Wavelets Made Easy – Some Linear Algebra

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Contents

Lecture 1

Vector Spaces over $\mathbb{R}$

1.1 Definition

Definition 1. A **vector space** over $\mathbb{R}$ is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* $+$ and *multiplication by scalars* $\cdot$, satisfying the following properties:

A1 (Closure of addition)
For all $u, v \in V, u + v$ is defined and $u + v \in V$.

A2 (Commutativity for addition)
$u + v = v + u$ for all $u, v \in V$.

A3 (Associativity for addition)
$u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.

A4 (Existence of additive identity)
There exists an element $\vec{0}$ such that $u + \vec{0} = u$ for all $u \in V$.

A5 (Existence of additive inverse)
For each $u \in V$, there exists an element -denoted by $-u$- such that $u + (-u) = \vec{0}$.

M1 (Closure for scalar multiplication)
For each number $r \in \mathbb{R}$ and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.

M2 (Multiplication by 1)
$1 \cdot u = u$ for all $u \in V$.

M3 (Associativity for multiplication)
$r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.

D1 (First distributive property)
$r \cdot (u + v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$.

D2 (Second distributive property)
$(r + s) \cdot u = r \cdot u + s \cdot u$ for all $r, s \in \mathbb{R}$ and all $u \in V$.

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0_1}, \vec{0_2} \in V$ are such that

$$u + \vec{0_1} = u + \vec{0_2} = u, \forall u \in V$$

then $\vec{0_1} = \vec{0_2}$.

Proof. We have $\vec{0_1} = \vec{0_1} + \vec{0_2} = \vec{0_2} + \vec{0_1} = \vec{0_2}$ $\square$

Lemma. Let $u \in V$, then $0 \cdot u = \vec{0}$.

Proof.

$$\begin{aligned} u + 0 \cdot u &= 1 \cdot u + 0 \cdot u \\ &= (1 + 0) \cdot u \\ &= 1 \cdot u \\ &= u \end{aligned}$$

$$\text{Thus} \qquad \begin{aligned} \vec{0} = u + (-u) &= (0 \cdot u + u) + (-u) \\ &= 0 \cdot u + (u + (-u)) \\ &= 0 \cdot u + \vec{0} \\ &= 0 \cdot u \end{aligned}$$ $\square$

Lemma. a) The element $-u$ is unique.

b) $-u = (-1) \cdot u$.

Proof of part (b).

$$\begin{aligned} u + (-1) \cdot u &= 1 \cdot u + (-1) \cdot u \\ &= (1 + (-1)) \cdot u \\ &= 0 \cdot u \\ &= \vec{0} \end{aligned}$$

$\square$

1.2 Examples

Before examining the axioms in more detail, let us discuss two examples.

Example. Let $V = \mathbb{R}^n$,considered as column vectors
$\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} | x_1, x_2, \ldots, x_n \in \mathbb{R}\}$ Then for

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad r \in \mathbb{R} :$$

Define

$$u + v = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad r \cdot u = \begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix}$$

Note that the zero vector and the additive inverse of u are given by:
$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad -u = \begin{pmatrix} -x_1 \\ \vdots \\ -x_2 \end{pmatrix}$

Remark. $\mathbb{R}^n$ can also be considered as the space of all row vectors.

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

The addition and scalar multiplication is again given coordinate wise

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$r \cdot (x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$$

Example. If $\vec{x} = (2, 1, 3), \vec{y} = (-1, 2, -2)$ and $r = -4$ find $\vec{x} + \vec{y}$ and $r \cdot \vec{x}$.

Solution.

$$\begin{aligned}\vec{x}+\vec{y} &= (2,1,3)+(-1,2,-2)\\ &= (2-1,1+2,3-2)\\ &= (1,3,1)\end{aligned}$$

$$r\cdot\vec{x} = -4\cdot(2,1,3) = (-8,-4,-12).$$

Remark.

$$\begin{aligned}(x_1,\ldots,x_n)+(0,\ldots,0) &= (x_1+0,\ldots,x_n+0)\\ &= (x_1,\ldots,x_n)\end{aligned}$$

So the additive identity is $\vec{0}=(0,\ldots,0)$.

Note also that

$$\begin{aligned}0\cdot(x_1,\ldots,x_n) &= (0x_1,\ldots,0x_n)\\ &= (0,\ldots,0)\end{aligned}$$

for all $(x_1,\ldots,x_n)\in\mathbb{R}^n$.

Example. Let A be the interval $[0,1)$ and V be the space of functions $f:A\longrightarrow\mathbb{R}$, i.e.,

$$V=\{f:[0,1)\longrightarrow\mathbb{R}\}$$

Define addition and scalar multiplication by

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (r\cdot f)(x) &= rf(x)\end{aligned}$$

For instance, the function $f(x)=x^4$ is an element of V and so are

$$g(x)=x+2x^2,\qquad h(x)=\cos x,\qquad k(x)=e^x$$

We have $(f+g)(x)=x+2x^2+x^4$.

Remark. (a) The zero element is the function $\vec{0}$ which associates to each x the number 0:

$$\vec{0}(x) = 0 \text{ for all } x \in [0,1)$$

Proof. $(f + \vec{0})(x) = f(x) + \vec{0}(x) = f(x) + 0 = f(x)$. $\square$

(b) The additive inverse is the function $-f : x \mapsto -f(x)$.

Proof. $(f + (-f))(x) = f(x) - f(x) = 0$ for all x. $\square$

Example. Instead of $A = [0,1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space V. We set

$$V^A = \{f : A \longrightarrow V\}$$

and set addition and scalar multiplication by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (r \cdot f)(x) &= r \cdot f(x) \end{aligned}$$

Remark. (a) The zero element is the function which associates to each x the vector $\vec{0}$:

$0 : x \mapsto \vec{0}$

Proof

$$\begin{aligned} (f+0)(x) &= f(x) + 0(x) \\ &= f(x) + \vec{0} = f(x) \quad \square \end{aligned}$$

Remark.

(b) Here we prove that $+$ is associative:

Proof. Let $f, g, h \in V^A$. Then

$$\begin{aligned} [(f+g)+h](x) &= (f+g)(x) + h(x) \\ &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) \quad \text{associativity in } V \\ &= f(x) + (g+h)(x) \\ &= [f + (g+h)](x) \end{aligned}$$

$\square$

1.3 Exercises

Let $V = \mathbb{R}^4$. Evaluate the following:

a) $(2, -1, 3, 1) + (3, -1, 1, -1)$.

b) $(2, 1, 5, -1) - (3, 1, 2, -2)$.

c) $10 \cdot (2, 0, -1, 1)$.

d) $(1, -2, 3, 1) + 10 \cdot (1, -1, 0, 1) - 3 \cdot (0, 2, 1, -2)$.

e) $x_1 \cdot (1, 0, 0, 0) + x_2 \cdot (0, 1, 0, 0) + x_3 \cdot (0, 0, 1, 0) + x_4 \cdot (0, 0, 0, 1)$.

Lecture 2

Subspaces

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Question: Do we have to test all the axioms to find out if W is a vector space?

The answer is NO.

Theorem. Let $W \neq \emptyset$ be a subset of a vector space V. Then W, with the same addition and scalar multiplication as V, is a vector space if and only if the following two conditions hold:

1. $u + v \in W$ for all $u, v \in W$ $\quad$ (or $W + W \subseteq W$)

2. $r \cdot u \in W$ for all $r \in \mathbb{R}$ and all $u \in W$ $\quad$ (or $\mathbb{R}W \subseteq W$).

In this case we say that W is a *subspace* of V.

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.
To show that W is a vector space we have to show that all the 10 axioms of Definition 1.1 hold for W. But that follows because the axioms hold for V and W is a subset of V:

A1 (Commutativity of addition)
For $u, v \in W$, we have $u + v = v + u$. This is because u, v are also in V and commutativity holds in V.

A4 (Existence of additive identity)
Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.

A5 (Existence of additive inverse)
If $u \in W$ then $-u = (-1) \cdot u \in W$.

One can check that the other axioms follow in the same way.

$\square$

2.1 Examples

Usually the situation is that we are given a vector space V and a subset of vectors W satisfying some conditions and we need to see if W is a subspace of V.

$$W = \{v \in V : \underline{\text{some conditions}} \text{ on } v\}$$

We will then have to show that

$$\begin{matrix} u, v \in W & u+v \\ r \in \mathbb{R} & r \cdot u \end{matrix} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{Satisfy the } \underline{\text{same conditions}}.$$

2.2 Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$\begin{aligned} V &= \mathbb{R}^2, \\ W &= \{(x, y) | y = kx\} \qquad \text{for a given } k \\ &= \text{ line through } (0,0) \text{ with slope } k. \end{aligned}$$

To see that W is in fact a subspace of $\mathbb{R}^2$:
Let $u = (x_1, y_1),\ v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and $y_2 = kx_2$

and

$$\begin{aligned} u+v &= (x_1+x_2, y_1+y_2) \\ &= (x_1+x_2, kx_1+kx_2) \\ &= (x_1+x_2, k(x_1+x_2)) \in W \end{aligned}$$

Similarly, $r \cdot u = (rx_1, ry_1) = (rx_1, krx_1) \in W$

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
2. Lines. But only those that contain $(0,0)$. Why?
3. $\mathbb{R}^2$

Remark (First test). If W is a subspace, then $\vec{0} \in W$.
Thus: If $\vec{0} \notin W$, then W is not a subspace.

This is why a line not passing through $(0,0)$ can not be a subspace of $\mathbb{R}^2$.

2.3 A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that W is a subspace.

Example. Lets consider the following subset of $\mathbb{R}^2$:

$$W = \{(x,y) | x^2 - y^2 = 0\}$$

Is W a subspace of $\mathbb{R}^2$? Why?

The answer is NO.

We have $(1,1)$ and $(1,-1) \in W$ but $(1,1)+(1,-1) = (2,0) \notin W$. i.e., W is not closed under addition.

Notice that $(0,0) \in W$ and W is closed under multiplication by scalars.

2.4 Subspaces of $\mathbb{R}^3$

What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$.

2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector (a, b, c) and its distance from $(0, 0, 0)$ or
$$W = \{(x, y, z) | \underbrace{ax + by + cz = p}_{\text{condition on } (x,y,z)} \}$$

For W to be a subspace, $(0, 0, 0)$ must be in W by the *first test*. Thus
$$p = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$$
or
$$p = 0$$

2.4.1 Planes containing the origin

A plane containing $(0, 0, 0)$ is indeed a subspace of $\mathbb{R}^3$.

Proof. Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W$. Then
$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned}$$

Then we have
$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) &= \underbrace{(ax_1 + by_1 + cz_1)}_{0} + \underbrace{(ax_2 + by_2 + cz_2)}_{0} \\ &= 0 \end{aligned}$$

and
$$\begin{aligned} a(rx_1) + b(ry_1) + c(rz_1) &= r(ax_1 + by_1 + cz_1) \\ &= 0 \quad \square \end{aligned}$$

2.5 Summary of subspaces of $\mathbb{R}^3$

1. $\{0\}$ and $\mathbb{R}^3$.

2. Planes containing $(0,0,0)$.

3. Lines containing $(0,0,0)$.
   (Intersection of two planes containing $(0,0,0)$)

2.6 Exercises

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

a) $W = \{(x,y) \in \mathbb{R}^2 |\ xy = 0\}$.

b) $W = \{(x,y,z) \in \mathbb{R}^3 |\ 3x + 2y^2 + z = 0\}$.

c) $W = \{(x,y,z) \in \mathbb{R}^3 |\ 2x + 3y - z = 0\}$.

d) The set of all vectors (x_1, x_2, x_3) satisfying

$$2x_3 = x_1 - 10x_2.$$

e) The set of all vectors in $\mathbb{R}^4$ satisfying the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + 5x_4 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

f) The set of all points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying

$$x_1 + 2x_2 + 3x_3 + x_4 = -1.$$

Lecture 3

Vector Spaces of Functions

3.1 Space of Continuous Functions $\mathcal{C}(I)$

Let $I \subseteq \mathbb{R}$ be an interval. Then I is of the form (for some $a < b$)

$$I = \begin{cases} \{x \in \mathbb{R} |\ a < x < b\}, & \text{an open interval;} \\ \{x \in \mathbb{R} |\ a \leq x \leq b\}, & \text{a closed interval;} \\ \{x \in \mathbb{R} |\ a \leq x < b\} \\ \{x \in \mathbb{R} |\ a < x \leq b\}. \end{cases}$$

Recall that the space of <u>all</u> functions $f : I \longrightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

Example (1). Let $\mathcal{C}(I)$ be the space of all continuous functions on I. If f and g are continuous, so are the functions $f + g$ and $rf \quad (r \in \mathbb{R})$. Hence $\mathcal{C}(I)$ is a vector space.

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:

a) Let $x_0 \in I$ and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $x \in I \cap (x_0 - \delta, x_0 + \delta)$ we have

$$|f(x) - f(x_0)| < \epsilon$$

This tells us that the value of f at nearby points is arbitrarily close to the value of f at x_0.

b) A reformulation of (a) is:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

3.2 Space of continuously differentiable functions $\mathcal{C}^1(I)$

Example (2). The space $\mathcal{C}^1(I)$. Here we assume that I is open. Recall that f is differentiable at x_0 if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0)$$

exists. If $f'(x_0)$ exists for all $x_0 \in I$, then we say that f is differentiable on I. In this case we get a new function on I

$$x \mapsto f'(x)$$

We say that f is continuously differentiable on I if f' exists and is continuous on I.

Recall that if f and g are differentiable, then so are

$$f+g \text{ and } rf \quad (r \in \mathbb{R})$$

moreover

$$(f+g)' = f' + g' \ ; \quad (rf)' = rf'$$

As $f'+g'$ and rf' are continuous by Example (1), it follows that $\mathcal{C}^1(I)$ is a vector space.

3.2.1 A continuous but not differentiable function

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$. We will show that f is not differentiable at $x_0 = 0$. For $h > 0$ we have

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{|h|-0}{h} = \frac{h}{h} = 1$$

hence

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = 1$$

But if $h < 0$, then

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{|h|-0}{h} = \frac{-h}{h} = -1$$

hence

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = -1$$

Therefore,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

does not exist. $\square$

3.3 Space of r-times continuously differentiable functions $\mathcal{C}^r(I)$

Example (3). The space $\mathcal{C}^r(I)$
Let $I=(a,b)$ be an open interval. and let $r\in\mathbb{N}=\{1,2,3,\cdots\}$.

Definition. The function $f: I \longrightarrow \mathbb{R}$ is said to be r-times continuously differentiable if all the derivatives $f',\ f'',\ \cdots,\ f^{(r)}$ exist and $f^{(r)}: I \longrightarrow \mathbb{R}$ is continuous.

We denote by $\mathcal{C}^r(I)$ the space of r-times continuously differentiable functions on I. $\mathcal{C}^r(I)$ is a subspace of $\mathcal{C}(I)$.

We have

$$\mathcal{C}^r(I) \subsetneqq \mathcal{C}^{r-1}(I) \subsetneq \cdots \subsetneqq \mathcal{C}^1(I) \subsetneqq \mathcal{C}(I).$$

3.3.1 $\mathcal{C}^r(I) \neq \mathcal{C}^{r-1}(I)$

We have seen that $\mathcal{C}^1(I)\neq\mathcal{C}(I)$. Let us try to find a function that is in $\mathcal{C}^1(I)$ but not in $\mathcal{C}^2(I)$.

