# MAT1002 Midterm Examination

Sunday, March 29, 2020.

Time: 9:00 AM - 9:00 PM

**Notes and Instructions**

*1. The total score of this examination is 100.*

*2. There are **twelve** questions (with parts) in total.*

*3. Show intermediate steps of your solution.*

*4. You are allowed to check the following material when writing the midterm exam: the textbook, the reference book (see course outline), material posted to Blackboard (MAT1002, Term 2 of 2019-2020 Academic Year), and notes you prepared **before the exam**. No other material is allowed. In particular, you are **not** allowed to refer to any note or web page on the Internet (except Blackboard).*

*5. **No collaboration is allowed**. During the the entire exam period, you are not allowed to talk to any person about any exam question.*

## Midterm Examination Questions

**Question 1** (5+5+5=15 Points)

For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2+3\cos n}{3e^n}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{n}-1\right)$.

**Question 2** (7 Points)

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbb{R}^2$, where $|\mathbf{v}_1| = 2$, $|\mathbf{v}_2| = 5$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 4$. Construct $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \ldots$ by $\mathbf{v}_{k+2} = \mathrm{proj}_{\mathbf{v}_k} \mathbf{v}_{k+1}$ for $k \geq 1$. Determine the value of

$$\sum_{n=1}^{\infty} |\mathbf{v}_n|.$$

**Question 3** (4+3+3=10 Points)

(a) Find the vector function $\mathbf{r} = \mathbf{r}(t)$ that satisfies the following conditions.

$$\begin{array}{ll} \text{Differential equation:} & \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j} + 5\mathbf{k} \\ \text{Initial condition:} & \mathbf{r}(0) = -10\mathbf{i}. \end{array}$$

(b) Find the unit tangent vector for the curve parametrized by $\mathbf{r}$.

(c) Find the arc length along the curve of $\mathbf{r}$ over $0 \leq t \leq \pi/2$.

**Question 4** (5 Points)

Let $s$ be a positive real number. Find all three-dimensional vectors $\mathbf{v}$ such that the length of $\mathbf{v} \times \mathbf{k}$ is equal to $s$. Here, $\mathbf{k}$ is a unit vector on the positive $z$-axis.

**Question 5** (3+3+4=10 Points)

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of $(x, y, z)$.

(a) Find a function $f(x, y, z)$ such that $\nabla f = \mathbf{r}$.

(b) Compute the gradient of $|\mathbf{r}|$.

(c) Compute the Laplacian of $|\mathbf{r}|$, where the Laplacian $\Delta f$ of a function $f(x, y, z)$ is defined as

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

**Question 6** (5+5=10 Points)

Find each of the following limits, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2 + (x-y)^2}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{x^2y^2}$.

**Question 7** (5 Points)

Find all the points on the surface $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$.

**Question 8** (2+3+5=10 Points)

Consider the function $f(x, y) = x^{1/3}y^{2/3}$.

(a) At the point $(1, 1)$, in which direction does $f$ increase the fastest?

(b) What is the derivative of $f$ at $(1, 1)$ in the direction in $(a)$?

(c) Find the directional derivative of $f$ at $(0, 0)$ in the direction of $3\mathbf{i} + 4\mathbf{j}$, or explain why it does not exist.

**Question 9** (6 Points)

Find the constant $a$ such that the limit

$$\lim_{x \to 0} \frac{\cos ax - \cos x + 2x^2}{x^4}$$

is finite. Justify your answer.

**Question 10** (6 Points)

Suppose that $\sum_{n=1}^{\infty} c_n$ is a convergence series, where $c_n \geq 0$ for all $n$. Show that

$$\sum_{n=1}^{\infty} \frac{\sqrt{c_n}}{n^{2/3}}.$$

also converges. $\left(\textit{Hint:} \text{ consider } \left(\sqrt{c_n} - n^{-2/3}\right)^2. \right)$

**Question 11** (6 Points)

Suppose that $x \in (0, 1)$. Show that for any $n = 1, 2, \ldots$,

$$\sum_{k=1}^{n} x^k (1 - x)^{2k} \leq \frac{4}{23}.$$

**Question 12** (7+3=10 Points)

(a) Show that

$$\int_0^1 x^x \, dx = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-n}.$$

(*Hint*: Use $x^x = e^{x \ln x}$ and $\int_0^1 x^m (\ln x)^n \, dx = -\frac{n}{m+1} \int_0^1 x^m (\ln x)^{n-1} \, dx$)

(b) Use the formula in (a) to estimate $\int_0^1 x^x \, dx$ with an error of magnitude less than 0.001.