MID-TERM EXAMINATION

Nov. 6, 2022

Question	Points	Score
True or False	15	
The Simplex Method and Simplex Tableau	18	
Duality	20	
Sensitivity Analysis	15	
Optimization Formulation	14	
Null Variables	18	
Total:	100	

- Please write down your **student ID** on the **answer paper**.
- Please justify your answers except Question 1.
- The exam time is 90 minutes.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

Question 1 [15 points]: True or False

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) [3 points] For any arbitrary nonempty polyhedron, it has at least one extreme point.

Solution: (The geometry of LP) **False.** Choose $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ which is a halfspace, so no vertex.

(b) [3 points] Define a square $S = \{x \in \mathbb{R}^2 \mid 0 \leq x_i \leq 1, i = 1, 2\}$ and a disk $D = \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$. Then $S \cap D$ is convex.

Solution: (Convex set) **True.** S and D are both convex and the intersection of convex set is still convex.

(c) [3 points] Consider a standard LP with n variables and m constraints. Suppose it has a finite optimal solution, then the optimal solution returned by the simplex method must have no more than m strictly positive entries.

Solution: (Simplex method) **True.**

(d) [3 points] Consider a standard form LP problem and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. If the problem is unbounded, then its dual problem is infeasible.

Solution: (Daul problem, simplex method) **True.**

(e) [3 points] For linear optimization problems, if the primal problem has a feasible solution, then the dual problem must also have a feasible solution.

Solution: (Duality) **False.**

Question 2 [18 points]: The Simplex Method and Simplex Tableau

Consider a LP problem with an unknown K ($0 < K < 6$):

$$\begin{aligned}\underset{x_1,x_2,x_3\in\mathbb{R}}{\text{minimize}}\ & -2x_1 - 4x_2 + 6x_3 \\ \text{s.t.}\ & 2x_2 + 2x_3 \leq K \\ & x_1 + x_2 + 3x_3 \leq 8 \\ & 2x_1 + 2x_2 - x_3 \leq 6 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\end{aligned}$$

(a) [4 points] Derive the standard form of the LP problem.

Solution: We transform the problem into the standard form:

$$\begin{aligned}\text{minimize}\quad & -2x_1 - 4x_2 + 6x_3 \\ \text{subject to}\quad & 2x_2 + 2x_3 + x_4 = K \\ & x_1 + x_2 + 3x_3 + x_5 = 8 \\ & 2x_1 + 2x_2 - x_3 + x_6 = 6 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0\end{aligned}$$

(b) [10 points] Use the simplex method and obtain the final simplex tableau.

$-c_B^\top x_B$

Solution: The problem is then already in standard form and the initial simplex tableau is given by:

B	-2	-4	6	0	0	0	0
4	0	2	2	1	0	0	K
5	1	1	3	0	1	0	8
6	2	2	-1	0	0	1	6

The pivot column is $\{1\}$; the pivot row is $\{6\}$; the pivot element is 2; after the row updates we obtain the new tableau:

B	0	-2	5	0	0	1	6
4	0	2	2	1	0	0	K
5	0	0	3.5	0	1	-0.5	5
1	1	1	-0.5	0	0	0.5	3

The pivot column is $\{2\}$; the pivot row is $\{4\}$; the pivot element is 2; after the row updates we obtain the new tableau:

B	0	0	7	1	0	1	6+K
2	0	1	1	0.5	0	0	0.5K
5	0	0	3.5	0	1	-0.5	5
1	1	0	-1.5	-0.5	0	0.5	3-0.5K

Since there are no negative reduced costs, we have already got the final tableau.

(c) [4 points] If the optimal solution is $\mathbf{x}^* = [1, 2, 0]^\top$. What is the value of K?

Solution: The optimal solution is $x^* = (3 - 0.5K, 0.5K, 0)$ based on the final tableau. $3 - 0.5K = 1, 0.5K = 2$, so we get $K = 4$.

Question 3 [20 points]: Duality

You have downloaded a program from a website of unknown quality to solve LP problems of the form

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \ & \mathbf{c}^\top \mathbf{x} \\ \text{s.t. } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

You test the program with the following data:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 3 & 3 & 2 \\ 2 & 4 & 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 & 3 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 14 \\ 16 \\ 10 \end{bmatrix}, \mathbf{c}^\top = [2, 3, 2, 2, 3, 2].$$

The program prints the following: "An optimal solution to the problem is $\mathbf{x} = [3, 2, 1, 0, 0, 0]^\top$, and an optimal solution to the corresponding dual problem is $\mathbf{y} = [0.25, 0.50, 0.25]^\top$".

(a) [5 points] Verify whether the result of the program is correct or not, and give your reason.