Assume $0\in I$ and let $f(x)=x^{\frac{5}{3}}$. Then f is differentiable and

$$f'(x)=\tfrac{5}{3}x^{\frac{2}{3}}$$

which is continuous.

If $x\neq 0$, then f' is differentiable and

$$f''(x)=\tfrac{10}{3}x^{-\frac{1}{3}}$$

But for $x = 0$ we have

$$\lim_{h\to 0} \frac{f'(h) - 0}{h} = \lim_{h\to 0} \frac{5}{3} \frac{h^{\frac{2}{3}}}{h} = \lim_{h\to 0} \frac{5}{3}\ h^{-\frac{1}{3}}$$

which does not exist.

Remark. One can show that the function

$$f(x) = x^{\frac{3r-1}{3}}$$

is in $\mathcal{C}^{r-1}(\mathbb{R})$, but not in $\mathcal{C}^r(\mathbb{R})$.

Thus, as stated before, we have

$$\mathcal{C}^r(I) \subsetneqq \mathcal{C}^{r-1}(I) \subsetneqq \cdots \subsetneqq \mathcal{C}^1(I) \subsetneqq \mathcal{C}(I).$$

3.4 Piecewise-continuous functions $\mathcal{PC}(I)$

Example (4). Piecewise-continuous functions on I

Definition. Let $I = [a, b)$. A function $f : I \longrightarrow \mathbb{R}$ is called piecewise-continuous if there exists finitely many points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that f is continuous on each of the sub-intervals (x_i, x_{i+1}) for $i = 0, 1, \cdots, n-1$.

Remark. If f and g are both piecewise-continuous, then

$$f + g \text{ and } rf \quad (r \in \mathbb{R})$$

are piecewise-continuous.

Hence the space of piecewise-continuous functions is a vector space. We denote this vector space by $\mathcal{PC}(I)$.

3.5 The indicator function χ_A

Important elements of $\mathcal{PC}(I)$ are the indicator functions χ_A, where $A \subseteq I$ is a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

So the values of χ_A tell us whether x is in A or not.
If $x \in A$, then $\chi_A(x) = 1$ and if $x \notin A$, then $\chi_A(x) = 0$.

We will work a lot with indicator functions so let us look at some of their properties.

3.5.1 Some properties of χ_A

Lemma. Let $A,\ B \subseteq I$. Then

$$\chi_{A\cap B}(x) = \chi_A(x)\ \chi_B(x) \tag{$*$}$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

If $x \in A$ and $x \in B$, that is $x \in A \cap B$, then
$\chi_{A\cap B}(x) = 1$ and,

since $\chi_A(x) = 1$ and $\chi_B(x) = 1$, we also have $\chi_A(x)\chi_B(x) = 1$.

Thus, the left and the right hand sides of $(*)$ agree.
On the other hand, if $x \notin A \cap B$, then there are two possibilities:

- $x \notin A$, then $\chi_A(x) = 0$, so $\chi_A(x)\chi_B(x) = 0$.
- $x \notin B$, then $\chi_B(x) = 0$, so $\chi_A(x)\chi_B(x) = 0$.

It follows that

$$0 = \chi_{A\cap B}(x) = \chi_A(x)\chi_B(x) \quad \square$$

3.5.2 $\chi_{A\cup B}$ with A, B disjoint

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A and χ_B?
If A and B are disjoint, that is $A\cap B=\emptyset$ then

$$\chi_{A\cup B}(x)=\chi_A(x)+\chi_B(x).$$

Let us prove this:

- If $x\notin A\cup B$, then $x\notin A$ and $x\notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.
- If $x\in A\cup B$ then either

  x is in A but not in B. In this case

  $$\chi_{A\cup B}(x)=1 \text{ and } \chi_A(x)+\chi_B(x)=1+0=1$$

  or

  x is in B but not in A. In this case

  $\chi_{A\cup B}(x)=1$ and $\chi_A(x)+\chi_B(x)=0+1=1$ □

3.5.3 $\chi_{A\cup B}$ with A, B arbitrary

Thus we have,

$$\text{If } A\cap B=\emptyset, \text{ then } \chi_{A\cup B}(x)=\chi_A(x)+\chi_B(x).$$

Now, what if $A\cap B\neq\emptyset$?

Lemma. $\chi_{A\cup B}(x)=\chi_A(x)+\chi_B(x)-\chi_{A\cap B}(x)$.

Proof.

- If $x\notin A\cup B$, then both of the LHS and the RHS take the value 0.
- If $x\in A\cup B$, then we have the following possibilities:
  1. If $x\in A,\ x\notin B$, then
     $\chi_{A\cup B}(x)=1$
     $\chi_A(x)+\chi_B(x)-\chi_{A\cap B}=1+0-0=1$
  2. Similarly for the case $x\in B,\ x\notin A$: LHS equals the RHS.

3. If $x \in A \cap B$, then
   $\chi_{A\cup B}(x) = 1$
   $\chi_A(x) + \chi_B(x) - \chi_{A\cap B} = 1 + 1 - 1 = 1$

As we have checked all possibilities, we have shown that the statement in the lemma is correct $\square$

3.6 Exercise

Write the product

$$(2\chi_{[0,3]} - 3\chi_{[3,6]})(\chi_{[0,2]} + 5\chi_{[2,4]} - 7\chi_{[4,6]})$$

as a linear combination of the indicator functions of intervals.

Lecture 4

Linear Maps

We have all seen linear maps before. In fact, most of the maps we have been using in Calculus are linear.

4.1 Two Important Examples

4.1.1 The Integral

To integrate the function $f(x) = x^2 + 3x - \cos x$ over the interval $[a, b]$, we first find the antiderivative of x^2, that is $\frac{1}{3}x^3$, then the antiderivative of x, which is $\frac{1}{2}x^2$, and then multiply that by 3 to get $\frac{3}{2}x^2$. Finally, we find the antiderivative of $\cos x$, which is $\sin x$, and then multiply that by -1 to get $-\sin x$. To finish the problem we insert the endpoints. Thus,

$$\begin{aligned}\int_{-1}^{1} x^2 + 3x - \cos x \, dx &= \int_{-1}^{1} x^2 \, dx + 3\int_{-1}^{1} x \, dx \\ &\quad - \int_{-1}^{1} \cos x \, dx \\ &= \left[\frac{1}{3}x^3\right]_{-1}^{1} + \left[\frac{3}{2}x^2\right]_{-1}^{1} - [\sin x]_{-1}^{1} \\ &= \frac{2}{3} - \sin 1 + \sin(-1).\end{aligned}$$

What we have used is the fact that the integral is a linear map $\mathcal{C}([a, b]) \longrightarrow \mathbb{R}$ and that

$$\int_a^b rf(x) + sg(x)\,dx = r\int_a^b f(x)\,dx + s\int_a^b g(x)\,dx.$$

4.1.2 The Derivative

Another example is differentiation $Df = f'$. To differentiate the function $f(x) = x^4 - 3x + e^x - \cos x$, we first differentiate each term of the function and then add:

$$\begin{aligned} D(x^4 - 3x + e^x - \cos x) &= Dx^4 - 3Dx + De^x \\ &\quad -D\cos x \\ &= 4x^3 - 3 + e^x + \sin x. \end{aligned}$$

Definition. Let V and W be two vector spaces. A map $T : V \longrightarrow W$ is said to be linear if for all $v, u \in V$ and all $r, s \in \mathbb{R}$ we have:

$$T(rv + su) = rT(v) + sT(u).$$

Remark: This can also be written by using two equations:

$$T(v + u) = T(v) + T(u)$$

$$T(rv) = rT(v).$$

Lemma. Suppose that $T : V \longrightarrow W$ is linear. Then $T(\vec{0}) = \vec{0}$.

Proof. We can write $\vec{0} = 0v$, where v is any vector in V. But then $T(\vec{0}) = T(0v) = 0T(v) = 0$ □

4.2 Linear Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Example. Let us find all the linear maps from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Any arbitrary vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as:

$$(x_1, x_2) = x_1(1, 0) + x_2(0, 1).$$

Hence,

$$T(x_1, x_2) = x_1T(1, 0) + x_2T(0, 1).$$

Write $T(1, 0)$ and $T(0, 1)$ as:

$$T(1, 0) = (a_{11}, a_{12}), \quad T(0, 1) = (a_{21}, a_{22}), \quad \text{where} \quad a_{ij} \in \mathbb{R}.$$

$$\begin{aligned}
T(x_1,x_2) &= x_1(a_{11},a_{12})+x_2(a_{21},a_{22})\\
&= (x_1a_{11}+x_2a_{21},x_1a_{12}+x_2a_{22})\\
&= (x_1,x_2)\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}.
\end{aligned}$$

Thus, all the information about T is given by the matrix:
$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$.

Example. Next, let us find all the linear maps $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$. As before we write $(x_1,x_2,x_3)\in\mathbb{R}^3$ as:

$$(x_1,x_2,x_3)=x_1(1,0,0)+x_2(0,1,0)+x_3(0,0,1)$$

where,

$$\begin{aligned}
T(1,0,0) &= (a_{11},a_{12},a_{13})\\
T(0,1,0) &= (a_{21},a_{22},a_{23})\\
T(0,0,1) &= (a_{31},a_{32},a_{33}).
\end{aligned}$$

Then,

$$\begin{aligned}
T(x_1,x_2,x_3) &= x_1(a_{11},a_{12},a_{13})+x_2(a_{21},a_{22},a_{23})\\
&\quad + x_3(a_{31},a_{32},a_{33})\\
&= (x_1a_{11}+x_2a_{21}+x_3a_{31},x_1a_{12}+x_2a_{22}\\
&\quad + x_3a_{32},x_1a_{13}+x_2a_{23}+x_3a_{33})\\
&= (x_1,x_2,x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}.
\end{aligned}$$

Example. All the linear maps from $\mathbb{R}^3 \longrightarrow \mathbb{R}$. Notice that $\mathbb{R}$ is also a vector space, so we can consider all the linear maps $\mathbb{R}^n$ to $\mathbb{R}$. We have :

$$\begin{aligned}
T(x_1,x_2,...,x_n) &= x_1T(1,0,...,0)+x_2T(0,1,...,0)\\
&\quad +...+x_nT(0,0,...,1)\\
&= x_1a_1+x_2a_2+...+x_na_n
\end{aligned}$$

where,

$$T(1,0,...,0)=a_1,\ T(0,1,...,0)=a_2,\dots\ T(0,0,...,1)=a_n.$$

4.2.1 Lemma

Lemma. A map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear if and only if there exists numbers $a_{ij}, i = 1, ..., n,\ j = 1, ..., m$, such that:

$$\begin{aligned} T(x_1, x_2, ..., x_n) &= (x_1 a_{11} + x_2 a_{21} + ... + x_n a_{n1}, ..., \\ &\qquad x_1 a_{1m} + x_2 a_{2m} + ... + x_n a_{nm}) \end{aligned}$$

This can also be written as:

$$T(x_1, x_2, ..., x_n) = (\sum_{i=1}^{n} x_i a_{i1}, \sum_{i=1}^{n} x_i a_{i2}, \sum_{i=1}^{n} x_i a_{im})$$

or by using matrix multiplication:

$$T(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n) \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{pmatrix}.$$

4.2.2 A Counterexample

Example. The map $T(x, y, z) = (2x + 3xy, z + y)$ is <u>not</u> linear because of the factor xy. Notice that:

$$T(1, 1, 0) = (5, 0)$$

but

$$T(2(1, 1, 0)) = T(2, 2, 0) = (16, 0)$$

and

$$2T(1, 1, 0) = (10, 0) \neq (16, 0)$$

4.2.3 Examples

Example. Evaluate the given following maps at a given point:

$$\begin{aligned} T(x, y) &= (3x + y, 3y), \quad (x, y) = (1, -1) \\ T(1, -1) &= (3 \cdot 1 - 1, 3(-1)) = (2, -3) \end{aligned}$$

$$\begin{aligned} T(x, y, z) &= (2x - y + 3z, 2x + z), \quad (x, z, y) = (2, -1, 1) \\ T(2, -1, 1) &= (4 + 1 + 3, 4 + 1) = (8, 5) \end{aligned}$$

Example. Some examples involving differentiation and integration:

$$D(3x^2 + 4x - 1) = 6x + 4$$

$$\begin{aligned} \int_1^2 x^2 - e^x \, dx &= \left[\frac{1}{3}x^3 - e^x\right]_1^2 \\ &= \frac{8}{3} - e^2 - \frac{1}{3} + e \\ &= \frac{7}{3} - e^2 + e \end{aligned}$$

4.3 Kernel

Definition. Let V and W be two vector spaces and $T : V \longrightarrow W$ a linear map.

- A1 The set $Ker(T) = \{v \in V : T(v) = 0\}$ is called the **kernel** of T.
- A2 The set $Im(T) = \{w \in W : \text{there exists a} v \in V : T(v) = w\}$ is called the **image** of T.

Remark: Notice that $Ker(T) \subseteq V$ and $Im(T) \subseteq W$.

Theorem. The kernel of T is a vector space.

Proof. Let $u, v \in Ker(T)$ and $r, s \in \mathbb{R}$ We have to show that $ru + sv \in Ker(T)$. Now,$u, v \in Ker(T)$ if and only if $T(u) = T(v) = 0$. Hence,

$$\begin{aligned} T(ru + sv) &= rT(u) + sT(v) \quad (T \text{is linear}) \\ &= r \cdot 0 + s \cdot 0 \quad (u, v \in Ker(T)) \\ &= 0 \end{aligned}$$

This shows that $ru + sv \in Ker(T)$. □

Remark: Let $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map:

$$T(x, y) = (x^2 + y, x + y).$$

Then,

$$T(1, -1) = (1 - 1, 1 - 1) = (0, 0).$$

But $T(2(1, -1)) = T(2, -2) = (4 - 2, 2 - 2) = (2, 0) \neq (0, 0)$.

So if T is <u>not</u> linear, then the set $v \in V : T(v) = 0$ is in general <u>not</u> a vector space.

Example. Let $\mathbb{R}^2 \longrightarrow \mathbb{R}$ be the map: $T(x, y) = 2x - y$. Describe the kernel of T. We know that (x, y) is in the kernel of T if and only if $T(x, y) = 2x - y = 0$. Hence, $y = 2x$. Thus, the kernel of T is a line through $(0, 0)$ with slope 2.

Example. Let $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the map: $T(x, y, z) = (2x-3y+z, x+2y-z)$. Describe the kernel of T.

We have that $(x, y, z) \in Ker(T)$ if and only if

$$2x - 3y + z = 0 \quad \text{and} \quad x + 2y - z = 0.$$

The equations describe planes through $(0, 0, 0)$ with normal vectors $(2, -3, 1)$ and $(1, 2, -1)$ respectively. The normal vectors are not parallel and therefore the planes are different. It follows that the intersection is a line.

Let us describe this line. Adding the equations we get:

$$3x - y = 0 \quad \text{or} \quad y = 3x.$$

Plugging this into the second equation we get:

$$0 = x + 2(3x) - z = 7x - z \quad \text{or} \quad z = 7x.$$

Hence, the line is given by: $x \cdot (1, 3, 7)$.

