

Practice Problems in Mathematical Analysis

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*Preface: A Challenging Course in Mathematical Analysis

Skipping Guidance: **If your course curriculum has not covered a particular topic please feel free to skip those problems entirely.** Focus your efforts on the core material related to Topology, Uniform Convergence, Differentiation, and Riemann Integration.

1. Foundations in \mathbb{R} and Point Set Topology (Q.1 - Q.6)

Q.1. Algebraic and Transcendental Numbers

1. Prove that the set of all algebraic numbers \mathbb{A} is countable. Use this to argue that transcendental numbers exist, without explicitly constructing one.
2. Given a polynomial $P(x) = a_n x^n + \cdots + a_0$ with integer coefficients, let $\epsilon > 0$. Show that there exists a $\delta > 0$ such that for any rational number p/q (in lowest terms) satisfying $|P(p/q)| < \delta$, we must have $q > (1/\delta)^{1/n} - 1$.
3. Let S be a subset of \mathbb{R} such that $S \cap [a, b]$ is countable for every compact interval $[a, b] \subset \mathbb{R}$. Must S be countable? Justify your answer thoroughly.

Q.2. Compactness and Covering

1. Let $E \subset \mathbb{R}$ be a set with the property that every infinite subset of E has a limit point in E . Prove that E is closed and bounded.
2. Given an open cover $\{G_\alpha\}$ of a compact set $K \subset \mathbb{R}$. Define the **Lebesgue Number** δ for this cover. Show that for any $x, y \in K$ with $|x - y| < \delta$, there exists some G_{α_0} such that $x \in G_{\alpha_0}$ and $y \in G_{\alpha_0}$.
3. Suppose K_n is a sequence of non-empty, compact, and connected subsets of a metric space (X, d) such that $K_{n+1} \subset K_n$ for all n . Prove that the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty, compact, and connected.

Q.3. Cantor's Set and Baire Category

1. Construct the Cantor set C . Show that C is a closed set, has Lebesgue measure zero, and is **nowhere dense** in $[0, 1]$.
2. Prove that the Cantor set C is uncountable.
3. State the **Baire Category Theorem** (BCI). Show that the set of rational numbers \mathbb{Q} is of the **first category** in \mathbb{R} , but the set of irrational numbers \mathbb{I} is not.

Q.4. Completeness and Subspaces

1. Let (X, d) be a metric space. Show that if $A \subset X$ is a complete subspace of X , then A is closed in X .
2. State and prove the **Cantor's Nested Interval Lemma** in a complete metric space (X, d) . Why is completeness a necessary condition?
3. Consider the space $C[0, 1]$ of continuous functions on $[0, 1]$ equipped with the metric $d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$. Show that $C[0, 1]$ is a complete metric space.

Q.5. Oscillation and Discontinuity

1. Define the **oscillation** of f at a point x_0 , denoted $\omega_f(x_0)$.
2. Prove that a function f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

3. Let D be the set of points where f is discontinuous. Show that D is an F_σ set (a countable union of closed sets).

Q.6. Connectedness and Intermediate Value Theorem

1. Let A be a connected subset of a metric space X . Prove that the closure \bar{A} is also connected.
2. State the **Intermediate Value Theorem** for a continuous function $f : X \rightarrow \mathbb{R}$, where X is a connected metric space.
3. Consider $E = \{(x, y) \in \mathbb{R}^2 \mid y = \sin(1/x) \text{ for } x > 0\} \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$. Is E connected? Is E path-connected? Justify your answer.

2. Metric Spaces, Topology, and Continuous Functions (Q.7 - Q.12)

Q.7. Stone-Weierstrass Theorem

1. State the **Stone-Weierstrass Theorem** for a compact metric space X . Define what it means for an algebra of continuous functions $\mathcal{A} \subset C(X)$ to *separate points*.
2. Let $X = [0, 1]^2$. Use the Stone-Weierstrass theorem to show that every continuous function $f(x, y)$ on X can be uniformly approximated by finite sums of the form $\sum_{i,j} a_{ij} x^i y^j$.
3. Show that the Stone-Weierstrass theorem fails if the space X is not compact. Specifically, consider $X = \mathbb{R}$ and the algebra of polynomials. Can $f(x) = \sin x$ be uniformly approximated by polynomials on \mathbb{R} ?

Q.8. Ascoli-Arzelà Theorem

1. State the necessary and sufficient conditions for a family of functions $\mathcal{F} \subset C[a, b]$ to have a uniformly convergent subsequence, as stated by the **Ascoli-Arzelà Theorem**.
2. Define **equicontinuity** and **uniform boundedness**.
3. Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[0, 1]$ such that $|f'_n(x)| \leq M$ for all $x \in [0, 1]$ and all $n \in \mathbb{N}$. If the sequence $\{f_n(0)\}$ is bounded, prove that $\{f_n\}$ has a uniformly convergent subsequence on $[0, 1]$.

Q.9. Mean Value Theorems in Multi-Variable Calculus

1. State the **Implicit Function Theorem** for a function $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$.
2. Use the Implicit Function Theorem to show that the system of equations

$$\begin{cases} xu - yv + x^2v^2 = 2 \\ xv + yu - y^2u^2 = 0 \end{cases}$$

can be locally solved for u and v as functions of x and y near $(1, 1, 1, 1)$, and compute $\frac{\partial u}{\partial x}$ at this point.

3. State and prove the **Cauchy Mean Value Theorem** for two functions f and g on $[a, b]$.

Q.10. Directional Derivatives and Differentiability

1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Show that the directional derivative $D_{\mathbf{u}}f(0, 0)$ exists in every direction $\mathbf{u} = (u_1, u_2)$, but f is **not differentiable** at $(0, 0)$.

2. Explain why the existence of all directional derivatives is not sufficient for differentiability.
3. How does the differentiability condition relate to the existence and continuity of the partial derivatives?

Q.11. Taylor's Theorem and Error Bounds

1. State **Taylor's Theorem with Lagrange remainder** for a function $f : \mathbb{R} \rightarrow \mathbb{R}$.
2. Use Taylor's theorem to prove **L'Hôpital's Rule** for the indeterminate form $0/0$ at $x = a$.
3. Show that the Maclaurin series for $\cos x$ converges for all $x \in \mathbb{R}$, and use Taylor's remainder to prove that the series **converges uniformly** to $\cos x$ on any compact interval $[-M, M]$.

Q.12. Critical Points and Second Derivative Test

1. Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^2 . State the criterion for a local minimum at a critical point \mathbf{x}_0 using the **Hessian matrix** $H(\mathbf{x}_0)$.
2. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Find all critical points of f and classify them using the Second Derivative Test (and direct analysis if inconclusive).
3. Show that for $f(x, y) = x^4 - y^4$, the Second Derivative Test is inconclusive at $(0, 0)$. Does f have a local extremum at $(0, 0)$?

3. Integration Theory and Convergence (Q.13 - Q.19)

Q.13. Riemann/Darboux Integrability

1. Define the **Darboux upper integral** $\overline{\int}_C f$ and the **Darboux lower integral** $\underline{\int}_C f$ for a bounded function f on a cell $C \subset \mathbb{R}^N$. State the condition for f to be Riemann integrable.
2. Prove that a bounded function f on a compact cell $C \subset \mathbb{R}^N$ is Riemann integrable if and only if the set of points of discontinuity of f has Lebesgue measure zero.
3. Let $f(x, y)$ be defined on $C = [0, 1] \times [0, 1]$ by

$$f(x, y) = \begin{cases} 1 & x \in \mathbb{Q} \\ 2y & x \notin \mathbb{Q} \end{cases}$$

Calculate the iterated integrals $\int_0^1 \left(\int_0^1 f(x, y) dy \right) dx$ and $\int_0^1 \left(\int_0^1 f(x, y) dx \right) dy$. Is f Riemann integrable on C ?

Q.14. Uniform Convergence and Integration

1. State the theorem for interchange of limit and Riemann integral (i.e., integration of a uniformly convergent sequence of functions).
2. Give an example of a sequence of Riemann integrable functions $f_n \rightarrow f$ *pointwise* on $[0, 1]$ such that f is Riemann integrable, but $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 f(x) dx$.
3. Define **equicontinuity** for a sequence of functions $\{f_n\}$ on $[a, b]$. Show that if $\{f_n\}$ is a sequence of continuously differentiable functions on $[a, b]$ such that $\{f'_n(x)\}$ is uniformly bounded, then $\{f_n\}$ is equicontinuous.

