FINAL EXAM

MAT 3007
Dec 2020

INSTRUCTIONS

a) Write ALL your answers in the ANSWER SHEET.

b) Two pieces of notes are allowed. No computer or cell phone is allowed.

c) The exam time is December 24 (Thursday) 1:00pm - 2:30pm.

d) There are 6 questions and 100 points in total. Except the true or false questions, write down the reasonings for your answers.

In taking this examination, I acknowledge and accept the instructions.

NAME (signed) ______________________________

NAME (printed) ______________________________

Problem 1: True or False (18pts)

State whether each of the following statements is True or False. For each part, only your answer, which should be one of True or False, will be graded. Explanations will not be read.

(a) Suppose function $f(x,y) = \frac{x^2}{y}$. Then f is convex on the set $S = \{(x,y) : y > 0\}$.

(b) For function f, define the level set of f to be $\text{lev}_{\leq\alpha}(f) := \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$. If for all $\alpha \in \mathbb{R}$ it holds that $\text{lev}_{\leq\alpha}(f)$ is convex, then f is convex.

(c) Suppose function f is continuous on $\mathbb{R}^n$, and all points x in $\mathbb{R}^n$ is a local minimizer of f then f must be constant. Moreover, if f is not necessarily continuous, then f can be a non-constant function.

(d) Consider a nonlinear optimization problem (minimization). Suppose this problem has a unique local minimizer and a unique KKT point, then this KKT point must be a local minimizer.

(e) If an optimization problem is of high dimension and the memory of the computer is only several fold of the dimension of the problem, then using gradient descent method will be more appropriate than using Newton's method to solve the problem.

(f) For an integer optimization problem, if one solves the LP relaxation of the problem and obtains that the optimal solution is integral, then the solution must be optimal to the original integer optimization problem.

Problem 2: KKT conditions. (20pts)

Consider the following optimization problem:

$$\begin{aligned} \min \quad & \frac{1}{2}x^T P x + q^T x + r \\ \text{subject to} \quad & -1 \leq x_i \leq 1, \;\; i = 1,2,3 \end{aligned}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \quad r = 1.$$

(a) (8 pts) Show that this problem is a convex optimization problem.

(b) (8 pts) Write down the KKT conditions.

(c) (4 pts) Show that $(1, 1/2, -1)$ is optimal for the problem.

Problem 3: Convexity (12pts)

(a) (6 pts) Let $C \subset \mathbb{R}^n$ be a convex set, with $x_1, x_2, \cdots, x_k \in C$, and let $\theta_1, \theta_2, \cdots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0, \theta_1 + \theta_2 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$.

(b) (6 pts) Show that the *hyperbolic* set $\{x \in \mathbb{R}^2_+ | x_1 x_2 \geq 1\}$ is convex. As a generalization, show that $\{x \in \mathbb{R}^n_+ | \Pi_{i=1}^n x_i \geq 1\}$ is convex. Hint: If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$.

Problem 4: Branch-and-Bound Algorithm (20pts)

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & 10x_1 + 7x_2 + 8x_3 \\ \text{subject to} & 5x_1 + 6x_2 + 4x_3 \leq 10 \\ & x_1, x_2, x_3 \in \{0, 1\}. \end{array}$$

Use branch-and-bound method to solve it (draw the branch-and-bound tree and mark the results on each node).

Problem 5: Integer Optimization Formulation (15pts)

Mike is considering investments into 6 projects: A, B, C, D, E, and F. That is, Mike needs to make a choose-or-not decision on each project. Each project has an initial cost, an expected profit rate (one year from now) expressed as a percentage of the initial cost, and an associated risk of failure. These numbers are given in the table below:

	A	B	C	D	E	F
Initial cost (in M)	1.3	0.8	0.6	1.8	1.2	2.4
Profit rate	10%	20%	20%	10%	10%	10%
Failure risk	6%	4%	6%	5%	5%	4%

(a) (7 pts) Provide a linear integer programming (LIP) formulation to choose the projects that maximize total expected profit, such that Mike does not invest more than 4M dollars and its average failure risk is not over 5%.

(b) (4 pts) Suppose that if A is chosen, B must be chosen. Modify your LIP formulation.

(c) (4 pts) Suppose that if C and D are chosen, E must be chosen. Modify your LIP formulation.

Problem 6: Gradient descent and Newton's method (15 pts)

Armijo rule is a successive reduction rule to choose a proper stepsize. To be specific, for fixed scalars s, β, and σ, with $\beta \in (0, 1)$, and $\sigma \in (0, 1)$. We set the stepsize for k-th iteration $\alpha_k = \beta^{m_k} s$ where m_k is the first nonnegative integer m for which

$$f(x_k) - f(x_k + \beta^m s \mathbf{d}_k) \geq -\sigma \beta^m s (\nabla f(x_k))^T \mathbf{d}_k.$$

In other words the stepsizes $\beta^m s$, $m = 0, 1, \cdots$, are tried successively until the above inequality is satisfies for $m = m_k$.

Consider a problem of minimizing the function of two variables $f(x, y) = 3x^2 + y^4$.

(a) (5 pts) Apply one iteration of the steepest descent method with $(1, -2)$ as the starting point and with the step size chosen by the Armijo rule with $s = 1$, $\sigma = 0.1$, and $\beta = 0.5$. (Note: For choosing the step size based on the Armijo rule, you do not need to write down the computation details.)

