MAT1012

Tutorial 9

HUANG Zhirui

CUHK(SZ)

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Outline

Midterm Exam

Question 1 (25 marks). Let

$$\mathbf{u}(t) = t\mathbf{i} + t\mathbf{j}, \qquad \mathbf{v}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k},$$
$$\mathbf{w}(t) = -t\mathbf{i} + (\cos t)\mathbf{k}, \quad -\infty < t < \infty.$$

Find

$$\frac{d}{dt}[(\mathbf{u}(t) \times \mathbf{v}(t)) \cdot \mathbf{w}(t)] \qquad \text{and} \qquad \int_0^1 \mathbf{u}(t) \times \mathbf{v}(t)\, dt.$$

Midterm Exam

Solution.

$$\mathbf{u}(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t & 0 \\ 1 & 1 & t \end{vmatrix} = t^2\mathbf{i} - t^2\mathbf{j}.$$

$$(\mathbf{u}(t) \times \mathbf{v}(t)) \cdot \mathbf{w}(t) = (t^2\mathbf{i} - t^2\mathbf{j}) \cdot (-t\mathbf{i} + (\cos t)\mathbf{k}) = -t^3.$$

Hence,

$$\frac{d}{dt}[(\mathbf{u}(t) \times \mathbf{v}(t)) \cdot \mathbf{w}(t)] = \frac{d}{dt}(-t^3) = -3t^2,$$

$$\int_0^1 \mathbf{u}(t) \times \mathbf{v}(t) dt = \int_0^1 (t^2\mathbf{i} - t^2\mathbf{j}) dt = \left(\frac{t^3}{3}\mathbf{i} - \frac{t^3}{3}\mathbf{j} \right) \Bigg|_0^1 = \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j}.$$

$\square$

Midterm Exam

Question 2 (30 marks). Let C be a curve given by

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad -\infty < t < \infty.$$

Find the unit tangent vector $\mathbf{T}$ and the curvature κ of the curve C.

Midterm Exam

Solution.

The curve C can be represented by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad -\infty < t < \infty.$$

Hence,

$$\begin{aligned}
\mathbf{r}'(t) &= -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}, \\
|\mathbf{r}'(t)| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}, \\
\mathbf{T} &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{\sin t}{\sqrt{2}}\mathbf{i} + \frac{\cos t}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k},
\end{aligned}$$

Midterm Exam

Solution.

and

$$\frac{d\mathbf{T}}{dt} = -\frac{\cos t}{\sqrt{2}}\mathbf{i} - \frac{\sin t}{\sqrt{2}}\mathbf{j}, \qquad \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{2}},$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{\left|\frac{d\mathbf{T}}{dt}\right|}{|\mathbf{r}'(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}.$$

□

Midterm Exam

Question 3 (20 marks). Find the equations of the tangent line and normal plane of the curve

$$\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + \sin t\mathbf{j} + e^t\mathbf{k}, \quad -\infty < t < \infty,$$

at P_0 corresponding to $\mathbf{r}(0)$.

Midterm Exam

Solution.

Since P_0 corresponds to

$$\mathbf{r}(0) = (1 + \cos 0)\mathbf{i} + (\sin 0)\mathbf{j} + e^0\mathbf{k} = 2\mathbf{i} + \mathbf{k},$$

then $P_0 = (2, 0, 1)$. Since

$$\mathbf{r}'(0) = (-\sin 0)\mathbf{i} + (\cos 0)\mathbf{j} + e^0\mathbf{k} = \mathbf{j} + \mathbf{k},$$

the tangent line at $P_0(2, 0, 1)$ has equations

$$\frac{x-2}{0} = \frac{y-0}{1} = \frac{z-1}{1}, \quad \text{i.e.} \quad x = 2, \quad y = z - 1.$$

Midterm Exam

Solution.

The normal plane at $P_0(2, 0, 1)$ is

$$0 \cdot (x-2) + 1 \cdot (y-0) + 1 \cdot (z-1) = 0, \quad \text{i.e.} \quad y + z = 1.$$

$\square$

Midterm Exam

Question 4 (30 marks). Let

$$f(x,y)=\begin{cases}\frac{\sqrt[3]{xy^2}}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

(a) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$. (16 marks)

(b) Is $f(x,y)$ continuous at $(0,0)$? Prove your conclusion. (14 marks)

Midterm Exam

Solution.

