MAT1011

Tutorial 8

CUHK(SZ)

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Outline

1. Midterm Exam
2. Additional Exercises (HW7-Sol)
3. Tutorial Exercises
4. Exercises from Thomas' Calculus (HW8-Sol, Cont.)

Question 1 (15 marks). Let

$$f(x) = (x-1)x^{5/3}.$$

(a) Find the derivatives $f'(x)$ and $f''(x)$ when they exist.

(b) Find the tangent line and normal line of the curve $y = f(x)$ at the point $P_0(0,0)$.

(c) Find all critical points of f. (15 marks)

Solution.

(a)

$$y'(x) = \frac{1}{3}(8x-5)x^{2/3},$$
$$y''(x) = \frac{10}{9}(4x-1)x^{-1/3}, \quad x \neq 0.$$

$y''(0)$ does not exist.

(b) $y'(0) = 0$. So the tangent line of the curve $y = f(x)$ at $P_0(0,0)$ is the x-axis, and the normal line is the y-axis.

(c) $y'(x)$ exists for all x, and $y'(x) = 0$ for $x = 0, \frac{5}{8}$. Hence the critical points of $f(x)$ are $x = 0$ and $x = \frac{5}{8}$.

Question 2 (25 marks). Find the following limits.

$$(a)\quad \lim_{x\to 0} \frac{\sin^3(\sin^2 x)}{\sin^2(\sin x^3)}.$$

$$(b)\quad \lim_{x\to 0^+} \frac{\cos(1+\sqrt{x}) - \cos 1}{\sqrt{\sin x}}.$$

$$(c)\quad \lim_{x\to 0} \frac{1}{x^2} \int_0^{x^2} \Big[\cos(t^2) + \cos^2(t^2)\Big] dt.$$

Solution.

(a)

$$\frac{\sin^3(\sin^2 x)}{\sin^2(\sin^3 x)} = \frac{\sin^3(\sin^2 x)}{(\sin x)^6} \cdot \frac{(\sin x)^6}{\sin^2(\sin^3 x)} = [\frac{\sin(\sin^2 x)}{\sin^2 x}]^3 \cdot [\frac{\sin^3 x}{\sin(\sin^3 x)}]^2.$$

Since

$$\lim_{x\to 0} \frac{\sin(\sin^2 x)}{\sin^2 x} = 1,$$
$$\lim_{x\to 0} \frac{\sin^3 x}{\sin(\sin^3 x)} = 1,$$

so

$$\lim_{x\to 0} \frac{\sin^3(\sin^2 x)}{\sin^2(\sin^3 x)} = \lim_{x\to 0}[\frac{\sin(\sin^2 x)}{\sin^2 x}]^3 \cdot [\frac{\sin^3 x}{\sin(\sin^3 x)}]^2 = 1 \cdot 1 = 1.$$

Solution.

(b)

$$\frac{\cos(1+\sqrt{x})-\cos 1}{\sqrt{\sin x}} = \frac{\cos 1\cos(\sqrt{x})-\sin 1\sin(\sqrt{x})-\cos 1}{\sqrt{\sin x}}$$

$$=\cos 1\cdot\frac{\cos(\sqrt{x})-1}{\sqrt{\sin x}}-\sin 1\cdot\frac{\sin(\sqrt{x})}{\sqrt{\sin x}}$$

$$=\cos 1\cdot\frac{\cos(\sqrt{x})-1}{x}\cdot\frac{x}{\sqrt{\sin x}}-\sin 1\cdot\frac{\sin(\sqrt{x})}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{\sin x}}.$$

□

Solution.

$$\lim_{x\to 0^+} \frac{\cos(\sqrt{x})-1}{x} = -\frac{1}{2},$$
$$\lim_{x\to 0^+} \frac{x}{\sqrt{\sin x}} = 0,$$
$$\lim_{x\to 0^+} \frac{\sin(\sqrt{x})}{\sqrt{x}} = 1,$$
$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = 1.$$

Hence

$$\lim_{x\to 0^+} \frac{\cos(1+\sqrt{x})-\cos 1}{\sqrt{\sin x}} = -\sin 1.$$

Solution.

(c) Let $f(x) = \cos(x^2) + \cos^2(x^2)$. The function $f(x)$ is continuous for all $x \in \mathbb{R}$, then

$$F(x) = \int_0^x [\cos(t^2) + \cos^2(t^2)]dt = \int_0^x f(t)dt$$

is differentiable for all $x \in \mathbb{R}$, and $F(0) = 0$. By the Fundamental theorem of calculus, we have

$$F'(x) = f(x) = \cos(x^2) + \cos^2(x^2).$$

Solution.

Hence

$$\lim_{x\to 0}\frac{1}{x^2}\int_0^{x^2}[\cos(t^2)+\cos^2(t^2)]dt=\lim_{x\to 0}\frac{F(x^2)}{x^2}$$
$$=\lim_{y\to 0}\frac{F(y)}{y}=\lim_{y\to 0}\frac{F(y)-F(0)}{y}=F'(0)=f(0)=2.$$

Question 3 (10 marks). Find the limit

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-1}{x}.$$

Then use the ε-δ language to prove your result.

Solution.

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-1}{x} = \lim_{x\to 0} \frac{2}{\sqrt{1+2x}+1} = 1.$$

Now we prove this limit by using the ε-δ language. We shall show that, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{\sqrt{1+2x}-1}{x} - 1\right| < \varepsilon \quad \text{whenever } 0 < |x| < \delta. \tag{1}$$

We compute

$$\frac{\sqrt{1+2x}-1}{x} - 1 = \frac{\sqrt{1+2x}-1-x}{x}$$
$$= \frac{(1+2x)-(1+x)^2}{x(\sqrt{1+2x}+1+x)} = -\frac{x}{\sqrt{1+2x}+1+x}.$$

Solution.

