \mathbb{C}$ and $k_i \in \mathbb{Z}$. $x \in \mathbb{C}$. $f(z)$ is meromorphic at ∞ if $f(1/z)$ is meromorphic at 0.

$$f(1/z) = \lambda \prod_i (1/z - a_i)^{k_i} = \lambda z^{-\sum k_i} \prod_i (1 - a_i z)^{k_i}$$

$$\mathrm{ord}_\infty(f) = \begin{cases} -\sum k_i & p = \infty \\ k_i & p = a_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{Note: } \sum_{p \in \mathbb{C}_\infty} \mathrm{ord}_p(f) = 0.$$

Theorem 2.12 (Meromorphic functions as holomorphic maps to $\mathbb{C}_\infty$)**.** *For a **Riemann Surface** X, there is a 1-1 correspondence:*

$$\mathcal{M}(X) = \{\textit{meromorphic functions on } X\} \longleftrightarrow \{\textit{holomorphic maps } F : X \to \mathbb{C}_\infty \mid F \not\equiv \infty\}$$

The correspondence is given by:

$$f \longmapsto F : X \to \mathbb{C}_\infty, \quad F(x) = \begin{cases} f(x), & x \notin Pole(f) \\ \infty, & x \in Pole(f) \end{cases}$$

$$f = \phi_\infty \circ F\,|_{F^{-1}(\mathbb{C})} \longleftarrow F : X \to \mathbb{C}_\infty.$$

and $f(x) = \infty$ *if* $F(x) = \infty$.

Proof. The map F associated to $f \in \mathcal{M}(X)$ is holomorphic on $X \setminus \mathrm{Pole}(f)$. **We want to show:** F is holomorphic at each $p \in \mathrm{Pole}(f)$. f has a pole at $p \iff f \circ \psi^{-1}$ has a pole at $\psi(p) = 0$ (by definition). $\iff \phi_\infty \circ F \circ \psi^{-1} = \frac{1}{f \circ \phi^{-1}}$ has a zero at $\psi(p) = 0$. This means F is holomorphic at p (by definition of holomorphicity for a map to $\mathbb{C}_\infty$). $\square$

Lemma 2.13 (Relating the order of $f \in \mathcal{M}(X)$ and the multiplicity of the corresponding map $F : X \to \mathbb{C}_\infty$)**.** *For* $f \in \mathcal{M}(X)$ *non-constant, and* $F : X \to \mathbb{C}_\infty$ *the corresponding holomorphic map at* $p \in X$*:*

i) If $f(p) = 0$, *then* $mult_p(F) = ord_p(f)$.

ii) If $f(p) = \infty$, *then* $mult_p(F) = -ord_p(f)$.

iii) Otherwise, $mult_p(F) = ord_p(f - f(p))$.

Theorem 2.14 (Meromorphic functions on $\mathbb{C}_\infty$)**.**

$$\mathcal{M}(\mathbb{C}_\infty) = \mathbb{C}(z)$$

Proof: We've seen $\mathbb{C}(z) \subset \mathcal{M}(\mathbb{C}_\infty)$. Let $f \in \mathcal{M}(\mathbb{C}_\infty)$. Let $p_1, p_2, \ldots, p_n$ be the zeros of f in $\mathbb{C} \subset \mathbb{C}_\infty$. Let $g(z) := \prod_{i=1}^{n}(z - p_i)^{r_i}$, where $r_i = \mathrm{ord}_{p_i}(f) \in \mathbb{Z}$. (Note that $g(z) \in \mathbb{C}(z)$). By construction, $\mathrm{ord}_p(f) = \mathrm{ord}_p(g)$ for all $p \in \mathbb{C}$. Then $h = f/g \in \mathcal{M}(\mathbb{C}_\infty)$ has no zeros and no poles in $\mathbb{C}$.

