MAT1001 Midterm Examination

Saturday, October 26, 2024

Time: 9:00 - 11:00 AM

Notes and Instructions

1. No books, no notes, no dictionaries, no calculators, and no phones.

*2. The maximum possible score of this examination is **129**.*

*3. There are **13** problems (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

*5. Write down your solutions on the **answer book**.*

*6. Show your intermediate steps **except Questions 1 and 2** — answers without intermediate steps will receive minimal (or even zero) marks.*

THIS PAGE IS LEFT BLANK INTENTIONALLY.

MAT1001 Midterm Questions

1. [**10**] True or False? No need to show intermediate steps.

   (i) If f is continuous on (a, b) then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

   (ii) If a car's speedometer reads 30 km/hour at 2:00 PM and it reads 50 km/hour at 2:10 PM, then at some moment between 2:00 and 2:10 PM the car's acceleration is exactly 120 km/(hour)^2. (Assume that the car's velocity is differentiable.)

   (iii) $f(x) = x^3 + 2x + \tan x$ does not have any local maximum or minimum values.

   (iv) If $x = 1$ is a critical point of a function $f(x)$, $f' < 0$ on the interval $(0, 1)$, and $f' > 0$ on the interval $(1, 2)$, then $f(1)$ is a local minimum of f.

   (v) Supposing the function $f(x)$ is continuous over the interval $[a, b]$ except at a point $c \in [a, b]$. Then, we can have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

   even though the discontinuity at point c is a finite jump.

2. [**18**] Short questions. No need to show intermediate steps.

   (i) Is the following function continuous at $x = 1$? If not, state the type of discontinuity.

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1}, & \text{if } x \neq 1, \\ 1, & \text{if } x = 1. \end{cases}$$

   (ii) Determine the vertical asymptotes of $f(x) = \dfrac{x}{x^2 - 1}$.

   (iii) Given a function of f, what is the definition of its derivative $f'(x)$ at the point x in its domain?

(iv) Let y be a function of u, and assume y is differentiable at $u = 1$ where the (instantaneous) rate of change of y with respect to u is 10; let us further assume that u in turn is a function of x such that $u = 1$ when $x = 26$, and u is differentiable at $x = 26$ where the (instantaneous) rate of change of u with respect to x is 2024. Find the (instantaneous) rate of change y with respect to x when $x = 26$.

(v) We wish to find the solution to the equation $f(x) = x^3 - x - 5 = 0$ using Newton's method. Supposing we take the initial estimate $x_0 = 0$, then we have for the next estimate

(A) $x_1 = -1$
(B) $x_1 = -5$
(C) $x_1 = -7$
(D) $x_1 = 4$

(vi) Choose the correct answer: If f is continuous and we know that $f(0) = -1$ and $f(2) = 1$, then

(A) There is a unique solution of the equation $f(x) = 0$.
(B) $-1 < f(x) < 1$ for any $x \in (0, 2]$.
(C) $f(1) = 0$.
(D) There is at least one solution $x \in (-1, 2)$ of $f(x) = 0$.

3. [**6**] Find the limit $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x}\right)$.

4. [**12**] Let a be a nonzero real number and g be a function satisfying $|g(x)| \leq 2$ for all $x > 0$. Consider the function

$$f(x) = \begin{cases} \frac{\sin(a^2 x) - x}{x}, & \text{if } x < 0, \\ a + 1, & \text{if } x = 0, \\ \cos^2\left(x + \frac{\pi}{2}\right) g\left(\frac{1}{x}\right), & \text{if } x > 0. \end{cases}$$

(i) Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

(ii) Determine all possible values of a for which $\lim_{x\to 0} f(x)$ exists. What are the values of a, if any, such that f is continuous at $x = 0$?

5. [**15**] Had Galileo dropped a cannonball from the Tower of Pisa, 98 m above the ground, the ball's height above the ground t seconds into the fall would have been $s(t) = 98 - 4.9t^2$ meters.

(i) What would have been the ball's velocity, speed, and acceleration at time t? (Hint: please remember to write down the unit.)

(ii) About how long would it have taken the ball to hit the ground?

(iii) What would have been the ball's velocity at the moment of impact?

6. [**12**] Compute the derivative of each of the following functions.

(i) $5\pi^2 + 2x\cos(2x) + \frac{3\tan x}{\sqrt{1+x^2}}$.

(ii) $y^4 - 4y^2 = x^4 - 9x^2$, find $\frac{dy}{dx}$ at $(x, y) = (3, 2)$.

7. [**8**] A boy 1.5 m tall is walking towards a 6 m lamppost at the rate of 2 m per second, as shown in Figure 1, where B indicates the position of the lamppost, and E indicates the position of the boy. How fast is the tip of the boy's shadow (cast by the lamp) moving?

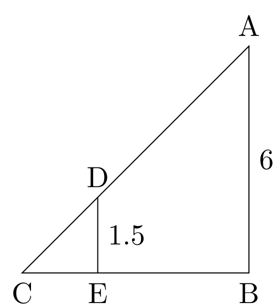

Figure 1: Boy at E walking towards a lamppost at B

8. [**6**] A coat of paint of thickness 0.01 cm is applied to the faces of a cube whose edge is 10 cm, thereby producing a slightly larger cube. Use a differential to estimate the number of cubic centimetres of paint used.

9. [**12**] Consider the function

$$f(x) = \begin{cases} x^6 - 3x^4, & x < 1 \\ -(\sqrt{x-1} + 2), & x \geq 1 \end{cases}$$

(i) Find all critical points of f.

(ii) Find all inflection points of f.

10. [**8**] Find the largest area for a rectangle whose diagonal has length $\sqrt{2}$.

11. [**6**] Solve the initial value problem $\begin{cases} \dfrac{dy}{dx} = 1 - \cos x, \\ y(0) = 1. \end{cases}$.

12. [**8**] Using known methods for computing areas, find the mean value (average value) of the function

$$f(x) = \sqrt{a^2 - x^2}$$

over the interval $-\frac{a}{2} \leq x \leq \frac{a}{2}$ (where $a > 0$).

13. [**8**] A particle travels along the x-axis with the velocity

$$v(t) = t(1-t) \text{ m/s}, \quad t \geq 0.$$

Show that over the interval $0 \leq t \leq 1$, the distance it covers is not more than 0.25 m.