MAT1001 Midterm Examination

Saturday, October 28, 2023

Time: 9:30 - 11:30 AM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The maximum possible score of this examination is ***110****.*

3. There are ***13*** *questions (with parts), which are worth 120 points in total.* ***This means that you do not have to answer all the questions in order to get the full score****.*

*4. The symbol [****N****] at the beginning of a question indicates that the question is worth N points.*

5. Write down your solutions on the ***answer book****.*

6. Show your intermediate steps ***except Questions 1, 2, and 3*** *— answers without intermediate steps will receive minimal (or even no) marks.*

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MAT1001 Midterm Questions

1. [**10**] True or False? No explanation is required.

(i) If $\lim_{x\to 0} |f(x)| = 0$, then $\lim_{x\to 0} f(x) = 0$.

(ii) If $y = (f(x))^2$ is continuous on the real line, then $y = f(x)$ is also continuous on the real line.

(iii) The graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

has a vertical tangent at the point $(0, 1)$.

(iv) Suppose that $y = f(x)$ is decreasing and concave up on the real line. Then for any x, if $\Delta x = dx > 0$, then $|\Delta y| < |dy|$.

(v) If f is continuous on (a, b) then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

2. [**9**] For each part of this question, there is only one correct answer. Choose the correct answer. No explanation is required.

(i) Consider the function $y = f(x)$ defined over the interval $[0,\ 1]$ as follows:

$$f(x) = \begin{cases} x - x^2, & 0 < x < 1 \\ 1, & x = 0 \text{ and } x = 1 \end{cases}$$

Which of the following is correct?

A) The function has three absolute maxima.
B) The function has no local minimum and three local maxima.
C) The function has one local minimum and two absolute maxima.
D) The function has two local minima and one local maximum.

(ii) Let f be twice-differentiable on $I = (a, b)$ and continuous on $[a, b]$, and let $f'' > 0$ on I. Which of the following must be correct?

A) For any $c \in I$, the tangent line to $y = f(x)$ at c lies below the graph of $y = f(x)$.
B) For any $c \in I$, the tangent line to $y = f(x)$ at c lies above the graph of $y = f(x)$.
C) The graph of f lies above the secant line joining $(a, f(a))$ and $(b, f(b))$.
D) None of the above (A), (B), (C) is true.

(iii) Consider using Newton's method to solve the equation $f(x) = 0$ where

$$f(x) = a - (x - b)^2.$$

Here, $a > 0$ and b is arbitrary. What initial guess x_0 below will always approximate the largest root?

A) Choose $x_0 < a$.
B) Choose $x_0 \leq b$.
C) Choose $x_0 > b$.
D) Any initial guess x_0 can guarantee the convergence.

3. [**15**] Short questions: no explanation is required.

(i) Calculate the derivative of $y = \sin\left(\cos\left(2t - \frac{\pi}{6}\right)\right)$ at $t = \frac{\pi}{3}$.

(ii) Given that $y = x^2 + 7x - 5$ and $\frac{dx}{dt} = \frac{1}{3}$ when $x = 1$, find $\frac{dy}{dt}$ at $x = 1$.

(iii) Find the linearization of the function $f(x) = \frac{1}{1 - x}$ centered at $x = 0$.

(iv) We wish to estimate the solution to the equation $x^3 - x - 5 = 0$ using Newton's method. Supposing we take the initial estimate $x_0 = 0$, find x_1.

(v) Which of the following statements are always true? (There could be one or more answers.)

A) If f is both left-continuous and right-continuous at $x = c$, then f is continuous at $x = c$.
B) If f is both left-differentiable and right-differentiable at $x = c$, then f is differentiable at $x = c$.
C) For any real numbers x and y, we have $|\cos(x) - \cos(y)| \leq |x - y|$.
D) The function

$$f(x) = \begin{cases} (x-1)\cos\left(\dfrac{1}{x-1}\right), & \text{if } x \neq 1; \\ 1, & \text{if } x = 1 \end{cases}$$

has a jump discontinuity at $x = 1$.

4. [**15**] Evaluate the following limits. Use only methods and theories from Chapters 2, 3, or 4 in the textbook.

(i) $\displaystyle\lim_{x\to 0} \frac{x\,\cot(5x)}{\sin^2(x)\cot^2(3x)}$

(ii) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{\sqrt{1+x} - 1}$

(iii) $\displaystyle\lim_{x\to+\infty} (\sin\sqrt{x+1} - \sin\sqrt{x-1})$

5. [**7**] Find all vertical and oblique asymptotes for the function

$$f(x) = \frac{x^3 + 5x^2 - 7}{x^2 - 1}.$$

6. [**7**] Determine the first and second derivative functions of

$$f(t) = \begin{cases} \dfrac{1}{2}(t-2)^2 + 4, & \text{if } 0 \leq t < 2; \\ -\dfrac{1}{2}(t-2)^2 + 4, & \text{if } t \geq 2. \end{cases}$$

If you think the derivative functions are not defined at some points, explain and specify these points.

7. [**8**] Is the derivative of

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? Is the derivative of $g(x) = xf(x)$ continuous at $x = 0$? Give reasons for your answers.

8. [**5+4**] Given the curve defined by the equation $x^2(2 - y) = y^3$:

(i) Find the equation of the tangent line to the curve at $(1, 1)$.

(ii) Find $\dfrac{d^2y}{dx^2}$ at $(1, 1)$.

9. [**6**] A car braked with a constant deceleration of 16 ft$/s^2$ (feet per second squared), producing skid marks (刹车痕) measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied? Answer this question using the theory of antiderivatives.

10. [**4+3+4**] Consider the function $f(x) = x^4 - 4x^3$ defined on the real line.

(i) Determine the intervals where $f(x)$ is concave up and where $f(x)$ is concave down.

(ii) Determine the points of inflection of this function.

(iii) Determine the locations of all local maxima and minima.

11. [**9**] Consider the equation

$$x^3 - 2x + c = 0,$$

where c is a constant. Without solving the equation, determine the range of values of c for which:

(i) the equation has only one solution,

(ii) the equation has exactly two solutions, and

(iii) the equation has three solutions.

12. [**7**] A string of length L cm is used to form a triangle ΔABC whose sides AB and AC are of the same length L_1 cm, where $2L_1 < L$. Find L_1 in terms of L so that the area of the triangle is maximized.

13. [**2+5**] Suppose the function $f(x)$ is continuous on $[0,1]$ and twice differentiable on $(0,1)$.

(i) Use standard linear approximation of f at $x=0$ to approximate $f(1)$.

(ii) Show that there exist $A \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $c \in (0,1)$ such that

$$f(1) = f(0) + f'\left(\frac{1}{2}\right) + Af''(c).$$