Let $h(z) = \sum c_n z^n$ be the Taylor expansion of h at $0 \in \mathbb{C}$. Let $w = 1/z$ be a local coordinate at $\infty \in \mathbb{C}_\infty$. Then

$$h(w) = \sum c_n w^{-n}$$

is the Laurent expansion of h at ∞. Since h is meromorphic at ∞, then $h(z) = \sum_{n=0}^{m} c_n z^n \in \mathbb{C}[z]$ (polynomial). If $\deg(h) > 0$, then h has a zero in $\mathbb{C}$. Thus $h(z) = \lambda$, a constant. And $f = \lambda g \in \mathbb{C}(z)$. ☐

Corollary 2.15. *For $f \in \mathcal{M}(\mathbb{C}_\infty)$, $\sum_{p \in \mathbb{C}_\infty} ord_p(f) = 0$.*

2.4 The degree of a holomorphic map

Theorem 2.16. *Let $F : X \to Y$ be a non-constant holomorphic map of compact **Riemann Surfaces**. For $q \in Y$, the quantity $deg_q(F) = \sum_{p \in F^{-1}(q)} mult_p(F)$ is independent of q.*

Definition 2.5. The **degree of F** is $\deg(F) = \deg_q(F)$ for any $q \in Y$.

Proof. For $q \in Y$, take $F^{-1}(q) = \{p_1, p_2, \ldots, p_s\}$ and let $k_i = \mathrm{mult}_{p_i}(F)$. By Theorem 2.2, $\exists$ local normal forms at each p_i, i.e., $\exists$ charts $\phi_i : U_i \to V_i$ on X and $\psi : U_i' \to V_i'$ on Y, such that $F(U_i) \subset U_i'$, $p_i \to 0$, $q \to 0$, and $\psi \circ F \circ \phi_i^{-1}(z) = z^{k_i}$. Assume the U_i are pairwise disjoint.

Claim. $\exists$ *open neighborhood W of q in Y such that $F^{-1}(W) \subset \bigcup_i U_i$.*

Proof of Claim: Let $\overline{W}$ be an open neighborhood of q, $\overline{W} = \{q\} \cup$something else. Let W be an open neighborhood of q such that $W \cap F(X \setminus \bigcup_i U_i) = \emptyset$. Thus $F^{-1}(W) \cap (X \setminus \bigcup_i U_i) = \emptyset$. Since X is compact, $F(X \setminus \bigcup_i U_i)$ is compact. $\exists$ an open neighborhood W_i of q in Y such that $F^{-1}(W_i) \cap (X \setminus U_i) = \emptyset$.Let $W = \bigcap_{i=1}^{s} W_i$, which is an open neighborhood of q. Then $F^{-1}(W) \subset \bigcup_{i=1}^{s} U_i$. ☐

For any $q' \in W$, we have $\deg_{q'}(F) = \deg_q(F)$. This is because, for $q' \in W$, $F^{-1}(q') \cap U_i$ consists of k_i points of multiplicity 1. Hence $\deg_q(F)$ is locally constant, and as Y is connected, $\deg_q(F)$ is constant. ☐

Remark 2.1. Let $f : X \to \mathbb{C}$. If f is locally constant and X is connected, then f is a constant.

Proof. Take $p \in X$. $\exists U_p \subset X$ such that $f|_{U_p}$ is a constant $f(p)$. Consider $\mathcal{O} = \{x \in X \mid f(x) = f(p)\}$.

1. $\mathcal{O}$ is open since f is locally constant.
2. $\mathcal{O}$ is closed since f is locally constant.

If $y \in X \setminus \mathcal{O}$, then $f(y) \neq f(p)$. Then $\exists U_y$ such that $U_y \subset X \setminus \mathcal{O}$. Since X is connected and $\mathcal{O}$ is non-empty (as $p \in \mathcal{O}$) and is both open and closed, we must have $\mathcal{O} = X$. Thus f is constant. ☐

Remark 2.2.
1. At a ramification point p, F looks locally like $\mathbb{C} \to \mathbb{C}, z \mapsto z^k$.
2. $F|_{X \setminus R(F)} : X \setminus R(F) \to Y \setminus B(F)$ is a d-sheeted covering, where $d = \deg(F)$.

Corollary 2.17. *i) If F is a degree 1 non-constant holomorphic map of compact **Riemann Surfaces** (***RS***), then F is a **biholomorphism** (surjectivity + injectivity).*

ii) If X is compact and has a meromorphic function with a single simple pole, then $X \cong \mathbb{C}_\infty$.

Proof. Left as an exercise ☐

Corollary 2.18. $\mathbb{CP}^1 \cong \mathbb{C}_\infty$.

