Advanced Linear Algebra (49 Problems)

Final Exam Review Questions collected by Wang Yuyao

Skipping Guidance: **If your course curriculum has not covered a particular topic please feel free to skip those problems entirely.** Focus your efforts on the core material related to universal property, determinant, module, Cayley-Hamilton theorem, and Jordan normal form.

I. Vector Space Structure and Decomposition

Problem 1 (Internal Direct Sum and Projections). Let V be a vector space, and $W_1, \dots, W_k$ be subspaces such that $V = W_1 \oplus \cdots \oplus W_k$ (internal direct sum).

(a) Prove that there exists a unique set of linear maps $\pi_i : V \to W_i$ (projections) such that $v = \sum_{i=1}^{k} \pi_i(v)$ for all $v \in V$, $\pi_i \circ \pi_j = 0$ for $i \neq j$, and $\pi_i^2 = \pi_i$.

(b) Consider the algebra $A \subseteq \mathrm{Hom}(V, V)$ generated by $\{\pi_1, \dots, \pi_k\}$. Prove that A is a commutative algebra generated by idempotents.

(c) Construct a canonical isomorphism $\Phi : V \to W_1 \times \cdots \times W_k$ (external direct product).

Problem 2 (External Direct Product vs. Sum (Infinite)). Let V_i be non-zero vector spaces over a field F for $i \in \mathbb{N}$. Let $P = \prod_{i=1}^{\infty} V_i$ (external direct product) and $S = \bigoplus_{i=1}^{\infty} V_i$ (external direct sum).

(a) Give the formal definitions of S and P as sets of sequences.

(b) Prove that S is a proper subspace of P.

(c) Prove that P and S are **never** isomorphic if $\dim(V_i) \geq 1$ for all i.

Problem 3 (The Splitting Lemma). A short exact sequence (SES) of vector spaces is $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$.

(a) Prove that any SES of vector spaces is **split**, meaning $V \cong U \oplus W$.

(b) Provide a counterexample of an SES of $\mathbb{Z}$-modules (Abelian groups) that is **not** split.

Problem 4 (Universal Property of the Quotient Space). Let W be a subspace of V, and $\pi : V \to V/W$ be the canonical projection.

(a) State the universal property that characterizes the quotient space $(V/W, \pi)$.

(b) Use this property to prove the First Isomorphism Theorem: $V/\mathrm{Ker}(T) \cong \mathrm{Im}(T)$ for any linear map $T : V \to U$.

(c) Prove that a linear operator $S : V \to V$ induces a well-defined map $\bar{S} : V/W \to V/W$ if and only if W is S-invariant ($S(W) \subseteq W$).

Problem 5 (Quotients and the Tower Law). Let $W_1 \subseteq W_2 \subseteq V$ be subspaces.

(a) Prove the Third Isomorphism Theorem (Tower Law): $(V/W_1)/(W_2/W_1) \cong V/W_2$.

(b) Using quotient spaces, derive the dimension formula: $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

II. Dual Spaces and Annihilators

Problem 6 (Canonical Isomorphism $V \cong V^{**}$). (a) Define $\Phi : V \to V^{**}$ by $\Phi(v)(f) = f(v)$. Prove that Φ is well-defined, injective, and linear.

(b) Prove that Φ is an isomorphism if and only if $\dim(V)$ is finite.

(c) Explain why Φ is called the **canonical** (basis-independent) isomorphism, while $V \cong V^*$ is not.

Problem 7 (Annihilators and Subspaces). Let W be a subspace of a finite-dimensional space V.

(a) Prove that $\dim(W) + \dim(Ann(W)) = \dim(V)$.

(b) Prove that $W = Ann(Ann(W))$.

(c) Let $T : V \to U$. Prove that $\mathrm{Ker}(T) = Ann(\mathrm{Im}(T^t))$.

Problem 8 (Duality of Sums and Intersections). Let W_1, W_2 be subspaces of a finite-dimensional space V.

(a) Prove that $Ann(W_1 \cap W_2) = Ann W_1 + Ann W_2$.

(b) Prove that $Ann(W_1 + W_2) = Ann W_1 \cap Ann W_2$.