Q.15. Improper Integrals and Special Functions

1. State the **Comparison Test** for improper integrals of the first kind (on an infinite interval).
2. Prove the convergence of the **Gamma Function** $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for all $x > 0$.
3. Use the approximation $n! \approx \sqrt{2\pi n} (n/e)^n$ (**Stirling's Formula**) to estimate the integral $\int_0^1 (\ln(1/x))^n dx$ for large n .

Q.16. Leibniz Rule for Integration under the Integral Sign

1. State the general **Leibniz Rule** for differentiating $\Phi(x) = \int_{a(x)}^{b(x)} f(x, t) dt$ with respect to x .
2. Prove that if f and f' are continuous on $[0, 1]$, then $\int_0^1 f(x) dx = \sum_{n=1}^\infty \frac{f(1)-f(0)}{n!}$.

Q.17. Iterated Integrals (Fubini's Theorem)

1. State Fubini's theorem (or the Darboux iterated integral theorem) for a continuous function $f(x, y)$ on a cell $C \subset \mathbb{R}^2$.
2. Let $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$. Show that

$$\int_0^1 \left(\int_0^1 f(x, y) dy \right) dx \neq \int_0^1 \left(\int_0^1 f(x, y) dx \right) dy$$

Explain why this does not contradict Fubini's Theorem.

Q.18. Wedge Product and Exterior Derivative (Warm-up)

1. Let $\omega = x^2 dx + yz dy - xz dz$ be a 1-form in \mathbb{R}^3 . Compute the exterior derivative $d\omega$.
2. Compute $d(d\omega)$. What fundamental principle does this result confirm?
3. Write down the vector field \mathbf{F} associated with ω and compute $\text{curl}(\mathbf{F})$. How does this compare to $d\omega$?

Q.19. Generalized Stokes' Theorem (Fundamental Theorem of Calculus)

1. Let $f(x)$ be a smooth 0-form on \mathbb{R} . Compute the exterior derivative df .
2. Let $M = [a, b]$. Define the boundary ∂M and its orientation.
3. Apply the Generalized Stokes' Theorem, $\int_M d\omega = \int_{\partial M} \omega$, to the 0-form $\omega = f$. Show that this yields the **Fundamental Theorem of Calculus**: $\int_a^b f'(x) dx = f(b) - f(a)$.

4. Vector Calculus and Differential Forms (Q.20 - Q.25)

Q.20. Closed and Exact Forms

1. In $\mathbb{R}^2 \setminus \{(0,0)\}$, consider the 1-form

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

Show that ω is **closed** ($d\omega = 0$).

2. Show that ω is **not exact** on $\mathbb{R}^2 \setminus \{(0,0)\}$ by computing $\oint_C \omega$ where C is the unit circle.
3. What theorem (related to topology and domain) explains the relationship between closed and exact forms?

Q.21. Surface Integrals as Flux (Divergence Theorem)

1. Let ω be the 2-form $\omega = z dx \wedge dy + x dy \wedge dz - y dz \wedge dx$ in \mathbb{R}^3 . Compute $d\omega$.
2. Let V be the unit cube $[0,1] \times [0,1] \times [0,1]$. Apply the Generalized Stokes' Theorem: $\int_V d\omega = \int_{\partial V} \omega$. Calculate $\int_V d\omega$.
3. State the **Divergence Theorem**. How does the calculation in (b) relate to the divergence theorem when identifying the corresponding vector field \mathbf{F} ?

Q.22. Change of Variables and Pullbacks

1. Let $\omega = f(x,y) dx \wedge dy$ be a 2-form. Let $\Phi(r, \theta)$ be the polar coordinate change: $x = r \cos \theta$, $y = r \sin \theta$. Compute the **pullback** $\Phi^*\omega$.
2. Show that $\Phi^*(dx \wedge dy) = \det(J) dr \wedge d\theta$, where J is the Jacobian matrix of Φ .
3. Use the result from (b) to show how the Jacobian arises in the standard **Change of Variables Formula** for double integrals.

Q.23. The Operator d and Its Properties

1. Prove the linearity of the exterior derivative: $d(\omega + \eta) = d\omega + d\eta$.
2. Prove the anti-derivation property (Leibniz rule for forms): $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, where ω is a k -form.
3. Use the anti-derivation property to explicitly show that $d(df) = 0$ for a 0-form $f(x,y)$.

Q.24. Mollifiers and Approximation

1. Define a **mollifier** $\phi_\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$. State the three essential properties it must satisfy.
2. Let $f \in L^1(\mathbb{R}^n)$. Define the convolution $f_\epsilon = f * \phi_\epsilon$. Show that f_ϵ is a smooth function (i.e., $f_\epsilon \in C^\infty(\mathbb{R}^n)$).
3. Prove that $\lim_{\epsilon \rightarrow 0} \|f_\epsilon - f\|_{L^1} = 0$. Why are mollifiers important for extending theorems from smooth functions to L^p functions?

Q.25. Density and Approximation in L^p Spaces

1. Define the **density** of a set A in a metric space (X, d) .
2. Let $X = C[0, 1]$ with the L^2 metric $d_2(f, g) = \sqrt{\int_0^1 |f(x) - g(x)|^2 dx}$. Show that the set of all polynomials $\mathcal{P}[0, 1]$ is dense in X .
3. Show that $C^1[0, 1]$ (continuously differentiable functions) is **not dense** in $C[0, 1]$ with respect to the L^∞ metric $d_\infty(f, g) = \sup |f(x) - g(x)|$.

5. Advanced Continuous Functions and Approximation (Q.26 - Q.30)

Q.26. Stone-Weierstrass Theorem (Complex Version)

1. State the **Complex Stone-Weierstrass Theorem** for an algebra \mathcal{A} of complex-valued continuous functions on a compact set X . What additional closure property is required?
2. Let $X = \{z \in \mathbb{C} \mid |z| = 1\}$. Show that the algebra \mathcal{A} spanned by $\{1, z, z^2, z^3, \dots\}$ is *not* dense in $C(X, \mathbb{C})$.
3. Show that the algebra \mathcal{B} spanned by $\{1, z, \bar{z}, z^2, \bar{z}^2, \dots\}$ *is* dense in $C(X, \mathbb{C})$.

Q.27. Uniform Convergence and Function Extension

1. Let $f : E \rightarrow Y$ be a uniformly continuous function from a metric space E to a complete metric space Y . Prove that f can be uniquely extended to a uniformly continuous function $\bar{f} : \bar{E} \rightarrow Y$.
2. Give an example of a function $g : (0, 1) \rightarrow \mathbb{R}$ that is continuous but not uniformly continuous, and thus cannot be continuously extended to $[0, 1]$.
3. Let X be a compact metric space and $f : X \rightarrow \mathbb{R}$ be continuous. Must f be uniformly continuous? Prove or provide a counterexample.

Q.28. Density and Nowhere Dense Sets

1. Let (X, d) be a complete metric space. If $A \subset X$ is a dense G_δ set, show that A is of the **second category**.
2. Prove that in \mathbb{R} , the set of irrational numbers \mathbb{I} is a dense G_δ set.
3. Show that the set of functions which are nowhere differentiable is a dense G_δ set in $C[0, 1]$ (with the supremum norm). State the main ideas of the proof without full rigor.

Q.29. Mollifiers and Smooth Approximation

1. Let $f \in C(\mathbb{R})$ have compact support. If ϕ_ϵ is a standard mollifier, prove that the convolution $f_\epsilon = f * \phi_\epsilon$ converges to f **uniformly** as $\epsilon \rightarrow 0$.
2. Construct a sequence of smooth functions g_n that approximate the characteristic function $\chi_{[0,1]}$ in L^1 norm, but do *not* converge pointwise on $[0, 1]$.

Q.30. Equicontinuity in Non-Compact Spaces

1. State the **Ascoli-Arzelà Theorem** emphasizing the necessary conditions for a set of functions in $C[a, b]$ to be **relatively compact**.
2. Consider the space $C_b(\mathbb{R})$ of bounded continuous functions on \mathbb{R} with the supremum norm. Show that the set $\mathcal{F} = \{f_n(x) = \sin(x/n) \mid n \in \mathbb{N}\}$ is uniformly bounded and equicontinuous on any compact interval $[-M, M]$, but **not** relatively compact in $C_b(\mathbb{R})$.
3. Explain why the Ascoli-Arzelà Theorem does not apply directly to the entire space $C_b(\mathbb{R})$.

