MAT1012

Tutorial 10

CUHK(SZ)

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Outline

Ex.1. Evaluate the integral

$$\iint_R f(x,y)dA,$$

where

$$f(x,y)=\begin{cases}1, & y\le e^x,\\ 0, & y>e^x,\end{cases}$$

and $R=[0,1]\times[0,e]$.

Solution.

$$
\begin{aligned}
\iint_R f(x,y)dA &= \int_0^1 dx \int_0^e f(x,y)dy \\
&= \int_0^1 \left(\int_0^{e^x} f(x,y)dy + \int_{e^x}^e f(x,y)dy \right) dx \\
&= \int_0^1 \left(\int_0^{e^x} 1dy + \int_{e^x}^e 0dy \right) dx \\
&= \int_0^1 dx \int_0^{e^x} dy = \int_0^1 e^x dx = e - 1.
\end{aligned}
$$

□

Ex.2. Evaluate the integral

$$\iint_D (e^{-x^2} + e^x \sin x)dxdy,$$

where D is the region enclosed by $y = 0$, $x = 1$ and $y = x$.

Solution.

By the Fubini's theorem, we have

$$\begin{aligned}
&\iint_D (e^{-x^2} + e^x \sin x)dxdy = \int_0^1 (e^{-x^2} + e^x \sin x)dx \int_0^x dy \\
=&\int_0^1 (e^{-x^2} + e^x \sin x)xdx = \int_0^1 xe^{-x^2}dx + \frac{1}{2}\int_0^1 2xe^x \sin x dx \\
=&-\frac{1}{2}e^{-x^2}\Big|_0^1 + \frac{1}{2}\left[xe^x(\sin x - \cos x)\Big|_0^1 - \int_0^1 e^x(\sin x - \cos x)dx\right] \\
=&\frac{1}{2}\left(1 - \frac{1}{e}\right) + \frac{1}{2}e(\sin 1 - \cos 1) - \frac{1}{2}(-e^x \cos x)\Big|_0^1 \\
=&\frac{1}{2}\left(1 - \frac{1}{e}\right) + \frac{1}{2}e(\sin 1 - \cos 1) + \frac{1}{2}(e\cos 1 - 1) = \frac{e^2 \sin 1 - 1}{2e}.
\end{aligned}$$

Solution.

Here we used the fact

$$\int e^x(\sin x - \cos x)dx = \int e^x \sin x dx - \int e^x \cos x dx$$

$$= \int e^x \sin x dx - \left(e^x \cos x - \int e^x(-\sin x)dx\right) = -e^x \cos x + C.$$

$\square$

Remark.

The other iterated integral,

$$\int_0^1 dy \int_y^1 (e^{-x^2} + e^x \sin x)dx,$$

is hard to calculate directly.

Ex.3. Evaluate the integral

$$\iint_D \frac{x^2 - y^2}{\sqrt{x + y + 3}} dxdy,$$

where $D = \{(x, y) : |x| + |y| \leq 1\}$.

Solution.

In the xy-plane, the region is

$$D = \{(x, y) : \quad |x| + |y| \leq 1\}.$$

For the transformation

$$u(x, y) = x + y, \quad v(x, y) = x - y,$$

we have

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2,$$

Solution.

and the region corresponding to D in the uv-plane is

$$G = \{(u,v) : \quad -1 \le u \le 1, \quad -1 \le v \le 1\}.$$

Since the substitution transformation is one-to-one with continuous first partial derivatives, it has an inverse transformation and there are equations $u = \alpha(x,y)$, $v = \beta(x,y)$ with continuous first partial derivatives transforming R back into G. Moreover, the Jacobian determinants of the transformations are related reciprocally by

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}. \tag{10}$$

Solution.

From (10), we have

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{2}.$$

Thus, we have

$$\iint_D \frac{x^2-y^2}{\sqrt{x+y+3}}dxdy = \iint_G \frac{uv}{\sqrt{u+3}}|J|dudv$$
$$= \int_{-1}^{1}\int_{-1}^{1} \frac{uv}{2\sqrt{u+3}}dudv = \frac{1}{2}\int_{-1}^{1}\frac{u}{\sqrt{u+3}}du\int_{-1}^{1} vdv = 0.$$

Solution.

Another Solution.

