MAT3040 Midterm Exam Paper 2024

Student Name: ____________ Student ID: ____________

No book, note, calculator or dictionary allowed. Show your steps or reasoning in detail. Please write down your solution on the answer paper. The total score is 30 out of 60 points including 17 bonus points.

Q1 [**18 points**]. Let $V = M_{n\times n}(\mathbb{C})$ be the $\mathbb{C}$-vector space of complex $n \times n$ matrices, we define

$$W_{\star,+} := \{A \in V :\ {}^t\bar{A} = A\} \text{ and } W_{\star,-} := \{A \in V :\ {}^t\bar{A} = -A\}.$$

Here ${}^t\bar{A} ={}^t \overline{(a_{ij})}_{i,j} := (\bar{a}_{ji})_{i,j}$ with $\bar{a}_{ij} = \overline{x + \sqrt{-1}y} := x - \sqrt{-1}y \in \mathbb{C}$.

(i) Prove or disprove that $W_{\star,-}$ is a $\mathbb{C}$-vector subspace of V.

(ii) Prove or disprove that $W_{\star,+}$ is a $\mathbb{R}$-vector subspace of V.

(iii) Show that $\{E_{ij} + E_{ji} :\ i \geq j\} \cup \{\sqrt{-1}(E_{ij} - E_{ji}) :\ i > j\}$ is a basis of $W_{\star,+}$ as $\mathbb{R}$-vector space. Here $E_{ij} := (a_{kl})_{k,l}$ with $a_{ij} = 1$ and the remaining entries are 0.

(iv) Show that $V = W_{\star,+} \oplus W_{\star,-}$ as $\mathbb{R}$-vector spaces.

(v) Show that $T :\ V \to V :\ A \mapsto {}^tA$ is a $\mathbb{C}$-linear map.

(vi) Determine the characteristic polynomial of T in (v).

(vii) Determine the minimal polynomial of T in (v).

(viii) Calculate $\dim_{\mathbb{C}}(\mathrm{Span}_{\mathbb{C}}(S))$. Here

$$S := \left\{ \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix} \right\} \subset V = M_{2\times 2}(\mathbb{C}).$$

(ix) Calculate $\dim_{\mathbb{R}}(\mathrm{Ann}(\mathrm{Span}_{\mathbb{C}}(S)))$. Here $S \subset V$ is given in (viii).

Q2 [**18 points**]. Let V be a $\mathbb{C}$-vector space and $T: V \to V$ be a $\mathbb{C}$-linear map.

(i) Show that $\mathrm{Ker}(T^i)$ and $\mathrm{Im}(T^i)$ are $\mathbb{C}$-vector subspaces of V for any positive integer $i \in \mathbb{N}_+$.

(ii) Show that $\mathrm{Im}(T^i) \subset \mathrm{Im}(T^j)$ for any postive integers $i > j \in \mathbb{N}_+$.

(iii) Show that $\mathrm{Ker}(T^i) \supset \mathrm{Ker}(T^j)$ for any positive integers $i > j \in \mathbb{N}_+$.

(iv) Show that $\mathrm{Im}(T^i) = \mathrm{Im}(T^j)$ for some positive integers $i > j \in \mathbb{N}_+$ implies $\mathrm{Im}(T^{j+n}) = \mathrm{Im}(T^j)$ for any non-negative integer $n \in \mathbb{N}$.

(v) Show that $\mathrm{Ker}(T^i) = \mathrm{Ker}(T^j)$ for some positive integers $i > j \in \mathbb{N}_+$ implies $\mathrm{Ker}(T^{j+n}) = \mathrm{Ker}(T^j)$ for any non-negative integer $n \in \mathbb{N}$.

(vi) If $\dim_{\mathbb{C}}(V) < \infty$, show that, for any $i, j \in \mathbb{N}_+$,

$$\mathrm{Im}(T^i) = \mathrm{Im}(T^j) \text{ implies } \mathrm{Ker}(T^i) = \mathrm{Ker}(T^j).$$

Vice versa, show that, for any $i, j \in \mathbb{N}_+$,

$$\mathrm{Ker}(T^i) = \mathrm{Ker}(T^j) \text{ implies } \mathrm{Im}(T^i) = \mathrm{Im}(T^j).$$

(vii) Prove or disprove the claims in (vi) for V with $\dim_{\mathbb{C}}(V) = \infty$.