(a) $f(0,0)=0$. If $(x,y)\neq(0,0)$,

$$f(x,y)=\frac{\sqrt[3]{xy^2}}{\sqrt{x^2+y^2}}.$$

$f(x,0)=0$ for all x, hence

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0.$$

$f(0,y)=0$ for all y, hence

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=0.$$

Midterm Exam

Solution.

(b) f is not continuous at $(0,0)$. In fact, for any $k>0$, along the ray $y=kx$, we have

$$\lim_{\substack{(x,y)\to(0,0),\\ y=kx, x>0}} f(x,y) = \lim_{x\to 0^+} \frac{\sqrt[3]{k^2x^3}}{\sqrt{x^2+k^2x^2}} = \frac{k^{2/3}}{\sqrt{1+k^2}},$$

which depends on k. So the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus f is not continuous at $(0,0)$. $\square$

Midterm Exam

Question 5 (30 marks). Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Find $f_x(0,0)$, $f_y(0,0)$, $f_{xx}(0,0)$ and $f_{yx}(0,0)$. Here $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)$. (18 marks)

(b) Is f differentiable at $(0,0)$? Prove your conclusion. (12 marks)

Midterm Exam

Solution.

(a) $f(0,0)=0$. If $(x,y)\neq(0,0)$,

$$f(x,y)=\frac{x^2y}{x^2+y^2}.$$

Then

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0,$$
$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=0.$$

If $(x,y)\neq(0,0)$, we have

$$f_x(x,y)=\frac{(x^2+y^2)2xy-x^2y\cdot 2x}{(x^2+y^2)^2}=\frac{2xy^3}{(x^2+y^2)^2},$$

Midterm Exam

Solution.

$$f_y(x,y) = \frac{(x^2+y^2)x^2 - x^2y \cdot 2y}{(x^2+y^2)^2} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}.$$

Hence,

$$f_x(0,y) = 0, \quad f_x(x,0) = 0.$$

Thus,

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h} = 0,$$

$$f_{yx}(0,0) = \frac{\partial}{\partial y} f_x(0,y)\Big|_{y=0} = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = 0.$$

Midterm Exam

Solution.

(b) The function f is not differentiable at $(0,0)$. Prove by contradiction. Suppose f is differentiable at $(0,0)$. Since $f_x(0,0) = f_y(0,0) = 0$, we have

$$f(x,y) = \varepsilon_1 x + \varepsilon_2 y,$$

here $\varepsilon_1, \varepsilon_2 \to 0$ as $x \to 0, y \to 0$. In particular, if $y = x$, we have

$$\frac{x}{2} = f(x,x) = \varepsilon_1 x + \varepsilon_2 x \quad \text{as } x \to 0,$$

hence

$$\frac{1}{2} = \varepsilon_1 + \varepsilon_2 \to 0 \quad \text{as } x \to 0,$$

which is a contradiction. Thus, f is not differentiable at $(0,0)$. □

Midterm Exam

Question 6 (30 marks). Find the radius and set of convergence of the given power series.

$$(a) \quad \sum_{n=1}^{\infty} \left(\sin \frac{1}{n}\right) x^n,$$

(15 marks)

$$(b) \quad \sum_{n=1}^{\infty} n^n x^{n(n+1)}.$$

(15 marks)

Midterm Exam

Solution.

(a) Denote $a_n = \sin \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\sin \frac{1}{n+1}} = 1.$$

Hence $R = 1$. So the series converges at x if $|x| < 1$.
When $x = 1$, the series is $\sum_{n=1}^{\infty} \sin \frac{1}{n}$, which diverges comparing with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$.
When $x = -1$, the series is $\sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{n}$, which converges by the alternating series test, because $\sin \frac{1}{n}$ decreases to 0 as $n \to \infty$.
Thus, the set of convergence is $[-1, 1)$.

Midterm Exam

Solution.

(b) Write the series as $\sum_{k=1}^{\infty} a_k x^k$, where

$$a_k = \begin{cases} n^n, & k = n(n+1), \\ 0, & n(n+1) < k < (n+1)(n+2). \end{cases}$$

Then

$$\limsup_{k\to\infty} |a_k|^{1/k} = \lim_{n\to\infty} |a_{n(n+1)}|^{\frac{1}{n(n+1)}} = \lim_{n\to\infty} (n^n)^{\frac{1}{n(n+1)}} = 1.$$

Midterm Exam

Solution.

