The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

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MAT 3007 – Optimization

Final Exam – Sample

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

- *The exam time is 120 minutes.*
- *There are six exercises on seven sheets (including this sheet).*
- *The total number of achievable points is* 100 *points.*
- *You are allowed to bring two sheets of A4 paper (with arbitrary notes on both sides of it) for your personal use in this exam.*
- *Please abide by the honor codes of CUHK-SZ.*

Good Luck!

Exercise 1 (KKT Conditions and Constrained Problems): (*20* points)
Consider the nonlinear optimization problem

$$\min_{x\in\mathbb{R}^2} \quad f(x) := \ln(1+x_2) + x_1x_2 - x_1^2x_2^2 \tag{1a}$$

$$\text{s.t.} \quad g(x) \le 0, \tag{1b}$$

where the constraint function $g : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$g_1(x) := x_1^2 + (x_2-1)^2 - 1, \quad g_2(x) := 1 - (x_1+1)^2 - x_2^2, \quad g_3(x) := x_1 - x_2.$$

Let us further set $\bar{x} := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *Here* ln *denotes the natural logarithm to the base e.*

a) Sketch the feasible set $\Omega := \{x \in \mathbb{R}^2 : g(x) \le 0\}$.

b) Is problem (1) convex? Explain.

c) Determine the active set $\mathcal{A}(\bar{x})$. The active set $\mathcal{A}(x)$ refers to the set of indices of constraints that hold at equality, i.e., are tight. For example, if the first two constraints are tight, the active set is $\{1, 2\}$.

d) Is $\bar{x}$ a KKT point of problem (1)? If yes, find all coresponding Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^3$ such that the pair $(\bar{x}, \bar{\lambda})$ is a KKT point of (1). Is the multiplier $\bar{\lambda}$ unique?

Exercise 2 (Convexity): (*15* points)

Consider the function $f(x,y) = -\log(x+e)\log(y+e)$ defined on the region $\Omega := \mathbb{R}^2_+ = \{(x,y) : x \geq 0,\ y \geq 0\}$. (Here, log denotes the natural logarithm and e is Euler's number).

a) For fixed y, is the mapping $f(x,y)$ a convex function of x for $x \geq 0$? Explain your answer.

b) For fixed x, is the mapping $f(x,y)$ a convex function of y for $y \geq 0$? Explain your answer.

c) Is f a convex function of (x,y) on Ω? Explain your answer!

Exercise 3 (Integer Programming Formulation): (*15* points)

A company wishes to put together an academic "package" for an executive training program. The package will consist of 6 courses. There are 4 fields and the 6 courses must cover all the 4 fields. There are 3 colleges, each offering one course in each of the 4 fields. The tuition (basic charge) assessed when at least one course is taken, at college j is T_j (independent of the number of courses taken). Moreover, each college imposes an additional charge (covering course materials, instructional aids, and so forth) for each course, the charge for taking course i (course in the field i) at college j is c_{ij}. Formulate an integer program that will provide the company with the minimum amount it must spend to meet the requirements of the program.

Exercise 4 (Branch-and-Bound Algorithm): (*20* points)
Consider the following integer program:

