UC Davis Analysis Preliminary Exam Solutions

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Contents

Spring 2020

Problem 1

Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Suppose $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \mathbb{R}$, show that $f \equiv 0$.

Solution by Kyle Chickering.
We begin by proving a small lemma.

(Convolution Preserves Pariety) Let $f \in \mathcal{S}(\mathbb{R})$, be even, and let $K \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ be even, then

$$\begin{aligned}(f * K)(x) &= \int_{\mathbb{R}} f(y)K(x-y)\,dy = \int_{\mathbb{R}} f(-y)K(x-y)\,dy \\ &= \int_{\mathbb{R}} f(y)K(x+y)\,dy = \int_{\mathbb{R}} f(y)K((-x)-y)\,dy \\ &= (f * K)(-x)\end{aligned}$$

let f be even with the same hypothesis on K, then

$$\begin{aligned}(f * K)(x) &= \int_{\mathbb{R}} f(y)K(x-y)\,dy = -\int_{\mathbb{R}} f(-y)K(x-y)\,dy \\ &= -\int_{\mathbb{R}} f(y)K(x+y)\,dy = -\int_{\mathbb{R}} f(y)K((-x)-y)\,dy \\ &= -(f * K)(-x)\end{aligned}$$

as desired. $\square$

Next recall the following facts

(i)
$$\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$$

(ii) For $f \in \mathcal{S}(\mathbb{R})$, the Fourier inversion formula says

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{-2\pi i x\xi}\,d\xi$$

(iii)
$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

(iv)
$$(e^{-\pi^2\xi^2})^{\vee} = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

We are now ready to solve our problem. We have

$$\begin{aligned}
\int_{\mathbb{R}} f(y) e^{2xy} e^{-y^2}\, dy &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i y\xi + 2xy - y^2}\, d\xi\, dy && \text{(ii)} \\
&= \int_{\mathbb{R}} \hat{f}(\xi) \int_{\mathbb{R}} e^{2\pi i y\xi + 2xy - y^2}\, dy\, d\xi && \text{(Fubini)} \\
&= e^{x^2} \int_{\mathbb{R}} \hat{f}(\xi) e^{-\pi^2\xi^2} e^{2\pi i x\xi} \int_{\mathbb{R}} e^{-(y-(\pi i\xi + x))^2}\, dy\, d\xi \\
& && \text{(Complete Square)} \\
&= \sqrt{\pi} e^{x^2} \int_{\mathbb{R}} \hat{f}(\xi) e^{-\pi^2\xi^2} e^{2\pi i x\xi}\, d\xi && \text{(i)} \\
&= \sqrt{\pi} e^{x^2} (f * \frac{1}{\sqrt{\pi}} e^{-x^2})(x) = 0 && \text{(iii)}
\end{aligned}$$

which implies that $(f * e^{-x^2})(x) = 0$. By (iv) we see that f is both even and odd, and hence $f = 0$.

Problem 4

Suppose that $f \in L^2(\mathbb{R})$ and $\hat{f}$ is continuous. Suppose that $\hat{f} = \mathcal{O}(|\xi|^{-1-\alpha})$ as $|\xi| \to \infty$. Show that $|f(x+h) - f(x)| \leqslant Ch^{\alpha}$ for all $h > 0$ and some constant C independent of h.

Solution by Kyle Chickering.
We apply the inversion formula

$$\frac{f(x+h) - f(x)}{h^{\alpha}} = \frac{1}{h^{\alpha}} \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x\xi} (e^{2\pi i h\xi} - 1)\, d\xi.$$

We estimate

$$\frac{|e^{2\pi i h\xi} - 1|}{h^{\alpha}} = \left(\frac{|e^{2\pi i h\xi} - 1|}{h}\right)^{\alpha} |e^{2\pi i h\xi} - 1|^{1-\alpha}$$
$$\leqslant 2^{1-\alpha} \left(\frac{|e^{2\pi i h\xi} - 1|}{h}\right)^{\alpha}$$

since $|e^{2\pi i x h} - 1| \leqslant 2$.

By the mean value theorem we have that

$$|e^{2\pi i h\xi} - 1| \leqslant 2\pi|\xi||e^{2\pi i \eta\xi}|h \leqslant 2\pi h|\xi|$$

where $0 < \eta < h$.

Combining the above two estimates shows that

$$\frac{|e^{2\pi ih\xi} - 1|}{h^\alpha} \leqslant \min\left\{2\pi^\alpha |\xi|^\alpha, 2/h^\alpha\right\}.$$

Note that whenever $|\xi| < 1/(\pi h)$, $2\pi^\alpha|\xi|^\alpha < 2$.

Recall that $\hat{f}(\xi) = \mathcal{O}(|\xi|^{-1-\alpha})$ means that there exist constants $M, B \in \mathbb{R}^{>0}$ such that whenever $|\xi| > M$, $|\hat{f}(\xi)| \leqslant B|\xi|^{-1-\alpha}$.

Choose $N = \max M, 1/(\pi h)$, then we estimate

$$\begin{aligned}
\left|\int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi ix\xi} \frac{(e^{2\pi ih\xi} - 1)}{h^\alpha} ;\xi\right| &\leqslant \left|\left(\int_{|\xi|<N} + \int_{|\xi|>N}\right) \hat{f}(\xi) e^{2\pi ix\xi} \frac{(e^{2\pi ih\xi} - 1)}{h^\alpha}\, d\xi\right| \\
&\leqslant 2\pi^\alpha \int_{|\xi|<N} |\hat{f}(\xi)||\xi|^\alpha\, d\xi + \frac{2B}{h^\alpha} \int_{|\xi|\geqslant N} |\xi|^{-1-\alpha}\, d\xi \\
&\leqslant 4\pi^\alpha N^\alpha \left(\int_{|\xi|<N} d\xi\right)^{1/2} \|\hat{f}\|_{L^2(\mathbb{R})} + \frac{4}{h^\alpha} N^{-\alpha} B \\
&= 4\pi^\alpha N (2N)^{1/2} \|f\|_{L^1(\mathbb{R})} + \frac{4}{h^\alpha} N^{-\alpha} B
\end{aligned}$$

By our choice of N we have $N^{-\alpha} \leqslant \pi^\alpha h^\alpha$ so that we have shown

$$\frac{f(x+h) - f(x)}{h^\alpha} \leqslant 2^{5/2} \pi^\alpha M^{3/2} \|f\|_{L^2(\mathbb{R})} + 4\pi^\alpha B =: C$$

as desired.

Problem 6

Find a sequence $\{f_k\}$ of continuous functions on $\mathbb{R}$ such that it is uniformly bounded and equicontinuous but fails to have a subsequence that converges uniformly on $\mathbb{R}$.

Solution by Kyle Chickering.

Let

$$f_k(x) = \begin{cases} 0 & x \leqslant k \\ x - k & k < x \leqslant k+1 \\ 1 & k+1 < x. \end{cases}$$

It is clear that each f_k is continuous, and it is also easy to see that that $\{f_k\}$ is uniformly bounded by 1, since $|f_k(x)| \leqslant 1$ for all $x \in \mathbb{R}$.

We show that $\{f_k\}$ is equi-continuous. Given $\epsilon > 0$, choose $0 < \delta < \epsilon$. For x, y, $|x - y| < \delta$, $x < y$, we have the following cases.

- $(x, y) \cap (k, k+1) = \varnothing$. In this case $|f_k(x) - f_k(y)| = 0 < \epsilon$.

- $(x, y) \subset (k, k+1)$ In this case we have

$$|f_k(x) - f_k(y)| = |x - k - y + k| = |x - y| < \delta < \epsilon.$$

- $(x, y) \not\subset (k, k+1)$ and $(x, y) \cap (k, k+1) \neq \varnothing$.

  We have two further sub cases. First, suppose $f_k(y) = 1$, then $f_k(x) = x - k$ and $k < y < k+1$ which means

$$|f_k(x) - f_k(y)| = |x - k - 1| < |x - y| < \delta < \epsilon$$

  Next, suppose $f_k(y) = y - k$, then $f_k(x) = 0$ and $k \leqslant y < k+1$ and we have that

$$|f_k(x) - f_k(y)| = |k - y| \leqslant |x - y| < \delta < \epsilon.$$

This proves that $\{f_k\}$ is equicontinuous.

Next note that $f_k(x) \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}$ since for any x, $f_k(x) = 0$ for all $k > x$.

Choose any subsequence f_j of $\{f_k\}$ and suppose for the sake of contradiction that f_j converges uniformly. Then for every $\epsilon > 0$ we have that $|f_j(x)| < \epsilon$ for all $x \in \mathbb{R}$, $j \geqslant J$ for some $J \in \mathbb{N}$. However it is easy to see that if we choose $x > j$, $|f_j(x)| = 1$ and hence $|f_j(x)| \not< \epsilon$ in general. Since $\{f_j\}$ was an arbitrary subsequence, no subsequence converges uniformly.

Spring 2019

Problem 1

Consider the Hilbert space $L^2(\mathbb{T})$ of complex valued square integrable functions with the inner product given by

$$(f, g) = \int_{\mathbb{T}} \overline{f(x)} g(x)\, dx.$$

(a) For all $\phi \in \mathbb{R}$, define $g_\phi(\theta) = \sin(\theta + \phi)$ for $\theta \in [0, 2\pi]$. Let V be the closed linear span of $\{g_\phi \mid \phi \in \mathbb{R}\}$. Show that V is two-dimensional.

(b) Find $k : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ such that for all $f \in L^2(\mathbb{T})$ the integral operator K defined by

$$Kf(x) = \int_{\mathbb{T}} k(x, y) f(y)\, dy$$

satisfies

$$\|Kf - f\| = \inf\{\|g - f\| \mid g \in V\}.$$

Solution by Kyle Chickering.

