MAT1002 Final Examination

Monday, May 16, 2022

Time: 4:00 - 7:00 PM

Notes and Instructions

1. This exam is closed-book. No book, note, dictionary, or calculator is allowed.

*2. The total score of this examination is **110**.*

*3. There are **twelve** questions (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

5. State your answers in exact form, e.g., write $\sqrt{2}$ instead of 1.414.

*6. Show your intermediate steps **except Questions 1 and 2** — answers without intermediate steps will receive minimal (or even no) marks.*

7. Onsite examinees should answer all questions in the answer book.

MAT1002 Final Examination Questions

1. [**6**] True (T) or False (F)? No explanation is required.

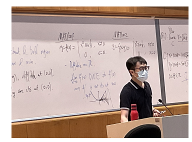
Cts function on closed and bounded region attain global max and min

T

(i) Consider $f(x, y, z) = xy^2z^6 - \sin(e^{yz}) - \ln(x^2)$ defined on

$$D = \{(x, y, z) : 4 \le x \le 8,\ 3 \le y \le 4,\ -1 \le z \le 1\}.$$

Then there must exist (x_1, y_1, z_1) and (x_2, y_2, z_2) in D such that $f(x_1, y_1, z_1) \ge f(x, y, z) \ge f(x_2, y_2, z_2)$ for all $(x, y, z) \in D$.



F (ii) If a function $f(x, y)$ is differentiable at $(0, 0)$, then the partial derivatives f_x and f_y must both be continuous at $(0, 0)$.

(iii) If the series $\sum_{n=1}^{\infty} u_n$ converges, then the series $\sum_{n=1}^{\infty}(u_{2n-1} - u_{2n})$ must also converge.

2. [**27**] Short questions. No explanation or intermediate steps are required.

$\nabla f = (2x+y)\,\vec{i} + (2x+z)\,\vec{j} + y\,k$ $\quad \mathrm{div}(\nabla f) = M_x + N_y + P_z = 2x + 0 + 0 = 2$

(i) Let $f(x, y, z) = x^2 + 2xy + yz$. Then $\mathrm{div}(\nabla f) = ($ 2 $)$.

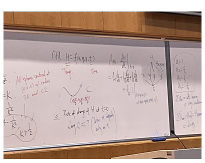
$f(x,y,z) = k$

$4 - x^2 - y^2 - z^2 = \frac{1}{k^2}$

$x^2 + y^2 + z^2 = 4 - \frac{1}{k^2}$ $\quad k \geq \frac{1}{2}$

(ii) Let $f(x, y, z) = \frac{1}{\sqrt{4-x^2-y^2-z^2}}$. Describe all the level surfaces of f.

(iii) The temperature H is described by a differentiable function $H = f(x, y, z, t)$, where (x, y, z) is the position in the space and t is time. A space curve C is described by a smooth parametrization $x = x(t)$, $y = y(t)$, and $z = z(t)$, where t is time (same as above). What is the rate of change of temperature with respect to time along C at $t = 0$? Describe with an algebraic expression.



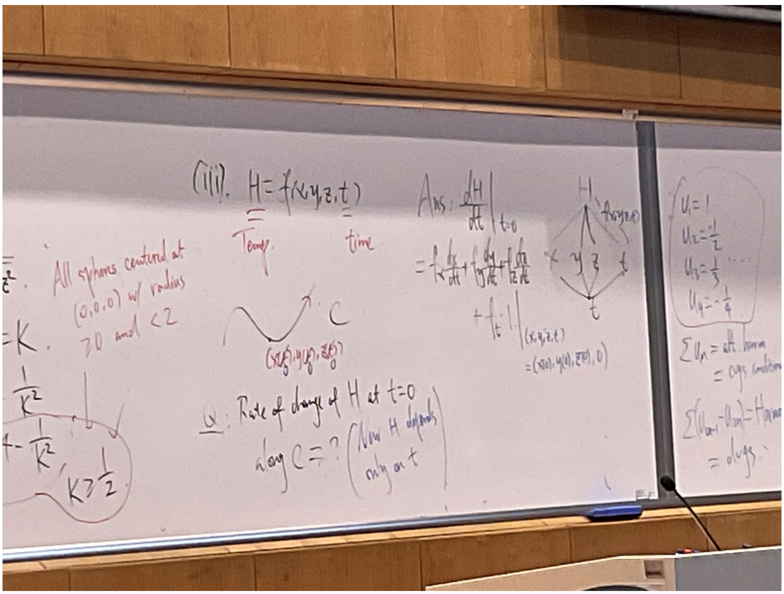
(iv) Find the curvature of the curve $y = \sin x$ at the point $P(\pi/2, 1)$.