Take first $\delta_1 = 1/2$. If $0 < |x| < \delta_1$ we have $x > -1/2$, $1 + 2x > 0$, $1 + x > 1/2$, so

$$\sqrt{1+2x} + 1 + x > 1/2.$$

Hence

$$\left|\frac{\sqrt{1+2x}-1}{x} - 1\right| = \left|\frac{x}{\sqrt{1+2x}+1+x}\right| \leq 2|x|.$$

Take $\delta = \min\{1/2, \varepsilon/2\}$. If $0 < |x| < \delta$ then

$$|\frac{\sqrt{1+2x}-1}{x} - 1| \leq 2|x| < \varepsilon.$$

Solution.

So (1) holds. Thus we have proved

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-1}{x}=1.$$

$\square$

Question 4 (25 marks).

(a) Evaluate the indefinite integral

$$\int |x| \cos x \, dx.$$

(b) For any non-negative integer n denote

$$J_n = \int \sin^n x \, dx.$$

Show that for any $n \geq 2$ it holds that

$$J_n = \frac{n-1}{n} J_{n-2} - \frac{1}{n} \sin^{n-1} x \cos x.$$

(c) Evaluate the following:

$$\int \left[\frac{d}{dx} f(\cos^2 x) \right] dx - \frac{d}{dx} \int f(\cos^2 x) dx,$$

where f is a continuously differentiable function on $\mathbb{R}$.

Solution.

(a) Denote

$$F(x) = \int |x| \cos x dx.$$

If $x > 0$, then

$$F(x) = \int x \cos x dx = \int x d \sin x = x \sin x - \int \sin x dx = x \sin x + \cos x + C_1.$$

If $x < 0$, then

$$F(x) = -\int x \cos x dx = -x \sin x - \cos x + C_2.$$

Solution.

We need $F(0^-) = F(0^+)$, so $1 + C_1 = -1 + C_2$, $C_2 = C_1 + 2$. Denote $C_1 = C$, then $C_2 = C + 2$.

$$F(x) = \begin{cases} x \sin x + \cos x + C & \text{if } x \geq 0, \\ -x \sin x - \cos x + C + 2 & \text{if } x < 0. \end{cases}$$

Then F is continuous. $F'(x) = |x| \cos x$ for $x \neq 0$.

Solution.

$$F'_+(0) = \lim_{h\to 0^+} \frac{F(h)-F(0)}{h} = \lim_{h\to 0^+} \frac{(h\sin h + \cos h + C) - (1+C)}{h}$$
$$= \lim_{h\to 0^+} \frac{h\sin h + \cos h - 1}{h} = 0,$$

$$F'_-(0) = \lim_{h\to 0-+} \frac{F(h)-F(0)}{h}$$
$$= \lim_{h\to 0^-} \frac{(-h\sin h - \cos h + C + 2) - (3+C)}{h}$$
$$= \lim_{h\to 0^-} \frac{-h\sin h - \cos h + 1}{h} = 0,$$

□

Solution.

So $F'(0) = 0$. Thus for all x,

$$\int |x| \cos x dx = F(x) = \begin{cases} x \sin x + \cos x + C & \text{if } x \geq 0, \\ -x \sin x - \cos x + C + 2 & \text{if } x < 0. \end{cases}$$

$\square$

Solution.

$$J_n = \int \sin^n x dx = -\int \sin^{n-1} x d\cos x = -\sin^{n-1} x \cos x + \int \cos x d\sin^{n-}$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) J_{n-2} - (n-1) J_n.$$

□

Solution.

(c) By the Fundamental Theorem of Calculus, we have

$$\int \frac{d}{dx} f(\cos^2 x)dx = f(\cos^2 x) + C,$$

$$\frac{d}{dx} \int f(\cos^2 x)dx = f(\cos^2 x).$$

Hence

$$\int \frac{d}{dx} f(\cos^2 x)dx - \frac{d}{dx} \int f(\cos^2 x)dx = C,$$

where C is an arbitrary constant.

$\square$

Question 5 (18 marks).

(a) Let

$$u(x) = g(x)D(x),$$

where g is differentiable at $x = 0$ and $g(0) = g'(0) = 0$, and $D(x)$ is the Dirichlet function,

$$D(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Is $u(x)$ differentiable at $x = 0$? If your answer is "YES" then find $u'(0)$; if your answer is "NO" then give your reason.

(b) Let $f(x) = \cos(\zeta(x))$, where $\zeta(x)$ is the Riemann function,

$$\zeta(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}\text{, an irreducible fraction.} \end{cases}$$

Is $f(x)$ integrable over the interval $[0, 1]$? If your answer is “YES” then find $\int_0^1 f(x)dx$; if your answer is “NO” then give your reason.

Solution.

(a) u is differentiable at $x = 0$ and $u'(0) = 0$.

To prove, note that $u(0) = g(0)D(0) = 0$. Then

$$u'(0) = \lim_{x \to 0} \frac{u(x) - u(0)}{x} = \lim_{x \to 0} \frac{g(x)D(x) - 0}{x} = \lim_{x \to 0} \frac{g(x)}{x} D(x) = 0$$

because

$$\lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = g'(0) = 0$$

and $D(x)$ is bounded.

Solution.

(b) The answer is "YES".

