MAT 1002 Final Exam, 4:00-6:30 pm, May 18, 2021

Your Name and Student ID:

Your Lecture Class(e.g, L1) and **your tutorial class** (e.g, T01):

Instruction: (i) This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones; (ii) Show your work unless otherwise instructed–a correct answer without showing your work when required shall be given **no credits**; (iii) Write down ALL your work and your answers(including the answers for short questions) in the Answer Book.

1. (30 points) **Short Questions** (for these questions, NO need to show your work, just write down your answers in the Exam Book; NO partial credits for each question)

(i). If $a_n \le b_n$ for all $n \ge N$ (for some ~~fixed~~ nonnegative integer N), and the series $\sum b_n$ converges, then $\sum a_n$ must also converge.

True   False ✓

$T: \langle \cos t, \sin t, 0 \rangle$

(ii). Find curvature of the curve given by

$V = \langle t\cos t, t\sin t, 0 \rangle$

$$\overrightarrow{r}(t) = <\cos t + t\sin t, \sin t - t\cos t, 3>, \quad 0 < t < \infty.$$

$k = \frac{1}{|V|}\left|\frac{dT}{dt}\right| = \frac{1}{t} \cdot |\langle -\sin t, \cos t \rangle| = \frac{1}{t}$.

(iii). If $f = f(x, y)$ has all directional derivatives at (a, b), then f must be differentiable at (a, b).

$\hookrightarrow$ may not be cts

True   False ✓

differentable
all $D_{\vec{u}} f$ exists
partial (f) exists
cts

$D_{\vec{u}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$

(iv). Let

$$f(x, y) = xy\frac{x^2 - 2y^2}{x^2 + y^2}.$$

Find $f_y(x, 0)$, where $x \ne 0$.

$f_y(x,0) = \lim_{h \to 0} \frac{f(x,h) - f(x_0)}{h} = \lim_{h \to 0} \frac{xh \cdot \frac{x^2 - 2h^2}{x^2 + h^2} - 0}{h} = \lim_{h \to 0} x \cdot \frac{x^2}{x^2} = x$.

(v). Suppose today's temperature function is given by $T(x, y) = 43 - y^2 - 2y + xy - x$, and you are taking this exam at CUHKSZ which is located at $(0, 0)$. To escape from the sweltering (hot) weather the fastest, in which direction should you head to?

$\nabla T = \langle x - 1, -2y - 2 + x \rangle = \langle -1, -2 \rangle$.

decent: $\langle 1, 2 \rangle = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$ (direction) (vi). If $M = M(x, y)$ and $N = N(x, y)$ both have continuous partial directives on an open region D, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ on D, then the vector field $\overrightarrow{F} = \langle M, N \rangle$ must be conservative on D.

True False

(vii). If $f = f(x, y)$ is continuous on a closed and bounded region D, then f must attain its absolute maximum and absolute minimum values in D. $f_{xx} = 24x^2 - 4$ $f_{xx} < 0$

True False $f_{xy} = 0$

$f_{yy} = 12y^2 - 4$

C (viii). For the critical points of the function $f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2$, which one of the following statements is correct?

$\nabla f(x,y) = \langle 8x^3 - 4x,\ 4y^3 - 4y \rangle = 0$

~~(a)~~ $(0, 0)$ is a local minimum point.

$x = 0$ 或 $\pm\sqrt{2}$ $f_{xx} \cdot f_{yy} - f_{xy}^2$

~~(b)~~ $(0, 1)$ is a local maximum point.

(c) $(0, -1)$ is a saddle point. $y = 0$ 或 ± 1 $=$

(d) There are no local maximum points among all the critical points.

True False

a (ix). Let $\overrightarrow{V}(x, y, z) = < x^2 - y, 4z, x^2 >$ be the velocity vector field of a gas flowing in space. At point $(1, 1, 1)$ which of the following is true?

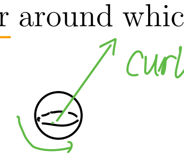
(a) The gas is expanding. $\nabla \cdot V = \langle 2x, 0, 0 \rangle > 0$

(b) The gas is contracting.

(c) Neither of the above.

(x). For the gas mentioned above and at point $(1, 1, 1)$, find a vector around which the gas rotates most rapidly: $\longrightarrow$ curl $\vec{F}$.