6. Advanced Differentiation and Implicit Function Theorem (Q.31 - Q.36)

Q.31. Critical Points and Extrema on Manifolds

1. Explain how the method of **Lagrange Multipliers** is derived by considering the level sets of the constraint function.
2. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that are closest to and farthest from the origin $(0, 0, 0)$.
3. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 and has a local maximum at \mathbf{a} , show that the Hessian matrix $H(\mathbf{a})$ is **negative semi-definite**.

Q.32. Second Derivative Test (Failure Case)

1. For a C^2 function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, state the criterion for a local extremum at a critical point \mathbf{x}_0 using D_1 and $D_2 = \det(H)$.
2. Let $f(x, y) = x^2 + kxy^2$. Find the critical points. Show that the Hessian test is inconclusive at $(0, 0)$ for $k \neq 0$.
3. Analyze the behavior of $f(x, y) = x^2 + xy^2$ near $(0, 0)$. Does it have a local extremum at $(0, 0)$?

Q.33. Local Diffeomorphisms and Inverse Function Theorem

1. State the **Inverse Function Theorem** for $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$.
2. Consider $F(x, y) = (e^x \cos y, e^x \sin y)$. Show that F is locally invertible everywhere, but **not globally invertible** on \mathbb{R}^2 .
3. Explain how the failure of global invertibility is related to the topological properties (e.g., injectivity) of F .

Q.34. The Rank Theorem and Implicit Function Theorem

1. State the **Rank Theorem** for a smooth map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$.
2. Use the Implicit Function Theorem to show that if $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is C^1 and its Jacobian matrix has rank 2 at a point \mathbf{a} , then the level set $F(\mathbf{x}) = F(\mathbf{a})$ locally defines a smooth curve (a 1-dimensional manifold) in \mathbb{R}^3 .

Q.35. Total Differential vs. Directional Derivatives

1. Define the differentiability of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \mathbf{x}_0 using the linear approximation error term $E(\mathbf{h})$.
2. Construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, but f is **not differentiable** at $(0,0)$.
3. Construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the partial derivatives exist, but the directional derivative $D_{\mathbf{u}}f(0,0)$ does **not** exist for some vector \mathbf{u} .

Q.36. Improper Integrals: Differentiation under the Integral Sign

1. State the conditions under which the differentiation of an improper integral $\Phi(x) = \int_a^\infty f(x,t)dt$ with respect to the parameter x is valid.
2. Define $I(x) = \int_0^\infty \frac{e^{-tx}-e^{-t}}{t}dt$ for $x > 0$. Use differentiation under the integral sign to compute $I'(x)$.
3. Find the closed-form expression for $I(x)$ using $I(1) = 0$.

7. Advanced Integration and Special Functions (Q.37 - Q.41)

Q.37. Stirling's Formula Derivation (Using Gamma)

1. Recall $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Show that $\Gamma(n+1) = n!$ for integers $n \geq 0$.
2. Let $f_n(t) = t^n e^{-t}$. Show that the maximum of $\ln f_n(t)$ occurs at $t = n$.
3. State the method of **Laplace's Approximation** for integrals. Explain how this method, applied to $\Gamma(n+1)$, leads to the $\sqrt{2\pi n}$ factor in Stirling's formula.

Q.38. Darboux Integrals and Countable Sets

1. Prove that if the set of discontinuities of a bounded function $f : [a,b] \rightarrow \mathbb{R}$ is finite, then f is Riemann integrable.
2. Let $A \subset [0,1]$ be a countable set. Define $f(x) = \chi_A(x)$. Calculate the upper and lower Darboux integrals of f over $[0,1]$.
3. Generalize the result: Prove that f is Riemann integrable on a compact cell $C \subset \mathbb{R}^N$ if and only if its set of discontinuities has **measure zero**.

Q.39. Uniform Convergence and Power Series

1. State **Abel's Theorem** regarding the uniform convergence of a power series $f(x) = \sum_{n=0}^\infty a_n x^n$ at the endpoints of its interval of convergence.
2. Show that $\sum_{n=1}^\infty \frac{(-1)^n}{n} x^n$ converges uniformly on $[0,1]$.
3. Use (b) and term-by-term integration to find the exact value of the integral $\int_0^1 \frac{\ln(1+x)}{x} dx$.

Q.40. Fubini-Tonelli (Non-Integrability Case)

1. State the general condition (related to $|f|$) for Fubini's theorem to guarantee the equality of iterated integrals over a compact cell $C \subset \mathbb{R}^N$.
2. Let $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$ on $C = [0, 1] \times [0, 1]$. Show that the iterated integrals are unequal.
3. Prove that $\iint_C |f(x, y)| dx dy$ is **divergent**. Explain why this confirms the failure of Fubini's Theorem.

Q.41. The Hodge Star Operator (Introduction)

1. In \mathbb{R}^3 , the **Hodge star operator** \star maps a k -form to an $(3-k)$ -form. Define $\star dx, \star dy, \star dz$.
2. Show that $\star(dx \wedge dy) = dz$.
3. Use the exterior derivative d and the codifferential $\delta = \pm \star d \star$ to define the **Laplace-de Rham operator** Δ_{dR} .

8. Vector Calculus and Differential Forms (Q.42 - Q.50)

Q.42. Stokes' Theorem (Surface Integral)

1. Let $\omega = yzdx + xzdy + xydz$ be a 1-form in \mathbb{R}^3 . Compute $d\omega$.
2. Let S be the open hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$. The boundary ∂S is the unit circle C .
3. Apply the Generalized Stokes' Theorem: $\int_S d\omega = \int_{\partial S} \omega$. Verify the theorem by calculating both sides.

Q.43. Divergence Theorem (Flux in \mathbb{R}^3)

1. Let $F(x, y, z) = (x, y, z)$. The associated 2-form for flux is $\omega_F = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Compute the exterior derivative $d\omega_F$.
2. Let V be the unit sphere $x^2 + y^2 + z^2 \leq 1$. Apply the Generalized Stokes' Theorem: $\int_V d\omega_F = \int_{\partial V} \omega_F$.
3. Calculate $\int_V d\omega_F$. Relate the 3-form $d\omega_F$ to the divergence $\nabla \cdot F$.

Q.44. Exactness and Potential Functions

1. Let $\omega = f_1 dx + f_2 dy + f_3 dz$ be a 1-form. Show that if ω is exact ($\omega = df$), then $d\omega = 0$. Relate this to the condition for a conservative vector field.
2. Consider $\omega = (y^2 + 2x)dx + 2xydy + dz$. Show that ω is closed ($d\omega = 0$).
3. Find the 0-form (potential function) f such that $df = \omega$.

Q.45. Closed Forms on Domains with Holes

1. Let ω be the closed 1-form $\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$ on $X = \mathbb{R}^2 \setminus \{(0, 0)\}$.
2. Define the **De Rham Cohomology Group** $H^1(X)$. State the relationship between $H^1(X)$ and the set of closed, non-exact 1-forms on X .

3. Show that $\oint_{C_1} \omega = \oint_{C_2} \omega$ for any two circles C_1, C_2 centered at the origin. Explain why this implies that ω represents a non-trivial cohomology class.

Q.46. Integrals of Forms (Orientation)

1. Let C be the curve $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Compute $\int_C \omega$, where $\omega = xdx + ydy$.
2. Let C' be the same curve with reverse orientation. Compute $\int_{C'} \omega$.
3. Explain how the orientation of the manifold M affects the sign of the integral $\int_M \omega$.

Q.47. Generalizing ∇ (Gradient, Curl, Divergence)

1. Summarize the correspondence between the vector calculus operators (grad, curl, div) and the exterior derivative d in terms of k -forms.
2. Use the formula $\text{curl}(\text{grad}(f)) = 0$ to derive the identity $d(df) = 0$.
3. Use the formula $\text{div}(\text{curl}(\mathbf{F})) = 0$ to derive the identity $d(d\omega) = 0$ for a 1-form ω .

Q.48. Differential Forms in \mathbb{R}^4

1. In \mathbb{R}^4 , how many basis 2-forms are there? List them.
2. Let $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_4$ be a 2-form in \mathbb{R}^4 . Compute the exterior derivative $d\omega$.
3. Let $\eta = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4$ be a 1-form. Compute $d\eta \wedge \eta$.

Q.49. Wedge Product and Geometry

1. Explain the geometric meaning of the 2-form $\omega = dx \wedge dy$ in \mathbb{R}^3 .
2. Show that $(dx \wedge dy)(\mathbf{u}, \mathbf{v})$ gives the signed area of the parallelogram spanned by \mathbf{u} and \mathbf{v} projected onto the xy -plane.
3. Prove that for any k -form ω , the wedge product $\omega \wedge \omega = 0$ if k is odd.

