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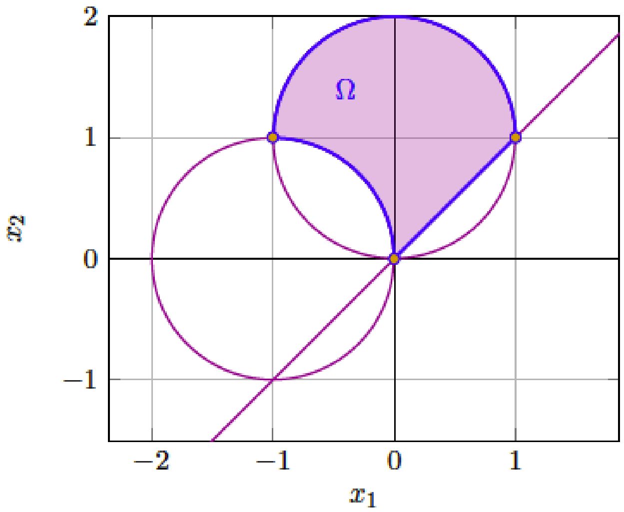
MAT 3007 – Optimization

Final Exam – Sample

Exercise 1 (KKT Conditions and Constrained Problems): (*20* points)

a) The following sketch shows the feasible region Ω



b) Based on our sketch in part a), we immediately see that the feasible region Ω is not convex. Hence, problem (1) is not a convex program.

c) We have $g_1(\bar{x}) = g_2(\bar{x}) = g_3(\bar{x}) = 0$ and hence, it follows $\mathcal{A}(\bar{x}) = \{1, 2, 3\}$.

d) We first calculate several derivatives:

$$\nabla f(x) = \begin{bmatrix} x_2 - 2x_1x_2^2 \\ \frac{1}{1+x_2} + x_1 - 2x_1^2x_2 \end{bmatrix}, \quad \nabla g_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 - 2 \end{bmatrix},$$

$$\nabla g_2(x) = \begin{bmatrix} -2x_1 - 2 \\ -2x_2 \end{bmatrix}, \quad \nabla g_3(x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Inserting $x = \bar{x}$, we obtain

$$\nabla f(\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla g_1(\bar{x}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad \nabla g_2(\bar{x}) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad \nabla g_3(\bar{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We need to find $\lambda \in \mathbb{R}^3$ such that

$$\nabla f(\bar{x}) + \nabla g_1(\bar{x})\lambda_1 + \nabla g_2(\bar{x})\lambda_2 + \nabla g_3(\bar{x})\lambda_3 = \begin{bmatrix} -2\lambda_2 + \lambda_3 \\ 1 - 2\lambda_1 - \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, we have $\lambda_3 = 1 - 2\lambda_1$ and $\lambda_2 = \frac{1}{2} - \lambda_1$. Since the multiplier need to be nonnegative, we need to have $\lambda_1 \leq \frac{1}{2}$. Furthermore, due to $\mathcal{A}(\bar{x}) = \{1, 2, 3\}$, the complementarity conditions are automatically satisfied. Thus, $\bar{x}$ is KKT point and all associated multiplier are given by $(\lambda, \frac{1}{2} - \lambda, 1 - 2\lambda)$ with $\lambda \in [0, \frac{1}{2}]$. This also shows that the multiplier is not unique in this case.

Exercise 2 (Convexity): (*15* points)

Consider the function $f(x, y) = -\log(x + e)\log(y + e)$ defined on the region $\Omega := \mathbb{R}^2_+ = \{(x, y) : x \geq 0,\ y \geq 0\}$. (Here, log denotes the natural logarithm and e is Euler's number).

a) For fixed y, is the mapping $f(x, y)$ a convex function of x for $x \geq 0$? Explain your answer.

b) For fixed x, is the mapping $f(x, y)$ a convex function of y for $y \geq 0$? Explain your answer.

c) Is f a convex function of (x, y) on Ω? Explain your answer!

