Week 1

Tutorial

Question Q1

Let C be a curve given by

$$x = e^t \cos t, \quad y = e^t \sin t, \quad z = e^t, \quad t \in \mathbb{R}.$$

(a) Find the unit tangent vector $\mathbf{T}$, the principal normal vector $\mathbf{N}$, and the curvature κ of the curve C for any $t \in \mathbb{R} \equiv (-\infty, \infty)$.

(b) Find the arc length L of the part of the curve C with $0 \leq t \leq 1$.

(c) Evaluate the integral

$$\int_0^{2\pi} \mathbf{T}(t) \times \mathbf{N}(t) dt.$$

Solution for Q1

(a) The curve C can be represented by

$$\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k}.$$

Hence, we have

$$\begin{aligned}
\mathbf{r}'(t) &= e^t(\cos t - \sin t)\mathbf{i} + e^t(\sin t + \cos t)\mathbf{j} + e^t\mathbf{k}, \\
|\mathbf{r}'(t)| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}} = \sqrt{3}e^t.
\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{T} =& \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{e^t(\cos t - \sin t)}{\sqrt{3}e^t}\mathbf{i} + \frac{e^t(\sin t + \cos t)}{\sqrt{3}e^t}\mathbf{j} + \frac{e^t}{\sqrt{3}e^t}\mathbf{k}\\ =& \frac{\cos t - \sin t}{\sqrt{3}}\mathbf{i} + \frac{\sin t + \cos t}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k},\\ \frac{d\mathbf{T}}{dt} =& -\frac{\sin t + \cos t}{\sqrt{3}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{3}}\mathbf{j},\\ \left|\frac{d\mathbf{T}}{dt}\right| =& \sqrt{\Big(-\frac{\sin t + \cos t}{\sqrt{3}}\Big)^2 + \Big(\frac{\cos t - \sin t}{\sqrt{3}}\Big)^2} = \sqrt{\frac{2}{3}},\\ \mathbf{N} =& \frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|} = -\frac{\sin t + \cos t}{\sqrt{2}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\mathbf{j},\\ \kappa =& \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{\left|\frac{d\mathbf{T}}{dt}\right|}{|\mathbf{r}'(t)|} = \frac{\sqrt{\frac{2}{3}}}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3e^t}.\end{aligned}$$

(b) The arc length L is given by

$$L = \int_0^1 |\mathbf{r}'(t)|dt = \sqrt{3}\int_0^1 e^t dt = \sqrt{3}(e-1).$$

(c)

$$\mathbf{T}(t)\times\mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\cos t - \sin t}{\sqrt{3}} & \frac{\sin t + \cos t}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{\sin t + \cos t}{\sqrt{2}} & \frac{\cos t - \sin t}{\sqrt{2}} & 0\end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos t - \sin t & \sin t + \cos t & 1\\ -\sin t - \cos t & \cos t - \sin t & 0\end{vmatrix} = \frac{\sin t - \cos t}{\sqrt{6}}\mathbf{i} - \frac{\sin t + \cos t}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k},$$

$$\int_0^{2\pi}\mathbf{T}(t)\times\mathbf{N}(t)dt = \frac{1}{\sqrt{6}}\int_0^{2\pi}\{(\sin t - \cos t)\mathbf{i} - (\sin t + \cos t)\mathbf{j} + 2\mathbf{k}\}dt$$

$$= \frac{4\pi}{\sqrt{6}}\mathbf{k} = \frac{2\sqrt{6}\pi}{3}\mathbf{k}.$$

Question Q2

Examine the existence of the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ and prove your conclusion.

$$(a) \quad f(x,y) = \frac{x^2 y}{x^2+y^2} \ln\left(\frac{1}{x^2+y^2}\right).$$

$$(b) \quad f(x,y) = \frac{x^4 \sin^2 y}{x^5+y^{10}}.$$

Solution for Q2

(a) Denote $\rho(x,y) = \sqrt{x^2+y^2}$. Then $|x| \le \rho(x,y)$.

$$|f(x,y)| = |x| \left| \frac{xy}{x^2+y^2} \ln(\frac{1}{\rho^2}) \right| \le |x| |\ln(\rho)| \le \rho |\ln(\rho)|.$$

