MAT1002 Midterm Examination

Saturday, March 23, 2019.

Time: 7:00 - 9:00 PM

Notes and Instructions

1. The total score of this examination is 100.

*2. There are **eight** questions (with parts) in total.*

*3. Answer all questions on the **answer book**.*

4. Show intermediate steps of your solution.

5. No book, calculator or dictionary is allowed.

6. One sheet of double-sided A4 handwritten note is allowed. No Scanned note is allowed.

Midterm Examination Questions

Question 1 (6+6+6=18 Points)

For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{5^n n!}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(1 - (-1)^n \frac{1}{2}\right)^{n/3} \sin \frac{n\pi}{2}$.

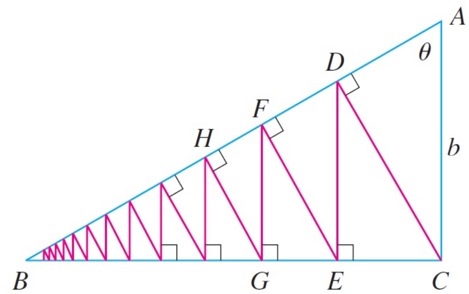
(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(1 + \frac{1}{n}\right)$.

Question 2 (6 Points)

As shown in the following figure, a right-angle triangle ABC is given with $\angle A = \theta$ (where $0 < \theta < \pi/2$), $|AC| = b$ and $AC \perp BC$. The line segment CD is drawn perpendicular to AB, then DE is drawn perpendicular to BC, and then $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Use series to find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \dots$$

in terms of b and θ. Simplify your answer (it should not contain the symbol $\sum$).



Question 3 (5+5=10 Points)

For any $m \in \mathbb{R}$ and $n \in \mathbb{N}$, the *binomial coefficient* $\binom{m}{n}$ is defined by

$$\binom{m}{0} := 1, \ \binom{m}{n} := \frac{m(m-1)(m-2)\dots(m-n+1)}{n!} \text{ if } n \geq 1.$$

You may use the following fact: for any $x \in (-1, 1)$,

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n.$$

The series on the right is the Maclaurin series of the function $f(x) := (1+x)^m$.

(a) Find the Maclaurin series for $1/\sqrt{1-x^2}$.

(b) Use part (a) to find the Maclaurin series for $\arcsin x$. (*Hint:* $\arcsin' x = 1/\sqrt{1-x^2}$.)

Question 4 (5+5+5=15 Points)

Consider the plane L with equation $3x + 3y - 2z = 5$.

(a) Find the constant c such that the vector $2c\mathbf{j} - \mathbf{k}$ lies on the plane L.

(b) Locate the projection of the origin onto the plane L.

(c) Find the equation of the plane that is perpendicular to the vector in your answer in (a) and contains the point in (b).

Question 5 (5+5+5=15 Points)

Let a and b be positive constants. A particle P moves on the cone

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2,\ z \geq 0\},$$

with the x-coordinate equal to $e^{-at}\cos(bt)$ and the y-coordinate equal to $e^{-at}\sin(bt)$, for $t \geq 0$.

(a) Find the z-coordinate of the particle as a function of the parameter t.

(b) What is the length of the path travelled by the particle for t from 0 to ∞?

(c) Re-parameterize the path in part (b) using the arc length as the parameter and the point $(1, 0, 1)$ as the base point.

Question 6 (5+5=10 Points)

A curve C is given by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

for $t \in \mathbb{R}$. Compute:

(a) The unit tangent vector $\mathbf{T}(t)$.

(b) The principal unit normal vector $\mathbf{N}(t)$.

Question 7 (6+5=11 Points)

Consider the function

$$g(x, y) := x^2y^3 + y + 2e^x.$$

(a) Compute $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$.

(b) Suppose that x and y are functions of t, where

$$x(t) = t^2 \quad \text{and} \quad y(t) = \cos(t).$$

Evaluate the first derivative of $g\big(x(t), y(t)\big)$ at $t = \pi$.

Question 8 (5+5+5=15 Points)

Consider the function

$$f(x,y) := \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0); \end{cases}$$

defined on $\mathbb{R}^2$.

(a) Is f continuous at $(0,0)$? Justify your answer.

(b) Compute the partial derivative $f_y(0,0)$ if it exists, or explain why it does not exist.

(c) Is f_y continuous at $(0,0)$? Justify your answer.