(a) We claim $\{\sin(\theta), \cos(\theta)\}$ is a basis for V, from which it follows that V is two dimensional.

Note that for any $g \in V$, we can write

$$\begin{aligned} g(\theta) &= \sum_{\phi \in \mathcal{A}} g_\phi(\theta) \\ &= \sin\theta \sum_{\phi \in \mathcal{A}} \cos\phi + \cos\theta \sum_{\phi \in \mathcal{A}} \sin\phi \end{aligned}$$

for some finite collection of real numbers $\mathcal{A}$ by the angle sum and difference formulas.

We show that given any $\alpha, \beta \in \mathbb{R}$, we may write

$$\alpha = \sum_{\phi \in \mathcal{A}_\alpha} \cos\phi \qquad \text{and} \qquad \beta = \sum_{\phi \in \mathcal{A}_\beta} \sin\phi.$$

Note that it is clear that $g_0(\theta) = \sin(\theta)$, $g_\pi(\theta) = -\sin(\theta)$, $g_{\pi/2}(\theta) = \cos(\theta)$, and $g_{-\pi/2}(\theta) = -\cos(\theta)$ so we can obtain any integer α, β by repeated addition of these elements.

Let $\tilde{\alpha}$ be the fractional part of α and $\tilde{\beta}$ be the fractional part of β. If we can obtain any fractional part, then we are done and $\{\sin\theta, \cos\theta\}$ is a basis of V.

Let $0 \leqslant \tilde{\alpha} \leqslant 1$ and choose ϕ such that $\cos(\phi) = \tilde{\alpha}/2$. Then

$$g_\phi(\theta) + g_{-\phi}(\theta) = 2\cos\phi = \tilde{\alpha}.$$

Similarly, let $0 \leqslant \tilde{\beta} \leqslant 1$ and choose ϕ such that $\sin(\phi) = \tilde{\beta}/2$. Then

$$g_\phi(\theta) + g_{\pi-\phi}(\theta) = 2\sin\phi = \tilde{\beta}$$

as desired.

Problem 2

Let $S = [0,1] \times [0,1]$ and consider the space $C(S)$ of continuous complex-valued functions on S equipped with the sup-norm. Define $F \subset C(S)$ by

$$F = \{f \in C(S) \,:\, \exists\, n \geq 1 \text{ and } g_1, \dots, g_n\,,\ h_1, \dots, h_n \in C([0,1]) \text{ such that } f(x,y) = \sum_{k=1}^{n} g_k(x) h_k(y)\}.$$

Show that F is dense in $C(S)$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let S and F be given as above. In order to use the Stone-Weierstrass theorem, we must show that F is an algebra that is nonvanishing, separates points, and for any $f \in F$, we must also have $\overline{f} \in F$.

We first note that F is nonvanishing because the constant function $\chi_S \in F$.

Next, take any $(x_1, y_1), (x_2, y_2) \in S$. Without loss of generality, suppose that $x_1 \neq x_2$. Let $\epsilon = d(x_1, x_2)$ and denote $A = B_\epsilon/2(x_1)$ and $B = S \backslash A$. Then there exists a continuous Urysohn function

$$\rho(x,y) = \frac{d(x,A)}{d(x,A) + d(x,B)} \in F$$

such that $\rho(x_1, y) = 1$ and $\rho(x_2, y) = 0$. Thus, F separates points.

Now if we examine $f(x,y) = \sum_{k=1}^n g_k(x) h_k(y) \in F$, we can see that $\overline{f}(x,y) = \overline{\sum_{k=1}^n g_k(x) h_k(y)} = \sum_{k=1}^n \overline{g_k}(x) \overline{h_k}(y) \in F$ since $g_k(x) \in C([0,1])$ implies $\overline{g_k}(x) \in C([0,1])$.

Finally, F is an algebra because $\left(\sum_{k=1}^n g_k(x) h_k(y)\right) \left(\sum_{j=1}^m g_j(x) h_j(y)\right)$ distributes, giving us another element of F.

Thus, by the Stone-Weierstrass theorem, F is dense in $C(S)$.

Problem 6

Let $\Omega = (0,1) \subset \mathbb{R}$. For $\overline{u} = \int_\Omega u(x)dx$ show that

$$\|u - \overline{u}\|_{L^\infty(\Omega)} \leq \|u'\|_{L^2(\Omega}$$

for all $u \in W^{1,1}(\Omega)$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let $x \in (0,1)$, and let $x_0 \in (0,1)$ be such that $\overline{u} = u(x_0)$. By the Fundamental Theorem of Calculus we have:

$$|u(x) - u(x_0)| = \left| \int_{x_0}^{x} u'(t)\, dt \right| \leq \int_0^1 |u'(t)|\, dt \leq \|u'\|_{L^2(0,1)}$$

where the last inequality comes from applying Holder's Inequality.

Solution by Kyle Chickering.
We have

$$\begin{aligned}
u(x) - \overline{u} &= \int_0^1 u(x) - u(y)\, dy \\
&= \int_0^1 \int_y^x u'(\eta)\, d\eta && (W^{1,p} \text{ FTC (Brezis 8.2)}) \\
&\leqslant \int_0^1 \int_y^x |u'(\eta)|\, d\eta \leqslant \int_0^1 \int_0^1 |u'(\eta)|\, d\eta && (\sup x \text{ and } y) \\
&= \|u'\|_{L^1} \leqslant \|u'\|_{L^2} && (\text{Hölder's with 1 and } u')
\end{aligned}$$

sup over x on both sides to obtain the inequality.

Spring 2018

Problem 2

Let H be a Hilbert space and let $P, Q \in B(H)$ be two orthogonal projections. Prove that $\ker PQ \subseteq \ker P + \ker Q$ always, and that $\ker PQ = \ker P + \ker Q$ when PQ is also an orthogonal projection.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Take $z \in ker(PQ)$. Then $PQ(z) = 0$. $\implies Q(z) \in ker(P)$. So $z \in ran(Q)$ or $z \in ker(Q)$. If $z \in ran(Q)$ then $z \in ker(P)$, otherwise $z \in ker(Q)$. Thus, if $z \in ker(PQ) \implies z \in ker(P) + ker(Q)$. Therefore, $ker(PQ) \subseteq ker(P) + ker(Q)$.

Now suppose PQ is also an orthogonal projection. Then $PQ = (PQ)^* = Q^*P^*$. Since P and Q are orthogonal projections, $P = P^*$ and $Q = Q^*$, so $PQ = QP$. Take $z \in ker(P) \implies PQ(z) = 0$. Since $PQ = QP \implies QP(z) = 0$, so $z \in ker(PQ)$. Thus, $ker(P) + ker(Q) = ker(PQ)$.

Problem 4

Let T be a bounded linear operator on a Hilbert space with an orthonormal basis of eigenvectors with eigenvalues $\Lambda = \{\lambda_n\}$. Show that the spectrum $\sigma(T)$ is exactly the closure of the set Λ.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
First note that because $\sigma(T)$ is closed, $\Lambda \subseteq \sigma(T) \implies \overline{\Lambda} \subseteq \sigma(T)$. For the reverse inclusion, first note that because T has an orthonormal basis of eigenvectors, henceforth denoted $\{e_n\}$, we may express T as:

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where P_n denotes the not necessarily finite-dimensional orthogonal projection on the eigenspace of λ_n. We first show that the point spectrum of T is exactly Λ. To see this, suppose there exists $x \neq 0$ such that $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$.

$x \neq 0$ implies that there exists some $m \in \mathbb{N}$ such that $\langle x, e_m \rangle \neq 0$. Then observe:

$$Tx = \sum_{n=1}^{\infty} \lambda_n P_n x = \sum_{n=1}^{\infty} \lambda_n P_n \left(\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \right) = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n = \sum_{n=1}^{\infty} \lambda \langle x, e_n \rangle e_n = \lambda x.$$

Hence we require that $\lambda_n \langle x, e_n \rangle = \lambda \langle x, e_n \rangle$ for all n. Since $langlex, e_m \rangle \neq 0$ this implies that $\lambda = \lambda_m$.

Note, for $x, y \in \mathcal{H}$ we have

$$\begin{aligned} \langle Tx, y \rangle = \langle \sum_{n=1}^{\infty} \lambda_n P_n x, y \rangle = \sum_{n=1}^{\infty} \langle \lambda_n P_n x, y \rangle &= \sum_{n=1}^{\infty} \langle x, \overline{\lambda_n} P_n y \rangle \\ &= \langle x, \sum_{n=1}^{\infty} \overline{\lambda_n} P_n y \rangle, \end{aligned}$$

so $T^* = \sum_{n=1}^{\infty} \overline{\lambda_n} P_n$.

Fall 2018

Problem 1

> Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that
> $$\lim_{n\to\infty} \int_0^1 f(x)\sin(n\pi x)dx = 0.$$

Solution by Kyle Chickering.
This is the well-known "Riemann-Lebesgue lemma", it is more generally a statement about the coeficient decay of the L^2 Fourier expansion for f. We will prove this result in three different ways. Note that all proofs hold for $e^{in\pi x}$ in place of $\sin(n\pi x)$ which is a more general result.

The first proof is the "unsophisticated" proof, presented before the notions of measure theory, using only undergraduate notions of continuity. The other two proofs use approximation properties of continuous functions.

