UC Davis Analysis Preliminary Exam Solutions

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Contents

Spring 2019

Problem 2

Let $S = [0,1] \times [0,1]$ and consider the space $C(S)$ of continuous complex-valued functions on S equipped with the sup-norm. Define $F \subset C(S)$ by

$$F = \{f \in C(S) : \exists n \geq 1 \text{ and } g_1, \dots, g_n, h_1, \dots, h_n \in C([0,1]) \text{ such that } f(x,y) = \sum_{k=1}^{n} g_k(x)h_k(y)\}.$$

Show that F is dense in $C(S)$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let S and F be given as above. In order to use the Stone-Weierstrass theorem, we must show that F is an algebra that is nonvanishing, separates points, and for any $f \in F$, we must also have $\overline{f} \in F$.

We first note that F is nonvanishing because the constant function $\chi_S \in F$.

Next, take any $(x_1, y_1), (x_2, y_2) \in S$. Without loss of generality, suppose that $x_1 \neq x_2$. Let $\epsilon = d(x_1, x_2)$ and denote $A = B_\epsilon/2(x_1)$ and $B = S \backslash A$. Then there exists a continuous Urysohn function

$$\rho(x,y) = \frac{d(x,A)}{d(x,A) + d(x,B)} \in F$$

such that $\rho(x_1, y) = 1$ and $\rho(x_2, y) = 0$. Thus, F separates points.

Now if we examine $f(x,y) = \sum_{k=1}^n g_k(x)h_k(y) \in F$, we can see that $\overline{f}(x,y) = \overline{\sum_{k=1}^n g_k(x)h_k(y)} = \sum_{k=1}^n \overline{g_k}(x)\overline{h_k}(y) \in F$ since $g_k(x) \in C([0,1])$ implies $\overline{g_k}(x) \in C([0,1])$.

Finally, F is an algebra because $(\sum_{k=1}^n g_k(x)h_k(y)) \left(\sum_{j=1}^m g_j(x)h_j(y)\right)$ distributes, giving us another element of F.

Thus, by the Stone-Weierstrass theorem, F is dense in $C(S)$.

Problem 6

Let $\Omega = (0,1) \subset \mathbb{R}$. For $\overline{u} = \int_\Omega u(x)dx$ show that

$$\|u - \overline{u}\|_{L^\infty(\Omega)} \leq \|u'\|_{L^2(\Omega}$$

for all $u \in W^{1,1}(\Omega)$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let $x \in (0,1)$, and let $x_0 \in (0,1)$ be such that $\overline{u} = u(x_0)$. By the Fundamental Theorem of Calculus we have:

$$|u(x) - u(x_0)| = \left| \int_{x_0}^{x} u'(t)\, dt \right| \leq \int_0^1 |u'(t)|\, dt \leq \|u'\|_{L^2(0,1)}$$

where the last inequality comes from applying Holder's Inequality.

Spring 2018

Problem 2

Let H be a Hilbert space and let $P, Q \in B(H)$ be two orthogonal projections. Prove that $\ker PQ \subseteq \ker P + \ker Q$ always, and that $\ker PQ = \ker P + \ker Q$ when PQ is also an orthogonal projection.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Take $z \in ker(PQ)$. Then $PQ(z) = 0$. $\implies Q(z) \in ker(P)$. So $z \in ran(Q)$ or $z \in ker(Q)$. If $z \in ran(Q)$ then $z \in ker(P)$, otherwise $z \in ker(Q)$. Thus, if $z \in ker(PQ) \implies z \in ker(P) + ker(Q)$. Therefore, $ker(PQ) \subseteq ker(P) + ker(Q)$.

Now suppose PQ is also an orthogonal projection. Then $PQ = (PQ)^* = Q^*P^*$. Since P and Q are orthogonal projections, $P = P^*$ and $Q = Q^*$, so $PQ = QP$. Take $z \in ker(P) \implies PQ(z) = 0$. Since $PQ = QP \implies QP(z) = 0$, so $z \in ker(PQ)$. Thus, $ker(P) + ker(Q) = ker(PQ)$.

Problem 4

Let T be a bounded linear operator on a Hilbert space with an orthonormal basis of eigenvectors with eigenvalues $\Lambda = \{\lambda_n\}$. Show that the spectrum $\sigma(T)$ is exactly the closure of the set Λ.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

First note that because $\sigma(T)$ is closed, $\Lambda \subseteq \sigma(T) \implies \overline{\Lambda} \subseteq \sigma(T)$. For the reverse inclusion, first note that because T has an orthonormal basis of eigenvectors, henceforth denoted $\{e_n\}$, we may express T as:

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where P_n denotes the not necessarily finite-dimensional orthogonal projection on the eigenspace of λ_n. We first show that the point spectrum of T is exactly Λ. To see this, suppose there exists $x \neq 0$ such that $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$.

$x \neq 0$ implies that there exists some $m \in \mathbb{N}$ such that $\langle x, e_m \rangle \neq 0$. Then observe:

$$Tx = \sum_{n=1}^{\infty} \lambda_n P_n x = \sum_{n=1}^{\infty} \lambda_n P_n \left(\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \right) = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n = \sum_{n=1}^{\infty} \lambda \langle x, e_n \rangle e_n = \lambda x.$$

Hence we require that $\lambda_n \langle x, e_n \rangle = \lambda \langle x, e_n \rangle$ for all n. Since $langlex, e_m \rangle \neq 0$ this implies that $\lambda = \lambda_m$.

Note, for $x, y \in \mathcal{H}$ we have

$$\begin{aligned}\langle Tx, y \rangle = \langle \sum_{n=1}^{\infty} \lambda_n P_n x, y \rangle = \sum_{n=1}^{\infty} \langle \lambda_n P_n x, y \rangle &= \sum_{n=1}^{\infty} \langle x, \overline{\lambda_n} P_n y \rangle \\ &= \langle x, \sum_{n=1}^{\infty} \overline{\lambda_n} P_n y \rangle,\end{aligned}$$

so $T^* = \sum_{n=1}^{\infty} \overline{\lambda_n} P_n$.