Note that $D = \{(x, y) : |x| + |y| \leq 1\}$, and

$$\iint_D \frac{x^2}{\sqrt{x+y+3}} dxdy = \iint_D \frac{y^2}{\sqrt{y+x+3}} dydx.$$

Thus, we have

$$\iint_D \frac{x^2 - y^2}{\sqrt{x+y+3}} dxdy = \iint_D \frac{x^2}{\sqrt{x+y+3}} dxdy - \iint_D \frac{y^2}{\sqrt{y+x+3}} dydx = 0.$$

$\square$

Question 1 (35 marks). Let C be a curve given by

$$x = a\cos t, \quad y = a\sin t, \quad z = bt, \quad t \in \mathbb{R},$$

where a, b are positive constants.

(a) Find the unit tangent vector $\mathbf{T}$, the principal normal vector $\mathbf{N}$, and the curvature κ of the curve C for any t.

(b) Evaluate the integral

$$\int_0^{\pi} \mathbf{T}(t) \times \mathbf{N}(t) dt.$$

Solution.

(a) The curve C can be represented by

$$\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}.$$

Hence, we have

$$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k},$$
$$|\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}.$$

Solution.

Thus,

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}),$$

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2+b^2}}(-a\cos t\mathbf{i} - a\sin t\mathbf{j}),$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}},$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = -\cos t\mathbf{i} - \sin t\mathbf{j},$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{\left|\frac{d\mathbf{T}}{dt}\right|}{\frac{ds}{dt}} = \frac{\left|\frac{d\mathbf{T}}{dt}\right|}{|\mathbf{r}'(t)|} = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}.$$

Proof.

(b)

$$\mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{a^2+b^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2+b^2}} (b\sin t\mathbf{i} - b\cos t\mathbf{j} + a\mathbf{k}),$$

$$\int_0^\pi \mathbf{T}(t) \times \mathbf{N}(t) dt = \frac{1}{\sqrt{a^2+b^2}} \int_0^\pi (b\sin t\mathbf{i} - b\cos t\mathbf{j} + a\mathbf{k}) dt$$

$$= \frac{1}{\sqrt{a^2+b^2}} (-b\cos t\mathbf{i} - b\sin t\mathbf{j} + at\mathbf{k}) \Big|_{t=0}^{t=\pi}$$

$$= \frac{1}{\sqrt{a^2+b^2}} (2b\mathbf{i} + a\pi\mathbf{k}).$$

$\square$

Question 2 (20 marks). Let

$$f(x,y)=\begin{cases}\frac{xy^2}{x^2+y^4}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

Examine continuity of $f(x,y)$ at $(0,0)$ and prove your conclusion.

Solution.

The function is not continuous at $(0,0)$. In fact the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

does not exist.

To prove, take $k>0$ and consider the limit along the curve $x=ky^2$.

Then

$$\lim_{\substack{(x,y)\to(0,0),\\ x=ky^2}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{ky^4}{k^2y^4+y^4} = \frac{k}{k^2+1},$$

which depends on k. Thus the limit does not exist, so the function is not continuous at $(0,0)$. □

Question 3 (40 marks). Let

$$f(x,y)=\begin{cases}\frac{\sqrt[3]{x^4-y^4}}{\sqrt[3]{x}+\sqrt[3]{y}} & \text{if } x\neq -y,\\ 0 & \text{if } x=-y.\end{cases}$$

(a) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(b) Find the directional derivative of $f(x,y)$ at $P(0,0)$ along the direction $\mathbf{u}=(\alpha,\beta)$, where $\alpha^2+\beta^2=1$.

(c) Examine the differentiability of $f(x,y)$ at $P(0,0)$ and prove your conclusion.

Solution.

(a) If $x \neq 0$,

$$f(x,0) = \frac{\sqrt[3]{x^4}}{\sqrt[3]{x}} = \frac{x^{4/3}}{x^{1/3}} = x.$$

$$\frac{\partial f}{\partial x}(0,0) = 1.$$

If $y \neq 0$,

$$f(0,y) = \frac{\sqrt[3]{-y^4}}{\sqrt[3]{y}} = -\frac{y^{4/3}}{y^{1/3}} = -y.$$

$$\frac{\partial f}{\partial y}(0,0) = -1.$$

Solution.