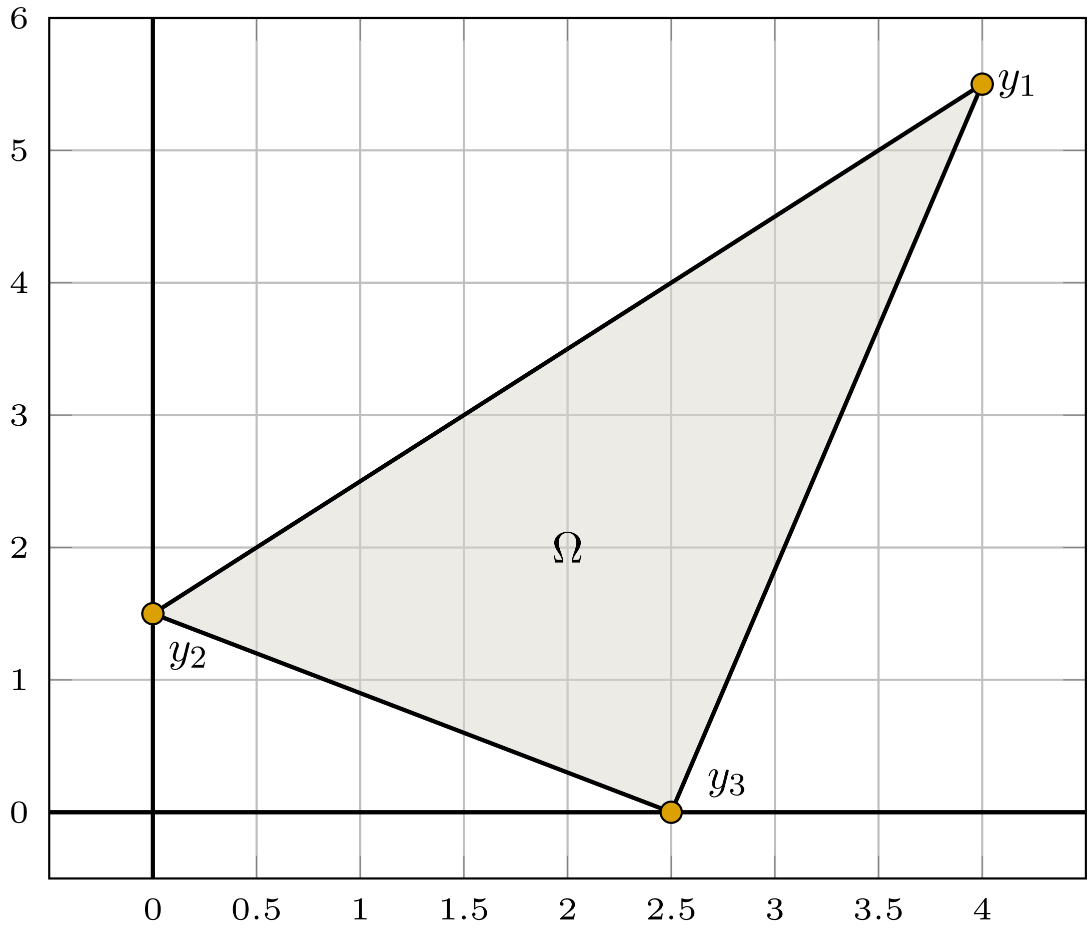
$$
\begin{array}{lrcrcl}
\text{maximize} & x_1 & - & x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1.5 \\
& -6x_1 & - & 10x_2 & \leq & -15 \\
& 22x_1 & - & 6x_2 & \leq & 55 \\
& x_1, & & x_2 & \in & \mathbb{Z}.
\end{array}
$$

Use the branch-and-bound method to solve the problem. Draw the branch-and-bound tree and mark the results on each node.

Hint: In order to solve the LP relaxations you can use a graphical approach or check the corresponding extreme points. The following sketch shows the feasible set

$$\Omega := \{x \in \mathbb{R}^2 : -x_1 + x_2 \leq 1.5,\ -6x_1 - 10x_2 \leq -15,\ 22x_1 - 6x_2 \leq 55\}.$$

The extreme points of Ω are given by $y_1 = (4, 5.5)^\top$, $y_2 = (0, 1.5)^\top$, and $y_3 = (2.5, 0)^\top$.