Fall 2018

Problem 2

Consider the function $f : [0, 1] \to$ defined by

$$f(x) = \begin{cases} x \log x & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

(a) Is f Lipschitz continuous on $[0, 1]$?

(b) Is f uniformly continuous on $[0, 1]$?

(c) Suppose (p_n) is a sequence of polynomial functions on $[0, 1]$, converging uniformly to f. Is the set $A = \{p_n : n \geq 1\} \cup \{f\}$ equicontinuous?

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

The function is not Lipschitz, to see this note that:

$$\lim_{n\to\infty} \frac{f(1/n) - f(0)}{(1/n) - 0} = \lim_{n\to\infty} \log(1/n) = -\infty,$$

so f cannot be Lipschitz. However the function is continuous as by L'Hopital's rule:

$$\lim_{x\to 0} x \log(x) = \lim_{n\to\infty} \frac{\log(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so f is continuous and hence uniformly continuous since we are working over a compact set.

For part (c.), fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be sufficiently large so that $\|p_n - f\| < \epsilon/3$. Next, since f is uniformly continuous let $\delta > 0$ be sufficiently small so that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/3$. Then for all $n > N$, if $|x - y| < \delta$ then:

$$|p_n(x)-p_n(y)| \leq |p_n(x)-f(x)|+|f(x)-f(y)|+|f(y)-p_n(y)| \leq 2\|p_n - f\|+|f(x)-f(y)| < \epsilon$$

Finally, since $p_1, \ldots, p_N$ are each uniformly continuous there exist corresponding $\delta_1, \ldots, \delta_N$ so $\delta_0 = \min\{\delta, \delta_1, \ldots, \delta_N\}$ works for all the $\{p_n\} \cup \{f\}$, hence the set is equicontinuous.

Fall 2017

Problem 1

Let X be a Banach space with dual space X^* and let $A \subset X$ be a linera subspace. Define the annihilator $A^\perp \subset X^*$ of A by

$$A^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in A\}.$$

Prove that A is dense in X if and only if $A^\perp = \{0\}$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
($\Longrightarrow$) Suppose A is dense and let $\phi \in A^\perp$. Let $x \in X$ be arbitrary. By the density of A there exists a sequence $(a_n) \subseteq A$ such that $a_n \to x$. Then by the continuity of ϕ we have that:

$$\phi(x) = \phi(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} \phi(a_n) = 0,$$

so ϕ is identically 0.

($\Longleftarrow$) Now suppose $A^\perp = \{0\}$, and let $x_0 \in X$ be arbitrary. Consider the map $T : \overline{Span\{A, x_0\}} \to$ defined so that $T(a) = 0$ for all $a \in A$, $T(x_0) = d(x_0, A) = \inf_{a\in A}\|x_0 - a\|$, and T extends linearly to the rest of the span. One can easily verify that T is a bounded linear functional so by the Hahn-Banach Theorem T extends to a bounded linear functional $\tilde{T}$ on all of X. But then this $\tilde{T}$ would annihilate A, so it must be that $\tilde{T}$ is identically zero. But then $\tilde{T}(x_0) = T(x_0) = d(x_0, A) = 0$, so x_0 is in the closed linear span of A. Since we chose x_0 to be arbitrary it follows that A is dense in X.

Problem 2

Prove that every metric subspace of a separable metric space is separable.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let (X, d) be a separable metric space and let $S \subseteq X$ be a metric subspace of X.

Since X is separable, there exists a countable dense subset $\{x_i\}_{i=1}^{\infty}$ such that

$$X \subseteq \bigcup_{i=1}^{\infty} B_\epsilon(x_i)$$

for any $\epsilon > 0$.

In particular, S is also contained in this union. For each x_i and every $n \in \mathbb{N}$ with $B_{1/n}(x_i) \cap S \neq 0$, choose $s_j \in B_{1/n}(x_i) \cap S$. We claim that $\{s_j\}_{j=1}^{\infty}$ is a countable dense subset of S. Indeed, $\{s_j\}_{j=1}^{\infty}$ is countable because it is a countable union of countable sets. Moreover, for any $s \in S$, we know that there exists x_i such that $d(x_i, s) < \epsilon/2$ because $\{x_i\}$ is dense in X. Choosing $s_j \in \{s_j\}$ such that $d(s_j, x_i) < \epsilon/2$ then gives us that

$$d(s_j, s) \leq d(s_j, x_i) + d(x_i, s) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, we have produced a countable, dense subset of S and it follows that S is separable.

Problem 3

Prove or disprove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there exists a sequence of polynomials (p_n) on $[0,1]$ such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every integer $k \geq 0$. Here $f^{(k)}$ denotes the k-th derivative of f.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

The statement is true. Let $\mathcal{P} = \mathcal{P}([0,1])$ be the space of polynomials on $[0,1]$ and for $k \geq 0$ let $C^k([0,1])$ be the space of functions f such that $f^{(i)}$ exists and is continuous for all $i = 0, \ldots, k$. Equip $C^k([0,1])$ with its usual norm:

$$\|f\|_{C^k} = \sum_{i=0}^{k} \|f^{(i)}\|_\infty.$$

We first show that $\mathcal{P}$ is dense in $C^k([0,1])$ for all $k \geq 0$. Fix $\epsilon > 0$ and let $f \in C^k([0,1])$ be arbitrary. Then $f^{(k)} \in C([0,1])$ so by the Stone-Weierstrass Theorem there exists a polynomial $p_k \in \mathcal{P}$ such that $||f^{(k)} - p_k||_\infty < \epsilon/k$. Now consider the function p_{k-1} given by

$$p_{k-1}(x) = f^{(k-1)}(0) + \int_0^x p_k(y)dy.$$

Note $p_{k-1} \in \mathcal{P}$ and by the fundamental theorem of calculus $p_{k-1}^{(1)} = p_k$. Next, note that for any $x \in [0,1]$ we have:

$$\begin{aligned}
|f^{(k-1)}(x) - p_{k-1}(x)| &= \left| f^{(k-1)}(0) + \int_0^x f^{(k)}(y)dy - \left(f^{(k-1)}(0) + \int_0^x p_k(y)dy \right) \right| \\
&\leq \int_0^x |f^{(k)}(y) - p_k(y)|dy \\
&\leq \int_0^1 ||f^{(k)} - p_k||_\infty dy \\
&\leq \epsilon/k.
\end{aligned}$$

Hence $||f^{(k-1)} - p_{k-1}||_{C^1} \leq 2\epsilon/k$. Now repeat this process, at each step defining a new polynomial p_i so that

$$p_i(x) = f^{(i)}(0) + \int_0^x p_{i+1}(y)dy.$$

It follows from the same argument as above that $||f - p_0||_{C^k} \leq \epsilon$. Since ϵ was taken to be arbitrary we have that $\mathcal{P}$ is dense in $C^k([0,1])$.