(viii) Define $\mathrm{Ker}(T^\infty) := \bigcup_{i \geq 1} \mathrm{Ker}(T^i)$ and $\mathrm{Im}(T^\infty) := \bigcap_{i \geq 1} \mathrm{Im}(T^i)$. Show that $V = \mathrm{Ker}(T^\infty) \oplus \mathrm{Im}(T^\infty)$ if $\dim_{\mathbb{C}}(V) < \infty$.

(ix) Prove or disprove the claim in (viii) for V with $\dim_{\mathbb{C}}(V) = \infty$.

Q3 [**12=7+5 points**]. Suppose V is a finite dimensional vector space over a field $\mathbb{F}$, e.g. $\mathbb{F} = \mathbb{R}$. Let $W_i,\ i = 1, 2, 3$, be three vector subspaces of V.

(i) Write down a definition of the sum of the subspaces W_1, W_2, W_3, i.e., $W_1 + W_2 + W_3$.

(ii) Write down a definition of the direct sum of the subspaces W_1, W_2, W_3, i.e., $W_1 \oplus W_2 \oplus W_3$.

(iii) Show that $\dim_{\mathbb{F}}(W_1+W_2+W_3) \leq \sum_{i=1}^{3} \dim_{\mathbb{F}}(W_i)$. Moreover, the equality holds if and only if $W_1 + W_2 + W_3 = W_1 \oplus W_2 \oplus W_3$.

(iv) Prove the following inequality.

$$\dim_{\mathbb{F}}(W_1+W_2+W_3) \leq \sum_{i=1}^{3} \dim_{\mathbb{F}}(W_i) - \sum_{1\leq i<j\leq 3} \dim_{\mathbb{F}}(W_i \cap W_j) + \dim_{\mathbb{F}}(W_1 \cap W_2 \cap W_3).$$

(v) (**Bonus question 5 points**) Prove the following three statements are equivalent.

(a) $(W_1 + W_2) \cap W_3 = W_1 \cap W_3 + W_2 \cap W_3$.

(b) $(W_2 + W_3) \cap W_1 = W_2 \cap W_1 + W_3 \cap W_1$.

(c) $(W_3 + W_1) \cap W_2 = W_3 \cap W_2 + W_1 \cap W_2$.

Q4 [**Bonus question 12 points**] Let

$$\mathbb{C}[x,y] := \left\{ f(x,y) = \sum_{0\le i\le n} \sum_{0\le j\le m} a_{i,j} x^i y^j \ : \ n,m \in \mathbb{N}, a_{i,j} \in \mathbb{C} \right\},$$

and $H_n[x,y]$ be the subset of $\mathbb{C}[x,y]$ consisting of homogeneous polynomials of degree n, i.e.,

$$H_n[x,y] := \left\{ f(x,y) = \sum_{0\le i\le n} a_i x^i y^{n-i} \ : \ a_i \in \mathbb{C} \right\}.$$

(i) Show that $\mathbb{C}[x,y]$ is a $\mathbb{C}$-vector space under the following two binary operations.

- Addition: $\sum_{i,j} a_{i,j} x^i y^j + \sum_{i,j} b_{i,j} x^i y^j := \sum_{i,j} (a_{i,j} + b_{i,j}) x^i y^j$.
- Scalar multiplication: $\lambda \cdot \sum_{i,j} a_{i,j} x^i y^j := \sum_{i,j} \lambda a_{i,j} x^i y^j$ for any $\lambda \in \mathbb{C}$.

(ii) Show that $\left\{ (x_0 x + y_0 y)^i (x_1 x + y_1 y)^{n-i} \ : \ 0 \le i \le n \right\}$ is a basis of $H_n[x,y]$ if and only if $\det \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \neq 0$.

(iii) For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. Consider the associated $\mathbb{C}$-linear map

$$T_g : \ H_n[x,y] \to H_n[x,y] : \ f(x,y) \mapsto f((x,y)g) := f(ax+cy, bx+dy).$$

Calculate $\mathrm{Tr}(T_g)$ and $\det(T_g)$ in terms of eigenvalues of g.