Theorem. Let V and W be vector spaces, and $T : V \longrightarrow W$ linear. Then, $Im(T) \subseteqq W$ is a vector space.

Proof. Let $w_1, w_2 \in Im(T)$. Then we can find $u_1, u_2 \in V$ such that $T(u_1) = w_1, T(u_2) = w_2$. Let $r, s \in \mathbb{R}$. Then,

$$\begin{aligned} rw_1 + sw_2 &= rT(u_1) + sT(u_2) \\ &= T(ru_1 + su_2) \in Im(T)). \end{aligned}$$

$\square$

Lecture 5

Inner Product

Let us start with the following problem. Given a point $P \in \mathbb{R}^2$ and a line $L \subsetneqq \mathbb{R}^2$, how can we find the point on the line closest to P?

Answer: Draw a line segment from P meeting the line in a right angle. Then, the point of intersection is the point on the line closest to P.

Let us now take a plane $L \subsetneqq \mathbb{R}^3$ and a point outside the plane. How can we find the point $u \in L$ closest to P?

The answer is the same as before, go from P so that you meet the plane in a right angle.

Observation

In each of the above examples we needed two things:

A1 We have to be able to say what the length of a vector is.

B1 Say what a right angle is.

Both of these things can be done by using the dot-product (or inner product) in $\mathbb{R}^n$.

Definition. Let $(x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n) \in \mathbb{R}^n$. Then, the **dot-product** of these vectors is given by the number:

$$((x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n)) = x_1y_1 + x_2y_2 + ... + x_ny_n.$$

The **norm** (or length) of the vector $\vec{u} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the non-negative number:

$$\|u\| = \sqrt{(u,u)} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

Examples

Example. (a) $((1,2,-3),(1,1,1)) = 1+2-3 = 0$

(b) $((1,-2,1),(2,-1,3)) = 2+2+3 = 7$

Perpendicular

Because,

$$|x_1y_1 + x_2y_2 + ... + x_ny_n| \leq \sqrt{x_1^2 + x_2^2 + ... + x_n^2}\sqrt{y_1^2 + y_2^2 + ... + y_n^2}$$

or

$$|(u,v)| \leq \|u\| \cdot \|v\|$$

we have that (for $u, v \neq 0$)

$$-1 \leq \frac{(u,v)}{\|u\| \cdot \|v\|} \leq 1.$$

Hence we can define:

$$\cos(\angle(u,v)) = \frac{(u,v)}{\|u\| \cdot \|v\|}.$$

In particular, $u \perp v$ (u is **perpendicular** to v) if and only if $(u,v) = 0$.

Questions

Example. Let L be the line in $\mathbb{R}^2$ given by $y = 2x$. Thus,

$$L = \{r(1,2) : r \in \mathbb{R}\}.$$

Let $P = (2,1)$. Consider the following questions.

Question 1: What is the point on L closest to P?

Answer: Because $u \in L$, we can write $\vec{u} = (r, 2r)$. Furthermore, $v - u = (2-r, 1-2r)$ is perpendicular to L. Hence,

$$0 = ((1,2),(2-r,1-2r)) = 2 - r + 2 - 4r = 4 - 5r.$$

Hence, $r = \frac{4}{5}$ and $\vec{v} = (\frac{4}{5}, \frac{8}{5})$. Question 2: What is the distance of P from the line?

Answer: The length of the vector $v - u$, i.e. $\|v - u\|$. First we have to find out what $v - u$ is. We have done almost all the work:

$$v - u = (2,1) - (\frac{4}{5}, \frac{8}{5}) = (\frac{6}{5}, \frac{-3}{5}).$$

The distance therefore is:

$$\sqrt{\frac{36}{25} + \frac{9}{25}} = \frac{3\sqrt{5}}{5}.$$

Properties of the Inner Product

1. (**positivity**)To be able to define the norm, we used that $(u, u) \geq 0$.
2. (**zero length**)All non-zero vectors should have a non-zero length. Thus, $(u, u) = 0$ only if $u = 0$.
3. (**linearity**)If the vector $v \in \mathbb{R}^n$ is fixed, then a map $u \mapsto (u, v)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is linear. That is,
$$(ru + sw, v) = r(u, v) + s(w, v).$$
4. (**symmetry**) For all $u, v \in \mathbb{R}^n$ we have: $(u, v) = (v, u)$.

We will use the properties above to define an inner product on arbitrary vector spaces.

Definition

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

1. (**positivity**) $(u, u) \geq 0$, for all $v \in V$.
2. (**zero length**) $(u, u) = 0$ only if $u = 0$.
3. (**linearity**)If $v \in V$ is fixed, then a map $u \mapsto (u, v)$ from V to $\mathbb{R}$ is linear.
4. (**symmetry**) $(u, v) = (v, u)$, for all $u, v \in V$.

Definition. We say that u and v are perpendicular if $(u, v) = 0$.

Definition. If (.,.) is an inner product on the vector space V, then the norm of a vector $v \in V$ is given by:

$$\|u\| = \sqrt{(u, u)}.$$

Properties of the Norm

Lemma. The norm satisfies the following properties:

1. $\|u\| \geq 0$, and $\|u\| = 0$ only if $u = 0$.
2. $\|ru\| = |r| \cdot \|u\|$.

Proof. We have that

$$\begin{aligned} \|ru\| &= \sqrt{(ru, rv)} \\ &= \sqrt{(r^2(u, v)} \\ &= |r|\sqrt{(u, v)} = |r| \cdot \|u\| . \end{aligned}$$

□

Examples

Example. Let $a < b$, $I = [a, b]$, and $V = PC([a, b])$. Define:

$$(f, g) = \int_a^b f(t)g(t)\,dt$$

Then, $(.,.)$ is an inner product on V.

Proof. Let $r, s \in \mathbb{R}, f, g, h \in V$. Then:

1. $(f, f) = \int_a^b f(t)^2\,dt$. As $f(t)^2 \geq 0$, it follows that $\int_a^b f(t)^2\,dt \geq 0$.
2. If $(f, f) = 0$, then $f(t)^2 = 0$ for all t, i.e $f = 0$.
3. $\int_a^b (rfsg)(t)h(t)\,dt = \int_a^b rf(t)h(t) + sg(t)h(t)\,dt$

   $$\begin{aligned} &= r\int_a^b f(t)h(t)\,dt + s\int_a^b g(t)h(t)\,dt \\ &= r(f, h) + s(g, h). \end{aligned}$$

   Hence, linear in the first factor.
4. As $f(t)g(t) = g(t)f(t)$, it follows that $(f, g) = (g, f)$.

Notice that the norm is:

$$\|f\| = \sqrt{\int_a^b f(t)^2\,dt}.$$

□

Example. Let $a = 0$, $b = 1$ in the previous example. That is, $f(t) = t^2$ and $g(t) = t - 3t^2$. Then:

$$\begin{aligned}
(f, g) &= \int_0^1 t^2(t - 3t^2)\, dt \\
&= \int_0^1 t^3 - 3t^4)\, dt \\
&= \frac{1}{4} - \frac{3}{5} \\
&= -\frac{7}{20}.
\end{aligned}$$

Also, the norms are:

$$\|f\| = \sqrt{\int_0^1 t^4\, dt} = \frac{1}{\sqrt{5}}.$$

$$\begin{aligned}
\|g\| &= \sqrt{\int_0^1 (t - 3t^2)^2\, dt} \\
&= \sqrt{\int_0^1 t^2 - 6t^3 + 9t^4\, dt} \\
&= \sqrt{\frac{1}{3} - \frac{3}{2} + \frac{9}{5}} \\
&= \sqrt{\frac{19}{30}}.
\end{aligned}$$

Example. Let $f(t) = \cos 2\pi t$ and $g(t) = \sin 2\pi t$. Then:

$$\begin{aligned}
(f, g) &= \int_0^1 \cos 2\pi t \sin 2\pi t\, dt \\
&= \frac{1}{4\pi}\left[(\sin 2\pi t)^2\right]_0^1 = 0.
\end{aligned}$$

So, $\cos 2\pi t$ is perpendicular to $\sin 2\pi t$ on the interval $[0, 1]$.

Example. Let $f(t) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ and $g(t) = \chi_{[0,1)}$. Then:

$$\begin{aligned}
(f, g) &= \int_0^1 (\chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t))(\chi_{[0,1)})\, dt \\
&= \int_0^1 \chi_{[0,1/2)}(t)\, dt - \int_0^1 \chi_{[1/2,1)}(t)\, dt \\
&= \int_0^{1/2} dt - \int_{1/2}^1 dt = \frac{1}{2} = \frac{1}{2} = 0.
\end{aligned}$$

One can also easily show that $\|f\| = \|g\| = 1$.

Problem

Problem: Find a polynomial $f(t) = a + bt$ that is perpendicular to the polynomial $g(t) = 1 - t$.

Answer: We are looking for numbers a and b such that:

$$\begin{aligned} 0 = (f, g) &= \int_0^1 (a + bt)(1 - t)\, dt \\ &= \int_0^1 a + bt - at - bt^2\, dt \\ &= a + \frac{b}{2} - \frac{a}{2} - \frac{b}{3} \\ &= \frac{a}{2} + \frac{b}{6}. \end{aligned}$$

Thus, $3a + b = 0$. So, we can take $f(t) = 1 - 3t$.

Important Facts

We state now two important facts about the inner product on a vector space V. Recall that in $\mathbb{R}^2$ we have:

$$\cos(\theta) = \frac{(u, v)}{\|u\| \cdot \|v\|}.$$

where u, v are two non-zero vectors in $\mathbb{R}^2$ and θ is the angle between u and v. In particular, because $-1 \leq \cos\theta \leq 1$, we must have:

$$\|(u, v)\| \leq \|u\| \cdot \|v\|.$$

We will show now that this comes from the positivity and linearity of the inner product.

Theorem. Let V be a vector space with inner product $(.,.)$. Then:

$$|(u, v)| \leq \|u\| \cdot \|v\|$$

for all $u, v \in V$.

Proof. We can assume that $u, v \neq 0$ because otherwise both the LHS and the RHS will be zero. By the positivity of the inner product we get:

$$\begin{aligned}
0 &\leq (v - \frac{(v,u)}{\|u\|^2}u, v - \frac{(v,u)}{\|u\|^2}u) \quad (\textit{positivity}) \\
&= (v,v) - \frac{(v,u)}{\|u\|^2}(u,v) - \frac{(v,u)}{\|u\|^2}(v,u) + \frac{(v,u)^2}{\|u\|^4}(u,u) \quad (\textit{linearity}) \\
&= \|v\|^2 - 2\frac{(u,v)^2}{\|u\|^2} + \frac{(v,u)^2}{\|u\|^2} \quad (\textit{symmetry}) \\
&= \|v\|^2 - \frac{(u,v)^2}{\|u\|^2}.
\end{aligned}$$

Thus,

$$\frac{(u,v)^2}{\|u\|^2} \leq \|v\|^2 \quad \text{or} \quad \|(u,v)\| \leq \|u\| \cdot \|v\|.$$

□

Note

Notice that:

$$0 = (v - \frac{(u,v)}{\|u\|^2}u, v - \frac{(u,v)}{\|u\|^2}u)$$

only if

$$v - \frac{(u,v)}{\|u\|^2}u = 0$$

i.e.

$$v = \frac{(u,v)}{\|u\|^2}u.$$

Thus, v and u have to be on the same line through 0.

A Lemma

We can therefore conclude:

Lemma. $\|(u,v)\| = \|u\| \cdot \|v\|$ if and only if u and v are on the same line through 0.

Theorem

The following statement is generalization of Pythagoras Theorem.

Theorem. Let V be a vector space with inner product $(.,.)$. Then:

$$\|u+v\| \leq \|u\| \cdot \|v\|$$

for all $u, v \in V$. Furthermore, $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $(u, v) = 0$.

Proof.

$$\begin{aligned}
\|u+v\|^2 &= (u+v, u+v) \\
&= (u,u) + 2(u,v) + (v,v) \quad (*) \\
&\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

If $(u, v) = 0$, then (*) reads:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

On the other hand, if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, we see from (*) that $(u, v) = 0$. $\square$

Example. Let $u = (1, 2, -1), v = (0, 2, 4)$. Then:

$$(u, v) = 4 - 4 = 0$$

and

$$\|u\|^2 = 1 + 4 + 1 = 6, \|v\|^2 = 4 + 16 = 20.$$

Also, $u + v = (1, 4, 3)$ and finally:

$$\|u+v\|^2 = 1 + 16 + 9 = 26 = 6 + 20 = \|u\|^2 + \|v\|^2.$$

Lecture 6

Generating Sets and Bases

Let V be the vector space $\mathbb{R}^2$ and consider the vectors $(1,0),(0,1)$. Then, every vector $(x,y) \in \mathbb{R}^2$ can be written as a combination of those vectors. That is:

$$(x,y) = x(1,0) + y(0,1).$$

Similarly, the two vectors $(1,1)$ and $(1,2)$ do not belong to the same line, and every vector in $\mathbb{R}^2$ can be written as a combination of those two vectors.

6.1 Introduction

In particular:

$$(x,y) = a(1,1) + (1,2)$$

gives us two equations

$$a + b = x \quad \text{and} \quad a + 2b = y$$

Thus, by substituting the first equation to the second, we get

$$b = -x + y$$

Inserting this into the first equation we get

$$a = 2x - y$$

Take for example the point $(4,3)$. Then:

$$\begin{aligned} (4,3) &= 5(1,1) + (-1)(1,2) \\ &= 5(1,1) - (1,2) \end{aligned}$$

We have similar situation for $\mathbb{R}^3$ and all of the spaces $\mathbb{R}^n$.

In the case of $\mathbb{R}^3$, for example, every vector can be written as combinations of $(1,0,0),(0,1,0)$ and $(0,0,1)$, i.e.,

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Or, as a combination of $(1,-1,0),(1,1,1)$ and $(0,1,-1)$, that is:

$$(x,y,z) = a(1,-1,0) + b(1,1,1) + c(0,1,-1).$$

The latter gives three equation:

$$\begin{array}{rcll} a+b & = & x & (1) \\ -a+b+c & = & y & (2) \\ b-c & = & z & (3). \end{array}$$

$(2)+(3)$ gives:

$$-a+2b = y+z \quad (4)$$

$(4)+(1)$ gives:

$$3b = x+y+z \text{or} b = \frac{x+y+z}{3}.$$

Then (1) gives:

$$\begin{array}{rcl} a & = & x-b \\ & = & x - \dfrac{x+y+z}{3} \\ & = & \dfrac{2x-y-z}{3}. \end{array}$$

Finally, (3) gives:

$$c = b - z = \frac{x+y-2z}{3}$$

Hence, we get:

$$(x,y,z) = \frac{2x-y-z}{3}(1,-1,0) + \frac{x+y+z}{3}(1,1,1) + \frac{x+y-2z}{3}(0,1,-1).$$

6.2 In general

Notice that we get only one solution, so there is only one way that we can write a vector in $\mathbb{R}^3$ as a combination of those vectors. In general, if we have k vectors in $\mathbb{R}^3$, then the equation:

$$x = (x_1, x_2, ..., x_n) = c_1v_1 + c_2v_2 + ... + c_kv_k \quad (*)$$

gives n-equations involving the n-coordinates of $v_1, v_2, ..., v_k$ and the unknowns $c_1, c_2, ..., c_k$. There are three possibilities:

6.3 Possibilities

A The equation(*) has no solution. Thus, x can not be written as a combination of the vectors $v_1, v_2, ..., v_k$.

