MAT 1001 Final Exam, 13:30-16:30 am, December 16, 2024

Instruction:

- This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones;
- Check to see if your paper has 6 pages; if not, report it to the proctor in the exam room;
- The full score of the exam is 232 points, to earn which you need to answer ALL questions correctly;
- Write down ALL your work and your answers(including the answers for short questions) in the Exam Book;
- Show your work unless otherwise instructed–a correct answer without showing your work when required shall be given no credits.

1. (54 points) **Short Questions**(for these questions, NO need to show your work, just write down your answers in the Exam Book; NO partial credits for each question)

   (i). True or False: If f is differentiable at c, then f is continuous at c.

   True False

   (ii). True or False: The derivative of an even and differentiable function is odd, and the derivative of an odd and differentiable function is even.

   True False

   (iii). True or False: Given a function $h(x) = \dfrac{|x|}{x}$, $x \in (-\infty, 0) \cup (0, \infty)$. By appropriately defining the value of $h(0)$, the extended function h is able to be continuous on $(-\infty, \infty)$.

   True False

   (iv). Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the point $(-3, 2)$.

   (v). True or False: Let $f(x) = x^3 + bx^2 + cx + d$ be a cubic function. If $b^2 < 3c$, then f has no local extreme values.

   True False

(vi). True or False: If the function f is continuous in $[a, b]$, then there is a point $a < c < b$ such that $f(c) = f(a) + (f(b) - f(a))/3$.

True False

(vii). Find the linearization $L(x)$ of $x + \frac{1}{x}$ at $x = 1$.

(viii). Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find $(f^{-1})'(-1)$.

(ix). For the function $f(x) = -2x^3 - x^2 + 6x + 10$, let M be a midpoint Riemann sum of f over the interval $[0, 1]$. Which of the following statements is true?

(a) $f(x)$ is concave down; $M > \int_0^1 f(x)\,dx$ always holds no matter what partition of $[0, 1]$ we use.

(b) $f(x)$ is concave down; $M < \int_0^1 f(x)\,dx$ always holds no matter what partition of $[0, 1]$ we use.

(c) None of the above.

(x). Which of the following functions grows at the fastest rate as $x \to \infty$?

(a) $(1314x)^{\pi+2024}$

(b) $e^x + 48$

(c) $e^{(x^2)}$

(d) $x^x/3$

(xi). For the following statements, consider $x \to \infty$. Choose all that are true.

(a) $\arctan x = O(1)$

(b) $\ln(\ln x) = O(\ln x)$

(c) $x^{-2} + x^{-4} = O(x^{-4})$

(d) $x = o(x + \ln x)$

(xii). Suppose a curve $y = f(x)$, $a \leq x \leq b$, is revolved about the x-axis, the formula that should be used for computing the surface area of the revolution is

(a) $S = 2\pi \displaystyle\int_a^b |y| \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

(b) $S = 2\pi \displaystyle\int_a^b |y| \sqrt{1 + (\frac{dy}{dx})^2}\, dx$

(c) $S = 2\pi \displaystyle\int_a^b |y| \sqrt{1 + \frac{dy}{dx}}\, dx$

(d) $S = 2\pi \displaystyle\int_a^b |y| \sqrt{1 + \frac{dx}{dy}}\, dy$

(xiii). The volume of the solid generated by rotating about the x-axis the region below the curve $y = \sqrt{x}$ and above the interval $[0, 1]$ is

(a) $\frac{\pi}{2}$
(b) $\frac{\pi}{3}$
(c) 2π
(d) π^2

(xiv). True or False: If $uv = (x-1)e^{x^2}$ and $\int_0^1 udv = -1$, then $\int_0^1 vdu = 0$.

True False

(xv). To find the integral $\displaystyle\int \sec^4 \theta d\theta$, we use the substitution:

(a) $u = \sin\theta$
(b) $u = \cos\theta$
(c) $u = \tan\theta$

(xvi). Estimate $I = \displaystyle\int_0^1 \frac{1}{1+x^2}dx$ by using Simpson's Rule with $n = 2$ steps.

(xvii). Use Euler's method to approximate $y(3)$, where $y(x)$ is the solution of the following initial value problem

$$y' = 1 - \frac{y}{x}, \quad y(2) = -1,$$

with the increment size $dx = \Delta x = 0.5$.

(xviii). Which of the following statements is NOT correct?

(a) $dH/dt = H_S - H$ can be used to model Newton's Law of Cooling, where $H(t)$ is the temperature of an object at time t and H_S is the constant temperature of the surrounding medium.
(b) The criticism on the Malthusian Model (Exponential Population Growth Model) $dP/dt = kP$ (k is a positive constant) is that the exponential growth of the population predicted by this model cannot be sustained in the real world, due to limited number of resources.
(c) $dP/dt = P^2 - P$ is a special case of the logistic model.