Q.50. Category Theorem and Smoothness

1. State the **Baire Category Theorem** (BCI).
2. Use Baire's Theorem to prove that $C[0, 1]$ (with the supremum norm) **cannot** be expressed as a countable union of its compact subsets.
3. Show that the set of polynomials $\mathcal{P}[0, 1]$ is of the **first category** in $C[0, 1]$.

9. Advanced Metric Spaces and Topology (Q.51 - Q.57)

Q.51. Totally Boundedness and Compactness

1. Define **totally bounded** (or precompact) for a subset E of a metric space (X, d) .
2. Prove that in any metric space, a compact set K is always totally bounded.
3. Give an example of a metric space X and a subset $E \subset X$ that is totally bounded but **not** compact.

Q.52. Perfect Sets and Cantor-Bendixson Theorem

1. Define a **perfect set** in a metric space X .
2. Prove that every non-empty perfect set $P \subset \mathbb{R}$ is uncountable.
3. State the **Cantor-Bendixson Theorem**.

Q.53. Separable Spaces and Density

1. Define a **separable metric space**.
2. Show that the metric space $C[0, 1]$ with the L^2 metric is separable.
3. Consider the space ℓ^∞ of bounded sequences, with the metric d_∞ . Prove that ℓ^∞ is **not separable**.

Q.54. Connectedness and Continuous Maps

1. Let $f : X \rightarrow Y$ be a continuous surjective map. If X is connected, prove that Y must also be connected.
2. Give an example of a continuous injective map $f : X \rightarrow Y$ where X is not compact, but $f(X)$ is closed and bounded (but not compact in Y).
3. Show that if a metric space X contains a dense, path-connected subset, X itself is not necessarily path-connected.

Q.55. Completion of a Metric Space

1. Define a **Cauchy sequence** and a **complete metric space**.
2. State the theorem regarding the existence and uniqueness of the **completion** (\tilde{X}, \tilde{d}) of a metric space (X, d) .
3. Describe the completion \tilde{X} of $X = \mathbb{Q}$ with the standard metric.

Q.56. G_δ and F_σ Sets

1. Define an F_σ set and a G_δ set.
2. Prove that every closed set is a G_δ set.
3. Show that the set of rational numbers \mathbb{Q} in \mathbb{R} is an F_σ set, but \mathbb{Q} is **not** a G_δ set.

Q.57. Homeomorphisms and Topological Equivalence

1. Define a **homeomorphism** between two metric spaces X and Y .
2. Prove that the open interval $(0, 1)$ is homeomorphic to \mathbb{R} .
3. Prove that the compact set $[0, 1]$ is **not** homeomorphic to \mathbb{R} .

10. Advanced Differentiation in \mathbb{R}^N (Q.58 - Q.62)

Q.58. Higher-Order Derivatives and Symmetry

1. State the theorem concerning the equality of mixed partial derivatives ($f_{xy} = f_{yx}$). Under what continuity assumption does this hold?
2. Show that the Hessian matrix $H(x, y)$ of a C^2 function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a **symmetric matrix**.
3. Construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ for which $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ exist but are **not** equal.

Q.59. Mean Value Theorem in \mathbb{R}^N (Failure Case)

1. State the **Mean Value Theorem** for a scalar-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.
2. Show that the Mean Value Theorem does **not** hold in general for vector-valued functions $F : \mathbb{R} \rightarrow \mathbb{R}^2$.
3. State the Mean Value *Inequality* for a vector-valued function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Q.60. Implicit Function Theorem (System of Equations)

1. Consider the system:

$$\begin{cases} x^2 + y^2 - u^2 - v^2 = 0 \\ xu + yv = 0 \end{cases}$$

Show that this system can be solved for u and v as C^1 functions of x and y in a neighborhood of $(1, 0, 1, 0)$.

2. Compute the partial derivative $\frac{\partial v}{\partial x}$ at $(1, 0, 1, 0)$.

Q.61. Chains of Differentiable Functions

1. State the **Chain Rule** for $h(\mathbf{x}) = f(g(\mathbf{x}))$ using Jacobian matrices.
2. Let $z = f(x, y)$ where $x = r \cos \theta$ and $y = r \sin \theta$. Use the Chain Rule to express $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.
3. Use the results from (b) to prove the identity:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

Q.62. Tangent Planes and Linear Approximations

1. Define the equation of the **tangent plane** to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) .
2. Find the equation of the tangent plane to $f(x, y) = x^3 + y^3 - 3xy$ at $(1, 2, 3)$.
3. Use the linear approximation of $f(x, y)$ at $(1, 2)$ to estimate the value of $f(1.01, 1.98)$.

11. Advanced Integration Theory (Q.63 - Q.68)

Q.63. Integrals over Unbounded Regions (Improper Integrals)

1. Define the improper integral $\int_a^\infty f(x)dx$.
2. Show that $\int_0^\infty \frac{\sin x}{x} dx$ **converges**, but $\int_0^\infty \left| \frac{\sin x}{x} \right| dx$ **diverges**.
3. Define **absolute convergence** for improper integrals.

Q.64. The Beta Function

1. Define the **Beta Function** $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$. State the convergence condition on x and y .
2. Show that $B(x, y)$ is symmetric ($B(x, y) = B(y, x)$).
3. Use the relation $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $\Gamma(1/2) = \sqrt{\pi}$ to compute $\int_0^1 \frac{dt}{\sqrt{t(1-t)}}$.

Q.65. Iterated Integrals (Change of Order)

1. Given the iterated integral $\int_0^1 \int_y^1 \frac{1}{\sqrt{1-x^2}} dx dy$. Sketch the region of integration.
2. Change the order of integration and compute the resulting integral.

Q.66. Riemann Sums and Norm of Subdivision

1. Define the **mesh** (or norm) of a subdivision \mathcal{P} of a cell $C \subset \mathbb{R}^N$.
2. Define the **Riemann integral** of f over C using the limit of Riemann sums as the mesh approaches zero.
3. Prove that if f is Darboux integrable on C , then it is also Riemann integrable, and the two integrals are equal (**Darboux Theorem**).

Q.67. Inner and Outer Volume (Jordan Measure)

1. Define the **inner volume** $v_*(A)$ and **outer volume** $v^*(A)$ of a bounded set $A \subset \mathbb{R}^N$. State the condition for A to be a **Jordan measurable** set.
2. Prove that a bounded set A is Jordan measurable if and only if its **boundary** ∂A has volume zero.
3. Give an example of a bounded set $A \subset \mathbb{R}^2$ that is **not** Jordan measurable.

Q.68. Integration and Discontinuity

1. Let $f(x) = \sum_{n=1}^\infty \frac{1}{n} \chi_{\{1/n\}}(x)$ on $[0, 1]$. Determine if f is Riemann integrable on $[0, 1]$.
2. Let $g : [0, 1] \rightarrow \mathbb{R}$ be the **Thomae function**. Prove that g is Riemann integrable on $[0, 1]$.

12. Advanced Differential Forms and Integrals (Q.69 - Q.75)

Q.69. Generalized Stokes' Theorem (Green's Theorem)

1. State **Green's Theorem** relating a line integral to a double integral.
2. Identify the 1-form ω and the 2-form $d\omega$ in the Generalized Stokes' Theorem that correspond to Green's Theorem.
3. Use Green's Theorem to find the area of a region D enclosed by a curve C .

Q.70. Calculating the Pullback of Forms

1. Let $\Phi(u, v) = (u, v, u^2 + v^2)$ be the parameterization of a paraboloid patch S .
2. Let $\omega = zdx \wedge dy$ be a 2-form in \mathbb{R}^3 . Compute the pullback $\Phi^*\omega$.
3. Use the result from (b) to compute the integral $\iint_S zdx \wedge dy$.

Q.71. Integrals over Manifolds (Surface Area)

1. The area element on a parameterized surface $\Phi(u, v)$ is $dA = \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du dv$. Relate this to the wedge product of 1-forms.
2. Let S be the surface of a cone parameterized by $\Phi(u, v) = (u \cos v, u \sin v, u)$. Compute the surface area element dA .
3. Calculate the lateral surface area of the cone for $0 \leq u \leq H, 0 \leq v \leq 2\pi$.

Q.72. Potential Forms in \mathbb{R}^3

1. Let $\omega = fdx + gdy + hdz$ be a 1-form. Define the condition for ω to be **closed** in terms of f, g, h .
2. Consider $\omega = \frac{xdx+gdy+zdz}{(x^2+y^2+z^2)^{3/2}}$ on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Show that ω is closed.
3. Find the 0-form (potential function) f such that $df = \omega$.