(v) Find the principle unit normal for $y = -x^2$ at the point $P(1/2, -1/4)$.

(vi) State the first three **nonzero** terms in the Maclaurin series of $\frac{x^2}{\sqrt{2 + x}}$.

(vii) Let $H = 5x^2 - 3xy + xyz$. At the point $P(3, 4, 5)$, in which direction does the value of H decrease the fastest? (You do not have to normalize your direction.)

Direction of fastest $\nearrow$ is $\nabla f \big|_{(x,y,z)}$



(iii). H = f(x,y,z,t)
Temp.
time
All spheres centered at (0,0,0) w/ radius ≥0 and <2
C
(x(0), y(0), z(0))
Q: Rate of change of H at t=0 along C = ?
(Now H depends only on t)
Ans: dH/dt |t=0
= fx dx/dt + fy dy/dt + fz dz/dt
+ ft · 1 |(x,y,z,t) = (x(0), y(0), z(0), 0)
H
f(x,y,z,t)
x y z t
t
u1 = 1
u2 = -1/2
u3 = 1/3
u4 = -1/4
Σ un = alt. harm
= cvgs conditionally
Σ(u2n-1 - u2n) = Harmonic
= dvgs.

(viii) Let E be the solid that is outside the sphere $x^2 + y^2 + z^2 = z$ and inside the sphere $x^2 + y^2 + z^2 = 2z$. Write the following triple integral in the spherical coordinates (you do not need to compute the value):

$$\iiint_E z\,dV = (\qquad).$$

(ix) Let C be the circle that is the intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $x + y + z = 0$. Find the line integral $\int_C x^2\,ds$.

symmetric

$\int_C x^2 ds = \int_C y^2 ds = \int_C z^2 ds$

$\int_C x^2 dx = \frac{1}{3}\int_C x^2+y^2+z^2\, ds$

$= \frac{1}{3}\int_C ds$

$= \frac{1}{3}\,\text{Length}(C) = \frac{2}{3}\pi$

3. [**6+4=10**] Consider the surface S given by $xz^2 - yz + \cos(xy) = 1$.

(i) Find the tangent plane M and normal line ℓ to the surface S at the point $P(0, 0, 1)$.

(ii) Show that the tangent line to the curve

$$\mathbf{r}(t) = (\ln t)\,\mathbf{i} + (t \ln t)\,\mathbf{j} + t\,\mathbf{k}$$

at $P(0, 0, 1)$ is lying on M.

4. [**6**] Let

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2 + (x - y)^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Determine whether f is continuous at the origin. Justify.

5. [**6**] Find the global extrema of the function $f(x, y, z) = x + y + z$ subject to the constraints

$$x^2 + z^2 = 2 \quad \text{and} \quad x + y = 1.$$

6. [**5+5**] Evaluate the following integrals:

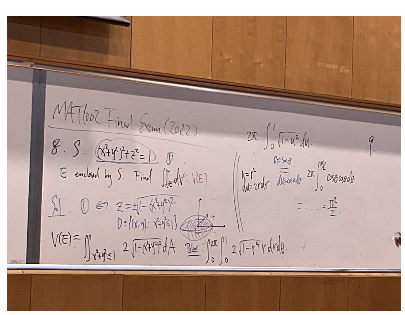
(i) $\displaystyle\int_0^1 \int_{3y}^3 e^{(x^2)}\,dx\,dy.$

(ii) $\displaystyle\iint_R v(u + v^2)^4\,dA$, where R is the rectangle $0 \le u \le 1$, $0 \le v \le 1$.

