Math 663-A1: Topics in Applied Mathematics I
Transform-based Methods for Data Science

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Class: Monday/Wednesay/Friday 1:00pm–1:50pm
Location: C E4-36

Fast Framelet Transforms (FFrT) and Fast Wavelet Transforms (FWT)

- Multilevel fast framelet/wavelet transform
- Stability of fast framelet transforms
- Subdivision schemes in computer graphics
- Some basics on wavelet theory in $L_2(\mathbb{R})$.
- Construction of wavelet filter banks.
- Construction of framelet filter banks.

Multi-level Fast Framelet Transform (FFrT)

- Let $\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}$ and $\{a; b_1, \ldots, b_s\}$ be filters in $l_0(\mathbb{Z})$.
- For a positive integer J, a J-level discrete framelet decomposition is given by

$$v_j := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{a}} v_{j-1}, \qquad w_{\ell,j} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{b}_\ell} v_{j-1}, \quad \ell = 1, \ldots, s, \ j = 1, \ldots, J,$$

  where $v_0 : \mathbb{Z} \to \mathbb{C}$ is an input signal.
- $\tilde{\mathcal{W}}_J v_0 := (w_{1,1}, \ldots, w_{s,1}, \ldots, w_{1,J}, \ldots, w_{s,J}, v_J)$.
- a J-level discrete framelet reconstruction is

$$v_{j-1} := \frac{\sqrt{2}}{2}\mathcal{S}_a v_j + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s} \mathcal{S}_{b_\ell} w_{\ell,j}, \quad j = J, \ldots, 1.$$

- $\mathcal{V}_J(w_{1,1}, \ldots, w_{s,1}, \ldots, w_{1,J}, \ldots, w_{s,J}, v_J) = v_0$.
- The perfect reconstruction property: $\mathcal{V}_J \tilde{\mathcal{W}}_J v_0 = v_0$ for all $J \in \mathbb{N}$, $v_0 \in l_2(\mathbb{Z})$.
- The fast framelet transform has the perfect reconstruction property if and only if $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ is a dual framelet filter bank satisfying

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}_1}(\xi) & \cdots & \widehat{\tilde{b}_s}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}_1}(\xi+\pi) & \cdots & \widehat{\tilde{b}_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}^{\star} = I_2.$$

- A fast framelet transform with $s = 1$ is called a fast wavelet transform.

Variants of FFrT: Undecimated FFrT

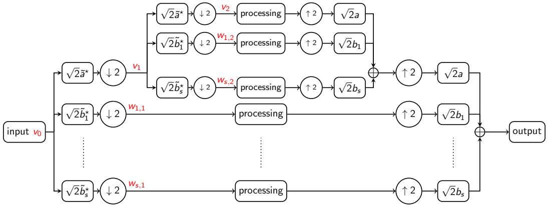
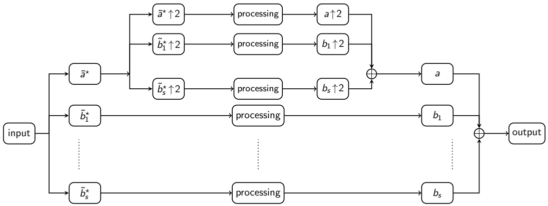


Figure: Diagram of a two-level discrete framelet transform using a pair of filter banks $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, (a; b_1, \dots, b_s\})$.



Undecimated DFrT using a framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, (a; b_1, \dots, b_s\})$, which is required to satisfy $\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi)} + \widehat{\tilde{b}_1}(\xi)\overline{\widehat{b_1}(\xi)} + \cdots + \widehat{\tilde{b}_s}(\xi)\overline{\widehat{b_s}(\xi)} = 1.$