$f(x) = \cos(\zeta(x))$ is bounded on $[0,1]$. For any $0 < \varepsilon < 1$, the set

$$\mathcal{M}(\varepsilon) = \{x \in [0,1] : \zeta(x) > \varepsilon\}$$

contains only a finite number of points $\{x_j\}_{j=1}^{m}$.

For all $x \in [0,1] \setminus \mathcal{M}(\varepsilon)$,

$$0 \leq 1 - \cos(\zeta(x)) = 2\sin^2\left(\frac{\zeta(x)}{2}\right) \leq 2\left(\frac{\zeta(x)}{2}\right)^2 = \frac{[\zeta(x)]^2}{2} < \varepsilon.$$

Solution.

Hence, we have

$$1 - \varepsilon < \cos(\zeta(x)) \leq 1, \quad \forall x \in [0,1] \setminus \mathcal{M}(\varepsilon). \tag{6}$$

Take

$$\delta = \frac{\varepsilon}{2m}.$$

Let P be any partition of $[0,1]$ with $\|P\| < \delta$. Then at most $2m$ sub-intervals contain some points from $\{x_1, \cdots, x_m\}$. Denote the union of these sub-intervals by J_1. Then

$$|J_1| \leq 2m\delta = \varepsilon.$$

Solution.

Denote $J_2 = [0,1] \setminus J_1$. Accordingly we split $P = P_1 \cup P_2$. Denote by $S(P_i)$ and $s(P_i)$ the upper and lower sums associated with the partition P_i of J_i for $i = 1, 2$, and $\Delta(P_i) = S(P_i) - s(P_i)$.

Since $|J_1| \leq \varepsilon$ and $0 \leq \cos(\zeta(x)) \leq 1$, we see that

$$0 = |J_1| \cdot 0 \leq s(P_1) \leq S(P_1) \leq |J_1| \cdot 1 = \varepsilon,$$

so

$$\Delta(P_1) \leq \varepsilon.$$

□

Solution.

Denote the sub-intervals associated with P_2 by $\{I_k\}$, and let

$$\omega_k = \sup_{I_k} \cos(\zeta(x)) - \inf_{I_k} \cos(\zeta(x)).$$

From (6), we have $1 - \varepsilon \leq \cos(\zeta(x)) \leq 1$ on J_2, then $0 \leq \omega_k \leq \varepsilon$. Hence

$$\Delta(P_2) = \sum_k \omega_k |I_k| \leq \varepsilon \sum_k |I_k| \leq \varepsilon.$$

Or,

$$s(P_2) = \sum_k \inf_{I_k} f(x)|I_k| \geq (1-\varepsilon) \sum_k |I_k| \geq (1-\varepsilon)|J_2| = (1-\varepsilon)(1-|J_1|) \geq (1-$$

$$S(P_2) = \sum_k \sup_{I_k} f(x)|I_k| \leq 1 \cdot \sum_k |I_k| \leq |J_2| = 1 - |J_1| = 1 - \varepsilon.$$

Solution.

Hence

$$\Delta(P_2) = S(P_2) - s(P_2) \leq (1-\varepsilon) - (1-\varepsilon)^2 < \varepsilon.$$

Hence

$$\Delta(P) = \Delta(P_1) + \Delta(P_2) \leq 2\varepsilon.$$

Thus

$$\lim_{\|P\|\to 0} \Delta(P) = 0.$$

By Theorem 2.1.6 we conclude that $f(x) = \cos(\zeta(x))$ is integrable over $[0,1]$.

Solution.

Moreover, from the above computation, we have

$$s(P) = s(P_1) + s(P_2) \geq 0 + (1-\varepsilon)^2 = (1-\varepsilon)^2,$$
$$S(P) = S(P_1) + S(P_2) \leq \varepsilon + (1-\varepsilon) = 1,$$

that is,

$$(1-\varepsilon)^2 \leq s(P) \leq S(P) \leq 1.$$

Hence

$$\lim_{\|P\|\to 0} s(P) = \lim_{\|P\|\to 0} S(P) = 1.$$

Thus, we have

$$\int_0^1 \cos(\zeta(x))dx = 1.$$

Question 6 (7 marks). Assume $f(x)$ is a continuously differentiable function on the interval $[0,1]$ and $f'(x) \neq 0$ for all $x \in [0,1]$.

(a) Prove that there exists a unique number $\xi_* \in [0,1]$ such that

$$\int_0^1 f(x)dx = f(\xi_*).$$

(b) For any positive integer n, let

$$S_n = \sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n}.$$

Prove that there exists a unique $\xi_n \in [0,1]$ such that

$$S_n = f(\xi_n).$$

(c) Prove that for any $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that the numbers ξ_* and ξ_n given in (a) and (b) above respectively satisfy

$$|\xi_n - \xi_*| < \varepsilon \quad \text{whenever } n > N.$$

Solution.

(a) Since f is continuous on $[0,1]$, by the first mean value theorem of definite integral, there exists $\xi_* \in [0,1]$ such that

$$\int_0^1 f(x)dx = f(\xi_*).$$

Since $f'(x) \neq 0$ for any $x \in [0,1]$, by the mean value theorem we conclude that ξ_* is unique.

Solution.

(b) Since f is continuous on $[0,1]$, it achieves the minimum m and maximum M on $[0,1]$, and

$$m \leq f(x) \leq M, \quad \forall x \in [0,1].$$

Let $S_n = \sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n}$ be a Riemann sum of f. Since $m \leq f(c_k) \leq M$, we see that

$$m = m\sum_{k=1}^{n}\frac{1}{n} \leq S_n = \sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n} \leq M\sum_{k=1}^{n}\frac{1}{n} = M.$$

□

Solution.