B The equation (*) has only one solution, so x can be written in exactly one way as a combination of $v_1, v_2, ..., v_k$.

C The system of equations has infinitely many solutions, so there are more than one way to write x as a combination of $v_1, v_2, ..., v_k$.

6.3.1 Case C

Let us look at the last case a little closer. If we write x in two different ways:

$$\begin{aligned} x &= c_1v_1 + c_2v_2 + ... + c_kv_k \\ x &= d_1v_1 + d_2v_2 + ... + d_kv_k \end{aligned}$$

Then, by subtracting, we get:

$$0 = (c_1 - d_1)v_1 + (c_2 - d_2)v_2 + ... + (c_k - d_k)v_k$$

where some of the numbers $c_i - d_i$ are non-zero.

Similarly, since we can write:

$$0 = a_1v_1 + a_2v_2 + ... + a_kv_k$$

and

$$x = c_1v_1 + c_2v_2 + ... + c_kv_k$$

then we also have:

$$x = (c_1 + a_1)v_1 + (c_2 + a_2)v_2 + ... + (c_k + a_k)v_k.$$

Thus, we can write x as a combination of the vectors $v_1, v_2, ..., v_k$ in several different ways (in fact ∞-many ways).

6.4 Definitions

We will now use this as a motivation for the following definitions.

Definition. Let V be a vector space and $v_1, v_2, ..., v_n \in V$.

1. Let $W \subseteq V$ be a subspace. We say that W is spanned by the vectors $v_1, v_2, ..., v_n$ if every vector in W can be written as a linear combination of $v_1, v_2, ..., v_n$. Thus, if $w \in W$, then there exist numbers $c_1, c_2, ..., c_n \in \mathbb{R}$ such that $w = c_1 v_1 + c_2 v_2 + ... + c_n v_n$.

2. The set of vectors $v_1, v_2, ..., v_n$ is linearly dependent if there exist $c_1, c_2, ..., c_n$, not all equal to zero, such that $c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$.

Definition.
1. The set of vectors $v_1, v_2, ..., v_n$ is linearly independent if the set is not linearly dependent(if and only if we can only write $c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$ with all $c_i = 0$).

2. The set of vectors $v_1, v_2, ..., v_n$ is a basis for W, if $v_1, v_2, ..., v_n$ is linearly independent and spans W.

Before we show some examples, let us make the following observations:

Lemma. Let V be a vector space with an inner product $(.,.)$. Assume that $v_1, v_2, ..., v_n$ is an orthogonal subset of vectors in V (thus $(v_i, v_j) = 0$ if $i \neq j$). If $v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$, then $c_i = \frac{(v, v_i)}{\|v_i\|^2}$, $i = 1, ..., n$.

Proof. Assume that $v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$. Take the inner product with v_1 in both sides of the equation. The LHS is (v, v_1). The RHS is:

$$\begin{aligned}
(c_1 v_1 + c_2 v_2 + ... + c_n v_n, v_1) &= c_1(v_1, v_1) + c_2(v_2, v_1) \\
&\quad + ... + c_n(v_n, v_1) \\
&= c_1(v_1, v_1) \\
&= c_1 \|v_1\|^2 .
\end{aligned}$$

Thus, $(v, v_1) = c_1 \|v_1\|^2$, or $c_1 = \frac{(v, v_1)}{\|v_1\|^2}$. Repeat this for $v_2, ..., v_n$. □

Corollary. If the vectors $v_1, v_2, ..., v_n$ are orthogonal, then they are linearly independent.

6.5 Examples

Example. Let $V = \mathbb{R}^2$. The vectors $(1,2)$ and $(-2,-4)$ are linearly dependent because:

$$(-2)(1,2) + 1(-2,-4) = 0.$$

The vectors $(1,2),(1,1)$ are linearly independent. In fact, $(1,2),(1,1)$ is a basis for $\mathbb{R}^2$.

Indeed, let $(x,y) \in \mathbb{R}^2$. Then,

$$\begin{aligned}(x,y) &= c_1(1,2) + c_2(1,1)\\ &= (c_1 + c_2, 2c_1 + c_2).\end{aligned}$$

Thus,

$$\begin{aligned}x &= c_1 + c_2\\ y &= 2c_1 + c_2.\end{aligned}$$

Subtracting we get: $x - y = -c_1$, or $c_1 = y - x$. Plugging this into the first equation we get:

$$c_2 = x - c_1 = x - (y - x) = 2x - y.$$

Thus, we can write any vector in $\mathbb{R}^2$ as a combination of those two. In particular, for $(0,0)$ we get $c_1 = c_2 = 0$. The vectors $(1,2),(-2,1)$ are orthogonal and hence linearly independent, and in fact a basis. Hence,

$$(x,y) = c_1(1,2) + c_2(-2,1).$$

Taking the inner product we get: $c_1 = \frac{x+2y}{\|v_1\|^2} = \frac{x+2y}{5}$ and $c_2 = \frac{-2x+y}{5}$.

Example. Let $V = \mathbb{R}^3$. One vector can only generate a line, two vectors can at most span a plane, so we need at least three vectors to span $\mathbb{R}^3$. The vectors $(1,2,1),(1,-1,1)$ are orthogonal but <u>not</u> a basis. In fact, those two vectors span the plane:

$$W = (x,y,z) \in \mathbb{R}^3 : x - z = 0$$

(explain why).

On the other hand, the vectors: $(1,2,1),(1,-1,1)$ and $(1,0,-1)$ are orthogonal, and hence a basis.

We have, for example:

$$(4,3,1) = c_1(1,2,1) + c_2(1,-1,1) + c_3(1,0,-1)$$

with

$$c_1 = \frac{4+6+1}{1+4+1}$$
$$c_2 = \frac{4-3+1}{3}$$
$$c_3 = \frac{4-1}{2}$$

In general, we have:

$$(x,y,z) = \frac{x+2y+z}{6}(1,2,1) + \frac{x-y+z}{3}(1,-1,1) + \frac{x-z}{2}(1,0,-1)$$

Let us now discuss some spaces of functions:

a) Let $v_0(x) = 1, v_1(x) = x$ and $v_2(x) = x^2$. Then, v_o, v_1 and v_2 are linearly independent.

$$\begin{aligned} 0 &= c_0 v_0(x) + c_1 v_1(x) + c_2 v_2(x) \text{ for all } x \\ &= c_0 + c_1 x + c_2 x^2 \end{aligned}$$

Take $x = 0$, then we get $c_0 = 0$

Differentiate both sides to get:

$$0 = c_1 + 2c_2 x$$

Take again $x = 0$ to find $c_1 = 0$. Differentiate one more time to get that $c_2 = 0$. Notice that the span of v_0, v_1, v_2 is in the space of polynomials of degree ≤ 2. Hence, the functions $1, x, x^2$ form a basis for this space. Notice that the functions $1+x, 1-2x, x^2$ are also a basis.

b) Are the functions $v_0(x) = x, v_1(x) = xe^x$ linearly independent/dependent on $\mathbb{R}$? Answer: No.

Assume that $0 = c_0 x + c_1 x e^x$. It does <u>not</u> help to put $x = 0$ now, but let us first differentiate both sides and get:

$$0 = c_0 + c_1 e^x + c_1 x e^x$$

Now, $x = 0$ gives:

$$0 = c_0 + c_1 \quad (1)$$

Differentiating again, we get: $0 = c_1e^x + c_1e^x + c_1xe^x$. Now, $x = 0$ gives $0 = 2c_1$, or $c_1 = 0$. Hence, (1) gives $c_1 = 0$.

c) The functions $\chi_{[0,\frac{1}{2})}, \chi_{[\frac{1}{2},1)}$ are orthogonal and hence linearly independent. Let us show this directly. Assume that

$$0 = c_1\chi_{[0,\frac{1}{2})} + c_2\chi_{[\frac{1}{2},1)}.$$

An equation like this means that every x we put into the function on the RHS, the result is always 0.

Let us take $x = \frac{1}{4}$. Then: $\chi_{[0,\frac{1}{2})}(\frac{1}{4}) = 1$, but $\chi_{[\frac{1}{2},1)}(\frac{1}{4}) = 0$. Hence, $0 = c_1 \cdot 1 + c_2 \cdot 0$, or $c_1 = 0$. Taking $x = \frac{3}{4}$ shows that $c_2 = 0$.

d) The functions $\chi_{[0,\frac{1}{2})}, \chi_{[0,1)}$ are not orthogonal, but linearly independent.

$$0 = c_1\chi_{[0,1)} + c_2\chi_{[0,\frac{1}{2})}.$$

Take x so that $\chi_{[0,\frac{1}{2})}(x) = 0$ but $\chi_{[0,1)}(x) = 1$. Thus, any $x \in [0,1) \setminus [0,\frac{1}{2}) = [\frac{1}{2},1)$ will do the job. So, take $x = \frac{3}{4}$. Then, we see that:

$$0 = c_1 \cdot 1 + c_2 \cdot 0, \mathrm{or} c_1 = 0.$$

Then take $x = \frac{1}{4}$ to see that $c_2 = 0$.

Lecture 7

Gram-Schmidt Orthogonalization

The "best" basis we can have for a vector space is an orthogonal basis. That is because we can most easily find the coefficients that are needed to express a vector as a linear combination of the basis vectors $v_1, ..., v_n$:

$$v = \frac{(v, v_1)}{\|v_1\|^2} v_1 + ... + \frac{(v, v_n)}{\|v_n\|^2} v_n.$$

But usually we are not given an orthogonal basis. In this section we will show how to find an orthogonal basis starting from an arbitrary basis.

7.1 Procedure

Let us start with two linear independent vectors v_1 and v_2 (i.e. not on the same line through zero). Let $u_1 = v_1$. How can we find a vector u_2 which is perpendicular to u_1 and that the span of u_1 and u_2 is the same as the span of v_1 and v_2? We try to find a number $a \in \mathbb{R}$ such that:

$$u_2 = a u_1 + v_2, \quad u_2 \perp u_1$$

Take the inner product with u_1 to get:

$$\begin{aligned} 0 = (u_2, u_1) &= a(u_1, u_1) + (v_2, u_1) \\ &= a \|u_1\|^2 + (v_2, u_1) \end{aligned}$$

or

$$a = -\frac{(v_2, u_1)}{\|u_1\|^2}$$

What if we have a third vector v_3? Then, after choosing u_1, u_2 as above, we would look for u_3 of the form:

$$u_3 = a_1 u_1 + a_2 u_2 + v_3$$

Take the inner product with u_1 to find a_1:

$$0 = (u_3, u_1) = a_1 \|u_1\|^2 + (v_3, u_1)$$

or

$$a_1 = -\frac{(v_3, u_1)}{\|u_1\|^2}$$

and the inner product with u_2 to find a_2:

$$0 = (u_3, u_2) = a_2 \|u_2\|^2 + (v_3, u_2)$$

or

$$a_2 = -\frac{(v_3, u_2)}{\|u_2\|^2}$$

Thus:

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ u_3 &= v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} u_2 \end{aligned}$$

7.2 Examples

Example. Let $v_1 = (1, 1), v_2 = (2, -1)$. Then, we set $u_1 = (1, 1)$ and

$$\begin{aligned} u_2 &= (2, -1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ &= (2, -1) - \frac{2-1}{2}(1, 1) \\ &= \frac{3}{2}(1, -1) \end{aligned}$$

Example. Let $v_1 = (2, -1), v_2 = (0, 1)$. Then, we set $u_1 = (2, -1)$ and

$$\begin{aligned} u_2 &= (0, 1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ &= (0, 1) - \frac{-1}{5}(2, -1) \\ &= \frac{2}{5}(1, 2) \end{aligned}$$

Note We could have also started with $v_2 = (0,1)$, and get first basis vector to be $(0,1)$ and second vector to be:

$$(2,-1) - \frac{(2,-1)\cdot(0,1)}{\|(0,1)\|^2}(0,1) = (2,0)$$

Example. Let $v_1 = (0,1,2), v_2 = (1,1,2), v_3 = (1,0,1)$. Then, we set $u_1 = (0,1,2)$ and

$$\begin{aligned} u_2 &= (1,1,2) - \frac{(0,1,2)\cdot(1,1,2)}{\|(0,1,2)\|^2}(0,1,2) \\ &= (1,1,2) - \frac{5}{2}(0,1,2) \\ &= (1,0,0) \end{aligned}$$

$$\begin{aligned} u_3 &= (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0) \\ &= \frac{1}{5}(0,-2,1) \end{aligned}$$

Example. Let $v_0 = 1, v_1 = x, v_2 = x^2$. Then, v_0, v_1, v_2 is a basis for the space of polynomials of degree $\leq$. But they are not orthogonal, so we start with $u_0 = v_0$ and $u_1 = v_1 - \frac{(v_1,u_0)}{\|u_0\|^2}u_0$. So we need to find:

$$(v_1, u_0) = \int_0^1 x\,dx = [\frac{1}{2}x^2]_0^1 = \frac{1}{2}$$

$$\|u_0\|^2 = \int_0^1 1\ dx = [x]_0^1 = 1$$

Hence, $u_1 = x - \frac{1}{2}$. Then:

$$u_1 = v_2 - \frac{(v_2,u_0)}{\|u_0\|^2}u_0 - \frac{(v_2,u_1)}{\|u_1\|^2}u_1\,.$$

We also find that:

$$\begin{aligned} (v_2,u_0) &= \int_0^1 x^2\,dx = \frac{1}{3} \\ (v_2,u_1) &= \int_0^1 x^2(x-\frac{1}{2})\,dx = \frac{1}{12} \\ \|u_1\|^2 &= \int_0^1 (x-\frac{1}{2})^2\,dx = \frac{1}{12}. \end{aligned}$$

Hence, $u_2 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$.

7.3 Theorem

Theorem. (Gram-Schmidt Orthogonalization)Let V be a vector space with inner product $(.,.)$. Let $v_1, ..., v_k$ be a linearly independent set in V. Then, there exists an orthogonal set $u_1, ..., u_k$ such that $(v_i, u_i) > 0$ and $\text{span}\{v_i, ..., v_i\} = \text{span}\{u_1, ..., u_i\}$ for all $i = 1, ..., k$.

Proof. See the book, p.$129 - 131$. □

Lecture 8

Orthogonal Projections

8.1 Introduction

We will now come back to our original aim: Given a vector space V, a subspace W, and a vector $v \in V$, find the vector $w \in W$ which is closest to v.

First let us clarify what the "closest" means. The tool to measure distance is the <u>norm</u>, so we want $\|v - w\|$ to be as small as possible.

Thus our problem is:
Find a vector $w \in W$ such that

$$\|v - w\| \leq \|v - u\|$$

for all $u \in W$.

Now let us recall that if $W = \mathbb{R}w_1$ is a line, then the vector w on the line W is the one with the property that $v - w \perp W$.
We will start by showing that this is always the case.

