MID-TERM EXAMINATION

Nov. 6, 2022

Question	Points	Score
1. True or False	15	
2. The Simplex Method and Simplex Tableau	18	
3. Duality	20	
4. Sensitivity Analysis	15	
5. Optimization Formulation	14	
6. Null Variables	18	
Total:	**100**	

Instructions:

- Please write down your student ID on the answer paper.
- Please justify your answers except Question 1.
- The exam time is 90 minutes.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

Question 1 [15 points]: True or False

State whether each of the following statements is True or False. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) **[3 points]** For any arbitrary nonempty polyhedron, it has at least one extreme point.

(b) **[3 points]** Define a square $S = \{x \in \mathbb{R}^2 \mid 0 \leq x_i \leq 1, i = 1, 2\}$ and a disk $D = \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$. Then $S \cap D$ is convex.

(c) **[3 points]** Consider a standard LP with n variables and m constraints. Suppose it has a finite optimal solution, then the optimal solution returned by the simplex method must have no more than m strictly positive entries.

(d) **[3 points]** Consider a standard form LP problem and assume that the rows of the matrix A are linearly independent. If the problem is unbounded, then its dual problem is infeasible.

(e) **[3 points]** For linear optimization problems, if the primal problem has a feasible solution, then the dual problem must also have a feasible solution.

Question 2 [18 points]: The Simplex Method and Simplex Tableau

Consider a LP problem with an unknown K $(0 < K < 6)$:

$$\begin{aligned}
\text{minimize} \quad & -2x_1 - 4x_2 + 6x_3 \\
\text{s.t.} \quad & 2x_2 + 2x_3 \leq K \\
& x_1 + x_2 + 3x_3 \leq 8 \\
& 2x_1 + 2x_2 - x_3 \leq 6 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(a) **[4 points]** Derive the standard form of the LP problem.

(b) **[10 points]** Use the simplex method and obtain the final simplex tableau.

(c) **[4 points]** If the optimal solution is $x^* = [1, 2, 0]^\top$, what is the value of K?

Question 3 [20 points]: Duality

You have downloaded a program from a website of unknown quality to solve LP problems of the form:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

You test the program with the following data:

$$A = \begin{bmatrix} 3 & 2 & 1 & 3 & 3 & 2 \\ 2 & 4 & 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 & 3 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 14 \\ 16 \\ 10 \end{bmatrix}, \quad c^\top = [2, 3, 2, 2, 3, 2].$$

The program prints the following: "An optimal solution to the problem is $x = [3, 2, 1, 0, 0, 0]^\top$, and an optimal solution to the corresponding dual problem is $y = [0.25, 0.50, 0.25]^\top$."

(a) **[5 points]** Verify whether the result of the program is correct or not, and give your reason.

(b) **[10 points]** Assume that the constraints $Ax = b$ above are changed to the constraints $Ax \geq b$. Find an optimal solution to the new problem.

(c) **[5 points]** Suppose that the constraints $Ax = b$ above are changed to $Ax \leq b$. Find an optimal solution to the new problem.

Question 4 [15 points]: Sensitivity Analysis

Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & \frac{1}{2}x_2 + x_3 = 3 \\ & x \geq 0 \end{aligned}$$

Table 1 gives the final simplex tableau when solving the standard form of the above problem (after adding variable $x_4 \geq 0$ in the first constraint i.e. $x_1 + x_2 + x_4 = 5$).

Table 1: Simplex Tableau for Sensitivity Analysis

Basis	x_1	x_2	x_3	x_4	...	RHS
z	0	0	...	...		1/2
...	...	...	...	...		...

(Note: Refer to the provided tableau in the class handout for specific coefficients if needed, as the source image was partially obscured.)

(a) **[3 points]** From Table 1, what is the optimal solution and the optimal value of the original problem?

(b) **[6 points]** The vector b is changed from $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Solve the new problem (Hint: use Table 1).

(c) **[6 points]** In what range can we change the objective coefficient $c_3 = 1$, so that the current optimal solution is still optimal to the resulting new problem?

Question 5 [14 points]: Optimization Formulation

A company has two grades of inspectors, I and II, to undertake quality control inspection. At least 1500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are Rs.5 per hour while those of grade II inspector are Rs.4 per hour. Any error made by an inspector costs Rs.3 to the company. There are, in all, 10 grade I inspectors and 15 grade II inspectors in the company.

(a) **[8 points]** Formulate an optimization problem for finding an optimal assignment of inspectors (Assignment of inspectors refers to determine how many grade I and grade II inspectors that should be assigned the job of quality control inspection) that minimizes the daily inspection cost.
Note: The assignment of inspectors should be modeled as integer variables. However, since we do not know how to deal with these integer constraints in general at this moment, you may just ignore them for now.

(b) **[6 points]** Transform it into a standard form. Determine whether it has an optimal solution. What is the type of this optimization problem (constrained vs unconstrained, continuous vs discrete)?

Question 6 [18 points]: Null Variables

Let $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ be a nonempty polyhedron in $\mathbb{R}^n$, and let m be the dimension of the vector b. We call x_j, the j-th entry of x, a null variable if $x_j = 0$ whenever $x \in P$.

(a) **[4 points]** For the coefficients A and b that define the polyhedron P, consider the following LP problem:

$$\begin{array}{ll} \text{minimize} & b^\top y \\ \text{s.t.} & A^\top y \geq e_j \end{array}$$

where e_j is a unit vector, of which the j-th entry is 1 and the other entries are 0. Write down its dual problem.

(b) **[8 points]** Suppose that there exists some $p \in \mathbb{R}^m$ for which $A^\top p \geq 0$, $p^\top b = 0$, and the j-th entry of $A^\top p$ is positive. Show that x_j is a null variable.

(c) **[6 points]** Prove that if x_j is a null variable then there exists some $p \in \mathbb{R}^m$ with the properties stated in (b).