Solution :

a) We have $f_x(x, y) = -\frac{\log(y+e)}{x+e}$ and $f_{xx}(x, y) = \frac{\log(y+e)}{(x+e)^2}$. For $y \geq 0$, we have $\log(y + e) \geq 1$. Hence $f_{xx}(x, y) \geq 0$ for all x. This shows convexity of $x \mapsto f(x, y)$.

b) Similarly, we have $f_y(x, y) = -\frac{\log(x+e)}{y+e}$ and $f_{yy}(x, y) = \frac{\log(x+e)}{(y+e)^2}$. For $x \geq 0$, it holds that $\log(x + e) \geq 1$. Hence $f_{yy}(x, y) \geq 0$ for all y. This establishes convexity of $y \mapsto f(x, y)$.

c) It holds that

$$\nabla f(x, y) = \begin{pmatrix} -\frac{\log(y+e)}{x+e} \\ -\frac{\log(x+e)}{y+e} \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x, y) = \begin{pmatrix} \frac{\log(y+e)}{(x+e)^2} & -\frac{1}{(x+e)(y+e)} \\ -\frac{1}{(x+e)(y+e)} & \frac{\log(x+e)}{(y+e)^2} \end{pmatrix}.$$

The trace of the Hessian is positive and its determinant is given by

$$\frac{\log(x + e)\log(y + e)}{(x + e)^2(y + e)^2} - \frac{1}{(x + e)^2(y + e)^2}$$

This expression is also nonnegative due to $\log(x + e) \geq 1$ and $\log(y + e) \geq 1$. Hence, the eigenvalues of $\nabla^2 f$ need to be nonnegative which implies that f is convex on $\mathbb{R}^2_+$.

Exercise 3 (Integer Programming Formulation): (*15* points)
A company wishes to put together an academic "package" for an executive training program. The package will consist of 6 courses. There are 4 fields and the 6 courses must cover all the 4 fields. There are 3 colleges, each offering one course in each of the 4 fields. The tuition (basic charge) assessed when at least one course is taken, at college j is T_j (independent of the number of courses taken). Moreover, each college imposes an additional charge (covering course materials, instructional aids, and so forth) for each course, the charge for taking course i (course in the field i) at college j is c_{ij}. Formulate an integer program that will provide the company with the minimum amount it must spend to meet the requirements of the program.

Solution : Let $y_j \in \{0,1\}$ denote whether to take any course in college j and let $x_{ij} \in \{0,1\}$ denote whether to take course i and college j.

Then the integer programming formulation for this problem is:

$$\begin{array}{lll} \text{minimize} & \sum_{j=1}^{3} T_j y_j + \sum_{j=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} & \\ \text{subject to} & \sum_{j=1}^{3} x_{ij} \geq 1 & \forall\, i = 1, \ldots, 4 \\ & \sum_{j=1}^{3} \sum_{i=1}^{4} x_{ij} = 6 & \\ & x_{ij} \leq y_j & \forall\, i, j \\ & x_{ij}, y_j \in \{0,1\}. & \end{array}$$

Exercise 4 (Branch-and-Bound Algorithm): (*20* points)
Consider the following integer program:

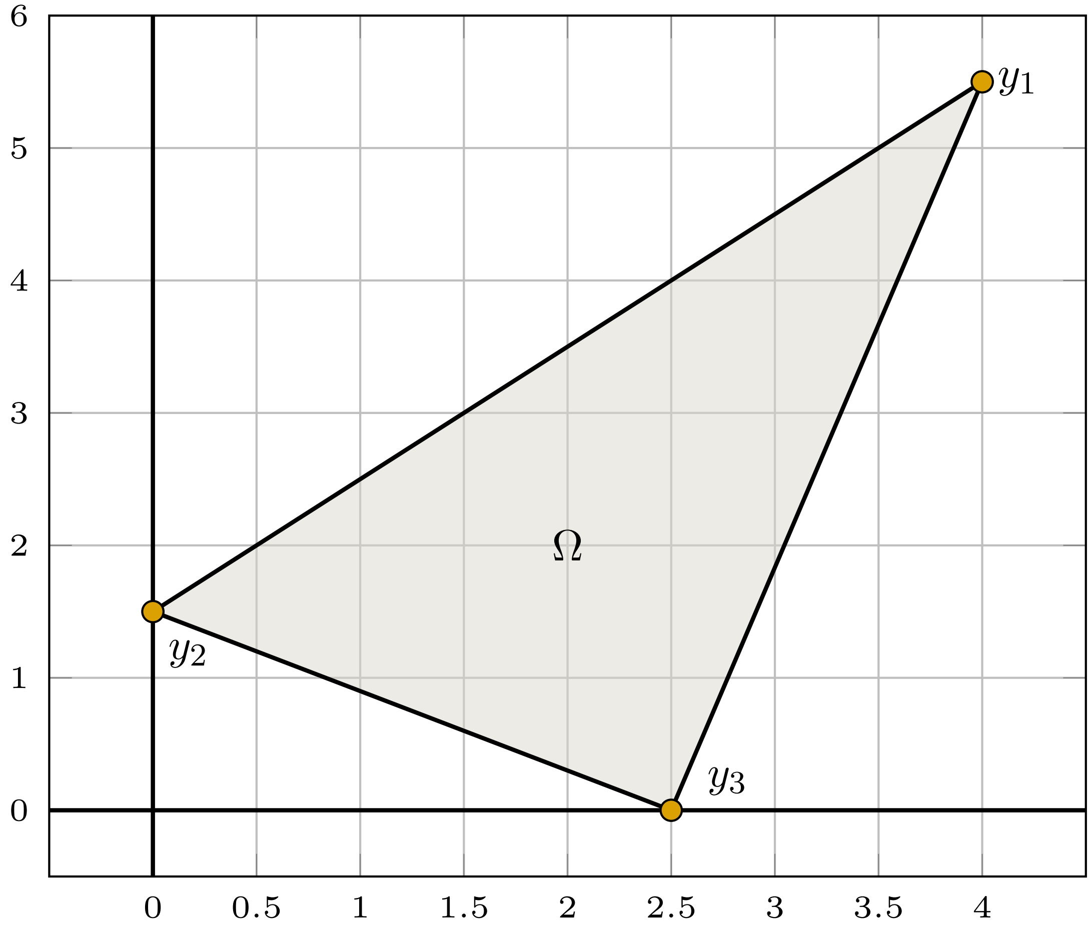
$$\begin{array}{lrcrcl} \text{maximize} & x_1 & - & x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 1.5 \\ & -6x_1 & - & 10x_2 & \leq & -15 \\ & 22x_1 & - & 6x_2 & \leq & 55 \\ & x_1, & & x_2 & \in & \mathbb{Z}. \end{array}$$