8.2 $w \in W$ is closest to v *iff* $v - w \perp W$

Theorem. Let V be a vector space with inner product $(\cdot, \cdot)$. Let $W \subset V$ be a subspace and $v \in V$. If $v - w \perp W$, then $\|v - w\| \leq \|v - u\|$ for all $u \in W$ and $\|v - w\| = \|v - u\|$ if and only if $w = u$. Thus w is the member of W closest to v.

Proof. First we remark that $\|v-w\| \leq \|v-u\|$ if and only if $\|v-w\|^2 \leq \|v-u\|^2$. Now we simply calculate

$$\begin{aligned}
\|v-u\|^2 &= \|(v-w)+(w-u)\|^2 \\
&= \|v-w\|^2 + \|w-u\|^2 \\
&\quad \text{because } v-w \perp W \text{ and } w-u \in W \\
(*) \quad &\geq \|v-w\|^2 \quad \text{because } \|w-u\|^2 \geq 0
\end{aligned}$$

So $\|v-u\| \geq \|v-w\|$. If $\|v-u\|^2 = \|v-w\|^2$, then we see - using $(*)$- that $\|w-u\|^2 = 0$, or $w = u$.
As $\|v-w\| = \|v-u\|$ if $u = w$, we have shown that the statement is correct. □

Theorem. Let V be a vector space with inner product $(\cdot,\cdot)$. Let $W \subset V$ be a subspace and $v \in V$. If $w \in W$ is the closest to v, then $v - w \perp W$.

Proof. We know that $\|v-w\|^2 \leq \|v-u\|^2$ for all $u \in W$. Therefore the function $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$F(t) := \|v - w + tx\|^2 \quad (x \in W)$$

has a minimum at $t = 0$. We have

$$\begin{aligned}
F(t) &= (v-w+tx, v-w+tx) \\
&= (v-w, v-w) + t(v-w, x) \\
&\quad + t(x, v-w) + t^2(x,x) \\
&= \|v-w\|^2 + 2t(v-w,x) + t^2\|x\|^2
\end{aligned}$$

Therefore

$$0 = F'(0) = 2(v-w, x).$$

As $x \in W$ was arbitrary, it follows that $v - w \perp W$. □

8.2.1 Construction of w

Our task now is to construct the vector w such that $v - w \perp W$. The idea is to use Gram-Schmidt orthogonalization.
Let $W = \mathbb{R}u$ and $v \in V$. Applying Gram-Schmidt to u and v gives:

$$v - \frac{(v,u)}{\|u\|^2}u \perp W$$

So that $w = \frac{(v,u)}{\|u\|^2}u$ is the vector (point) on the line W closest to v.

What if the dimension of W is greater than one? Let $v_1, \ldots, v_n$ be an orthogonal basis for W. Applying the Gram-Schmidt to the vectors $v_1, \ldots, v_n, v$ shows that

$$v - \sum_{j=1}^{n} \frac{(v, v_j)}{\|v_j\|^2} v_j$$

is orthogonal to each one of the vectors $v_1, \ldots, v_n$. Since

$$v - \sum_{j=1}^{n} \frac{(v, v_j)}{\|v_j\|^2} v_j$$

is orthogonal to v_j for all j, it is orthogonal to any linear combination of them $c_1v_1 + \ldots + c_nv_n = \sum_{j=1}^{n} c_jv_j$, and hence it is orthogonal to W. Therefore our vector w closest to v is given by

$$w = \sum_{j=1}^{n} \frac{(v, v_j)}{\|v_j\|^2} v_j.$$

Let us look at another motivation; Let $w \in W$ be the closest to v and let $v_1, \ldots, v_n$ be a basis for W. Then there are scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$w = \sum_{k=1}^{n} c_kv_k.$$

So what are these scalars? As $v - w \perp v_j$ for $j = 1, \ldots, n$ and $v_k \perp v_j$ for

$k \neq j$ we get:

$$\begin{aligned} 0 &= (v-w, v_j) \\ &= (v, v_j) - (w, v_j) \\ &= (v, v_j) - \sum_{k=1}^{n} c_k (v_k, v_j) \\ &= (v, v_j) - c_j (v_j, v_j) \\ &= (v, v_j) - c_j \|v_j\|^2. \end{aligned}$$

Solving for c_j we get

$$c_j = \frac{(v, v_j)}{\|v_j\|^2}.$$

Thus

$$w = \sum_{j=1}^{n} \frac{(v, v_j)}{\|v_j\|^2} v_j.$$

8.3 The main theorem

We collect the results of the above computations in the following (main)theorem:

Theorem. Let V be a vector space with inner product $(\cdot, \cdot)$. Let $W \subset V$ be a subspace and assume that $\{v_1, \ldots, v_n\}$ is an orthogonal basis for W. For $v \in V$ let $w = \sum_{j=1}^{n} \frac{(v, v_j)}{\|v_j\|^2} v_j \in W$. Then $v - w \perp W$ (or equivalently, w is the vector in W closest to v).

Proof. We have

$$\begin{aligned} (v-w, v_j) &= (v, v_j) - (w, v_j) \\ &= (v, v_j) - \sum_{k=1}^{n} \frac{(v, v_k)}{\|v_k\|^2} (v_k, v_j) \\ &= (v, v_j) - \frac{(v, v_j)}{\|v_j\|^2} \|v_j\|^2 \\ &= (v, v_j) - (v, v_j) \\ &= 0 \end{aligned}$$

Hence $v - w \perp v_j$. But, as we saw before, this implies that $v - w \perp W$ because $v_1, \ldots, v_n$ is a basis. $\square$

8.4 Orthogonal projections

Let us now look at what we just did from the point of view of linear maps. What is given in the beginning is a vector space with an inner product and a subspace W. Then for each $v \in V$ we associated a unique vector $w \in W$. Thus we got a map

$$P : V \longrightarrow W, \quad v \mapsto w$$

We even have an explicit formula for $P(v)$: Let (if possible) $v_1, \ldots, v_n$ be an orthogonal basis for W, then

$$P(v) = \sum_{k=1}^{n} \frac{(v, v_k)}{\|v_k\|^2} v_k$$

This shows that P is <u>linear</u>.

We showed earlier that if $v \in W$, then

$$v = \sum_{k=1}^{n} \frac{(v, v_k)}{\|v_k\|^2} v_k$$

So $P(v) = v$ for all $v \in W$. In particular, we get

Lemma. $P^2 = P$.

The map P is called the <u>orthogonal</u> <u>projection</u> onto W. The <u>projection</u> part comes from $P^2 = P$ and <u>orthogonal</u> from the fact that $v - P(v) \perp W$.

8.5 Summary

The result of this discussion is the following:

To find the vector w closest to v we have to:

1. Find (if possible) a basis $u_1, \ldots, u_n$ for W.

2. If this is not an orthogonal basis, then use Gram-Schmidt to construct an orthogonal basis $v_1, \ldots, v_n$.

3. Then $w = \sum_{k=1}^{n} \frac{(v, v_k)}{\|v_k\|^2} v_k$.

8.6 Examples

Example. Let W be the line $W = \mathbb{R}(1,2)$. Then $u = (1,2)$ is a basis (orthogonal!) for W. It follows that the orthogonal projection is given by

$$P(x,y) = \frac{x+2y}{5}(1,2).$$

Let $(x,y) = (3,1)$. Then

$$P(3,1) = (1,2).$$

Example. Let W be the line given by $y = 3x$. Then $(1,3) \in W$ and hence $W = \mathbb{R}(1,3)$.It follows that

$$P(x,y) = \frac{x+3y}{10}(1,3).$$

Example. Let W be the plane generated by the vectors $(1,1,1)$ and $(1,0,1)$. Find the orthogonal projection $P : \mathbb{R}^3 \longrightarrow W$.

Solution. We notice first that $((1,1,1),(1,0,1)) = 2 \neq 0$, so this is not an orthogonal basis. Using Gram-Schmidt we get:
$v_1 = (1,1,1)$
$v_2 = (1,0,1) - \frac{2}{3}(1,1,1) = (\frac{1}{3}, -\frac{2}{3}), \frac{1}{3} = \frac{1}{3}(1,-2,1)$.

To avoid fractions, we can use $(1,-2,1)$ instead of $\frac{1}{3}(1,-2,1)$. Thus the orthogonal projection is:

$$\begin{aligned} P(x,y,z) &= \frac{x+y+z}{3}(1,1,1) + \frac{x-2y+z}{6}(1,-2,1) \\ &= \left(\frac{2x+2y+2z}{6} + \frac{x-2y+z}{6},\right. \\ &\qquad \frac{2x+2y+2z}{6} - 2\frac{x-2y+z}{6}, \\ &\qquad \left.\frac{2x+2y+2z}{6} + \frac{x-2y+z}{6}\right) \\ &= \left(\frac{x+z}{2}, y, \frac{x+z}{2}\right). \end{aligned}$$

Example. Let W be the plane $\{(x,y,z) \in \mathbb{R}^3 | x + y + 2z = 0\}$. Find the orthogonal projection $P : \mathbb{R}^3 \longrightarrow W$.

Solution. We notice that our first step is to find an orthogonal basis for W. The vectors $(1,-1,0)$ and $(2,0,-1)$ are in W, but are not orthogonal.We have

$$(2,0,-1)-\tfrac{2}{2}(1,-1,0)=(1,1,-1)\in W$$

and orthogonal to $(1,-1,0)$. So we get:

$$\begin{aligned} P(x,y,z) &= \frac{x-y}{2}(1,-1,0)+\frac{x+y-z}{3}(1,1,-1) \\ &= \left(\frac{5x-y-2z}{6},\frac{-x+5y-2z}{6},\frac{-x-y+z}{3}\right). \end{aligned}$$

8.7 Exercises

1. Let $V\subset\mathbb{R}^2$ be the line $V=\mathbb{R}(1,-1)$.

   (a) Write a formula for the orthogonal projection $P:\mathbb{R}^2\to V$.

   (b) What is: i) $P(1,1)$, ii) $P(2,1)$, iii) $P(2,-2)$?

2. Let $W\subset\mathbb{R}^3$ be the plane

   $$W=\{(x,y,z)\in\mathbb{R}^3:\ x-2y+z=0\}.$$

   (a) Find the orthogonal projection $P:\mathbb{R}^3\to W$.

   (b) What is: i) $P(1,1,2)$, ii) $P(1,-2,1)$, iii) $P(2,1,1)$?

3. Let $W\subset\mathbb{R}^3$ be the plane generated by the vectors $(1,1,1)$ and $(1,-1,1)$.

   (a) Find the orthogonal projection $P:\mathbb{R}^3\to W$.

   (b) What is: i) $P(1,1,2)$, ii) $P(2,0,1)$?

4. Let W be the space of continuous functions on $[0,1]$ generated by the constant function 1 and x. Thus $W=\{a_0+a_1x:\ a_0,a_1\in\mathbb{R}\}$. Find the orthogonal projection of the following functions onto W:
   i) $P(x^2)$, ii) $P(e^x)$, iii) $P(1+x^2)$.

5. Let W be the space of piecewise continuous functions on $[0,1]$ generated by $\chi_{[0,1/2)}$ and $\chi_{[1/2,1)}$. Find orthogonal projections of the following functions onto W:
   i) $P(x)$, ii) $P(x^2)$, iii) $P(\chi_{[0,3/4]})$.

Lecture 9

2-D Haar Wavelet Transform

9.1 Tensor Product of Functions

In 1-dimension we used step functions of the form:

$$p = \sum_{j=1}^{2^n-1} s_j \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$$

to approximate "arbitrary" functions.

Thus we made the following steps:
(1) Decide up on the resolution, and divide the interval $[0,1)$ into subintervals $I_j = [\frac{j}{2^n}, \frac{j+1}{2^n})$.
(2) Take the orthogonal projection

$$f \to p = \sum_{j=1}^{2^n-1} s_j \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$$

(3) Replace p by the vector $[s_0, ..., s_{2^n-1})$
(4) Apply the fast wavelet transform to the sequence $[s_0, ..., s_{2^n-1})$.

In 2-D we have now two directions, x and y. We will therefore have to divide the square:

$$I = [0,1) \times [0,1) = \{(x,y) \in \mathbb{R}^2 : 0 \leq x < 1, 0 \leq y < 1\}$$

into smaller squares:

$$I_{ij}^n = \{(x,y) \in \mathbb{R}^2 : \frac{i}{2^n} \leq x < \frac{i+1}{2^n}, \frac{j}{2^n} \leq y < \frac{j+1}{2^n}\}$$

Then we approximate functions of two variables by functions that are constants on each of the squares I_{ij}. That is,

$$f \to p = \sum s_j \chi_{I_{ij}}$$

We start by discussing special functions of two variables.

Definition. Let $p, q : [0, 1) \longrightarrow \mathbb{R}$. Then, $p \otimes q$ is the function on $[0, 1) \times [0, 1)$ given by:

$$p \otimes q(x, y) = p(x)q(y).$$

Example. Let $A, B \subseteq [0, 1)$. Then, $\chi_A \otimes \chi_B = \chi_{A \times B}$.

PROOF: We have $\chi_A \otimes \chi_B(x, y) = \chi_A(x)\chi_B(y)$. Thus, $\chi_A \otimes \chi_B(x, y) = 1$ if and only if $x \in A$ and $y \in B$. But that is the same as saying that $(x, y) \in A \times B$.

Example. Let $\varphi_{ij} = \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$ and

$$\psi_{ij} = \chi_{[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}})} - \chi_{[\frac{j+2}{2^{i+1}}, \frac{j+3}{2^{i+1}})}$$

Recall that φ_{ij} is used to express the averages, and ψ_{ij} is used to express the details.

We have:

$$\varphi_{ni} \otimes \varphi_{nj} = \varphi_{I_{ij}^n}.$$

Can be used to express the "average values" on I_{ij}^n and

$$\varphi_{ni} \otimes \psi_{nj}, \psi_{ni} \otimes \varphi_{nj}, \psi_{ni} \otimes \psi_{nj}.$$

can be used to express details going from one column to the next, one row to the next, and going diagonally.

Let f be a step function in $[0, 1) \times [0, 1)$. Instead of associating to f an array, we have to use a matrix. Thus if:

$$f(x, y) = \sum_{i,j=1}^{2^n - 1} s_{ij} \chi_{I_{ij}^n}$$

then, f corresponds to the matrix:

$$f \leftrightarrow \begin{pmatrix} s_{00} & s_{01} & \cdot & \cdot & \cdot & s_{0,2^n-1} \\ s_{10} & s_{11} & \cdot & \cdot & \cdot & s_{1,2^n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{2^n-1,0} & s_{2^n-1,1} & \cdot & \cdot & \cdot & s_{2^n-1,2^n-1} \end{pmatrix}$$

Notice that the first indices (the ones corresponding to the x-variable) indicate the row, and the second indices correspond to the column.

Example. 2×2-matrix

$$\chi_{[0,\frac{1}{2})} \otimes \chi_{[0,1)} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Example. Find the 2×2-matrix corresponding to the function:

$$\chi_{[0,\frac{1}{2})} \otimes (\chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}) + 2\chi_{[\frac{1}{2},1)} \otimes \chi_{[0,1)}$$

The matrix is:

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$$

9.2 The two-dim. Haar Wavelet Transform

In two dimensions we apply the Haar wavelet transform on the columns and on the rows.

