MAT4002 Midterm Examination (120 min)

Name: __________ ID: __________ Date: 2024-Spring

1 (20 points) Let $X = \mathbb{R}$ and let $\mathcal{T}_1$ be the usual topology on X.

1. (10 points)Let $\mathcal{T}_2$ be the topology generated by

$$\{(a, b] : a, b \in \mathbb{R}\}.$$

Show that $f : (\mathbb{R}, \mathcal{T}_2) \to (\mathbb{R}, \mathcal{T}_1)$ given by

$$f(x) = \begin{cases} x - 1, & \text{for } x \geq 0 \\ x + 1, & \text{for } x < 0 \end{cases}$$

is continuous.

2. (10 points)Let $\mathcal{T}_3$ be the topology with basis

$$\mathcal{T}_3 = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\varnothing, X\}.$$

Show that $f : (\mathbb{R}, \mathcal{T}_3) \to (\mathbb{R}, \mathcal{T}_1)$ is continuous if and only if f is constant.

2 (20 points) Let $C_0 = [0,1]$ with the usual topology. Set

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

by removing middle interval $(\frac{1}{3}, \frac{2}{3})$. Successively set

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

and so on to define C_n from C_{n-1} by removing the open middle third of each subinterval. Define the Cantor set C to be be following subset of $\mathbb{R}$ with the subspace topology:

$$C = \bigcap_{n=1}^{\infty} C_n.$$

ⓐ (5 points) Let $X = \{0,1\} \subset \mathbb{R}$ with discrete topology. Set $Y = \prod_{N=1}^{\infty} X$, the infinite product of X with product topology. Show that C is homeomorphic to Y.

Hint: Show that every element of C can be written as $\sum_{n=1}^{\infty} \frac{X_n}{3^n}$ for $X_n \in \{0,2\}$.

ⓑ (5 points) Show that C is homeomorphic to $\prod_{N=1}^{\infty} C$.

ⓒ (5 points) Show that there exists a continuous surjection $f : C \to \prod_{N=1}^{\infty} [0,1]$.

Hint: Map $\sum_{n=1}^{\infty} \frac{X_n}{3^n}$ to $\sum_{n=1}^{\infty} \frac{X_n}{2^{n+1}}$.

ⓓ (5 points) Let Z be a compact metrizable topological space. Show that there exists $f : Z \to \prod_{N=1}^{\infty} [0,1]$ homeomorphic to $f(Z)$, hence every such Z is a quotient of C.

Hint: In lecture we showed metric space has a countable base.

3 (15 points) Let X be a topological space such that there exists collection of continuous maps $f_n : X \to [0,1]$, for $n \in \mathbb{N}$, with the following property: for all $x \in X$ and all open U such that $x \in U$ there exists $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n(X \setminus U) = 0$.

ⓐ (10 points) Define $\overline{f} : X \to \prod_{n=1}^{\infty} \mathbb{R}$ by $\overline{f}(x) = \big(f_1(x), f_2(x), \dots\big)$, for $x \in X$. Suppose $\prod_{n=1}^{\infty} \mathbb{R}$ is given the product topology of usual topology of $\mathbb{R}$. Show that $\overline{f}$ is a homeomorphism onto its image $\overline{f}(X) \subseteq \prod_{n=1}^{\infty} \mathbb{R}$.

ⓑ (5 points) Use ⓐ to show that X is metrizable.

Hint: Show $\prod_{n=1}^{\infty} \mathbb{R}$ *is metrizable.*

4 (15 points) Let X be a topologoical space and let $\sim$ be an equivalence relation on X, and let $\mathcal{T}$ be a topology on the set $X/\sim$ which satisfies the following property :

- The canonical surjection $\pi : X \to (X/{\sim})$ is continuous (with respect to $\mathcal{T}$). Moreover, for any topological space Y and any continuous map $f : X \to Y$ such that $f(x) = f(x')$ whenever $x \sim x'$ in X, there exists a unique continuous map $\overline{f} : (X/{\sim}) \to Y$ such that $f = \overline{f} \circ \pi$.

Then $\mathcal{T}$ is the quotient topology on $X/\sim$.

5 (15 points) Let $f : X \to Y$ be a continuous map with X and Y Hausdorff, and suppose that every closed ball of Y is compact and Y is metrizable. Show that the following are equivalent:

ⓐ If $V \subseteq Y$ is compact then $f^{-1}(V)$ is compact. We call such f satisfying the property proper map.

ⓑ For any topological space Z, the map $f \times Id_Z : X \times Z \to Y \times Z$ sends closed subsets to closed subsets.

ⓒ $f^{-1}(y)$ is compact for every $y \in Y$ and f maps closed subsets to closed subsets.

Hint: For ⓑ $\Rightarrow$ ⓒ, you can use the fact that a topological space C is compact iff for any topological space Z, the projection $C \times Z \to Z$ is closed. Extra 5 points if you prove this

6 (15 points) Circle your answer for the following multiple choice questions:

ⓐ $(\mathbb{R} \times \mathbb{R}, \mathcal{T}_1 \times \mathcal{T}_2)$ is metrizable, where $\mathcal{T}_1$ is the indiscrete topology and $\mathcal{T}_2$ is the infinite topology.

True False

ⓑ $(\mathbb{R} \times \mathbb{R}, \mathcal{T}_3 \times \mathcal{T}_4)$ is not metrizable, where $\mathcal{T}_3$ is the cofinite topology and $\mathcal{T}_4$ is the topology induced by Euclidean distance .

True False

ⓒ $(C(\mathbb{R}), \mathcal{T}_5)$ is not metrizable, where $\mathcal{T}_5$ is the pointwise convergence topology on the space $C(\mathbb{R})$ of continuous functions $\mathbb{R} \to \mathbb{R}$.

True False

ⓓ Every Hausdorff topological space is metrizable.

True False

ⓔ The countable product of path-connected spaces is path-connected.

True False

ⓕ A quotient space of a compact, Hausdorff space is again compact and Hausdorff.

True False

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Then $\mathcal{T}$ is the quotient topology on $X/\sim$.