The radius of convergence

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = 1.$$

So the series converges at x if $|x| < 1$. If $x = \pm 1$, the series is $\sum_{n=1}^{\infty} n^n$ which diverges since $n^n \to \infty$ as $n \to \infty$. Thus, the set of convergence is the open interval $(-1, 1)$. $\square$

Midterm Exam

Question 7 (15 marks). Denote by D the set of the points at which the following series of functions converges:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^x}. \tag{$*$}$$

(a) Find the set D. (5 marks)

(b) Show that the series of functions uniformly converges on any closed subinterval of D. (5 marks)

(c) Show that the series of functions does not uniformly converge on the set D. (5 marks)

Midterm Exam

Solution.

(a) Denote $u_n(x) = \frac{(x-1)^n}{n^x}$.

Case 1. If $x = 1$, the series converges since $u_n(1) = 0$ for each n.

Case 2. If $x \neq 1$,

$$\lim_{n\to\infty} \frac{|u_{n+1}(x)|}{|u_n(x)|} = \lim_{n\to\infty} \frac{\frac{|x-1|^{n+1}}{(n+1)^x}}{\frac{|x-1|^n}{n^x}} = \lim_{n\to\infty} |x-1| \left(\frac{n}{n+1}\right)^x = |x-1|.$$

By the ratio test, the series converges when $0 < |x-1| < 1$, and diverges when $|x-1| > 1$. If $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges. If $x = 0$, the series is $\sum_{n=1}^{\infty}(-1)^n$, which diverges by the nth term test. Thus, the set of convergence is $D = (0, 2]$.

Midterm Exam

Solution.

(b) We show that for any $0 < c < 1$, the series of functions converges uniformly on $[c, 2]$.

On the set $c \leq x \leq 2 - c$,

$$|u_n(x)| = \frac{|x-1|^n}{n^x} \leq |x-1|^n \leq (1-c)^n,$$

and $\sum_{n=1}^{\infty}(1-c)^n$ converges because $0 < 1 - c < 1$. By the M-test, the series of functions uniformly converges on $[c, 2-c]$.

Midterm Exam

Solution.

On the set $2-c \le x \le 2$,

$$|u_n(x)| = \frac{|x-1|^n}{n^x} \le \frac{1}{n^{2-c}}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^{2-c}}$ converges because $2-c>1$. By the M-test, the series of functions uniformly converges on $[2-c, 2]$.

Thus, the series of functions uniformly converges on $[c, 2]$.

Midterm Exam

Solution.

(c) Now we show that the series does not uniformly converge on D.

Prove by contradiction. Suppose the series uniformly converges on D.

Then for

$$\varepsilon = \frac{1}{2\sqrt{e}}$$

there exists an integer $N > 1$ such that

$$\left|\sum_{n=p}^{m} \frac{(x-1)^n}{n^x}\right| < \frac{1}{2\sqrt{e}}, \quad \forall x \in D, \quad \forall m > p \geq N.$$

Midterm Exam

Solution.

In particular,

$$\left|\frac{(x-1)^m}{m^x}\right| < \frac{1}{2\sqrt{e}}, \quad \forall x \in D, \quad m \geq N. \tag{2}$$

Take $m > N$ large so that

$$\frac{m}{m^2-1} < \frac{1}{2}, \qquad m^{\frac{1}{m^2}} < 2.$$

Set

$$x_m = \frac{1}{m^2}.$$

Midterm Exam

Solution.

Recall that

$$\frac{-x}{1-x} < \ln(1-x) < -x, \quad x < 1, \quad x \neq 0.$$

So

$$\ln\left(1-\frac{1}{m^2}\right) > \frac{-\frac{1}{m^2}}{1-\frac{1}{m^2}} = -\frac{1}{m^2-1},$$

$$m\ln\left(1-\frac{1}{m^2}\right) > -\frac{m}{m^2-1} > -\frac{1}{2},$$

$$\left(1-\frac{1}{m^2}\right)^m = e^{m\ln\left(1-\frac{1}{m^2}\right)} > e^{-\frac{m}{m^2-1}} > e^{-\frac{1}{2}}.$$

Midterm Exam

Solution.

From (2), we have

$$\frac{1}{2\sqrt{e}} > \left|\frac{(x_m - 1)^m}{m^{x_m}}\right| = \frac{(1 - \frac{1}{m^2})^m}{m^{\frac{1}{m^2}}} > \frac{e^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{e}}.$$

This contradiction shows that (2) can not hold. Thus, the series does not converge uiformly on D. $\square$

Midterm Exam

Question 8 (10 marks). Let $f(x, y)$ be a function defined in an open disc containing $P_0(0,0)$. Assume f satisfies the following conditions.