(b) (5 pts) Repeat (a) using $s = 1$, $\sigma = 0.1$, and $\beta = 0.1$. How does the objective value of the new iterate compare to that obtained in (a). Comment on the tradeoffs involved in the choice of β.

(c) (5 pts) Apply one iteration of Newton's method with the same starting point and stepsize rule as in (a). How does the objective value of the new iterate compare to that obtained in (a)? How does the amount of work involved in finding the new iterate?

Problem 1.

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Problem 2.

(a) $\det(A) = 100 \quad \det(A_1) = 77 \Rightarrow PSD$

or

$eig(A) = [0.2589, \ 13.8429, \ 27.8981] \Rightarrow PSD$

convex domain + convex objective $\Rightarrow$ convex.

(b)
$$\begin{cases} Px + q + \lambda_1 - \lambda_2 = 0 \\ \lambda_1^T (x-1) = 0 \\ \lambda_2^T (-x-1) = 0 \\ \lambda_1 \geq 0, \quad \lambda_2 \geq 0 \\ x - 1 \leq 0 \\ -x - 1 \leq 0 \end{cases}$$

(c) $\lambda_1 = (\lambda_1, 0, 0) \qquad \lambda_2 = (0, 0, \lambda_2)$

KKT $\Rightarrow \lambda_1 = 1, \ \lambda_2 = 2$

KKT is satisfied + convex $\Rightarrow$ optimal

Problem 3

(a) ① $k=2$, this is obviously true.

② $k>2$, suppose this is true for some $k=p$

$$\sum_{i=1}^{k+1} \theta_i x_i = \sum_{i=1}^{k} \theta_i \left(\sum_{i=1}^{k} \frac{\theta_i}{\sum_{j=1}^{k} \theta_j} x_i \right) + \theta_{k+1} x_{k+1}$$

since $\sum_{i=1}^{p} \frac{\theta_i}{\sum_{j=1}^{p} \theta_j} x_i \in X$, thus $\sum_{i=1}^{k+1} \theta_i x_i \in X$.

(b) for $k=2$ $\quad \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1-\lambda) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 + (1-\lambda) y_1 \\ \lambda x_2 + (1-\lambda) y_2 \end{bmatrix}$

$$[\lambda x_1 + (1-\lambda) y_1][\lambda x_2 + (1-\lambda) y_2]$$
$$= \lambda^2 x_1 x_2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1] + (1-\lambda)^2 y_1 y_2$$
$$\geq \lambda^2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1] + (1-\lambda)^2$$
$$\geq \lambda^2 + \lambda(1-\lambda)\left[\frac{1}{x_2 y_1} + x_2 y_1\right] + (1-\lambda)^2$$
$$\geq \lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2$$
$$= 1$$

for $k \geq 3$, $\quad \prod_{i=1}^{n} (\alpha x_i + (1-\alpha) y_i) \geq \prod_{i=1}^{n} x_i^{\alpha} \prod_{i=1}^{n} y_i^{1-\alpha}$

$$\geq 1$$

Problem 4.

- step 1 $\qquad \max \quad 10x_1 + 7x_2 + 8x_3$

  $\qquad \text{s.t.} \quad 5x_1 + 6x_2 + 4x_3 \leq 10$

  $x = (0.6664, 1, 1.6669)$

- step 2

$$\begin{cases} (0, 1, 1.6669) \Rightarrow 13.3355 \\ (1, 1, 1.6669) \Rightarrow 23.3355 \end{cases}$$

- step 3

  $(1, 0, 1) \Rightarrow 18$ ✓

Problem 5

(a) decision variable $x \in R^6$.

$$c = [1.3,\ 0.8,\ 0.6,\ 1.8,\ 1.2,\ 2.4]^T$$
$$p' = [0.1,\ 0.2,\ 0.2,\ 0.1,\ 0.1,\ 0.1]^T$$
$$r = [0.06,\ 0.04,\ 0.06,\ 0.05,\ 0.05,\ 0.04]^T$$

$$P = (1+p) * (1-r) * c - c =$$
$$= [0.0442,\ 0.1216,\ 0.0768,\ 0.0810,\ 0.0540,\ 0.1344]^T$$

Formulation: $\max\limits_{x} \ P^T x$

$$c^T x \le 4$$
$$x \in \{0,1\}^6$$
$$a^T x \le 0$$

$a = [1, -1, 1, 0, 0, -1]$

(b) $x_2 \geq x_1$

(c) $x_5 + 1 \geq x_3 + x_4$

Problem 6:

(a) $\nabla f(x) = \begin{bmatrix} 6x \\ 4y^3 \end{bmatrix}$ $\quad \nabla f(x_0) = \begin{bmatrix} 6 \\ -32 \end{bmatrix}$

$-\nabla f(x_0)^T d = 1060$

$m=0, \quad x_1 = [-5,\ 30]$

$m=1, \quad x_2 = [-2,\ 14]$

$m=2, \quad x_3 = [-0.5,\ 6]$

$m=3, \quad x_4 = [0.25,\ 2]$

$m=4, \quad x_5 = [0.625,\ 0]$

(b) $m=1, \quad x_1 = (0.4,\ 1.2)$

(c) $m=0, \quad x_1 = (0,\ -\frac{4}{3})$

$$H = \begin{bmatrix} 6 & 0 \\ 0 & 48 \end{bmatrix}$$

$$d = -H^{-1} \nabla f(x_0) = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}$$