MAT4033 Final Examination (120 min)

name: __________ ID: __________ December 15, 3:30 pm - 5:30 pm

Note: No books, notes or calculators are allowed.

1.(20 pts) Let $S \subset R^3$ be a regular surface homeomorphic to a sphere. Let $\alpha \subset S$ be a simple closed geodesic in S. Let A and B be the regions of S which have α as a common boundary. Let $N : S \to S^2$ be the Gauss map of S. Prove that N(A) and N(B) have the same area.

2.(15 pts) Given surface $X(u,v) = (u, v, f(u,v))$, let $X^t(u,v)$ be a normal variational family of X^t, precisely

$$X^t(u,v) = X(u,v) - \frac{1}{2}tN(u,v),$$

where $N(u,v)$ is the normal vector at (u,v). Let I_p^t be the first fundamental form of X^t. Show that the second fundamental form is given by

$$II_p = \left.\frac{dI_p^t}{dt}\right|_{t=0}.$$

3.(15 pts) Let $F : S \to \mathbb{R}^3$ be smooth vectors on closed regular oriented surface S. Suppose $R \subset S$ is a region bounded by $\{C_i\}_{i=1}^n$ a collection of simple closed curves parametrized by arc length, such that $F(x) \neq 0$ for all $x \in R$ and is tangent vector at x.

(a) Show that there exists an orthonormal frame $\{e_1, e_2, N\}$ on R with N normal, such that

$$\iint_R K dA = \sum_1^n \int_{C_i} \langle e_1, \dot{e}_2 \rangle ds,$$

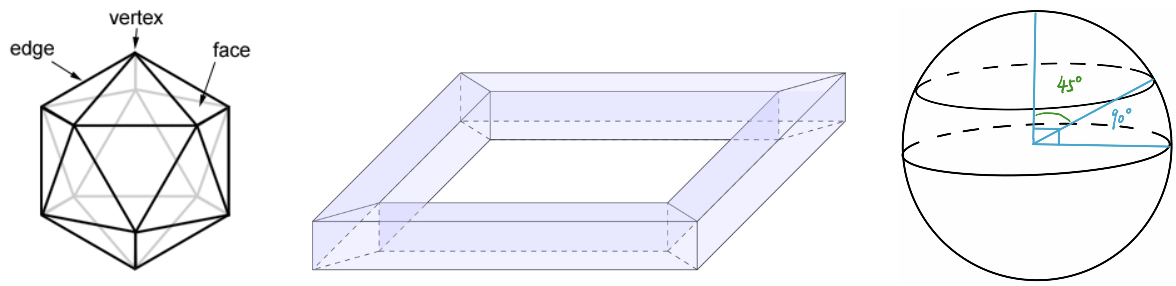
where $\dot{e}_2$ denotes derivative of e_2 along the curve C_i for $i = 1, \dots, n$.
(Hint: Use Green's theorem.)
Note: Given an orthogonal parametrization

$$\begin{aligned} K &= -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\} \\ K_g(s) &= \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \theta'(s) \end{aligned}$$

(b) Prove that if there exists $F \neq 0$ for $\forall x \in S$, then S is a torus.

(c) Describe such F on a torus.

4.(25 pts) Let S_{ico} be icosahedron surface which is closed platonic surface composed of 20 equilateral triangles, 12 vertices, 30 edges, see Figure 1(a). Let T^2_{sq} be the square torus that was given in the lecture example, see Figure 1(b).



(a) Icosahedron (b) Square torus (c) S_c

Figure 1:

(a) Determine Gaussian curvature $K(v)$ at vertex $v \in S_{ico}$?

(b) Let S'_{ico} be the surface S_{ico} with two disjoint triangles removed. Find

$$\sum_{v \in int(S'_{ico})} K(v)$$

where $int(S'_{ico})$ denotes the interior vertices of S'_{ico} and

$$\sum_{v \in \partial S'_{ico}} \theta_{ext}(v),$$

where $\theta_{ext}(v)$ denotes the exterior angle at boundary vertex v.

(c) Let $T^2_{sq} \# T^2_{sq}$ be the closed surface created by removing one of outer face from each T^2_{sq} and glue along the boundary curve. Find $K(v)$ for each vertex v of $T^2_{sq} \# T^2_{sq}$ and compute $\sum_{v \in T^2_{sq} \# T^2_{sq}} K(v)$.

(d) Let S_c be the annulus bounded by two circles on unite sphere S^2 which sites at $\pi/4$ and $\pi/2$ with respect to the north pole, see Figure 1(c). Find $\int_{\partial S_c} K_g ds$.

(e) Which of these surfaces: S_{ico}, S'_{ico}, T^2_{sq} and $T^2_{sq} \# T^2_{sq}$, S_c have the same $\chi(S)$ Euler characteristic?

5.(25 pts) Let S be a minimal regular surface. Let $N : S \to S^2$ be the Gauss map.

(a) Suppose S is given by:

$$f = e^z, g = e^{-z}.$$

Use Weierstrass-Enneper representation to find a parametrization of S.

(hint: $\phi = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix}$)

(b) Show that $N : S \to S^2\backslash\{p, q\}$ is homeomorphic angle preserving map, where $S^2\backslash\{p, q\}$ is unit 2-sphere deleting two points.

(c) Find

$$\int_S KdA$$

where K is the Gaussian Curvature of S in (a).
(Hint: Gauss-Bonnet theorem is not applicable. Use 5b.)

(d) Show that there exists unique closed geodesic in surface defined by (a).

(e) Let α be the geodesic in (d). Is N(α) a geodesic in S^2? Prove your statement. Is there any other geodesic such that its image on S^2 is geodesic?