$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix} = \langle 4, -2x, -1 \rangle = \langle 4, -2, -1 \rangle$.



2. (8 points) Find all values of x for which the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ converges; and indicate if the convergence is absolute or conditional.

3. (6 points) Find the following limit

$$\lim_{x \to 0} \frac{2x^2(1 - \cos(x^2)) - x^6}{\sin(x^{10})}.$$

4. (12 points) The parallelogram shown below has vertices $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D. Find

2. $\sum \frac{(-3)^n x^n}{\sqrt{n+1}}$, $\frac{a_{n+1}}{a_n} = \frac{(-3)^{n+1} \cdot x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n \cdot x^n} = |-3x| \le 1$, $-\frac{1}{3} \le x \le \frac{1}{3}$

$x = -\frac{1}{3}$, $\sum = \frac{1}{\sqrt{n+1}}$ diverge $(\frac{1}{n^p}, p<1)$

$x = \frac{1}{3}$, $\sum = -\frac{1}{\sqrt{n+1}}$ conditional

$x \in (-\frac{1}{3}, \frac{1}{3})$ absolutely.

3. $\lim_{x\to 0} \frac{2x^2(1-\cos(x^2)) - x^6}{\sin(x^{10})}$. $= \lim_{x\to 0} \frac{\cancel{2x^2} - 2x^2\cos(x^2) - x^6}{x^{10} + O(x^{30})}$

$\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, $\cos x = \frac{(-1)^n}{(2n)!} \cdot x^{4n}$

$\sin(x^{10}) = \sum \frac{(-1)^n}{(2n+1)!} x^{20n+10}$

$n=0$

$1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})$

$\cancel{2x^2} - \cancel{2x^2 \cdot \frac{x^4}{2}} + 2x^2 \cdot \frac{x^8}{24} - \cancel{x^6}$

$\lim_{x\to 0} \frac{-\frac{x^{10}}{12}}{x^{10}} = -\frac{1}{12}$

4. (12 points) The parallelogram shown below has vertices $A(2,-1,4)$, $B(1,0,-1)$, $C(1,2,3)$ and D. Find

3

(1) $\vec{BA} = \langle 1,-1,5\rangle$

$\vec{BC} = \langle 0,2,4\rangle$

$\cos B = \frac{\vec{BC}\cdot\vec{BA}}{|\vec{BC}|\cdot|\vec{BA}|} = \frac{-2+20}{\sqrt{27}\times\sqrt{20}}$

$= \sqrt{\frac{3}{5}}$

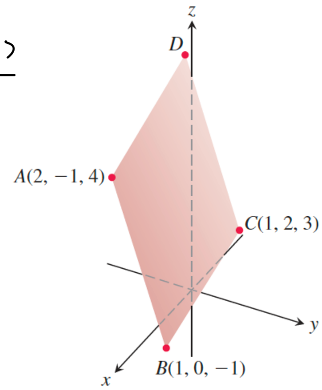
(2)

$\text{Proj}_{\vec{BA}}\vec{BC} = \vec{BA}\cdot\frac{\vec{BC}}{|\vec{BC}|}\cdot\frac{\vec{BC}}{|\vec{BC}|} = \langle 1,-1,5\rangle\cdot\frac{\langle 0,2,4\rangle}{\sqrt{20}}$

$= \frac{1}{\sqrt{20}}(-2+20)\cdot\frac{\langle 0,2,4\rangle}{\sqrt{20}}$

$= \frac{9}{10}\langle 0,2,4\rangle =$

$\frac{18}{20}$



(a) The cosine of the interior angle at B.

(b) The vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$.

(c) The area of the parallelogram.

(d) An equation for the plane containing the parallelogram.

(3) $S = |\vec{AB}\times\vec{BC}| = \begin{vmatrix} i & j & k \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{vmatrix} = |-14i - 4j + 2k|$

$= \sqrt{196+16+4} = \sqrt{56} = 2\sqrt{14}$

(4) $\vec{n} = \langle -14, -4, 2\rangle$

5. (15 points) Consider the surface $S: \cos(\pi yz) + 4xz^2 = 1$.

(a) Find an equation of the tangent plane at $(1/2, 1, -1)$.

(b) Let $z = f(x,y)$ be the function implicitly defined by $\cos(\pi yz) + 4xz^2 = 1$. Find the derivative of $f(x,y)$ at the point $(1/2, 1)$ in the direction of $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$.