Returning to the original problem, let $f \in C^\infty([0,1])$. For $k \geq 0$, let $(p_n^k)_{n=1}^\infty \subset \mathcal{P}$ be a sequence such that $p_n^k \to f$ in $C^k([0,1])$, these sequences are guaranteed to exist by the prior argument. Now define a "diagonal" sequence $(q_m)_{m=1}^\infty \subset \mathcal{P}$ by setting q_m to be the first term from (p_n^m) such that $||f - p_n^m||_{C^m} < 2^{-m}$. It follows that for any fixed k, $||f^{(k)} - q_m^{(k)}||_\infty \to 0$ as $m \to \infty$ so (q_m) is our desired sequence.

Spring 2017

Spring 2017

Problem 2

Suppose that (X,d) is a metric space such that every continuous function $f : X \to$ is bounded. Prove that X is complete.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let (x_n) be a Cauchy sequence in X, so that (x_n) converges to some $y \in \tilde{X}$, the completion of X. We want to show that $y \in X$, so suppose by way of contradiction that $y \notin X$. Recall that the metric d is a continuous function from $X \times X \to_{\geq 0}$. Hence the function $f : X \to$ defined by $f(x) = d(x,y)$ is continuous. What's more, since $y \notin X$ we have that $f(x) = d(x,y) \neq 0$ for all $x \in X$. Thus the function $g(x) = {}^{1}/_{f(x)}$ is also a continuous function on X. But since $x_n \to y$ we have that $d(x_n, y) \to 0$, so $g(x_n) \to \infty$, contradicting that every continuous function on X is bounded. Thus it must be the case that $y \in X$.

Fall 2016

Problem 3

Let $y = \{a_n\}_{n=1}^{\infty}$ be a sequence of real-valued scalars and assume that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$. Show that $y \in \ell^2(\mathbb{N})$

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

For $n \in \mathbb{N}$ define the linear functional $A_n : \ell^2(\mathbb{N}) \to \mathbb{C}$ so that $A_n(x) = \sum_{k=1}^{n} a_k x_k$. Note that this is in fact a bounded linear functional as we may express A in the form $Ax = \langle x, y|_n \rangle$, where $y|_n$ denotes the truncated sequence $(a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in \ell^2(\mathbb{N})$. From this it is also clear that $\|A_n\| = \sum_{k=1}^{n} |a_k|^2$.

Now fix an arbitrary $x \in \ell^2(\mathbb{N})$. Since $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$, it follows that the set of complex numbers $\{A_n(x) : n \in \mathbb{N}\}$ is bounded. Since this holds for all x, the Uniform Boundedness Theorem gives us that the set $\{\|A_n\| : n \in \mathbb{N}\}$ is bounded, i.e. that $\sup_{n \in \mathbb{N}} \|A_n\| = \sum_{k=1}^{\infty} |a_k|^2 < \infty$. Therefore, $y \in \ell^2(\mathbb{N})$.

Spring 2015

Problem 3

Let $P(x) : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. Show that there exists a constant C depending onl yon n such that $|P(\xi)| \leq C \int_{-1}^{1} |P(x)|^2 dx$ for all $\xi \in (-1, 1)$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
The problem as stated is incorrect. To see this simply consider the polynomials $p_n(x) = \frac{1}{n}$, $n \in \mathbb{N}$, each of degree 0. Then we have $|p_n(\xi)| = \frac{1}{n}$ while $\int_{-1}^{1} |p_n(x)|^2 \, dx = \frac{2}{n^2}$, so clearly no such C can exist.

However, if we replace the right hand side with $C\left(\int_{-1}^{1} |P(x)|^2 \, dx \right)^{1/2}$ then the problem becomes feasible. Note that the space of polynomials of degree $\leq n$ on the domain $[-1, 1]$, denoted $P_n([-1, 1])$ is a vector space of degree $n+1 < \infty$. What's more, the supremum norm and the L^2-norm are norms on this space. Then because any two norms on a finite dimensional vector space are equivalent we have that there exist constants c_1, c_2 depending only on n such that:

$$c_1 \|P\|_{L^2} \leq \|P\|_\infty \leq c_2 \|P\|_{L^2} \, \forall P \in P_n([-1, 1])$$

which gives the desired result.

Problem 4

Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of measurable functions. Suppose

1. $\int_0^1 |f_n(x)|^2 dx \leq 1 \forall n \in \mathbb{N}$, and
2. $f_n \to 0$ almost everywhere.

Show that

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx = 0.$$

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
By Egorov's Theorem, for every $\epsilon > 0$ there exists a set $\Omega \subseteq [0,1]$ such that $f_n \to 0$ uniformly on Ω and $\mu(\Omega^c) < \epsilon$, where naturally $\Omega^c = [0,1] \setminus \Omega$. Then we have:

$$\begin{aligned}\left| \int_0^1 f_n(x)\,dx \right| \le \int_0^1 |f_n(x)|\,dx &= \int_\Omega |f_n(x)|\,dx + \int_{\Omega^c} |f_n(x)|\,dx \\ &\le \mu(\Omega) \sup_{x\in\Omega} |f_n(x)| + \int_{\Omega^c} |f_n(x)|\,dx \\ &\le \sup_{x\in\Omega} |f_n(x)| + \int_{\Omega^c} |f_n(x)|\,dx\end{aligned}$$

