MIDTERM EXAM

MAT 3007

Nov 6, 2019

INSTRUCTIONS

a) Write ALL your answers in this exam paper.

b) One piece of note is allowed. No computer or calculator is allowed.

c) The exam time is 10:30am - 12:00pm.

d) There are 6 questions and 100 points in total. Except the true or false questions, write down the reasonings for your answers.

In taking this examination, I acknowledge and accept the instructions.

NAME (signed) ______________________________

NAME (printed) ______________________________

For grading use. Don't write in this part

	1 (18pts)	2 (20pts)	3 (16pts)	4 (22pts)	5 (14pts)	6 (10pts)	Total
Points							

Problem 1: True/False (18pts)

State whether each of the following statements is True or False. For each part, only your answer, which should be one of True or False, will be graded. Explanations will not be read.

(a) For a linear optimization problem in the standard form, suppose it has a finite optimal solution, then if one increases an objective coefficient (e.g., from 1 to 2), the optimal value will not decrease.

(b) In a linear optimization problem, two different basis must correspond to two different vertices.

(c) For linear optimization, if a primal problem is feasible, then the dual problem must be feasible too.

(d) Let $\boldsymbol{x}$ be an optimal solution to a linear optimization problem. Now we add another constraint. If $\boldsymbol{x}$ satisfies the additional constraint, then it must still be optimal.

(e) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solutions to a linear optimization problem and its dual, respectively. Then if they satisfy all the complementarity conditions, they must both be optimal solutions.

(f) In the simplex method for linear optimization problem, degeneracy happens when there are multiple optimal solutions.

Problem 2: Solve Linear Optimization Problem (20pts)

Use two-phase simplex method to solve the following linear optimization problem:

$$
\begin{array}{rrrrcl}
\text{maximize} & x_1 & +2x_2 & +x_3 & & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & = & 7 \\
 & 2x_1 & -2x_2 & +x_3 & \geq & 10 \\
 & x_1, & x_2, & x_3 & \geq & 0.
\end{array}
$$

Problem 3: Duality and Complementarity Conditions (16pts)

Continue to consider the linear program in the previous question:

$$\begin{array}{rrrrcl} \text{maximize} & x_1 & +2x_2 & +x_3 & & \\ \text{s.t.} & x_1 & +x_2 & +x_3 & = & 7 \\ & 2x_1 & -2x_2 & +x_3 & \geq & 10 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

(a) Write down its dual problem. (5pts)

(b) Write down the complementarity conditions. (6pts)

(c) Use the complementarity conditions and the answer to the previous question to find out the dual optimal solution. (5pts)

Problem 4: Sensitivity Analysis (22pts)

Consider the following linear optimization problem:

$$\begin{array}{lrrrrl}
\text{minimize} & 2x_1 & +x_2 & +2x_3 & -3x_4 & \\
\text{subject to} & 8x_1 & -4x_2 & -x_3 & +3x_4 & \leq 10 \\
 & 2x_1 & +3x_2 & +x_3 & -x_4 & \leq 7 \\
 & & -2x_2 & -x_3 & +4x_4 & \leq 12 \\
 & x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B	12/5	0	7/5	0	0	1/5	4/5	11
5	10	0	1/2	0	1	1	-1/2	11
2	4/5	1	3/10	0	0	2/5	1/10	4
4	2/5	0	-1/10	1	0	1/5	3/10	5

(a) What is the optimal solution and the optimal value to the original problem? (4pts)

(b) What is the optimal solution to the dual problem? (4pts)

(c) In what range can we change the second right hand side number 7 so that the current optimal basis is still the optimal basis? (7pts)

(d) In what range can we change the second objective coefficient 1 so that the current optimal basis is still the optimal basis? (7pts)

Problem 5: Cutting Stock Problem (14pts)

Consider a factory that produces certain lengths of tubes. The raw materials are tubes of W meters and there are I types of products each with length w_i $(i = 1, ..., I)$. The demand of product type i is d_i. The objective is to use the minimal number of raw materials to meet the demand.

To solve this question, we consider a set of P cutting patterns (we consider the set of cutting patterns as given). A cutting pattern p can cut a raw material into a_{pi} number of product i. For example, if $W = 20$, $w_1 = 5$, $w_2 = 7$, then there are 3 patterns which are $\{a_{11} = 4, a_{12} = 0\}$, $\{a_{21} = 2, a_{22} = 1\}$ and $\{a_{31} = 1, a_{32} = 2\}$ (other patterns are not efficient in using materials). In the following, please write an optimization formulation to determine the best way of producing. That is, to write an optimization problem to determine how many each patterns to use in order to use the minimal number of materials to meet the demand.

Problem 6: Robust Linear Program (10pts)

Consider the following so-called robust linear program:

$$
\begin{aligned}
\text{maximize}_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{a}^T \boldsymbol{x} \leq b + \boldsymbol{u}^T \boldsymbol{x}; \quad \forall \boldsymbol{u} \in \{\boldsymbol{v} : A\boldsymbol{v} \leq \boldsymbol{w}\} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

where $\boldsymbol{c} \in R^n$, $\boldsymbol{a} \in R^n$, $\boldsymbol{w} \in R^k$ are given column vectors; b is a given scalar; and $A \in R^{k \times n}$ is a given matrix. In this problem, $\boldsymbol{u} \in R^n$ is an uncertain state vector and varies in the hypercube $\{\boldsymbol{v} : A\boldsymbol{v} \leq \boldsymbol{w}\}$. The problem is to find an $\boldsymbol{x} \geq 0$ such that $\boldsymbol{a}^T \boldsymbol{x} \leq b + \boldsymbol{u}^T \boldsymbol{x}$ for all possible $\boldsymbol{u}$ in the hypercube and the objective function $\boldsymbol{c}^T \boldsymbol{x}$ is maximized. Reformulate the problem as a single linear optimization problem with a finite number of constraints (currently it has infinite many constraints indexed by $\boldsymbol{u}$) and variables and without the state vector $\boldsymbol{u}$ appearing in the problem. (Hint: Apply duality theory on $\boldsymbol{u}^T \boldsymbol{x}$.)