(i) For any positive integer n and any constant $a > 0$, the composite function

$$\phi(x) = f(x, ax^n)$$

has derivative at $x = 0$ and $\phi'(0) = 0$.

(ii) For any positive integer m and any constant $b > 0$, the composite function

$$\psi(y) = f(by^m, y)$$

has derivative at $y = 0$ and $\psi'(0) = 0$.

Midterm Exam

Can we conclude that f is continuous at P_0? If your answer is "Yes" then give a proof; if your answer is "No", then give a counterexample.

Midterm Exam

Solution.

The answer is "NO". Counterexample: Let

$$f(x,y)=\begin{cases}1, & \text{if } x\neq 0, \quad 0<y<e^{-\frac{1}{x^2}},\\ 0, & \text{otherwise.}\end{cases}$$

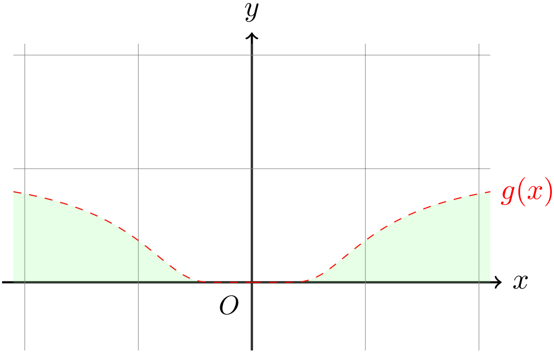
First, we shall show that f satisfies the conditions. Denote

$$g(x)=\begin{cases}e^{-\frac{1}{x^2}}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

Then $g^{(k)}(x)$ is continuous at $x=0$, and $g^{(k)}(0)=0$ for all integer $k\geq 0$.

Midterm Exam

The illustration (sketch).



$$f(x,y) = \begin{cases} 1, & \text{if } x \neq 0, \quad 0 < y < e^{-\frac{1}{x^2}}, \\ 0, & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Midterm Exam

Solution.

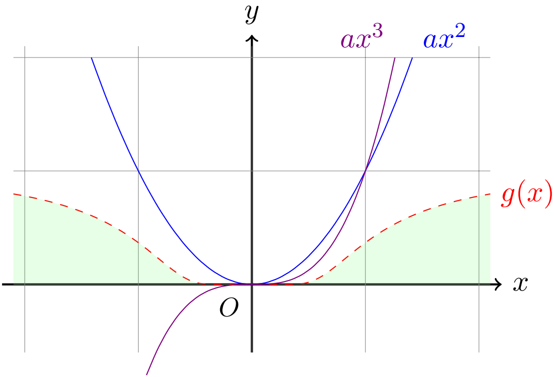
Step 1. Let $n > 0$ be an integer and $a > 0$, $u(x) = ax^n$. We have

$$\lim_{x\to 0} \frac{g(x)}{u(x)} = \frac{1}{a} \lim_{x\to 0} \frac{g(x)}{x^n} = \frac{1}{a} \lim_{x\to 0} \frac{g'(x)}{nx^{n-1}} = \cdots = \frac{1}{a} \lim_{x\to 0} \frac{g^{(n)}(x)}{n!} = 0.$$

Hence there exists $\delta_1 > 0$ depending on n and a such that $g(x) < |u(x)|$ if $|x| < \delta_1$. Then for $|x| < \delta_1$, the inequality $0 < u(x) < g(x)$ can not hold. Let $\phi(x) = f(x, u(x))$, then $\phi(x) = f(x, u(x)) = 0$ for $|x| < \delta_1$. Thus the composite function $\phi(x)$ has derivative at $x = 0$ and $\phi'(0) = 0$. So f satisfies the condition (i).

Midterm Exam

The illustration (sketch).



$$u(x) = ax^n, \quad a > 0$$

Midterm Exam

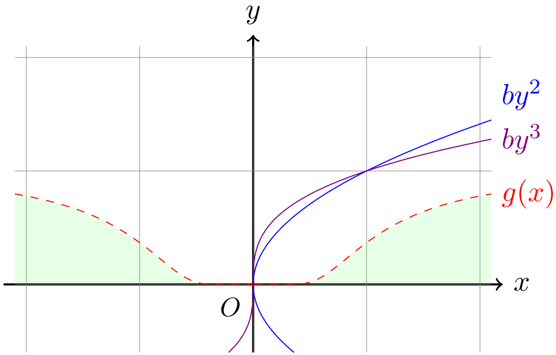
Solution.