Solution: If the primal problem is on the form

$$\begin{aligned} \text{minimize} \mathbf{c}^\top \mathbf{x} \\ \text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

then the corresponding dual problem is

$$\begin{aligned} \text{maximize} \mathbf{b}^\top \mathbf{y} \\ \text{s.t. } \mathbf{A}^\top \mathbf{y} \le \mathbf{c}. \end{aligned}$$

It is well-known that if

(i). $\mathbf{x}$ is a feasible solution to the primal problem,

(ii). $\mathbf{y}$ is a feasible solution to the dual problem, and

(iii). $\mathbf{c}^\top \mathbf{x} = \mathbf{b}^\top \mathbf{y}$,

then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions to their respective problem. The program proposed $\mathbf{x}$ and $\mathbf{y}$ fulfill $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}, \mathbf{A}^\top \mathbf{y} \le \mathbf{c}$ and $\mathbf{c}^\top \mathbf{x} = \mathbf{b}^\top \mathbf{y}$, so they are optimal to the primal and the dual respectively.

(b) [10 points] Assume that the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ above are changed to the constraints $\mathbf{A}\mathbf{x} \ge \mathbf{b}$. Find an optimal solution to the new problem.

Solution: If the primal problem has the form of

$$\begin{aligned} \text{minimize } \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \mathbf{A}\mathbf{x} \ge \mathbf{b} \\ \mathbf{x} \ge \mathbf{0} \end{aligned}$$

then the corresponding dual problem is

$$\begin{aligned} \text{maximize} \mathbf{b}^\top \mathbf{y} \\ \text{s.t. } \mathbf{A}^\top \mathbf{y} \le \mathbf{c} \\ \mathbf{y} \ge \mathbf{0} \end{aligned}$$

But $\mathbf{x}$ and $\mathbf{y}$ from the (a)-task above fulfill that $\mathbf{A}\mathbf{x} \ge \mathbf{b}, \mathbf{x} \ge \mathbf{0}, \mathbf{A}^\top \mathbf{y} \le \mathbf{c}, \quad \mathbf{y} \ge \mathbf{0}$ and $\mathbf{c}^\top \mathbf{x} = \mathbf{b}^\top \mathbf{y}$ Hence they are optimal also to these two problems.

(c) [5 points] Suppose that the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ above are changed to $\mathbf{A}\mathbf{x} \le \mathbf{b}$. Find an optimal solution to the new problem.

Solution: If the problem is on the form

$$\begin{aligned} \text{minimize } \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ \mathbf{x} \ge \mathbf{0} \end{aligned}$$

with the given $\mathbf{A}, \mathbf{b}$ and $\mathbf{c}$, then it is realized by "inspection" that $\hat{\mathbf{x}} = (0,0,0,0,0,0)^\top$ is the unique optimal solution. This is a feasible solution with the objective function value $\mathbf{c}^\top \hat{\mathbf{x}} = 0$, and for every other feasible solution $\mathbf{x}$ it holds that $\mathbf{c}^\top \mathbf{x} > 0$, since $\mathbf{c} > \mathbf{0}, \mathbf{x} \ge \mathbf{0}$ and at least one $x_j > 0$. As an alternative to "inspection" you can introduce slack variables and let these be starting basic variables in the simplex method. You then immediately realize that the starting basic solution is optimal.

Question 4 [15 points]: Sensitivity Analysis

B	$\frac{3}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
2	1	1	0	1	5
3	$-\frac{1}{2}$	0	1	-$\frac{1}{2}$	$\frac{1}{2}$

Table 1: Simplex Tableau for Sensitivity Analysis

$\bar{C}_N^{new} = \bar{C}_N - \Delta C_B^T A_B^{-1} A_N$

$= [\frac{3}{2}, \frac{1}{2}] - [0, \lambda] [\begin{smallmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{smallmatrix}$

$= [\frac{3}{2} + \frac{1}{2}\lambda, \frac{1}{2} + \frac{1}{2}\lambda] \geq 0.$

Consider the following linear program:

$$\begin{aligned} \underset{x_1,x_2,x_3 \in \mathbb{R}}{\text{minimize}} \quad & x_1 + \quad x_3 \\ \text{s.t.} \quad & x_1 + x_2 \quad \leq 5 \\ & \frac{1}{2}x_2 + x_3 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Table 1 gives the final simplex tableau when solving the standard form of the above problem (after adding variable $x_4 \geq 0$ in the first constraint i.e. $x_1 + x_2 \leq 5$).

(a) [3 points] From Table 1, what is the optimal solution and the optimal value of the original problem?

Solution: Directly from simplex tableau, the optimal solution is $x^* = (0, 5, \frac{1}{2})^T$ and the optimal value is $\frac{1}{2}$.