By the intermediate value theorem of continuous functions we see that there exists $\xi_n \in [0, 1]$ such that

$$f(\xi_n) = S_n.$$

Since $f'(x) \neq 0$ for all $x \in [0, 1]$, by the mean value theorem we know that ξ_n is unique.

$\square$

Solution.

(c) Since f is continuous on $[0,1]$, so it is integrable on $[0,1]$. Thus the Riemann sum S_n associated with the equal length division converges to the integral $\int_0^1 f(x)dx$. Since $f'(x)$ is continuous and never zero, it does not change its sign, and $c_0 = \min_{0\le x\le 1} |f'(x)| > 0$. By the mean value theorem, there exists η_n lying in between ξ_* and ξ_n such that

$$f(\xi_n) - f(\xi_*) = f'(\eta_n)(\xi_n - \xi_*).$$

Hence

$$|\xi_n - \xi_*| = |\frac{f(\xi_n) - f(\xi_*)}{f'(\eta_n)}| \le \frac{1}{c_0}|f(\xi_n) - f(\xi_*)| = \frac{1}{c_0}|S_n - \int_0^1 f(x)dx| \to 0$$

as $n \to \infty$.

*Ex.1.** Assume $f(x)$ and $g(x)$ are integrable over a bounded interval $[a, b]$. Show that $f(x) + g(x)$ and $f(x)g(x)$ are also integrable over $[a, b]$.

Remark.

By the assumption that f and g are integrable over $[a,b]$, f and g are bounded on $[a,b]$, say, there exist $C_f > 0$ and $C_g > 0$ such that

$$|f(x)| \leq C_f, \quad |g(x)| \leq C_g, \quad \forall x \in [a,b]. \tag{1}$$

Denote by $S(f,P)$ and $s(f,P)$ the upper and lower sums of f associated with the partition P, and $\Delta(f,P) = S(f,P) - s(f,P)$. Let $\varepsilon > 0$ be given.

Solution.

Let $\varepsilon > 0$ be given. By the assumption that f and g are integrable over $[a, b]$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for any partition P of $[a, b]$, we have

$$\Delta(f, P) < \frac{\varepsilon}{2C_g} \quad \text{whenever} \quad \|P\| < \delta_1, \tag{2}$$

and

$$\Delta(g, P) < \frac{\varepsilon}{2C_f} \quad \text{whenever} \quad \|P\| < \delta_2. \tag{3}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, and assume $P = \{x_0, \cdots, x_n\}$ is a partition of $[a, b]$ with $\|P\| < \delta$. Set $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.

Solution.

Denote the amplitude of the oscillation of f over I_i by $\omega_i(f)$. For any $x, x' \in I_i$,

$$|f(x)g(x) - f(x')g(x')| \leq |f(x)||g(x) - g(x')| + |g(x')||f(x) - f(x')|.$$

Hence, from this and (1), we have

$$\omega_i(fg) \leq C_f \cdot \omega_i(g) + C_g \cdot \omega_i(f).$$

Solution.

So

$$\Delta(fg,P) = \sum_{i=1}^{n} \omega_i(fg)\Delta x_i$$
$$\leq C_f \sum_{i=1}^{n} \omega_i(g)\Delta x_i + C_g \sum_{i=1}^{n} \omega_i(f)\Delta x_i = C_f \Delta(g,P) + C_g \Delta(f,P).$$

From this and (2)(3), we see that

$$\Delta(fg,P) < \varepsilon \quad \text{whenever} \quad \|P\| < \delta.$$

Therefore, $f(x)g(x)$ is integrable over $[a,b]$. □

Solution.

For any $x, x' \in I_i$, we have

$$\left|\Big(f(x)+g(x)\Big)-\Big(f(x')+g(x')\Big)\right| \leq |f(x)-f(x')|+|g(x)-g(x')|,$$

say,

$$\omega_i(f+g) \leq \omega_i(f)+\omega_i(g).$$

So

$$\begin{aligned}\Delta(f+g, P) &= \sum_{i=1}^{n} \omega_i(f+g)\Delta x_i \\ &\leq \sum_{i=1}^{n} \omega_i(f)\Delta x_i + \sum_{i=1}^{n} \omega_i(g)\Delta x_i = \Delta(f, P)+\Delta(g, P).\end{aligned}$$

Solution.

From this and (2)(3), we see that

$$\Delta(f+g,P) < \left(\frac{1}{2C_g}+\frac{1}{2C_f}\right)\varepsilon \quad \text{whenever} \quad \|P\| < \delta.$$

Therefore, $f(x)+g(x)$ is integrable over $[a,b]$.

*Ex.2.** Examine integrability of the following functions over $I = [0,1]$.

$$\text{(a)} \quad f(x) = \begin{cases} 1 - \frac{1}{n}, & x \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}], \quad n = 1, 2, \cdots, \\ 1, & x = 0. \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} x, & x \text{ is rational}, \\ 2x, & x \text{ is irrational}. \end{cases}$$

$$\text{(c)} \quad f(x) = u'(x), \quad u(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$\text{(d)} \quad f(x) = v'(x), \quad v(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Review.

Example 2.1.9.(a) (in Lecture PPT-13). Let f be a bounded function on a bounded interval $[a, b]$, and f is continuous on $[a, b]$ except at a sequence of points $\{a_n : n = 1, 2, \cdots\} \subset [a, b]$. Assume $\lim_{n\to\infty} a_n = c$. Show that f is integrable over $[a, b]$.