Corollary 2.19. *Let X be a compact **Riemann Surface** (**RS**) and $f \in \mathcal{M}(X)$ non-constant. Then $\sum_{p\in X} ord_p(f) = 0$.*

Proof. Let $F : X \to \mathbb{C}_\infty$ be the associated holomorphic map. We know that:

$$\sum_{p\in X} \mathrm{ord}_p(f) = \sum_{p\in \mathrm{Zero}(f)} \mathrm{ord}_p(f) + \sum_{p\in \mathrm{Pole}(f)} \mathrm{ord}_p(f)$$

We use the relationship between order and multiplicity

$$\sum_{p\in \mathrm{Zero}(f)} \mathrm{ord}_p(f) = \sum_{p\in F^{-1}(0)} \mathrm{mult}_p(F)$$

And

$$\sum_{p\in \mathrm{Pole}(f)} \mathrm{ord}_p(f) = \sum_{p\in F^{-1}(\infty)} (-\mathrm{mult}_p(F)) = -\sum_{p\in F^{-1}(\infty)} \mathrm{mult}_p(F)$$

Substituting these back:

$$\sum_{p\in X} \mathrm{ord}_p(f) = \sum_{p\in F^{-1}(0)} \mathrm{mult}_p(F) - \sum_{p\in F^{-1}(\infty)} \mathrm{mult}_p(F)$$

Since $\sum_{p\in F^{-1}(q)} \mathrm{mult}_p(F) = \deg(F)$ for any $q \in \mathbb{C}_\infty$:

$$= \deg(F) - \deg(F) = 0.$$

□

2.5 Germs of holomorphic functions

Definition 2.6. For a complex manifold X and $p \in X$, we define the ring of germs of holomorphic functions at p:

$$\mathcal{O}_{X,p} = \{(U, f) \mid U \text{ is an open neighborhood of } p, f : U \to \mathbb{C} \text{ is holomorphic}\} / \sim$$

where $\sim$ is the equivalence relation defined by:

$$(U, f) \sim (V, g) \iff \exists \text{ a neighborhood } W \text{ of } p \text{ s.t. } f|_W = g|_W$$

The equivalence class $[(U, f)]$ is called the **germ of f at p**.

Remark 2.3.

1. $\mathcal{O}_{X,p}$ is a ring whose non-invertible elements form an ideal.
2. The maximal ideal $\mathfrak{m}_p$ is given by:

$$\mathfrak{m}_p = \{[(U, f)] \mid f(p) = 0\}. \quad \text{(germs vanishing at } p\text{)}$$

This is the kernel of the evaluation map:

$$\ker(\mathrm{ev}_p : \mathcal{O}_{X,p} \to \mathbb{C}).$$

Thus $\mathcal{O}_{X,p}/\mathfrak{m}_p \cong \mathbb{C}$. This means $\mathfrak{m}_p$ is a **maximal ideal** (local ring).

Example 2.4.

1. $\mathcal{O}_{\mathbb{C}^n,0} \cong \mathbb{C}\{x_1, \ldots, x_n\}$.

$$[(U, f)] \mapsto \text{Taylor expansion of } f \text{ at } 0.$$

2. If X is an n-dimensional complex manifold and $p \in X$, then a local chart $\phi : U \to V$,

centered at p, induces an isomorphism:

$$\phi^* : \mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{X,p}$$

which maps the germ $[(V, \psi)]$ to the germ $[(U, \psi \circ \phi)]$.

$\mathcal{O}_{X,p}$ for a Riemann Surface

The order of a **holomorphic** function at p descends to a map

$$\mathrm{ord}_p : \mathcal{O}_{X,p} \to \mathbb{N} \cup \{\infty\} \text{ satisfying}$$

3. $\mathrm{ord}_p(f) = \infty \iff f \equiv 0$.
4. $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$.
5. $\mathrm{ord}_p(f + g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$.

This is known as a **discrete valuation**.
We can extend ord_p to $\mathrm{Frac}(\mathcal{O}_{X,p})$ by $\mathrm{ord}_p(f/g) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$.

Lemma 2.20. *For a **Riemann Surface** X, $\mathcal{O}_{X,p}$ is a **Discrete Valuation Ring** (***DVR***) with valuation given by $\mathrm{ord}_p : \mathrm{Frac}(\mathcal{O}_{X,p}) \to \mathbb{Z} \cup \{\infty\}$. The **uniformizer** (element with valuation 1) is given by a local chart centered at p.*