Let $x = \xi + \frac{1}{n}$ for an integer n, then

$$\int_0^1 f(x)\sin(n\pi x)\,dx = -\int_{-1/n}^{1-1/n} f\left(\xi + \frac{1}{n}\right)\sin(n\pi\xi)\,d\xi.$$

Consequently

$$\begin{aligned}
2\int_0^1 f(x)\sin(n\pi x)\,dx &= \int_0^1 f(x)\sin(n\pi x)\,dx - \int_{-1/n}^{1-1/n} f\left(\xi+\frac{1}{n}\right)\sin(n\pi\xi)\,d\xi \\
&= \int_0^{1-1/n} (f(x) - f\left(x+\frac{1}{n}\right))\sin(n\pi x)\,dx \\
&\quad + \int_{1-1/n}^1 f(x)\sin(n\pi x)\,dx \\
&\quad - \int_{-1/n}^0 f\left(x+\frac{1}{n}\right)\sin(n\pi x)\,dx \\
&=: I + J - K.
\end{aligned}$$

Since f is continuous on $[0,1]$, it is uniformly continuous and hence bounded by a constant $M > 0$. Then

$$|I| \leqslant M \int_{1-1/n}^{1} |\sin(n\pi x)|\, dx \leqslant \frac{M}{n}$$

and

$$|J| \leqslant M \int_{-1/n}^{0} |\sin(n\pi x)|\, dx \leqslant \frac{M}{n}.$$

Since f is uniformly continuous, given any $\epsilon > 0$ we may choose n large enough so that $|f(x) - f(x+1/n)| < \epsilon/3$ for all $x \in [0,1]$. This shows that

$$\begin{aligned} |K| &\leqslant \int_0^{1-1/n} \left|f(x) - f\left(1 + \frac{1}{n}\right)\right| |\sin(n\pi x)|\, dx \\ &< \frac{\epsilon}{3} \int_0^{1-1/n} |\sin(n\pi x)|\, dx < \frac{\epsilon}{3}. \end{aligned}$$

Choose n to be the smallest such n such that $|I|, |J|$ and $|K|$ are all less than $\epsilon/3$. Then

$$\left|\int_0^1 f(x)\sin(n\pi x)\, dx\right| \leqslant |I| + |J| + |K| < 3\frac{\epsilon}{3} = \epsilon$$

as desired. ■

By the Weierstrauss approximation theorem f can be approximated by a sequence of polynomials $p_k : [0,1] \to \mathbb{R}$ such that $\|f - p_k\|_{L^\infty} \to 0$ as $k \to \infty$. Let ϵ be given and choose N such that $\|f - p_k\|_{L^\infty} < \epsilon/2$ whenever $k \geqslant N$. Then a simple estimate shows

$$\begin{aligned} \left|\int_0^1 (p_k(x) - f(x))\sin(n\pi x)\right| &\leqslant \int_0^1 |p_k(x) - f(x)||\sin(n\pi x)|\, dx \\ &\leqslant \|p_k - f\|_{L^\infty} \|\sin(n\pi x)\|_{L^1} \\ &< \epsilon. \end{aligned}$$

Next, for any fixed k we may choose an n depending on k such that

$$\frac{1}{n\pi}\left[|p_k(0) - p_k(1)| + \int_0^1 |p_k'(x)|\, dx\right] < \epsilon/2$$

since p_k is a polynomial and hence bounded on $[0,1]$.

Note that a polynomial p_k can be differentiated and integrated on the interval $[0,1]$. Hence we estimate, integrating by parts and choosing a sufficient n depending on k as above

$$\begin{aligned} \left|\int_0^1 f(x)\sin(n\pi x)\, dx\right| &\leqslant \left|\int_0^1 (f(x) - p_k(x))\sin(n\pi x)\right| + \left|\int_0^1 p_k(x)\sin(n\pi x)\right| \\ &< \frac{\epsilon}{2} + \left|-\frac{1}{n\pi} p_k(x)\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 p_k'(x)\cos(n\pi x)\, dx\right| \\ &\leqslant \frac{\epsilon}{2} + \frac{1}{n\pi}\left[|p_k(0) - p_k(1)| + \int_0^1 |p_k'(x)|\, dx\right] \\ &< \epsilon. \end{aligned}$$

as desired. ■

Since f is continuous on $[0,1]$ it is $L^1([0,1])$ and can be approximated by a sequence of simple functions $\phi_k(x) \nearrow f(x)$ as $k \to \infty$. Since simple functions are dense in L^1, given any $\epsilon > 0$ we may choose an N large enough that $\|f - \phi_k\|_{L^1} < \epsilon/2$ whenever $k \geqslant N$ and hence by Hölder's inequality

$$\left| \int_0^1 (f(x) - \phi_k(x)) \sin(n\pi x)\, dx \right| \leqslant \|f - \phi_k\|_{L^1} \|\sin(n\pi x)\|_{L^\infty} < \epsilon/2$$

Given any $\epsilon > 0$ and any simple function $\phi_k(x) := \sum_{j=1}^M \alpha_j^k \chi_{A^j}(x)$, choose n such that $\frac{2}{n\pi} \sum_j |alpha_j^k| < \epsilon/2$. Then we have the estimate

$$\begin{aligned}
\left| \int_0^1 \phi_k(x) \sin(n\pi x)\, dx \right| &= \left| \sum_{j=1}^M \alpha_j^k \int_{A^j} \sin(n\pi x)\, dx \right| \\
&\leqslant \left| \sum_{j=1}^M \alpha_j^k \int_0^1 \sin(n\pi x)\, dx \right| \\
&= \left| -\frac{1}{n\pi} \sum_{j=1}^M \alpha_j^k \cos(n\pi x)\Big|_0^1 \right| \\
&\leqslant \frac{2}{n\pi} \sum_{j=1}^M |\alpha_j^k| < \frac{\epsilon}{2}.
\end{aligned}$$

By the two estimates above we have that

$$\begin{aligned}
\left| \int_0^1 f(x) \sin(n\pi x)\, dx \right| &\leqslant \left| \int_0^1 (f(x) - \phi_k(x)) \sin(n\pi x)\, dx \right| + \left| \int_0^1 \phi_k(x) \sin(n\pi x) \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

as desired. ■

Problem 2

Consider the function $f : [0,1] \to$ defined by

$$f(x) = \begin{cases} x \log x & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

(a) Is f Lipschitz continuous on $[0,1]$?

(b) Is f uniformly continuous on $[0,1]$?

(c) Suppose (p_n) is a sequence of polynomial functions on $[0,1]$, converging uniformly to f. Is the set $A = \{p_n : n \geq 1\} \cup \{f\}$ equicontinuous?

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
The function is not Lipschitz, to see this note that:

$$\lim_{n\to\infty} \frac{f(1/n) - f(0)}{(1/n) - 0} = \lim_{n\to\infty} \log(1/n) = -\infty,$$

so f cannot be Lipschitz. However the function is continuous as by L'Hopital's rule:

$$\lim_{x\to 0} x\log(x) = \lim_{n\to\infty} \frac{\log(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so f is continuous and hence uniformly continuous since we are working over a compact set.

For part (c.), fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be sufficiently large so that $\|p_n - f\| < \epsilon/3$. Next, since f is uniformly continuous let $\delta > 0$ be sufficiently small so that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/3$. Then for all $n > N$, if $|x - y| < \delta$ then:

$$|p_n(x)-p_n(y)| \leq |p_n(x)-f(x)|+|f(x)-f(y)|+|f(y)-p_n(y)| \leq 2\|p_n - f\|+|f(x)-f(y)| < \epsilon$$

Finally, since $p_1, \ldots, p_N$ are each uniformly continuous there exist corresponding $\delta_1, \ldots, \delta_N$ so $\delta_0 = \min\{\delta, \delta_1, \ldots, \delta_N\}$ works for all the $\{p_n\} \cup \{f\}$, hence the set is equicontinuous.

Solution by Kyle Chickering.

(a) No. $f'(x) = \log x + 1$ on $(0, 1]$, and $\lim_{x\to 0^+} f'(x) = -\infty$. Suppose FSOC that f is Lipschitz with Lipschitz constant L, then $|f(x)-f(y)| \leqslant L|x-y|$ for all $x, y \in [0, 1]$. Choose $x = 0$, then we have

$$\frac{|f(y)|}{y} \leqslant L$$

for all $y \in [0, 1]$. Choose $y = e^{L+1}$, then

$$\frac{|e^{L+1}\log e^{L+1}|}{e^{L+1}} = L + 1 > L$$

a contradiction. Therefore f is not Lipschitz.

(b) Yes. We will show that f is continuous, and therefore since f is continuous on a compact set $[0, 1]$, it is uniformly continuous on $[0, 1]$ (Note that this is an explicit example of uniform continuity being weaker than Lipschitz continuity).

Clearly $x \log x$ is continuous on $(0, 1]$ since the product of two continuous functions is continuous. By L'Hopital's rule

$$\lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} x = 0$$

so f is continuous on $[0, 1]$.

(c) Yes. Let $\epsilon > 0$, $x \in [0,1]$ be given and choose $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/3 < \epsilon$ whenever $|x-y| < \delta$. We will show that $|p_k(x)-p_k(y)| < \epsilon$ whenever $|x-y| < \delta$, for all k. Choose $N \in \mathbb{N}$ such that

$$\|p_k - f\|_{L^\infty} < \frac{\epsilon}{3} \quad \Longrightarrow \quad |p_k(x) - f(x)| < \frac{\epsilon}{3}.$$

for all $x \in [0,1]$ whenever $k \geqslant N$. We set

$$A_N := \{p_k \mid k < N\} \subset A.$$

It is clear that A_N is equicontinuous since it is a finite collection of uniformly continuous functions (take the infimum of the δ's). Because of this it is clear that if $A \backslash A_N$ is equicontinuous, then so is A. WLOG we assume N above is 1 and $A_N = \varnothing$.

We have

$$\begin{aligned} |p_k(x) - p_k(y)| &\leqslant |p_k(x) - f(x)| + |f(x) - f(y)| + |f(y) - p_k(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

whenever $|x-y| < \epsilon$ as desired. Therefore A is equicontinuous.