Recall that

$$\lim_{t\to 0^+} t \ln t = 0,$$

and $\rho \to 0$ as $(x,y) \to (0,0)$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} \ln\left(\frac{1}{x^2+y^2}\right) = 0.$$

(b) For any $k \neq 0$, along the parabola $x = ky^2$, we have

$$\lim_{\substack{(x,y)\to(0,0),\\ x=ky^2}} \frac{x^4 \sin^2 y}{x^5+y^{10}} = \lim_{y\to 0} \frac{k^4 y^8 \sin^2 y}{k^5 y^{10} + y^{10}}$$

$$= \lim_{y\to 0} \frac{k^4}{k^5+1} \cdot \frac{\sin^2 y}{y^2} = \frac{k^4}{k^5+1},$$

which depends on k. Thus, $\lim_{(x,y)\to(0,0)} \frac{x^4 \sin^2 y}{x^5+y^{10}}$ does not exist.

Question Q3

Let

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(b) Examine the differentiability of $f(x,y)$ at $P(0,0)$ and prove your conclusion.

(c) Find the directional derivative of $f(x,y)$ at $P(0,0)$ along the direction $\mathbf{u} = (\alpha, \beta)$, where $\alpha^2 + \beta^2 = 1$.

Solution for Q3

(a) $f(x,0)=0$ for all $x\in\mathbb{R}$, hence

$$\frac{\partial f}{\partial x}(0,0)=0.$$

Similarly, $f(0,y)=0$ for all $y\in\mathbb{R}$, hence

$$\frac{\partial f}{\partial y}(0,0)=0.$$

(b) The function f is not differentiable at $P(0,0)$.
Suppose f is differentiable at $P(0,0)$, then

$$f(x,y)=f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y=o(\rho)$$

as $\rho\to 0$, where $\rho=\sqrt{x^2+y^2}$. However, along $y=x>0$, we have

$$\lim_{\substack{(x,y)\to(0,0),\\ y=x>0}}\frac{f(x,y)}{\rho}=\lim_{\substack{(x,y)\to(0,0),\\ y=x>0}}\frac{\sin(xy)}{x^2+y^2}=\lim_{x\to 0^+}\frac{\sin(x^2)}{2x^2}=\frac{1}{2}>0,$$

which is a contradiction. Thus, f is not differentiable at $(0,0)$.
(c) Note that $f(0,0)=0$. Thus, the directional derivative of $f(x,y)$ at $P(0,0)$ along the direction $\mathbf{u}=(\alpha,\beta)$, where $\alpha^2+\beta^2=1$, is given by

$$\begin{aligned}\frac{\partial f}{\partial \mathbf{u}}(0,0)&=\lim_{s\to 0^+}\frac{f(\alpha s,\beta s)-f(0,0)}{s}\\ &=\lim_{s\to 0^+}\frac{\sin(\alpha\beta s^2)}{s\sqrt{(\alpha s)^2+(\beta s)^2}}=\lim_{s\to 0^+}\frac{\sin(\alpha\beta s^2)}{s^2}=\alpha\beta.\end{aligned}$$

Question Q4

Find the radius of convergence and the set of convergence of the given power series:

$(a)\ \displaystyle\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n}x^{n^2},$ $\qquad$ $(b)\ \displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{n}x^{2n}.$

