MAT4002 Introduction to Geometry and Topology
Midterm Examination

1. Short questions. Simply write down the answer, no justification is required.

   (a) Give an example of a space that is not metrizable.

   (b) Give an example of a space that is connected but not path-connected.

   (c) Let X and Y be topological spaces and $X \times Y$ be equipped with product topology. Then a map $f : Z \longrightarrow X \times Y$ is continuous if and only if $p_X \circ f : Z \longrightarrow X$ and $p_Y \circ f : Z \longrightarrow Y$ are continuous, where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are projection maps.

   True False

   (d) If $f : X \longrightarrow Y$ is a continuous surjective map between topological spaces, then we have an induced homeomorphism between the quotient space $X/\sim$ and Y, where the equivalence relation on X is given by $x \sim x' \Leftrightarrow f(x) = f(x')$.

   True False

2. (a) Show X is Hausdorff if and only if the diagonal

   $$\Delta := \{(x, x) \mid x \in X\}$$

   is closed in $X \times X$.

   (b) Suppose that $q : X \longrightarrow Y$ is an open quotient map. Show that Y is a Hausdorff if and only if the set

   $$\mathcal{R} = \{(x_1, x_2) \mid q(x_1) = q(x_2)\}$$

   is closed in $X \times X$.

3. Let $X_1 = \{(x, 1) \mid x \in \mathbb{R}\}$, $X_2 = \{(x, 2) \mid x \in \mathbb{R}\}$. Define the partition on $X_1 \cup X_2$ given by the equivalence relation $(x, 1) \sim (x, 2)$ for every $x \in \mathbb{R} \backslash \{0\}$.

   (a) Show $(X_1 \cup X_2)/\sim$ is not Hausdorff.

   (b) Show $(X_1 \cup X_2)/\sim$ is locally compact.

4. The comb space is the subset of $\mathbb{R}^2$ defined by

   $$C = (\{0\} \times [0, 1]) \cup (K \times [0, 1]) \cup ([0, 1] \times \{0\})$$

where $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show the comb space C is not triangulable.
(You can quote results from lecture or homework, provided that you fully state the result you wish to use).

5. Let $K = (V, \Sigma)$ be a simplicial complex.

   (a) In the case where $V = \{1, 2, 3, 4\}$, provide a complete description of K when K is a triangulation of the 2-sphere.

   (b) In the case where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, provide a complete description of K when K is a triangulation of the Klein bottle.
   (You may want to use the following polygonal representation of Klein bottle)

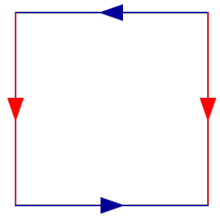

Figure 1: Klein bottle

   (c) Calculate the Euler characteristic $\chi(|K|)$ for part (b).

6. **Closed-projection characterization of compactness**

   Suppose X is a topological space. We say X has property $(\mathbf{P})$ if, for every topological space Y, the projection map $\pi : X \times Y \longrightarrow Y$ is a closed map, where $X \times Y$ is endowed with product topology.

   (a) Show that if a topological space X is compact, then X has property $(\mathbf{P})$.

   (b) Show that if a topological space X has property $(\mathbf{P})$, then X is compact.

Hint (one possible solution): Show that if X is not compact, then we can find a sequence of open sets $U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \cdots$ such that $\bigcup_{i \in \mathbb{N}} U_i = X$.

Use the sequence $U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \cdots$ to construct a closed set $C \subset X \times \mathbb{R}$ that projects to a set in $\mathbb{R}$ which is not closed.