Use the branch-and-bound method to solve the problem. Draw the branch-and-bound tree and mark the results on each node.

Hint: In order to solve the LP relaxations you can use a graphical approach or check the corresponding extreme points. The following sketch shows the feasible set

$$\Omega := \{x \in \mathbb{R}^2 : -x_1 + x_2 \leq 1.5,\ -6x_1 - 10x_2 \leq -15,\ 22x_1 - 6x_2 \leq 55\}.$$

The extreme points of Ω are given by $y_1 = (4, 5.5)^\top$, $y_2 = (0, 1.5)^\top$, and $y_3 = (2.5, 0)^\top$.



Solution : We denote the original problem by (S0). We first solve the LP relaxation of (S0) which is $\max_{x\in\Omega} f(x)$, $f(x) := x_1 - x_2$. Using the sketch of Ω or by checking the objective function values

$$f(y_1) = -1.5, \quad f(y_2) = -1.5, \quad f(y_3) = 2.5,$$

it is easy to see that y_3 is the optimal solution. Since the optimal value needs to be an integer number, this implies that the optimal objective function value is bounded by 2.

We branch on x_1. We consider the two branches:

- (S1): $x_1 \leq 2$.
- (S2): $x_1 \geq 3$.

For (S1), we calculate the two new extreme points $x_1 = 2$ and $-12-10x_2 = -15$ and $-2+x_2 = 1.5$. This gives $(2, 0.3)^\top$ and $(2, 3.5)^\top$ with function values 1.7 and -1.5, respectively.

We need to further branch on x_2. We consider the two branches:

- (S3): $x_2 \leq 0$.

- (S4): $x_2 \geq 1$.

Problem (S3) is infeasible. For (S4), we need to additionally the two new extreme points $(2,1)^\top$ and $x_2 = 1$, $-6x_1 - 10 = -15$ which gives $(\frac{5}{6}, 1)^\top$. The corresponding objective function values are 1 and $-\frac{1}{6}$. Hence, the optimal of (S4) is $(2,1)^\top$ with optimal value 1. This is an integer solution and we obtain the lower bound 1 for the entire problem.

For (S2), using the sketch, the optimal solution must satisfy $x_1 = 3$ and $66 - 6x_2 = 55$. This gives $(3, \frac{11}{6})^\top$ with optimal value $\frac{18-11}{6} = \frac{7}{6} < 2$.