(b) Note that $f(0,0)=0$. Let $\mathbf{u}=\alpha\mathbf{i}+\beta\mathbf{j}$, where $\alpha^2+\beta^2=1$.

If $\alpha \neq -\beta$, then $\alpha s \neq -\beta s$ for any $s>0$.

$$\frac{\partial f}{\partial \mathbf{u}}(0,0)=\lim_{s\to 0^+}\frac{f(\alpha s,\beta s)-f(0,0)}{s}$$
$$=\lim_{s\to 0^+}\frac{\sqrt[3]{(\alpha s)^4-(\beta s)^4}}{s(\sqrt[3]{\alpha s}+\sqrt[3]{\beta s})}=\frac{\sqrt[3]{\alpha^4-\beta^4}}{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}=\frac{\sqrt[3]{\alpha^2-\beta^2}}{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}.$$

In the last equality, since $\alpha^2+\beta^2=1$,

$\alpha^4-\beta^4=(\alpha^2-\beta^2)(a^2+\beta^2)=\alpha^2-\beta^2$.

If $\alpha=-\beta$, then $\alpha s=-\beta s$ for any $s>0$, $f(\alpha s,\beta s)=0$, so

$\frac{\partial f}{\partial \mathbf{u}}(0,0)=0$.

Solution.

(c) The function f is not differentiable at $P(0,0)$. Otherwise we have, for $\mathbf{u} = \alpha\mathbf{i} + \beta\mathbf{j}$ with $\alpha^2 + \beta^2 = 1$,

$$\frac{\partial f}{\partial \mathbf{u}}(0,0) = \nabla f(0,0) \cdot \mathbf{u} = (1,-1) \cdot (\alpha, \beta) = \alpha - \beta.$$

This is not true if $\alpha = -\beta$.

Direct proof. Suppose f is differentiable at $P(0,0)$, then

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y = o(\rho)$$

as $\rho \to 0$, where $\rho = \sqrt{x^2 + y^2}$. So

$$f(x,y) - x + y = o(\rho).$$

Solution.

However, along $x = 2t$, $y = t$, $t > 0$, we have

$$f(x,y) = f(2t,t) = \frac{\sqrt[3]{16t^4 - t^4}}{\sqrt[3]{2t} + \sqrt[3]{t}} = \frac{\sqrt[3]{15}t^{4/3}}{(\sqrt[3]{2}+1)t^{1/3}} = \frac{\sqrt[3]{15}}{\sqrt[3]{2}+1}t,$$

$$f(x,y) - x + y = f(2t,t) - 2t + t = f(2t,t) - t = (\frac{\sqrt[3]{15}}{\sqrt[3]{2}+1} - 1)t$$

$$\rho = \sqrt{x^2+y^2} = \sqrt{4t^2+t^2} = \sqrt{5}t.$$

Since

$$\frac{f(x,y) - x + y}{\rho} = \frac{(\frac{\sqrt[3]{15}}{\sqrt[3]{2}+1} - 1)t}{\sqrt{5}t} = \frac{(\frac{\sqrt[3]{15}}{\sqrt[3]{2}+1} - 1)}{\sqrt{5}}$$

which does not tend to zero as $t \to 0^+$, we reach a contradiction. Thus, f is not differentiable at $(0,0)$. □

Question 4 (15 marks). Let $f(x, y)$ be a continuous function defined on the plane and assume

$$\lim_{|x|+|y|\to\infty} f(x, y) = -\infty.$$

Prove that f attains its maximum value.

Solution.

Since $\lim_{|x|+|y|\to\infty} f(x,y) = -\infty$, there exists $R > 0$ such that

$$f(x,y) < f(0,0)$$

if $|x| + |y| \geq R$. The set

$$D = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq R\}$$

is a bounded and closed set. Let

$$M = \sup_{(x,y)\in D} f(x,y).$$

Then $f(0,0) \leq M$, hence for any $(x,y) \notin D$ we have $f(x,y) < M$.

Solution.