$\lambda \geq -1.$

$\lambda \geq -3$

$\lambda \geq -1.$

(b) [6 points] The vector $\mathbf{b}$ is changed from $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Solve the new problem (Hint: use Table 1).

range $[0, +\infty]$

Solution: After changing b from $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, we have

$$\tilde{b} = A_B^{-1} b = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}^{-1} * \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} * \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\tilde{z} = c_B^T * \tilde{b} = (0, 1) * \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2$$

(2 pt) The simplex tableau becomes the following table.

B	$\frac{3}{2}$	0	0	$\frac{1}{2}$	-2
2	1	1	0	1	-2
3	$-\frac{1}{2}$	0	1	-$\frac{1}{2}$	2

(1 pt) From the table, we conclude that there is no solution for new problem ($b_1 = -2$ but $a_{1j} > 0$).

(c) [6 points] In what range can we change the objective coefficient $c_3 = 1$, so that the current optimal solution is still optimal to the resulting new problem?

Solution: (2 pt) Since $j = 3 \in B$, the condition to keep optimal solution is

$$0 \leq r_N^T - \lambda(0, 1) A_B^{-1} A_N$$

s.t. $|x_1 - x_2| \leq 10$

$x_1 - x_2 \geq -10 \Rightarrow -x_1 + x_2 \leq 10$

$\Rightarrow$ s.t. $x_1 - x_2 + x_3 = 10$

$-x_1 + x_2 + x_4 = 10$

(2 pt) From simplex tableau and linear programming, we have

$$= (\frac{3}{2}, \frac{1}{2}) - \lambda(0,1) * \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= (\frac{3}{2}, \frac{1}{2}) + \lambda(\frac{1}{2}, \frac{1}{2}) \geq 0$$

(1 pt) Then, the condition on λ is $\lambda \geq -1$.
(1 pt) Thus, we can choose $c_3 \in [0, \infty)$.

Question 5 [14 points]: Optimization Formulation

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are Rs.5 per hour while those of grade II inspector are Rs.4 per hour. Any error made by an inspector costs Rs.3 to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company.

(a) [8 points] Formulate an optimization problem for finding an optimal assignment of inspectors (Assignment of inspectors refers to determine how many grade I and grade II inspectors that should be assigned the job of quality control inspection.) that minimizes the daily inspection cost.
Note: The assignment of inspectors should be modeled as integer variables. However, since we do not know how to deal with these integer constraints in general at this moment, you may just ignore them for now.

Solution: Let x_1 and x_2 denote the number of grade I and grade II inspectors that may be assigned the job of quality control inspection.
The objective is to minimise the daily cost of inspection. Now the company has to incur two types of costs; wages paid to the inspectors and the cost of their inspection errors. The cost of grade I inspector/hour is

$$\text{Rs. } (5 + 3 * 0.04 * 20) = \text{Rs. } 7.4$$

Similarly, cost of grade II inspector/hour is

$$\text{Rs. } (4 + 3 * 0.08 * 14) = \text{Rs. } 7.36$$

The objective function is

$$\min Z = 8(7.4x_1 + 7.36x_2)$$

Constrains are: on the number of grade I inspectors: $x_1 \leq 10$.
on the number of grade II inspectors: $x_2 \leq 15$.
on the number of pieces to be inspected daily: $20 * 8x_1 + 14 * 8x_2 \geq 1500$, where $x_1 \geq 0, x_2 \geq 0$.
In conclusion, we have the optimization problem:

$$\begin{aligned} \min \quad & 8(7.4x_1 + 7.36x_2) \\ \text{subject to} \quad & x_1 \leq 10 \\ & x_2 \leq 15 \\ & 160x_1 + 112x_2 \geq 1500 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(b) [6 points] Transform it into a standard form. Determine whether it has an optimal solution. What is the type of this optimization problem (constrained vs unconstrained, continuous vs discrete)?

Solution: The standard form:

$$\begin{aligned} \min \quad & 8(7.4x_1 + 7.36x_2) \\ \text{subject to} \quad & x_1 + s_1 = 10 \\ & x_2 + s_2 = 15 \\ & 160x_1 + 112x_2 - s_3 = 1500 \\ & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0 \end{aligned}$$

Denote A as

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 160 & 112 & 0 & 0 & -1 \end{bmatrix}$$

The optimization problem is feasible and bounded, it has an optimal solution. It is constrained, continuous optimization problem.

Question 6 [18 points]: Null Variables

Let $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ be a nonempty polyhedron in $\mathbb{R}^n$, and let m be the dimension of the vector $\mathbf{b}$. We call x_j, the jth entry of $\mathbf{x}$, a null variable if $x_j = 0$ whenever $\mathbf{x} \in P$.

