Spring 2019

Fall 2018

Problem 2

Consider the function $f : [0, 1] \to$ defined by

$$f(x) = \begin{cases} x \log x & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

(a) Is f Lipschitz continuous on $[0, 1]$?

(b) Is f uniformly continuous on $[0, 1]$?

(c) Suppose (p_n) is a sequence of polynomial functions on $[0, 1]$, converging uniformly to f. Is the set $A = \{p_n : n \geq 1\} \cup \{f\}$ equicontinuous?

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

The function is not Lipschitz, to see this note that:

$$\lim_{n\to\infty} \frac{f(1/n) - f(0)}{(1/n) - 0} = \lim_{n\to\infty} \log(1/n) = -\infty,$$

so f cannot be Lipschitz. However the function is continuous as by L'Hopital's rule:

$$\lim_{x\to 0} x \log(x) = \lim_{n\to\infty} \frac{\log(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so f is continuous and hence uniformly continuous since we are working over a compact set.

For part (c.), fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be sufficiently large so that $\|p_n - f\| < \epsilon/3$. Next, since f is uniformly continuous let $\delta > 0$ be sufficiently small so that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/3$. Then for all $n > N$, if $|x - y| < \delta$ then:

$$|p_n(x)-p_n(y)| \leq |p_n(x)-f(x)|+|f(x)-f(y)|+|f(y)-p_n(y)| \leq 2\|p_n - f\|+|f(x)-f(y)| < \epsilon$$

Finally, since $p_1, \ldots, p_N$ are each uniformly continuous there exist corresponding $\delta_1, \ldots, \delta_N$ so $\delta_0 = \min\{\delta, \delta_1, \ldots, \delta_N\}$ works for all the $\{p_n\} \cup \{f\}$, hence the set is equicontinuous.

Fall 2017

Problem 1

Prove that every metric subspace of a separable metric space is separable.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let (X, d) be a separable metric space and let $S \subseteq X$ be a metric subspace of X.

Since X is separable, there exists a countable dense subset $\{x_i\}_{i=1}^{\infty}$ such that

$$X \subseteq \bigcup_{i=1}^{\infty} B_\epsilon(x_i)$$

for any $\epsilon > 0$.

In particular, S is also contained in this union. For each x_i and every $n \in \mathbb{N}$ with $B_{1/n}(x_i) \cap S \neq 0$, choose $s_j \in B_{1/n}(x_i) \cap S$. We claim that $\{s_j\}_{j=1}^{\infty}$ is a countable dense subset of S. Indeed, $\{s_j\}_{j=1}^{\infty}$ is countable because it is a countable union of countable sets. Moreover, for any $s \in S$, we know that there exists x_i such that $d(x_i, s) < \epsilon/2$ because $\{x_i\}$ is dense in X. Choosing $s_j \in \{s_j\}$ such that $d(s_j, x_i) < \epsilon/2$ then gives us that

$$d(s_j, s) \leq d(s_j, x_i) + d(x_i, s) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, we have produced a countable, dense subset of S and it follows that S is separable.

Problem 3

Prove or disprove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there exists a sequence of polynomials (p_n) on $[0,1]$ such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every integer $k \geq 0$. Here $f^{(k)}$ denotes the k-th derivative of f.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

The statement is true. Let $\mathcal{P} = \mathcal{P}([0,1])$ be the space of polynomials on $[0,1]$ and for $k \geq 0$ let $C^k([0,1])$ be the space of functions f such that $f^{(i)}$ exists and is continuous for all $i = 0, \ldots, k$. Equip $C^k([0,1])$ with its usual norm:

$$\|f\|_{C^k} = \sum_{i=0}^{k} \|f^{(i)}\|_\infty.$$

We first show that $\mathcal{P}$ is dense in $C^k([0,1])$ for all $k \geq 0$. Fix $\epsilon > 0$ and let $f \in C^k([0,1])$ be arbitrary. Then $f^{(k)} \in C([0,1])$ so by the Stone-Weierstrass Theorem there exists a polynomial $p_k \in \mathcal{P}$ such that $||f^{(k)} - p_k||_\infty < \epsilon/k$. Now consider the function p_{k-1} given by