Hence

$$M = \sup_{(x,y)\in\mathbb{R}^2} f(x,y).$$

By the definition of supremum, there exists $(x_n, y_n) \in D$ such that $f(x_n, y_n) \to M$. Since D is bounded, so $\{(x_n, y_n)\}$ is bounded, hence there exists a subsequence $\{(x_{n_j}, y_{n_j})\} \to (x_0, y_0)$. Since f is continuous,

$$f(x_0, y_0) = \lim_{j\to\infty} f(x_{n_j}, y_{n_j}) = M.$$

Thus f achieves the maximum at (x_0, y_0). $\square$

Question 5 (25 marks). Find the set of convergence of the series

$$\sum_{n=1}^{\infty} \frac{\ln(1+nx)}{n} (\sin x)^n, \quad x \geq 0,$$

Solution.

The set of convergence is

$$\begin{aligned} D =& \{x \geq 0 : x \neq 2k\pi + \frac{\pi}{2} \text{ for any integer } k \geq 0\} \\ =& [0, \infty) \setminus \bigcup_{k=0}^{\infty} \{2k\pi + \frac{\pi}{2}\}. \end{aligned}$$

The series converges at $x = 0$.

Solution.

Let $x > 0$. Denote $u_n(x) = \frac{\ln(1+nx)}{n}(\sin x)^n$. Note that

$$\ln(1+(n+1)x) = \ln((1+nx)+x) = \ln[(1+nx)1+\frac{x}{1+nx}]$$

$$= \ln(1+nx) + \ln(1+\frac{x}{1+nx}).$$

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{\frac{\ln(1+(n+1)x)}{n+1}(\sin x)^{n+1}}{\frac{\ln(1+nx)}{n}(\sin x)^n} = \frac{n}{n+1}\sin x\frac{\ln(1+nx)+\ln(1+\frac{x}{1+nx})}{\ln(1+nx)}$$

$$= \frac{n}{n+1}\sin x[1+\frac{\ln(1+\frac{x}{1+nx})}{\ln(1+nx)}] \to \sin x$$

as $n \to \infty$. By the ratio test the series converges if $|\sin x| < 1$, namely if $x > 0$, $x \neq k\pi + \frac{\pi}{2}$ for any integer $k \geq 0$.

Solution.

If $x = 2k\pi + \frac{\pi}{2}$, then $\sin x = 1$, then $u_n(x) = \frac{\ln(1+nx)}{n} > \frac{1}{n}$ for large n. By comparison test the series diverges.

If $x = (2k+1)\pi + \frac{\pi}{2}$, then $\sin x = -1$, the series is

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln(1+nx)}{n},$$

which is an alternating series. We show the sequence $a_n = \frac{\ln(1+nx)}{n}$ decreases to zero as $n \to \infty$.

Solution.

Fix $x > 0$ and consider the function

$$g(t) = \frac{\ln(1+xt)}{t}.$$

$$g'(t) = \frac{1}{t^2}\left(\frac{tx}{1+xt} - \ln(1+xt)\right) < 0$$

because

$$\ln(1+xt) > \frac{xt}{1+xt}$$

for $xt > 0$. Hence $a_n = g(n)$ is decreasing in n. Obviously $a_n \to 0$ as $n \to \infty$. By Lebnitz test we conclude that $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. □

Question 6 (20 marks). Let $\{a_n\}$ be a sequence of positive numbers. Assume there exists $p > 0$ such that

$$\lim_{n\to\infty} \frac{\ln(na_n)}{\ln(\ln n)} = -p. \qquad (*)$$

(a) Prove that if $p > 1$ then the series $\sum_{n=3}^{\infty} a_n$ converges, and if $0 < p < 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) Can you conclude the series converges or diverges if $p = 1$? Give your reason.

Solution.

(a) From $(*)$, for any $0 < \varepsilon < p$, there exists N such that for any $n \geq N$,

$$p - \varepsilon < -\frac{\ln(na_n)}{\ln \ln n} < p + \varepsilon,$$

$$\ln((\ln n)^{p-\varepsilon}) = (p - \varepsilon) \ln \ln n < -\ln(na_n) < (p + \varepsilon) \ln \ln n = \ln((\ln n)^{p+\varepsilon}),$$

$$-\ln((\ln n)^{p+\varepsilon}) < \ln(na_n) < -\ln((\ln n)^{p-\varepsilon}),$$

$$\ln \frac{1}{(\ln n)^{p+\varepsilon}} < \ln(na_n) < \ln \frac{1}{(\ln n)^{p-\varepsilon}},$$

$$\frac{1}{(\ln n)^{p+\varepsilon}} < na_n < \frac{1}{(\ln n)^{p-\varepsilon}},$$

$$\frac{1}{n(\ln n)^{p+\varepsilon}} < a_n < \frac{1}{n(\ln n)^{p-\varepsilon}}.$$

Solution.