We can make the first term in that sum arbitrarily small for large n since f_n converges to 0 uniformly on Ω. For the second term, applying Holder's Inequality gives:

$$\int_{\Omega^c} |f_n(x)|\,dx \le \|f_n\|_{L^2(\Omega^c)} \|1\|_{L^2(\Omega^c)} \le \mu(\Omega^c)^{1/2} < \sqrt{\epsilon}.$$

Since this holds for all $\epsilon > 0$ we have that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

Spring 2014

Problem 2

Show that a linear operator T from a Banach space X to a Hilbert space H is bounded if and only if $x_n \rightharpoonup x$ imples that $Tx_n \rightharpoonup Tx$ for every weakly convergent sequence $(x_n) \subset X$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

First suppose that T is bounded and let $x_n \rightharpoonup x$ be a weakly convergent sequence in X. We wish to show that $Tx_n \rightharpoonup Tx$. Let $\phi \in H^*$ be arbitrary. Since T is bounded, it follows that $(\phi \circ T) : X \to \mathbb{R}$ is a bounded linear functional. Hence $(\phi \circ T) \in X^*$ which implies that $(\phi \circ T)(x_n) = \phi(Tx_n) \to (\phi \circ T)(x) = \phi(Tx)$. Since ϕ was taken to be arbitrary it follows that $Tx_n \rightharpoonup Tx$, as desired.

Now suppose instead that $x_n \rightharpoonup x$ imples that $Tx_n \rightharpoonup Tx$ for every weakly convergent sequence $(x_n) \subset X$. We want to show that T is bounded. We will use the closed graph theorem. Let $(x_n, Tx_n) \subset X \times H$ be a convergent sequence on the graph of T, so $(x_n, Tx_n) \to (x, y)$ for some $x \in X$ and $y \in H$. If we can show that $y = Tx$ then the graph of T is closed and hence T is bounded. To show this, note that $(x_n, Tx_n) \to (x, y) \implies x_n \to x \implies x_n \rightharpoonup x \implies Tx_n \rightharpoonup Tx$. However, we also have that $Tx_n \to y \implies Tx_n \rightharpoonup y$. Since weak limits are unique it follows that $Tx = y$, completing the proof.

Problem 3

Let $f,\ f_k : E \to [0, \infty)$ be non-negative, Lebesgue integrable functions on a measurable set $E \subseteq \mathbb{R}^n$. If (f_k) converges to f pointwise a.e. and

$$\int_E f_k dx \to \int_E f dx,$$

show that

$$\int_E |f - f_k| dx \to 0.$$

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
By the triangle inequality we have that $|f - f_k| \leq |f| + |f_k| = f + f_k$, since $f, f_k \geq 0$. Therefore for all $k \in \mathbb{N}$, $g_k \coloneqq f + f_k - |f - f_k| \geq 0$ and $g_k \to 2f$ almost everywhere. Now, applying Fatou's Lemma we see that:

$$\int_E \liminf g_k = \int_E 2f \leq \liminf \int_E g_k = \liminf \int_E f + f_k - |f - f_k|$$
$$\implies \int_E 2f \leq \int_E 2f - \limsup \int_E |f - f_k|$$

where the right hand side of the second inequality comes from leveraging the fact that $\lim \int_E f_k$ exists and equals $\int_E f$, and by using standard manipulations of $\liminf$ and $\limsup$. If this limit had not existed we would not have been able to split the $\liminf$ in this way. From this equation it follows that

$$\limsup \int_E |f - f_k| = 0 \implies \lim \int_E |f - f_k| = 0.$$

Problem 4

Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that P_1P_2 is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that P_1P_2 is the orthogonal projection onto $H_1 \cap H_2$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
($\Rightarrow$) Suppose P_1P_2 is an orthogonal projection. Then $P_1P_2 = (P_1P_2)^* = P_2^*P_1^*$. By P_1, P_2 each orthogonal projections, $P_1 = P_1^*$ and $P_2 = P_2^*$, so $P_1P_2 = P_2P_1$.

($\Leftarrow$) Suppose $P_1P_2 = P_2P_1$. Then $P_1P_2 = P_2^*P_1^*$ since each is an orthogonal projection, and $P_1P_2 = P_2^*P_1^* = (P_1P_2)^*$. Observe $(P_1P_2)^2 = P_1^2P_2^2$ by commutative assumption,and since each is an orthogonal projection, $(P_1P_2)^2 = P_1P_2$. Thus, P_1P_2 is an orthogonal projection.

Assume P_1P_2 is an orthogonal projection. Take $z \in ran(P_1P_2)$. Since they commute $\implies$ $z \in ran(P_2P_1)$. Thus $z \in ran(P_1)$ and $z \in ran(P_2)$, so P_1P_2 projects onto $H_1 \cap H_2$.

Problem 5

Show that if X is a separable Hilbert space with orthonormal basis $\{f_i\}_{i\geq 1}$ and $T \in B(X)$ is defined by $T(f_k) = \frac{1}{k} f_{k+1}$ then T is compact and has no eigenvalues.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Recall that an operator on a Hilbert space is called Hilbert-Schmidt if there exists an orthonormal basis e_n such that:

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

Observe,

$$\sum_{n=1}^{\infty} \|Af_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{k^2} < \infty,$$

so A is Hilbert-Schmidt. It is a theorem that any Hilbert-Schmidt operator is compact, so A is compact. To see that T has no eigenvalues, suppose $Tv = \lambda v$ for some $v \in X$ and $\lambda \in \mathbb{C}$. Since $\{f_n : n \in \mathbb{N}\}$ is an orthonormal basis we may express v as

$$v = \sum_{n=1}^{\infty} \langle v, f_n \rangle f_n$$

So we have that:

$$Tv = \sum_{n=1}^{\infty} \langle v, f_n \rangle T f_n = \sum_{n=1}^{\infty} \frac{1}{n} \langle v, f_n \rangle f_{n+1} = \sum_{n=1}^{\infty} \lambda \langle v, f_n \rangle f_n.$$

Therefore, for all $n \in \mathbb{N}$ we require that: $(1/n)\langle v, f_n \rangle = \lambda \langle v, f_{n+1} \rangle$. If $\langle v, f_1 \rangle \neq 0$ then the series

$$\sum_{n=1}^{\infty} |\langle v, f_n \rangle|^2$$

does not converge, which is a contradiction. Hence it must be the case that $\langle v, f_1 \rangle = 0 \implies \langle v, f_n \rangle = 0$ for all $n \in \mathbb{N}$, so $v = 0$. Thus T has no eigenvalues.

