Final Examination of MAT 2060

Question 1: True or False

(1) If f_x and f_y exist at P_0, then $D_{\vec{v}}f$ exists at P_0 and equals $\nabla f \cdot \vec{v}$.

(2) Let Ω be a bounded and open set in $\mathbb{R}^n$ and let $u \in \mathcal{C}^2(\Omega)$. If u has a minimum at $x_0 \in \Omega$, then $\partial_i u(x_0) = 0$ and $\partial_{ii}^2 u(x_0) \geq 0$.

(3) If f_x and f_y exist, then f is differentiable.

(4) If f has second order derivatives at P_0, then $f_{xy} = f_{yx}$ at P_0.

(5) Suppose S is a figure in $\mathbb{R}^n$. If f is integrable on S, then f is bounded.

(6) Let $\Omega \subseteq \mathbb{R}^2$ be smooth. Let $L = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ where $(x, y) \in D$ and $D := \Omega \backslash \{(0, 0)\}$. Then the generalized Stokes' Theorem applies and implies:

$$\int_{\partial D} L = \int_D dL = 0$$

(7) If $f_n \rightrightarrows f$ uniformly on $(0, 1)$, then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

(8) Give a sufficient condition for the existence of $y' = y'(x)$ in the Implicit Function Theorem (IFT).

(9) Determine whether the functions below are Riemann integrable:

- (a) Dirichlet function
- (b) Riemann function
- (c) Increasing function on $[\alpha, \beta]$
- (d) Continuous functions except for finite points on $[\alpha, \beta]$

(10) Determine whether the sets below are of the 2nd category:

- (a) $\mathcal{N}$(the set of all nowhere differentiable functions)
- (b) $\mathbb{Q}$
- (c) $D(f)$ (The set of discontinuities of a function f)

Question 2

Prove or disprove: Suppose f, g are continuous on $(0, \infty)$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. Then $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0$.

Question 3

Prove or disprove: If f is a continuous monotone function on $\mathbb{R}$, then f is differentiable.

Question 4

State the Arzela-Ascoli Theorem and explain the concepts in the theorem.

Question 5

Consider the equation:

$$x^2 + y + \sin(xy) = 0 \qquad (*)$$

(i) Verify that in the neighborhood near $(0,0)$, $y = y(x)$ is uniquely determined and $y(x)$ is continuous in the neighborhood near $(0,0)$ and $y(0) = 0$.

(ii) Determine the differentiability of $y(x)$ near $x = 0$.

(iii) Determine the monotonicity of $y(x)$ near $x = 0$.

(iv) In the sufficiently small neighborhood near $(0,0)$, can (*) uniquely determine $x = x(y)$? Why?

Question 6

Let f and ψ be smooth and define:

$$u(t,x) = \int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds$$

(i) Find $\partial_t u$ and $\partial_x u$.

(ii) Find $\partial_{tt}^2 u$ and $\partial_{xx}^2 u$.

Question 7

Let $I := \{x \mid x \in [a,b]\}$ and $S = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b], y \in [a,b]\}$. Suppose $K : S \to \mathbb{R}$ is continuous on S. Define $\mathcal{G} := \{\text{all functions uniformly bounded on } I\}$.

(i) State the definition of uniformly bounded.

(ii) Prove or disprove: $\forall g \in \mathcal{G}$, $T[g] := \int_a^b K(x,y)g(y)dy$ is well-defined and continuous on I.

(iii) Prove or disprove: $\forall$ well-defined $\{f_n\} \subseteq \mathcal{F} := \{T[g] \mid g \in \mathcal{G}\}$, there exists a uniformly convergent subsequence of $\{f_n\}$.

Question 8

Suppose $\Omega \subset \mathbb{R}^3$ is bounded and open, and that $\partial\Omega$ is smooth and $\vec{0} \in \Omega$.

(i) Let $u, v \in \mathcal{C}^2(\bar{\Omega})$. Then prove that:

$$\iiint_\Omega (v\Delta u - u\Delta v)dV = \iint_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) dS$$

(ii) Set $r := |\vec{x}|$, the radial variable. Compute $\nabla(r^{-1})$ and show that $\Delta(\frac{1}{r}) = 0$ for all $\vec{x} \neq \vec{0}$.

(iii) Let u be a function defined on $\mathbb{R}^3$ whose support is compact in $\mathbb{R}^3$. Then show that:

$$\iiint_{\mathbb{R}^3} \frac{1}{r}\Delta u dV \quad \text{is well-defined and equals } -4\pi u(\vec{0}).$$