Solution for q4

(a) Write the series as
$$\sum_{k=1}^{\infty} a_k x^k,$$
where for $n \geq 1$,
$$a_k = \begin{cases} \frac{\ln(n+1)}{n} & \text{if } k = n^2, \\ 0 & \text{if } n^2 < k < (n+1)^2. \end{cases}$$
$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{n\to\infty} \sqrt[n^2]{|a_{n^2}|} = \lim_{n\to\infty} \frac{\sqrt[n^2]{\ln(n+1)}}{\sqrt[n^2]{n}} = 1.$$
This is because
$$\lim_{n\to\infty} \ln(\sqrt[n^2]{\ln(n+1)}) = \lim_{n\to\infty} \frac{1}{n^2} \ln\ln(n+1) = 0,$$
$$\lim_{n\to\infty} \ln(\sqrt[n^2]{n}) = \lim_{n\to\infty} \frac{1}{n^2} \ln n = 0,$$
hence
$$\lim_{n\to\infty} \sqrt[n^2]{\ln(n+1)} = 1, \quad \lim_{n\to\infty} \sqrt[n^2]{n} = 1.$$
Thus the radius of convergence
$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = 1.$$
When $x = 1$, the series is
$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n},$$
which diverges by comparing with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$.
When $x = -1$, since n^2 is odd if and only if n is odd, so $(-1)^{n^2} = (-1)^n$, the series is
$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n+1)}{n},$$
which is alternating and can be written as
$$\sum_{n=1}^{\infty} (-1)^n u_n,$$
where $u_n = \frac{\ln(1+n)}{n}$ for each $n \geq 1$. Set $f(x) = \frac{\ln(1+x)}{x}$ for $x > 0$.

Hence, we have

$$f'(x) = \frac{\frac{x}{1+x} - \ln(1+x)}{x^2} < 0$$

because $\ln(1+x) > \frac{x}{1+x}$ for all $x > 0$. Therefore, f is decreasing, and

$$u_n = f(n) > f(n+1) = u_{n+1}$$

for all $n \geq 1$. Note that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\ln(1+n)}{n} = 0.$$

So we can apply the Leibniz Test to conclude that the given series converges.
Therefore the set of convergence is $[-1, 1)$.

(b) Denote

$$a_n = \frac{\sin^2 n}{n}.$$

The power series can be written as

$$\sum_{n=1}^{\infty} a_n x^n.$$

Note that $|a_n| \leq 1$.
(i) Denote by R the radius of convergence of the series.
If $0 < a < 1$, and $|x| \leq a$. Then comparing the series with the geometric series $\sum_{n=1}^{\infty} a^n$ we see that the given series converges. Hence $R \geq a$. This is true for any $0 < a < 1$, so $R \geq 1$.
(ii) When $x = \pm 1$, the series is

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1-\cos 2n}{n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
By the Dirichlet test the series $\sum_{n=1}^{\infty} \frac{\cos 2n}{n}$ converges. In fact, $\frac{1}{n}$ monotonically decreases to zero as $n \to \infty$. Recall that

$$\sum_{n=1}^{m} \cos 2n + i\sum_{n=1}^{m} \sin 2n = \sum_{n=1}^{m} (e^{2i})^n = \frac{(e^{2i})^{m+1} - e^{2i}}{e^{2i} - 1}$$
$$= \frac{e^{i(m+2)}(e^{im} - e^{-im})}{e^i(e^i - e^{-i})} = \frac{e^{i(m+2)}}{e^i} \cdot \frac{\sin m}{\sin 1}.$$

So for all $m \geq 1$ we have

$$\left|\sum_{n=1}^{m} \cos 2n\right|^2 + \left|\sum_{n=1}^{m} \sin 2n\right|^2 \leq \frac{\sin^2 m}{\sin^2 1} \leq \frac{1}{\sin^2 1}.$$

By the Dirichlet Test, the series $\sum_{n=1}^{\infty} \frac{\cos 2n}{n}$ converges.
Hence the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$ diverges. By the Abel theorem $R \leq 1$.
(iii) Combining (i) and (ii) we see that $R = 1$. The analysis in (ii) shows that the set of convergence is $(-1, 1)$.

Question Q5

Show that the series of functions

$$\sum_{n=1}^{\infty} \frac{(\sin x)^n}{n}$$

uniformly converges on the closed interval $[-\frac{\pi}{2}, c]$ for any given constant $0 < c < \frac{\pi}{2}$; but it does not uniformly converge on the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution for Q5

(a) For any given $0 < c < \frac{\pi}{2}$, we have $0 < \sin c < 1$. So the geometric series $\sum_{n=1}^{\infty} (\sin c)^n$ converges. For all $x \in [-c, c]$, we have

$$\left|\frac{(\sin x)^n}{n}\right| \leq (\sin c)^n.$$

By the M-test, the series $\sum_{n=1}^{\infty} \frac{(\sin x)^n}{n}$ uniformly absolutely converges on $[-c, c]$.
When $x = -\frac{\pi}{2}$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the Leibniz test.

