

MAT2040 Linear Algebra Final Exam

SSE, CUHK(SZ)

17 Dec 2019

Seat No.: _____ Student ID: _____

- i. The exam contains 15 questions.
- ii. Put answers in the space after each question. Ask for pages if needed.
- iii. Unless otherwise specified, be sure to give **full explanations** for your answers. The **correct reasoning** alone is worth **more credit** than the correct answer by itself.
- iv. A table of notations is given in the last page, which you can check-out before the exam.

| Question | Points | Score |
|----------|--------|-------|
| 1 | 6 | |
| 2 | 6 | |
| 3 | 9 | |
| 4 | 6 | |
| 5 | 6 | |
| 6 | 8 | |
| 7 | 5 | |
| 8 | 6 | |

| Question | Points | Score |
|----------|--------|-------|
| 9 | 4 | |
| 10 | 8 | |
| 11 | 4 | |
| 12 | 5 | |
| 13 | 10 | |
| 14 | 7 | |
| 15 | 10 | |
| Total: | 100 | |

Question 1 *6 points*

Judge each of the following statements is TRUE or FALSE in general. Give an example for each statement to support your judgement.

- (a) (2 points) A set of orthogonal vectors is linearly independent.
- (b) (2 points) For a linear transformation L , if vectors \mathbf{x} and \mathbf{y} are linearly independent, so is $L(\mathbf{x})$ and $L(\mathbf{y})$.
- (c) (2 points) The rank of a matrix A is equal to the number of non-zero eigenvalues of A .

5.

Question 2 6 points

Let \mathcal{V} and \mathcal{W} be vector spaces with bases \mathcal{E} and \mathcal{F} , respectively. Let $L : \mathcal{V} \rightarrow \mathcal{W}$ be a linear transformation with the matrix representation A with respect to \mathcal{E} and \mathcal{F} , i.e., for any $\mathbf{x} \in \mathcal{V}$,

$$[L(\mathbf{x})]_{\mathcal{F}} = A[\mathbf{x}]_{\mathcal{E}}.$$

Show that

- (a) (3 points) A vector $\mathbf{v} \in \mathcal{V}$ is in the kernel of L if and only if $[\mathbf{v}]_{\mathcal{E}} \in \text{Null}(A)$; and
- (b) (3 points) A vector $\mathbf{w} \in \mathcal{W}$ is in the range of L if and only if $[\mathbf{w}]_{\mathcal{F}} \in \text{Col}(A)$.

Question 3 9 points

Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \quad (1)$$

and

$$\mathbf{v}_1 = \begin{bmatrix} -5 \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad (2)$$

and let L be a linear operator on \mathbb{R}^2 whose matrix representation with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$.

- (2 points) Determine the transition matrix from the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.
- (3 points) Find the matrix representation of L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- (4 points) Let $\mathbf{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and for $k \geq 1$, $\mathbf{x}_k = L(\mathbf{x}_{k-1})$. Calculate \mathbf{x}_{99} . (It is okay to keep the power of a scalar in your answer, e.g., 7^{100} .)

Question 4 6 points

Consider the following given data points

| | | | | | |
|---|----|----|---|---|---|
| x | -2 | -1 | 0 | 1 | 2 |
| y | 4 | 1 | 0 | 1 | 4 |

- (a) (3 points) Find the best least squares fit by a linear function

$$y = c_0 + c_1 x.$$

- (b) (3 points) Find the best least squares fit by a quadratic function

$$y = c_0 + c_1 x + c_2 x^2.$$

Question 5 6 points

Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix}.$$

(a) (4 points) Find an orthonormal basis of $\text{Col}(A)^\perp$.(b) (2 points) Calculate the projection of $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ on to $\text{Col}(A)^\perp$.

Question 6 8 pointsGiven the vector space $C[-1, 1]$ with inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$$

and norm

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

- (a) (1 point) Show that the vectors 1 and x are orthogonal, where 1 is regarded as the function takes value 1 for all inputs in $[-1, 1]$.
- (b) (3 points) Compute $\|1\|$, $\|x\|$ and $\|x + 1\|$.
- (c) (1 point) Compute $\cos(\theta)$ where θ is the angle between x and $1 + x$.
- (d) (3 points) Find the best least squares approximation to $x^{1/3}$ on $[-1, 1]$ by a linear function $l(x) = c_01 + c_1x$.

Question 7 5 points

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \quad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

be bases of \mathbb{R}^2 . Find the coordinates of the vector $\begin{bmatrix} 4 \\ -1 \end{bmatrix}_{\mathcal{B}}$ in the standard basis and \mathcal{C} .

Question 8 *6 points*

Let A and B be $n \times n$ real matrices. Prove the following statements.

- (a) (3 points) If $AB = \mathbf{0}$, then $\text{rank}(A) + \text{rank}(B) \leq n$; and
- (b) (3 points) $\text{rank}(AB) \leq \min \{\text{rank}(A), \text{rank}(B)\}$.