Q.73. The Codifferential δ (Introduction)

1. In \mathbb{R}^3 , the codifferential $\delta = \pm \star d \star$ takes a k -form to a $(k - 1)$ -form.
2. Define the codifferential δ acting on a 3-form $\eta = fdx \wedge dy \wedge dz$ in \mathbb{R}^3 .
3. Show that if ω is the 1-form associated with \mathbf{F} , then $\delta\omega$ corresponds (up to sign) to the **divergence** $\nabla \cdot \mathbf{F}$.

Q.74. Harmonic Forms

1. A k -form ω is called **harmonic** if ω is both closed ($d\omega = 0$) and co-closed ($\delta\omega = 0$).
2. Show that a 0-form f is harmonic ($\Delta_{dR}f = 0$) if and only if it is harmonic in the standard sense ($\nabla^2 f = 0$).

3. Show that the 1-form $\omega = -ydx + xdy$ is harmonic on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Q.75. Integration of Non-Orientable Surfaces (Conceptual)

1. Define an **orientable manifold** and a **non-orientable manifold**.
2. Explain why the Generalized Stokes' Theorem requires the manifold M to be **orientable** to guarantee that $\int_M d\omega$ is well-defined.
3. What difficulty arises when applying Stokes' Theorem to the Möbius strip M , which has a single boundary curve ∂M ?

13. Advanced Topology and Measure Theory Synthesis (Q.76 - Q.81)

Q.76. Completeness and Product Spaces

1. Define a metric d on the product space $X \times Y$.
2. Prove that $X \times Y$ with a standard product metric is complete if and only if both (X, d_X) and (Y, d_Y) are complete.
3. Consider $C[0, 1]$ with the L^1 metric. Is $(C[0, 1], d_1)$ a complete metric space? Justify your answer.

Q.77. Baire Category Theorem Applications

1. State the **Baire Category Theorem** (BCI) for a complete metric space X .
2. Use the Baire Category Theorem to prove that \mathbb{R} cannot be written as a countable union of nowhere dense closed sets.
3. Let X be a complete metric space. If $X = \bigcup_{n=1}^{\infty} A_n$, where A_n are closed sets, prove that at least one of the sets A_n must have a non-empty interior.

Q.78. Pointwise and Uniform Boundedness

1. Let \mathcal{F} be a family of continuous functions on a complete metric space X . If \mathcal{F} is **pointwise bounded**, prove that there exists a non-empty open set $U \subset X$ on which \mathcal{F} is **uniformly bounded**.
2. Give an example of a sequence of continuous functions $f_n : (0, 1) \rightarrow \mathbb{R}$ that is pointwise bounded but **not** uniformly bounded on any non-empty open subinterval of $(0, 1)$.

Q.79. Stone-Weierstrass and Function Separation

1. State the conclusion of the Stone-Weierstrass Theorem for an algebra \mathcal{A} that separates points and contains the constants on a compact set X .
2. Let $X = [0, 1]$. Consider the algebra of even polynomials $\mathcal{A} = \{P(x^2) \mid P \text{ is a polynomial}\}$. Is \mathcal{A} dense in $C[0, 1]$?
3. Now consider $X = [-1, 1]$. Is the algebra of even polynomials \mathcal{A} dense in $C[-1, 1]$? Explain the difference.

Q.80. Uniform Convergence and Approximation

1. Let $f_n \rightarrow f$ uniformly on a compact set E . Show that if f_n are continuous, f is continuous.
2. Give an example of a sequence of continuous functions $f_n : (0, 1) \rightarrow \mathbb{R}$ that converges uniformly to a continuous function f , but f_n are **not** uniformly bounded.
3. Explain the necessary role of compactness in proving that a continuous function can be uniformly approximated by polynomials using Stone-Weierstrass.

Q.81. Oscillation and Riemann Integrability

1. State the condition for a bounded function $f : [a, b] \rightarrow \mathbb{R}$ to be Riemann integrable in terms of its set of discontinuities D .
2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a bounded function whose oscillation $\omega_f(x)$ is continuous for all $x \in [0, 1]$. Prove that f must be continuous on $[0, 1]$.
3. Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{(q_n, 1]}(x)$, where $\{q_n\}$ is an enumeration of $\mathbb{Q} \cap (0, 1)$. Where is f continuous, and where is it discontinuous?

14. Advanced Differentiation and Implicit Mappings (Q.82 - Q.87)

Q.82. Taylor's Theorem and Local Extrema

1. Use Taylor's theorem to prove that if $f'(a) = \cdots = f^{(n)}(a) = 0$ and $f^{(n+1)}(a) \neq 0$, then f has a local extremum at a if and only if $n + 1$ is even.
2. State the multi-variable version of Taylor's Theorem with the Peano remainder term up to order 2.
3. Use the multi-variable Taylor expansion to rigorously prove the Second Derivative Test for a local minimum at a critical point \mathbf{x}_0 when $H(\mathbf{x}_0)$ is positive definite.

Q.83. Implicit Function Theorem (Mapping Geometry)

1. Let $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Show that $F(x, y, z) = 0$ locally defines z as a function of x and y , except at two singular points.
2. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial^2 z}{\partial x^2}$ on the upper hemisphere $z > 0$.
3. Explain the geometric significance of the singular points where the Implicit Function Theorem fails.

Q.84. Differentiability and the Zeroes of a Function

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be C^1 . If $f(a) = 0$ and $f'(a) \neq 0$, show that a is an isolated zero of f .
2. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be C^1 with $f(0, 0) = 0$. If $\nabla f(0, 0) = \mathbf{0}$, must $(0, 0)$ be an isolated zero of f ? Provide a proof or a counterexample.
3. State the **Rolle's Theorem** and use it to prove that a polynomial $P(x)$ of degree n has at most n real roots.

Q.85. The Rank Theorem and Submanifolds

1. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be C^1 with $m < n$. If $DF(\mathbf{a})$ has rank m , state the local structure of the level set $F(\mathbf{x}) = F(\mathbf{a})$ guaranteed by the Rank Theorem.
2. Consider $F(x, y, z) = x^2 + y^2 - z^2$. Identify the singular point(s) where the rank of DF drops.
3. Describe the geometry of the level set $F(x, y, z) = 0$ near the singular point. Does this level set locally define a smooth 2-dimensional manifold at the singular point?

Q.86. Lipschitz Condition and C^1 Functions

1. Define the **Lipschitz condition** for a function $f : E \rightarrow \mathbb{R}^m$.
2. Prove that if $f : E \rightarrow \mathbb{R}^m$ is C^1 on an open convex set $E \subset \mathbb{R}^n$, and $\|DF(\mathbf{x})\| \leq M$ on E , then f satisfies the Lipschitz condition with constant M .
3. Give an example of a function $f : [0, 1] \rightarrow \mathbb{R}$ that satisfies the Lipschitz condition but is **not** C^1 .

Q.87. Directional Derivatives and Gradient

1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable at \mathbf{a} . Show that the directional derivative $D_{\mathbf{u}}f(\mathbf{a})$ is given by $\nabla f(\mathbf{a}) \cdot \mathbf{u}$.
2. Prove that the direction of the greatest rate of increase of f at \mathbf{a} is the direction of $\nabla f(\mathbf{a})$.
3. Construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that f has all directional derivatives at $(0, 0)$, but the function fails to be continuous at $(0, 0)$.

15. Advanced Differential Forms and Integration (Q.88 - Q.92)

Q.88. Integrating Forms (Path Independence)

1. Let ω be a 1-form in a simply connected region $U \subset \mathbb{R}^n$. If ω is closed ($d\omega = 0$), show that the line integral $\int_C \omega$ is **path-independent**.
2. Let $\omega = yzdx + xzdy + xydz$. Calculate $\int_{C_1} \omega$ and $\int_{C_2} \omega$ where C_1 is the line segment from $(1, 0, 0)$ to $(0, 1, 0)$ and C_2 is a quarter circle between the same points.
3. Explain the relation of the result in (b) to the potential function f such that $df = \omega$.

Q.89. Fundamental Theorem of Calculus for Vector Fields (Curl/Stokes)

1. State the **classical Stokes' Theorem** for a vector field \mathbf{F} .
2. Identify the 1-form ω corresponding to \mathbf{F} and the 2-form $d\omega$ corresponding to $\text{curl}(\mathbf{F})$.
3. Let $\mathbf{F} = (-y, x, 0)$ and S be the unit disk in the xy -plane. Calculate both sides of Stokes' Theorem for this case to verify the theorem.