Example. Suppose that the function f is represented by the 2×2 matrix:

$$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}$$

Then each row represents a one-dimensional vector of length 2. Namely, $[4, 2]$ and $[0, 6]$.

Applying the $1 - D$ wavelet transform to these vectors gives:

$$[4, 2] \rightarrow [3, 1]$$

and

$$[0, 6] \rightarrow [3, -3]$$

Thus,

$$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 \\ 3 & -3 \end{pmatrix}$$

We now apply the $1-D$ wavelet transform to the rows:

$$[3,3]^T \rightarrow [3,0]^T$$

and

$$[1,-3]^T \rightarrow [-1,2]^T$$

This results to the new matrix,

$$\begin{pmatrix} 3 & 1 \\ 3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 \\ 0 & -3 \end{pmatrix}$$

Notice that the element 3 represents the average value of all values:

$$\frac{4+2+0+6}{4} = \frac{12}{4} = 3.$$

To understand the other matrix elements, we will do this for a general matrix.

So, we have:

$$\begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \xrightarrow{1^{st}-\text{step}} \begin{pmatrix} \frac{s_{00}+s_{01}}{2} & \frac{s_{00}-s_{01}}{2} \\ \frac{s_{10}+s_{01}}{2} & \frac{s_{10}-s_{01}}{2} \end{pmatrix}$$

We apply the 1-D transform to each column and we get:

$$\xrightarrow{2^{st}-\text{step}} \begin{pmatrix} \frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} + \frac{s_{10}+s_{11}}{2}\right] & \frac{1}{2}\left[\frac{s_{00}-s_{01}}{2} + \frac{s_{10}-s_{11}}{2}\right] \\ \frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} - \frac{s_{10}+s_{11}}{2}\right] & \frac{1}{2}\left[\frac{s_{00}-s_{01}}{2} - \frac{s_{10}-s_{11}}{2}\right] \end{pmatrix}$$

Remarks: 1. The number $\frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} + \frac{s_{10}+s_{11}}{2}\right] = \frac{s_{00}+s_{01}+s_{10}+s_{11}}{4}$ is the average value.

2. The number $\frac{1}{2}\left[\frac{s_{00}-s_{01}}{2} + \frac{s_{10}-s_{11}}{2}\right]$ is the average change moving from the first to the second column.

3. The number $\frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} - \frac{s_{10}+s_{11}}{2}\right]$ is the average change moving from the first to the second row (or the details for the average).

4. The number $\frac{1}{2}\left[\frac{s_{00}-s_{01}}{2} - \frac{s_{10}-s_{11}}{2}\right]$ represents the average changes along the diagonals.

The method for bigger for matrices is the same:

$$\begin{pmatrix} 5 & 7 & -1 & -5 \\ 1 & -1 & -3 & -3 \\ 7 & 3 & 2 & 2 \\ 2 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{1-D \text{ on rows}} \begin{pmatrix} 6 & -3 & -1 & 2 \\ 0 & -3 & 1 & 0 \\ 5 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{1-D \text{ on columns}} \begin{pmatrix} 3 & -3 & 0 & 1 \\ 3 & 1 & 2 & 0 \\ 3 & 0 & -1 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

Note Now notice that the 2×2 matrix $\begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix}$ located in the upper-half corner, consists of <u>average</u> values. We apply the 2-D wavelet transform again to this matrix and get:

$$\begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

9.3 The Inverse Transform

Given a 2×2 matrix (or a 4×4 matrix) knowing that this matrix is a result of a $2-D$ Haar wavelet transform, how can we reconstruct the original matrix?

Let us start with a 2×2 matrix. Recall that the 2-D Haar wavelet transform consists of 2-steps:

1^{st}-step: ($\rightrightarrows$) 1-D Haar Wavelet Transform on the rows.
2^{nd}-step: ($\downdownarrows$) 1-D Haar Wavelet Transform on the columns.

Therefore to invert this we need to:

1^{st}-step: Apply the inverse 1-D Haar Wavelet Transform on the columns.
2^{st}-step: Apply the inverse 1-D Haar Wavelet Transform on the rows.

Example. Given the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, find the initial matrix.

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \xrightarrow{1-D \text{ inverse on columns}} \begin{pmatrix} 2-1 & 1+0 \\ 2-(-1) & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\xrightarrow{1-D\,\text{inverse on rows}} \begin{pmatrix} 1+1 & 1-1 \\ 3+1 & 3-1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$$

9.3.1 Bigger Matrices

Now look at 4×4 matrices. Recall that the last step was to apply the 2-D wavelet transform on the upper left corner (2×2 matrix). Here is an overview of the steps:

1^{st}-step:1-D on the rows.
2^{nd}-step:1-D on the columns.
3^{rd}-step:Take the 2×2 matrix in the left upper corner and apply step 1 and 2 to this matrix, keeping all other numbers the same.

For the inverse we need to invert each of the previous steps. Thus:

1^{st}-step: Pick the 2×2 matrix in the left upper corner. Apply the inverse 1-D on the columns.
2^{nd}-step: Apply the 1-D inverse transform on the rows on the 2×2 matrix from step 1.
3^{rd}-step: We have now a new 4×4 matrix. Apply the inverse 1-D to the columns.
4^{th}-step: Apply the inverse 1-D to the rows.

Example. Given the matrix $\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 3 \end{pmatrix}$, find the initial matrix.

We start by the 2×2 matrix $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and we apply the inverse transform to this matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{columns} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \xrightarrow{rows} \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}.$$

We have now the new matrix:

$$\begin{pmatrix} 4 & 2 & 0 & 3 \\ 2 & 0 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 3 \end{pmatrix}$$

Example. To this new matrix, we will apply the inverse row-column transform:

$$\begin{pmatrix} 4 & 2 & 0 & 3 \\ 2 & 0 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 3 \end{pmatrix} \xrightarrow{\text{inverse on columns}} \begin{pmatrix} 5 & 4 & -1 & 5 \\ 3 & 0 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 2 & 1 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow{\text{inverse on rows}} \begin{pmatrix} 4 & 6 & 9 & -1 \\ 4 & 2 & 1 & -1 \\ 5 & -1 & 1 & -3 \\ 3 & 1 & -3 & 5 \end{pmatrix}$$

So the final answer is: $\begin{pmatrix} 4 & 6 & 9 & -1 \\ 4 & 2 & 1 & -1 \\ 5 & -1 & 1 & -3 \\ 3 & 1 & -3 & 5 \end{pmatrix}$

Lecture 10

Complex Vector Spaces

10.1 Introduction

Up to now we have only considered vector spaces over the field of real numbers. For the Fourier transform we will need vector spaces where we can multiply vectors by complex numbers. If $x \in \mathbb{R}$, then we know that $x^2 \geq 0$ and $x^2 = 0$ only if $x = 0$. Therefore there is no <u>real</u> number such that $x^2 = -1$, or: There is no <u>real</u> solution to the equation

$$x^2 + 1 = 0.$$

More generally let us look at the equation

$$x^2 + bx + c = 0$$

By completing the square we get

$$x = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$$

There are now 3 possibilities:

1. $b^2 - 4c > 0$. Then the quadratic equation above gives two solutions

$$x = -\frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4c}$$

and

$$x = -\frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4c}$$

2. $b^2 - 4c = 0$. Then we have one solution

$$x = -\frac{b}{2}$$

3. $b^2 - 4c < 0$. Then there is no real solution to quadratic equation $x^2 + bx + c = 0$.

10.2 Complex numbers

We now introduce a new number $i = \sqrt{-1}$ such that

$$i^2 = -1$$

The complex numbers are all expressions of the form

$$z = x + iy\ ,\ \ x, y \in \mathbb{R}.$$

The set of complex numbers is denoted by $\mathbb{C}$. We say that $x = \Re z$ is the real part of z and $y = \Im z$ is the imaginary part of z.

Recall that the set of real numbers can be thought of as a line, the real line. To picture the set of complex numbers we use the plane. A vector $\vec{v} = (x, y)$ corresponds to the complex number $z = x + iy$.

10.2.1 Addition and multiplication on $\mathbb{C}$

The addition of two complex numbers $z = x + iy$ and $w = s + it$ then corresponds to the addition of the corresponding vectors. Thus

$$(x + iy) + (s + it) = (x + s) + i(y + t).$$

To find out what the product of z and w is, we use the familiar rules along with $i^2 = -1$. Thus

$$\begin{aligned}(x + iy) \cdot (s + it) &= \ xs + xit + iys + iyit \\ &= \ xs + i^2yt + i(xt + ys) \\ &= \ (xs - yt) + i(xt + ys).\end{aligned}$$

10.2.2 Conjugate and absolute value of z

Before we find the inverse -or reciprocal- of $z = x + iy$, we need to introduce the complex conjugate:

$$\overline{x + iy} = x - iy$$

Thus, complex conjugate corresponds to a reflection around the x-axis.

Now multiply z by $\overline{z}$:

$$\begin{aligned} z \cdot \overline{z} &= (x + iy)(x - iy) \\ &= x^2 - (iy)^2 \\ &= x^2 + y^2. \end{aligned}$$

Thus $\sqrt{z\overline{z}} := |z|$ is the length of the vector (x, y) or the absolute value of the complex number z.

10.2.3 Reciprocal of z

Lemma. Let $z = x + iy$ be a complex number with $|z| \neq 0$. Then

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

Proof. We have $z\frac{x-iy}{x^2+y^2} = \frac{(x+iy)(x-iy)}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$ □

10.2.4 Examples

- $(2 + 3i) + (5 - 2i) = 7 + i$.
- $(2 - 3i)(1 + i) = (2 + 3) + (-3 + 2)i = 5 - i$.
- $\frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i$.
- $\frac{2+3i}{1+5i} = \frac{(2+3i)(1-5i)}{26} = \frac{(2+15)+(3-10)i}{26} = \frac{17-7i}{26}$.

10.3 The complex exponential function

The idea behind the Fourier transform is to represent a function (or a signal) in the frequency domain using the complex exponential function.

Definition. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that z_n converges to the complex number w if for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|z_n - w| < \epsilon$$

for all $n \geq N$. We write $z_n \to w$ or $\lim_{n\to\infty} z_n = w$ if z_n converges to w.

Let $z \in \mathbb{C}$ and define $z_n = \sum_{k=0}^{n} \frac{z^k}{k!}$. Then it can be shown that the sequence $\{z_n\}$ converges. We denote the limit by

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{z^k}{k!}.$$

It can be shown that

$$e^{z+w} = e^z e^w.$$

In particular

$$\frac{1}{e^z} = e^{-z}.$$

Theorem. (The Euler formula)
Let $z = x + iy \in \mathbb{C}$. Then

$$e^z = e^x(\cos y + i \sin y).$$

10.3.1 Examples

- $e^{\pi i} = \cos\pi + i\sin\pi = -1.$
- $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$
- $e^{2+i\frac{\pi}{4}} = e^2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \frac{e^2}{\sqrt{2}}(1+i).$

Lemma. Let $z = x + iy$. Then

$$\overline{e^z} = e^{\overline{z}}$$

Proof.

$$\begin{aligned}\overline{e^z} &= \overline{e^x(\cos y + i\sin y)}\\ &= e^x(\cos y - i\sin y)\\ &= e^{x-iy}\\ &= e^{\overline{z}}\end{aligned}$$ □

10.4 Complex-valued functions

Let $F : I \longrightarrow \mathbb{C}$, $I \subseteq \mathbb{R}$ an interval, be a function. Then we can write

$$F(t) = f(t) + ig(t)$$

where $f, g : I \longrightarrow \mathbb{R}$.

The function F is continuous if and only if f and g are both continuous. In that case we have

$$\begin{aligned}\lim_{t\to t_0} F(t) &= \lim_{t\to t_0} f(t) + (\lim_{t\to t_0} g(t))i\\ &= f(t_0) + g(t_0)i\\ &= F(t_0).\end{aligned}$$

10.4.1 Integration and differentiation

We integrate and differentiate F by integrating $f(t)$ and $g(t)$ (differentiating $f(t)$ and $g(t)$). Thus

$$\int_a^b F(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt\;i$$

$$\frac{dF}{dt}(t) = \frac{df}{dt}(t) + \frac{dg}{dt}(t)\;i$$

Examples

1)

$$\begin{aligned}\int_0^1 2t + 3t^2\;i\;dt &= t^2]_0^1 + t^3]_0^1 i\\ &= 1 + i.\end{aligned}$$

2)

$$\int_0^{2\pi} e^{(1+i)t}\, dt = \int_0^{2\pi} e^t \cos t + e^t \sin t\, dt$$

We have

$$\begin{aligned}\int_0^{2\pi} e^t \cos t\, dt &= \cos t e^t]_0^{2\pi} + \int_0^{2\pi} e^t \sin t\, dt \\ &= e^{2\pi} - 1 - \int_0^{2\pi} e^t \cos t\, dt\end{aligned}$$

Thus

$$\int_0^{2\pi} e^t \cos t\, dt = \frac{1}{2}(e^{2\pi} - 1).$$

Similarly we have

$$\int_0^{2\pi} e^t \sin t\, dt = -\frac{1}{2}(e^{2\pi} - 1).$$

Thus,

$$\int_0^{2\pi} e^{(1+i)t}\, dt = \frac{e^{2\pi} - 1}{2}(1 - i).$$

But we could have also used the rule

$$\int e^{at}\, dt = \frac{1}{a}e^{at} + C$$

where a is any complex number, $a \neq 0$. Then

$$\begin{aligned}\int_0^{2\pi} e^{(1+i)t}\, dt &= \frac{1}{1+i}e^{(1+i)t}]_0^{2\pi} \\ &= \frac{1}{1+i}[e^{2\pi + i2\pi} - e^0] \\ &= \frac{1}{1+i}[e^{2\pi} - 1]\end{aligned}$$

Furthermore

$$\frac{1}{1+i} = \frac{1}{2}(1-i)$$

Thus we have

$$\int_0^{2\pi} e^{(1+i)t}\, dt = \frac{e^{2\pi}-1}{2}(1-i)$$

Example. Evaluate the integral $\int_0^1 (t+2it^2)(t^2-3it)\, dt$.

Solution: First we have to carry out the multiplication

$$\begin{aligned} (t+2it^2)(t^2-3it) &= t^3+6t^3-2it^4-3it^2 \\ &= 7t^3-(2t^4+3t^2)i \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 (t+2it^2)(t^2-3it)\, dt &= \int_0^1 7t^3\, dt - (\int_0^1 2t^4+3t^2)i \\ &= \frac{7}{4}t^4]_0^1 - [\frac{2}{5}t^5+t^3]_0^1 i \\ &= \frac{7}{4} - \frac{7}{5}i. \end{aligned}$$

10.5 Complex vector spaces

The axioms for complex vector spaces are the same as those for real vector spaces except the scalars are now complex numbers.

