Final Exam: MAT3006 Real Analysis

December 8, 2025

Instructions:

- Attempt all problems.
- Each problem is presented on a separate page.
- Justify your answers with rigorous proofs.
- Unless otherwise stated, m denotes the Lebesgue measure.

(Turn the page to begin)

Problem 1.

(i) Does there exist a pair of sets E and G such that $m^*(E) = m^*(G)$ but $m^*(G \backslash E) > 0$? Justify your answer.

(ii) Suppose $E \subset \mathbb{R}$ is a set of positive outer measure ($m^*(E) > 0$). Show that for every $\alpha \in (0,1)$, there exists an open interval I such that:

$$m^*(E \cap I) \geq \alpha \cdot m^*(I).$$

Problem 2.

(i) **Prove or Disprove:** If f and g are absolutely continuous functions on $[a, b]$, then their product $f \cdot g$ is absolutely continuous on $[a, b]$.

(ii) **Prove or Disprove:** If f, g are absolutely continuous on $[a, b]$, then the integration by parts formula holds:

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Problem 3.

Let f be a function of bounded variation on $[a, b]$. Let $TV(f)$ denote the total variation of f on $[a, b]$.

(i) Show that:

$$\int_a^b |f'(x)|\, dx \leq TV(f).$$

(ii) Show that the equality holds in the inequality above if and only if f is absolutely continuous on $[a, b]$.

Problem 4.

(i) Let f be an absolutely continuous function on $[a, b]$. **Prove or Disprove:** Given $\epsilon > 0$, there exists $\delta > 0$ such that for **any** countable collection of disjoint open intervals $(a_k, b_k) \subset [a, b]$ satisfying $\sum(b_k - a_k) < \delta$, we have:

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon.$$

(ii) Show that an increasing absolutely continuous function maps sets of measure zero to sets of measure zero.

Problem 5.

Let f be a bounded measurable function on a set E . Show that there exist a sequence of continuous functions $\{f_n\}$ converges to f pointwise a.e.

Problem 6.

(i) Let $f \in L^1([0,1])$ with $f(x) > 0$ and $\int_0^1 f(x)\,dx = 1$. **Prove or Disprove:**

$$\int_0^1 \log f(x)\,dx \le 0.$$

(ii) Let f, g be non-negative, bounded measurable functions on $[0,1]$ with $fg \ge 1$ **Prove or Disprove:**

$$\left(\int_0^1 f\right)\left(\int_0^1 g\right) \ge 1.$$

Problem 7.

Let $h \in L^\infty(\mathbb{R})$ be a periodic function with period 1 such that $\int_0^1 h(x)\,dx = 0$. Define the sequence $f_n(x) = h(nx)$ for $x \in [0,1]$.

(i) Does $\{f_n\}$ have a subsequence that converges weakly in L^1? If it does, what is the limit?

(ii) Does $\{f_n\}$ have a subsequence that converges pointwise a.e.? If it does, what is the limit?

(iii) Does $\{f_n\}$ have a subsequence that converges in measure? If it does, what is the limit?

Problem 8.

Let $1 < p_1 < p_2 < \infty$. Suppose $\{f_n\}$ is a sequence of functions bounded in $L^{p_2}([a,b])$ (i.e., $\sup_n \|f_n\|_{p_2} < \infty$) and $f_n \to f$ in measure.

Prove or Disprove: There exists a subsequence $\{f_{n_k}\}$ converges to f in $L^{p_1}([a,b])$.

Problem 9.

(i) Let $\{f_n\} \subset L^\infty([a,b])$. If $\int_a^x f_n(t)\,dt \to 0$ for all $x \in [a,b]$, can we conclude that $\int_a^b f_n g \to 0$ for any $L^1([a,b])$ function g?

(ii) Let $\{f_n\} \subset L^1([a,b])$. If $\int_a^x f_n(t)\,dt \to 0$ for all $x \in [a,b]$, can we conclude that $\int_a^b f_n g \to 0$ for any $g \in L^1([a,b])$?