Theorem 2.1.6 (in Lecture PPT-13). If a function f is continuous over the bounded and closed interval $[a, b]$, or if f is bounded and has only a finite number of discontinuity points on $[a, b]$, then f is integrable over $[a, b]$.

Lemma 2.1.5.(a) (in Lecture PPT-13). A bounded function on a bounded interval $I = [a, b]$ is integrable if and only if

$$\lim_{\|P\|\to 0} \Delta(P) = 0$$

Solution.

(a) The given function

$$f(x) = \begin{cases} 1 - \frac{1}{n}, & x \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}], \quad n = 1, 2, \cdots, \\ 1, & x = 0. \end{cases}$$

is bounded on $[0, 1]$ and is continuous on $[0, 1]$ except at a sequence of points $\{x_n = \frac{1}{2^n} : n = 1, 2, \cdots\}$. For discontinuity points $x_n = \frac{1}{2^n}$, we have $x_n \to 0$ as $n \to \infty$. From the Example 2.1.9.(a), we know that f is integrable over $[0, 1]$.

Solution.

(b) The given function

$$f(x) = \begin{cases} x, & x \text{ is rational,} \\ 2x, & x \text{ is irrational.} \end{cases}$$

is not integrable over $[0, 1]$.

Take equal-length partition $P_n = \{x_0, \cdots, x_n\}$ with $x_k = \frac{k}{n}$. Denote $I_k = \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1, \cdots, n$. If in each subinterval I_k, we choose $c_k = \frac{k-1}{n}$ which is rational, then $f(c_k) = c_k = \frac{k-1}{n}$. Denote $\xi = \{c_1, \cdots, c_n\}$.

Solution.

Then the corresponding Riemann sum

$$S(f, P_n, \xi) = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \left(\frac{k-1}{n} \cdot \frac{1}{n} \right)$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} (k-1) = \frac{1}{n^2} \left[\frac{n(n+1)}{2} - n \right] \to \frac{1}{2} \quad \text{as} \quad n \to \infty.$$

So there exists a positive integer N_1 such that

$$S(f, P_n, \xi) < \frac{5}{8}, \quad \forall n > N_1.$$

Solution.

If in each subinterval I_k, we choose c_k' to be an irrational number and $c_k' \in \left(\frac{k-\frac{1}{2}}{n}, \frac{k}{n}\right)$, then $f(c_k') = 2c_k' \geq \frac{2k-1}{n}$. Denote $\xi' = \{c_1', \cdots, c_n'\}$. Then the corresponding Riemann sum

$$S(f, P_n, \xi') = \sum_{k=1}^{n} f(c_k')\Delta x_k \geq \sum_{k=1}^{n} \left(\frac{2k-1}{n} \cdot \frac{1}{n}\right)$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} (2k-1) = \frac{1}{n^2}[n(n+1) - n] \to 1 \quad \text{as} \quad n \to \infty.$$

So there exists N_2 such that

$$S(f, P_n, \xi') > \frac{7}{8}, \quad \forall n > N_2.$$

Solution.

For any $\delta > 0$, we take $N = \max\left\{N_1, N_2, \frac{1}{\delta}\right\}$. For $n > N$, we have

$$\|P_n\| = \frac{1}{n} < \frac{1}{N} \le \delta,$$

but

$$S(f, P_n) - s(f, P_n) \ge S(f, P_n, \xi') - S(f, P_n, \xi) > \frac{7}{8} - \frac{5}{8} = \frac{1}{4},$$

say, $\lim_{\|P\|\to 0} \Delta(P) > 0$. Thus, from the Lemma 2.1.5.(a), we conclude that f is not integrable over $[0, 1]$.

Solution.

(c) For the given function

$$u(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

we have

$$u'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

When $0 < x \leq 1$, we have

$$u'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \leq 2|x| \cdot \left|\sin \frac{1}{x}\right| + \left|\cos \frac{1}{x}\right| \leq 3.$$

Solution.

$f(x) = u'(x)$ is bounded over $[0, 1]$ and has only one discontinuity point $x = 0$. By the Theorem 2.1.6, we conclude that $f(x) = u'(x)$ is integrable over $[0, 1]$.

Solution.

(d) For the given function

$$v(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

we have

$$v'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Take

$$x_k = \frac{1}{\sqrt{(2k+1)\pi}}, \quad k \in \mathbb{N},$$

Solution.

then

$$\sin\frac{1}{x_k^2} = 0, \qquad \cos\frac{1}{x_k^2} = -1,$$

and

$$\lim_{k\to\infty} v'(x_k) = \lim_{x_k\to 0^+} v'(x_k) = \lim_{x_k\to 0^+} \frac{2}{x_k} = \infty.$$

$f(x) = v'(x)$ is unbounded over $[0,1]$, hence is not integrable over $[0,1]$.

Remark: f has an antiderivative over $[0,1]$, but f is not integrable over $[0,1]$.

Ex.3. (a) Assume g is continuous on $[a,b]$ and f is continuous on the range of g. Show that $f \circ g$ is continuous on $[a,b]$, hence is integrable over $[a,b]$.

(b) Assume f is integrable over $[a,b]$ and $f \circ f$ is well-defined on $[a,b]$. Must $f \circ f$ be integrable over $[a,b]$?

More general, assume f and g are integrable over $[a,b]$ and $f \circ g$ is well-defined on $[a,b]$. Must $f \circ g$ be integrable over $[a,b]$?

(c) Assume f is bounded but not integrable over $[a,b]$. Assume $f \circ f$ is well-defined on $[a,b]$. Must $f \circ f$ be non-integrable over $[a,b]$?

