MAT4033 Mid-term Examination (120 min)

name: __________ ID: __________ November 6, 1:30 pm - 3:30 pm

Note: No books, notes or calculators are allowed.

1.(15 pts) Let

$$\alpha(t) = \big((1 + \delta \cos t) \cos t, (1 + \delta \cos t) \sin t\big),$$

for $t \in [0, 2\pi]$ be a plane curve, where δ is some given constant.

(a) Show that α is a simple closed curve for $|\delta| > 1$.

(b) Show that

$$\mathbb{R}^2 \setminus \big\{\alpha(t) : t \in [0, 2\pi]\big\}$$

is a disjoint union of 3 connected subsets of $\mathbb{R}^2$.

(c) Show that

$$\int_0^{2\pi} \frac{\delta(\cos t + \delta)}{1 + 2\delta \cos t + \delta^2} dt = \begin{cases} 0, & \text{for } |\delta| < 1 \\ 2\pi, & \text{for } |\delta| > 1. \end{cases}$$

(Hint: Use the proof of the Hopf Index Theorem shown in class.)

2.(15 pts) Let $\gamma(s)$ be a regular unit-speed simple curve with nowhere-vanishing curvature. The *tube* of radius $a > 0$ around γ is parametrized by

$$\sigma(s, \theta) = \gamma(s) + a(\mathbf{n}(s)\cos\theta + \mathbf{b}(s)\sin\theta),$$

where $\mathbf{n}(s)$ and $\mathbf{b}(s)$ denote the normal vector and binormal vector respectively.

(a) Show that σ is regular if the curvature κ of γ is less than $\frac{1}{a}$ everywhere.

(b) Show that the Gaussian curvature is

$$K = \frac{-k\cos\theta}{a(1 - ka\cos\theta)}$$

.

(c) Fix s_0. Show that the curves given by $\alpha(t) = \sigma(s_0, t)$ have $k_g = 0$.

3.(15 pts) Let $\alpha(u)$ be a regular, p.a.l. curve in $\mathbb{R}^3$ and let $b(u)$ be a regular curve with $b(u)$ a unit vector for all u. Define a surface S by $X(u,v) = \alpha(u) + vb(u)$.

(a) If $b(u)$ is the binormal of $\alpha(u)$, compute I_p.

(b) Show that the geodesic curvature of α is

$$k_g = -\dot{\varphi} - \frac{\langle \alpha', b' \rangle}{\sin \varphi}$$

where $\varphi(t)$ is the angle between $\alpha'(u)$ and $b(u)$.

$$k_g(\beta) = -\frac{v_0 \tau^2}{1 + v_0^2 \tau^2}$$

$$k_g(\gamma) = 0$$

(c) Assume $b(u)$ is the normal of α at u. Show that $k_n = 0$ for α.

(d) Show the Gaussian curvature $k \leq 0$ and if k is *constant* then $k = 0$.

4.(20 pts) Let $f : S_1 \to S_2$ be a diffeomorphism between regular surfaces.

(a) Show that df preserves the angle between T_pS_1 and $T_{f(p)}S_2$ for any $p \in S_1$ if and only if

$$I_{f(p)}\big(df_p(v_1), df_p(v_2)\big) = \lambda(p)I_p(v_1, v_2) \text{ for any } v_1, v_2 \in T_pS,$$

$$\big(\text{i.e. } \langle df_p(v_1), df_p(v_2)\rangle_{f(p)} = \lambda(p)\langle v_1, v_2\rangle_p \text{ for any } v_1, v_2 \in T_pS\big)$$

and some scalar $\lambda(p)$ depending on p.

(b) Let $a(t)$ be a regular p.a.l. curve in $\mathbb{R}^3$ and $b(t)$ a regular curve such that $b(t)$ is a unit vector for all t. Define a surface S by the parameterization $X(u, v) = a(u) + vb(u)$.

Show that the map $X : \mathbb{R}^2 \to S$ satisfies the condition of part (a) iff $b(t)$ is constant and $a(t)$ is perpendicular to $b(t)$.

5.(20 pts)

(a) Show that if all the normal lines to a surface S pass through a fixed point, then the surface is (a subset of) a sphere.

(By the normal line to S at P we mean the line passing through P with tangent vector equal to the unit normal to S at P).

(b) Show that if there exists a curve $\alpha(t)$ on surface S passing through a point P such that $k_n = 0$, then the Gaussian curvature K at P is $K \leq 0$.

(c) Assume the Gaussian curvature of S satisfies $K < 0$. Show that at every point P of S there exists exactly two curves passing through P with $k_n = 0$ and the angle between them is $2\tan^{-1}\left(\sqrt{-\frac{\lambda_1}{\lambda_2}}\right)$ for λ_1, λ_2 the principal curvatures of S at P.