(c) Find parametric equations of the tangent line of the contour curve $f(x,y) = -1$ in the plane $z = -1$, with the point of tangency being $(1/2, 1, -1)$.

6. (9 points) Let $f(x,y)$ be such that f and its partial derivatives up to order 2 are continuous in the rectangle

$$R = \{(a,b) \mid -1 < a, b < 1\}$$

Use Taylor's theorem for functions of a single variable to prove that for any point $(x,y) \in R$ there exists $c \in (0,1)$ such that

令 $g(t) = f(tx, ty)$

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right)\Big|_{(cx,cy)}.$$

7. (8 points) Find the maximum value of $f(x,y,z) = x + 2y + 5z$ on the sphere $x^2+y^2+z^2 = 1$ by the method of Lagrange multipliers.

8. (8 points) Consider the integral

$$\int_0^9 \int_0^1 \int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}}\, dx\, dy\, dz.$$

5. (a) $S: \cos(\pi yz) + 4xz^2 = 1$ $\qquad g(x) = \cos(\pi yz) + 4xz^2 - 1$

$\nabla g = \langle 4z^2, -\pi z\sin(\pi yz), -\pi y\sin(\pi yz) + 8xz \rangle$

at $(1/2, 1, -1)$, $\nabla g = \langle 4, 0, -4 \rangle$

tangent: $4(x - \frac{1}{2}) - 4(z+1) = 0$

(b) $\nabla f(x,y) = \langle -\frac{F_x}{F_z}, -\frac{F_y}{F_z} \rangle = \langle 1, 0 \rangle$

$D_{\vec{v}} f(x,y) = \nabla f \cdot \vec{v} = \langle 1,0 \rangle \cdot \langle 2,-1 \rangle \cdot \frac{1}{\sqrt{5}} = 2/\sqrt{5}$

(c) $f'(x,y) = \langle 1,0 \rangle$, $\qquad \langle 1, 0, -1 \rangle$.

7. $f(x,y,z) = x + 2y + 5z$, $\qquad x^2 + y^2 + z^2 = 1$,

$\nabla f = \langle 1, 2, 5 \rangle$. $\nabla g = \lambda \nabla f$ 且 $g(x) = 0$, $\nabla g = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = 5\lambda \end{cases}$$

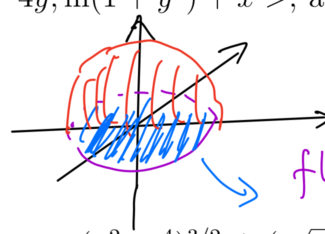
(a) Sketch the solid on which the triple integral of the integrand is equal to the above iterated integral.

(b) Find a way to evaluate the integral.

9. (6 points) Consider the solid ball B of radius 2 in the xyz-space with equation $x^2 + y^2 + z^2 \leq 4$. If we take out from B the portion inscribed by the cylinder $x^2 + y^2 = 1$, what is the volume of the remaining solid?

10. (6 points) Compute $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where $\overrightarrow{F} =< \arctan e^x + 4y, \ln(1 + y^2) + x >$, and C is the circle $x^2 + y^2 = 1$, oriented counter-clockwise.

11. (6 points) Let S be the unit upper hemisphere



$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

flux( )
= flux( )

oriented by the unit outer normal vector field $\overrightarrow{n}$; let $\overrightarrow{F} =< y, x, (x^2+y^4)^{3/2} \sin(e^{\sqrt{xyz}}) >$. Compute

$$\int \int_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n} \, d\sigma.$$

12. (6 points) Let Ω be the part of the unit ball $x^2 + y^2 + z^2 \leq 1$ inside the first octant; let S be the boundary of Ω, oriented by the unit outer normal vector field $\overrightarrow{n}$. Compute

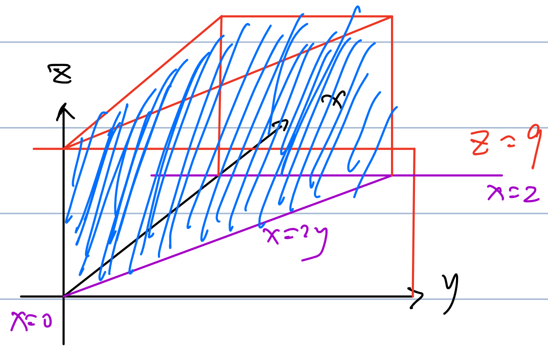
$$\int \int_S \overrightarrow{F} \cdot \overrightarrow{n} \, d\sigma,$$

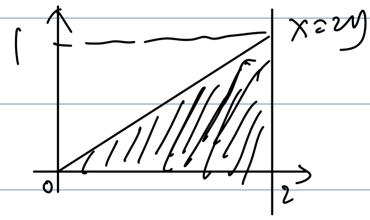
对各个方向都投影

where $\overrightarrow{F} =< x^2, -2xy, xz >$.