Problem 4

Consider the functions $f_N(x) = (2\pi)^{-1} \sum_{|k| \leq N} e^{ikx}$. Show that if $g \in L^2(\mathbb{T})$, then $(f_N * g) \to g$ in L^2.

Solution by Kyle Chickering.
A simple calculuation gives

$$\begin{aligned} (f_N * g)(x) &= \int_{\mathbb{T}} \frac{1}{2\pi} \sum_{|k| \leqslant N} e^{ik(x-y)} g(y)\, dy \\ &= \frac{1}{\sqrt{2\pi}} \sum_{|k| \leqslant N} e^{ikx} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-iky} g(y)\, dy \\ &= \frac{1}{\sqrt{2\pi}} \sum_{|k| \leqslant N} e^{ikx} \hat{g}_k = S_N(x). \end{aligned}$$

Since $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$, we can expand g as

$$g(x) = \sum_{k \in \mathbb{Z}} (e^{ikx}, g)_{L^2} e^{ikx}$$

where equality is taken in the L^2 sense. Because $(e^{ikx}, g)_{L^2} = \int_{\mathbb{T}} g(y) e^{-iky}\, dy = \hat{g}_k$ we can see that $S_N \to g$ in the L^2 sense as $N \to \infty$.

Problem 5

Show that for any $u \in L^1(\mathbb{R}^d)$,

$$\lim_{h\to 0}\|u(x+h) - u(x)\|_{L^1(\mathbb{R}^d)} = 0.$$

Solution by Kyle Chickering.
We begin by proving a well-known lemma for definiteness.

Lemma: If $f \in L^1$ and $f_k \in L^1$ with $f_k \to f$ pointwise, then $\|f - f_k\|_{L^1} \to 0$ as $k \to \infty$.

Proof: Clearly $f - f_k \in L^1$. We have $|f - f_k| \leqslant |f| - |f_k|$ so by dominated convergence

$$\lim_{k\to\infty}\|f - f_k\|_{L^1} = \|f - \lim_{k\to\infty} f_k\|_{L^1} = 0.$$

■

We now prove the proposition. Since $C_0^\infty(\mathbb{R}^d)$ dense in $L^1(\mathbb{R}^d)$, let $\psi_k(x) \to u(x)$ as $k \to \infty$ for all $x \in \mathbb{R}^d$. Then by the lemma we have that $\|\phi_k - u\|_{L^1} \to 0$ as $k \to 0$. Furthermore, since ψ_k is continuous $\psi_k(x+h) \to \psi_k(x)$ pointwise as $h \to 0$. Consequently $\|\psi_k(x+h) - \psi_k(x)\|_{L^1} \to 0$ as $h \to \infty$. Finally, note that $\psi_k(x+h) \to u(x+h)$ by a simple change of variable and the translational invarience of the Lebesgue measure on $\mathbb{R}^d$.

Let $\epsilon > 0$ be given and fix some $h \in \mathbb{R}^d$. Choose k such that

$$\|u(x+h) - \psi_k(x+h)\|_{L^1} = \|\psi_k(x) - u(x)\|_{L^1} < \epsilon.$$

Then we have that

$$\begin{aligned}\|u(x+h) - u(x)\|_{L^1} &\leqslant \|u(x+h) - \psi_k(x+h)\|_{L^1} + \|\psi_k(x+h) - \psi_k(x)\|_{L^1} \\ &\quad + \|\psi_k(x) - u(x)\|_{L^1} \\ &< 2\epsilon + \|\psi_k(x+h) - \psi_k(x)\|_{L^1}.\end{aligned}$$

Taking the limit as $h \to 0$ of the above inequality we arrive at

$$\lim_{h\to 0}\|u(x+h) - u(x)\|_{L^1} < 2\epsilon$$

by the convergence of $\psi(x+h)$ to $\psi(x)$ in L^1. Since $\epsilon > 0$ was arbitrary the result is immediate.

Problem 6

Let $\Omega = \{(x,y) \mid y \geq 0, x \in \mathbb{R}\}$. Let $f \in C_0^1(\mathbb{R}^2)$. Show that

$$\int_{\mathbb{R}} |f(x,0)|^2 dx \leq 2(\int_\Omega |f(x,y)|^2 dxdy + \int_\Omega |\frac{\partial f}{\partial y}(x,y)|^2 dxdy).$$

Solution by Kyle Chickering.
First, let $\phi \in C_0^1(\mathbb{R}^2)$ be non-negative. Then by the fundamental theorem of calculus and the vanishing of ϕ at infinity we find that

$$\begin{aligned}\int_{\mathbb{R}} \phi^2(x,0)\,dx &= -\int_{\mathbb{R}}\int_0^\infty \frac{\partial}{\partial y}\left[\phi^2\right]\,dy\,dx \\ &= -2\int_{\mathbb{R}}\int_0^\infty \phi\frac{\partial}{\partial y}\phi\,dy\,dx \\ &\leqslant 2\int_\Omega |\phi|\left|\frac{\partial\phi}{\partial y}\right|\,dy\,dx.\end{aligned}$$

Applying young's inequality finally yields

$$\int_{\mathbb{R}} \phi^2(x,0)\,dx \leqslant \int_\Omega |\phi|^2\,dy\,dx + \int_\Omega \left|\frac{\partial\phi}{\partial y}\right|\,dy\,dx. \tag{$*$}$$

We now consider an arbitrary function $f \in C_0^1(\mathbb{R}^2)$. We have that $|f| = f_+ + f_-$ where f_+ and f_- are respectivly the positive and negative parts of f. We also have

$$|f|^2 = f_+^2 + 2f_+f_- + f_-^2 \leqslant 2(f_+^2 + f_-^2) \leqslant 2|f|^2. \tag{$**$}$$

we also note the following

$$\begin{aligned}\left|\frac{\partial f}{\partial y}\right| &= \left(\frac{\partial f}{\partial y}\right)_+ + \left(\frac{\partial f}{\partial y}\right)_- \\ &= \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_+ + \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_- \\ &= \left|\frac{\partial}{\partial y}(f_+ - f_-)\right| - \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_- + \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_- \\ &\leqslant \left|\frac{\partial f_+}{\partial y}\right| + \left|\frac{\partial f_-}{\partial y}\right|\end{aligned}$$

where we have used the fact that $|\psi| = \psi_+ + \psi_-$. This implies that

$$\left|\frac{\partial f}{\partial y}\right|^2 \leqslant 2\left(\left|\frac{\partial f_+}{\partial y}\right| + \left|\frac{\partial f_-}{\partial y}\right|\right) \leqslant 2\left|\frac{\partial f}{\partial y}\right|^2 \tag{$***$}$$

using our estimate from above.

We now estimate

$$\begin{aligned}
\int_{\mathbb{R}} |f(x,0)|^2\,dx &\leqslant 2\int_{\mathbb{R}} f_+^2(x,0) + f_-^2(x,0)\,dx \qquad \text{by } (**) \\
&\leqslant 2\int_\Omega f_+^2\,dy\,dx + 2\int_\Omega \left(\frac{\partial f_+}{\partial y}\right)^2 dy\,dx + 2\int_\Omega f_-^2\,dy\,dx \\
&\quad + 2\int_\Omega \left(\frac{\partial f_-}{\partial y}\right)^2 dy\,dx \qquad \text{by } (*) \\
&= 2\int_\Omega f_+^2 + f_-^2\,dy\,dx + 2\int_\Omega \left(\frac{\partial f_+}{\partial y}\right)^2 + \left(\frac{\partial f_-}{\partial y}\right)^2 dy\,dx \\
&\leqslant 2\int_\Omega |f|^2\,dy\,dx + 2\int_\Omega \left|\frac{\partial f}{\partial y}\right|^2 dy\,dx \qquad \text{by } (*) \text{ and } (***)
\end{aligned}$$

as desired.

Fall 2017

Problem 1

Let X be a Banach space with dual space X^* and let $A \subset X$ be a linera subspace. Define the annihilator $A^\perp \subset X^*$ of A by

$$A^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in A\}.$$

Prove that A is dense in X if and only if $A^\perp = \{0\}$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
($\Longrightarrow$) Suppose A is dense and let $\phi \in A^\perp$. Let $x \in X$ be arbitrary. By the density of A there exists a sequence $(a_n) \subseteq A$ such that $a_n \to x$. Then by the continuity of ϕ we have that:

$$\phi(x) = \phi(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} \phi(a_n) = 0,$$

so ϕ is identically 0.

($\Longleftarrow$) Now suppose $A^\perp = \{0\}$, and let $x_0 \in X$ be arbitrary. Consider the map $T : \overline{Span\{A, x_0\}} \to$ defined so that $T(a) = 0$ for all $a \in A$, $T(x_0) = d(x_0, A) = \inf_{a\in A}\|x_0 - a\|$, and T extends linearly to the rest of the span. One can easily verify that T is a bounded linear functional so by the Hahn-Banach Theorem T extends to a bounded linear functional $\tilde{T}$ on all of X. But then this $\tilde{T}$ would annihilate A, so it must be that $\tilde{T}$ is identically zero. But then $\tilde{T}(x_0) = T(x_0) = d(x_0, A) = 0$, so x_0 is in the closed linear span of A. Since we chose x_0 to be arbitrary it follows that A is dense in X.

Problem 2

Prove that every metric subspace of a separable metric space is separable.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let (X, d) be a separable metric space and let $S \subseteq X$ be a metric subspace of X.

Since X is separable, there exists a countable dense subset $\{x_i\}_{i=1}^{\infty}$ such that

$$X \subseteq \bigcup_{i=1}^{\infty} B_\epsilon(x_i)$$

for any $\epsilon > 0$.