Step 2. Let $m > 0$ be an integer and $b > 0$, $v(y) = by^m$. Then there exists $\delta_2 > 0$ such that the piece of the curve $x = by^m$, $|y| < \delta_2$, does not lie between the curve $y = g(x)$ and the x-axis. Thus, for $|y| < \delta_2$ the inequality $0 < y < g(x)$ can not hold. Let $\psi(y) = f(v(y), y)$, then $\psi(y) = f(v(y), y) = 0$. Thus the composite function $\psi(y)$ has derivative at $y = 0$ and $\psi'(0) = 0$. So f satisfies the condition (ii).

Now we show f is not continuous at $(0, 0)$. We only need to show that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Midterm Exam

The illustration (sketch).



$$v(y) = by^m, \quad b > 0$$

Midterm Exam

Solution.

Step 3. Obviously for any $k > 0$, along the ray $y = kx$, we have

$$\lim_{\substack{(x,y)\to(0,0),\\ y=kx}} f(x,y) = 0.$$

However, along the curve $y = \frac{1}{2}e^{-\frac{1}{x^2}}$, we have

$$\lim_{\substack{(x,y)\to(0,0),\\ y=\frac{1}{2}e^{-\frac{1}{x^2}},\ x>0}} f(x,y) = 1.$$

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus, we conclude that f is not continuous at $(0,0)$. $\square$

Midterm Exam

Remark.

For $a > 0$ and $b > 0$, the function

$$f(x,y) = \begin{cases} 1, & \text{if } xy = 0 \text{ and } (x,y) \neq (0,0), \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

is also a correct counterexample.

However, if we modify the conditions of constants a and b to $a \geq 0$ and $b \geq 0$, then the function (1) is not a correct counterexample.

The counterexample in our solution is still correct.

Midterm Exam

Question 9 (10 marks). Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions defined on the interval $[0,1]$. Assume that

(i) For every $n \geq 1$, $f_n(x)$ is continuous and monotone on $[0,1]$.

(ii) Both series $\sum_{n=1}^{\infty} f_n(0)$ and $\sum_{n=1}^{\infty} f_n(1)$ absolutely converge.

Prove that both series $\sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} \int_0^x f_n(t)dt$ uniformly converge on $[0,1]$, and

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} \int_0^x f_n(t)dt\right) = \sum_{n=1}^{\infty} f_n(x), \qquad x \in (0,1).$$

Midterm Exam

Proof.

Step 1. By the condition (i), for each $n \geq 1$, $f_n(x)$ is monotone on $[0,1]$, hence either $f_n(0) \leq f_n(x) \leq f_n(1)$ for all $x \in [0,1]$, or $f_n(1) \leq f_n(x) \leq f_n(0)$ for all $x \in [0,1]$. So we have

$$|f_n(x)| \leq \max\{|f_n(0)|,\ |f_n(1)|\} \leq |f_n(0)| + |f_n(1)|, \quad \forall x \in [0,1],\ \ n \geq 1.$$

By the condition (ii), both $\sum_{n=1}^{\infty} |f_n(0)|$ and $\sum_{n=1}^{\infty} |f_n(1)|$ converge, hence $\sum_{n=1}^{\infty}(|f_n(0)| + |f_n(1)|)$ converges. Taking $\sum_{n=1}^{\infty}(|f_n(0)| + |f_n(1)|)$ as an M-series, by the M-test, we see that the series $\sum_{n=1}^{\infty} |f_n(x)|$ uniformly converges on $[0,1]$, hence $\sum_{n=1}^{\infty} f_n(x)$ uniformly converges on $[0,1]$.

Midterm Exam

Proof.

Step 2. Denote the sum function by $f(x)$, namely,

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in [0,1]. \tag{1}$$

By the condition (i) that $f_n(x)$ is continuous for each n, then $f_n(x)$ is integrable on $[0,1]$. Denote

$$u_n(x) = \int_0^x f_n(t)dt, \quad \forall n \geq 1.$$

Midterm Exam

Proof.