Q.90. Wedge Product in Higher Dimensions

1. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ and $\eta = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ be 2-forms in \mathbb{R}^4 . Compute the 4-form $\omega \wedge \eta$.
2. In \mathbb{R}^4 , what is the maximum degree k for which a non-zero k -form can exist?
3. Given 1-forms $\omega_1, \dots, \omega_k$, show that $\omega_1 \wedge \dots \wedge \omega_k$ is non-zero if and only if the set $\{\omega_1, \dots, \omega_k\}$ is linearly independent.

Q.91. Integration by Parts for Differential Forms

1. State the product rule for the exterior derivative: $d(\omega \wedge \eta)$.
2. Apply Stokes' Theorem to $\omega \wedge \eta$ to derive the generalized **Integration by Parts** formula:

$$\int_M d\omega \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta + \int_{\partial M} \omega \wedge \eta$$

3. Show that this formula reduces to the standard integration by parts for 1-forms on $[a, b]$.

Q.92. Improper Integrals: Uniform Convergence

1. State the condition for an improper integral $\int_a^\infty f(x, t) dt$ to converge **uniformly** with respect to the parameter x on a set E .
2. Show that $I(x) = \int_0^\infty e^{-t^2} \cos(xt) dt$ converges uniformly on any compact interval $[-M, M]$.
3. Use differentiation under the integral sign and the property $I'(x) = -\frac{x}{2}I(x)$ to find the closed-form expression for $I(x)$.

16. Synthesis and Counterexamples (Q.93 - Q.100)

Q.93. Continuity and Uniform Continuity

1. Give a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ that is not uniformly continuous.
2. Show that if $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous and $\lim_{x \rightarrow \infty} f(x) = L$ (finite), then f is uniformly continuous on $[0, \infty)$.
3. State the **Heine-Borel Theorem** and use it to prove that every continuous function on a compact set is uniformly continuous.

Q.94. Completeness and Contraction Mapping

1. Define a **contraction mapping** $T : X \rightarrow X$ on a metric space (X, d) .
2. State the **Contraction Mapping Theorem** (Banach Fixed Point Theorem). Why is completeness a crucial assumption?
3. Consider the incomplete space $X = (0, 1)$ and $T(x) = x^2$. Show that T is a contraction mapping but has **no fixed point** in X .

Q.95. Connectedness and Intermediate Value Property

1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function that satisfies the **Intermediate Value Property** (IVP). Must f be continuous?

2. State the properties of a Darboux function (one that satisfies the IVP but is highly discontinuous).
3. Prove that if $f : [a, b] \rightarrow \mathbb{R}$ satisfies the IVP and is monotonic, then f must be continuous.

Q.96. Uniform Convergence and Differentiability

1. Give an example of a sequence of differentiable functions $f_n : [0, 1] \rightarrow \mathbb{R}$ such that f_n converges uniformly to f , but f is **not differentiable** at some point.
2. Give an example of a sequence f_n that converges uniformly to f , and f'_n converges pointwise to g , but f' exists and $f' \neq g$.
3. State the theorem that provides sufficient conditions for $\lim_{n \rightarrow \infty} f'_n(x) = f'(x)$.

Q.97. Nowhere Dense and Closed Sets

1. Define a **nowhere dense** set.
2. Prove that every non-empty, nowhere dense, closed set $F \subset \mathbb{R}$ must have an empty interior.
3. Give an example of a nowhere dense set $A \subset \mathbb{R}$ that is **not** closed.

Q.98. Differentiation of Integrals (Leibniz Rule Failure)

1. State the conditions for the simple Leibniz Rule: $\frac{d}{dx} \int_a^b f(x, t) dt = \int_a^b \frac{\partial f}{\partial x}(x, t) dt$ to hold.
2. Consider $I(x) = \int_0^1 \frac{e^{-xt^2}}{t^2} dt$. Show that the conditions of the generalized Leibniz Rule fail at $x = 0$.

Q.99. Volume and Figure (Jordan Measure)

1. Let $A \subset [0, 1]^2$ be the set where y is rational. Calculate the inner volume $v_*(A)$ and the outer volume $v^*(A)$. Is A a Jordan measurable figure?
2. Let F be a Jordan measurable figure in \mathbb{R}^N . Prove that \bar{F} , F° , and ∂F are all Jordan measurable.
3. Show that the union of two Jordan measurable sets A and B is also Jordan measurable.

Q.100. Synthesis: \mathbb{R} Properties and Density

1. State the definition of an **algebraic number**.
2. Show that the set of rational numbers \mathbb{Q} is dense in the set of irrational numbers \mathbb{I} (with the standard metric inherited from \mathbb{R}).
3. Use the definition of the **supremum** to prove the existence of an irrational number between any two distinct rational numbers $a < b$.

17. Advanced Real Analysis and Set Theory (Q.101 - Q.105)

Q.101. Lebesgue Number and Covering

1. Define the **Lebesgue number** δ for an open cover $\mathcal{G} = \{G_\alpha\}$ of a compact set $K \subset \mathbb{R}^N$.
2. Prove that if \mathcal{G} is a finite open cover of a compact set K , a Lebesgue number $\delta > 0$ always exists.
3. Give an example of an open cover of a non-compact set $E \subset \mathbb{R}$ for which **no Lebesgue number** exists.

Q.102. Algebraic vs. Transcendental Numbers (Diophantine Approximation)

1. State **Liouville's Theorem** regarding the approximation of algebraic numbers by rational numbers.
2. Define a **Liouville number**. Use Liouville's Theorem to prove that every Liouville number is **transcendental**.
3. Show that $L = \sum_{k=1}^{\infty} 10^{-k!}$ is a Liouville number.

Q.103. Jordan Measure vs. Lebesgue Measure (Conceptual)

1. Define a **Jordan measurable set** $A \subset \mathbb{R}^N$ in terms of its boundary ∂A .
2. Show that the set of rational numbers $\mathbb{Q} \cap [0, 1]$ has Jordan measure zero and Lebesgue measure zero.
3. Explain the difference in measure for a set like the Smith-Volterra-Cantor set, which has Jordan measure zero but positive Lebesgue measure.

Q.104. Connectedness and Homeomorphism (Advanced)

1. Let $f : X \rightarrow Y$ be a continuous map. If X is connected, show that $f(X)$ is connected.
2. Prove that $\mathbb{R}^2 \setminus \{(0, 0)\}$ is connected.
3. Prove that \mathbb{R}^n is **not** homeomorphic to \mathbb{R} for any $n > 1$.

Q.105. Cauchy Sequences and Subsequences

1. Let (X, d) be a metric space. Show that if a Cauchy sequence $\{x_n\}$ in X has a convergent subsequence, then $\{x_n\}$ itself converges.
2. Give an example of a sequence $\{x_n\}$ in \mathbb{R} that is bounded but has **no Cauchy subsequence**. Why does this not contradict the Bolzano-Weierstrass theorem?
3. Prove that if $f : X \rightarrow Y$ is continuous and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy if and only if f is **uniformly continuous** on the domain of the sequence.

18. Advanced Approximation and Convergence (Q.106 - Q.110)

Q.106. Stone-Weierstrass on Non-Compact Intervals

1. State the Stone-Weierstrass theorem for $C[a, b]$.
2. Consider $C_0(\mathbb{R})$ (continuous functions vanishing at infinity). Show that the algebra $\mathcal{A} = \{\sum_{k=1}^n c_k e^{-kx^2}\}$ is dense in $C_0(\mathbb{R})$.
3. Show that the algebra of all polynomials $\mathcal{P}(\mathbb{R})$ is **not** dense in $C_b(\mathbb{R})$.

Q.107. Ascoli-Arzelà and Differentiable Functions

1. Show that if $\{f_n\} \subset C[a, b]$ and $\{f'_n(x)\}$ is uniformly bounded, and $\{f_n(x_0)\}$ is bounded for some x_0 , then \mathcal{F} is **equicontinuous**.
2. Prove that if $\{f_n\}$ satisfies the conditions in (a), it has a uniformly convergent subsequence.
3. Construct a sequence $\{g_n\} \subset C[0, 1]$ that is uniformly bounded but **not** equicontinuous.

