# A Quick Introduction to Differential Forms

## And Generalized Stokes's Theorem

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# Unification of Calculus

The primary goal of differential forms is to provide a **unified framework** for the various integration theorems of vector calculus.

- Green's Theorem (2D)
- Stokes's Theorem (3D Curl)
- Divergence Theorem (3D)

All these are specific instances of the **Generalized Stokes's Formula**:

$$\int_{\partial D} \omega = \int_D d\omega$$

# Classical Examples: 2D

## Green's Formula (Line Integrals)

$$\int_{\partial D} Pdx + Qdy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{1}$$

## 2D Flux

$$\int_{\partial D} -Qdx + Pdy = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy \tag{2}$$

# Classical Examples: 3D

## Divergence Theorem

$$\iint_{\partial D} Pdydz + Qdxdz + Rdxdy = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz \tag{3}$$

In this theory, we work in $\mathbb{R}^n$ with coordinates $x^1, x^2, \ldots, x^n$.

# 0-Forms and 1-Forms

## 0-Forms

Smooth functions $f : \mathbb{R}^n \to \mathbb{R}$, i.e., $f \in C^\infty(\mathbb{R}^n)$. *Example:* $f(\mathbf{x}) = \sum_{i=1}^n (x^i)^2$.

## 1-Forms

Vector space spanned by $\{dx^1, \ldots, dx^n\}$. *Example:* $\omega = (x^1)^2 dx^1 + (x^2) dx^2$.

# 2-Forms and the Wedge Product

A $\binom{n}{2}$ dimensional space spanned by $\{dx^i \wedge dx^j\}$.

## Properties of $\wedge$

1. **Anti-symmetry:** $dx^i \wedge dx^j = -dx^j \wedge dx^i$
2. **Nilpotency:** $dx^i \wedge dx^i = 0$

*Example:* $\omega = f dx^1 \wedge dx^2 + g dx^2 \wedge dx^3$.

# $k$-Forms $(1 \leq k \leq n)$

A space spanned by $\{dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}\}$ with dimension $\binom{n}{k}$.

- **Anti-symmetry:** Swapping any two indices flips the sign.
- **Vanishing:** If $k > n$, the form is identically zero because at least one $dx^i$ must repeat.

# The Exterior Derivative $d$

The map $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ is defined as:

- **On 0-forms:** $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$.
- **On $k$-forms:** If $\omega = \sum a_I dx^I$, then:

$$d\omega = \sum da_I \wedge dx^I$$

# Examples of Exterior Differentiation

**Example (n=2, 1-form $\to$ 2-form)**

$$\omega = Pdx + Qdy \implies d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

**Example (n=3, 2-form $\to$ 3-form)**

$$\omega = Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy$$
$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$$

# Integration of $k$-Forms

Suppose $D \subseteq \mathbb{R}^n$ is a $k$-dimensional region and $\omega = \sum a_I dx^I$ is a $k$-form.

$$\int_D \omega := \sum \int_D a_{i_1 \ldots i_k} dx^{i_1} \ldots dx^{i_k} \tag{4}$$

*Example (n=3):* Surface integral of a 2-form:

$$\int_S \omega = \int_S P dydz + \int_S Q dxdz + \int_S R dxdy$$

# The Generalized Stokes's Theorem

## Theorem

*Let $D$ be a $(k+1)$-dimensional region with boundary $\partial D$. If $\omega$ is a $k$-form, then:*

$$\int_D d\omega = \int_{\partial D} \omega \tag{5}$$

- **Dimension $n = 3$:** Recovers the Divergence and Curl theorems.
- **Dimension $n = 2$:** Recovers Green's Theorem.
- **Dimension $n = 1$:** Recovers the Fundamental Theorem of Calculus.

# Case $n = 1$: Fundamental Theorem of Calculus

Let $D = [a, b]$ be a 1-dimensional region in $\mathbb{R}^1$. The boundary is $\partial D = \{b\} - \{a\}$.

## Derivation

Let $\omega = f(x)$ be a 0-form. Then $d\omega = f'(x)dx$. Applying Generalized Stokes:

$$\int_D d\omega = \int_{[a,b]} f'(x)dx$$

$$\int_{\partial D} \omega = f(b) - f(a)$$

This yields the standard Fundamental Theorem of Calculus:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

# Case $n = 2$: Green's Theorem

Let $D \subset \mathbb{R}^2$ be a region with boundary curve $\partial D$.

## Derivation

Let $\omega = Pdx + Qdy$ be a 1-form. We calculate the 2-form $d\omega$:

$$
\begin{aligned}
d\omega &= dP \wedge dx + dQ \wedge dy \\
&= \left( \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) \wedge dy \\
&= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\
&= \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy
\end{aligned}
$$

Generalized Stokes $\int_D d\omega = \int_{\partial D} \omega$ recovers Green's Theorem.

# Case $n = 3$: Classical Stokes's Theorem

Let $S \subset \mathbb{R}^3$ be a 2D surface with boundary curve $\partial S = C$.

## Derivation

Let $\omega = F_1 dx + F_2 dy + F_3 dz$ (a 1-form representing vector field $\mathbf{F}$). The exterior derivative $d\omega$ is a 2-form:

$$d\omega = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy$$

Integrating $d\omega$ over $S$ is equivalent to the flux of $\nabla \times \mathbf{F}$, yielding:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

# Case $n = 3$: Divergence Theorem

Let $V \subset \mathbb{R}^3$ be a 3D solid with boundary surface $\partial V = S$.

## Derivation

Let $\omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$ be a 2-form.

$$
\begin{aligned}
d\omega &= dP \wedge dy \wedge dz + dQ \wedge dz \wedge dx + dR \wedge dx \wedge dy \\
&= \frac{\partial P}{\partial x} dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dz \wedge dx + \frac{\partial R}{\partial z} dz \wedge dx \wedge dy \\
&= \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz
\end{aligned}
$$

Generalized Stokes $\int_V d\omega = \int_S \omega$ recovers the Divergence Theorem:

$$\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$