(c) Prove the isomorphism $\mathrm{Hom}(V, U)^* \cong \mathrm{Hom}(U, V)$.

Problem 9 (Dual Basis and Coordinates). Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V, and $\mathcal{B}^* = \{f_1, \ldots, f_n\}$ be the dual basis.

(a) Prove that for any $v \in V$, $v = \sum_{i=1}^{n} f_i(v)v_i$.

(b) Prove that for any $f \in V^*$, $f = \sum_{i=1}^{n} f(v_i)f_i$.

III. Category Theory Basics

Problem 10 (Initial and Terminal Objects). (a) Define what an **initial object** I and a **terminal object** T are in a category $\mathcal{C}$.

(b) Prove that if an initial object exists, it is unique up to a **unique isomorphism**.

(c) Identify the initial object and the terminal object in the category **Set** (sets and functions).

Problem 11 (Products and Coproducts). (a) In the category **Set**, show that the Cartesian product $A \times B$ serves as both the **product** and the **coproduct**.

(b) In $\mathbf{Vect}_F$, explain why the external direct sum $V_1 \oplus V_2$ is simultaneously the **product** and the **coproduct**.

(c) In **Ab** ($\mathbb{Z}$-modules), show that the direct sum $A \oplus B$ serves as both the product and the coproduct.

IV. Tensor Products and Exterior Products

Problem 12 (Universal Property of Tensor Product). Let V, W be vector spaces over F.

(a) State the universal property that characterizes the tensor product $(V \otimes W, \otimes)$.

(b) Use the universal property to construct the canonical isomorphism $\theta : V \otimes F \to V$.

(c) Prove the canonical isomorphism $V^* \otimes W \cong \mathrm{Hom}(V, W)$ (finite dimensional case).

Problem 13 (Pure Tensors and Rank 1 Matrices). Let $\{v_i\}$ and $\{w_j\}$ be bases for V and W. Let $t = \sum_{i,j} c_{ij}(v_i \otimes w_j)$.

(a) Prove that $\{v_i \otimes w_j\}_{i,j}$ is a basis for $V \otimes W$.

(b) Prove that t is a **pure tensor** if and only if the coefficient matrix $C = (c_{ij})$ has rank 1 (or 0).

(c) Give an explicit example of an element in $\mathbb{R}^2 \otimes \mathbb{R}^2$ that is **not** a pure tensor.

Problem 14 (Tensor Product of Linear Maps). Let $T_1 : V_1 \to V_2$ and $T_2 : W_1 \to W_2$ be linear maps.

(a) Show that the map $B(v_1, w_1) = T_1(v_1) \otimes T_2(w_1)$ is bilinear.

(b) Define the induced linear map $T_1 \otimes T_2 : V_1 \otimes W_1 \to V_2 \otimes W_2$.

(c) Describe the matrix representation of $T_1 \otimes T_2$ (the Kronecker product).

Problem 15 (Exterior Product and Linear Independence). Let $\bigwedge^k V$ be the k-th exterior product space, $\dim(V) = n$.

(a) Prove that $v_1 \wedge v_2 \wedge \cdots \wedge v_k = 0$ if and only if the set $\{v_1, \ldots, v_k\}$ is linearly dependent.

(b) Prove that $\dim(\bigwedge^k V) = \binom{n}{k}$.

(c) Explain the connection between the 1-dimensional space $\bigwedge^n V \cong F$ and the determinant function.

V. Module Theory

Problem 16 (Torsion Modules and Free Modules). Let R be an Integral Domain.

(a) Define the **torsion submodule** $\mathrm{Tor}(M)$ of an R-module M.

(b) Define a **free module**.

(c) Prove that for $M = \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$ as a $\mathbb{Z}$-module, $\mathrm{Tor}(M) \cong \mathbb{Z}/3\mathbb{Z}$.

Problem 17 (Modules and Vector Spaces). (a) Prove that an R-module M is a vector space over the field F if and only if R is a field F.

(b) Explain the fundamental difference between $\mathbb{Z}$-modules (Abelian groups) and vector spaces (e.g., in terms of torsion).