Exercise 5 (Algorithms for Unconstrained Problems): (*15* points)
We consider the unconstrained minimization problem

$$\min_{x\in\mathbb{R}^3} \ f(x) := \frac{1}{2}x_1^4 + (x_1^2-1)x_2^2 + 2x_3 + x_3^2.$$

The gradient and Hessian of f are given by (*you don't need to verify this)*:

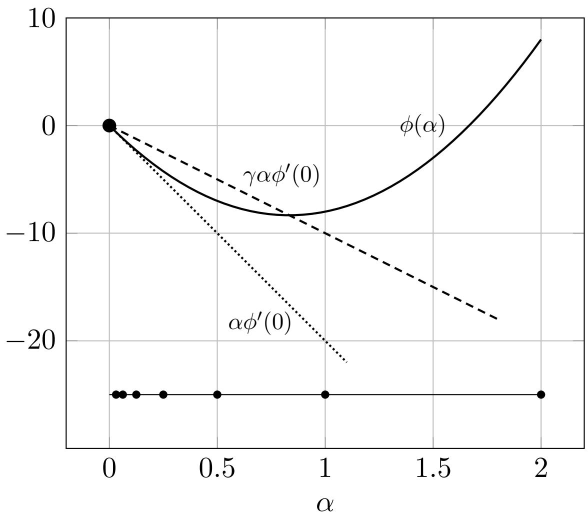
$$\nabla f(x) = \begin{pmatrix} 2x_1^3 + 2x_1x_2^2 \\ 2(x_1^2-1)x_2 \\ 2+2x_3 \end{pmatrix}, \quad \nabla^2 f(x) = \begin{pmatrix} 6x_1^2+2x_2^2 & 4x_1x_2 & 0 \\ 4x_1x_2 & 2(x_1^2-1) & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We want to apply Newton's method and the gradient descent method with backtracking to solve the problem $\min_x f(x)$. We choose the initial point x^0 and the Armijo parameter as follows:

$$x^0 = (0,1,1)^\top, \quad \gamma = 0.5, \quad \sigma = 0.5.$$

a) Compute the Newton direction $d_{\mathtt{n}}^0$ and the step $x_{\mathtt{n}}^1 = x^0 + d_{\mathtt{n}}^0$. Is $d_{\mathtt{n}}^0$ is a descent direction of f at x^0?

b) We now choose $d_{\mathtt{g}}^0 = -\nabla f(x^0)$ and set $\phi(\alpha) := f(x^0 + \alpha d_{\mathtt{g}}^0) - f(x^0)$. Compute the gradient iterate $x_{\mathtt{g}}^1$ and the stepsize α_0 using backtracking and the following plot:



c) Is the function f coercive?

Exercise 6 (True & False): (*15* points)

State whether each of the following statements are *True* or *False*. For each part, only your answer, which should be either *True* or *False*, will be graded. Explanations will not be read.

a) We consider a nonlinear program with $f(x) = x_1^2 + x_2^2 - 2x_3^4$, $g(x) = \|x\|^2 - 5$, and $h(x) = x_1^2 + x_2 + 2x_3$. (The feasible set is given by $\Omega = \{x : g(x) \leq 0,\, h(x) = 0\}$). This problem possesses a global solution.

b) Consider the nonlinear program $\min_{x \in \Omega}\ f(x)$ with linear inequality constraints $\Omega := \{x : Ax \leq b\}$. Let x^* be a local solution of this problem, then x^* satisfies the KKT conditions.

c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let us apply the gradient descent method with backtracking to solve $\min_x\ f(x)$. Suppose the method stops at iteration k with $\nabla f(x^k) = 0$. Then, x^k is a global minimizer of f.

d) Consider an integer optimization problem and its LP relaxation. If the integer program is feasible, i.e., the feasible set is nonempty, then the LP relaxation also must be feasible.

e) Consider the integer program

$$\min_x\ c^\top x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0, \quad x \in \mathbb{Z}^n, \tag{2}$$

where $A \in \mathbb{R}^{m \times n}$ is a totally unimodular matrix and $b \in \mathbb{Z}^m$ is a given integer vector. We apply the interior-point method to solve the associated LP relaxation of problem (2). Then, the method is guaranteed to return an integer solution.