More general, assume f and g are bounded but not integrable over $[a,b]$. Assume $f \circ g$ is well-defined on $[a,b]$. Must $f \circ g$ be non-integrable over $[a,b]$?

Solution.

(a) Denote I be the range of g, and $y = g(x) \in I$. Let $x_0 \in [a, b]$ arbitrary, and $y_0 = g(x_0) \in I$. Let $\varepsilon > 0$ be given.

By the assumption that g is continuous on $[a, b]$, there exists $\delta_1 > 0$ such that

$$|g(x) - g(x_0)| < \varepsilon \quad \text{whenever} \quad x \in [a, b], \quad |x - x_0| < \delta_1.$$

By the assumption that f is continuous on I, there exists $\delta_2 > 0$ such that

$$|f(y) - f(y_0)| < \varepsilon \quad \text{whenever} \quad y \in I, \quad |y - y_0| < \delta_2.$$

Solution.

Take $\delta = \min\{\delta_1, \delta_2\}$. For the given ε, there exists $\delta > 0$ such that

$$\left| f\Big(g(x)\Big) - f\Big(g(x_0)\Big) \right| < \varepsilon \quad \text{whenever} \quad x \in [a,b], \quad |x - x_0| < \delta.$$

That is, $f \circ g$ is continuous at $x_0 \in [a,b]$. Hence, $f \circ g$ is continuous on $[a,b]$ since x_0 is arbitrary. Thus, from the Theorem 2.1.6, the continuous function $f \circ g$ is integrable over $[a,b]$.

Solution.

(b) The integrability of f over $[a, b]$ does not imply integrability of $f \circ f$ over $[a, b]$. Counterexample: Let $f(x) = \zeta(x)$ be the Riemann function, which is integrable over $[0, 1]$. If $x = p/q$ is a rational number written in the irreducible form, then $\zeta(x) = 1/q$ which is also in the irreducible form. Hence $\zeta \circ \zeta(x) = \zeta(1/q) = 1/q$. If x is irrational number, then $\zeta(x) = 0 = 0/1$, hence $\zeta \circ \zeta(x) = \zeta(0) = 1$. Thus

$$u(x) \equiv \zeta \circ \zeta(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \text{ in irreducible form,} \\ 1, & x \text{ is irrational,} \end{cases}$$

which is well-defined on $[0, 1]$.

Now we show $u(x)$ is not integrable over $[0, 1]$.

Solution.

Take $0 < \varepsilon < 1/2$, and set

$$E = \{x \in [0,1] : \zeta(x) > \varepsilon\}.$$

The set E contains only a finite number of points. Hence for any non-empty open interval I, we can always take a rational number $x = p/q \in I \setminus E$. Then $\zeta(x) = 1/q \leq \varepsilon$, $u(x) = \zeta(1/q) = 1/q \leq \varepsilon$. Let $P = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[0,1]$. Each subinterval $[x_{k-1}, x_k]$ contains an irrational number on which $u(x) = 1$. So the upper sum $S(P) = 1$.

Solution.

On the other hand, taking $c_k \in [x_{k-1}, x_k] \setminus E$ to be a rational number, we have $u(c_k) \le \varepsilon$, hence the lower sum $s(P) \le \varepsilon$. So

$$\Delta(P) = S(P) - s(P) \ge 1 - \varepsilon > \frac{1}{2}.$$

By the Lemma 2.1.5, we conclude that u is not integrable over $[0, 1]$.

Solution.

(c) The non-integrability of f over $[a, b]$ does not imply non-integrability of $f \circ f$ over $[a, b]$.

Counterexample: Let $f(x) = D(x)$ be the Dirichlet function. The Dirichlet function D is non-integrable over $[0, 1]$. For any $x \in [0, 1]$, $D(x)$ is either 0 or 1, so $D(x)$ is rational, hence $u(x) = D \circ D(x) = 1$ for all $x \in [0, 1]$. Thus, $u(x)$ is integrable over $[0, 1]$.

*Ex.4.** Assume $f(x)$ is integrable on $[a,b]$ and $g(x)=f(x)$ for all x in (a,b) except for a finite number of points. Show that $g(x)$ is integrable on $[a,b]$, and

$$\int_a^b f(x)dx=\int_a^b g(x)dx.$$

Solution.

Since f is integrable, denote $\int_a^b f(x)dx = J$. From the definition of the definite integral, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ s.t. for any partition T and point set $\{\xi_i\}$ submit to the partition, when $\|T\| < \delta_1$, we have

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - J\right| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\left|\sum_{i=1}^{n} g(\xi_i)\Delta x_i - J\right| &\leqslant \left|\sum_{i=1}^{n} g(\xi_i)\Delta x_i - \sum_{i=1}^{n} f(\xi_i)\Delta x_i\right| + \left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - J\right| \\ &\leqslant \sum_{i=1}^{n} |g(\xi_i) - f(\xi_i)| \cdot \|T\| + \frac{\varepsilon}{2}.\end{aligned}$$

Solution.

Since f is integrable, f is bounded. Thus g is bounded (because finite number sets is always bounded). Set $|f|, |g| \leqslant M$, then $|f-g| \leqslant 2M$. Assume there are k points where $f \neq g$, then there are at most k subintervals in the partition such that $|g(\xi_i) - f(\xi_i)| > 0$. Let

$$\delta = \min\left\{\delta_1, \frac{\varepsilon}{4kM}\right\},$$

for all $\|T\| < \delta$,

$$\left|\sum_{i=1}^{n} g(\xi_i)\Delta x_i - J\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence g is integrable and $\int_a^b f(x)dx = \int_a^b g(x)dx$.