Example. Let

$$\mathbb{C}^n = \{(z_1,\ldots,z_n) : z_1,\ldots,z_n\}.$$

The addition is given by

$$\begin{gathered} u=(z_1,\ldots,z_n), v=(w_1,\ldots,w_n). \\ u+v=(z_1+w_1,\ldots,z_n+w_n) \end{gathered}$$

The scalar multiplication is given by

$$\lambda u = (\lambda z_1,\ldots,\lambda z_n).$$

Example. Let $I \subset \mathbb{R}$ be an interval. Let V be the space of functions $f : I \longrightarrow \mathbb{C}$. The addition and scalar multiplication are given by

$$\begin{gathered} (f+g)(t) = f(t)+g(t) \\ (\lambda f)(t) = \lambda f(t). \end{gathered}$$

10.5.1 Complex Inner Product

Let V be a complex vector space. A map $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$ is called an inner product if

1. $(u,u) \geq 0$ for all $u \in V$.

2. $(u,u) = 0$ if and only if $u = 0$.

3. For fixed $v \in V$, the map $u \mapsto (u,v)$ is linear,
   i.e., $(\lambda u + \mu w, v) = \lambda(u,v) + \mu(w,v)$
   for all $\lambda, \mu \in \mathbb{C}$ and all $u, w \in V$.

4. For all $u, v \in V$ we have $(u,v) = \overline{(v,u)}$.

Lemma. Let $(\cdot,\cdot)$ be an inner product on the complex vector space V. Then

$$(v, \lambda u + \mu w) = \overline{\lambda}(u,v) + \overline{\mu}(v,w)$$

for all $\lambda, \mu \in \mathbb{C}$ and all $u, v, w \in V$.

Proof. We have

$$\begin{aligned}
(v, \lambda u + \mu w) &= \overline{(\lambda u + \mu w, v)} \\
&= \overline{\lambda(u,v) + \mu(w,v)} \\
&= \overline{\lambda}\ \overline{(u,v)} + \overline{\mu}\ \overline{(w,v)} \\
&= \overline{\lambda}(v,u) + \overline{\mu}(v,w)
\end{aligned}$$

$\square$

Examples

Example (1). Let $V = \mathbb{C}^n$. Define for $u = (z_1, \ldots, z_n)$ and $v = (w_1, \ldots, w_n)$

$$(u,v) = z_1\overline{w_1} + \cdots + z_n\overline{w_n}.$$

Then

$$\begin{aligned}
(u,u) &= z_1\overline{z_1} + \cdots + z_n\overline{z_n} \\
&= |z_1|^2 + \cdots + |z_n|^2 \geq 0.
\end{aligned}$$

If $(u,u) = 0$, then we must have $|z_1| = |z_2| = \cdots |z_n| = 0$, so $u = 0$. Let $y = (t_1, \ldots, t_n) \in V$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} (u+y,v) &= (z_1+t_1)\overline{w_1} + \cdots + (z_n+t_n)\overline{w_n} \\ &= (z_1\overline{w_1} + \cdots + z_n\overline{w_n}) + (t_1 + \cdots + t_n)\overline{w_n} \\ &= (u,v) + (y,v). \end{aligned}$$

Similarly,

$$\begin{aligned} (\lambda u, v) &= (\lambda z_1)\overline{w_1} + \cdots + (\lambda z_n)\overline{w_n} \\ &= \lambda(z_1\overline{w_1}) + \cdots + \lambda(z_n\overline{w_n}) \\ &= \lambda(u,v). \end{aligned}$$

Finally,

$$\begin{aligned} (u,v) &= z_1\overline{w_1} + \cdots + z_n\overline{w_n} \\ &= \overline{\overline{z_1}w_1 + \cdots + \overline{z_n}w_n} \\ &= \overline{(v,u)}. \end{aligned}$$

Example (2). Let V be the space of piecewise continuous functions $f : [0,1] \longrightarrow \mathbb{C}$. Define

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt.$$

10.5.2 Norm in a complex vector space

Definition. Let V be a complex vector space with inner product $(\cdot,\cdot)$. Then the norm (or length) of a vector $u \in V$ is defined by

$$\|u\| := \sqrt{(u,u)}.$$

Remark. Notice that $\|u\| = 0$ if and only if $u = 0$ and that $\|\lambda u\| = |\lambda|\ \|u\|$.

Examples

1. $V = \mathbb{C}^2$ and $u = (1,i)$. Then

$$\|u\|^2 = 1 + i\bar{i} = 1 + i(-i) = 1 + 1 = 2.$$

2. $V = \mathbb{C}^2$ and $u = (1+i, 2+3i)$:

$$\|u\|^2 = (1+i)(1-i) + (2+3i)(2-3i) = 1+1+4+9 = 15$$

or $\|u\| = \sqrt{15}$.

3. Let V be the space of piecewise continuous functions on $[0,1]$. Let $a \in \mathbb{R}$ and

$$f(t) = e^{ait} = \cos(at) + i\sin(at).$$

Then

$$\begin{aligned} f(t)\overline{f(t)} &= |f(t)|^2 = (\cos(at) + i\sin(at))(\cos(at) - i\sin(at)) \\ &= (\cos(at))^2 + (\sin(at))^2 \\ &= 1. \end{aligned}$$

Hence,

$$\|f\| = \sqrt{\int_0^1 |f(t)|^2\, dt} = \sqrt{\int_0^1 1\, dt} = 1.$$

4. Let V be the space of piecewise continuous functions on $[0,1]$. Let

$$f(t) = t + it^2.$$

Then

$$\begin{aligned} |f(t)|^2 &= (t+it^2)(t-it^2) \\ &= t^2 + t^4. \end{aligned}$$

Hence,

$$\|f\| = \sqrt{\int_0^1 t^2 + t^4\, dt} = \sqrt{\frac{1}{3} + \frac{1}{5}} = \sqrt{\frac{8}{15}}.$$

5. Let V be the space of piecewise continuous functions on $[0,1]$. Let

$$f(t) = t^2 + 1 + 2i(t-3).$$

Then

$$\begin{aligned} |f(t)|^2 &= (1+t^2)^2 + 4(t-3)^2 \\ &= 1 + 2t^2 + t^4 + 4t^2 - 24t + 36 \\ &= t^4 + 6t^2 - 24t + 37. \end{aligned}$$

Hence,

$$\begin{aligned} \|f\|^2 &= \int_0^1 t^4 + 6t^2 - 24t + 37 \, dt \\ &= \frac{1}{5} + 2 - 12 + 37 \\ &= \frac{136}{5}. \end{aligned}$$

Lecture 11

Discrete and Fast Fourier Transforms

11.1 Introduction

The goal of the chapter is to study the Discrete Fourier Transform (DFT) and the Fast Fourier Transform (FFT). In the course of the chapter we will see several similarities between Fourier series and wavelets, namely

- Orthonormal bases make it simple to calculate coefficients,
- Algebraic relations allow for fast transform, and
- Complete bases allow for arbitrarily precise approximations.

There is, however, a very important difference between Fourier series and wavelets, namely

Wavelets have compact support, Fourier series do not.

11.2 The Discrete Fourier Transform (DFT)

11.2.1 Definition and Inversion

Let $\vec{\mathbf{e}}_0, \ldots, \vec{\mathbf{e}}_{N-1}$ denote the usual standard basis for $\mathbb{C}^n$. A vector $\vec{\mathbf{f}} = (f_0, \ldots, f_{N-1}) \in \mathbb{C}^N$ may then be written as $\vec{\mathbf{f}} = f_0\vec{\mathbf{e}}_0 + \cdots + f_{N-1}\vec{\mathbf{e}}_{n-1}$.

Important example: Assume the array $\vec{\mathbf{f}}$ is a sample from a function $f : \mathbb{R} \to \mathbb{C}$, that is, we use the sample points $x_0 := 0, \ldots, x_\ell := \ell \cdot (2\pi/N), \ldots, x_{N_1} := (N-1)\cdot(2\pi/N)$ with values $\vec{\mathbf{f}} = (f(x_0), \ldots, f(x_\ell), \ldots, f(x_{N-1}))$.

The Discrete Fourier Transform expresses such an array $\vec{\mathbf{f}}$ with linear combinations of arrays of the type

$$\begin{aligned}\vec{\mathbf{w}}_k &:= \left(e^{ikx_\ell}\right)_{\ell=0}^{N-1} = \left(1, e^{ik2\pi/N}, \ldots, e^{ik\ell\cdot 2\pi/N}, \ldots, e^{ik(N-1)\cdot 2\pi/N}\right) \\ &= (\vec{\mathbf{w}}_k)_\ell = \left(e^{i\cdot 2\pi/N}\right)^{k\ell} =: \omega_N^{k\ell}.\end{aligned}$$

Definition For each positive integer N, we define an inner product on $\mathbb{C}^N$ by

$$\langle \vec{z}, \vec{\mathbf{w}} \rangle_N = \frac{1}{N} \sum_{m=0}^{N-1} z_m \cdot \overline{w_m}.$$

Lemma For each positive integer N, the set

$$\{\vec{\mathbf{w}}_k \,|\, k \in \{0, \ldots, N-1\}\}$$

is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle_N$.

In fact $\{\vec{w}_0, \ldots, \vec{w}_{N-1}\}$ is an *orthonormal basis* for $\mathbb{C}^N$.

Sketch of Proof For all $k, \ell \in \mathbb{Z}$ so that $\ell = k + JN$ for some J, we have

$$\begin{aligned}\langle \vec{w}_k, \vec{w}_\ell \rangle_N &= \frac{1}{N} \sum_{m=0}^{N-1} (\vec{w}_k)_m \overline{(\vec{w}_\ell)_m} = \frac{1}{N} \sum_{m=0}^{N-1} e^{ikm\cdot 2\pi/N} e^{-i\ell m\cdot 2\pi/N} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} e^{ikm\cdot 2\pi/N} e^{-i(k+JN)m\cdot 2\pi/N} = \frac{1}{N} \sum_{m=0}^{N-1} 1 = 1.\end{aligned}$$

For the remaining $k, \ell \in \mathbb{Z}$ we use the geometric series to see that

$$\begin{aligned}\langle \vec{\mathbf{w}}_k, \vec{\mathbf{w}}_\ell \rangle_N &= \frac{1}{N} \sum_{m=0}^{N-1} (\vec{\mathbf{w}}_k)_m \overline{(\vec{\mathbf{w}}_\ell)_m} = \frac{1}{N} \sum_{m=0}^{N-1} e^{i[k-\ell]m\cdot 2\pi/N} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \left(e^{i[k-\ell]\cdot 2\pi/N}\right)^m = \frac{1}{N} \frac{1 - \left(e^{i[k-\ell]m\cdot 2\pi/N}\right)^N}{1 - e^{i[k-\ell]\cdot 2\pi/N}} \\ &= \frac{1}{N} \frac{1 - \left(e^{2\pi i}\right)^{[k-\ell]}}{1 - e^{i[k-\ell]\cdot 2\pi/N}} = \frac{1}{N} \frac{1 - 1}{1 - e^{i[k-\ell]\cdot 2\pi/N}} = 0.\end{aligned}$$

11.2.2 The Fourier matrix

Definition For each positive integer N, define the *Fourier matrix* ${}^N_F\Omega$ by

$$ {}^N_F\Omega_{k,\ell} = (\vec{\mathbf{w}}_\ell)_k = e^{ik\ell 2\pi/N} = \omega_N^{k\ell}. $$

Example: If $N = 1$, then $\omega_N = \omega_1 = 1$, and ${}^N_F\Omega = 1$. If $N = 2$, then $\omega_N = \omega_2 = -1$, and

$$ {}^2_F\Omega = \begin{pmatrix} (\omega_2^0)^0 & (\omega_2^1)^0 \\ (\omega_2^0)^1 & (\omega_2^1)^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{(the Haar matrix)} $$

If $N = 4$, then $\omega_N = \omega_4 = e^{i2\pi/4} = i$, and

$$ {}^4_F\Omega = \begin{pmatrix} 1 & (i^1)^0 & (i^2)^0 & (i^3)^0 \\ 1 & (i^1)^1 & (i^2)^1 & (i^3)^1 \\ 1 & (i^1)^2 & (i^2)^2 & (i^3)^2 \\ 1 & (i^1)^3 & (i^2)^3 & (i^3)^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}. $$

11.3 Discrete Fourier Transform

Definition For each positive integer N and each array $\vec{f} \in \mathbb{C}^N$, the *Discrete Fourier Transform* of $\vec{f}$ is the array $\hat{\mathbf{f}}$ defined by

$$ \hat{f}_k = \langle \vec{f}, \vec{\mathbf{w}}_k \rangle_N = \frac{1}{N} \sum_{m=0}^{N-1} f_m \cdot e^{-ikm\cdot 2\pi/N}. $$

It is very important to understand this definition. The left hand side is simply the *orthogonal projection of* $\vec{\mathbf{f}}$ onto the basis vector $\vec{\mathbf{w}}_k$.

Since the DFT consists of the coefficients of $\vec{\mathbf{f}}$ expressed with respect to the new basis $(\vec{\mathbf{w}}_k)_{k=0}^{N-1}$, the DFT simply "rotates" the coordinates in $\mathbb{C}^N$.

11.3.1 Two Results

Proposition The DFT corresponds to a multiplication by the transposed conjugate matrix $\frac{1}{N}\overline{{}^N_F\Omega^T}$.

Proof: Simply note that

$$\begin{aligned}\hat{f}_{N,k} = \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_k \rangle_N &= \frac{1}{N} \sum_{m=0}^{N-1} f_\ell e^{-ik\ell \cdot 2\pi/N} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \overline{{}_F^N\Omega^T} \cdot f_\ell = \frac{1}{N} \cdot \left(\overline{{}_F^N\Omega^T} \cdot \vec{\mathbf{f}} \right)_k .\end{aligned}$$

Proposition 13 For each positive integer N and each array $\vec{\mathbf{f}} \in \mathbb{C}^N$, we have the following inversion formulas:

$$\vec{\mathbf{f}} = \sum_{m=0}^{N-1} \hat{f}_k \vec{\mathbf{w}}_k \quad , \quad \hat{f}_k = \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_k \rangle_N = \frac{1}{N} \sum_{m=0}^{N-1} f_m \cdot e^{-ikm \cdot 2\pi/N} .$$

One proof The easiest way to prove Proposition 13 is to note that $(\vec{\mathbf{w}}_k)_{k=0}^{N-1}$ forms an orthonormal basis for $\mathbb{C}^N$. Therefore, the formula

$$\vec{\mathbf{f}} = \sum_{k=0}^{N-1} \hat{f}_k \vec{\mathbf{w}}_k = \sum_{k=0}^{N-1} \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_k \rangle_N \vec{\mathbf{w}}_k$$

represents the orthogonal projection of $\vec{\mathbf{f}}$ on $\mathbb{C}^N$. But the projection of $\vec{\mathbf{f}}$ on $\mathbb{C}^N$ **is** $\vec{\mathbf{f}}$, since $\vec{\mathbf{f}}$ already lies in the range of the projection (the range being $\mathbb{C}^N$).

