1.1

$$P(w_m, w_n) = k(w_m - w_n)^2 \quad k>0$$

$$P(\beta,\gamma) = k(\beta-\gamma)^2$$
$$P(\alpha,\delta) = k(\alpha-\delta)^2$$
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To meet the requirement to be submodular:

$$P(\beta,\gamma) + P(\alpha,\delta) - P(\beta,\delta) - P(\alpha,\gamma) \geq 0$$

applying the parameters:

$$\Rightarrow k(\beta-\gamma)^2 + k(\alpha-\delta)^2 - k(\beta-\delta)^2 - k.(\alpha-\gamma)^2 \geq 0$$

$$= k(\cancel{\beta^2} - 2\beta\gamma + \cancel{\gamma^2} + \cancel{\alpha^2} - 2\alpha\delta + \cancel{\delta^2} - \cancel{\beta^2} + 2\beta\delta - \cancel{\delta^2} - \cancel{\alpha^2} + 2\alpha\gamma - \cancel{\gamma^2}) \geq 0$$

$$= k(-2\beta\gamma - 2\alpha\delta + 2\beta\delta + 2\alpha\gamma) \geq 0$$

$$= k(2\beta(\delta-\gamma) + 2\alpha(\gamma-\delta)) \geq 0$$

We know that $\beta > \alpha$ and $\delta > \gamma$, also $k>0$

so, $(\delta-\gamma) > 0$ and $(\gamma-\delta) < 0$ always

Because $\beta > \alpha$ and the multiplier of positive part $(\delta-\gamma)$ is β and multiplier of negative part, $(\gamma-\delta)$ is α, the sum will be positive always.

k is also positive so function is submodular.

1.2 $\rho(w_m, w_n) = k(1 - \delta(w_m - w_n))$

$$\rho(\beta, \gamma) = k(1 - \delta(\beta - \gamma))$$
$$\rho(\alpha, \delta) = k(1 - \delta(\alpha - \delta))$$
$$\rho(\beta, \delta) = k(1 - \delta(\beta - \delta))$$
$$\rho(\alpha, \gamma) = k(1 - \delta(\alpha - \gamma))$$

$$\rho(\beta, \gamma) + \rho(\alpha, \delta) + \rho(\beta, \delta) + \rho(\alpha, \gamma) \geq 0$$

expanding the left part,

$$k(1 - \delta(\beta - \gamma) + k(1 - \delta(\alpha - \delta)) - k(1 - \delta(\beta - \delta)) - k(1 - \delta(\alpha - \gamma))$$

$$= k(\cancel{1} - \delta(\beta - \gamma) + \cancel{1} - \delta(\alpha - \delta) - \cancel{1} + \delta(\beta - \delta) - \cancel{1} + \delta(\alpha - \gamma))$$

We know that $\beta > \alpha$, $\delta > \gamma$ and $k > 0$

$$= k(\delta(\beta - \delta) + \delta(\alpha - \gamma) - \delta(\beta - \gamma) - \delta(\alpha - \delta))$$