$$p_{k-1}(x) = f^{(k-1)}(0) + \int_0^x p_k(y)dy.$$

Note $p_{k-1} \in \mathcal{P}$ and by the fundamental theorem of calculus $p_{k-1}^{(1)} = p_k$. Next, note that for any $x \in [0,1]$ we have:

$$\begin{aligned}
|f^{(k-1)}(x) - p_{k-1}(x)| &= \left| f^{(k-1)}(0) + \int_0^x f^{(k)}(y)dy - \left(f^{(k-1)}(0) + \int_0^x p_k(y)dy \right) \right| \\
&\leq \int_0^x |f^{(k)}(y) - p_k(y)|dy \\
&\leq \int_0^1 ||f^{(k)} - p_k||_\infty dy \\
&\leq \epsilon/k.
\end{aligned}$$

Hence $||f^{(k-1)} - p_{k-1}||_{C^1} \leq 2\epsilon/k$. Now repeat this process, at each step defining a new polynomial p_i so that

$$p_i(x) = f^{(i)}(0) + \int_0^x p_{i+1}(y)dy.$$

It follows from the same argument as above that $||f - p_0||_{C^k} \leq \epsilon$. Since ϵ was taken to be arbitrary we have that $\mathcal{P}$ is dense in $C^k([0,1])$.

Returning to the original problem, let $f \in C^\infty([0,1])$. For $k \geq 0$, let $(p_n^k)_{n=1}^\infty \subset \mathcal{P}$ be a sequence such that $p_n^k \to f$ in $C^k([0,1])$, these sequences are guaranteed to exist by the prior argument. Now define a "diagonal" sequence $(q_m)_{m=1}^\infty \subset \mathcal{P}$ by setting q_m to be the first term from (p_n^m) such that $||f - p_n^m||_{C^m} < 2^{-m}$. It follows that for any fixed k, $||f^{(k)} - q_m^{(k)}||_\infty \to 0$ as $m \to \infty$ so (q_m) is our desired sequence.

Spring 2017

Problem 2

Suppose that (X, d) is a metric space such that every continuous function $f : X \to$ is bounded. Prove that X is complete.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let (x_n) be a Cauchy sequence in X, so that (x_n) converges to some $y \in \tilde{X}$, the completion of X. We want to show that $y \in X$, so suppose by way of contradiction that $y \notin X$. Recall that the metric d is a continuous function from $X \times X \to_{\geq 0}$. Hence the function $f : X \to$ defined by $f(x) = d(x, y)$ is continuous. What's more, since $y \notin X$ we have that $f(x) = d(x, y) \neq 0$ for all $x \in X$. Thus the function $g(x) = {}^1/_{f(x)}$ is also a continuous function on X. But since $x_n \to y$ we have that $d(x_n, y) \to 0$, so $g(x_n) \to \infty$, contradicting that every continuous function on X is bounded. Thus it must be the case that $y \in X$.

Fall 2014

Problem 4

Let $C_0()$ denote the Banach space of continuous functions $f :\to$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm.

(a) For $n \in \natural$, define $f_n \in C_0()$ by

$$f_n = \begin{cases} 1 & |x| \leq n \\ \frac{n}{|x|} & |x| > n \end{cases}$$

Show that $F = \{f_n : n \in \mathbb{N}\}$ is a bounded, equicontinuous subset of $C_0()$, but that the sequence (f_n) has no uniformly convergent subsequence. Why doesn't this example contradict the Arzelà-Ascoli theorem?

(b) A family of functions $F \subset C_0()$ is said to be tight if for every $\epsilon > 0$ there exists a constant $M > 0$ such that $|f(x)| < \epsilon$ for all $x \in$ with $|x| \geq M$ and all $f \in F$. Prove that $F \subset C_0()$ is pre-compact in $C_0()$ if it is bounded, equicontinuous, and tight.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