Since the series $\sum_{n=1}^{\infty} f_n(x)$ uniformly converges on $[0,1]$ and and each $f_n(x)$ is continuous on $[0,1]$, by Theorem 4.2.28 (Term-by-term integration theorem) in Lecture Notes, for any $0 < x \leq 1$, we can integrate term-by term to get,

$$\int_0^x f(t)dt = \int_0^x \left(\sum_{n=1}^{\infty} f_n(t) \right) dt = \sum_{n=1}^{\infty} \int_0^x f_n(t)dt = \sum_{n=1}^{\infty} u_n(x),$$

and the series

$$\sum_{n=1}^{\infty} u_n(x)$$

uniformly converges on $[0,1]$.

Midterm Exam

Proof.

Step 3. By the condition (i) that $f_n(x)$ is continuous for each n, then $u_n(x)$ is continuously differentiable on $(0,1)$, and $u_n'(x) = f_n(x)$ for all $x \in [0,1]$ and $n \geq 1$ (understood as one-side derivatives at $x = 0, 1$). Hence,

$$\sum_{n=1}^{\infty} u_n'(x) = \sum_{n=1}^{\infty} f_n(x)$$

uniformly converges on $[0,1]$. Denote the sum function of this series by $g(x)$:

$$g(x) = \sum_{n=1}^{\infty} u_n(x), \quad 0 \leq x \leq 1.$$

Midterm Exam

Proof.

Since the series $\sum_{n=1}^{\infty} u_n(x)$ converges on $[0,1]$, and each $u_n(x)$ is continuously differentiable with the derived series $\sum_{n=1}^{\infty} u_n'(x)$ uniformly converges on $[0,1]$, by Theorem 4.2.29 (Term-by-term differentiation theorem) in Lecture Notes, we can differentiate term by term to get

$$g'(x) = \frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u_n'(x), \quad x \in (0,1),$$

which can be written as

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty}\int_0^x f_n(t)dt\right) = \sum_{n=1}^{\infty} f_n(x), \qquad x \in (0,1).$$

This is exactly wheat we need to prove. □

Tutorial Exercises

Ex.1. Show that the function

$$f(x,y)=\begin{cases}\frac{2xy^3}{x^2+y^4}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

is continuous and has directional derivative along any direction at $(0,0)$, but not differentiable at $(0,0)$.

Tutorial Exercises

Solution.

$$\left|\frac{2xy^3}{x^2+y^4}\right| = \left|\frac{2xy^2}{x^2+y^4}y\right| \leq \left|\frac{x^2+y^4}{x^2+y^4}y\right| = |y|,$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2+y^4} = 0 = f(0,0).$$

Hence, $f(x,y)$ is continuous at $(0,0)$. The directional derivative at $(0,0)$ for any unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ with $0 \leq \theta \leq 2\pi$ is given by

$$\left.\frac{\partial f}{\partial \mathbf{u}}\right|_{(0,0)} = \lim_{s\to 0^+} \frac{f(s\cos\theta, s\sin\theta) - f(0,0)}{s}$$

$$= \lim_{s\to 0^+} \frac{2s^4\cos\theta\sin^3\theta}{s(s^2\cos^2\theta + s^4\sin^4\theta)} = \lim_{s\to 0^+} \frac{2s\cos\theta\sin^3\theta}{\cos^2\theta + s^2\sin^4\theta} = 0.$$

Tutorial Exercises

Solution.

Hence, $f_x(0,0) = f_y(0,0) = 0$. Denote $x = \Delta x$, $y = \Delta y$, and $\rho = \sqrt{\Delta x^2 + \Delta y^2}$. Suppose $f(x, y)$ is differentiable at $(0, 0)$, then

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y = f(x,y) = o(\rho).$$

However, consider the limit along the curve $x = y^2$ with $y > 0$,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2, y>0}} \frac{f(x,y)}{\rho} = \lim_{\substack{(x,y)\to(0,0)\\ x=y^2, y>0}} \frac{\frac{2xy^3}{x^2+y^4}}{\sqrt{x^2+y^2}} = \lim_{y\to 0^+} \frac{\frac{2y^5}{y^4+y^4}}{\sqrt{y^4+y^2}}$$
$$= \lim_{y\to 0^+} \frac{1}{\sqrt{1+y^2}} = 1 \neq 0$$

which is a contradiction. Thus, $f(x, y)$ is not differentiable at $(0, 0)$ □

Tutorial Exercises

Ex.2. Let

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$

where $a > b > 0$.

At any point $P(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$, point out what direction f increases fastest.

Tutorial Exercises

Solution.

For the function

$$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad a > b > 0,$$

we have

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{b^2}.$$

Since $f(x,y)$ has all continuous derivative at any point $P(x,y) \in \mathbb{R}^2$, $f(x,y)$ is differentiable at any point P.