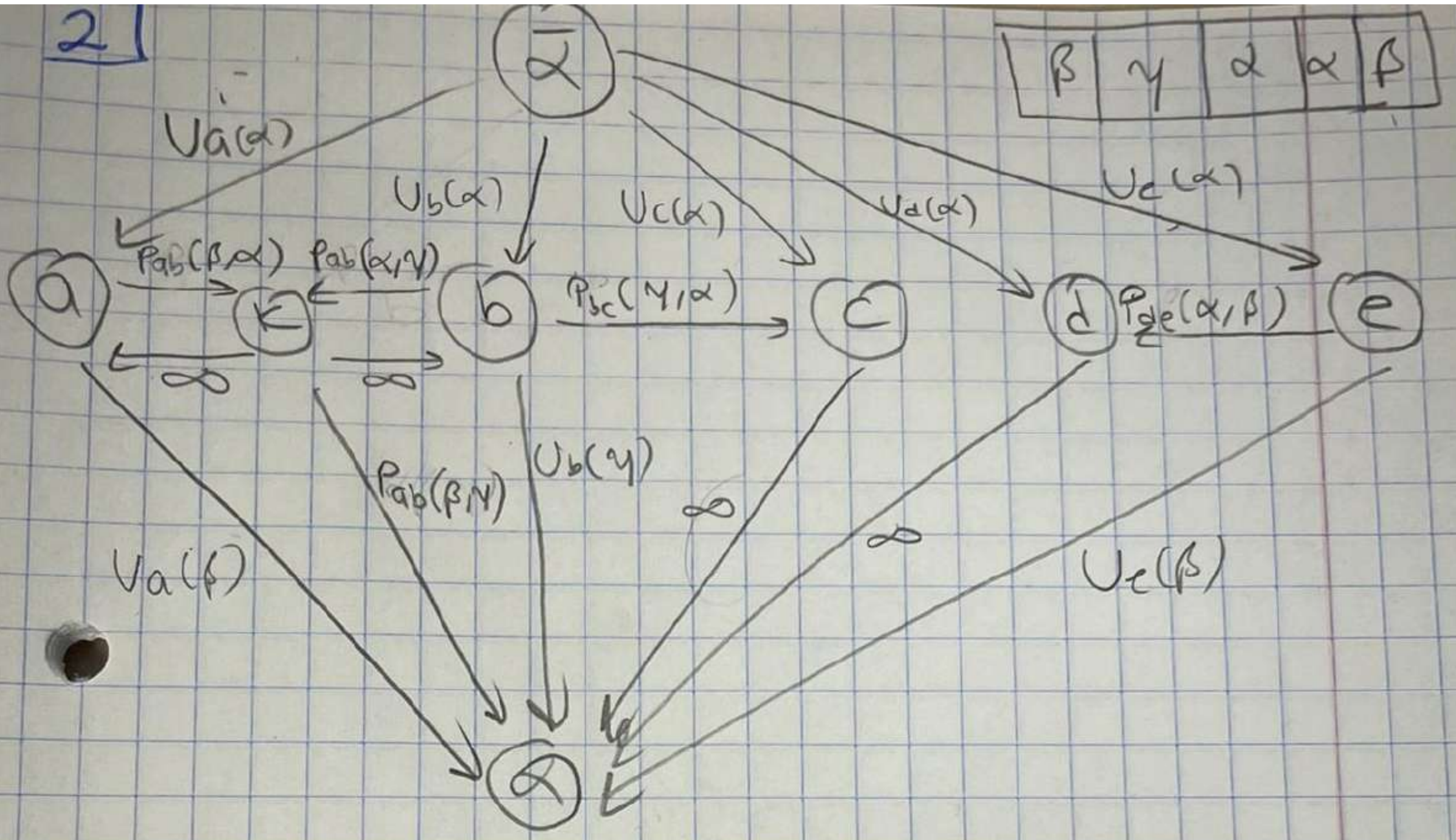
If $\alpha = \delta$, then $\beta > \alpha = \delta > \gamma$