8.

(b) $\int_0^9 \int_0^1 \int_{2y}^{2} \frac{4\sin x^2}{\sqrt{z}}\, dx\, dy\, dz =$





$\int_0^9 \frac{1}{\sqrt{z}} \int_0^1 \int_{2y}^{2} 4\sin x^2\, dx\, dy$

$= \int_0^9 \frac{1}{\sqrt{z}} \cdot \int_0^2 \int_0^{\frac{1}{2}x} 4\sin x^2\, dx\, dy$

$= (2\sqrt{z})\Big|_{z=0}^{9}$

$= 6$

$= \int_0^2 2x\sin x^2\, dx = \int_0^4 \sin u\, du$

$z^{-\frac{1}{2}}$

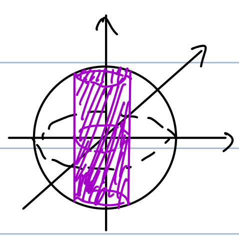
$6 \cdot (1-\cos 4)$

$z^{\frac{1}{2}}$

底面: $x^2+y^2=1$

9.

$z^2 = 4-x^2-y^2 = 4-r^2$, $z = \sqrt{4-r^2}$



$\iiint dz\, dy\, dx$

$\Rightarrow \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

$= 2\pi \cdot \int_0^1 r\sqrt{4-r^2}\, dr\, d\theta$

$= 2\pi \cdot \frac{1}{2} \int_3^4 \sqrt{u}\, du\, d\theta \ldots$

10. $\oint_C \vec{F}\, d\vec{r}$, $\quad \vec{F} = \langle \arctan e^x + 4y, \quad \ln(1+y^2) + x \rangle$

C : $(x^2 4y^2) = 1$, $\quad \vec{r} = \langle r\cos\theta, r\sin\theta \rangle$

$$= \iint_C \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA = \iint_C (1-4)\, dA = -3 \times \pi = -3\pi$$

11. $\vec{n}$ outer $\quad \vec{F} = \langle y, x, (x^2+y^4)^{3/2} \sin(e^{\sqrt{xyz}}) \rangle$

$$\iint_S (\nabla \times \vec{F}) \vec{n}\, d\sigma = \oint_C \vec{F}\, d\vec{r},$$

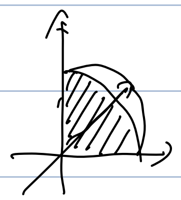
$x = \cos\theta$, $y = \sin\theta$, $z = 0$

$$= \int \vec{F}(\vec{r})\, \vec{r}\,' : \langle \sin\theta, \cos\theta, (\cos^2\theta + \sin^4\theta)^{3/2} \cdot \sin 1 \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle\, d\theta$$

$$: \int_1^{2\pi} \cdot \ -\sin^2\theta + \cos^2\theta = \cos 2\theta = 0$$

12. Ω : $x^2 + y^2 + z^2 \leq 1$.

S : boundary



$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma : \iiint \operatorname{div} \vec{F}\, dV$$

$$\operatorname{div} F = \langle 2x + -2x + x \rangle$$

$\phi \in [0, \frac{\pi}{2}]$, $\theta \in [0, \frac{\pi}{2}]$, $\rho \in [0, 1]$

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho \sin\phi \cos\theta \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^3 \sin^2\phi \cos\theta\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \frac{1}{4} \sin^2\phi\, d\phi = \int_0^{\pi/2} \frac{1}{4} \cdot \frac{1}{2}(1 - \cos 2\theta)\, d\phi$$

$$= \int_0^{\pi/2} \frac{1}{8} - \frac{1}{8} \cos 2\theta\, d\phi$$

$$= \frac{1}{16}\pi - \left(\frac{1}{16} \sin 2\theta\right) \Big|_{\theta=0}^{\pi/2} = \frac{\pi}{16}$$