If $p > 1$, we take $\varepsilon > 0$ such that $p - \varepsilon > 1$. Since the series $\sum_{n=N}^{\infty} \frac{1}{n(\ln n)^{p-\varepsilon}}$ converges and $0 < a_n < \frac{1}{n(\ln n)^{p-\varepsilon}}$, we conclude that $\sum_{n=N}^{\infty} a_n$ converges.

If $p < 1$, we take $\varepsilon > 0$ such that $p + \varepsilon < 1$. Since the series $\sum_{n=N}^{\infty} \frac{1}{n(\ln n)^{p+\varepsilon}}$ diverges and $a_n > \frac{1}{n(\ln n)^{p+\varepsilon}}$, we conclude that $\sum_{n=N}^{\infty} a_n$ diverges.

Solution.

(b) No. Let

$$a_n = \frac{1}{n(\ln n)(\ln\ln n)^q}, \quad q > 0.$$

$$\ln(na_n) = -\ln[(\ln n)(\ln\ln n)^p] = -\ln\ln n - q\ln\ln\ln n,$$

$$\frac{\ln(na_n)}{\ln\ln n} = -1 - q\frac{\ln\ln\ln n}{\ln\ln n} \to -1.$$

The series $\sum_{n=N}^{\infty} a_n$ converges if $q > 1$ and diverges if $0 < q \leq 1$. □

Question 7 (20 marks). Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions and $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Assume there exists a positive number c such that the following conditions hold:

(1) Every $u_n(x)$ is continuously differentiable on $[0, c]$.

(2) $|u_n(x)| \le a_n c^n$ for all $x \in [0, c]$ and $n \ge 1$.

(3) $|u_n'(x)| \le n a_n x^{n-1}$ for all $x \in (0, c)$ and $n \ge 1$.

(4) The series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c$.

Prove the following conclusions:

(a) $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on the interval $[0, c]$.

(b) Let $f(x) = \sum_{n=1}^{\infty} u_n(x)$. Then f is continuously differentiable in the open interval $(0, c)$.

Solution.

By condition (4) the power series converges at $x = c$, hence it converges uniformly on $[0, c]$ (see Exercise 4.3.8). From this and condition (2), we can use the M-test to conclude that the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[0, c]$.

From (4) the radius of converges of the power series $R \geq c$. Hence the radius of convergence of $\sum_{n=1}^{\infty} na_n x^{n-1}$ is also R, thus it converges for $|x| < c$. Let $x_0 \in (0, c)$. For any $|x_0| < b < c$ the series $\sum_{n=1}^{\infty} na_n b^{n-1}$ converges. From (3), for all $x \in (0, b]$ we have

$$|u_n'(x)| \leq na_n x^{n-1} \leq na_n b^{n-1}.$$

Solution.

By the M-test we see that $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $(0, b]$, hence we can differentiate the power series to get

$$f'(x) = \sum_{n=1}^{\infty} u_n'(x)$$

for $x \in (0, b]$. Since this power series converges uniformly on $(0, b]$ and each $u_n'(x)$ is continuous in $(0, b]$ we see that $f'(x)$ is continuous on $(0, b]$, in particular $f'(x)$ is continuous at x_0. This is true for any $x \in (0, c)$, so we conclude that f' is continuous on $(0, c)$. $\square$

Question 8 (25 marks). Let $f(x, y)$ be a function defined on an open disc $B(x_0, y_0, R)$ with center $P_0(x_0, y_0)$ and radius $R > 0$.

(a) Prove that if f is differentiable at $P_0(x_0, y_0)$, then the following conclusions hold:

(a1) For any unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, and for any smooth curve

$$\begin{cases} x = x(t), \quad y = y(t), \quad -1 \le t \le 1, \\ x(0) = x_0, \quad y(0) = y_0, \quad x'(0) = u_1, \quad y'(0) = u_2, \end{cases} \tag{1}$$

it holds that $\left(\frac{d}{dt}\right)_+ f(x(t), y(t))\Big|_{t=0} = \frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$, (2)

where $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ denotes the directional derivative of f at $P_0(x_0, y_0)$ along the direction $\mathbf{u}$.