Example With $N = 4$, the fourth root of unity is $\omega_N = \omega_4 = i$. The N arrays $\vec{\mathbf{w}}_k$ thus become

$$\begin{aligned}\vec{\mathbf{w}}_0 &= \left([i]^{0\cdot 0}, [i]^{0\cdot 1}, [i]^{0\cdot 2}, [i]^{0\cdot 3}\right) = (1, 1, 1, 1), \\ \vec{\mathbf{w}}_1 &= \left([i]^{1\cdot 0}, [i]^{1\cdot 1}, [i]^{1\cdot 2}, [i]^{1\cdot 3}\right) = (1, i, -1, -i), \\ \vec{\mathbf{w}}_2 &= \left([i]^{2\cdot 0}, [i]^{2\cdot 1}, [i]^{2\cdot 2}, [i]^{2\cdot 3}\right) = (1, -1, 1, -1), \\ \vec{\mathbf{w}}_3 &= \left([i]^{3\cdot 0}, [i]^{3\cdot 1}, [i]^{3\cdot 2}, [i]^{3\cdot 3}\right) = (1, -i, -1, i).\end{aligned}$$

Assume $\vec{\mathbf{f}} = (f_0, f_1, f_2, f_3) = (9, 7, 5, 7)$. Then (remember the complex conjugation!):

$$\begin{aligned}\hat{f}_0 &= \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_0 \rangle_N = \langle (9, 7, 5, 7), (1, 1, 1, 1) \rangle_4 = 7, \\ \hat{f}_1 &= \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_1 \rangle_N = \langle (9, 7, 5, 7), (1, i, -1, -i) \rangle_4 = 1, \\ \hat{f}_2 &= \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_2 \rangle_N = \langle (9, 7, 5, 7), (1, -1, 1, -1) \rangle_4 = 0, \\ \hat{f}_3 &= \langle \vec{\mathbf{f}}, \vec{\mathbf{w}}_3 \rangle_N = \langle (9, 7, 5, 7), (1, -i, -1, i) \rangle_4 = 1.\end{aligned}$$

Therefore

$$\begin{aligned}\vec{\mathbf{f}} &= (9,7,5,7)\\ &= 7\cdot(1,1,1,1)+1\cdot(1,i,-1,-i)\\ &\quad +0\cdot(1,-1,1,-1)+1\cdot(1,-i,-1,i)\end{aligned}$$

is the Inverse Discrete Fourier Transform of the array $\vec{\mathbf{f}} = (9,7,5,7)$.

11.4 Unitary Operators

We have just seen that the Discrete Fourier Transform is a linear operator

$$DFT : \mathbb{C}^N \to \mathbb{C}^N \quad , \quad \vec{\mathbf{f}} \mapsto \widehat{\mathbf{f}} = \frac{1}{N}\overline{{}^N_F\Omega^T}\vec{\mathbf{f}}.$$

Its inverse is given by

$$DFT^{-1} : \mathbb{C}^N \to \mathbb{C}^N \quad , \quad \widehat{\mathbf{f}} \mapsto \vec{\mathbf{f}} = {}^N_F\Omega\vec{\mathbf{f}}.$$

Using the multiplicative constant $1/\sqrt{N}$ instead of $1/N$, the Discrete Fourier Transform thus is multiplication by the matrix ${}^N_FU := \frac{1}{\sqrt{N}}\overline{{}^N_F\Omega^T}$.

In the exercises you are asked to show that $({}^N_FU)^{-1} = \overline{({}^N_FU)}^T$. Such an operator has a special name:

Definition 17 For each linear space V with an inner product $\langle\cdot,\cdot\rangle$, a linear operator $L : V \to V$ is **unitary** if

$$\langle Lv, Lw\rangle = \langle v, w\rangle$$

for all $v, w \in V$.

11.5 The Fast Fourier Transform

11.5.1 Introduction

We have seen how to convert a sample $\vec{\mathbf{f}} = (f_0, \ldots, f_{N-1})$ to a frequency sample $\vec{\hat{\mathbf{f}}} = (\hat{f}_0, \ldots, \hat{f}_{N-1})$ and back again. But there is an aspect we haven't touched upon yet, namely

How long does it take to perform these calculations?

Let us try to estimate the number of calculations that are required. So let us start with N complex numbers $(f_0, \ldots, f_{N-1})$.

- Each number $f_j = \mathrm{Re} f_j + i \mathrm{Im} f_j$ is being multiplied by $e^{-2\pi ijn/N} = \cos\frac{2\pi ijn}{N} - i\sin\frac{2\pi ijn}{N}$; this gives $\boxed{4N}$ operations.
- This has to be done for each of the numbers f_j, totalling $4N^2$ real multiplications, or N^2 complex multiplications.
- The numbers should then be added, but that would merely amount to $2N$ operations. Our number of operations is thus $\boxed{4N^2}$.

So how big is this number?

An Example Assume we need one minute to compute the Fourier transform for a sequence with 4 samples. It would therefore take us approximately $\dfrac{60}{4 \times 4^2} = \dfrac{60}{64} \approx 0.94$sec per operation.

- With $N = 8$ we would need $\dfrac{4 \times 8^2 \times 15}{16 \times 60} = 4$ min,
- With $N = 16$ we would need $\dfrac{4 \times 16^2 \times 15}{16 \times 60} = 16$ min,
- With $N = 32$ we would need $\dfrac{4 \times 32^2 \times 15}{16 \times 60} = 64$ min, and
- With $N = 2^8 = 256$ we would need $\dfrac{4 \times 256^2 \times 15}{16 \times 60} = 4096$ min, (almost three days).
- A standard TV need roughly 10000000 pixel values every second to preserve relevant information. This would take us around 118900256 years (!!!) to analyze all this information.

11.5.2 The Forward FFT

The point about the Fast Fourier Transform (FFT) is that it gives the same coefficients as the Discrete Fourier Transform (DFT) but requires less calculations. For each integer N of the form $N = 2^k$ for some k and for each array

$$\vec{\mathbf{f}} = (f_0, \ldots, f_{N-1}) = (f(0), \ldots, f(N-1)) \in \mathbb{C}^N,$$

we define two arrays

$$\begin{aligned} {}_{\text{even}}\vec{\mathbf{f}} &= (f(0), f(2), \ldots, f(2j), \ldots, f(N-2)) \text{ and} \\ {}_{\text{odd}}\vec{\mathbf{f}} &= (f(1), f(3), \ldots, f(2j+1), \ldots, f(N-1)). \end{aligned}$$

We want to know how the DFT of ${}_{\text{even}}\vec{\mathbf{f}}$ and the DFT of ${}_{\text{odd}}\vec{\mathbf{f}}$ are related to the DFT of $\vec{\mathbf{f}}$. The answer is given in the next result:

Central result

Lemma 19 For each $k \in \{0, 1, \ldots, \frac{N}{2}-1\}$ and all arrays $\vec{\mathbf{f}} = (f(0), \ldots, f(N-1)) \in \mathbb{C}^N$

$$\begin{aligned} \hat{f}_k &= \frac{1}{2} \cdot \left({}_{\text{even}}\hat{f}_k + [e^{-i\cdot 2\pi/N}]^k \cdot [{}_{\text{odd}}\hat{f}_k] \right), \\ \hat{f}_{k+\frac{N}{2}} &= \frac{1}{2} \cdot \left({}_{\text{even}}\hat{f}_k - [e^{-i\cdot 2\pi/N}]^k \cdot [{}_{\text{odd}}\hat{f}_k] \right). \end{aligned}$$

We will not give a proof but merely see what the Lemma says in the case $N = 2$. Then $N/2 = 1$ and $\vec{\mathbf{f}} = (f_0, f_1)$. Thus ${}_{\text{even}}\vec{\mathbf{f}} = (f_0)$ and ${}_{\text{odd}}\vec{\mathbf{f}} = (f_1)$. Using that $e^{-i\cdot 2\pi/2} = -1$, the Lemma simply says that

$$\hat{f}_0 = \frac{f_0 + f_1}{2} \quad \text{and} \quad \hat{f}_1 = \frac{f_0 - f_1}{2}.$$

Time required?

We will only count *complex* operations.

- We start with an array $\vec{\mathbf{f}} = (f_0, \ldots, f_{N-1})$ where $N = 2^k$.
- The DFT needed approximately N^2 complex multiplications.

- Using the FFT (where we decompose $\vec{\mathbf{f}}$ into two smaller arrays, divide each of these into two smaller arrays, and so on), we end up with k arrays each of length 2.

We thus need around kN complex multiplications instead of N^2.

Is this significant?

In particular, taking $N = 2^{10} \approx 1.024 \times 10^3$ and calculate the time, the DFT takes

$$\frac{4 \times 2^{20} \times 15}{16 \times 60 \times 60 \times 24 \times 365} \approx 1247 \text{ years}$$

but the FFT only takes

$$\frac{4 \times 10 \times 2^{10} \times 15}{16 \times 60 \times 60} \approx 10.67 \text{ min.}$$

Obviously this is a very distinct reduction in the time required to perform the calculations :-)

11.5.3 The Inverse FFT (IFFT)

For each integer N, the FFT admits an inverse map that we call the Inverse Fast Fourier Transform (IFFT). Starting with an array $\hat{\mathbf{f}} \in \mathbb{C}^N$ we thus contruct the array of data $\vec{\mathbf{f}} \in \mathbb{C}^N$ such that $\hat{\mathbf{f}}$ is the DFT of $\vec{\mathbf{f}}$.

Using that $\vec{\mathbf{f}} = \hat{f}_0 \vec{\mathbf{w}}_0 + \cdots + \hat{f}_k \vec{\mathbf{w}}_k + \cdots + \hat{f}_{N-1} \vec{\mathbf{w}}_{N-1}$ we thus get for the coordinates with index ℓ that

$$f_\ell = \sum_{k=0}^{N-1} \hat{f}_k (\vec{\mathbf{w}}_k)_\ell = \sum_{k=0}^{N-1} \hat{f}_k e^{ik\ell \cdot 2\pi/N}.$$

We may interpret this formula as yet another FFT:

$$\begin{aligned} f_\ell &= \sum_{k=0}^{N-1} \hat{f}_k e^{ik\ell \cdot 2\pi/N} = \overline{\sum_{k=0}^{N-1} \overline{\hat{f}_k} \overline{e^{ik\ell \cdot 2\pi/N}}} \\ &= \overline{N \cdot \frac{1}{N} \sum_{k=0}^{N-1} \overline{\check{f}_k} e^{-ik\ell k \cdot 2\pi/N}} = N \cdot \overline{\hat{\overline{\hat{f}}}_\ell} \end{aligned}$$

So in order to calculate the data $\vec{\mathbf{f}}$ from $\hat{\mathbf{f}}$, it suffices to form the complex conjugate of $\hat{\mathbf{f}}$, take its FFT multiplied by N, and then take the complex conjugate one more time.

11.5.4 Interpolation by the IFFT

Thus far we have used the sample points $x_\ell = \ell \cdot \frac{2\pi}{N}$, but the IFFT provides a way to interpolate the function f in *other* points. To see how it works, we recall that (by definition of the DFT),

$$f(x_\ell) = \sum_{k=0}^{N-1} \hat{f}_k e^{ikx_\ell}.$$

One way to interpolate f at x is therefore to approximate $f(x)$ by

$$f(x) \approx \sum_{k=0}^{N-1} \hat{f}_k e^{ikx}.$$

However, if we consider several values of x spaced by multiples of $\frac{2\pi}{N}$, the Fast Fourier Transform is a more convenient tool. So let $u = x - x_\ell$ for any ℓ. Thus

$$\begin{aligned} f(x) \approx \sum_{k=0}^{N_1} \hat{f}_k e^{ikx} &= \sum_{k=0}^{N-1} \hat{f}_k e^{ik[u+x_\ell]} \\ = \sum_{k=0}^{N-1} \left(\hat{f}_k e^{iku}\right) e^{ikx_\ell} &= \sum_{k=0}^{N-1} \left(\hat{f}_k e^{iku}\right) e^{ik\ell\cdot 2\pi/N}, \end{aligned}$$

where the sums thus obtained are just the IFFT applied to the coefficients $(\hat{f}_k e^{iku})_{k=0}^{N-1}$. Therefore, to interpolate f at the points $x = u + \ell \cdot \frac{2\pi}{N}$, $\ell \in \{0, \ldots, N-1\}$, it suffices to multiply each coefficient $\hat{f}_k$ by $e^{iku} = (e^{iu})^k$ and then calculate the IFFT of the array $(\hat{f}_k e^{iku})_{k=0}^{N-1}$.

11.5.5 Bit Reversal

One way to better facilitate a recursive calculation of the Fast Fourier Transform is to rearrange the initial date in such a way that the Fast Fourier Transform operates on adjacent pairs of arrays in each step of the recursion. More precisely:

Definition 23 *Bit reversal* transforms a finite sequence $(p_{k-1}, p_{k-2}, \ldots, p_1, p_0)$ of k (binary) integers $p_j \in \{0, 1\}$ into the *bit-reversed* sequence $(p_0, p_1, \ldots, p_{k-2}, p_{k-1})$.

So bit reversal transforms a binary integer

$$p = p_{k-1}2^{k-1} + p_{k-2}2^{k-2} + \cdots + p_1 2 + p_0$$

into the binary integer

$$q = B(p) := p_{k-1} + p_{k-2}2 + \cdots + p_1 2^{k-2} + p_0 2^{k-1}.$$

Proposition 24 For each index $N = 2^n$, the FFT amounts to the following chain of operations:

(0) For each binary index $p \in \{0, \ldots, N-1\}$, calculate the bit-reversed index $q = B(p)$ and arrange the data $\vec{\mathbf{z}} = (z_0, \ldots, z_{N-1})$ in the order $\vec{\mathbf{z}}_B := (z_{B(0)}, \ldots, z_{B(n-1)})$;

(1) For each $k \in \{0, \ldots, n-1\}$, perform one step of the FFT on each of the $2^{(n-k)-1}$ pairs of adjacent sequences of 2^k elements.

We can therefore perform the Fast Fourier Transform in the following way:

- For each $p \in \{0, \ldots, N-1\}$, calculate $q = B(p)$ and arrange the data $\vec{\mathbf{z}} = (z_0, \ldots, z_{N-1})$ as

$$\vec{\mathbf{z}}_B = (z_{B(0)}, \ldots, z_{B(N-1)})$$

- Compute the $N/2$ Fast Fourier Transform steps from one to two points of each of the $N/2$ pairs:

$$([\hat{z}_{B(0)}, \hat{z}_{B(1)}], \ldots, [\hat{z}_{B(N-2)}, \hat{z}_{B(N-1)}]).$$

- Compute the $N/4$ Fast Fourier Transform steps from two to four points of each of the $N/4$ sequences of four numbers:

$$([\hat{z}_{B(0)}, \hat{z}_{B(1)}, \hat{z}_{B(2)}, \hat{z}_{B(3)}], \ldots, \\ [\hat{z}_{B(N-3)}, \hat{z}_{B(N-2)}, \hat{z}_{B(N-1)}, \hat{z}_{B(N-1)}]).$$

- Compute the Fast Fourier Transform step from $N/2$ to N points of the sequence of N numbers:

$$([\hat{z}_{B(0)}, \ldots, \hat{z}_{B(N/2)-1)}], [\hat{z}_{B(N/2)}, \ldots, \hat{z}_{B(N-1)}]).$$

11.5.6 Applications of the FFT

Noise Reduction Through FFT

The idea is that if we are given a periodic signal with a larger amplitude added to noise, the Fast Fourier Transform can be used to decompose the the superposition of noise and the signal itself into a linear combination if terms with different frequencies. This will identify and preserve the contribution of the signal due to the larger coefficients and rejects the smaller coefficients from the noise. The original signal thus reemerges.

See p.165-167 in the book.

11.5.7 Multidimensional DFT and FFT