Fall 2014

Problem 4

Let $C_0()$ denote the Banach space of continuous functions $f :\to$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm.

(a) For $n \in \natural$, define $f_n \in C_0()$ by

$$f_n = \begin{cases} 1 & |x| \leq n \\ \frac{n}{|x|} & |x| > n \end{cases}$$

Show that $F = \{f_n : n \in \mathbb{N}\}$ is a bounded, equicontinuous subset of $C_0()$, but that the sequence (f_n) has no uniformly convergent subsequence. Why doesn't this example contradict the Arzelà-Ascoli theorem?

(b) A family of functions $F \subset C_0()$ is said to be tight if for every $\epsilon > 0$ there exists a constant $M > 0$ such that $|f(x)| < \epsilon$ for all $x \in$ with $|x| \geq M$ and all $f \in F$. Prove that $F \subset C_0()$ is pre-compact in $C_0()$ if it is bounded, equicontinuous, and tight.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

(a) We get boundededness immediately as $|f_n| \leq 1$ for all n. For equicontinuity let $x \in$ and fix $\epsilon > 0$. Since the f_n's are even we may assume without loss of generality that $x \geq 0$. Hence let $N = \lfloor x \rfloor$, so $x \in [N, N+1]$. Note that for all $n > N+1$ $f_n(u) = 1$ for all $|u| \leq N+2$, hence if $|u - x| < 1$ then $|f(u) - f(x)| = 0$. Then since $f_1, f_2, \ldots, f_{N+1}$ are continuous there exist $\delta_1, \delta_2, \ldots, \delta_{N+1}$ such that if $|u - x| < \delta_i$ then $|f_i(u) - f_i(x)| < \epsilon$ for each $i = 1, 2, \ldots N+1$. Hence letting $\delta = \min\{1, \delta_1, \ldots, \delta_{N+1}\}$ we have that $|u - x| < \delta \implies |f_n(u) - f_n(x)| < \epsilon$ for all $n \in \mathbb{N}$. Hence F is equicontinuous.

To see that F has no uniformly convergent subsequence note that for all n, $f_n(x) \to 0$ as $|x| \to \infty$ but that f_n converges pointwise to 1 as $n \to \infty$. However this does not contradict Arzelà-Ascoli because that theorem presupposes that we are working with functions over a compact domain.

(b) We will show that F is totally bounded since that is equivalent to pre-compact in a metric space. Fix $\epsilon > 0$. Because F is tight there exists an $M > 0$ such that $|f(x)| < \epsilon/2$ for all $|x| > M$. Now consider the set $G = \{g_n = f_n|_{[-M,M]} : n \in \mathbb{N}\} \subset C([-M, M])$. Note that G is bounded and equicontinuous, this is inherited from F. Then since $[-M, M]$ is compact we have that G is compact in $C([-M, M])$ by the Arzelà-Ascoli theorem.

Hence G is totally bounded so there exist $h_1, \dots h_N \in G$ such that $G \subseteq \bigcup_{i=1}^N B(h_i, \epsilon)$, where $B(h_i, \epsilon)$ is the ball of radius ϵ centered at h_i, taken with respect to the sup norm on $[-M, M]$. Now let $f_1, \dots f_N$ be the corresponding funcitons in F and consider an arbitrary $f \in F$ and $x \in$. Since the h's form a finite ϵ-net there exists some f_i in our finite collection such that $|f(x) - f_i(x)| < \epsilon$ for all $|x| < M$. Then for $|x| > M$ we have that $|f(x) - f_i(x)| \leq |f(x)| + |f_i(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\bigcup_{i=1}^N B(f_i, \epsilon)$ forms an ϵ-net cover of F, so F is totally bounded and pre-compact.

Spring 2013

Problem 1

Find the closest element to the constant function $g(x) = 1 \in L^2[0,1]$ in $V = \{f \in L^2[0,1] | \int_0^1 xf(x)dx = 0\} \subseteq L^2[0,1]$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Observe that $V = Span\{x\}^\perp$, and hence is a closed linear subspace of $L^2([0,1])$. Hence the desired function is simply the projection of g onto V, which we will denote by $P_v g(x)$. Recall that this projection is the unique function in the V satisfying: $g - P_v g \in V^\perp$. But $V^\perp = (Span\{x\}^\perp)^\perp = \overline{Span\{x\}} = Span\{x\}$, since $Span\{x\}$ is 1-dimensional and hence closed. Thus we have that $P_v g$ satisfies:

$$g(x) - P_v g(x) = cx \implies 1 - P_v g(x) = cx \implies P_v g(x) = 1 - cx,$$

for some scalar c. To determine the value of c recall that $P_v g \in V$ so we have that:

$$\int_0^1 x(1-cx)\,dx = \frac{1}{2} - \frac{c}{3} = 0.$$

Thus we have that $c = 3/2$ and the desired function is $P_v g(x) = 1 - (3/2)x$.

Fall 2013

Problem 1

Find $\inf \int_0^1 |f(x)-x|^2 dx$ where the infimum is taken over all $f \in L^2([0,1])$ such tha $\int_0^1 f(x)(x^2-1)dx = 1$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let $g(x) = x^2 - 1$ and let $U = \{u \in L^2([0,1]) : \langle u, g\rangle = 1\}$, so we can formulate the problem as finding $\inf_{u\in U} \|u(x) - x\|_{L^2}^2$. Now, let $\hat{g} = \frac{1}{\|g\|_{L^2}^2} g$ and consider the map $T : L^2([0,1]) \to L^2([0,1])$ defined by:

$$T(f) = f - \hat{g}.$$

Note T is a bijection and a distance preserving map, since for all $f_1, f_2 \in L^2([0,1])$ we have

$$\|Tf_1 - Tf_2\|_{L^2} = \|f_1 - \hat{g} - f_2 + \hat{g}\|_{L^2} = \|f_1 - f_2\|_{L^2}.$$