Q.108. Uniform Convergence and Differentiation Failure

1. Let $f_n(x) = \frac{1}{n} \arctan(n^2 x)$. Show that $\{f_n\}$ converges uniformly to $f(x) = 0$ on \mathbb{R} .
2. Compute $f'_n(x)$ and determine its pointwise limit $g(x)$.
3. Show that $\lim_{n \rightarrow \infty} f'_n(x) \neq f'(x)$ for all x . Explain why this does not contradict the theorem for the interchange of limit and derivative.

Q.109. Weierstrass's Continuous, Nowhere Differentiable Function

1. State the conditions on a and b for the Weierstrass function $W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ to be continuous and nowhere differentiable.
2. Prove that $W(x)$ converges uniformly on \mathbb{R} .
3. State the main idea of the proof that $W(x)$ is nowhere differentiable, focusing on the growth of the derivatives of the cosine terms.

Q.110. Extension of Functions (Tietze Extension Theorem)

1. Define a **normed space**.
2. State the **Tietze Extension Theorem** for a continuous function $f : A \rightarrow \mathbb{R}$ defined on a closed subset A of a metric space X .
3. Consider $A = \{0, 1\}$ in $X = [0, 1]$. Let $f(0) = 0$ and $f(1) = 1$. Find a continuous extension $F : [0, 1] \rightarrow \mathbb{R}$ such that $\|F\|_{\infty} = \|f\|_{\infty}$.

19. Advanced Integration and Volume (Q.111 - Q.115)

Q.111. Iterated Integrals and Fubini's Theorem (Complex Domain)

1. State the condition (on the function $|f|$) for Fubini's theorem to guarantee the equality of iterated integrals over a compact cell $C \subset \mathbb{R}^N$.
2. Let $f(x, y) = \frac{x-y}{(x+y)^3}$ on $C = [0, 1] \times [0, 1]$. Show that the iterated integrals are **not** equal.
3. Explain why this failure does not contradict Fubini's Theorem.

Q.112. Improper Integrals and Uniform Convergence (Comparison)

1. State the **Dirichlet Test** for the convergence of an improper integral $\int_a^\infty f(x)g(x)dx$.
2. Show that $I(\alpha) = \int_1^\infty \frac{\sin x}{x^\alpha} dx$ converges for $\alpha > 0$.
3. Determine the values of α for which $I(\alpha)$ converges **absolutely**.

Q.113. Volume of N -Dimensional Ball

1. State the formula for the volume $V_N(R)$ of an N -dimensional ball of radius R in terms of the Gamma function Γ .
2. Prove the recurrence relation $V_N(R) = \frac{2\pi R^2}{N} V_{N-2}(R)$.
3. Use the recurrence relation to compute $V_4(R)$ and $V_5(R)$.

Q.114. Change of Variables in \mathbb{R}^N (Jacobian)

1. State the Change of Variables Theorem for a multiple integral $\int_A f(\mathbf{x})d\mathbf{x}$ using a C^1 transformation Φ .
2. Let $\Phi(u, v) = (u^2, v^2)$. Compute the Jacobian $J_\Phi(u, v)$.
3. Use the Change of Variables formula to compute the integral $\iint_R \sin(\sqrt{x} + \sqrt{y})dxdy$, where $R = [0, 1] \times [0, 1]$.

Q.115. Darboux Integrability and Oscillation

1. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Let $D_\epsilon = \{x \in [a, b] \mid \omega_f(x) \geq \epsilon\}$. Show that D_ϵ is a closed set.
2. Prove that f is Riemann integrable if and only if for every $\epsilon > 0$, the set D_ϵ has **measure zero**.
3. Use (b) to show that the Dirichlet function $\chi_{\mathbb{Q}}(x)$ is **not** Riemann integrable on $[0, 1]$.

20. Advanced Differential Forms and Geometry (Q.116 - Q.125)

Q.116. Poincaré Lemma and Non-Star-Shaped Domains

1. State the **Poincaré Lemma** for an open star-shaped set $U \subset \mathbb{R}^N$.
2. Define a **star-shaped set**.
3. Give an example of a closed 1-form ω on $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ that is **not exact**. Explain why this does not contradict the Poincaré Lemma.

Q.117. Pullback and Chain Rule

1. Let $F(x, y, z) = (u, v, w)$ and $\omega = f(u, v, w)du$. Compute the pullback $F^*\omega$.
2. If $\eta = dw$ is an exact form, compute $F^*\eta$. Show that $d(F^*\omega) = F^*(d\omega)$.

Q.118. Wedge Product and Orientation (3-forms)

1. Define the 3-form volume element $dV = dx \wedge dy \wedge dz$ in \mathbb{R}^3 .
2. If $\Phi(x, y, z) = (y, x, z)$, compute the pullback $\Phi^*(dx \wedge dy \wedge dz)$.
3. Explain how the result from (b) shows that the transformation Φ **reverses orientation**.

Q.119. Exact Forms and Path Independence (Revisited)

1. Prove that $\int_C \omega = 0$ for every closed path $C \subset U$ if and only if ω is **exact** on U .
2. Let $\omega = ydx - xdy$. Show that ω is not exact on \mathbb{R}^2 .
3. Show that ω is exact on the half-plane $U = \{(x, y) \mid x > 0\}$. Find the potential function f such that $df = \omega$ on U .

Q.120. Codifferential and Divergence (Revisited)

1. Let $\omega_{\text{flux}} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$. Show that $d\omega_{\text{flux}}$ corresponds to $\text{div}(\mathbf{F})dx \wedge dy \wedge dz$.
2. Define the codifferential δ acting on a 3-form $\eta = f dx \wedge dy \wedge dz$ in \mathbb{R}^3 .
3. Show that if ω is the 1-form associated with \mathbf{F} , then $\delta\omega$ corresponds to $-\text{div}(\mathbf{F})$.

Q.121. Fundamental Theorem of Calculus for Differential Forms (The Core)

1. State the **Generalized Stokes' Theorem**.
2. Show that the standard integration by parts formula is an application of Stokes' theorem.
3. Show that the integral of an exact k -form $\omega = d\eta$ over a closed k -manifold M (a k -cycle) must be zero.

Q.122. Baire's Theorem and C^1 Functions

1. Show that the set of functions $\mathcal{D} = C^1[0, 1]$ is of the **first category** in $C[0, 1]$.

2. Use the Baire Category Theorem to argue that $C[0, 1] \setminus \mathcal{D}$ (non- C^1 functions) is dense in $C[0, 1]$.

Q.123. Oscillation and Continuity Sets

1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Let $C(f)$ be the set of points where f is continuous. Show that $C(f)$ is a G_δ set.
2. Show that any G_δ set $E \subset \mathbb{R}$ can be the set of continuity points of some function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Q.124. Mollifiers and Convergence (Non-Compact Support)

1. Define the support of a function, $\text{supp}(f)$.
2. Let $f(x) = \frac{1}{1+x^2}$. Let $f_\epsilon = f * \phi_\epsilon$. Does $f_\epsilon \rightarrow f$ uniformly on all of \mathbb{R} ? Prove or provide a counterexample.
3. Prove that $f_\epsilon \rightarrow f$ uniformly on every **compact subset** $K \subset \mathbb{R}$.

Q.125. The Hessian of a Potential Function

1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 scalar field. Show that the 1-form $\omega = df$ is always closed ($d\omega = 0$).
2. Relate the condition $d\omega = 0$ to the Hessian matrix $H(\mathbf{x})$ of f .
3. Explain the connection between $d(df)$ and the symmetry of the second partial derivatives.

21. Advanced Topology and Measure Theory Synthesis (Q.126 - Q.130)

Q.126. Compactness and Continuous Image

1. Let $f : X \rightarrow Y$ be a continuous function. If $K \subset X$ is compact, prove that $f(K)$ is compact in Y .
2. Give an example of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a closed, bounded set $E \subset \mathbb{R}$ such that $f(E)$ is closed but **not bounded**.
3. Define a **proper map**. Prove that if $f : X \rightarrow Y$ is a proper map and X is complete, then Y is also complete.

Q.127. Density and Topological Bases

1. Define a **topological base** for a metric space (X, d) .
2. Prove that a metric space (X, d) is separable if and only if it admits a **countable base** for its topology.
3. Consider the space $X = \mathbb{R}$ with the discrete metric. Determine if (X, d) is separable.

Q.128. Baire's Theorem and Function Properties

1. Show that the set of polynomials $\mathcal{P}[0, 1]$ is a dense subset of $C[0, 1]$.
2. Define a set of the **first category**. If X is a complete metric space, and A is of the first category, what can you conclude about the interior of A ?
3. Use Baire's Category Theorem to argue that there must exist continuous functions that **cannot** be uniformly approximated by polynomials on any subinterval of $[0, 1]$.