In particular, S is also contained in this union. For each x_i and every $n \in \mathbb{N}$ with $B_{1/n}(x_i) \cap S \neq 0$, choose $s_j \in B_{1/n}(x_i) \cap S$. We claim that $\{s_j\}_{j=1}^{\infty}$ is a countable dense subset of S. Indeed, $\{s_j\}_{j=1}^{\infty}$ is countable because it is a countable union of countable sets. Moreover, for any $s \in S$, we know that there exists x_i such that $d(x_i, s) < \epsilon/2$ because $\{x_i\}$ is dense in X. Choosing $s_j \in \{s_j\}$ such that $d(s_j, x_i) < \epsilon/2$ then gives us that

$$d(s_j, s) \leq d(s_j, x_i) + d(x_i, s) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, we have produced a countable, dense subset of S and it follows that S is separable.

Problem 3

Prove or disprove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there exists a sequence of polynomials (p_n) on $[0,1]$ such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every integer $k \geq 0$. Here $f^{(k)}$ denotes the k-th derivative of f.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

The statement is true. Let $\mathcal{P} = \mathcal{P}([0,1])$ be the space of polynomials on $[0,1]$ and for $k \geq 0$ let $C^k([0,1])$ be the space of functions f such that $f^{(i)}$ exists and is continuous for all $i = 0, \ldots, k$. Equip $C^k([0,1])$ with its usual norm:

$$\|f\|_{C^k} = \sum_{i=0}^{k} \|f^{(i)}\|_\infty.$$

We first show that $\mathcal{P}$ is dense in $C^k([0,1])$ for all $k \geq 0$. Fix $\epsilon > 0$ and let $f \in C^k([0,1])$ be arbitrary. Then $f^{(k)} \in C([0,1])$ so by the Stone-Weierstrass Theorem there exists a polynomial $p_k \in \mathcal{P}$ such that $||f^{(k)} - p_k||_\infty < \epsilon/k$. Now consider the function p_{k-1} given by

$$p_{k-1}(x) = f^{(k-1)}(0) + \int_0^x p_k(y)dy.$$

Note $p_{k-1} \in \mathcal{P}$ and by the fundamental theorem of calculus $p_{k-1}^{(1)} = p_k$. Next, note that for any $x \in [0,1]$ we have:

$$\begin{aligned}
|f^{(k-1)}(x) - p_{k-1}(x)| &= \left| f^{(k-1)}(0) + \int_0^x f^{(k)}(y)dy - \left(f^{(k-1)}(0) + \int_0^x p_k(y)dy \right) \right| \\
&\leq \int_0^x |f^{(k)}(y) - p_k(y)|dy \\
&\leq \int_0^1 ||f^{(k)} - p_k||_\infty dy \\
&\leq \epsilon/k.
\end{aligned}$$

Hence $||f^{(k-1)} - p_{k-1}||_{C^1} \leq 2\epsilon/k$. Now repeat this process, at each step defining a new polynomial p_i so that

$$p_i(x) = f^{(i)}(0) + \int_0^x p_{i+1}(y)dy.$$

It follows from the same argument as above that $||f - p_0||_{C^k} \leq \epsilon$. Since ϵ was taken to be arbitrary we have that $\mathcal{P}$ is dense in $C^k([0,1])$.

Returning to the original problem, let $f \in C^\infty([0,1])$. For $k \geq 0$, let $(p_n^k)_{n=1}^\infty \subset \mathcal{P}$ be a sequence such that $p_n^k \to f$ in $C^k([0,1])$, these sequences are guaranteed to exist by the prior argument. Now define a "diagonal" sequence $(q_m)_{m=1}^\infty \subset \mathcal{P}$ by setting q_m to be the first term from (p_n^m) such that $||f - p_n^m||_{C^m} < 2^{-m}$. It follows that for any fixed k, $||f^{(k)} - q_m^{(k)}||_\infty \to 0$ as $m \to \infty$ so (q_m) is our desired sequence.

Spring 2017

Spring 2017

Problem 2

Suppose that (X,d) is a metric space such that every continuous function $f : X \to$ is bounded. Prove that X is complete.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let (x_n) be a Cauchy sequence in X, so that (x_n) converges to some $y \in \tilde{X}$, the completion of X. We want to show that $y \in X$, so suppose by way of contradiction that $y \notin X$. Recall that the metric d is a continuous function from $X \times X \to_{\geq 0}$. Hence the function $f : X \to$ defined by $f(x) = d(x,y)$ is continuous. What's more, since $y \notin X$ we have that $f(x) = d(x,y) \neq 0$ for all $x \in X$. Thus the function $g(x) = {}^{1}/_{f(x)}$ is also a continuous function on X. But since $x_n \to y$ we have that $d(x_n, y) \to 0$, so $g(x_n) \to \infty$, contradicting that every continuous function on X is bounded. Thus it must be the case that $y \in X$.

Fall 2016

Problem 3

Let $y = \{a_n\}_{n=1}^{\infty}$ be a sequence of real-valued scalars and assume that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$. Show that $y \in \ell^2(\mathbb{N})$

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

For $n \in \mathbb{N}$ define the linear functional $A_n : \ell^2(\mathbb{N}) \to \mathbb{C}$ so that $A_n(x) = \sum_{k=1}^{n} a_k x_k$. Note that this is in fact a bounded linear functional as we may express A in the form $Ax = \langle x, y|_n \rangle$, where $y|_n$ denotes the truncated sequence $(a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in \ell^2(\mathbb{N})$. From this it is also clear that $\|A_n\| = \sum_{k=1}^{n} |a_k|^2$.

Now fix an arbitrary $x \in \ell^2(\mathbb{N})$. Since $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$, it follows that the set of complex numbers $\{A_n(x) : n \in \mathbb{N}\}$ is bounded. Since this holds for all x, the Uniform Boundedness Theorem gives us that the set $\{\|A_n\| : n \in \mathbb{N}\}$ is bounded, i.e. that $\sup_{n \in \mathbb{N}} \|A_n\| = \sum_{k=1}^{\infty} |a_k|^2 < \infty$. Therefore, $y \in \ell^2(\mathbb{N})$.

Spring 2015

Problem 3

Let $P(x) : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. Show that there exists a constant C depending onl yon n such that $|P(\xi)| \leq C \int_{-1}^{1} |P(x)|^2 dx$ for all $\xi \in (-1, 1)$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
The problem as stated is incorrect. To see this simply consider the polynomials $p_n(x) = \frac{1}{n}$, $n \in \mathbb{N}$, each of degree 0. Then we have $|p_n(\xi)| = \frac{1}{n}$ while $\int_{-1}^{1} |p_n(x)|^2 \, dx = \frac{2}{n^2}$, so clearly no such C can exist.

However, if we replace the right hand side with $C\left(\int_{-1}^{1} |P(x)|^2 \, dx \right)^{1/2}$ then the problem becomes feasible. Note that the space of polynomials of degree $\leq n$ on the domain $[-1, 1]$, denoted $P_n([-1, 1])$ is a vector space of degree $n+1 < \infty$. What's more, the supremum norm and the L^2-norm are norms on this space. Then because any two norms on a finite dimensional vector space are equivalent we have that there exist constants c_1, c_2 depending only on n such that:

$$c_1 \|P\|_{L^2} \leq \|P\|_\infty \leq c_2 \|P\|_{L^2} \, \forall P \in P_n([-1, 1])$$

which gives the desired result.

Problem 4

Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of measurable functions. Suppose

1. $\int_0^1 |f_n(x)|^2 dx \leq 1 \forall n \in \mathbb{N}$, and
2. $f_n \to 0$ almost everywhere.

Show that

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx = 0.$$

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
By Egorov's Theorem, for every $\epsilon > 0$ there exists a set $\Omega \subseteq [0,1]$ such that $f_n \to 0$ uniformly on Ω and $\mu(\Omega^c) < \epsilon$, where naturally $\Omega^c = [0,1] \setminus \Omega$. Then we have:

$$\begin{aligned}\left|\int_0^1 f_n(x)\,dx\right| \leq \int_0^1 |f_n(x)|\,dx &= \int_\Omega |f_n(x)|\,dx + \int_{\Omega^c} |f_n(x)|\,dx \\ &\leq \mu(\Omega)\sup_{x\in\Omega}|f_n(x)| + \int_{\Omega^c} |f_n(x)|\,dx \\ &\leq \sup_{x\in\Omega}|f_n(x)| + \int_{\Omega^c} |f_n(x)|\,dx\end{aligned}$$

We can make the first term in that sum arbitrarily small for large n since f_n converges to 0 uniformly on Ω. For the second term, applying Holder's Inequality gives:

$$\int_{\Omega^c} |f_n(x)|\,dx \leq \|f_n\|_{L^2(\Omega^c)}\|1\|_{L^2(\Omega^c)} \leq \mu(\Omega^c)^{1/2} < \sqrt{\epsilon}.$$

Since this holds for all $\epsilon > 0$ we have that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = 0.$$

Spring 2014

Problem 2

Show that a linear operator T from a Banach space X to a Hilbert space H is bounded if and only if $x_n \rightharpoonup x$ imples that $Tx_n \rightharpoonup Tx$ for every weakly convergent sequence $(x_n) \subset X$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

First suppose that T is bounded and let $x_n \rightharpoonup x$ be a weakly convergent sequence in X. We wish to show that $Tx_n \rightharpoonup Tx$. Let $\phi \in H^*$ be arbitrary. Since T is bounded, it follows that $(\phi \circ T) : X \to \mathbb{R}$ is a bounded linear functional. Hence $(\phi \circ T) \in X^*$ which implies that $(\phi \circ T)(x_n) = \phi(Tx_n) \to (\phi \circ T)(x) = \phi(Tx)$. Since ϕ was taken to be arbitrary it follows that $Tx_n \rightharpoonup Tx$, as desired.