For $-\frac{\pi}{2} < x \leq -c$, the series is

$$\sum_{n=1}^{\infty} \frac{(-|\sin x|)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (|\sin x|)^n.$$

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, and the sequence $|\sin x|^n$ is monotone in n for $x \in (-\frac{\pi}{2}, -c]$ and is uniformly bounded. Then by the Abel test the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (|\sin x|)^n$$

uniformly converges on $(-\frac{\pi}{2}, -c]$.

Combining the above we see that the series uniformly converges on $[-\frac{\pi}{2}, -c]$.

(b) Since the above conclusion is true for any $c \in (0, \frac{\pi}{2})$, and it diverges at $x = \frac{\pi}{2}$, we conclude that the series converges on $[-\frac{\pi}{2}, \frac{\pi}{2})$.

Now we show the series does not uniformly converge on $[0, \frac{\pi}{2})$.

Take $\varepsilon_0 = \frac{1}{4}$. For any integer $N > 0$, take $n > N$ and

$$x_n = \sin^{-1}\left(2^{-\frac{1}{n}}\right) \in (0, \frac{\pi}{2}).$$

Then

$$\sin(x_n)^n = (2^{-\frac{1}{n}})^n = \frac{1}{2}.$$

Let $m > N$. Then

$$\sum_{n=m+1}^{2m} \frac{\sin(x_{2m})^n}{n} \geq \sum_{n=m+1}^{2m} \frac{\frac{1}{2}}{n} = \frac{1}{2} \sum_{n=m+1}^{2m} \frac{1}{n} \geq \frac{1}{4} = \varepsilon_0.$$

By the Cauchy criterion of uniform convergence of series of functions, we conclude that the series of functions $\sum_{n=1}^{\infty} \frac{(\sin x)^n}{n}$ does not uniformly converge on $[-\frac{\pi}{2}, \frac{\pi}{2})$.

Question Q6

Let $f(x, y)$ be a function defined on an open disc B with center $P_0(x_0, y_0)$ and radius $R > 0$. Assume that

(a) f has second partial derivatives $f_{xx}(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$ and $f_{yy}(x, y)$ in B; and

(b) $f_{xy}(x, y) = f_{yx}(x, y)$ holds for all $(x, y) \in B$.

Can you conclude that f is differentiable at $P_0(x_0, y_0)$? If your answer is "Yes" then give a proof; if your answer is "No" then give a counterexample.

Solution for Q6

Under the assumption we can not conclude f is differentiable at (x_0, y_0).

Counterexample: Let

$$f(x,y) = \begin{cases} \frac{x^2y^2}{(x^2+y^2)^{3/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$f(x,y)$ is defined on $\mathbb{R}^2$.
We compute

$$f_x(x,y) = \begin{cases} \frac{2xy^4-x^3y^2}{(x^2+y^2)^{5/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$$f_y(x,y) = \begin{cases} \frac{2x^4y-x^2y^3}{(x^2+y^2)^{5/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$f_x(x,0) = 0$ for all x, so

$$f_{xx}(0,0) = 0.$$

Hence

$$f_{xx}(x,y) = \begin{cases} \frac{2x^4y^2-11x^2y^4+2y^6}{(x^2+y^2)^{7/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$f_x(0,y) = 0$ for all y, so

$$f_{xy}(0,0) = 0.$$

Hence

$$f_{xy}(x,y) = \begin{cases} \frac{-2x^5y+11x^3y^3-2xy^5}{(x^2+y^2)^{7/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$f_y(x,0) = 0$ for all x, so

$$f_{yx}(0,0) = 0.$$

Hence

$$f_{yx}(x,y) = \begin{cases} \frac{-2x^5y+11x^3y^3-2xy^5}{(x^2+y^2)^{7/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Therefore

$$f_{xy}(x,y) = f_{yx}(x,y) \quad \text{holds for all } (x,y) \in \mathbb{R}^2.$$

$f_y(0,y) = 0$ for all y, so

$$f_{yy}(0,0) = 0.$$

Hence

$$f_{yy}(x,y) = \begin{cases} \frac{2x^6 - 11x^4y^2 + 2x^2y^4}{(x^2+y^2)^{7/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

So f satisfies conditions (a) and (b) on any disc centered at $(0,0)$.