Problem 18 (Equivalent Definitions of Noetherian). Let M be an R-module. Prove that the following are equivalent:

(i) M is a **Noetherian module** (every submodule is finitely generated).

(ii) M satisfies the **Ascending Chain Condition (ACC)** on submodules.

Problem 19 (Properties of Noetherian Modules). Let M be an R-module, and N be a submodule of M.

(a) Prove that if M is Noetherian, then N and M/N are both Noetherian.

(b) Prove that if N and M/N are both Noetherian, then M is Noetherian.

Problem 20 (Hilbert's Basis Theorem). (a) State Hilbert's Basis Theorem.

(b) Use it to prove that $F[x, y]$ (polynomial ring) is a Noetherian ring.

(c) Give an example of a ring R (non-Noetherian) and a finitely generated, non-Noetherian R-module M.

Problem 21 (Classification of Modules over a PID). Let R be a Principal Ideal Domain (PID) and M be a finitely generated R-module.

(a) State the structure theorem for M in terms of **invariant factors**.

(b) State the structure theorem for M in terms of **elementary divisors** (primary decomposition).

(c) Prove that the torsion-free part $M/\mathrm{Tor}(M)$ is a free module R^r.

Problem 22 (Uniqueness of Invariant Factors). Let $M \cong R/(a_1) \oplus \cdots \oplus R/(a_k)$, where $a_1 | \ldots | a_k$ are the invariant factors.

(a) Prove that a_k generates the annihilator ideal $\mathrm{Ann}(\mathrm{Tor}(M))$.

(b) Explain why the invariant factors are unique up to multiplication by a unit.

VI. Linear Operators as $F[x]$-Modules

Problem 23 (The $F[x]. -ModuleStructure]LetVbeafinite-dimensionalvectorspaceover\mathrm{F}, and\mathrm{T}$ $\in \mathrm{Hom}(V, V)$. Define the $F[x]$-module structure on V by $p(x) \cdot v = p(T)(v)$.

(a) Prove that V is a finitely generated $F[x]$-module.

(b) Prove that V is a **torsion module**.

(c) Find the annihilator of V, $\mathrm{Ann}(V) = \{p(x) \in F[x] \mid p(T) = 0\}$. How does this ideal relate to the minimal polynomial $\mu_T(x)$?

Problem 24 (Rational Canonical Form (RCF)). Let V's invariant factor decomposition be $V \cong F[x]/(a_1) \oplus \cdots \oplus F[x]/(a_k)$.

(a) Define the **companion matrix** $C(a(x))$ for a monic polynomial $a(x)$.

(b) Explain how this decomposition leads to T's **Rational Canonical Form**.

(c) Prove that the RCF is invariant under field extensions.

Problem 25 (Cayley-Hamilton Theorem from Module Theory). (a) Use the module structure to show that $\mu_T(x)$ divides $\chi_T(x)$.

(b) Prove that the characteristic polynomial $\chi_T(x)$ is the product of all invariant factors $\prod_{i=1}^{k} a_i(x)$.

(c) Use the relationship between $\mu_T(x)$ and $\chi_T(x)$ to prove $\chi_T(T) = 0$.

Problem 26 (Jordan Canonical Form (JCF) and Primary Decomposition). Assume F is algebraically closed. V's elementary divisor decomposition is $V \cong \bigoplus F[x]/((x-\lambda_i)^{e_{ij}})$.

(a) Show that each $F[x]/((x-\lambda)^e)$ corresponds to a single λ-Jordan block $J_e(\lambda)$.

(b) Relate the number of times the elementary divisor $(x-\lambda)^e$ appears to the structure of the JCF.

(c) Prove that T is diagonalizable if and only if all elementary divisors are of the form $x-\lambda$ (i.e., $e = 1$).

VII. Deeper Connections: Duality, Quotients, and Tensor Products

Problem 27 (Universal Property of the Internal Direct Sum). Let $V = W_1 \oplus W_2$ (internal direct sum).

(a) Prove that V satisfies the universal property of a **coproduct** (via inclusions $i_k : W_k \to V$).

