MAT4033 Mid-term Examination (120 min)

name: ___________ ID: ___________ 2024 Fall, 1:30 pm - 3:30 pm

Note: No books, notes or calculators are allowed.

1.(20 pts) Let S be a connected, regular surface. Let N be the Gauss map on S. Assume $I_{N(p)}(dN_p(v)) = \lambda(p)I_p(v)$ for any $v \in T_pS$, and some scalar $\lambda(p)$ depends on p.

(a) (10 pts) Show that for $p \in S$ with $H(p) \neq 0$ then there is an open $U \subseteq S$ of p such that H is constant on U.

(b) (10 pts) Show that if S is not in a sphere, then $H = 0$.

2.(15 pts) Define the n-th fundamental form I_p^n of a regular surface S as the quadratic form given by the bilinear form $\langle(-dN)^n v, w\rangle$ for $v, w \in T_pS$, show:

$$I_p^n = 2HI_p^{n-1} - KI_p^{n-2} \quad \text{for } n \geq 3$$

3.(25 pts) Let $U \subset \mathbb{R}^2$ be an open set and $f : U \to \mathbb{R}^2$ be a C^1-map, and we consider images of f in $\mathbb{R}^2$ as vectors. Let α be a regular simple closed curve parametrized by arc length in U.

(a) (10 pts) Suppose $\alpha'(t) = f(\alpha(t))$ on α. Show that there exists $(x_0, y_0) \in U$ such that $f(x_0, y_0) = 0$.

(b) (5 pts) Suppose $\alpha(t)$ is given as ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $f(x, y) = (-x, y)$. Determine the index of α relative to f counterclockwise, i.e. $\frac{1}{2\pi} \oint_\alpha d\theta$, where θ is the angle of vector f along α with respect to x-axis. (Hint: consider to integrate over another curve to compute the index.)

(c) (10 pts) By parametrize the ellipse α in (b), show

$$\frac{1}{\sqrt{ab}} \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt \geq \left| \int_\alpha \kappa ds \right|,$$

where the κ is the curvature of α, and the equality holds if and only if $a = b$. (Hint: use isoperimetric inequality.)

4.(20 pts) Let $\alpha(s)$ be a space regular curve p.a.l with $k_\alpha \neq 0$ and torsion $\tau \neq 0$ along the curve. Suppose $b(s)$ is the binormal of $\alpha(s)$ and $S_{\alpha,b}$ is a surface defined by:

$$X(s,t) = \alpha(s) + tb(s), \quad s \in [0,1], \quad t \in (-\epsilon, \epsilon), \quad \epsilon > 0$$

(a) (5 pts) Show that the geodesic curvature of α is 0 ($k_g = 0$).

(b) (5 pts) Show that the Gaussian curvature $K < 0$

5.(20 pts) Suppose that a surface S has no umbilical point and one of its principal curvatures is a non-zero constant $\lambda \neq 0$.

(a) (5 pts) Show that there is a parametrization $\mathbf{X}(u, v)$ at $p \in S$ such that:

$$I_p \quad \text{with} \quad E_p = 1, \quad F_p = 0, \quad \text{and} \quad II_p \quad \text{with} \quad e_p = \lambda, \quad f_p = 0.$$

(b) (5 pts) Show that curves given by $v =$ constant are circles of radius $\frac{1}{|\lambda|}$.

(c) (10 pts) Show that there is some curve $\alpha(v)$, and unit vectors $\mathbf{c}_1(v), \mathbf{c}_2(v)$ with $\mathbf{c}_1 \perp \mathbf{c}_2$ such that:

$$\mathbf{X}(u, v) = \alpha(v) + \frac{1}{|\lambda|} \left(\mathbf{c}_1(v) \cos(|\lambda| u) + \mathbf{c}_2(v) \sin(|\lambda| u) \right).$$