Question 9 4 points

For any inner product space \mathcal{V} , show that the function $\|\mathbf{x}\|$ defined as $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ satisfies the triangular inequality, i.e., for any $\mathbf{x}, \mathbf{y} \in \mathcal{V}$,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Question 10 *8 points*

Consider the quadratic form $f(x, y, z) = 2xy + 2xz + 2yz$.

- (a) (2 points) Find the matrix A associated with the above quadratic form.
- (b) (1 point) Show that -1 is an eigenvalue of the matrix A obtained above.
- (c) (5 points) Change the variables to obtain a quadratic form associated with a diagonal matrix.

Question 11 4 points

Let C be a symmetric positive definite matrix.

- (a) (2 points) Show that C is invertible.
- (b) (2 points) Show that C^{-1} is also positive definite.

(a) $\det(C) > 0$

(b) $\frac{1}{\lambda} > 0$

Question 12 5 points

The singular-value decomposition (SVD) of an $m \times n$ matrix A with $m \geq n$ is a factorization of the form $U\Sigma V^T$ where

- U is an $m \times m$ orthogonal matrix;
- V is an $n \times n$ orthogonal matrix;
- Σ is an $m \times n$ matrix whose off-diagonal entries are all 0's, and whose (i,i) entries σ_i satisfy $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$.

Justify the following matrix equalities. Please include detailed intermediate steps of matrix operations.

- (a) (2 points) $A\mathbf{v}_j = \sigma_j \mathbf{u}_j$, $j = 1, \dots, n$.
(b) (3 points) $A = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T$.

Question 13 10 points

Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

- (a) (7 points) Find the singular-value decomposition of A .
- (b) (3 points) Give an orthonormal basis for each of the following subspaces: $\text{Col}(A)$ and $\text{Col}(A^T)^\perp$.

Question 14 7 points

Let A be an $m \times n$ real matrix with $m \leq n$. Denote by $p_1(\lambda)$ the characteristic polynomial of AA^T and $p_2(\lambda)$ the characteristic polynomial of A^TA .

- (a) (3 points) Show the following two matrices are similar

$$M = \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times m} \\ A & AA^T \end{bmatrix}, \quad N = \begin{bmatrix} A^T A & \mathbf{0}_{n \times m} \\ A & \mathbf{0}_{m \times m} \end{bmatrix}.$$

Hint: construct an invertible matrix B such that $BM = NB$.

- (b) (4 points) Show that

$$p_2(\lambda) = \lambda^{n-m} p_1(\lambda).$$

Hints: one approach is to consider the characteristic polynomials of M and N .

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Question 15 10 points

(Cayley transform) For an $n \times n$ matrix S that is real and skew-symmetric, i.e. $S^T = -S$, prove the following statements:

- (a) (4 points) Each eigenvalue of S is either 0 or a purely imaginary number. (Hint: a non-zero number c is purely imaginary if $c = -\bar{c}$.)
- (b) (3 points) $I + S$ is non-singular.
- (c) (3 points) $(I - S)(I + S)^{-1}$ is an orthogonal matrix.

Hints: You can use the following formula without proof, though the proof is rather straightforward:

- $(I + S)(I - S) = (I - S)(I + S)$ and
- $(A^T)^{-1} = (A^{-1})^T$ where A is an invertible square matrix.

This page has no questions, but a table of notations.

Table of Notations

| | |
|--|--|
| \mathbb{R} | the set of real numbers |
| | Without otherwise specified, all matrices have entries from \mathbb{R} |
| \mathbb{R}^n | the set of all (column) vectors of n entries from \mathbb{R} |
| | the n -dimensional Euclidean vector spaces |
| $\bar{\lambda}$ | complex conjugate. If $\lambda = a + bi$, where a and b are real, $\bar{\lambda} = a - bi$ |
| \bar{A} | the complex conjugate of a matrix A , i.e., $\bar{A} = (\bar{a}_{ij})$ for $A = (a_{ij})$ |
| $\mathbf{0}$ | the zero vector or the all zero matrix, whose size is implied in the context or specified in the subscript |
| I_n | the $n \times n$ identity matrix |
| A^T | the transpose of matrix A |
| A^H | the conjugate transpose of a complex matrix A , i.e., $A^H = \bar{A}^T$ |
| $\det A$, $\det(A)$ | the determinant of matrix A |
| $\text{Col}A$, $\text{Col}(A)$ | the column space of matrix A |
| $\text{Null}A$, $\text{Null}(A)$ | the null space of matrix A |
| $\langle \mathbf{x}, \mathbf{y} \rangle$ | the inner product (scalar product) of vectors \mathbf{x} and \mathbf{y} |
| $\ \mathbf{x}\ $ | the norm of vector \mathbf{x} |
| $\dim V$, $\dim(V)$ | the dimension of a vector space V |
| V^\perp | the orthogonal complement of subspace V |
| $[\mathbf{x}]_{\mathcal{B}}$ | the coordinate vector of \mathbf{x} with respect to the basis \mathcal{B} |
| $C[a, b]$ | the set of all the continuous functions defined on the closed interval $[a, b]$ |