Therefore we can express the desired quantity as:

$$\inf_{u\in U} \|u(x) - x\|_{L^2}^2 = \inf_{f\in T(U)} \|f(x) - Tx\|_{L^2}^2.$$

But note, if $u \in U$ then

$$\langle Tu, g\rangle = \langle u - \hat{g}, g\rangle = \langle u, g\rangle - \langle \hat{g}, g\rangle = 0,$$

so in fact $T(U) = Span\{g\}^\perp$. Thus we have reformulate the problem as finding $\inf_{f\in Span\{g\}^\perp} \|f(x) - Tx\|_{L^2}^2$. Since $Span\{g\}^\perp$ is linear subspace, this infimum is equal to the norm squared of the projection of Tx onto $(Spang^\perp)^\perp = \overline{Span\{g\}} = Span\{g\}$, since any finite dimensional subspace is closed. Hence we only have to calculate the norm-squared of this projection, which is given by:

$$\left(\frac{\langle Tx, g\rangle}{\|g\|_{L^2}}\right)^2 = \left(\frac{\langle x - \hat{g}, g\rangle}{\|g\|_{L^2}}\right)^2.$$

Problem 4

Let H be a separable infinite dimensional Hilbert space and suppose that $e_1, e_2, \ldots$ is an orthonormal system in H. Let $f_1, f_2, \ldots$ be another orthonormal system which is complete (i.e., the closure of the span of $\{f_i\}_i$ is all of H). Prove that if $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ then $\{e_1\}_i$ is also a complete orthonormal system.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Let $v \in \overline{Span\{e_n\}_{n=1}^{\infty}}$, it suffices to show that $v = 0$. Note that $\langle v, e_n \rangle = 0$ for all $n \in \mathbb{N}$. Also, because $\{f_n\}$ is a complete orthonormal system, we have from Parseval's Identity that:

$$\|v\|^2 = \sum_{n=1}^{\infty} \langle v, f_n \rangle = \sum_{n=1}^{\infty} \langle v, f_n - e_n \rangle^2 \leq \sum_{n=1}^{\infty} \|v\|^2 \|e_n - f_n\|^2 = \|v\|^2 \sum_{n=1}^{\infty} \|e_n - f_n\|^2,$$

where the inequality comes from applying Cauchy-Schwarz. Now if $\|v\|$ were not zero, then dividing both sides by $\|v\|^2$ and using the given inequality would give us that $1 < 1$, which is absurd. So it must be the case that $\|v\| = 0 \implies v = 0$, as desired.

Spring 2011

Problem 1

Show that the sequence $f_n = ne^{-nx}$ does not converge weakly in $L^1([0,1])$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
For some given $\epsilon > 0$ consider the map $T_\epsilon : L^1([0,1]) \to$ defined by

$$T_\epsilon f = \int_0^\epsilon f(x)\,dx = \int_0^1 f(x)\chi_{[0,\epsilon]}(x)\,dx,$$

Note that since $\chi_{[0,\epsilon]}(x) \in L^\infty([0,1])$ this defines a linear functional on $L^1([0,1])$ (it is also not hard to verify this directly). Now, observe that

$$T_\epsilon f_n = \int_0^\epsilon ne^{-nx}\,dx = 1 - \frac{1}{e^{-n\epsilon}},$$

so $T_\epsilon f_n \to 1$ for all $\epsilon > 0$. Hence if there did exist some $f \in L^1([0,1])$ such that $f_n \rightharpoonup f$ we would require that

$$\int_0^\epsilon f(x)\,dx = 1$$

for all $\epsilon > 0$, which is absurd (can prove via DCT). Hence f_n cannot converge weakly to any $f \in L^1([0,1])$.

Problem 6

For $\alpha \in (0,1]$, the space of Hölder continuous functions on the interval $[0,1]$ is defined as

$$C^{9,\alpha}([0,1]) = \{u \in C[0,1] : |u(x) - u(y) \leq C|x-y|^{\alpha} \ \forall x, y \in [0,1]\}$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x \in [0,1]} |u(x)| + \sup_{x,y \in [0,1]} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}}.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Let $U = \{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$, and let $u \in U$ be arbitrary. Note that for all $x \in [0,1]$, $|u(x)| \leq \|u\|_{\infty} \leq \|u\|_{C^{0,\alpha}} \leq 1$, so U is uniformly bounded. Now fix $\epsilon > 0$ and note that if $y \in [0,1]$ with $|x-y| < \epsilon^{1/\alpha}$ then we have:

$$\frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \leq \|u\|_{C^{0,\alpha}} \leq 1 \implies |u(x) - u(y)| < \epsilon,$$

so U is equicontinuous. Then by the Arzelà-Ascoli Theorem we have that U is pre-compact in $C([0,1])$. Thus we only have left to show that U is closed in $C([0,1])$, which follows from ...

Fall 2011

Problem 3

1. Define the compactness of an operator $A \in B(X)$ with X a Hilbert space in terms of properties of the images of bounded sets.
2. Suppose that X is separable with orthonormal basis $\{e_i\}_{i\geq 0}$ and write P_i for the orthogonal projection onto the span on $\{e_0, \dots, e_i\}$. Show that $A \in B(X)$ is compact iff $\{P_i A\}$ converges to A in norm.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

We say that an operator $A \in B(X)$ is compact if it maps bounded sets to pre-compact sets, i.e., if $A(B)$ is pre-compact for all $B \subseteq X$ bounded. Now suppose $A \in B(X)$ is a compact operator, we wish to show that $(P_n A)$ converges to A in norm. Fix $\epsilon > 0$ and let $\overline{B}$ denote the closed unit ball in X. Since this set it bounded, it follows that $A(\overline{B})$ is pre-compact. Hence, by Theorem 9.17 in H&N, there exists an $N \in \mathbb{N}$ such that for all $a \in A(B)$ and $n > N$ we have that:

$$\sum_{k=n}^{\infty} |\langle a, e_k \rangle|^2 < \epsilon^2.$$

Now let $x \in X$ such that $\|x\| = 1$. Since $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal basis we may write

$$Ax = \sum_{k=1}^{\infty} \langle Ax, e_k \rangle e_k \implies P_i Ax = \sum_{k=1}^{i} \langle Ax, e_k \rangle e_k.$$

Hence for all $n > N$ we have that:

$$\|Ax - P_n Ax\|^2 = \sum_{k=n}^{\infty} \langle Ax, e_k \rangle e_k < \epsilon^2 \implies \|Ax - P_n Ax\| < \epsilon,$$

so we have that $(P_n A)$ converges to A in norm.