Q.129. Bolzano-Weierstrass in \mathbb{R}^N

1. State the **Bolzano-Weierstrass Theorem** for a sequence in \mathbb{R}^N .
2. Prove the Bolzano-Weierstrass Theorem in \mathbb{R}^2 using the d_2 metric, by applying the one-dimensional result twice.
3. Give an example of a closed and bounded set E in a metric space (X, d) that does **not** satisfy the Bolzano-Weierstrass property.

Q.130. Closure and Boundary (Topological Properties)

1. Let $A, B \subset X$. Prove that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
2. Give a counterexample to show that $\overline{A \cap B} = \bar{A} \cap \bar{B}$ is **not** generally true.
3. Prove that the **boundary** of a set A , $\partial A = \bar{A} \cap \overline{X \setminus A}$, is always a closed set.

22. Advanced Differentiation and Extrema (Q.131 - Q.135)

Q.131. Inverse Function Theorem and Global Invertibility

1. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C^1 . If $\det(J_F(\mathbf{x})) \neq 0$ everywhere, must F be globally injective? Prove or provide a counterexample.
2. State the **Hadamard-Cacciopoli Theorem** (or Global Inverse Function Theorem).
3. Consider $F(x) = x + \sin x$ on \mathbb{R} . Is F globally invertible?

Q.132. Implicit Function Theorem (High-Dimensional)

1. Let $F : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ be given by $F(\mathbf{x}) = (x_1^2 + x_2^2 + x_3^2 - 1, x_4^2 + x_5^2 - 1)$.
2. Show that near $P = (1, 0, 0, 1, 0)$, the level set $F(\mathbf{x}) = \mathbf{0}$ defines a smooth 3-dimensional manifold.
3. Compute the partial derivative $\frac{\partial x_4}{\partial x_2}$ at P .

Q.133. Mean Value Theorem for Integrals

1. State the **First Mean Value Theorem for Integrals**.
2. State the **Second Mean Value Theorem for Integrals** (Bonnet's form).
3. Use the second mean value theorem to prove the convergence of the improper integral $\int_1^\infty \frac{\sin x}{x} dx$.

Q.134. Critical Points and Global Extrema

1. State the theorem that guarantees the existence of global extrema for a continuous function f on a compact set K .
2. Let $f(x, y) = x^2 - 2xy + 2y^2 - 2y$ on the square region $K = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$. Find the absolute maximum and minimum values of f on K .
3. Explain why finding extrema on the boundary ∂K is necessary.

Q.135. Hessian and Directional Information

1. The second directional derivative is $D_{\mathbf{u}}^2 f(\mathbf{a}) = \mathbf{u}^T H(\mathbf{a}) \mathbf{u}$.
2. If \mathbf{a} is a critical point, show that if $H(\mathbf{a})$ is **indefinite**, then \mathbf{a} is a **saddle point**.
3. Construct a function $f(x, y)$ such that $H(0, 0)$ has $\det(H) = 0$ but $(0, 0)$ is a local minimum.

23. Advanced Differential Forms and Stokes' Theorem (Q.136 - Q.145)

Q.136. Exactness and Topology (Simply Connected)

1. Define a **simply connected** region U in terms of closed and exact forms.
2. Show that the annulus $A = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$ is **not** simply connected by finding a closed, non-exact 1-form ω on A .
3. Show that any contractible domain (like a star-shaped set) is simply connected.

Q.137. Pullback and Vector Fields

1. Let $F(r, \theta) = (x, y)$. Let $\mathbf{G} = (x, -y)$. Find the 1-form ω associated with \mathbf{G} .
2. Compute the pullback $F^*\omega$ and interpret the result in the (r, θ) coordinate system.

Q.138. The Operator $d^2 = 0$ (Formal Proof)

1. Let $\omega = Pdx + Qdy$ be a 1-form in \mathbb{R}^2 . Compute $d\omega$.
2. Compute $d(d\omega)$ for the 1-form in (a). Show that the result is 0 by invoking the equality of mixed partials.

Q.139. Flux and Volume Integral (Gauss/Divergence)

1. Let $\mathbf{F} = \text{curl}(\mathbf{G})$. Use the Divergence Theorem to prove that the flux of \mathbf{F} out of ∂V is zero.
2. Use the $d^2 = 0$ property and the Generalized Stokes' Theorem to prove the same result using differential forms.

Q.140. Non-Differentiable Path and Line Integrals

1. Explain how the line integral $\int_C \omega$ can be extended to a rectifiable curve C .
2. Is the Koch curve K a rectifiable curve? Justify your answer.
3. If $\omega = df$ is an exact form, and C is any rectifiable path from A to B , show that $\int_C \omega = f(B) - f(A)$.

Q.141. Change of Variables and Non-Invertible Jacobian

1. State the condition on the Jacobian determinant for the Change of Variables Theorem to be valid.
2. Let $F(r, \theta) = (r \cos \theta, r \sin \theta)$. Compute $\det(J_F)$.
3. Explain why the Change of Variables formula remains valid for integrating over the unit disk B , even though J_F is zero along the line segment $r = 0$.

Q.142. Integration by Parts for Multi-Variable Functions

1. State the **Integration by Parts formula** for the inner product $\int_U \nabla f \cdot \mathbf{G} d\mathbf{x}$.
2. Write the result in (a) using differential forms and the generalized Integration by Parts formula.

Q.143. Convergence of Integrals (Non-Uniform Case)

1. Let $f_n(x) = nxe^{-nx^2}$ on $[0, 1]$. Show that $f_n \rightarrow 0$ pointwise on $[0, 1]$.
2. Compute $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$. Show that this limit is not equal to $\int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx$.
3. Explain why the convergence is not uniform.

Q.144. Equicontinuity and Closed Sets

1. Let X be a compact metric space. Show that if $\mathcal{F} \subset C(X)$ is closed, uniformly bounded, and equicontinuous, then \mathcal{F} is **compact** in $C(X)$.
2. Give an example of a set of equicontinuous and uniformly bounded functions \mathcal{F} in $C[0, 1]$ that is **not closed**.

Q.145. Mollifiers and Density (Advanced)

1. Prove that the space of smooth functions with compact support $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$.
2. Show that $C_c^\infty(\mathbb{R}^N)$ is **not** dense in $L^\infty(\mathbb{R}^N)$.

24. Synthesis and Final Challenges (Q.146 - Q.150)

Q.146. Riemann Integrability and Countable Discontinuities

1. Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{\{1/n\}}(x)$. Find the set of discontinuities of f .
2. Prove that f is Riemann integrable on $[0, 1]$.
3. Calculate the value of the Riemann integral $\int_0^1 f(x) dx$.

Q.147. Density and Algebraic Closure

1. Show that the set of transcendental numbers \mathbb{T} is dense in \mathbb{R} .
2. Prove that the set of algebraic numbers \mathbb{A} has an empty interior in \mathbb{R} .
3. Use the Baire Category Theorem (BCI) to argue that \mathbb{R} is **not** a countable union of closed sets with empty interiors.

Q.148. Generalized Fundamental Theorem of Calculus (Boundary)

1. Let $M = [0, 1]^2$. Describe the boundary ∂M and its induced orientation.
2. Let $\omega = x^2 dy$ be a 1-form. Compute $d\omega$.
3. Use the Generalized Stokes' Theorem, $\int_M d\omega = \int_{\partial M} \omega$, to verify the theorem for this ω and M .

Q.149. The Critical Point Test (Advanced)

1. For a C^3 function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f'(a) = f''(a) = 0$, state the condition on $f'''(a)$ that determines if a is a point of inflection.
2. Generalize the Second Derivative Test for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ when $H(\mathbf{a})$ is **positive semi-definite** (but not definite). Must \mathbf{a} be a local minimum? Provide a counterexample.
3. Construct a function $f(x, y)$ such that $H(0, 0)$ is positive semi-definite, and $(0, 0)$ is **not** a local minimum.

Q.150. Stone-Weierstrass and Function Norms (Final Synthesis)

1. Use the Stone-Weierstrass theorem to show that the algebra of trigonometric polynomials is dense in $C[0, 2\pi]$ (with $f(0) = f(2\pi)$).
2. Show that if a sequence of functions $\{f_n\}$ in $C[a, b]$ converges to f in the L^1 norm, the convergence does **not** imply pointwise convergence almost everywhere.
3. State the relationship between the L^∞ norm (sup norm), the L^p norm ($1 \leq p < \infty$), and uniform convergence on a compact set K .