Tutorial Exercises

Solution.

Hence, for any unit vector $\mathbf{u}$, the directional derivative $\left.\frac{\partial f}{\partial \mathbf{u}}\right|_P$ exists and

$$\left.\frac{\partial f}{\partial \mathbf{u}}\right|_P = (\nabla f)_P \cdot \mathbf{u} = |(\nabla f)_P| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $(\nabla f)_P$. When the gradient $(\nabla f)_P$ is not zero, the direction of the gradient $(\nabla f)_P$ is the direction along which f increases the fastest. Hence, at any point $P(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$, the direction of the gradient

$$(\nabla f)_P = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j}$$

is the expected direction. □

Tutorial Exercises

Remark.

At the origin $(0,0)$, the gradient $(\nabla f)_{(0,0)}$ is zero. We need to use another method to find the expected direction.

Tutorial Exercises

Ex.3. (a). Let

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0. \end{cases}$$

Find the directional derivative $\frac{\partial f}{\partial \mathbf{u}}$ at $P_0(0,0)$ for any unit vector $\mathbf{u}$.
(b). Summarize the steps of finding the directional derivative $\frac{\partial f}{\partial \mathbf{r}}$ of a function $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$ for any vector $\mathbf{r} = \langle a,b,c \rangle$.

Tutorial Exercises

Solution.

(a). Denote $\mathbf{u} = \langle \cos\alpha, \sin\alpha \rangle$. According to the definition of directional derivative, we have

$$\left.\frac{\partial f}{\partial \mathbf{u}}\right|_{P_0} = \lim_{s \to 0^+} \frac{f(s\cos\alpha, s\sin\alpha) - f(0,0)}{s} = \cos\alpha \sin\alpha.$$

Remark.

We have that $f_x(0,0) = f_y(0,0) = 0$ but $f(x,y)$ is not differentiable at $P_0(0,0)$. If we use the formular $\left.\frac{\partial f}{\partial \mathbf{u}}\right|_{P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}$ to find the directional derivative $\left.\frac{\partial f}{\partial \mathbf{u}}\right|_{P_0}$, we will get the incorrect result that $\left.\frac{\partial f}{\partial \mathbf{u}}\right|_{P_0} = 0$ for any unit vector $\mathbf{u}$.

Tutorial Exercises

Solution.

(b). For any function $f(x, y, z)$ (differentiable or non-differentiable) at P_0 for any vector $\mathbf{r}$, we can find the directional derivative $\frac{\partial f}{\partial \mathbf{r}}\Big|_{P_0}$ by the definition. For instance,

$$\frac{\partial f}{\partial \mathbf{r}}\Big|_{P_0} = \lim_{\rho \to 0^+} \frac{f(x_0 + \rho \cos\alpha, y_0 + \rho\cos\beta, z_0 + \rho\cos\gamma) - f(x_0, y_0, z_0)}{\rho}$$

where $\mathbf{u} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ is the unit vector of $\mathbf{r}$, which is given by

$$\mathbf{u} = \left\langle \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \right\rangle$$

where α, β, γ are the direction angles of the vector $\mathbf{r}$.

Tutorial Exercises

Solution.

If f is differentiable at P_0, we can find $\left.\frac{\partial f}{\partial \mathbf{r}}\right|_{P_0}$ as follows.

(i). Find the unit vector $\mathbf{u} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$.

(ii). Find the gradient $(\nabla f)_{P_0}$ which is given by

$$(\nabla f)_{P_0} = \Big\langle f_x(P_0), f_y(P_0), f_z(P_0) \Big\rangle.$$

Tutorial Exercises

Solution.

(iii). Find $\frac{\partial f}{\partial \mathbf{r}}(P_0)$ by the formular

$$\frac{\partial f}{\partial \mathbf{r}}\bigg|_{P_0} = \frac{\partial f}{\partial \mathbf{u}}\bigg|_{P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}$$

$$= f_x(P_0)\cos\alpha + f_y(P_0)\cos\beta + f_z(P_0)\cos\gamma$$

$$= \frac{af_x(P_0) + bf_y(P_0) + cf_z(P_0)}{\sqrt{a^2+b^2+c^2}}.$$

□

Tutorial Exercises

Ex.4. Let $f(x, y) = x^3 + 2x^2 - 2xy + y^2$ be a function defined on a region $D = [-2, 2] \times [-2, 2]$. Find the global maximum and the global minimum values (if any) of $f(x, y)$.

