1

$$\mathrm{Norm}_x[a,A] = \frac{1}{(2\pi)^{\frac{k}{2}}|A|^{\frac{1}{2}}} \cdot \exp\left(\frac{-1}{2}(x-a)^T A^{-1}(x-a)\right)$$

$$\mathrm{Norm}_x[b,B] = \frac{1}{(2\pi)^{\frac{k}{2}} B^{\frac{1}{2}}} \cdot \exp\left(\frac{-1}{2}(x-b)^T B^{-1}(x-b)\right)$$

$$\mathrm{Norm}_x[a,A] \cdot \mathrm{Norm}_x[b,B] = \frac{1}{2\pi\sqrt{AB}} \exp\left(\frac{-1}{2}\left[(x-a)^T A^{-1}(x-a) + (x-b)^T B^{-1}(x-b)\right]\right)$$

$$(x-a)^T A^{-1}(x-a) = x^T A^{-1} x - 2a^T A^{-1} x + a^T A^{-1} a$$

$$(x-b)^T B^{-1}(x-b) = x^T B^{-1} x - 2b^T B^{-1} x + b^T B^{-1} b$$

Adding them;

$$x^T A^{-1} x - 2a^T A^{-1} x + a^T A^{-1} a + x^T B^{-1} x - 2b^T B^{-1} x + b^T B^{-1} b$$

grouping

$$\underbrace{x^T(A^{-1}+B^{-1})x - 2(a^T A^{-1} + b^T B^{-1})x} + (a^T A^{-1} a + b^T B^{-1} b)$$

$\cancel{= x^T(A^{-1}+B^{-1})}$

this part can be written as

$$(x - \Sigma^*(A^{-1}a + B^{-1}b))^T (A^{-1}+B^{-1})(x - \Sigma^*(A^{-1}a+B^{-1}b))$$

Remaining terms

$$a^T A^{-1} a + b^T B^{-1} b - (A^{-1}a + B^{-1}b)^T \Sigma^*(A^{-1}a + B^{-1}b)$$

$$\mathrm{Norm}_x[a,A] \cdot \mathrm{Norm}_x[b,B] = \frac{\exp\left(\frac{-1}{2}\left[a^T A^{-1} a + b^T B^{-1} b - (A^{-1}a + B^{-1}b)^T \Sigma^*(A^{-1}a + B^{-1}b)\right]\right)}{2\pi^k \cdot |A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} \cdot$$

$$\rightarrow \cdot \frac{1}{|\Sigma^*|} \exp\left(\frac{-1}{2}\left[x - \Sigma^*(A^{-1}a + B^{-1}b)\right]^T (\Sigma^*)^{-1}\left[x - \Sigma^*(A^{-1}a + B^{-1}b)\right]\right)$$

$$\mu^* = (A^{-1} + B^{-1})^{-1} \cdot (A^{-1}a + B^{-1}b)$$

then,

$$Norm_x[a, A]\ Norm_x[b, B] = C \cdot \exp\left(-\frac{1}{2}(x-\mu^*)^T \Sigma^{*-1}(x-\mu^*)\right)$$

where C is the term not dependent to x.

$$\int \exp\left(-\frac{1}{2}(x-\mu^*)^T \Sigma^{*-1}(x-\mu^*)\right) = (2\pi)^{\frac{k}{2}} |\Sigma^*|^{\frac{1}{2}} \cdot C$$

$$C = \frac{1}{(2\pi)^{\frac{k}{2}} |A||B|} \cdot \exp\left(-\frac{1}{2}\left[a^T A^{-1} a + b^T B^{-1} b - \mu^{*T} \Sigma^{*-1} \mu^*\right]\right)$$

$$\boxed{\frac{|\Sigma^*|^{\frac{1}{2}}}{\sqrt{|A||B|}} = \frac{1}{\sqrt{A+B}}}$$

$$Norm_a[b, A+B] = \frac{1}{(2\pi)^{\frac{k}{2}} |A+B|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(a-b)^T (A+B)^{-1}(a-b)\right)$$

exponential term in C simplifies to,

$$(a-b)^T (A+B)^{-1} (a-b)$$

Integration of $Norm_x\left[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*\right]$ will be 1

because it's a normalized Gaussian

so $\int Norm_x[a, A]\ Norm_x[b, B]\,dx = Norm_a[b, A+B] \int Norm_x\left[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*\right] dx$