Express J-level FFrT using Discrete Wavelets in $l_2(\mathbb{Z})$

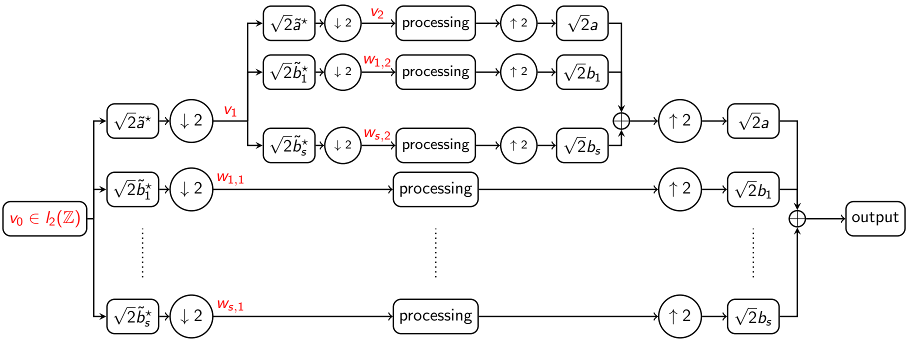


Figure: Diagram of a two-level discrete framelet transform using a pair of filter banks $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$.

Property of FFrT: Stability

Definition: A multi-level discrete framelet transform employing a dual framelet filter bank $\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})$ has stability in the space $l_2(\mathbb{Z})$ if there exists $C > 0$ such that for all $J \in \mathbb{N}_0$,

$$\|\tilde{\mathcal{W}}_J v\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}} \leqslant C\|v\|_{l_2(\mathbb{Z})}, \qquad \forall\ v \in l_2(\mathbb{Z}),$$
$$\|\mathcal{V}_J \vec{w}\|_{l_2(\mathbb{Z})} \leqslant C\|\vec{w}\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}}, \qquad \forall\ \vec{w} \in (l_2(\mathbb{Z}))^{1\times(sJ+1)}.$$

Theorem

Let $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})$ be a dual framelet filter bank with $\widehat{a}(0) = \widehat{\tilde{a}}(0) = 1$ (i.e., $\sum_{k\in\mathbb{Z}} a(k) = \sum_{k\in\mathbb{Z}} \tilde{a}(k) = 1$). Define

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \quad \widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\tilde{a}}(2^{-j}\xi), \qquad \xi \in \mathbb{R}.$$

Then a multi-level discrete framelet transform has stability in the space $l_2(\mathbb{Z})$ if and only if $\phi, \tilde{\phi} \in L_2(\mathbb{R})$ and

$$\widehat{\tilde{b}_1}(0) = \cdots = \widehat{\tilde{b}_s}(0) = \widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = 0,$$

that is, all $\tilde{b}_1, \dots, \tilde{b}_s, b_1, \dots, b_s$ have one vanishing moment.

Theory of Discrete Wavelets and Framelets

- J-level discrete affine system $\mathrm{DAS}_J(a; b_1, \ldots, b_s) :=$:

$$\{a_{J;k} \ : \ k \in \mathbb{Z}\} \cup \{b_{\ell,j;k} \ : \ k \in \mathbb{Z}, \ell = 1, \ldots, s, j = 1, \ldots, J\},$$

where $a_{j;k} := 2^{j/2} a_j(\cdot - 2^j k)$ and $b_{\ell,j;k} := 2^{j/2} b_{\ell,j}(\cdot - 2^j k)$ with

$$\widehat{a_j}(\xi) := \widehat{a}(\xi) \cdots \widehat{a}(2^{j-1}\xi), \quad \widehat{b_{\ell,j}}(\xi) := \widehat{a}(\xi) \cdots \widehat{a}(2^{j-2}\xi)\widehat{b_\ell}(2^{j-1}\xi).$$

- That is,

$$a_j = a * (a\uparrow 2) * \cdots * (a\uparrow 2^{j-1}) = 2^{-j}\mathcal{S}_a^j \boldsymbol{\delta}, \quad b_{\ell,j} = 2^{-j}\mathcal{S}_a^{j-1}\mathcal{S}_{b_\ell}\boldsymbol{\delta}.$$

Note that $a_1 = a$, $b_{\ell,1} = b_\ell$ and $a_{1;0} = 2^{1/2}a$, $b_{\ell,1;0} = 2^{1/2}b_\ell$.

