MAT2060 2022 Midterm

June 3, 2025

1. (10 pts) Prove or disprove: A bounded continuous function on $(0, 1]$ can be extended continuously to $\mathbb{R}$.

2. (20 pts) Show that a metric space is compact if and only if it is complete and totally bounded.

3. (10 pts) Let f_n be a monotone sequence of continuous functions on a compact metric space S. Suppose that f_n converges pointwise to a continuous function f on S, prove or disprove that the convergence is uniform.

4. (10 pts) Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is continuous on an interval I, then the set $\{(x, y) : x \in I, y = f(x)\}$ is a connected subset of $\mathbb{R}^2$.

5. (10 pts) Prove that two continuous functions on $[0, 1]$ are identical if they have the same sequence of moments. (The moments of a function f on $[0, 1]$ are the numbers $\int_0^1 x^n f(x)dx$, $n = 0, 1, 2 \ldots$.)
   Hint: Use the Stone-Weirstrass theorem

6. (10 pts) Prove or disprove: $\mathbb{R} \setminus \mathbb{Q}$ can be written as a countable union of closed sets.
   Hint: Use the Baire Category theorem

7. (30 pts) The distance between two disjoint sets A and B in a complete metric space S is defined to be
   $$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$
   (i) (10 pts) If both A and B are compact, then $d(A, B) > 0$.
   (ii) (10 pts) If both A and B are closed, then $d(A, B) > 0$.
   (iii) (10 pts) If A is compact and B is closed, then $d(A, B) > 0$.