Tutorial Exercises

Solution.

Step 1. Find critical points. Solving

$$\begin{cases} f_x(x,y) = 3x^2 + 4x - 2y = 0, \\ f_y(x,y) = -2x + 2y = 0. \end{cases}$$

The critical points of f on D are $(0,0)$ and $(-\frac{2}{3}, -\frac{2}{3})$.

Step 2. Determine the classification of critical points (local maximum point, local minimum point, saddle point or inconclusive point).

Tutorial Exercises

Solution.

Consider the Hessian matrix H_f of $f(x, y)$,

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x+4 & -2 \\ -2 & 2 \end{pmatrix},$$

and its determinant

$$\det(H_f) = |H_f| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 12x + 4.$$

At the point $(0, 0)$, $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} = 4 > 0$. The point $(0, 0)$ is a local minimum point. At the point $(-\frac{2}{3}, -\frac{2}{3})$, $f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$. The point $(-\frac{2}{3}, -\frac{2}{3})$ is a saddle point.

Tutorial Exercises

Solution.

Step 3. Discusses the values of $f(x, y)$ at the boundary of $D = [-2, 2] \times [-2, 2]$.

$$\min_{y\in[-2,2]} f(2, y) = \min_{y\in[-2,2]} [(y-2)^2 + 12] = f(2, 2) = 12,$$

$$\min_{y\in[-2,2]} f(-2, y) = \min_{y\in[-2,2]} [(y+2)^2 - 4] = f(-2, -2) = -4,$$

$$\min_{x\in[-2,2]} f(x, 2) = \min_{x\in[-2,2]} [x^3 + 2x^2 - 4x + 4] = f\left(\frac{2}{3}, 2\right) = \frac{68}{27},$$

$$\min_{x\in[-2,2]} f(x, -2) = \min_{x\in[-2,2]} [x^3 + 2x^2 + 4x + 4] = f(-2, -2) = -4,$$

Tutorial Exercises

Solution.

and

$$\max_{y\in[-2,2]} f(2,y) = \max_{y\in[-2,2]}[(y-2)^2+12] = f(2,-2) = 28,$$

$$\max_{y\in[-2,2]} f(-2,y) = \max_{y\in[-2,2]}[(y+2)^2-2] = f(-2,2) = 12,$$

$$\max_{x\in[-2,2]} f(x,2) = \max_{x\in[-2,2]}[x^3+2x^2-4x+4] = f(\pm 2,2) = 12,$$

$$\max_{x\in[-2,2]} f(x,-2) = \max_{x\in[-2,2]}[x^3+2x^2+4x+4] = f(2,-2) = 28.$$

Tutorial Exercises

Solution.

Step 4. The global aximum and global minimum values are obtained by comparison. That is,

$$\begin{aligned}\max_{(x,y)\in D} f(x,y) &= \max\{f(2,-2), f(-2,2), f(2,2)\}\\ &= f(2,-2) = 28,\\ \min_{(x,y)\in D} f(x,y) &= \min\left\{f(0,0), f(2,2), f(-2,-2), f\left(\frac{2}{3},2\right)\right\}\\ &= f(-2,-2) = -4.\end{aligned}$$

$\square$

Tutorial Exercises

Ex.5. Let $f(x)$ be a continuous function defined on a closed and bounded interval $I = [a, b]$. If $x = x_0$ is a unique point such that $x_0 \in (a, b)$ and $f(x_0)$ is either a local maximum value or a local mimum value on I, then $f(x_0)$ is a global maximum (minimum) value on I if $f(x_0)$ is a local maximum (minimum) value on I.
Can we generalize the above conclusion to the case when f is a function of several variables? If your answer is "Yes", then give a proof. If your answer is "No", then give a counterexample.

Tutorial Exercises

Solution.

The answer is "No". Here is a counterexample.

Let $f(x, y) = x^3 + 2x^2 - 2xy + y^2$ be a function defined on a region $D = [-2, 2] \times [-2, 2]$. $f(x, y)$ has two critical points $(0, 0), (-\frac{2}{3}, -\frac{2}{3})$ on D, but $(-\frac{2}{3}, -\frac{2}{3})$ is a saddle point. And also, $f(0, 0) = 0$ is a local minimum value on D.

However, a global minimum value of f on D is $f(-2, -2) = -4$. $\square$

Tutorial Exercises

The illustration (sketch).