For the reverse direction suppose that (P_nA) converges to A in norm. One may take it as a fact that the uniform limit of compact operators is compact, but to be thorough we will prove this statement. Suppose $T_n \in B(X, y)$ is a sequence of compact operators such that $T_n \to T$ in the operator norm. Let B be a bounded set and let Tx_n be a sequence in $T(B)$. We wish to show that Tx_n has a convergent subsequence. First note that since T_1 is compact, T_1x_n has a convergent subsequence, which we will denote $T_1x_{n,1}$. Now since T_2 is compact $T_2x_{n,1}$ also contains its own convergent subsequence $T_2x_{n,2}$. Repeating this process for all $n \in \mathbb{N}$ and taking the diagonal sequence of inputs, $x_{n,n}$, we see that $T_ix_{n,n}$ converges for all $i \in \mathbb{N}$. Now fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be sufficiently large so that $\|T_N - T\| < \epsilon/3$ and let M be sufficiently large so that $||T_Nx_{n,n} - T_Nx_{k,k}|| < \epsilon/3$ for all $k, n > M$. Then for all $k, n > M$ we have that:

$$\begin{aligned} \|Tx_{n,n} - Tx_{k,k}\| &\leq \|Tx_{n,n} - T_Nx_{n,n}\| + \|T_Nx_{n,n} - T_Nx_{k,k}\| + \|T_Nx_{k,k} - Tx_{k,k}\| \\ &\leq 2\|T_N - T\| + \|T_Nx_{n,n} - T_Nx_{k,k}\| \\ &< \epsilon. \end{aligned}$$

Thus the subsequence $Tx_{n,n}$ is Cauchy and hence convergent, so $T(B)$ is sequentially pre-compact which is equivalent to pre-compact as we are in a metric space.

Spring 2009

Problem 3

1. Show that if (x_n) is a sequence of elements of a Hilbert space $\mathcal{H}$ converging weakly to x and $||x_n||$ also converges to $||x||$ then the sequence converges to x in norm
2. Find a sequence of elements (x_n) of a Hilbert space converging weakly to x with $\liminf_{n\to\infty} ||x_n|| > ||x||$.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
Suppose $x_n \rightharpoonup x$ in a Hilbert space H and furthermore $\lim_{n\to\infty}\|x_n\| = \|x\|$. Then observe:

$$\lim_{n\to\infty}\|x_n-x\|^2 = \lim_{n\to\infty}\langle x_n-x, x_n-x\rangle = \lim_{n\to\infty}\langle x_n, x_n\rangle - 2\langle x_n, x\rangle + \langle x, x\rangle = 2\|x\|^2 - 2\|x\|^2 = 0.$$

For part 2, consider the sequence $(e_n) \subseteq \ell^2(\mathbb{N})$ where e_i is the i-th element of the standard basis of $\ell^2(\mathbb{N})$, so the k-th term of e_i, denoted e_i^k, is 0 if $k \neq i$ and 1 if $k = i$. Then observe for arbitrary $y \in \ell^2(\mathbb{N})$ we have $\langle e_n, y\rangle = y^n \to 0$ as $n \to \infty$, so $e_n \rightharpoonup 0$ but $\liminf_{n\to\infty}\|e_n\| = 1 > 0$.

Problem 4

Let $\mathcal{H}$ be a Hilbert space (not necessarily separable) and let $B \subset \mathcal{H}$ denote the closed unit ball.

1. Show that B is weakly sequentially compact.
2. If $T \in B(\mathcal{H})$ is compact, prove that $T(B)$ is closed.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
For the first problem, let $(x_n) \subseteq B$. Define $U = \overline{Span\{x_n\}_{n=1}^{\infty}}$. Note that U is a separable Hilbert space (since it has a countable basis) and so by Banach

Alaglou the closed unit ball of U^* is weak-$*$ compact. But U being Hilbert means it is isomorphic to its dual under the embedding:

$$u \mapsto \tilde{u}(v) = \langle v, u \rangle.$$

Hence U^* is separable so the closed unit ball in U^* is weak-$*$ sequentially compact. By identifying each x_n with its embedding in the dual this means that there exists a subsequence (x_{n_k}) such that $x_{n_k} \rightharpoonup x$ in U. That is for all $u \in U$ we have that:

$$\langle x_{n_k}, u \rangle \to \langle x, u \rangle.$$

Now let $y \in \mathcal{H}$ be arbitrary. Recall from the Riesz Representation Theorem that the map $T_y : \mathcal{H} \to \mathbb{C}$ defined by $T_y(x) = \langle x, y \rangle$ is a bounded linear functional on $\mathcal{H}$. Hence $T_y|_U$ is a bounded linear functional on U. Applying Riesz again this means that $T_y|_U(x) = \langle x, u \rangle$ for some $u \in U$ (in particular u is the projection of y on U). Hence we have that:

$$\langle x_{n_k}, y \rangle = \langle x_{n_k}, u \rangle \to \langle x, u \rangle = \langle x, y \rangle,$$

so in fact $x_{n_k} \rightharpoonup x$ in $\mathcal{H}$ as desired.

Now for the second problem, let $(Tx_n) \subseteq T(B)$ be a convergent sequence, so $Tx_n \to y \in \mathcal{H}$. We want to show $y \in T(B)$. From the previous part, B is weakly sequentially compact so there exists a subsequence (x_{n_k}) of (x_n) converging weakly to some $x \in B$. That is, for all $\phi \in \mathcal{H}^*$ we have that $\phi x_{n_k} \to \phi x$. But note that for all $\phi \in \mathcal{H}^*$, the map $\phi \circ T$ defines a bounded linear functional on $\mathcal{H}$ so it must be the case that $\phi \circ Tx_{n_k} \to \phi \circ Tx$. Hence $Tx_{n_k} \rightharpoonup Tx$. But $Tx_n \to y \implies Tx_n \rightharpoonup y \implies Tx_{n_k} \rightharpoonup y$. Since weak limits are unique it follows that $y = Tx$, so $T(B)$ is closed.