Now suppose instead that $x_n \rightharpoonup x$ imples that $Tx_n \rightharpoonup Tx$ for every weakly convergent sequence $(x_n) \subset X$. We want to show that T is bounded. We will use the closed graph theorem. Let $(x_n, Tx_n) \subset X \times H$ be a convergent sequence on the graph of T, so $(x_n, Tx_n) \to (x, y)$ for some $x \in X$ and $y \in H$. If we can show that $y = Tx$ then the graph of T is closed and hence T is bounded. To show this, note that $(x_n, Tx_n) \to (x, y) \implies x_n \to x \implies x_n \rightharpoonup x \implies Tx_n \rightharpoonup Tx$. However, we also have that $Tx_n \to y \implies Tx_n \rightharpoonup y$. Since weak limits are unique it follows that $Tx = y$, completing the proof.

Problem 3

Let f, $f_k : E \to [0, \infty)$ be non-negative, Lebesgue integrable functions on a measurable set $E \subseteq \mathbb{R}^n$. If (f_k) converges to f pointwise a.e. and

$$\int_E f_k dx \to \int_E f dx,$$

show that

$$\int_E |f - f_k| dx \to 0.$$

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
By the triangle inequality we have that $|f - f_k| \le |f| + |f_k| = f + f_k$, since $f, f_k \ge 0$. Therefore for all $k \in \mathbb{N}$, $g_k := f + f_k - |f - f_k| \ge 0$ and $g_k \to 2f$ almost everywhere. Now, applying Fatou's Lemma we see that:

$$\int_E \liminf g_k = \int_E 2f \le \liminf \int_E g_k = \liminf \int_E f + f_k - |f - f_k|$$
$$\implies \int_E 2f \le \int_E 2f - \limsup \int_E |f - f_k|$$

where the right hand side of the second inequality comes from leveraging the fact that $\lim \int_E f_k$ exists and equals $\int_E f$, and by using standard manipulations of lim inf and lim sup. If this limit had not existed we would not have been able to split the lim inf in this way. From this equation it follows that

$$\limsup \int_E |f - f_k| = 0 \implies \lim \int_E |f - f_k| = 0.$$

Problem 4

Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that P_1P_2 is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that P_1P_2 is the orthogonal projection onto $H_1 \cap H_2$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
($\Rightarrow$) Suppose P_1P_2 is an orthogonal projection. Then $P_1P_2 = (P_1P_2)^* = P_2^*P_1^*$. By P_1, P_2 each orthogonal projections, $P_1 = P_1^*$ and $P_2 = P_2^*$, so $P_1P_2 = P_2P_1$.

($\Leftarrow$) Suppose $P_1P_2 = P_2P_1$. Then $P_1P_2 = P_2^*P_1^*$ since each is an orthogonal projection, and $P_1P_2 = P_2^*P_1^* = (P_1P_2)^*$. Observe $(P_1P_2)^2 = P_1^2P_2^2$ by commutative assumption,and since each is an orthogonal projection, $(P_1P_2)^2 = P_1P_2$. Thus, P_1P_2 is an orthogonal projection.

Assume P_1P_2 is an orthogonal projection. Take $z \in ran(P_1P_2)$. Since they commute $\implies$ $z \in ran(P_2P_1)$. Thus $z \in ran(P_1)$ and $z \in ran(P_2)$, so P_1P_2 projects onto $H_1 \cap H_2$.

Problem 5

Show that if X is a separable Hilbert space with orthonormal basis $\{f_i\}_{i\ge 1}$ and $T \in B(X)$ is defined by $T(f_k) = \frac{1}{k} f_{k+1}$ then T is compact and has no eigenvalues.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Recall that an operator on a Hilbert space is called Hilbert-Schmidt if there exists an orthonormal basis e_n such that:

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

Observe,

$$\sum_{n=1}^{\infty} \|Af_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{k^2} < \infty,$$

so A is Hilbert-Schmidt. It is a theorem that any Hilbert-Schmidt operator is compact, so A is compact. To see that T has no eigenvalues, suppose $Tv = \lambda v$ for some $v \in X$ and $\lambda \in \mathbb{C}$. Since $\{f_n : n \in \mathbb{N}\}$ is an orthonormal basis we may express v as

$$v = \sum_{n=1}^{\infty} \langle v, f_n \rangle f_n$$

So we have that:

$$Tv = \sum_{n=1}^{\infty} \langle v, f_n \rangle T f_n = \sum_{n=1}^{\infty} \frac{1}{n} \langle v, f_n \rangle f_{n+1} = \sum_{n=1}^{\infty} \lambda \langle v, f_n \rangle f_n.$$

Therefore, for all $n \in \mathbb{N}$ we require that: $(1/n)\langle v, f_n \rangle = \lambda \langle v, f_{n+1} \rangle$. If $\langle v, f_1 \rangle \neq 0$ then the series

$$\sum_{n=1}^{\infty} |\langle v, f_n \rangle|^2$$

does not converge, which is a contradiction. Hence it must be the case that $\langle v, f_1 \rangle = 0 \implies \langle v, f_n \rangle = 0$ for all $n \in \mathbb{N}$, so $v = 0$. Thus T has no eigenvalues.

Fall 2014

Problem 4

Let $C_0()$ denote the Banach space of continuous functions $f :\to$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm.

(a) For $n \in \natural$, define $f_n \in C_0()$ by

$$f_n = \begin{cases} 1 & |x| \leq n \\ \frac{n}{|x|} & |x| > n \end{cases}$$

Show that $F = \{f_n : n \in \mathbb{N}\}$ is a bounded, equicontinuous subset of $C_0()$, but that the sequence (f_n) has no uniformly convergent subsequence. Why doesn't this example contradict the Arzelà-Ascoli theorem?

(b) A family of functions $F \subset C_0()$ is said to be tight if for every $\epsilon > 0$ there exists a constant $M > 0$ such that $|f(x)| < \epsilon$ for all $x \in$ with $|x| \geq M$ and all $f \in F$. Prove that $F \subset C_0()$ is pre-compact in $C_0()$ if it is bounded, equicontinuous, and tight.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

(a) We get boundededness immediately as $|f_n| \leq 1$ for all n. For equicontinuity let $x \in$ and fix $\epsilon > 0$. Since the f_n's are even we may assume without loss of generality that $x \geq 0$. Hence let $N = \lfloor x \rfloor$, so $x \in [N, N+1]$. Note that for all $n > N+1$ $f_n(u) = 1$ for all $|u| \leq N+2$, hence if $|u - x| < 1$ then $|f(u) - f(x)| = 0$. Then since $f_1, f_2, \ldots, f_{N+1}$ are continuous there exist $\delta_1, \delta_2, \ldots, \delta_{N+1}$ such that if $|u - x| < \delta_i$ then $|f_i(u) - f_i(x)| < \epsilon$ for each $i = 1, 2, \ldots N+1$. Hence letting $\delta = \min\{1, \delta_1, \ldots, \delta_{N+1}\}$ we have that $|u - x| < \delta \implies |f_n(u) - f_n(x)| < \epsilon$ for all $n \in \mathbb{N}$. Hence F is equicontinuous.

To see that F has no uniformly convergent subsequence note that for all n, $f_n(x) \to 0$ as $|x| \to \infty$ but that f_n converges pointwise to 1 as $n \to \infty$. However this does not contradict Arzelà-Ascoli because that theorem presupposes that we are working with functions over a compact domain.

(b) We will show that F is totally bounded since that is equivalent to pre-compact in a metric space. Fix $\epsilon > 0$. Because F is tight there exists an $M > 0$ such that $|f(x)| < \epsilon/2$ for all $|x| > M$. Now consider the set $G = \{g_n = f_n|_{[-M,M]} : n \in \mathbb{N}\} \subset C([-M, M])$. Note that G is bounded and equicontinuous, this is inherited from F. Then since $[-M, M]$ is compact we have that G is compact in $C([-M, M])$ by the Arzelà-Ascoli theorem.

Hence G is totally bounded so there exist $h_1, \dots h_N \in G$ such that $G \subseteq \bigcup_{i=1}^{N} B(h_i, \epsilon)$, where $B(h_i, \epsilon)$ is the ball of radius ϵ centered at h_i, taken with respect to the sup norm on $[-M, M]$. Now let $f_1, \dots f_N$ be the corresponding funcitons in F and consider an arbitrary $f \in F$ and $x \in$. Since the h's form a finite ϵ-net there exists some f_i in our finite collection such that $|f(x) - f_i(x)| < \epsilon$ for all $|x| < M$. Then for $|x| > M$ we have that $|f(x) - f_i(x)| \leq |f(x)| + |f_i(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\bigcup_{i=1}^{N} B(f_i, \epsilon)$ forms an ϵ-net cover of F, so F is totally bounded and pre-compact.

Spring 2013

Problem 1

Find the closest element to the constant function $g(x) = 1 \in L^2[0,1]$ in $V = \{f \in L^2[0,1] | \int_0^1 xf(x)dx = 0\} \subseteq L^2[0,1]$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Observe that $V = Span\{x\}^{\perp}$, and hence is a closed linear subspace of $L^2([0,1])$. Hence the desired function is simply the projection of g onto V, which we will denote by $P_v g(x)$. Recall that this projection is the unique function in the V satisfying: $g - P_v g \in V^{\perp}$. But $V^{\perp} = (Span\{x\}^{\perp})^{\perp} = \overline{Span\{x\}} = Span\{x\}$, since $Span\{x\}$ is 1-dimensional and hence closed. Thus we have that $P_v g$ satisfies:

$$g(x) - P_v g(x) = cx \implies 1 - P_v g(x) = cx \implies P_v g(x) = 1 - cx,$$

for some scalar c. To determine the value of c recall that $P_v g \in V$ so we have that:

$$\int_0^1 x(1 - cx)\, dx = \frac{1}{2} - \frac{c}{3} = 0.$$

Thus we have that $c = 3/2$ and the desired function is $P_v g(x) = 1 - (3/2)x$.