(a2) $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ is continuous in $\mathbf{u}$.

(b) Assume f has directional derivative $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ for all unit vector $\mathbf{u}$, and assume $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ is continuous in $\mathbf{u}$.

(b1) Can you conclude that f is differentiable at $P_0(x_0, y_0)$?

(b2) Can you conclude that the equality (2) still holds true?

For each of (b1) and (b2), if your answer is "YES" then give a proof; if your answer is "NO" then give a counterexample.

Solution.

(a1) Assume f is differentiable at (x_0, y_0), there exists $\delta > 0$ such that

$$\begin{aligned} f(x,y) = & f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ & + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) \end{aligned} \tag{3}$$

whenever $|x-x_0|^2 + |y-y_0|^2 < \delta^2$, where $\varepsilon_1, \varepsilon_2 \to 0$ as both $x - x_0, y - y_0 \to 0$.

Solution.

For any unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, and any smooth curve satisfying (1), we have

$$x(t) = x_0 + u_1 t + o(t), \quad y(t) = y_0 + u_2 t + o(t).$$

Plugging $x = x(t)$ and $y = y(t)$ into (3) we get

$$\begin{aligned} & f(x(t), y(t)) - f(x_0, y_0) \\ =& f_x(x_0, y_0)[u_1 t + o(t)] + f_y(x_0, y_0)[u_2 t + o(t)] \\ & + \varepsilon_1(x(t) - x_0) + \varepsilon_2(y(t) - y_0) \\ =& [f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2]t + o(t) + \varepsilon_1(x(t) - x_0) + \varepsilon_2(y(t) - y_0). \end{aligned}$$

Solution.

Hence

$$
\begin{aligned}
\left(\frac{d}{dt}\right)_+ f(x(t), y(t))\Big|_{t=0} &= \lim_{t\to 0^+} \frac{f(x(t), y(t)) - f(x_0, y_0)}{t} \\
&= \lim_{t\to 0^+} \Big\{ [f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2] \\
&\qquad + \frac{o(t) + \varepsilon_1(x(t) - x_0) + \varepsilon_2(y(t) - x_0)}{t} \Big\} \\
&= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \\
&= \nabla f(x_0, y_0) \cdot \mathbf{u} = \frac{\partial f}{\partial \mathbf{u}}(x_0, y_0).
\end{aligned}
$$

Solution.

Here we used the fact that as $t \to 0^+$,

$$\Delta x = x(t) - x_0 \to 0, \quad \Delta y = y(t) - y_0 \to 0,$$

hence $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. We also used the fact that since f is differentiable at (x_0, y_0), the directional derivative $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ exists and it is equal to $\nabla f(x_0, y_0) \cdot \mathbf{u}$. Therefore the directional derivative exists and (2) holds.

Solution.

(a2) Since f is differentiable at (x_0, y_0), for any unit vector $\mathbf{u}$ we have

$$\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}.$$

If $\mathbf{u}_j = \alpha_j \mathbf{i} + \beta_j \mathbf{j}$ are unit vectors, then

$$\begin{aligned}
&|\frac{\partial f}{\partial \mathbf{u}_1}(x_0, y_0) - \frac{\partial f}{\partial \mathbf{u}_2}(x_0, y_0)| = |\nabla f(x_0, y_0) \cdot \mathbf{u}_1 - \nabla f(x_0, y_0) \cdot \mathbf{u}_2| \\
=&|\nabla f(x_0, y_0) \cdot (\mathbf{u}_1 - \mathbf{u}_2)| = |f_x(x_0, y_0)(\alpha_1 - \alpha_2) + f_y(x_0, y_0)(\beta_1 - \beta_2)| \\
\leq&\sqrt{f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2}\sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2} \\
=&|\nabla f(x_0, y_0)||\mathbf{u}_1 - \mathbf{u}_2|.
\end{aligned}$$

Solution.

