

The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Midterm Exam
MAT 3007 – Optimization
Spring Semester 2023

Prof. Xiao Li, Prof. Andre Milzarek, Prof. Zizhuo Wang
March 25th, 2023

Please read the following instructions carefully:

- You have **90 minutes** to complete the exam.
- Examination Time: **10:00 to 11:30 am**.
- The exam consists of **six problems** in total.
- You are **(only)** allowed to bring one self-made sheet of A4 paper (with arbitrary notes on both sides of it) for your personal use in this exam. Usage of electronic devices or other tools is not allowed.
- Please abide by the honor codes of CUHK-SZ.
- Please make sure to present your answers in a comprehensible way and give explanations of your steps and results. Write down all necessary steps when answering the questions.
- Violation of the exam policies will be considered as cheating and reported. Consequences of such violation include zero points for the exam and disciplinary actions.

Good luck!

Exercise 1 (The Simplex Method)

(25 points)

We want to use the two-phase method to solve the following linear programming problem:

$$\begin{array}{ll}\text{maximize} & 5x_1 + 8x_2 + 11x_3 + x_4 \\ \text{subject to} & 2x_1 - 2x_2 + x_3 - x_4 = 6 \\ & 2x_1 + 2x_2 + 2x_3 - x_4 = 16 \\ & x_1, x_2, x_3, x_4 \geq 0.\end{array}$$

- Apply Phase I of the two-phase method to find an initial basic feasible solution (BFS). For each step, clearly mark the current basis, the current basic solution, and the corresponding objective value.
- Show that the initial BFS found in Phase I is already an optimal solution by checking its reduced costs. What is the optimal value of the problem?

Exercise 2 (Duality)**(16 points)**

Consider the following linear programming problem:

$$\begin{array}{ll}\text{maximize} & 2x_1 + x_2 - 3x_3 - x_4 \\ \text{subject to} & 2x_1 - x_2 + x_3 \leq 2 \\ & x_1 + 2x_2 + x_3 - x_4 = 1 \\ & x_1 - x_2 + 2x_3 + 2x_4 \leq 2 \\ & x_1 \text{ free}, x_2, x_3, x_4 \geq 0.\end{array}$$

- a) Derive the dual problem.
- b) Use the complementarity-based optimality conditions for LPs to show that $x = (\frac{3}{2}, 1, 0, \frac{1}{2})^\top$ is an optimal solution.

Exercise 3 (True or False)**(12 points)**

State whether each of the following statements is true or false. Explanations are not required.

- a) We consider an unbounded linear program. Then, the LP remains unbounded if a new variable is added to the problem.
- b) The simplex tableau can contain a row vector r with $r_i < 0$ for all i .
- c) We consider the standard LP polyhedron $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ with $A \in \mathbb{R}^{m \times n}$ having full rank. Let x be a basic feasible solution with basis B . Then, there exists an extreme point $y \in P$ with $x \neq y$ and $x_i = y_i = 0$ for all $i \notin B$.
- d) We consider an infeasible primal linear optimization problem. Then its associated dual must be infeasible as well.

Exercise 4 (Sensitivity Analysis)**(21 points)**

A coal company converts raw coals to low, medium and high grade coal mix. The coal requirements for each mix, the availability of each raw coal (ZX, SH, GF), and the selling price are shown below:

	Low grade	Medium grade	High grade	Available (tons)
ZX coal	2	2	1	180
SH coal	1	2	3	120
GF coal	1	1	2	160
Price	\$9	\$10	\$12	

Let x_1, x_2, x_3 denote the amount of low, medium, and high grade mix to produce. The LP is:

$$\begin{aligned}
 &\text{maximize} && 9x_1 + 10x_2 + 12x_3 \\
 &\text{subject to} && 2x_1 + 2x_2 + x_3 \leq 180 \\
 & && x_1 + 2x_2 + 3x_3 \leq 120 \\
 & && x_1 + x_2 + 2x_3 \leq 160 \\
 & && x_1, x_2, x_3 \geq 0.
 \end{aligned}$$

- In what range can the price of medium grade mix vary without changing the optimal basis?
- In what range can the price of low grade mix vary without changing the optimal basis?
- In what range can the availability of ZX coal vary without changing the optimal basis?

Exercise 5 (Inventory Planning Problem)**(12 points)**

A manufacturing company forecasts the demand over the next n months to be d_1, \dots, d_n . In any month, the company can produce up to C units using regular production at a cost of b dollars per unit. The company may also produce using overtime under which case it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month. Formulate a linear optimization problem to determine the production schedule that meets the demand while minimizing the cost.

Exercise 6 (Relaxing a Binary Optimization Problem)**(14 points)**

In this exercise, we investigate the binary optimization problem:

$$\begin{aligned}
 (1) \quad &\text{maximize} && c^\top x \\
 &\text{subject to} && \mathbf{1}^\top x = k \\
 & && x_i \in \{0, 1\} \text{ for all } i,
 \end{aligned}$$

where $c \in \mathbb{R}^n$ and $k \in \mathbb{N}, k < n$ are given. We consider the associated relaxed linear program:

$$\begin{aligned} (2) \quad & \text{maximize} && c^\top x \\ & \text{subject to} && \mathbf{1}^\top x = k \\ & && x \geq 0, \quad x \leq \mathbf{1}. \end{aligned}$$

- a) Derive the dual problem of (2).
- b) Prove that problem (2) has a binary optimal solution x^* satisfying $x_i^* \in \{0, 1\}$ for all i . (Hint: You may assume $c_1 \geq c_2 \geq \dots \geq c_n$).