MAT3040 Final Exam Paper 2024

Student Name: ____________ Student ID: ____________

No book, note, calculator or dictionary allowed. Show your steps or reasoning in detail. Please write down your solution on the answer paper. The total score is 40 out of 70 points.

Q1 [**25 points**] Let V be a finite dimensional vector space over a field $\mathbb{F}$, e.g., $\mathbb{R}$ or $\mathbb{C}$, and $T: V \to V$ be a $\mathbb{F}$-linear map.

(i) For any $v \in V$, show that the T-cyclic subset $\langle v \rangle_T := \{h(T)v : h(x) \in \mathbb{F}[x]\}$ is a T-invariant $\mathbb{F}$-subspace.

(ii) For a given $0 \neq v \in V$ and a polynomial $g(x) \in \mathbb{F}[x]$, show that

$$\langle v \rangle_T = \langle g(T)v \rangle_T \text{ if and only if g.c.d}\,(g(x),\ m_{T,v}(x)) = 1.$$

Here $m_{T,v}(x)$ is the associated minimal polynomial of v w.r.t. T.

(iii) For any $0 \neq v \in V$, show that $\dim_{\mathbb{F}}(\langle v \rangle_T) = \deg(m_{T,v}(x))$.

(iv) Given an $\mathbb{F}$-subspace $W \subset V$, show that the following quotient map

$V/W \xrightarrow{\bar{T}} V/W : \ \bar{v} := v + W \mapsto \overline{Tv}$ is well-defined iff W is T-invariant.

(v) As in (iv), show that those minimal polynomials satisfy $m_{\bar{T}}(x) | m_T(x)$.

(vi) Show that the set $D := \{\deg(m_{T,v}(x)) : \ 0 \neq v \in V\}$ is bounded in $\mathbb{Z}$.

(vii) Let $v_1 \in V$ such that $\deg(m_{T,v_1}(x)) = \max\{d : \ d \in D\}$, show that

$$m_{T,v_1}(x) = m_T(x).$$

(viii) For $V/\langle v_1 \rangle_T \xrightarrow{\bar{T}_1} V/\langle v_1 \rangle_T : \ \bar{v} \mapsto \overline{Tv}$,

show that there exists $v_2 \in V$ such that $m_{T,v_2}(x) = m_{\bar{T}_1,\bar{v}_2}(x) = m_{\bar{T}_1}(x)$.

(ix) As in (viii), set $V_1 := \langle v_1 \rangle_T + \langle v_2 \rangle_T$. Consider $V/V_1 \xrightarrow{\bar{T}_2} V/V_1 : \ \bar{v} \mapsto \overline{Tv}$,

show that there exists $v_3 \in V$ such that $m_{T,v_3}(x) = m_{\bar{T}_2,\bar{v}_3}(x) = m_{\bar{T}_2}(x)$.

(x) As above, show that $\langle v_1 \rangle_T + \langle v_2 \rangle_T + \langle v_3 \rangle_T = \langle v_1 \rangle_T \oplus \langle v_2 \rangle_T \oplus \langle v_3 \rangle_T$.

Q2 [**24 points**] Let V be a finite dimensional vector space over $\mathbb{C}$, and $T: V \to V$ be a $\mathbb{C}$-linear map.

(i) State a necessary and sufficient condition for T to be diagonalizable.

(ii) If $T^m = T$ for some integer $m > 1$, show that T is diagonalizable.

(iii) State a necessary and sufficient condition for T to be nilpotent.

(iv) If $T^4 = 0$ and $\dim_{\mathbb{C}}(V) = 5$, classify the equivalence classes of T under similarity.

(v) If $\dim_{\mathbb{C}}(V) = 9$ and the eigenvalues of T are $\{1, 1, 1, 0, 0, 0, 0, -1, -1\}$, determine the list of eigenvalues of T^2.

(vi) As in (v), determine all possible minimal polynomials of T^2.

(vii) Set $\mathrm{End}_{\mathbb{C}}(V) := \mathrm{Hom}_{\mathbb{C}}(V, V)$. For any subset $\Sigma \subset \mathrm{End}_{\mathbb{C}}(V)$, set $C_{\mathrm{End}_{\mathbb{C}}(V)}(\Sigma) := \{S \in \mathrm{End}_{\mathbb{C}}(V) : S \circ T = T \circ S, \ \forall T \in \Sigma\}$, show that the following two statements are equivalent.

  (a) $C_{\mathrm{End}_{\mathbb{C}}(V)}(T) = \{f(T) : f(x) \in \mathbb{C}[x]\}$.

  (b) the geometric multiplicity $m_g(\lambda) = 1$ for any eigenvalue λ of T.

(viii) Show that $C_{\mathrm{End}_{\mathbb{C}}(V)}\left(C_{\mathrm{End}_{\mathbb{C}}(V)}(T)\right) = \{f(T) : f(x) \in \mathbb{C}[x]\}$.

Q3 [**17 points**] Let $V := \mathrm{Span}_{\mathbb{R}}\left\{\prod_{i=1}^{4} x_i^{a_i} : \sum_{i=1}^{n} a_i = 2\right\} \subset \mathbb{R}[x_1, \cdots, x_4]$ be the $\mathbb{R}$-subspace of degree 2 homogeneous polynomials. Consider the bilinear form $V \times V \xrightarrow{(\cdot,\cdot)} \mathbb{R} : \ (f(x_1, \cdots, x_4),\ g(x_1, \cdots, x_4)) \mapsto f\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_4}\right) g(x_1, \cdots, x_4)$.

(i) Show that the bilinear form is an inner product.

(ii) Find an basis $\mathcal{B} := \{e_1, \cdots, e_{10}\}$ of V and write down the matrix representation $M_{\mathcal{B}}$ of this inner product under $\mathcal{B}$.

(iii) Given a set of basis $\mathcal{C} := \{v_1, \cdots, v_{10}\}$ of V, show that the Gram–Schmidt process gives rise to an orthogonal basis $\{w_1, \cdots, w_{10}\}$ of this inner product. Here $w_i := v_i - \mathrm{Proj}_{V_{i-1}}(v_i)$, with $V_{i-1} := \mathrm{Span}_{\mathbb{R}}\{v_1, \cdots v_{i-1}\}$.

(iv) Write down the change of basis formula for the matrix representation $M_{\mathcal{B}}$ under basis $\mathcal{B}$ in terms of $M_{\mathcal{C}}$ under $\mathcal{C}$, and state a definition of isomorphism of inner product structures on $V \times V$.

(v) Classify the equivalence classes of inner product structures under the isomorphism in (iv).

(vi) As in (iii), let $M_{\mathcal{C}} := (a_{k,t})_{1 \le k,t \le 10}$ and $M_{\mathcal{C},i} := (a_{k,t})_{1 \le k,t \le i}$. Show that, for any $i > 1$, $(w_i, w_i) = \det(M_{\mathcal{C},i}) \cdot \det(M_{\mathcal{C},i-1})^{-1}$.

Q4 [**4 points**]

(i) Please write down the distribution of grades for this course based on your observation of your classmates.

(ii) Please write down your expected grade with some justified reasons.