Fall 2013

Problem 1

Find $\inf \int_0^1 |f(x) - x|^2 dx$ where the infimum is taken over all $f \in L^2([0,1])$ such tha $\int_0^1 f(x)(x^2-1)dx = 1$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let $g(x) = x^2 - 1$ and let $U = \{u \in L^2([0,1]) : \langle u, g \rangle = 1\}$, so we can formulate the problem as finding $\inf_{u \in U} \|u(x) - x\|_{L^2}^2$. Now, let $\hat{g} = \frac{1}{\|g\|_{L^2}^2} g$ and consider the map $T : L^2([0,1]) \to L^2([0,1])$ defined by:

$$T(f) = f - \hat{g}.$$

Note T is a bijection and a distance preserving map, since for all $f_1, f_2 \in L^2([0,1])$ we have

$$\|Tf_1 - Tf_2\|_{L^2} = \|f_1 - \hat{g} - f_2 + \hat{g}\|_{L^2} = \|f_1 - f_2\|_{L^2}.$$

Therefore we can express the desired quantity as:

$$\inf_{u \in U} \|u(x) - x\|_{L^2}^2 = \inf_{f \in T(U)} \|f(x) - Tx\|_{L^2}^2.$$

But note, if $u \in U$ then

$$\langle Tu, g \rangle = \langle u - \hat{g}, g \rangle = \langle u, g \rangle - \langle \hat{g}, g \rangle = 0,$$

so in fact $T(U) = Span\{g\}^\perp$. Thus we have reformulate the problem as finding $\inf_{f \in Span\{g\}^\perp} \|f(x) - Tx\|_{L^2}^2$. Since $Span\{g\}^\perp$ is linear subspace, this infimum is equal to the norm squared of the projection of Tx onto $(Spang^\perp)^\perp = \overline{Span\{g\}} = Span\{g\}$, since any finite dimensional subspace is closed. Hence we only have to calculate the norm-squared of this projection, which is given by:

$$\left(\frac{\langle Tx, g \rangle}{\|g\|_{L^2}} \right)^2 = \left(\frac{\langle x - \hat{g}, g \rangle}{\|g\|_{L^2}} \right)^2.$$

Problem 4

Let H be a separable infinite dimensional Hilbert space and suppose that $e_1, e_2, \dots$ is an orthonormal system in H. Let $f_1, f_2, \dots$ be another orthonormal system which is complete (i.e., the closure of the span of $\{f_i\}_i$ is all of H). Prove that if $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ then $\{e_1\}_i$ is also a complete orthonormal system.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let $v \in \overline{Span\{e_n\}_{n=1}^{\infty}}$, it suffices to show that $v = 0$. Note that $\langle v, e_n \rangle = 0$ for all $n \in \mathbb{N}$. Also, because $\{f_n\}$ is a complete orthonormal system, we have from Parseval's Identity that:

$$\|v\|^2 = \sum_{n=1}^{\infty} \langle v, f_n \rangle = \sum_{n=1}^{\infty} \langle v, f_n - e_n \rangle^2 \leq \sum_{n=1}^{\infty} \|v\|^2 \|e_n - f_n\|^2 = \|v\|^2 \sum_{n=1}^{\infty} \|e_n - f_n\|^2,$$

where the inequality comes from applying Cauchy-Schwarz. Now if $\|v\|$ were not zero, then dividing both sides by $\|v\|^2$ and using the given inequality would give us that $1 < 1$, which is absurd. So it must be the case that $\|v\| = 0 \implies v = 0$, as desired.

Spring 2011

Problem 1

Show that the sequence $f_n = ne^{-nx}$ does not converge weakly in $L^1([0,1])$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
For some given $\epsilon > 0$ consider the map $T_\epsilon : L^1([0,1]) \to$ defined by

$$T_\epsilon f = \int_0^\epsilon f(x)\,dx = \int_0^1 f(x)\chi_{[0,\epsilon]}(x)\,dx,$$

Note that since $\chi_{[0,\epsilon]}(x) \in L^\infty([0,1])$ this defines a linear functional on $L^1([0,1])$ (it is also not hard to verify this directly). Now, observe that

$$T_\epsilon f_n = \int_0^\epsilon ne^{-nx}\,dx = 1 - \frac{1}{e^{-n\epsilon}},$$

so $T_\epsilon f_n \to 1$ for all $\epsilon > 0$. Hence if there did exist some $f \in L^1([0,1])$ such that $f_n \rightharpoonup f$ we would require that

$$\int_0^\epsilon f(x)\,dx = 1$$

for all $\epsilon > 0$, which is absurd (can prove via DCT). Hence f_n cannot converge weakly to any $f \in L^1([0,1])$.

Problem 6

For $\alpha \in (0,1]$, the space of Hölder continuous functions on the interval $[0,1]$ is defined as

$$C^{9,\alpha}([0,1]) = \{u \in C[0,1] : |u(x) - u(y) \leq C|x-y|^{\alpha} \ \forall x, y \in [0,1]\}$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x\in[0,1]} |u(x)| + \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let $U = \{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$, and let $u \in U$ be arbitrary. Note that for all $x \in [0,1]$, $|u(x)| \leq \|u\|_\infty \leq \|u\|_{C^{0,\alpha}} \leq 1$, so U is uniformly bounded. Now fix $\epsilon > 0$ and note that if $y \in [0,1]$ with $|x-y| < \epsilon^{1/\alpha}$ then we have:

$$\frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq \|u\|_{C^{0,\alpha}} \leq 1 \implies |u(x)-u(y)| < \epsilon,$$

so U is equicontinuous. Then by the Arzelà-Ascoli Theorem we have that U is pre-compact in $C([0,1])$. Thus we only have left to show that U is closed in $C([0,1])$, which follows from ...

Fall 2011

Problem 3

1. Define the compactness of an operator $A \in B(X)$ with X a Hilbert space in terms of properties of the images of bounded sets.
2. Suppose that X is separable with orthonormal basis $\{e_i\}_{i \geq 0}$ and write P_i for the orthogonal projection onto the span on $\{e_0, \ldots, e_i\}$. Show that $A \in B(X)$ is compact iff $\{P_i A\}$ converges to A in norm.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

We say that an operator $A \in B(X)$ is compact if it maps bounded sets to pre-compact sets, i.e., if $A(B)$ is pre-compact for all $B \subseteq X$ bounded. Now suppose $A \in B(X)$ is a compact operator, we wish to show that $(P_n A)$ converges to A in norm. Fix $\epsilon > 0$ and let $\overline{B}$ denote the closed unit ball in X. Since this set it bounded, it follows that $A(\overline{B})$ is pre-compact. Hence, by Theorem 9.17 in H&N, there exists an $N \in \mathbb{N}$ such that for all $a \in A(B)$ and $n > N$ we have that:

$$\sum_{k=n}^{\infty} |\langle a, e_k \rangle|^2 < \epsilon^2.$$

Now let $x \in X$ such that $\|x\| = 1$. Since $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal basis we may write

$$Ax = \sum_{k=1}^{\infty} \langle Ax, e_k \rangle e_k \implies P_i Ax = \sum_{k=1}^{i} \langle Ax, e_k \rangle e_k.$$

Hence for all $n > N$ we have that:

$$\|Ax - P_n Ax\|^2 = \sum_{k=n}^{\infty} \langle Ax, e_k \rangle e_k < \epsilon^2 \implies \|Ax - P_n Ax\| < \epsilon,$$

so we have that $(P_n A)$ converges to A in norm.

For the reverse direction suppose that (P_nA) converges to A in norm. One may take it as a fact that the uniform limit of compact operators is compact, but to be thorough we will prove this statement. Suppose $T_n \in B(X, y)$ is a sequence of compact operators such that $T_n \to T$ in the operator norm. Let B be a bounded set and let Tx_n be a sequence in $T(B)$. We wish to show that Tx_n has a convergent subsequence. First note that since T_1 is compact, T_1x_n has a convergent subsequence, which we will denote $T_1x_{n,1}$. Now since T_2 is compact $T_2x_{n,1}$ also contains its own convergent subsequence $T_2x_{n,2}$. Repeating this process for all $n \in \mathbb{N}$ and taking the diagonal sequence of inputs, $x_{n,n}$, we see that $T_ix_{n,n}$ converges for all $i \in \mathbb{N}$. Now fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be sufficiently large so that $\|T_N - T\| < \epsilon/3$ and let M be sufficiently large so that $||T_Nx_{n,n} - T_Nx_{k,k}|| < \epsilon/3$ for all $k, n > M$. Then for all $k, n > M$ we have that:

$$\begin{aligned}\|Tx_{n,n} - Tx_{k,k}\| &\leq \|Tx_{n,n} - T_Nx_{n,n}\| + \|T_Nx_{n,n} - T_Nx_{k,k}\| + \|T_Nx_{k,k} - Tx_{k,k}\| \\ &\leq 2\|T_N - T\| + \|T_Nx_{n,n} - T_Nx_{k,k}\| \\ &< \epsilon.\end{aligned}$$

Thus the subsequence $Tx_{n,n}$ is Cauchy and hence convergent, so $T(B)$ is sequentially pre-compact which is equivalent to pre-compact as we are in a metric space.