7. [**5**] **Use a double integral** to find the area of the region inside the circle $(x - 1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$.

8. [**5**] Find the volume of the solid enclosed by the surface $(x^2+y^2)^2+z^2 = 1$.

$① \Leftrightarrow z = \pm\sqrt{1-(x^2+y^2)^2}$



9. [**5+5**] Suppose that the vector field

$$\mathbf{F} = (e^{kx}\ln y)\,\mathbf{i} + \left(\frac{e^{kx}}{y} + \sin z\right)\mathbf{j} + (my\cos z)\,\mathbf{k}$$

is conservative on $\{(x,y,z) : y > 0\}$, where k and m are two constants.

(i) Find the values of k and m.

(ii) Find one potential function of $\mathbf{F}$.

RHS: $\oint_C x\,dy = \int_C 0\,dx + x\,dy \overset{\text{Green}}{=} \iint_R (N_x - M_y)\,dA$
$= \iint_R (1-0)\,dA$
$= \iint_R dA = A(R)$

10. [**5+3+5**] Green's Theorem

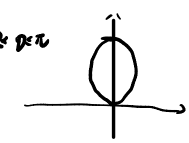
(i) Suppose that C is a piecewise smooth, simple closed curve that is counterclockwise. Show that the area $A(R)$ of the region R enclosed by C is given by Double integral $\Longleftrightarrow$ Line Integral formula

$$A(R) = \oint_C x\,dy.$$ [idea]

(ii) Now consider the simple closed curve C in the xy-plane given by the polar equation $r = \sqrt{\sin\theta}$. State a parametrization of C.

$x = r\cos\theta = \cos\theta\sqrt{\sin\theta}$
$y = r\sin\theta = \sin\theta\sqrt{\sin\theta}$

$0 \le \theta \le \pi$



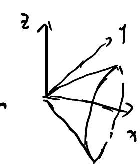
(iii) **Use the formula in part (i)** to find the area of the region enclosed by the curve C in part (ii). $A(R) = \oint_C x\,dy = \int_0^{\pi} \cos\theta\sqrt{\sin\theta}\,\frac{3}{2}\sqrt{\sin\theta}\,\cos\theta\,d\theta = \cdots\cdots = 1$

$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,d\sigma$

11. [**6**] Let S be the cone $x = \sqrt{y^2+z^2}$, $x \le 2$. Given the vector field

0

$$\mathbf{F} = (\sin(x^3y^2z))\,\mathbf{i} + \left(x^2y\right)\,\mathbf{j} + (x^2z^2)\,\mathbf{k},$$

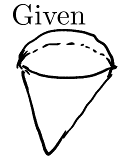


find the flux done by the curl of $\mathbf{F}$ across S, Stroke's Theorem

$$\iint_S \text{curl}(\mathbf{F})\cdot\mathbf{n}\,d\sigma,$$

with unit normals pointing to the positive x-direction.

12. [**6**] Let E be the solid that lies above the cone $z = \sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2+y^2+z^2 = 1$, and let S be the boundary surface of E. Given the vector field



$$\mathbf{F} = (\cos(z^2))\,\mathbf{i} + \left(z^3(y+x)\right)\,\mathbf{j} + (e^{x+y}x)\,\mathbf{k},$$

find the value of the outward flux across S: divergence Theorem

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

1 $\text{div}\,\vec{F} = z^3$

2 $z = \sqrt{\frac{x^2+y^2}{3}} \Longleftrightarrow \sqrt{3}\rho\cos\phi = \rho\sin\phi$
$\Longrightarrow \tan\phi = \sqrt{3}$ $\phi = \frac{\pi}{3}$

E in spherical $0 \le \theta \le 2\pi$ $0 \le \phi \le \frac{\pi}{3}$ $0 < \rho < 1$

$\iint_S \vec{F}\cdot\vec{n}\,d\sigma \overset{\text{div Thm}}{=} \iiint_E z^3\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^3\cos^3\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
$= \frac{5\pi}{64}$