Remark.

Beware that when proving integrability, you should prove that for any partition P with $\|P\| < \delta$, the Darboux sums S and s satisfies

$$|S - s| < \varepsilon.$$

We might not have marked out this mistake this time, but you should be careful when dealing with such problems.

Solution.

(c) Denote by $S(f, P)$ and $s(f, P)$ the upper and lower sums of f associated with a partition P, and $\Delta(f, P) = S(f, P) - s(f, P)$.
Let $\varepsilon > 0$ be given. By the assumption that f is integrable over $[a, b]$, there exists $\delta > 0$ such that for any partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$, we have

$$\Delta(f, P) < \varepsilon. \tag{7}$$

Set $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Denote the amplitude of the oscillation of f over I_i by $\omega_i(f)$.

*Ex.5.** Assume that $f(x)$ is integrable on $[0,1]$, $f''(x) \geqslant 0$ in $[0,1]$. Prove that

$$\int_0^1 f(x)dx \geqslant f(\frac{1}{2}).$$

Solution.

Take the partition as uniform partition with $2n$ subintervals, with the same length, and let ξ_i be the mid point of each subinterval($i = 1, 2, ..., 2n$). Then $\frac{\xi_i + \xi_{2n-i+1}}{2} = \frac{1}{2}, i = 1, 2, ..., 2n$. Since $f''(x) \geqslant 0$, we can conclude that for any $x_1, x_2 \in [0.1]$, $f(\frac{x_1+x_2}{2}) \leqslant \frac{1}{2}(f(x_1) + f(x_2))$. Thus

$$\sum_{i=1}^{2n} f(\xi_i) = \sum_{i=1} n(f(\xi_i) + f(\xi_{2n-i+1})) \geqslant 2nf(\frac{1}{2}).$$

Thus

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{2n} f(\xi_i) \cdot \frac{1}{2n} \geqslant \lim_{n\to\infty} \frac{1}{2n} f(\frac{1}{2}) \cdot 2n = f(\frac{1}{2}).$$

Ex.1. Assume that f is continuously differentiable on $[a, b]$ and $f(a) = 0$. Prove the following inequality.

$$\int_a^b |f(x)f'(x)|\, dx \leq \frac{b-a}{2} \int_a^b |f'(x)|^2\, dx$$

Proof.

By the assumption that f is continuously differentiable on $[a, b]$ and $f(a) = 0$, let

$$g(x) = \int_a^x |f'(t)| \mathrm{d}t, \tag{1}$$

then $g'(x) = |f'(x)|$.

By the assumption that $f(a) = 0$, we have

$$|f(x)| = |f(x) - f(a)| = \left| \int_a^x f'(t) \mathrm{d}t \right| \leq \int_a^x |f'(t)| \mathrm{d}t = g(x). \tag{2}$$

Proof.

From (1)(2) and the Cauchy inequality, we have

$$\begin{aligned}
\int_a^b |f(x)f'(x)|\,\mathrm{d}x &\leq \int_a^b g(x)g'(x)\mathrm{d}x = \int_a^b g(x)\mathrm{d}[g(x)] = \frac{1}{2}g^2(x)\Big|_a^b \\
&= \frac{1}{2}\left(\int_a^b |f'(t)|\mathrm{d}t\right)^2 = \frac{1}{2}\left(\int_a^b 1\cdot|f'(x)|\mathrm{d}x\right)^2 \\
&\leq \frac{1}{2}\int_a^b 1^2\mathrm{d}x\int_a^b |f'(x)|^2\mathrm{d}x \\
&= \frac{b-a}{2}\int_a^b |f'(x)|^2\mathrm{d}x.
\end{aligned}$$

□

Ex.2. Prove the Wallis product formula

$$\lim_{n\to\infty}\left(\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\cdot\frac{1}{2n+1}\right)=\frac{\pi}{2}$$

by using definite integrals

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\ \mathrm{d}x=\frac{(2n)!!}{(2n+1)!!},\quad \int_0^{\frac{\pi}{2}}\sin^{2n}x\ \mathrm{d}x=\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2},$$

where the integer n is positive.

Solution.

For any positive integer n, for all $x \in [0, \pi/2]$, we have

$$\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x,$$

then

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, \mathrm{d}x \le \int_0^{\frac{\pi}{2}} \sin^{2n} x \, \mathrm{d}x \le \int_0^{\frac{\pi}{2}} \sin^{2n-1} x \, \mathrm{d}x.$$

And also, for any positive integer n, we have

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, \mathrm{d}x = \frac{(2n)!!}{(2n+1)!!}, \quad \int_0^{\frac{\pi}{2}} \sin^{2n} x \, \mathrm{d}x = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}.$$

Solution.

Hence, we have

$$\frac{(2n)!!}{(2n+1)!!} \le \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \le \frac{(2n-2)!!}{(2n-1)!!}.$$

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n+1} \le \frac{\pi}{2} \le \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n}.$$

Here we denote

$$A_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n+1}, \quad \text{and } B_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n},$$

then $A_n \le \frac{\pi}{2} \le B_n$ for any positive integer n.

Solution.

Therefore,

$$0 \le \frac{\pi}{2} - A_n \le B_n - A_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n(2n+1)} = \frac{1}{2n} A_n \le \frac{1}{2n} \cdot \frac{\pi}{2}.$$

Since $\lim_{n\to\infty} \left(\frac{1}{2n} \cdot \frac{\pi}{2}\right) = 0$, by the Sandwich Theorem,

$$\lim_{n\to\infty} \left(\frac{\pi}{2} - A_n\right) = \lim_{n\to\infty} (B_n - A_n) = 0.$$

Thus, we have $\lim_{n\to\infty} A_n = \frac{\pi}{2}$. □

Extension (Not Require).