(b) Prove that V simultaneously satisfies the universal property of a **product** (via projections $\pi_k : V \to W_k$).

Problem 28 (Annihilators and Direct Sums). Let $V = W_1 \oplus W_2$ (internal direct sum, finite dimensional).

(a) Prove that $V^* = W_1^\circ \oplus W_2^\circ$ (internal direct sum in V^*).

(b) Prove the canonical isomorphisms $W_1^\circ \cong W_2^*$ and $W_2^\circ \cong W_1^*$.

Problem 29 (Duality of Quotient Spaces (Infinite Dim)). Let $W \subseteq V$.

(a) Construct the canonical isomorphism $\Psi : (V/W)^* \to W^\circ$.

(b) If V is infinite dimensional, is it still true that $V^*/W^\circ \cong W^*$? (Prove or provide a counterexample).

Problem 30 (Tensor Product and Duality). (a) Prove the canonical isomorphism $V^* \otimes W^* \cong (V \otimes W)^*$.

(b) State the **Hom-Tensor Adjunction** relation.

(c) Prove $V \otimes V^* \cong \mathrm{Hom}(V, V)$ (finite dimensional).

Problem 31 (Tensor Product of Quotient Modules). Let $R = \mathbb{Z}$.

(a) Prove that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ is the zero module.

(b) Generalize: $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

(c) Let $R = F[x]$. Prove that $F[x]/(x) \otimes_{F[x]} F[x]/(x-1)$ is the zero module.

Problem 32 (Exterior Product and Characteristic Polynomial). Let $T : V \to V$, $\dim(V) = n$.

(a) Define the induced map $\bigwedge^k T : \bigwedge^k V \to \bigwedge^k V$.

(b) Prove that $\det(T) = \mathrm{tr}(\bigwedge^n T)$.

(c) Prove the characteristic polynomial formula: $\chi_T(\lambda) = \sum_{k=0}^{n} (-1)^k \mathrm{tr}(\bigwedge^k T)\lambda^{n-k}$.

VIII. Module Theory: Torsion, Annihilators, and Exact Sequences

Problem 33 (Primary Decomposition and Eigenvalues). Let V be the T-module with primary decomposition $V \cong \bigoplus F[x]/((x-\lambda_i)^{e_i})$.

(a) Prove that the set of distinct λ_i's are the eigenvalues of T.

(b) Prove that $\mu_T(x) = \mathrm{lcm}((x-\lambda_1)^{e_1}, \ldots, (x-\lambda_k)^{e_k})$.

(c) Show that T is nilpotent ($T^m = 0$) if and only if the only elementary divisor is of the form x^e.

Problem 34 (Injective and Projective Modules (PID Case)). Let R be a PID.

(a) Define **injective module** Q and **projective module** P.

(b) Prove that an R-module M is projective if and only if it is **free**.

(c) Prove that a $\mathbb{Z}$-module M is injective if and only if M is **divisible**.

Problem 35 (Rank of a Module and Short Exact Sequences). Let R be a PID, M be a finitely generated R-module.

(a) Define the **rank** r of M.

(b) Prove that $r = \dim_K(M \otimes_R K)$ where K is the field of fractions of R.

(c) Prove that if $0 \to M' \to M \to M'' \to 0$ is an SES, then $\mathrm{rank}(M) = \mathrm{rank}(M') + \mathrm{rank}(M'')$.

IX. Advanced Linear Operator Theory (Module Perspective)

Problem 36 (Noetherian Ring vs. Module). (a) Prove that R is a Noetherian ring if and only if every finitely generated R-module is Noetherian.

(b) Give an example of a ring R (non-Noetherian) and a finitely generated, non-Noetherian R-module M.

Problem 37 (Primary Decomposition and Generalized Eigenspaces). Let V be the T-module.

(a) Prove that the primary decomposition $V \cong \bigoplus_{i=1}^{m} V_i$ is exactly the decomposition into T's **generalized eigenspaces**.

(b) Prove that the minimal polynomial of $T|_{V_i}$ is $(x - \lambda_i)^{k_i}$.