Spring 2008

Problem 3

Show that if V is a closed subspace of a Hilbert space $\mathcal{H}$ and ϕ is a bounded linera map from V to $\mathbb{C}$ then tehre is a unique bounded map from $\mathcal{H}$ to $\mathbb{C}$ which is an extension of ϕ and has the same operator norm as ϕ.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.
I assume we can't simply apply the Hahn-Banach theorem to this question. Note that V is itself a Hilbert space as it is a closed linear subspace of $\mathcal{H}$. Hence by the Riesz Representation Theorem we may express ϕ as $\phi(v) = \langle v, u \rangle$ for some fixed $u \in V$. Note that by the Cauchy Schwarz inequality, $\|\phi(v)\| \leq \|v\|\|u\|$, and $\|\phi(u)\| = \|u\|^2$, so $\|\phi\| = \|u\|$.

Now since $u \in \mathcal{H}$, we can define a bounded linear functional Φ on $\mathcal{H}$ so that $\Phi(x) = \langle x, u \rangle$. Clearly Φ agrees with ϕ on V and

$$\|\Phi\| = \sup_{\|x\|=1, x \in \mathcal{H}} |\langle x, u \rangle| \geq \sup_{\|v\|=1, v \in V} |\langle v, u \rangle| = \|\phi\|.$$

At the same time, $\|\Phi(x)\| \leq \|u\|\|x\| = \|\phi\|\|x\|$, again by Cauchy-Schwarz, so in fact $\|\Phi\| = \|\phi\|$, as desired.

To show this is the unique linear functional extending ϕ and having the same norm, suppose there existed some other functional, ψ with these properties. By Riesz, we may express ψ as $\psi(x) = \langle x, y \rangle$ for some $y \in \mathcal{H}$. Since V is a closed linear subspace, we may write $y = a + b$ where $a \in V$ and $b \in V^{\perp}$. Then note for all $v \in V$:

$$\langle v, u \rangle = \langle v, y \rangle = \langle v, a + b \rangle = \langle v, a \rangle + \langle v, b \rangle = \langle v, a \rangle,$$

therefore it must be the case that $a = u$. But note, by a similar argument as before, that

$$\|\psi\| = \|y\| = \sqrt{\langle a + b, a + b \rangle} = \sqrt{\|a\|^2 + \|b\|^2} = \sqrt{\|u\|^2 + \|b\|^2}.$$

In order for $\|\psi\| = \|\phi\| = \|u\|$, it must be the case that $b = 0 \implies y = u \implies \psi = \Phi$.

Fall 2008

Problem 2

Show that if $\{f_n\}$ is a sequence of continuously differentiable function on $[0, 1]$ and both the original sequence and the sequence of derivatives are uniformly bounded, then (f_n) has a uniformly convergent subsequence.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

Since the sequence $\{f_n\}$ is uniformly bounded, we can get the equicontinuity of $\{f_n\}$. By the equicontinuity and uniform boundedness of $\{f_n\}$, appling Arzela-Ascoli, $\{f_n\}$ is precompact and there exists a subsequence $\{f_{n_k}\}$ converging uniformly.

Spring 2006

Problem 5

> Take $\{u_k\}_{k\in\mathbb{N}}$ to be an orthonormal set in the Hilbert space X. Characterize those sequences of scalars (a_k) such that $(a_k u_k)$ is compact in X.

Solution by Esha Datta, James Hughes, Edgar Jaramillo Rodriguez, Jeonghoon Kim, Van Vinh Nguyen, Qianhui Wan.

We claim it is necessary and sufficient that $(a_n) \to 0$ and that at least one $a_k = 0$. First consider what happens if (a_n) does not converge to 0. Then there exists an $\epsilon > 0$ such that $|a_n| > \epsilon$ for infinitely many $n \in \mathbb{N}$. Consider this subsequence, denoted (a_{n_k}) and note that for $k_1 \neq k_2$:

$$\|a_{n_{k_1}} u_{n_{k_1}}^2\|_{-a_{n_{k_2}} u_{n_{k_2}}} = |a_{n_{k_1}}|^2 + |a_{n_{k_2}}|^2 > \epsilon^2 + \epsilon^2 = 2\epsilon^2.$$

$$\implies \|a_{n_{k_1}} u_{n_{k_1}}\|_{-a_{n_{k_2}} u_{n_{k_2}}} > \sqrt{2}\epsilon > \epsilon$$

Therefore $(a_{n_k} u_{n_k})$ cannot be covered by a finite $\epsilon/2$-net, hence it is not pre-compact and thus not compact.

Now consider the situation where $(a_n) \to 0$, and fix $\epsilon > 0$. Note that there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have that $|a_n| < \epsilon/\sqrt{2}$. Now consider the collection of open balls $\mathcal{B} = \bigcup_{k=1}^{N+1} B(a_k u_k, \epsilon)$. We claim that this ϵ-net covers $(a_n u_n)$. Cleary $a_n u_n \in \mathcal{B}$ for all $n \leq N+1$. Then if $n > N+1$ note that:

$$\|a_n u_n - a_{N+1} u_{N+1}\|^2 = |a_n|^2 + |a_{N+1}|^2 < \epsilon^2/2 + \epsilon^2/2 = \epsilon^2$$

$$\implies \|a_n u_n - a_{N+1} u_{N+1}\| < \epsilon,$$

so $a_n u_n \in B(a_{N+1} u_{N+1}, \epsilon)$, and hence in $\mathcal{B}$. So we have that $\mathcal{B}$ is a finite ϵ-net covering $(a_n u_n)$, so the set is pre-compact. We only have left to show that it is closed. But note that if $a_n \to 0$ then $(a_n u_n) \to 0$ in norm. Hence the only limit points of $(a_n u_n)$ are the terms of the sequence and zero. Thus we require that at least one $a_k = 0$ so that 0 is an element in the set.