The fomous Wallis product formula for π, pulished in 1656 by the English mathematician John Wallis, states that

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{2\times 2}{1\times 3} \cdot \frac{4\times 4}{3\times 5} \cdot \frac{6\times 6}{5\times 7} \cdots$$

It is equivalent to the following limit expression

$$\lim_{n\to\infty} \left(\left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{1}{2n+1} \right) = \frac{\pi}{2}.$$

The Wallis product formula shows a relationship between π and integers.

Extension (Not Require).

Its proof method and applications have attracted much attention of mathematicians. For instance, we can use it to estimate the value of π (*Thomas' Calculus*, Exercises 10.10, Ex.63), to prove the Stirling's approximation for the factorial function $n!$:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$$

(refer to *Thomas' Calculus*, Chapter 8 Additional and Advanced Exercises, Ex.52), and so on.

Please peruse the following content after this tutorial.

Exercise-5.3 Ex.86. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.
(a). If $m_k = \min\{f(x)$ for x in the kth subinterval $\}$, explain the connection between the lower sum

$$L = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

and the shaded regions in the first part of the figure.

(b). If $M_k = \max\{f(x)$ for x in the kth subinterval $\}$, explain the connection between the upper sum

$$U = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n$$

and the shaded regions in the second part of the figure.

(c). Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.

Solution.

(a) The area of the shaded region in the first part is $\sum_{i=1}^{n} m_i \Delta x_i$ which is equal to L.

(b) The area of the shaded region in the second part is $\sum_{i=1}^{n} M_i \Delta x_i$ which is equal to U.

(c) The area of the shaded region in the third partis is the difference between areas in the second and the first part. Thus, this area is $U - L$. $\square$

Exercises-5.3 Ex.87. We say f is uniformly continuous on $[a, b]$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \leq \epsilon \cdot (b - a)$ by making the largest of the Δx_k 's sufficiently small.

Solution.

Let $\varepsilon > 0$ be given. We shall show that there exists $\delta > 0$ such that for any partition P of $[a, b]$, if $\|P\| < \delta$, then the upper sum U and lower sum L associated with the partition P (see Ex.86) satisfy
$U - L < \varepsilon(b - a)$.

Since f is continuous on a closed and bounded interval $[a, b]$, it is uniformly continuous on $[a, b]$. For the given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } x, y \in [a, b], \ \ |x - y| < \delta. \tag{1}$$

Solution.

Assume that $P : a = x_0 < x_1 < \cdots < x_n = b$ and $\|P\| < \delta$. Then for any $1 \leq i \leq n$, $\Delta x_i = (x_i - x_{i-1}) \leq \|P\| < \delta$. Therefore from (1), we have

$$|M_i - m_i| < \varepsilon,$$

where

$$M_i = \max\{f(x) \text{ for x in the i-th subinterval }\},$$

$$m_i = \min\{f(x) \text{ for x in the i-th subinterval }\}.$$

Solution.

From this and using Ex.86, we have the following:

$$U-L=\sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i < \sum_{i=1}^{n} \varepsilon \Delta x_i = \varepsilon(b-a).$$

Ex.72. Another proof of the Evaluation Theorem.

(a). Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$ and let F be any antiderivative of f. Show that

$$F(b) - F(a) = \sum_{i=1}^{n} \left[F\left(x_i\right) - F\left(x_{i-1}\right)\right].$$

(b). Apply the Mean Value Theorem to each term to show that $F\left(x_i\right) - F\left(x_{i-1}\right) = f\left(c_i\right)\left(x_i - x_{i-1}\right)$ for some c_i in the interval $\left(x_{i-1}, x_i\right)$. Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.

(c). From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Solution.

(a).

$$
\begin{aligned}
&\sum_{i=1}^{n}\left[F\left(x_i\right)-F\left(x_{i-1}\right)\right] \\
=&\left[F\left(x_1\right)-F\left(x_0\right)\right]+\left[F\left(x_2\right)-F\left(x_1\right)\right]+\left[F\left(x_3\right)-F\left(x_2\right)\right] \\
&+\cdots+\left[F\left(x_{n-1}\right)-F\left(x_{n-2}\right)\right]+\left[F\left(x_n\right)-F\left(x_{n-1}\right)\right] \\
=&F\left(x_n\right)-F\left(x_0\right)=F(b)-F(a).
\end{aligned}
$$

Solution.

(b). Since F is an antiderivative of f on $[a,b]$, then F is differentiable and continuous on $[a,b]$.

Consider any subinterval $[x_{i-1}, x_i] \subseteq [a,b]$. By the Mean Value Theorem, there exists $c_i \in (x_{i-1}, x_i)$ such that

$$[F(x_i) - F(x_{i-1})] = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ for each i. Therefore,

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x_i \tag{1}$$

is a Riemann sum for f on $[a,b]$. □

Solution.

(c). Since f is continuous on $[a, b]$, so it is integrable on $[a, b]$, and

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \left(\sum_{i=1}^{n} f(c_i) \Delta x_i \right).$$

Taking the limit as $\|P\| \to 0$ in (1), we find

$$F(b) - F(a) = \lim_{||P|| \to 0} (F(b) - F(a))$$
$$= \lim_{||P|| \to 0} \left(\sum_{i=1}^{n} f(c_i) \Delta x_i \right) = \int_a^b f(x)dx.$$

□