Problem 38 (Cyclic Submodules and RCF). Let V be the T-module.

(a) Define a T-**cyclic submodule** W.

(b) Prove that W is T-cyclic if and only if $\mu_{T|W}(x) = \chi_{T|W}(x)$.

(c) Explain how the invariant factor decomposition corresponds to decomposing V into an **internal direct sum** of T-cyclic subspaces.

Problem 39 (Jordan Form for Real Matrices). Let A be a real matrix.

(a) If A has non-real eigenvalues $\lambda = a + bi$, show the corresponding real invariant factor must be a power of an irreducible quadratic $p(x)$.

(b) Describe the structure of the **Real Jordan Block** corresponding to $p(x)^e$.

Problem 40 (Commuting Linear Operators). Let $S, T \in \mathrm{Hom}(V, V)$ such that $ST = TS$.

(a) Prove that S is an $F[x]$**-module homomorphism** on the T-module V.

(b) Prove that S preserves T's **generalized eigenspaces**.

(c) If T and S are both diagonalizable, prove they are **simultaneously diagonalizable**.

Problem 41 (Diagonalizability and $\mu_T(x)$). Let $\mu_T(x)$ be the minimal polynomial and $\chi_T(x)$ be the characteristic polynomial.

(a) Prove that T is diagonalizable if and only if $\mu_T(x)$ is a product of distinct linear factors.

(b) Prove that V is a T-cyclic module if and only if $\mu_T(x) = \chi_T(x)$.

(c) Give an example where $\mu_T(x) = \chi_T(x)$ but T is **not** diagonalizable.

X. Advanced Categorical Structures and Constructions

Problem 42 (Exactness of the Dual Functor). (a) Define a **Functor** and a **Contravariant Functor**.

(b) Prove that the Dual Functor $(\_)^* : \mathbf{Vect}_F \to \mathbf{Vect}_F$ is a contravariant functor.

(c) Prove that the Dual Functor is **exact**.

Problem 43 (The Tensor Functor). (a) For a fixed W, prove that $(\_)\otimes W : \mathbf{Vect}_F \to \mathbf{Vect}_F$ is a **covariant functor**.

(b) Prove that the Tensor Functor $(\_) \otimes W$ is **exact**.

Problem 44 (Hom-Tensor Adjunction). (a) Write down the **Hom-Tensor Adjunction** relation.

(b) Explain how this relation recovers the universal property of the tensor product.

Problem 45 (Initial and Terminal Objects in Cyclic Modules). (a) Identify the initial object and the terminal object in the category $\mathbf{Mod}_{F[x]}$.

(b) Consider the subcategory of **cyclic** $F[x]$-modules. Does this subcategory have an initial object or a terminal object?

Problem 46 (Exterior Product and Subspaces). Let W be a k-dimensional subspace of V.

(a) Prove that W can be canonically identified with a one-dimensional subspace of $\bigwedge^k V$.

(b) Prove that $W_1 = W_2$ if and only if $\bigwedge^k W_1 = \bigwedge^k W_2$.

Problem 47 (Tensor Product and Basis Change). Let $\Phi : V \otimes W \to W \otimes V$ be the canonical isomorphism defined by $\Phi(v \otimes w) = w \otimes v$.

(a) Prove that Φ is a well-defined linear isomorphism.

(b) If $T \in \mathrm{Hom}(V \otimes V, F)$ is a bilinear form, explain how Φ relates to the definitions of symmetric and anti-symmetric forms.

Problem 48 (Dual Space and Eigenvalues). Let $T : V \to V$, $T^t : V^* \to V^*$ be the transpose operator.

(a) Prove that T and T^t have the **same eigenvalues**.

(b) Prove that the minimal polynomial of T equals the minimal polynomial of T^t.

Problem 49 (Annihilators of Cyclic Modules). Let $M = F[x]/(p(x)^e)$ be a cyclic module annihilated by $(p(x))^e$, where $p(x)$ is irreducible.

(a) Prove that M has exactly one maximal submodule N, and $N \cong F[x]/(p(x)^{e-1})$.

(b) Prove that $M/N \cong F[x]/(p(x))$.