Thus for any $\varepsilon > 0$, if $\mathbf{u}_1, \mathbf{u}_2$ are unit vectors and

$$|\mathbf{u}_1 - \mathbf{u}_2| < \frac{\varepsilon}{1 + |\nabla f(x_0, y_0)|},$$

then

$$|\frac{\partial f}{\partial \mathbf{u}_1}(x_0, y_0) - \frac{\partial f}{\partial \mathbf{u}_2}(x_0, y_0)| < \varepsilon.$$

Thus $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ is continuous in $\mathbf{u}$.

Solution.

(b) Assume f has directional derivative $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ for all unit vector $\mathbf{u}$ and $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0)$ is continuous in $\mathbf{u}$, we can not conclude f is differentiable at $P_0(x_0, y_0)$, we can not conclude (2) holds.

(b1) We can not conclude f is differentiable at $P_0(x_0, y_0)$.

Counterexample:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

$\nabla f(0,0) = (0,0)$.

Solution.

For any unit vector $\mathbf{u} = (\alpha, \beta)$,

$$f(\alpha t, \beta t) = \alpha^2 \beta t, \quad \frac{\partial f}{\partial \mathbf{u}}(0,0) = \alpha^2 \beta,$$

which is continuous in (α, β), hence continuous in $\mathbf{u}$. Hence the directional derivative exists for any unit vector $\mathbf{u}$ and is continuous in $\mathbf{u}$.

But f is not differentiable at $P_0(0,0)$.

Solution.

To prove, suppose f is differentiable at $P_0(0,0)$. Since $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, we have, for $(x,y) \neq (0,0)$,

$$\frac{x^2y}{x^2+y^2} = f(x,y) = \varepsilon_1 x + \varepsilon_2 y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(x,y) \to (0,0)$. Take $y = x$. We have

$$x = (\varepsilon_1 + \varepsilon_2)x,$$

$$1 = \varepsilon_1 + \varepsilon_2,$$

which is a contradiction.

Solution.

Other counterexample:

$$f(x,y)=\begin{cases}1, & 0<y<x^2,\\ 0, & \text{otherwise.}\end{cases}$$

At $P_0=(0,0)$, f has directional derivative in any direction $\mathbf{u}$ and for any $\mathbf{u}$

$$\frac{\partial f}{\partial \mathbf{u}}(0,0)=0.$$

Hence $\frac{\partial f}{\partial \mathbf{u}}(0,0)$ is continuous in $\mathbf{u}$. See PPT-Lecture 21, Example 5.4.15 (c). f is not continuous at P_0, hence f is not differentiable at P_0.

Solution.

(b2) We can not conclude (2) holds. Counterexample:

$$f(x,y) = \begin{cases} x+y, & 0 < y < x^2, \\ 0, & \text{otherwise.} \end{cases}$$

Given any unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, the line segment $x = u_1 t$ and $y = u_2 t$ does not lie in the region $0 < y < x^2$ for small $t > 0$, so

$$f(u_1 t, u_2 t) = 0$$

for small $t \geq 0$. Hence we have again $\frac{\partial f}{\partial \mathbf{u}}(0,0) = 0$. Thus $\frac{\partial f}{\partial \mathbf{u}}(0,0)$ exists for any unit $\mathbf{u}$ and is continuous in $\mathbf{u}$.

Solution.

In particular if $\mathbf{u} = \mathbf{i}$, then $\frac{\partial f}{\partial \mathbf{u}}(0, 0) = 0$.

However is we choose the curve

$$x(t) = t, \quad y(t) = t^3, \quad -1 \leq t \leq 1,$$

then $x(0) = 0$, $y(0) = 0$. $x'(0) = 1$, $y'(0) = 0$, hence the curve has tangent vector $\mathbf{u} = \mathbf{i}$ at $t = 0$. However the curve satisfies $0 < y(t) < x^2(t)$ for all $0 < t < 1$.

Solution.

Hence

$$f(x(t), y(t)) = x(t) + y(t) = t + t^3, \quad 0 \le t < 1.$$

Hence

$$\left(\frac{d}{dt}\right) f(x(t), y(t))\Big|_{t=0} = \left(\frac{d}{dt}\right)\Big|_{t=0} (t + t^3) = 1 \neq \frac{\partial f}{\partial \mathbf{u}}(0,0).$$

$\square$