MAT2060 2024 Midterm

June 3, 2025

1. (10 pts) Suppose that $\{A_n\}_{n=1,2,\ldots}$ are subsets of a metric space S. Prove or disprove:
   (i) (5 pts) $\overline{A_1} \cap \overline{A_2} = \overline{A_1 \cap A_2}$.
   (ii) (5 pts) $\bigcup_{n=1}^{\infty} \overline{A_n} = \overline{\bigcup_{n=1}^{\infty} A_n}$.

2. (10 pts) Define
$$f(x) = \begin{cases} 0, & x = 0 \text{ or } x \notin \mathbb{Q}, \\ \frac{1}{|q|}, & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ with } \gcd(p, q) = 1. \end{cases}$$
   Determine the set of all points in $\mathbb{R}$ at which the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. (Justify your answer with an ϵ-δ proof.)

3. (10 pts) Suppose that f is a uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$m. Prove or disprove:
$$\lim_{x\to\infty} \frac{f(x)}{x^2} = 0$$

4. (10 pts) Suppose that A is a subset of $\mathbb{R}^2$ with smooth boundary ∂A. For $x \in A$ we set $d(x) = d(x, \partial A)$. Prove or disprove:
   (i) (5 pts) $d(x)$ is continuous in A.
   (ii) (5 pts) $d(x)$ is differentiable in A.

5. (15 pts) Let A and B be two disjoint closed subsets in a metric space S
   (i) (5 pts) Find a continuous function $f\colon S \to \mathbb{R}$ such that $f \equiv 0$ on A and $f \equiv 1$ on B.
   (ii) (10 pts) Prove or disprove: There are two disjoint open subsets V and W of S such that $A \subseteq V$ and $B \subseteq W$.

6. (10 pts) Every compact metric space have a countable dense subset

7. (15 pts) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate-value property; i.e., if $f(a) < \alpha < f(b)$, then $\exists c \in (a, b)$ such that $f(c) = \alpha$.

(i) (5 pts) Is f necessarily continuous?

(ii) (10 pts) If, in addition, $f^{-1}(r)$ is closed for every $r \in \mathbb{Q}$, then is f necessarily continuous?

8. (10 pts) Show that a metric space is sequential compact if and only if it is complete and totally bounded

9. (10 pts) Prove that two continuous functions on $[0, 1]$ are identical if they have the same sequence of moments. (The moments of a function f on $[0, 1]$ are the numbers $\int_0^1 x^n f(x)dx, n = 0, 1, 2, 3, \cdots$)
Hint:Stone-Weirstrass