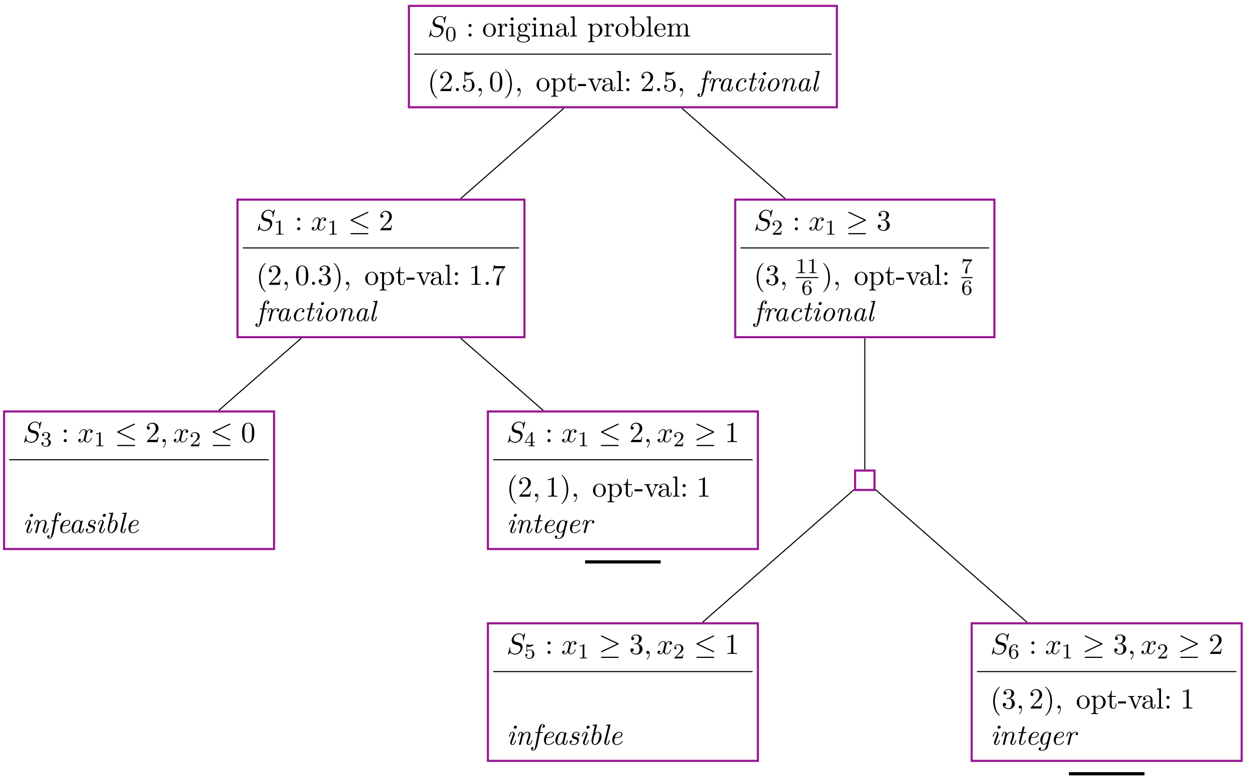
We can infer that the branch (S2) does not contain an optimal integer solution with function value greater than 1. We can stop here with the optimal solution $x^* = (2,1)^\top$.

(We can continue branching:

- (S5): $x_2 \leq 1$.
- (S6): $x_2 \geq 2$.

Problem (S5) is infeasible. For (S6), using the sketch, the optimal solution must satisfy $x_1 = 3$ and $x_2 = 2$. This is an integer solution with optimal value 1. Hence, the problem has the two global solutions $(2,1)^\top$ and $(3,2)^\top$.)

A complete picture of the procedure given by:



Exercise 5 (Algorithms for Unconstrained Problems): (*15* points)
We consider the unconstrained minimization problem

$$\min_{x\in\mathbb{R}^3} f(x) := \frac{1}{2}x_1^4 + (x_1^2 - 1)x_2^2 + 2x_3 + x_3^2.$$

The gradient and Hessian of f are given by (*you don't need to verify this)*:

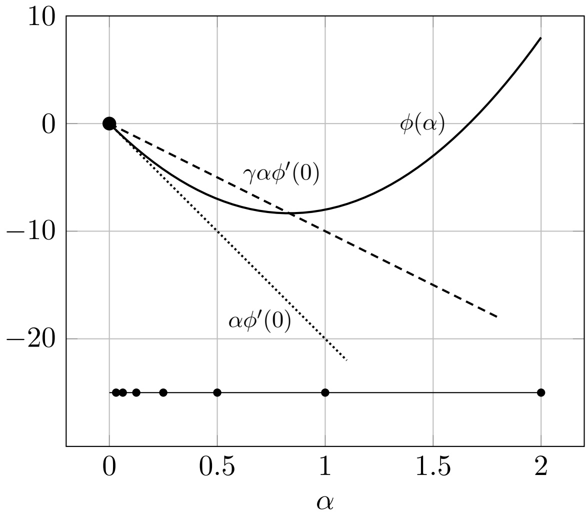
$$\nabla f(x) = \begin{pmatrix} 2x_1^3 + 2x_1x_2^2 \\ 2(x_1^2 - 1)x_2 \\ 2 + 2x_3 \end{pmatrix}, \quad \nabla^2 f(x) = \begin{pmatrix} 6x_1^2 + 2x_2^2 & 4x_1x_2 & 0 \\ 4x_1x_2 & 2(x_1^2 - 1) & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We want to apply Newton's method and the gradient descent method with backtracking to solve the problem $\min_x f(x)$. We choose the initial point x^0 and the Armijo parameter as follows:

$$x^0 = (0, 1, 1)^\top, \quad \gamma = 0.5, \quad \sigma = 0.5.$$

a) Compute the Newton direction $d_{\mathtt{n}}^0$ and the step $x_{\mathtt{n}}^1 = x^0 + d_{\mathtt{n}}^0$. Is $d_{\mathtt{n}}^0$ is a descent direction of f at x^0?

b) We now choose $d_{\mathtt{g}}^0 = -\nabla f(x^0)$ and set $\phi(\alpha) := f(x^0 + \alpha d_{\mathtt{g}}^0) - f(x^0)$. Compute the gradient iterate $x_{\mathtt{g}}^1$ and the stepsize α_0 using backtracking and the following plot:



c) Is the function f coercive?

Solution :

a) The Newton direction is given by

$$d_{\mathtt{n}}^0 = -\nabla^2 f(x^0)^{-1}\nabla f(x^0) = -\begin{pmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Hence, the Newton step is $x^1_{\mathtt{n}} = (0, 0, -1)^\top$. Moreover, we have $\nabla f(x^0)^\top d^0_{\mathtt{n}} = -(8-2) < 0$ which shows that $d^0_{\mathtt{n}}$ is a descent direction.

b) The Armijo condition is satisfied whenever $\phi(\alpha) \le \gamma\alpha\phi'(0)$. Since $\sigma = \frac{1}{2}$, this is the case for $\alpha_0 = \frac{1}{2}$. We then obtain $x^1_{\mathtt{g}} = x^0 + \alpha_0 d^0_{\mathtt{g}} = (0, 2, -1)^\top$.

c) Consider the family of points $x_\alpha = (0, \alpha, 0)^\top$ with $\alpha \in \mathbb{R}$. It follows $f(x_\alpha) = -\alpha^2$ and hence, we obtain $\|x_\alpha\| \to \infty$ and $f(x_\alpha) \to -\infty$ as $\alpha \to \infty$. This implies that f is not coercive.

Exercise 6 (True & False): (*15* points)

State whether each of the following statements are *True* or *False*. For each part, only your answer, which should be either *True* or *False*, will be graded. Explanations will not be read.

a) We consider a nonlinear program with $f(x) = x_1^2 + x_2^2 - 2x_3^4$, $g(x) = \|x\|^2 - 5$, and $h(x) = x_1^2 + x_2 + 2x_3$. (The feasible set is given by $\Omega = \{x : g(x) \le 0,\ h(x) = 0\}$). This problem possesses a global solution.

b) Consider the nonlinear program $\min_{x\in\Omega}\ f(x)$ with linear inequality constraints $\Omega := \{x : Ax \le b\}$. Let x^* be a local solution of this problem, then x^* satisfies the KKT conditions.

c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let us apply the gradient descent method with backtracking to solve $\min_x\ f(x)$. Suppose the method stops at iteration k with $\nabla f(x^k) = 0$. Then, x^k is a global minimizer of f.

d) Consider an integer optimization problem and its LP relaxation. If the integer program is feasible, i.e., the feasible set is nonempty, then the LP relaxation also must be feasible.

e) Consider the integer program

$$\min_x\ c^\top x \quad \text{s.t.} \quad Ax = b, \quad x \ge 0, \quad x \in \mathbb{Z}^n, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ is a totally unimodular matrix and $b \in \mathbb{Z}^m$ is a given integer vector. We apply the interior-point method to solve the associated LP relaxation of problem (1). Then, the method is guaranteed to return an integer solution.

Solution :

a) True.

b) True.

c) True.

d) True.

e) False.