MAT2040: Linear Algebra

Final Exam (2017-18, Summer)

Instructions:

1. This exam consists of 9 questions (3 pages). This exam is 3 hour long, and worth 100 points.
2. This exam is in closed book format. No books, calculators, dictionaries or blank papers are allowed. Any cheating will be given **ZERO** mark. **Please show your steps.**

Student Number: ____________________ **Name:** _______________

Problem 1 (10 points) Determinant
Given the matrix

$$\mathbf{A} = \begin{bmatrix} \alpha & -1 & -1 \\ -1 & \alpha & -1 \\ -1 & -1 & \alpha \end{bmatrix}$$

where α is a real number.

(a) Compute the determinant for the above matrix $\mathbf{A}$. **[6 marks]**

(b) Find α such that the matrix $\mathbf{A}$ is singular. **[4 marks]**

Problem 2 (10 points) Linear transformation
Define a map $L : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ by

$$L(p) = (x-1)\frac{dp}{dx}$$

where $\mathbb{P}_2 = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$.

(a) Show that L is a linear transformation. **[2 marks]**

(b) Write down a matrix representation of L with respect to basis $\{1, x, x^2\}$ for the input and output vector spaces. **[8 marks]**

Problem 3 (16 points) Least square problem
Given the linear system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

(a) Find the least square solution of the linear system. **[6 marks]**

(b) Find the projection matrix and projection vector corresponding to the least square solution in (a). **[6 marks]**

(c) Find the distance between $\mathbf{b}$ and column space $C(\mathbf{A})$. **[4 marks]**

Problem 4 (15 points) True or False. No justifications are required

(a) If $\mathbf{A}$ is an $n \times n$ matrix with characteristic polynomial $p_A(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n$, then $\mathbf{A} = \mathbf{O}$. **[3 marks]**

(b) If $\mathbf{Q} \in \mathcal{R}^{n\times n}$ and $\|\mathbf{Qx}\| = \|\mathbf{x}\|$ for every column vector $\mathbf{x} \in \mathcal{R}^{\mathbf{n}}$, then $\mathbf{Q}$ is an orthogonal matrix, where orthogonal matrix means square matrix with orthonormal columns. **[3 marks]**

(c) If $\mathbf{A}$ is the sum of 6 rank one matrices, then rank($\mathbf{A}$)≤ 6. **[3 marks]**

(d) If $\mathbf{A}, \mathbf{B} \in \mathcal{R}^{n\times n}$ and λ is the eigenvalue of $\mathbf{AB}$, then λ is also the eigenvalue of $\mathbf{BA}$. **[3 marks]**

(e) If $\mathbf{A} \in \mathcal{R}^{n\times n}$ and the eigenvalues of $\mathbf{A}$ are not distinct, then $\mathbf{A}$ must be non-diagonalizable. **[3 marks]**

Problem 5 (12 points) SVD
Given matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$. **[4 marks]**

(b) Find the SVD decomposition of $\mathbf{A} = \mathbf{U\Sigma V^T}$in two steps: **[8 marks]**
1) First, compute $\mathbf{V}$ and $\mathbf{\Sigma}$ using the matrix $\mathbf{A^TA}$.
2) Second, find the (orthonormal) columns of $\mathbf{U}$.

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Problem 6 (12 points) Orthogonality
Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -2 \\ 1 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} = [\mathbf{a}_1|\mathbf{a}_2|\mathbf{a}_3]$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are three column vectors of $\mathbf{A}$.

(a) Using Gram-Schmidt process for $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ to obtain three orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. **[8 marks]**

(b) Suppose that $\mathbf{A} = \mathbf{QR}$ be the $\mathbf{QR}$ factorization, find $\mathbf{Q}$ and $\mathbf{R}$. **[4 marks]**

Problem 7 (8 points) Eigenvalues and eigenvectors

(a) Let $\mathbf{A}, \mathbf{B}$ be two $n \times n$ real symmetric matrices, if the eigenvalues of $\mathbf{A}, \mathbf{B}$ are the same, show that $\mathbf{A}, \mathbf{B}$ are similar. **[4 marks]**

(b) Let $\mathbf{A}$ be any $m \times n$ real matrix, show that the eigenvalues of $\mathbf{AA^T}$ must be nonnegative. **[4 marks]**

Problem 8 (9 points) Positive definite matrix

(a) Given matrix $\mathbf{A} = \begin{bmatrix} \lambda & -\sqrt{2} \\ -\sqrt{2} & 3-\lambda \end{bmatrix}$, where λ is a real number, find the condition for λ such that $\mathbf{A}$ is positive definite. **[3 marks]**

(b) Let $\mathbf{A},\ \mathbf{B}$ be two $n \times n$ real symmetric matrices, and suppose $\mathbf{A}$ is positive definite. Show that there exists an $n \times n$ nonsingular matrix $\mathbf{C}$ such that $\mathbf{C^TAC}$ and $\mathbf{C^TBC}$ are both diagonal matrices. **[6 marks]**

Problem 9 (8 points) Vector space

Suppose $\mathbf{U}, \mathbf{V}$ are two subspaces of $\mathcal{R}^n$, define:

$$\mathbf{U} + \mathbf{V} = \{a + b | a \in \mathbf{U}, b \in \mathbf{V}\}$$

(a) Show that $\mathbf{U} + \mathbf{V}$ is a subspace of $\mathcal{R}^n$. **[2 marks]**

(b) If $\mathbf{U} \cap \mathbf{V} = \{\mathbf{0}\}$, show that $\dim(\mathbf{U} + \mathbf{V}) = \dim \mathbf{U} + \dim \mathbf{V}$. **[6 marks]**