then $\delta(\alpha - \delta) = 1$, rest of them will be 0

$$k(0 + 0 - 1 - 0) < 0$$

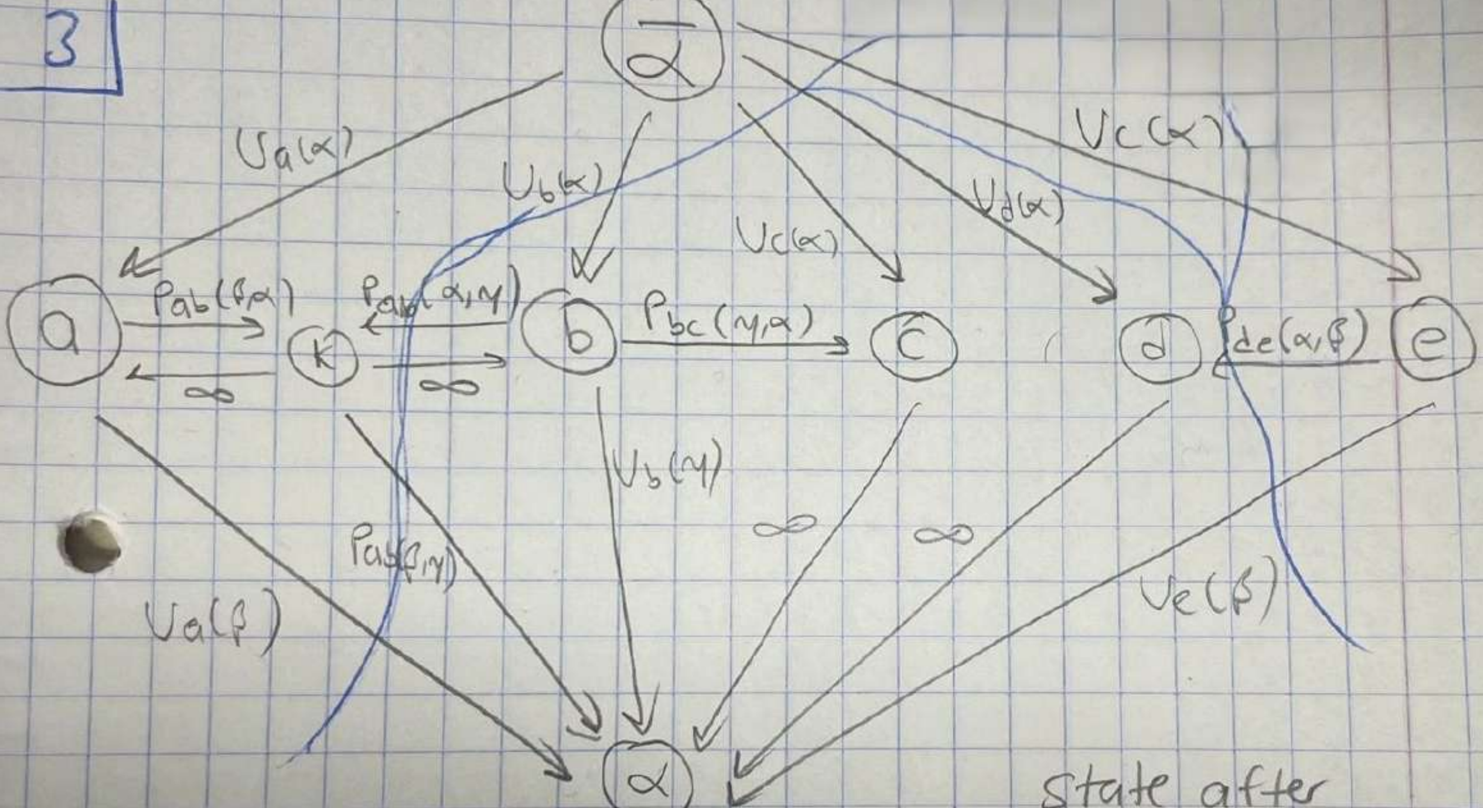
k is (+) number so the function doesn't satisfy the requirement. Thus, it's not submodular.

2



β γ α α β
$\bar{\alpha}$
$U_a(\alpha)$
$U_b(\alpha)$
$U_c(\alpha)$
$U_d(\alpha)$
$U_e(\alpha)$
a
k
b
c
d
e
$P_{ab}(\beta,\alpha)$
$P_{ab}(\alpha,\gamma)$
$P_{bc}(\gamma,\alpha)$
$P_{de}(\alpha,\beta)$
∞
∞
$P_{ab}(\beta,\gamma)$
$U_b(\gamma)$
∞
∞
$U_a(\beta)$
$U_e(\beta)$
α

3



state after

β	α	α	α	β

Cost:

$$U_a(\beta) + P_{ab}(\beta,\gamma) + U_b(\alpha) + U_c(\alpha) + U_d(\alpha) + P_{de}(\alpha,\beta) + U_e(\beta)$$