Final Exam: Math 4002

Name: ______________________________

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1. Short questions. Simply write down the answer, no justification is required.

   (a) Let $X \sqcup Y$ be the disjoint union of X and Y. Then a map $f : X \sqcup Y \longrightarrow Z$ is continuous if and only if $f \circ i_X : X \longrightarrow Z$ and $f \circ i_Y : Y \longrightarrow Z$ are continuous, where $i_X : X \longrightarrow X \sqcup Y$ and $i_Y : Y \longrightarrow X \sqcup Y$ are inclusions.

   True False

   (b) A continuous map $f : S^n \longrightarrow X$ is homotopic to a constant map if and only if it extends to D^{n+1}, i.e., there is a continuous map $\tilde{f} : D^{n+1} \longrightarrow X$ such that $\tilde{f} \mid S^n = f$:

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, hook] & X. \\ D^{n+1} \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

   True False

   (c) Let $f : |K| \to |L|$ be a continuous map between topological realizations of simplicial complexes. Then there is a simplicial map $\varphi : K \to L$ such that $f \simeq |\varphi|$.

   True False

   (d) Let X be a contractible space. Then for any $x \in X$ we have $\mathrm{id}_X \simeq c_x$, where c_x is the constant map onto x.

   True False

   (e) Let T be the torus and p a point of T, and K be the Klein bottle and q a point of K, then $(T - p) \times S^1$ is homotopy equivalent to $(K - q) \times (\mathbb{R}^2 - 0)$.

   True False

   (f) The infinite product $\prod_{i \in I} X_i$, with the product topology, is compact if and only if each X_i is compact.

   True False

2. Let P be a regular polyhedron that has n-polygonal faces and m edges incident on each vertex.

   (a) Show that
   $$\frac{1}{n} + \frac{1}{m} = \frac{1}{e} + \frac{1}{2},$$
   where e denotes the total number of edges.

   (b) Find all solutions to the above equation with m, n, e positive integers.

   (c) Conclude that there only exist 5 platonic solids.

   (d) Using the edge loop group, show that S^n is simply connected for $n \geq 2$.

3. Let $f : X \to Y$ be a map from a topological space X to a compact, Hausdorff space Y. Let

$$\Gamma = \{(x, f(x)) : x \in X\}.$$

Show that f is continuous if and only if Γ is closed in $X \times Y$.

[*Hint: Recall a question in the midterm where it was shown that A is compact if and only if $A \times B \xrightarrow{\pi} B$ is closed for all B.*]

4. Let $p : X \to Y$ be a continuous map.

   (a) Suppose p is a quotient map. Show that if Y is connected and $p^{-1}(\{y\})$ is connected for each $y \in Y$, then X is connected.

   (b) Suppose p is closed and surjective. Show that if Y is compact and $p^{-1}(\{y\})$ is compact for each $y \in Y$, then X is compact.

5. Let $C \subset \mathbb{R}^2$ be a convex subset, containing a nonempty open set, and let $f : C \to C$ be a continuous map.

   (a) If C is compact, show that f has a fixed point.
   [*Hint: Show that there cannot exist a retraction* $r : C \to \partial C$.]

   (b) Show that if C is not compact, then there there is an $f : C \to C$ without a fixed point.

6. For a space X, let $m(X)$ be the **minimum** integer m satisfying the following condition: For every open covering $\mathcal{A}$ of X, there is an open covering $\mathcal{B}$ of X such that

   - for each $B \in \mathcal{B}$, there is an $A \in \mathcal{A}$ such that $B \subset A$, and
   - no point of X lies in more than $m + 1$ elements of $\mathcal{B}$.

   (a) Show that if X is a discrete set, then $m(X) = 0$.

   (b) Show that if $X = [0, 1]$, then $m(X) = 1$.

   (c) Show that if C is a closed subspace of X, then $m(C) \leq m(X)$.

   (d) Show that if K is a finite simplicial complex of dimension m, then $m(|K|) \leq m$.