(a) [4 points] For the coefficients $\mathbf{A}$ and $\mathbf{b}$ that define the polyhedron P, consider the following LP problem.

$$\begin{aligned} \underset{\mathbf{y} \in \mathbb{R}^m}{\text{minimize}} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^\top \mathbf{y} \geq \mathbf{e}_j, \end{aligned}$$

where $\mathbf{e}_j$ is a unit vector, of which the jth entry is 1 and the other entries are 0. Write down its dual problem.

Solution: Its dual problem is as follows:

$$\begin{aligned} \underset{\mathbf{x} \in \mathbb{R}^n}{\text{maximize}} \quad & x_j \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned}$$

(b) [8 points] Suppose that there exists some $\mathbf{p} \in \mathbb{R}^m$ for which $\mathbf{A}^\top \mathbf{p} \geq 0,\ \mathbf{p}^\top \mathbf{b} = 0$, and the jth entry of $\mathbf{A}^\top \mathbf{p}$ is positive. Show that x_j is a null variable.

Solution: From the existence of $\mathbf{p}$, we know that problem in (a) is feasible and at the feasible point the objective value is 0. From its dual problem, we also known that feasible region of the dual problem is P, implying that the dual problem is also feasible. By weak duality theorem, for all $\mathbf{x} \in P$, $x_j \leq 0$. On the other hand, $x_j \geq 0$ for any $\mathbf{x} \geq 0$, hence we have that $x_j = 0$ whenever $\mathbf{x} \in P$.

(c) [6 points] Prove that if x_j is a null variable then there exists some $\mathbf{p} \in \mathbb{R}^m$ with the properties stated in (b).

Solution: The dual problem in (a) has an optimal solution, and any feasible point is its optimal solution. From strong duality theorem, we known that the problem in (a) has an optimal solution $\mathbf{y}^*$ and $\mathbf{b}^\top \mathbf{y}^* = 0$, which completes the proof.

(b) [6 points] Transform it into a standard form. Determine whether it has an optimal solution. What is the type of this optimization problem (constrained vs unconstrained, continuous vs discrete)?

Solution: The standard form:

$$\begin{aligned} \min \quad & 8(7.4x_1 + 7.36x_2) \\ \text{subject to} \quad & x_1 + s_1 = 10 \\ & x_2 + s_2 = 15 \\ & 160x_1 + 112x_2 - s_3 = 1500 \\ & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0 \end{aligned}$$

Denote A as

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 160 & 112 & 0 & 0 & -1 \end{bmatrix}$$

The optimization problem is feasible and bounded, it has an optimal solution. It is constrained, continuous optimization problem.

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Let $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ be a nonempty polyhedron in $\mathbb{R}^n$, and let m be the dimension of the vector $\mathbf{b}$. We call x_j, the jth entry of $\mathbf{x}$, a null variable if $x_j = 0$ whenever $\mathbf{x} \in P$.

(a) [4 points] For the coefficients $\mathbf{A}$ and $\mathbf{b}$ that define the polyhedron P, consider the following LP problem.

$$\begin{aligned} \underset{\mathbf{y} \in \mathbb{R}^m}{\text{minimize}} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^\top \mathbf{y} \geq \mathbf{e}_j, \end{aligned}$$

where $\mathbf{e}_j$ is a unit vector, of which the jth entry is 1 and the other entries are 0. Write down its dual problem.

Solution: Its dual problem is as follows:

$$\begin{aligned} \underset{\mathbf{x} \in \mathbb{R}^n}{\text{maximize}} \quad & x_j \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned}$$

(b) [8 points] Suppose that there exists some $\mathbf{p} \in \mathbb{R}^m$ for which $\mathbf{A}^\top \mathbf{p} \geq 0,\ \mathbf{p}^\top \mathbf{b} = 0$, and the jth entry of $\mathbf{A}^\top \mathbf{p}$ is positive. Show that x_j is a null variable.

Solution: From the existence of $\mathbf{p}$, we know that problem in (a) is feasible and at the feasible point the objective value is 0. From its dual problem, we also known that feasible region of the dual problem is P, implying that the dual problem is also feasible. By weak duality theorem, for all $\mathbf{x} \in P$, $x_j \leq 0$. On the other hand, $x_j \geq 0$ for any $\mathbf{x} \geq 0$, hence we have that $x_j = 0$ whenever $\mathbf{x} \in P$.

(c) [6 points] Prove that if x_j is a null variable then there exists some $\mathbf{p} \in \mathbb{R}^m$ with the properties stated in (b).

Solution: The dual problem in (a) has an optimal solution, and any feasible point is its optimal solution. From strong duality theorem, we known that the problem in (a) has an optimal solution $\mathbf{y}^*$ and $\mathbf{b}^\top \mathbf{y}^* = 0$, which completes the proof.