1) Compute the fourier transform

$$r(t) = \begin{cases} k, & 3a/2 \leq t \leq 5a/2 \\ 0, & \text{otherwise} \end{cases}$$

$$F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$

$$R(f) = \int_{-\infty}^{\infty} r(t) e^{-j2\pi f t} dt$$

only nonzero in the interval $\left[\frac{3a}{2}, \frac{5a}{2}\right]$

$$R(f) = \int_{\frac{3a}{2}}^{\frac{5a}{2}} r(t) e^{-jwft} dt$$

evaluation

$$R(f) = k \int_{\frac{3a}{2}}^{\frac{5a}{2}} e^{-j2\pi f t} dt$$

integral of $\int e^{-j2\pi f t} dt \equiv \dfrac{e^{-j2\pi f t}}{-j2\pi f}$

$$R(f) = \frac{k}{-j2\pi f} \left[e^{-j2\pi \frac{5a}{2} f} - e^{-j2\pi f \cdot \frac{3a}{2}} \right]$$

1b) high value on both horizontal and vertical directions mean that the original image has something repetitive both horizontal and vertical. (Might be a rubic's cube)

1c)

$h(t) = s(t) * g(t)$

then $H(f) = S(f) \cdot G(f)$

To keep the $s(t)$ same, $G(f)$ must be

$$G(f) = 1 \quad \text{for all } f$$

It can be true if $g(t)$ is dirac delta function

$$g(t) = \delta(t)$$

$$(s * \delta)(t) = \int_{-\infty}^{\infty} s(\tau)\,\delta(t-\tau)\,d\tau = s(t)$$

1)d

If rank(A) is 1, it's seperable. A matrix with rank 1 can be written as the outer product of 2 vectors

rank(A) = 5

so it's not seperable

e) rank is 1 so seperable

$$A = u v^T$$

$$v^T = [-21, 6, 3, 12, 3]$$

$$u[1] = \frac{A[1,5]}{v[5]} = 1$$

$$u[2] = \frac{A[2,5]}{v[5]} = \frac{7}{-21} = \frac{-1}{3}$$

$$u[3] = 0$$

$$u[4] = \frac{A[4,5]}{v[5]} = \frac{35}{-21} = \frac{-5}{3}$$

$$u[5] = \frac{-14}{21} = \frac{2}{3}$$

$$u = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ 0 \\ -\frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$$

1f)

Foreground: 0

Initialization: background: ∞

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

$$\begin{matrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty & \infty & 0 & 0 & 0 \\ 0 & 0 & 0 & \infty & \infty & 0 & 0 & 0 \\ 0 & 0 & \infty & \infty & \infty & \infty & 0 & 0 \\ 0 & \infty & \infty & 0 & 0 & \infty & \infty & 0 \\ \infty & \infty & 0 & 0 & 0 & 0 & \infty & \infty \\ \infty & 0 & 0 & 0 & 0 & 0 & 0 & \infty \end{matrix}$$

Forward Pass:

Checking minimum of left, top left, top and top-right neighbors.

Result after forward pass:

∞	∞	∞	∞	∞	∞	∞	∞
∞	0	0	1	2	0	0	1
0	0	0	1	1	0	0	0
0	0	0	1	1	0	0	0
0	0	1	1	1	1	0	0
0	1	1	0	0	1	1	0
1	1	0	0	0	0	1	1
2	0	0	0	0	0	0	1

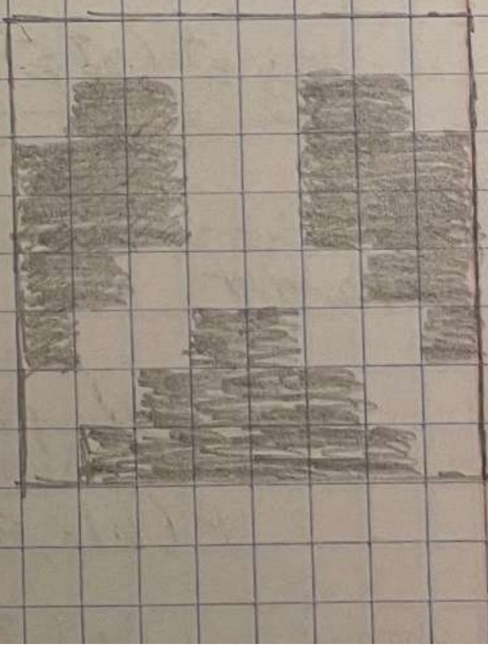
Backward Pass:

Checking right, bottom, bottom right, bottom-left neighbors

Result after backward pass:

1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1
0	0	0	1	1	0	0	0
0	0	0	1	1	0	0	0
0	0	1	1	1	1	0	0
0	1	1	0	0	1	1	0
1	1	0	0	0	0	1	1
1	0	0	0	0	0	0	1

Image:



2a) $\hat{q}_i(h_i) = P(h_i | x_i, \theta)$

1)

$$P_r(x|h) = N_x(\mu + \Phi h, \Sigma)$$

$$P_r(h) = N_h(0, I)$$

$$P(x,h) = P(x|h)\,P(h)$$

PDF

$$P(x|h) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{\frac{1}{2}}} \exp\left(\frac{-1}{2}(x-\mu-\Phi h)^T \Sigma^{-1} (x-\mu-\Phi h)\right)$$

$$P(h) = \frac{1}{(2\pi)^{\frac{K}{2}}} \exp\left(\frac{-1}{2} h^T h\right)$$

Then, (ignoring constants)

$$P(x,h) \propto \left(\frac{-1}{2}(x-\mu-\Phi h)^T \Sigma^{-1}(x-\mu-\Phi h) - \frac{1}{2} h^T h\right)$$

2)

To find $P(h|x,\theta)$, grouping by h^T

$$\frac{-1}{2} h^T \underbrace{(\Phi^T \Sigma^{-1} \Phi + I)}_{\text{precision}} h + h^T \underbrace{\Phi^T \Sigma^{-1} (x-\mu)}_{\text{mean term}} + \text{constant}$$

3) Posterior

$P(h|x,\theta)$ posterior is Gaussian distribution

$$P(h|x,\theta) = N(h; \hat{h}_i, \Lambda^{-1})$$

mean: $\hat{h}_i = \Lambda^{-1} \Phi^T \Sigma^{-1} (x_i - \mu)$

$$\Lambda^{-1} = (\Phi^T \Sigma^{-1} \Phi + I)^{-1}$$

for $\hat{q}_i\, h_i$,

Posterior mean:

$$E(h_i) = \hat{h}_i = (\Phi^T \Sigma^{-1} \Phi + I)^{-1} \Phi^T \Sigma^{-1} (x_i - \mu)$$

Posterior covariance

$$E(h_i h_i^T) = (\Phi^T \Sigma^{-1} \Phi + I)^{-1} + E(h_i) E(h_i)^T$$

In each step, these ones are computed.

In M step, they are used to updata parameter

$\theta = (\mu, \Phi, \Sigma)$

2 c)

$\mu^{(1)}$ becomes

$$\mu^{(1)} = \frac{1}{I} \sum_{i=1}^{I} (x_i - \Phi^{(0)} E(h_i))$$

Update for Φ is dependent to $x_i - \mu^{(t+1)}$

Covariance matrix Σ will be updated based on

$x_i - \mu^{(t+1)}$.

Initializing $\mu^{(0)}$ with the empirical mean increases

the EM convergence.

Updates rules for μ, Φ and Σ won't be changed.