Now we show f is not differentiable at $P_0(0,0)$. Suppose f is differentiable at $P_0(0,0)$. Since $f(0,0) = 0$, $f_x(0,0) = 0$ and $f_y(0,0) = 0$, we have

$$f(x,y) = \varepsilon\rho(x,y),$$

where $\rho(x,y) = \sqrt{x^2+y^2}$, $\varepsilon \to 0$ as $\rho \to 0$. Hence for $(x,y) \neq (0,0)$ we have

$$\frac{x^2y^2}{(x^2+y^2)^{3/2}} = \varepsilon\sqrt{x^2+y^2}, \quad \text{i.e.} \quad x^2y^2 = \varepsilon(x^2+y^2)^2.$$

Letting $x = y > 0$ we have

$$x^4 = \varepsilon(2x^2)^2 = 4\varepsilon x^4,$$

so

$$1 = 4\varepsilon \to 0$$

as $(x,y) \to (0,0)$, which is a contradiction. This shows f is not differentiable at $(0,0)$.

Question Q7

Let f be a continuous function defined on a bounded and closed interval $[a, b]$. Prove that f has the least global maximum point on $[a, b]$, namely there exists $\alpha \in [a, b]$ such that
(1) α is a global maximum point of f; and
(2) if there exists another global maximum point $c \in [a, b]$ then $c \geq \alpha$.

Solution for Q7

Let E be the set of all global maximum points on $[a, b]$. Since f is continuous on a bounded and closed interval $[a, b]$, so $E \neq \emptyset$, and

$$f(x) = M \equiv \max_{[a,b]} f$$

for all $x \in E$.
Let

$$\alpha = \inf E.$$

By the infimum and supremum theorem α exists. Obviously $a \leq \alpha \leq b$.
Now we show that
(i) $f(\alpha) = M$. So α is a maximum point of f on $[a, b]$.
(ii) If $a < \alpha$ then $f(x) < M$ for all $a \leq x < \alpha$. Therefore α is the least global maximum point of f on $[a, b]$.
Proof of (i). By the definition of infimum, for any integer $n > 0$, there exists $x_n \in E \cap [\alpha, \alpha + \frac{1}{n})$. Then $f(x_n) = M$. By continuity of f we have $f(\alpha) = \lim_{n\to\infty} f(x_n) = M$. So (i) is true.
Proof of (ii). Assume $a < \alpha$. Since α is a lower bound of E, we see that $[a, \alpha) \cap E = \emptyset$. That is, $f(x) < M$ for all $a \leq x < \alpha$ if $a < \alpha$. So α is the least global maximum point of f. So (ii) is true.

Question Q8

Let Ω be a bounded, open and path-connected set in $\mathbb{R}^2$, and f be a continuous function on $\overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω. Recall that

$$\overline{\Omega} = \Omega \cup \partial\Omega = \Omega \cup \Omega',$$

where $\partial\Omega$ denotes the boundary of Ω, and Ω' denotes the set of accumulating points of Ω. Denote

$$m = \inf_{P \in \Omega} f(P), \qquad M = \sup_{P \in \Omega} f(P).$$

Prove the following conclusions.

(a) There exist $P_1, P_2 \in \overline{\Omega}$ such that $f(P_1) = m$ and $f(P_2) = M$.

(b) For any constant c satisfying $m < c < M$, there exists a point $P \in \Omega$ such that $f(P) = c$.

Solution for Q8

Step 1. We show $\overline{\Omega}$ is bounded.

Since Ω is bounded, there exists $R > 0$ such that any $P \in \Omega$ satisfies $\text{dist}(P, O) < R$, where O is the origin.

If $P \in \Omega'$, then there exists a sequence $\{P_j\} \subset \Omega$ such that $P_j \to P$. Then

$$\text{dist}(P, O) = \lim_{j\to\infty} \text{dist}(P_j, O) \leq R.$$

This shows Ω' is also bounded.