2. (24 points) Find each of the following limits or explain why the limit does not exist.

(a) $\displaystyle\lim_{x\to-\infty} \sqrt{x^2+2024} + x$.

(b) $\displaystyle\lim_{x\to 0} \frac{\sin(3x) - 3x + x^2}{\sin(x)\sin(2x)}$.

(c) $\displaystyle\lim_{x\to 0^+} [\cos(\sqrt{x})]^{1/\sin x}$.

(d) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\frac{k}{n}}$.

3. (18 points) Find each of the following derivatives or explain why the derivative does not exist.

   (a) $\dfrac{d}{dx}x^{\sin x}$

   (b) $f'(x)$, where
$$f(x) = \int_{x^4}^{x^2} \log_{10}(\sqrt{u})du$$

   (c) $f'(1)$, where $f(x) = |x^2 - 1|$.

4. (16 points) Let $f(x) = (x^2 + 2x + 2)e^{-x}$.

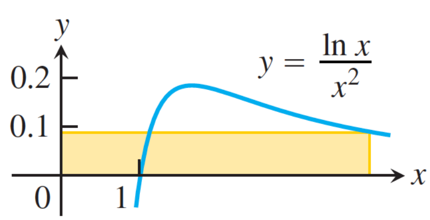
   (a) Find the open intervals on which f is increasing and decreasing.

   (b) Find the coordinates of its local maximums and local minimums (if any).

   (c) Find the open intervals on which f is concave up and concave down.

   (d) Find the coordinates of its inflection points (if any).

5. (8 points) The rectangle shown below has one side on the positive y-axis, one side on the positive x-axis, and its upper-right vertex on the curve $y = (\ln x)/x^2$, $x \geq 1$. What value of x gives the rectangle its largest area, and what is that area?



6. (8 points) Find the area of the region in the first quadrant bounded by the graphs of the functions $f(x) = e^x$, $g(x) = \cos x$ and the line $x = \pi/2$.

7. (8 points) Let R be the (finite) region bounded by $y = (x - 1)^2$ and $y = x + 1$. Find the volume of the solid obtained by rotating R about the x-axis.

8. (36 points) Compute the following integrals

   (i)
$$\int \frac{\cos\sqrt{x}}{\sqrt{x}\sin\sqrt{x}}dx.$$

   (ii)
$$\int \frac{dx}{(x+3)(x^2+2x+1)}.$$

(iii)
$$\int \frac{dx}{\sqrt{4x - x^2}}$$

(iv)
$$\int x \ln(x+1)dx$$

(v)
$$\int \frac{e^x dx}{\sqrt{e^{2x} - 4}}$$

(vi)
$$\int \sin(2x)\sin(6x)dx$$

9. (10 pts) The graph of $y = x^3$ on $0 \leq x \leq 1$ is revolved about the y-axis to form a tank that is then filled with salt water from the Dead Sea (density is approximately $11400N/m^3$), where both x and y are measured in meters. Set up an integral for the work needed to pump all the water to the top of the tank. **Do not spend your time computing the integral!**

10. (12 points) Determine the convergence of each of the following improper integrals (**you are not required to find its value if the integral is convergent**).

(i)
$$\int_1^\infty \frac{2 + \sin x}{x} dx.$$

(ii)
$$\int_0^1 \frac{1}{\sin(\sqrt{x})} dx.$$

11. (12 pts) Solve the following differential equations

(a) $\dfrac{dy}{dx} = xe^{x^2 - y}$;

(b) $x\dfrac{dy}{dx} - 2y = x^3, \quad x > 0, \quad y(1) = 2.$

12. (10 points) A tank initially contains 200 L of fresh water. A solution containing 0.08 kg/L of soluble lawn fertilizer runs into the tank at the rate of 3 L/min, and the well-mixed mixture is pumped out of the tank at the same rate. **Derive** a differential equation for the amount $F(t)$ of fertilizer inside the tank at time t, and specify the initial condition $F(0)$. **DO NOT SPEND TIME SOLVING the DIFFERENTIAL EQUATION**!

13. (16 points) Consider the autonomous differential equation

$$\frac{dP}{dt} = 3P(9 - P)(P - 0.07),$$

where $P(t)$, measured in thousands, is the number of graduate students at CUHKSZ at time t.

(i) Find all equilibrium solutions of the differential equation.

(ii) Perform phase-line analysis to the equation.

(iii) We know that currently CUHKSZ has 4.1 thousands graduate students. Use the result from (ii) to predict the long-time population of graduate students on the campus, i.e., predict $\lim_{t\to\infty} P(t)$.

(iv) What would be the long-time population of graduate students if the university started off with only 60 graduate students?