MAT4002 Final Examination (120 min)

Name: __________ ID: __________ 2024 Spring

Note: No books, notes or calculators are allowed.

1. (15 pts) Given $f : X \to Y$, there is a natural mapping, which we denote (by abuse of notation) $f : X \times \{0\} \to Y$. One may define the quotient spaces (called the mapping cylinder and mapping cone) :

$$M_f = ((X \times [0,1]) \sqcup Y) / \sim \quad \text{where } (x,0) \sim f(x)$$

and

$$C_f = ((X \times [0,1]) \sqcup Y) / \sim, \text{ where } (x,0) \sim f(x) \text{ and } (x_1,1) \sim (x_2,1) \text{ for all } x, x_1, x_2 \in X.$$

(a) (10 pts) Show that if $f, g : X \to Y$ are homotopic mappings, then M_f and M_g are homotopy equivalent. Likewise, show that C_f and C_g are homotopy equivalent.

(b) (5 pts) Show that if both X and Y are Hausdorff, then so are M_f and C_f.

2. (15 pts) Let K be the Klein bottle and let T^2 be the torus.

(a) (5 pts) Find $\pi_1(K)$.

(b) (5 pts) Find $\pi_1(T^2)$.

(c) (5 pts) Find $\pi_1(K \# T^2)$.

3. (15 pts) Let $T_{2,3}$ denote the torus knot shown below.

(a) (5 pts) Determine $\pi_1(R^3 \setminus T_{2,3})$ using the Wirtinger presentation.

(b) (5 pts) Determine $\pi_1(R^3 \setminus T_{2,3})$ using the Seifert-van Kampen theorem.

(c) (5 pts) Show that your answers in (a) and (b) are isomorphic.

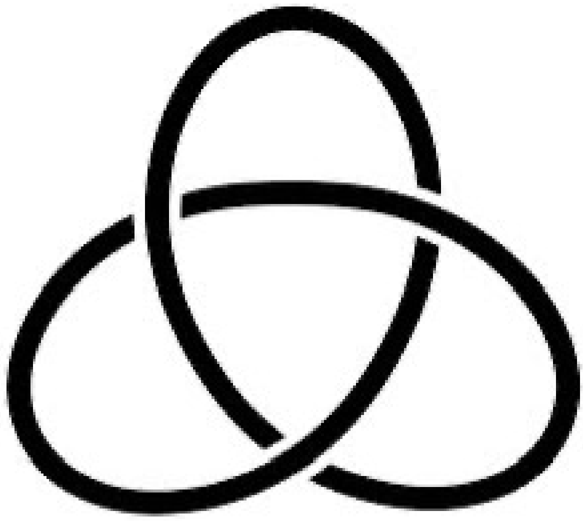


Figure 1: Trefoil Knot

4. (20 pts) Let K and L be simplicial complexes and let $|K|$ and $|L|$ denote their topological realizations, respectively. Suppose that $|K| = |L| = S^1$, parameterized by the angle coordinate and that both K and L are given by $V = \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ and

$$\Sigma = \left\{ \{0\}, \left\{\frac{2\pi}{3}\right\}, \left\{\frac{4\pi}{3}\right\}, \left\{0, \frac{2\pi}{3}\right\}, \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}, \left\{0, \frac{4\pi}{3}\right\} \right\}.$$

(a) Define $f : |K| \to |L|$ by $f(\theta) = 2\theta$. Show that there does not exist a simplicial map $f : K \to L$ such that $f(\mathrm{st}(v)) \subset \mathrm{st}(\hat{f}(v))$ for all vertices $v \in V$.

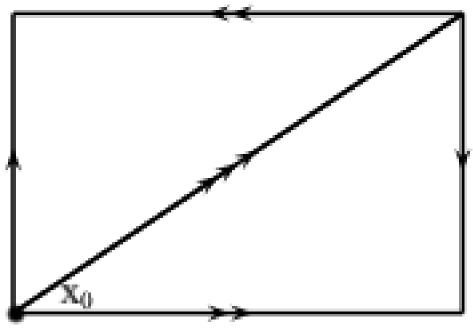
(b) Find a barycentric subdivision K' of K, so that there exists a simplicial approximation $\hat{f} : K' \to L$ of f from part (a).

(c) Use simplicial approximation to show that there does not exist a homotopically nontrivial map from $S^n \to S^m$ for $n < m$.

5. (20 pts) Let $\pi_1(X, x_0)$ denote the fundamental group of a topological space X based at x_0.

(a) Let
$$A = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + y^2 = 1, z = 0\}$$
and
$$B = \{(x, y, z) \in \mathbb{R}^3 : (x+1)^2 + z^2 = 1, y = 0\}.$$
Let $X = \mathbb{R}^3 \setminus (A \cup B)$. Find $\pi_1(X, x_0)$.

(b) Let X be the topological realization of the following space, where we have identified the edges as indicated by the arrows :



Find $\pi_1(X, x_0)$.

(c) Use the Seifert-van Kampen theorem to compute $\pi_1(S^n, x_0)$, $n \geq 2$.

(d) Given path-connected topological spaces X and Y, let $x_0 \in X$ and $y_0 \in Y$ be basepoints. We let $X \vee Y$ denote the quotient space obtained from the disjoint union $X \sqcup Y$ under the identification $x_0 \sim y_0$.

Use the Seifert-van Kampen theorem to compute $\pi_1(S^2 \vee T^2)$.

(e) Determine which of the following spaces are homotopically equivalent :

(i) The space X defined in part 1).
(ii) The space X defined in part 2).
(iii) The sphere S^2.
(iv) The space $S^2 \vee T^2$ defined in part 4).

6. (20 pts) For each of the following pairs of spaces X and Y, does there exist a homotopically nontrivial $f : X \to Y$? Prove your answer.

(a) If $X = \mathbb{R}P^2$, $Y = S^1 \times S^1$.

(b) If $Y = S^1 \times S^1$, $X = \mathbb{R}P^2$.

(c) If $X =$ Klein bottle, $Y = S^1 \times S^1$.

(d) If $X = \mathbb{R}P^2$, $Y =$ Klein bottle.