Step 2. We show that

$$\begin{aligned} \inf_{P\in\Omega} f(P) &= \inf_{P\in\overline{\Omega}} f(P), \\ \sup_{P\in\Omega} f(P) &= \sup_{P\in\overline{\Omega}} f(P). \end{aligned} \tag{1}$$

Since $\overline{\Omega} = \Omega \cup \Omega'$, we have

$$\inf_{P\in\overline{\Omega}} f(P) = \inf_{P\in\Omega\cup\Omega'} f(P) = \min\{\inf_{P\in\Omega} f(P), \quad \inf_{P\in\Omega'} f(P)\}.$$

Now we show

$$\inf_{P\in\Omega'} f(P) \geq \inf_{P\in\Omega} f(P).$$

Denote

$$m' = \inf_{P\in\Omega'} f(P).$$

Take any $\varepsilon > 0$. By the definition of infimum, there exists $P \in \Omega'$ such that

$$f(P) < m' + \varepsilon.$$

By the definition of accumulating points, there exists a sequence $Q_j \in \Omega$ such that $Q_j \to P$. Since g is continuous on $\overline{\Omega}$, we have

$$f(P) = \lim_{j\to\infty} f(Q_j).$$

Hence we can take Q_j such that

$$f(Q_j) < f(P) + \varepsilon < m' + 2\varepsilon.$$

Thus

$$\inf_{P\in\Omega} f(P) \leq f(Q_j) < m' + 2\varepsilon.$$

Since ε is arbitrary, we have

$$\inf_{P\in\Omega} f(P) \le m'.$$

It follows that

$$\min\{\inf_{P\in\Omega} f(P), \quad \inf_{P\in\Omega'} f(P)\} = \inf_{P\in\Omega} f(P).$$

So the first equality in (1) is proved.
The second equality in (1) is proved in a same manner.

Step 3. Denote

$$m = \inf_{P\in\Omega} f(P), \qquad M = \sup_{P\in\Omega} f(P).$$

Let

$$m \le c \le M.$$

Case 1. $c = M$. Since $\overline{\Omega}$ is bounded and closed, and f is continuous on $\overline{\Omega}$, hence f achieves on $\overline{\Omega}$ the maximum value M.
Direct proof: Let

$$M = \sup_{(x,y)\in\Omega} f(x,y).$$

By the $\inf$-$\sup$ theorem M is well-defined (maybe infinity).
By the definition of supremum we can find a sequence of points $\{P_k\} \subset \Omega$ such that $f(P_k) \to M$ as $k \to \infty$.
Since Ω is bounded, the sequence $\{P_k\}$ is bounded. Then there exists a convergent subsequence P_{k_j} and $P_{k_j} \to P_0$. Since $\overline{\Omega}$ is closed, $P_0 \in \overline{\Omega}$. Since f is continuous,

$$f(P_0) = \lim_{j\to\infty} f(P_{k_j}) = M.$$

Thus M is achieved, which shows M is finite, hence it is maximum value of f on $\overline{\Omega}$.

Case 2. $c = m$. The reason is similar.
Now (a) is proved.

Step 4. Case 3. $m < c < M$. From (2) we know that

$$m = \inf_{P\in\Omega} f(P) < c < \sup_{P\in\Omega} f(P) = M.$$

By the definition of infimum and supremum, we can take $P_0, P_1 \in \Omega$ such that

$$m \leq f(P_0) < c < f(P_1) \leq M.$$

Since Ω is path-connected, there exists a continuous path $\gamma(t)$ in Ω, $0 \leq t \leq 1$, such that $\gamma(0) = P_0$ and $\gamma(1) = P_1$.
Since f is continuous on Ω and $\gamma(t)$ is continuous on $[0, 1]$, so $u(t) = f(\gamma(t))$ is continuous on $[0, 1]$. Since

$$u(0) = f(\gamma(0)) < c < f(\gamma(1)) = u(1),$$

applying the intermediate value theorem to $u(t)$ we conclude that there exists $t_0 \in (0, 1)$ such that

$$u(t_0) = c.$$

Denote $P_{t_0} = \gamma(t_0)$. Then $P_{t_0} \in \Omega$ and $f(P_{t_0}) = c$. Now (b) is proved.