

MAT 2040 Linear Algebra Final Exam

SSE, CUHK(SZ)

May 21, 2021

Seat No.: _____ Student ID: _____

- i. The exam contains 11 questions.
- ii. Answer all the questions in your answer Book. Ask for additional pages if needed.
- iii. Unless otherwise specified, be sure to give **full explanations** for your answers. The **correct reasoning** alone is worth **more credit** than the correct answer by itself.
- iv. A table of notations is given in the last page, which you can check-out before the exam.

Question	Points	Score
1	10	
2	10	
3	4	
4	15	
5	10	
6	10	

Question	Points	Score
7	5	
8	10	
9	6	
10	10	
11	10	
Total:	100	

Question 1 10 points

Judge each of the following statements is TRUE or FALSE in general. No explanation is necessary.

- (a) (1 point) If \mathbf{A} and \mathbf{B} are $n \times n$ matrices that have the same rank, then the rank of \mathbf{A}^2 must equal the rank of \mathbf{B}^2 .
- (b) (1 point) An $n \times n$ matrix that is diagonalizable must be symmetric.
- (c) (1 point) Let \mathbf{Q} be an orthogonal matrix, then its determinant is equal to 1
- (d) (1 point) If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}, \mathbf{x}_k$ are vectors in a vector space \mathcal{V} and

$$\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}, \mathbf{x}_k) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1})$$

then $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}, \mathbf{x}_k$ are linearly dependent.

- (e) (1 point) If $\mathbf{A}^T = \mathbf{A}$, then its two eigenvectors \mathbf{u} and \mathbf{v} must satisfy $\mathbf{u}^T \mathbf{v} = 0$.
- (f) (1 point) If all the entries of a square matrix \mathbf{A} are positive, then \mathbf{A} is positive definite.
- (g) (1 point) If two matrices are similar, they have the same eigenvectors.
- (h) (1 point) If \mathcal{U}, \mathcal{V} , and \mathcal{W} are subspaces of \mathbb{R}^3 and if $\mathcal{U} \perp \mathcal{V}$ and $\mathcal{V} \perp \mathcal{W}$, then $\mathcal{U} \perp \mathcal{W}$.
- (i) (1 point) If $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$ then the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ will have a unique least squares solution.
- (j) (1 point) Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation, and let \mathbf{A} be the standard matrix representation of L . If L^2 is defined by

$$L^2(\mathbf{x}) = L(L(\mathbf{x}))$$

for all $\mathbf{x} \in \mathbb{R}^2$, then L^2 is a linear transformation and its standard matrix representation is \mathbf{A}^2 .

Question 2 10 points

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}.$$

Find the bases for the following subspaces :

- (a) (2 points) $\text{Col}(\mathbf{A})$
- (b) (2 points) $\text{Null}(\mathbf{A})$
- (c) (2 points) $\text{Row}(\mathbf{A})$
- (d) (2 points) $\text{Null}(\mathbf{A}^T)$
- (e) (2 points) $\text{Row}(\mathbf{A}^T)$

Question 3 4 points

An $n \times n$ matrix \mathbf{A} is said to be an orthogonal matrix if the column vectors of \mathbf{A} form an orthonormal set in \mathbb{R}^n . Find all solutions of a, b, c and d such that

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{2} & -\frac{1}{2} & 0 & b \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & c \\ \frac{1}{2} & -\frac{1}{2} & 0 & d \end{bmatrix}$$

is orthogonal matrix.

Question 4 15 points

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

- (a) (5 points) Use the Gram-Schmidt process to find an orthonormal basis for the column space of \mathbf{A} .
- (b) (5 points) Factor \mathbf{A} into a product \mathbf{QR} , where \mathbf{Q} has an orthonormal set of column vectors and \mathbf{R} is upper triangular;
- (c) (5 points) Using the QR decomposition of \mathbf{A} , solve the least squares problem $\mathbf{Ax} = \mathbf{b}$. We are *NOT* allowed to directly compute \mathbf{A}^{-1} .

Question 5 10 points

Let

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and let L be the linear operator on \mathbb{R}^3 defined by

$$L(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3) = (c_1 + c_2 + c_3)\mathbf{y}_1 + (c_1 + 2c_3)\mathbf{y}_2 - (3c_2 + c_3)\mathbf{y}_3$$

- (a) (5 points) Find a matrix representing L with respect to the ordered basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.
- (b) (5 points) write the vector $\mathbf{x} = [6, 3, 1]^T$ as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ and use the matrix from part (a) to determine $L(\mathbf{x})$.

Question 6 10 points

Consider an arbitrary complex matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$. We wish to decompose \mathbf{A} as

$$\mathbf{A} = \mathbf{BC} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \end{bmatrix}$$

where $\mathbf{B} \in \mathbb{C}^{m \times 2}$ and $\mathbf{C} \in \mathbb{R}^{2 \times n}$. Furthermore, it is required that every entry b_{ij} of \mathbf{B} should satisfy $|b_{ij}| = 1$. For instance, if $n = 1$, the above decomposition becomes

$$\mathbf{A} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- (a) (5 points) Decompose $\mathbf{A} = [3 + 4i, -2, 8]^T$ into \mathbf{B} and \mathbf{C} when $n = 1$.
- (b) (5 points) Prove that the decomposition $\mathbf{A} = \mathbf{BC}$ does NOT always exist when $n \geq 3$.