Spring 2009

Problem 3

1. Show that if (x_n) is a sequence of elements of a Hilbert space $\mathcal{H}$ converging weakly to x and $||x_n||$ also converges to $||x||$ then the sequence converges to x in norm

2. Find a sequence of elements (x_n) of a Hilbert space converging weakly to x with $\liminf_{n\to\infty} ||x_n|| > ||x||$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Suppose $x_n \rightharpoonup x$ in a Hilbert space H and furthermore $\lim_{n\to\infty}\|x_n\| = \|x\|$. Then observe:

$$\lim_{n\to\infty}\|x_n-x\|^2 = \lim_{n\to\infty}\langle x_n-x, x_n-x\rangle = \lim_{n\to\infty}\langle x_n, x_n\rangle - 2\langle x_n, x\rangle + \langle x, x\rangle = 2\|x\|^2 - 2\|x\|^2 = 0.$$

For part 2, consider the sequence $(e_n) \subseteq \ell^2(\mathbb{N})$ where e_i is the i-th element of the standard basis of $\ell^2(\mathbb{N})$, so the k-th term of e_i, denoted e_i^k, is 0 if $k \neq i$ and 1 if $k = i$. Then observe for arbitrary $y \in \ell^2(\mathbb{N})$ we have $\langle e_n, y\rangle = y^n \to 0$ as $n \to \infty$, so $e_n \rightharpoonup 0$ but $\liminf_{n\to\infty}\|e_n\| = 1 > 0$.

Problem 4

Let $\mathcal{H}$ be a Hilbert space (not necessarily separable) and let $B \subset \mathcal{H}$ denote the closed unit ball.

1. Show that B is weakly sequentially compact.

2. If $T \in B(\mathcal{H})$ is compact, prove that $T(B)$ is closed.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
For the first problem, let $(x_n) \subseteq B$. Define $U = \overline{Span\{x_n\}_{n=1}^{\infty}}$. Note that U is a separable Hilbert space (since it has a countable basis) and so by Banach

Alaglou the closed unit ball of U^* is weak-$*$ compact. But U being Hilbert means it is isomorphic to its dual under the embedding:

$$u \mapsto \tilde{u}(v) = \langle v, u \rangle.$$

Hence U^* is separable so the closed unit ball in U^* is weak-$*$ sequentially compact. By identifying each x_n with its embedding in the dual this means that there exists a subsequence (x_{n_k}) such that $x_{n_k} \rightharpoonup x$ in U. That is for all $u \in U$ we have that:

$$\langle x_{n_k}, u \rangle \to \langle x, u \rangle.$$

Now let $y \in \mathcal{H}$ be arbitrary. Recall from the Riesz Representation Theorem that the map $T_y : \mathcal{H} \to \mathbb{C}$ defined by $T_y(x) = \langle x, y \rangle$ is a bounded linear functional on $\mathcal{H}$. Hence $T_y|_U$ is a bounded linear functional on U. Applying Riesz again this means that $T_y|_U(x) = \langle x, u \rangle$ for some $u \in U$ (in particular u is the projection of y on U). Hence we have that:

$$\langle x_{n_k}, y \rangle = \langle x_{n_k}, u \rangle \to \langle x, u \rangle = \langle x, y \rangle,$$

so in fact $x_{n_k} \rightharpoonup x$ in $\mathcal{H}$ as desired.

Now for the second problem, let $(Tx_n) \subseteq T(B)$ be a convergent sequence, so $Tx_n \to y \in \mathcal{H}$. We want to show $y \in T(B)$. From the previous part, B is weakly sequentially compact so there exists a subsequence (x_{n_k}) of (x_n) converging weakly to some $x \in B$. That is, for all $\phi \in \mathcal{H}^*$ we have that $\phi x_{n_k} \to \phi x$. But note that for all $\phi \in \mathcal{H}^*$, the map $\phi \circ T$ defines a bounded linear functional on $\mathcal{H}$ so it must be the case that $\phi \circ Tx_{n_k} \to \phi \circ Tx$. Hence $Tx_{n_k} \rightharpoonup Tx$. But $Tx_n \to y \implies Tx_n \rightharpoonup y \implies Tx_{n_k} \rightharpoonup y$. Since weak limits are unique it follows that $y = Tx$, so $T(B)$ is closed.

Spring 2008

Problem 3

> Show that if V is a closed subspace of a Hilbert space $\mathcal{H}$ and ϕ is a bounded linera map from V to $\mathbb{C}$ then tehre is a unique bounded map from $\mathcal{H}$ to $\mathbb{C}$ which is an extension of ϕ and has the same operator norm as ϕ.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

I assume we can't simply apply the Hahn-Banach theorem to this question. Note that V is itself a Hilbert space as it is a closed linear subspace of $\mathcal{H}$. Hence by the Riesz Representation Theorem we may express ϕ as $\phi(v) = \langle v, u \rangle$ for some fixed $u \in V$. Note that by the Cauchy Schwarz inequality, $\|\phi(v)\| \leq \|v\|\|u\|$, and $\|\phi(u)\| = \|u\|^2$, so $\|\phi\| = \|u\|$.

Now since $u \in \mathcal{H}$, we can define a bounded linear functional Φ on $\mathcal{H}$ so that $\Phi(x) = \langle x, u \rangle$. Clearly Φ agrees with ϕ on V and

$$\|\Phi\| = \sup_{\|x\|=1, x \in \mathcal{H}} |\langle x, u \rangle| \geq \sup_{\|v\|=1, v \in V} |\langle v, u \rangle| = \|\phi\|.$$

At the same time, $\|\Phi(x)\| \leq \|u\|\|x\| = \|\phi\|\|x\|$, again by Cauchy-Schwarz, so in fact $\|\Phi\| = \|\phi\|$, as desired.

To show this is the unique linear functional extending ϕ and having the same norm, suppose there existed some other functional, ψ with these properties. By Riesz, we may express ψ as $\psi(x) = \langle x, y \rangle$ for some $y \in \mathcal{H}$. Since V is a closed linear subspace, we may write $y = a + b$ where $a \in V$ and $b \in V^{\perp}$. Then note for all $v \in V$:

$$\langle v, u \rangle = \langle v, y \rangle = \langle v, a + b \rangle = \langle v, a \rangle + \langle v, b \rangle = \langle v, a \rangle,$$

therefore it must be the case that $a = u$. But note, by a similar argument as before, that

$$\|\psi\| = \|y\| = \sqrt{\langle a + b, a + b \rangle} = \sqrt{\|a\|^2 + \|b\|^2} = \sqrt{\|u\|^2 + \|b\|^2}.$$

In order for $\|\psi\| = \|\phi\| = \|u\|$, it must be the case that $b = 0 \implies y = u \implies \psi = \Phi$.

Fall 2008

Problem 2

Show that if $\{f_n\}$ is a sequence of continuously differentiable function on $[0, 1]$ and both the original sequence and the sequence of derivatives are uniformly bounded, then (f_n) has a uniformly convergent subsequence.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Since the sequence $\{f_n\}$ is uniformly bounded, we can get the equicontinuity of $\{f_n\}$. By the equicontinuity and uniform boundedness of $\{f_n\}$, appling Arzela-Ascoli, $\{f_n\}$ is precompact and there exists a subsequence $\{f_{n_k}\}$ converging uniformly.

Spring 2006

Problem 5

Take $\{u_k\}_{k\in\mathbb{N}}$ to be an orthonormal set in the Hilbert space X. Characterize those sequences of scalars (a_k) such that $(a_k u_k)$ is compact in X.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
We claim it is necessary and sufficient that $(a_n) \to 0$ and that at least one $a_k = 0$. First consider what happens if (a_n) does not converge to 0. Then there exists an $\epsilon > 0$ such that $|a_n| > \epsilon$ for infinitely many $n \in \mathbb{N}$. Consider this subsequence, denoted (a_{n_k}) and note that for $k_1 \neq k_2$:

$$\|a_{n_{k_1}} u^2_{n_{k_1}}\|_{-a_{n_{k_2}} u_{n_{k_2}}} = |a_{n_{k_1}}|^2 + |a_{n_{k_2}}|^2 > \epsilon^2 + \epsilon^2 = 2\epsilon^2.$$

$$\implies \|a_{n_{k_1}} u_{n_{k_1}}\|_{-a_{n_{k_2}} u_{n_{k_2}}} > \sqrt{2}\epsilon > \epsilon$$

Therefore $(a_{n_k} u_{n_k})$ cannot be covered by a finite $\epsilon/2$-net, hence it is not pre-compact and thus not compact.

Now consider the situation where $(a_n) \to 0$, and fix $\epsilon > 0$. Note that there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have that $|a_n| < \epsilon/\sqrt{2}$. Now consider the collection of open balls $\mathcal{B} = \bigcup_{k=1}^{N+1} B(a_k u_k, \epsilon)$. We claim that this ϵ-net covers $(a_n u_n)$. Cleary $a_n u_n \in \mathcal{B}$ for all $n \leq N+1$. Then if $n > N+1$ note that:

$$\|a_n u_n - a_{N+1} u_{N+1}\|^2 = |a_n|^2 + |a_{N+1}|^2 < \epsilon^2/2 + \epsilon^2/2 = \epsilon^2$$

$$\implies \|a_n u_n - a_{N+1} u_{N+1}\| < \epsilon,$$

so $a_n u_n \in B(a_{N+1} u_{N+1}, \epsilon)$, and hence in $\mathcal{B}$. So we have that $\mathcal{B}$ is a finite ϵ-net covering $(a_n u_n)$, so the set is pre-compact. We only have left to show that it is closed. But note that if $a_n \to 0$ then $(a_n u_n) \to 0$ in norm. Hence the only limit points of $(a_n u_n)$ are the terms of the sequence and zero. Thus we require that at least one $a_k = 0$ so that 0 is an element in the set.