(a) We get boundededness immediately as $|f_n| \leq 1$ for all n. For equicontinuity let $x \in$ and fix $\epsilon > 0$. Since the f_n's are even we may assume without loss of generality that $x \geq 0$. Hence let $N = \lfloor x \rfloor$, so $x \in [N, N+1]$. Note that for all $n > N+1$ $f_n(u) = 1$ for all $|u| \leq N+2$, hence if $|u - x| < 1$ then $|f(u) - f(x)| = 0$. Then since $f_1, f_2, \dots, f_{N+1}$ are continuous there exist $\delta_1, \delta_2, \dots, \delta_{N+1}$ such that if $|u - x| < \delta_i$ then $|f_i(u) - f_i(x)| < \epsilon$ for each $i = 1, 2, \dots N+1$. Hence letting $\delta = \min\{1, \delta_1, \dots, \delta_{N+1}\}$ we have that $|u - x| < \delta \implies |f_n(u) - f_n(x)| < \epsilon$ for all $n \in \mathbb{N}$. Hence F is equicontinuous.

To see that F has no uniformly convergent subsequence note that for all n, $f_n(x) \to 0$ as $|x| \to \infty$ but that f_n converges pointwise to 1 as $n \to \infty$. However this does not contradict Arzelà-Ascoli because that theorem presupposes that we are working with functions over a compact domain.

(b) We will show that F is totally bounded since that is equivalent to pre-compact in a metric space. Fix $\epsilon > 0$. Because F is tight there exists an $M > 0$ such that $|f(x)| < \epsilon/2$ for all $|x| > M$. Now consider the set $G = \{g_n = f_n|_{[-M,M]} : n \in \mathbb{N}\} \subset C([-M, M])$. Note that G is bounded and equicontinuous, this is inherited from F. Then since $[-M, M]$ is compact we have that G is compact in $C([-M, M])$ by the Arzelà-Ascoli theorem.

Hence G is totally bounded so there exist $h_1, \ldots h_N \in G$ such that $G \subseteq \bigcup_{i=1}^{N} B(h_i, \epsilon)$, where $B(h_i, \epsilon)$ is the ball of radius ϵ centered at h_i, taken with respect to the sup norm on $[-M, M]$. Now let $f_1, \ldots f_N$ be the corresponding funcitons in F and consider an arbitrary $f \in F$ and $x \in$. Since the h's form a finite ϵ-net there exists some f_i in our finite collection such that $|f(x) - f_i(x)| < \epsilon$ for all $|x| < M$. Then for $|x| > M$ we have that $|f(x) - f_i(x)| \leq |f(x)| + |f_i(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\bigcup_{i=1}^{N} B(f_i, \epsilon)$ forms an ϵ-net cover of F, so F is totally bounded and pre-compact.

Spring 2011

Problem 6

For $\alpha \in (0,1]$, the space of Hölder continuous functions on the interval $[0,1]$ is defined as

$$C^{9,\alpha}([0,1]) = \{u \in C[0,1] : |u(x) - u(y) \leq C|x-y|^\alpha \ \forall x, y \in [0,1]\}$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x\in[0,1]} |u(x)| + \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let $U = \{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$, and let $u \in U$ be arbitrary. Note that for all $x \in [0,1]$, $|u(x)| \leq \|u\|_\infty \leq \|u\|_{C^{0,\alpha}} \leq 1$, so U is uniformly bounded. Now fix $\epsilon > 0$ and note that if $y \in [0,1]$ with $|x-y| < \epsilon^{1/\alpha}$ then we have:

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq \|u\|_{C^{0,\alpha}} \leq 1 \implies |u(x)-u(y)| < \epsilon,$$

so U is equicontinuous. Then by the Arzelà-Ascoli Theorem we have that U is pre-compact in $C([0,1])$. Thus we only have left to show that U is closed in $C([0,1])$, which follows from ...

Fall 2008

Problem 2

Show that if $\{f_n\}$ is a sequence of continuously differentiable function on $[0,1]$ and both the original sequence and the sequence of derivatives are uniformly bounded, then (f_n) has a uniformly convergent subsequence.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Since the sequence $\{f_n\}$ is uniformly bounded, we can get the equicontinuity of $\{f_n\}$. By the equicontinuity and uniform boundedness of $\{f_n\}$, appling Arzela-Ascoli, $\{f_n\}$ is precompact and there exists a subsequence $\{f_{n_k}\}$ converging uniformly.