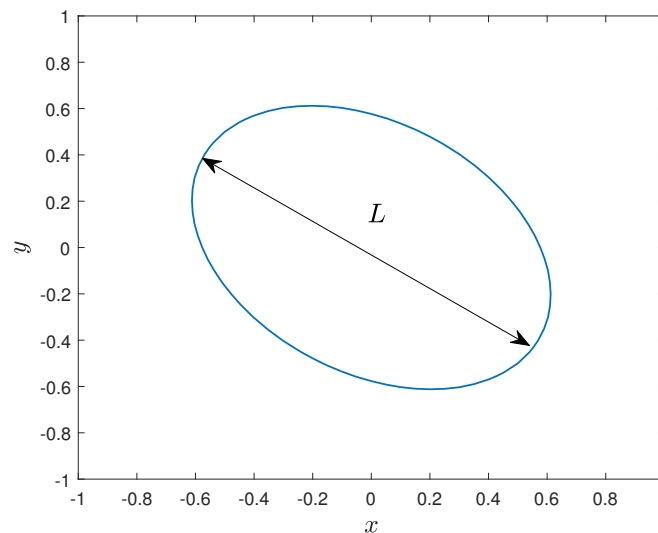
Question 7 5 points

If an ellipse is expressed in the standard form as

$$\frac{x^2}{\ell_1^2} + \frac{y^2}{\ell_2^2} = 1,$$

with $\ell_1 > \ell_2 > 0$, then the length of the major axis of the ellipse is given by $2\ell_1$, which stands for the longest distance between any two points on the ellipse.

The following figure displays an ellipse $3x^2 + 2xy + 3y^2 = 1$ that is NOT in the standard form. Use the quadratic form to first convert the expression into its standard form before computing the length L of the major axis.



Question 8 10 points

Consider two $n \times n$ square matrices \mathbf{A} and \mathbf{B} . Prove the following two statements:

- (a) (5 points) If $\lambda \neq 0$ is an eigenvalue of \mathbf{AB} , then it is also an eigenvalue of \mathbf{BA} .
- (b) (5 points) If $\lambda = 0$ is an eigenvalue of \mathbf{AB} , then it is also an eigenvalue of \mathbf{BA} .

Question 9 6 points

Let $\mathbf{A} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$. Determine the type for each of the following matrices, i.e., positive definite, negative definite, or indefinite.

- (a) (2 points) \mathbf{A} .
- (b) (2 points) \mathbf{A}^2 .
- (c) (2 points) \mathbf{A}^{2021} .

Question 10 10 points

Consider a reserve park with a particular species of birds that we wish to protect. The birds are free to cross the boundary, both from the inside out and from the outside in. For each year, 10% of the birds leave the park and in the meanwhile 1% of the birds from the outside find their way in. Assume that the overall population of birds for the park and the rest of the world stays constant over the time. For the n th year, let $0 \leq x_n \leq 1$ be the proportion of birds in the park and let $0 \leq y_n \leq 1$ be that in the rest of the world.

- (a) (2 points) Find the relationship between $[x_{n+1}, y_{n+1}]^T$ and $[x_n, y_n]^T$.
- (b) (8 points) We now compute $[x_{n+1}, y_{n+1}]^T$ as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}.$$

Please compute the entries $c_{11}, c_{12}, c_{21}, c_{22}$ in terms of n .

Question 11 10 points

We fit the circle $(x-c_1)^2 + (y-c_2)^2 = r^2$ to m sample points $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ by the method of least squares. You may assume that the three vectors $[x_1, \dots, x_m]^T$, $[y_1, \dots, y_m]^T$, and $[1, \dots, 1]^T$ are linearly independent.

- (a) (5 points) Estimate the radius r when the center (c_1, c_2) is already known.
- (b) (5 points) Estimate the radius r when the center (c_1, c_2) is unknown.

This page has no questions, but a table of notations.

Table of Notations

\mathbb{R}	the set of real numbers
	Without otherwise specified, all matrices have entries from \mathbb{R}
\mathbb{R}^n	the set of all (column) vectors of n entries from \mathbb{R}
	the n -dimensional Euclidean vector spaces
$\bar{\lambda}$	complex conjugate. If $\lambda = a + bi$, where a and b are real, $\bar{\lambda} = a - bi$
$\overline{\mathbf{A}}$	the complex conjugate of a matrix \mathbf{A} , i.e., $\overline{\mathbf{A}} = (\overline{a_{ij}})$ for $\mathbf{A} = (a_{ij})$
$\mathbf{0}$	the zero vector or the all zero matrix, whose size is implied in the context or specified in the subscript
I_n	the $n \times n$ identity matrix
\mathbf{A}^T	the transpose of matrix \mathbf{A}
\mathbf{A}^H	the conjugate transpose of a complex matrix \mathbf{A} , i.e., $\mathbf{A}^H = \overline{\mathbf{A}}^T$
$\det \mathbf{A}$, $\det(\mathbf{A})$	the determinant of matrix \mathbf{A}
$\text{Col}\mathbf{A}$, $\text{Col}(\mathbf{A})$	the column space of matrix \mathbf{A}
$\text{Null}\mathbf{A}$, $\text{Null}(\mathbf{A})$	the null space of matrix \mathbf{A}
$\langle \mathbf{x}, \mathbf{y} \rangle$	the inner product (scalar product) of vectors \mathbf{x} and \mathbf{y}
$\ \mathbf{x}\ $	the norm of vector \mathbf{x}
$\dim \mathcal{V}$, $\dim(\mathcal{V})$	the dimension of a vector space \mathcal{V}
\mathcal{V}^\perp	the orthogonal complement of subspace \mathcal{V}
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of \mathbf{x} with respect to the basis \mathcal{B}
$C[a, b]$	the set of all the continuous functions defined on the closed interval $[a, b]$