Taylor Expansions and Taylor Series

1. Taylor Expansion with Peano Remainder

1. Expand $\sqrt[3]{2-cos(x)}$ till x^5
2. Expand e^{cos^2x} till x^5
3. Expand $tanx$ till x^7
4. Expand $ln(cosx)$ till x^7
5. Expand $sec(x)$ till x^6
6. Expand $sech(x) = \dfrac{2}{e^{-x}+e^{x}}$ till x^5
7. Consider a function

$$f(x) = \begin{cases} ln(\dfrac{sinx}{x}) & \text{if } x > 0 \\ b & \text{if } x = 0 \end{cases}$$

(1) Decide the constant b to make the function continuous at $x = 0$

(2) Expand $f(x)$ at $x = 0$ till x^6

8. Consider a function

$$f(x) = \begin{cases} (1+x)^{\frac{1}{x}} & \text{if } x > 0 \\ c & \text{if } x = 0 \end{cases}$$

(1) Decide the constant c to make the function continuous at $x = 0$

(2) Expand $f(x)$ till x^3

(3) What is the value of $f^{(n)}(0), n = 1, 2, 3$ (The Maclaurin Expansion is Unique, hence we can calculate the value of the derivatives)

Remark: The difference between "Analytic","C^∞"
Analytic: f can be expressed as $\Sigma a_n(x-x_0)^n$ in a neighborhood of x_0.

C^∞: f has derivatives of any order at x_0

Analytic implies C^∞, and we have $a_n = \dfrac{f^{(n)(x_0)}}{n!}$
while C^∞ does not imply Analytic.

2. Application of Taylor Expansion with Peano Remainder

1. Decide a,b, such that $f(x)$ is of higher order as $x \to 0$
(1) $f(x) = x - (a + bcosx)sinx$
(2) $f(x) = e^x - \dfrac{1+ax}{1+bx}$
(3) $f(x) = cotx - \dfrac{1+ax^2}{x+bx^3}$

(4) $f(x) = cosx - \dfrac{1+ax^2}{1+bx^2}$

2. Calculate the following limit

(1)$\lim\limits_{x\to 0} \dfrac{e^{x^2}-1-x^3}{sin^6 2x}$

(2) $\lim\limits_{n\to\infty} n^2 ln(nsin(\dfrac{1}{n}))$

(3) $\lim\limits_{n\to\infty} (-1)^n nsin(\sqrt{n^2+2}\pi)$

(4) $\lim\limits_{x\to 0} \dfrac{sin(tanx)-tan(sinx)}{x^7}$

(5) $\lim\limits_{x\to 0} \dfrac{xsin(x)-sin^2 x}{x^6}$

(6) $\lim\limits_{x\to 0} \dfrac{a^{-x}+a^{-x}-2}{x^2}$,where $a > 0$.

(7) $\lim\limits_{x\to 0} x^{-3}((\dfrac{2+cosx}{3})^x - 1)$

(8) $\lim\limits_{x\to\infty} x - x^2 ln(1+\dfrac{1}{x})$

(9) $\lim\limits_{x\to 0} \dfrac{sinx - arctanx}{tanx - sinx}$

(10) $\lim\limits_{n\to\infty} n(e-(1+\dfrac{1}{n})^n)$

(11) $\lim\limits_{n\to\infty} (\dfrac{\sqrt[n]{a}+\sqrt[n]{b}}{2})^n$

(12) $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{\sqrt{1-x}-cosx}$

(13)$\lim\limits_{x\to 0} \dfrac{1-(cosx)^{sinx}}{x^3}$

(14)$\lim\limits_{x\to 0} \dfrac{cos(sinx)-cosx}{x^4}$

More challenging Problems

3. $\lim\limits_{n\to\infty} \prod_{k=1}^{n}(1+\dfrac{k}{n^2})$

4. $\lim\limits_{n\to\infty} \prod_{k=1}^{n}(1+\dfrac{k}{n^{\alpha+2}})^{n^\alpha}$, where $\alpha > -1$

5. $\lim\limits_{n\to\infty} nsin(2\pi n!e)$

6. $\lim\limits_{n\to\infty} \prod_{k=1}^{n} cos(\dfrac{k}{n\sqrt{n}})$

7. Suppose $f \in C[0,1]$ satisfies $f(0)=0, f'(0)=1$, find the following limit

$$\lim_{n\to\infty}[f(\frac{1}{n^2})+f(\frac{2}{n^2})+\cdots+f(\frac{n}{n^2})]$$

8. Determine the range of x such that

$$\Sigma_{n=1}^{\infty}(2-x)(2-x^{\frac{1}{2}})\cdots(2-x^{\frac{1}{n}})$$

converges.

9. Let

$$f(x)=\lim_{n\to\infty} n^x[(1+\frac{1}{n+1})^{n+1}-(1+\frac{1}{n})^n]$$

Determine the range of x so that f can be defined as a real-valued function , and then determine the range of $f(x)$.

10. Consider

$$A_n = \sum_{k=1}^{n} \frac{1}{n+k}$$

(1) Find $\lim\limits_{n\to\infty} A_n$

(2) Determine α, such that $\lim\limits_{n\to\infty} n^{\alpha}(\ln 2 - A_n) = L$ is finite, and calculate L

3. Taylor Series

1. Expand the following functions as series at x=0 and figure out the range of x where the formula holds.

   (1) $arctanx$

   (2) $arcsinx$

   (3) $\dfrac{1}{1+x+x^2}$

   (4) $\sqrt{4-x}$

   (5) $a^x, a > 0$

   (6) $ln(1+3x+2x^2)$

   (7)$\arctan(\dfrac{2x}{1-x^2})$

   (8) $\dfrac{1}{1+x+x^2+x^3+x^4}$

   (9) $\dfrac{d}{dx}(\dfrac{e^x-1}{x})$

   (10)$\displaystyle\int_0^x \frac{1-cost}{t^2}dt$

2. Expand the function at given point

   (1) $\dfrac{1}{3+x}, x = 2$

   (2) $ln(\dfrac{1}{3+2x+x^2}), x = 1$

   (3) $sinx, x = \dfrac{\pi}{6}$

3. Expand $f(x) = lnx$ as the power series of $\dfrac{x-1}{x+1}$