MAT4033 Final Examination (120 min)

name: ___________ ID: ___________ 2024 Fall, 1:30 pm - 3:30 pm

Note: No books, notes or calculators are allowed.

Student	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Total**
Grade							

1.(15 pts) Let S be a surface of revolution given by

$$X(u,v) = (a\cos u\cos v, a\cos u\sin v, \int_0^u \sqrt{1-a^2\sin^2\theta}d\theta)$$

. Let γ be a geodesic which passes through the point (a,0,0) and makes angle σ with the parallel through this point. Show that γ is given by :

$$\sin av\tan\sigma = \pm\tan u$$

2.(20 pts) Let S be a minimal surface. Recall in class, using complex variable $z = x + \sqrt{-1}y$

$$\begin{cases} \phi^1 = \dfrac{1}{2} f(1 - g^2) \\ \phi^2 = \dfrac{i}{2} f(1 + g^2) \\ \qquad \phi^3 = fg \end{cases}$$

(a) (5 pts) Let N be Gauss map of minimal surface S. Let $Q : S^2 \to \overline{C}$ be the stereographic projection from unit sphere to closure of complex plane. Show that $Q \circ N : S \to \overline{C}$ is one of the f, g the functions for the Weirstrass Enneper representation.

(b) (5 pts) Given $y \cos(\frac{z}{a}) = x \sin(\frac{z}{a})$; $a \neq 0$. Find the Gaussian curvature K and mean curvature H.

(c) (5 pts) Let $p \in S$ be a point on the minimal surface. Let $\vec{v_p} \neq 0$ be a tangent vector at point p. Show that the Gaussian curvature K_p at point p is given by:

$$K_p = -\frac{III_p(\vec{v_p}, \vec{v_p})}{I_p(\vec{v_p}, \vec{v_p})}$$

(d) Show that every umbilical point on S is a planar point.

3.(20 pts) Let $\vec{v}$ be smooth vector field on regular oriented surface S. Let X be orthogonal coorddinate patch with $X(0,0) = p$. Let $\gamma : [0,l] \to S$ be a p.a.l simple closed curve bounds simple connected region containing $p \in R \subseteq S$. Assume all orientation are positive.

(a) Set $\theta(t) = \angle(X_u, \vec{v}(\gamma(t))$. In class we proved there exists index I_p of $\vec{v}$ relative to γ. Show that:

$$I_p = \frac{1}{2\pi} \int_0^l d\theta$$

(b) Show that I_p is independent of parametrization X or choice of γ.

(c) Assume $\{p \in S | \vec{v}(p) = 0\}$ is finite and S is a compact surface without boundary. Use local Gauss-Bonnet theorem(or local Hopf index theorem), show that:

$$\sum_{\{p \in S | \vec{v}(p) = 0\}} I_p = \chi(S)$$

(d) Assume $\vec{v}(p)$ is tangent vector at $p \in S^2$, the unit sphere. Show that there exists p such that $\vec{v}(p) = 0$.

4.(15 pts) Prove the Codazzi equations:

$$\Pi_{ij,k} - \Pi_{ik,j} + \Gamma^r_{ij}\Pi_{rk} - \Gamma^r_{ik}\Pi_{rj} = 0$$

is equivalent to $N_{jk} = N_{kj}$. Here $i, j, k \in \{u, v\}$ as in lecture. (Note: Repeated indices are implicitly summed over. Precisely, repeated indices mean sum from 1 to 2 where 1 represents u and 2 stands for v. Comma means derivative.)
If you like, you can use the following notation for Codazzi equations.

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}$$

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12},$$

and show that it is equivalent to $N_{uv} = N_{vu}$.

5.(10 pts) Given smooth family of parametrizations of surface

$$X^t : U \times (-\epsilon, \epsilon) \to \mathbb{R}^3$$

with

$$X^0(u_1, u_2)|_{\partial U} = X(u_1, u_2)|_{\partial U} = X^t(u_1, u_2)|_{\partial U}$$

fixed boundary deformation.
Show that:

$$\frac{d}{dt} area(X^t)|_{t=0} = \int_U < X^t_t, \vec{H} > |_{t=0} dA$$

where $X^t_t = \frac{\partial}{\partial t} X^t$ restricted to S, $\vec{H} = H\vec{N}$, and H denotes mean curvature and $\vec{N}$ denotes the normal vector of S. ($S = X^0$) (Note : X^t is a general deformation, not the "normal" deformation we did in the lecture)

6. (20 pts) Let S_{ico} be surface which is closed surface composed of 12 pentagons(polyhedron soccer ball), see Figure 1. Let T^2_{sq} be the triangular torus see Figure 2.

(a) Determine Gaussian curvature $K(v)$ at vertex $v \in S_{\text{ico}}$.

(b) Let S'_{ico} be the surface S_{ico} with two disjoint **pentagons** removed. Find

$$\sum_{v \in \text{int}(S'_{\text{ico}})} K(v),$$

where $\text{int}(S'_{\text{ico}})$ denotes the interior vertices of S'_{ico}, and

$$\sum_{v \in \partial S'_{\text{ico}}} \theta_{\text{ext}}(v),$$

where $\theta_{\text{ext}}(v)$ denotes the exterior angle at boundary vertex v.

(c) Let $T^2_{\text{sq}} \# T^2_{\text{sq}}$ be the closed surface created by removing one of the outer faces from each T^2_{sq} and glue along the boundary curve. Find $K(v)$ for each vertex v of $T^2_{\text{sq}} \# T^2_{\text{sq}}$ and compute

$$\sum_{v \in T^2_{\text{sq}} \# T^2_{\text{sq}}} K(v).$$

(d) Which of these surfaces: $S_{\text{ico}}, S'_{\text{ico}}, T^2_{\text{sq}}, T^2_{\text{sq}} \# T^2_{\text{sq}}$ have the same $\chi(S)$ Euler characteristic?

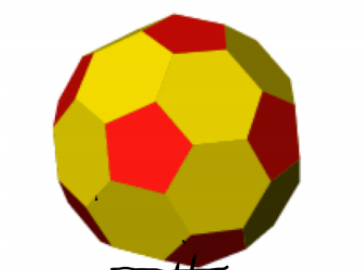

Figure 1: polyhedron soccer ball

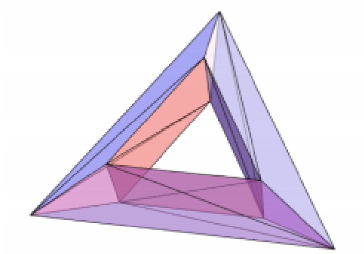

Figure 2: triangular torus