

# MAT1001 Midterm Examination

Saturday, October 30, 2021

Time: 9:30 - 11:30 AM

## Notes and Instructions

1. *No books, no notes, no dictionaries, and no calculators.*
2. *The total score of this examination is 140.*
3. *There are **11** questions (with parts) in total.*
4. *The symbol  $[N]$  at the beginning of a question indicates that the question is worth  $N$  points.*
5. *Answer all questions on the **answer book**.*
6. *Show your intermediate steps **except Questions 1, 2 and 3** — answers without intermediate steps will receive minimal (or even no) marks.*



# MAT1001 Midterm Questions

1. [15] Multiple Choice. No explanation is required.

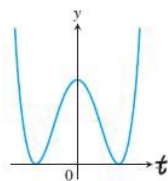
(i)  $\lim_{x \rightarrow 1} 2^{\frac{3}{x-1}} = \underline{\hspace{2cm}}.$

- A) 0
- B) 1
- C)  $\infty$
- D) None of the above

(i)  $\lim_{x \rightarrow \infty} \frac{4x^2 - 3x + 2}{x^4 - 2x^2 + x - 5} = \underline{\hspace{2cm}}.$

- A) 0
- B) 4
- C)  $\infty$
- D)  $-\infty$

(iii) Given the graph of the velocity of a particle moving along a horizontal line, which of the following could be the position function  $s = f(t)$  for the particle?



- A)  $s = \sin |t|$
- B)  $s = \frac{t^4}{4} - 2t^2 + 4$
- C)  $s = \frac{3}{4}(t^2 - 1)^{\frac{2}{3}}$
- D)  $s = \frac{1}{5}t^5 - \frac{8}{3}t^3 + 16t$

(iv) For  $x \neq 0$ , we have  $\frac{d}{dx} \left( \sqrt{|x|} \right) = \underline{\hspace{2cm}}.$

- A)  $\frac{1}{2\sqrt{|x|}}$
- B)  $\frac{x}{2\sqrt{|x|}}$
- C)  $\frac{\sqrt{|x|}}{2x}$
- D)  $\frac{\sqrt{|x|}}{-2x}$

(v) What is the normal line of  $y = \sqrt{1-x}$  through the point  $(-3, 2)$ ?

- A)  $x + 4y - 5 = 0$
- B)  $4x - y - 14 = 0$
- C)  $x + 4y + 5 = 0$
- D)  $4x - y + 14 = 0$

2. [10] True or False (in general)? No explanation is required.

- (i) Let  $f, g$  be functions defined for all real numbers and both not continuous at  $x = 0$ . Then  $f + g$  must be non-differentiable at  $x = 0$ .
- (ii) Let  $y = f(x)$  be defined for all real numbers such that the left-hand derivative and the right-hand derivative at  $x = a$  both exist. Then  $y = f(x)$  is differentiable at  $x = a$ .
- (iii) Let  $f$  and  $g$  be functions having the same domain  $D$ . If  $f'(x) = g'(x)$  for all  $x \in D$ , then there exists a constant  $C$  such that  $f(x) = g(x) + C$  for all  $x \in D$ .
- (iv) If  $\lim_{x \rightarrow a} (f(x) - g(x))$  exists and  $\lim_{x \rightarrow a} f(x)$  exists, then  $\lim_{x \rightarrow a} g(x)$  exists.
- (v) Let  $f$  be differentiable on the interval  $(0, 6)$ . Then  $\lim_{x \rightarrow 3} f'(x)$  must exist.

3. [21] Short questions: no explanation is required.

- (i) Find the values of  $a$  and  $b$  that make  $f$  continuous on  $\mathbb{R}$ , where

$$f(x) = \begin{cases} x + 2, & \text{if } x < 2 \\ ax^2 - bx + 3, & \text{if } 2 \leq x < 3 \\ 2x - a + b, & \text{if } x \geq 3 \end{cases}.$$

- (ii) If  $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$ , find  $\lim_{x \rightarrow 2} f(x)$ .
- (iii) Let  $f(x) = x^3 + 2x - 4$ . Starting with  $x_0 = 2$ , find  $x_1$  using Newton's method.
- (iv) Estimate the area under the curve  $y = x^2 + (1/x)$  for  $x$  from 1 to 5 using the midpoint sum  $S$  with two subintervals.

- (v) In (iv) above, is  $S$  greater than, smaller than, or equal to the exact area under the curve?
- (vi) Find the function  $y = f(x)$  that satisfies  $y' = \sqrt{x} + \frac{2}{x^4} - \sin(\pi x)$  and  $y(1) = \pi$ .
- (vii) Let  $f$  and  $g$  be differentiable functions such that

$$f(3) = 3, \quad g(3) = -4, \quad f'(3) = 2\pi \quad \text{and} \quad g'(3) = 5.$$

Find the derivative of  $\sqrt{(f(x))^2 + (g(x))^2} + 5^\pi$  at  $x = 3$ .

4. [24] Evaluate the following limits, or explain why they do not exist.

- (i)  $\lim_{x \rightarrow -6} \frac{2x + 12}{|x + 6|}$
- (ii)  $\lim_{x \rightarrow \infty} \sqrt{4x^2 + 3x} + 2x$
- (iii)  $\lim_{x \rightarrow 0} \frac{|2x - 1| - |2x + 1|}{x}$
- (iv)  $\lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{x}$

5. [6] A curve is given by  $y^3 - 4 \sin(xy) = 8$ . Find the tangent line to the curve at  $x = 0$ .

6. [2+6+6=14] Consider the function rule  $f(x) = 3x^2 + \frac{x^2 - 1}{(x - 1)(x - \sin x)}$ .

- (i) Find its natural domain  $D$  (that is, the biggest domain in  $\mathbb{R}$ ).
- (ii) Extend the function to have domain  $\mathbb{R}$  by giving the function some values at the points missing in  $D$  (in part (i)). At which of these points can the function be extended continuously?
- (iii) Find all asymptotes (horizontal/vertical/oblique) for  $y = f(x)$ .

7. [6+6+4+4=20] Consider the function  $f(x) = x^{2/3}(6-x)^{1/3}$  defined on  $\mathbb{R}$ .

- (i) Find all intervals on which the function is increasing/decreasing.
- (ii) Find all intervals on which the function is concave up/concave down.
- (iii) Find all inflection points. (State their  $x$ -coordinates.)
- (iv) Find all local extrema and global extrema by stating their  $x$ -coordinates, or explain why they do not exist.

8. [6] A projectile is fired from a canon over horizontal ground and lands a distance  $s$  away from the canon, where  $s$  is given by the equation

$$s = \frac{v_0^2}{9.8} \sin 2\alpha,$$

where  $v_0$  is the initial velocity of the projectile when it is fired, and  $\alpha$  is the angle to the horizontal at which it is fired. At what angle should the canon be fired to maximize the distance travelled by the projectile?

9. [8] Let

$$f(x) = \begin{cases} x \cos \frac{\pi}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Show that  $f(x)$  is continuous but not differentiable at  $x = 0$ .

10. [8] Prove that  $\sin x < x$  for all  $x \in (0, 2\pi]$ .

11. [8] A Ferris wheel with radius 10 m is rotating at a rate of one cycle every 2 minutes. During the rising process, how fast is the rider rising when her seat is 16 m above the ground? You may assume that the base of the wheel is just touching the ground level.