- A J-level discrete framelet decomposition for $v = v_0 \in l_2(\mathbb{Z})$ just becomes

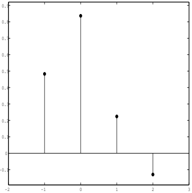
$$v_j(k) = \langle v, \tilde{a}_{j;k}\rangle, \qquad w_{\ell,j}(k) = \langle v, \tilde{b}_{\ell,j;k}\rangle, \quad \ell = 1, \ldots, s.$$

- A J-level fast framelet transform using a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ becomes: for every $v \in l_2(\mathbb{Z})$,

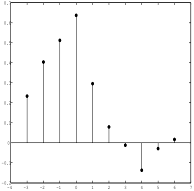
$$v = \sum_{w \in \mathrm{DAS}_J(a; b_1, \ldots, b_s)} \langle v, \tilde{w}\rangle w = \sum_{k\in\mathbb{Z}} \langle v, \tilde{a}_{J;k}\rangle a_{J;k} + \sum_{j=1}^{J}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle v, \tilde{b}_{\ell,j;k}\rangle b_{\ell,j;k}.$$

An Example: Daubechies Orthogonal Wavelets

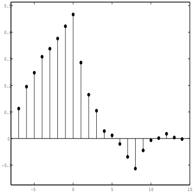
$$a = \left\{\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\right\}, \quad b = \left\{-\frac{1-\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, -\frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8}\right\}.$$



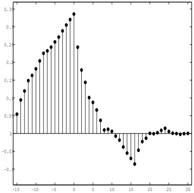
(a) a_1



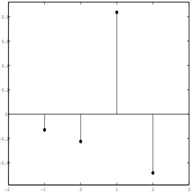
(b) a_2



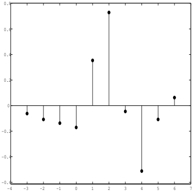
(c) a_3



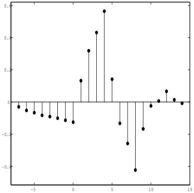
(d) a_4



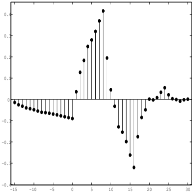
(e) b_1



(f) b_2



(g) b_3



(h) b_4

Figure: $\mathrm{DAS}_J(a; b)$ is an orthonormal basis of $l_2(\mathbb{Z})$ for all $J \in \mathbb{N}$

Discrete Dual Framelets in $l_2(\mathbb{Z})$

- $\mathrm{DAS}(a; b_1, \ldots, b_s)$ is a discrete framelet in $l_2(\mathbb{Z})$ if there exist positive constants C_1 and C_2 such that

$$C_1\|v\|_2^2 \leqslant \sum_{w \in \mathrm{DAS}_J(a;b_1,\ldots,b_s)} |\langle v, w\rangle|^2 \leqslant C_2\|v\|_2^2, \qquad \forall v \in l_2(\mathbb{Z}), J \in \mathbb{N}.$$

- If $C_1 = C_2 = 1$, it is called a discrete tight framelet in $l_2(\mathbb{Z})$.
- $\mathrm{DAS}(a; b_1, \ldots, b_s)$ is a discrete orthogonal wavelet in $l_2(\mathbb{Z})$ is each $\mathrm{DAS}_J(a; b_1, \ldots, b_s)$ is an orthonormal basis of $l_2(\mathbb{Z})$ for all $J \in \mathbb{N}$. $\mathrm{DAS}(a; b_1, \ldots, b_s)$ is a discrete orthogonal wavelet in $l_2(\mathbb{Z})$ if and only if it is a discrete tight framelet in $l_2(\mathbb{Z})$ and $\|2^{1/2}a\|_{l_2(\mathbb{Z})} = 1$ and $\|2^{1/2}b_1\|_{l_2(\mathbb{Z})} = \cdots = \|2^{1/2}b_s\|_{l_2(\mathbb{Z})} = 1$.
- $(\mathrm{DAS}(\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s), \mathrm{DAS}(a; b_1, \ldots, b_s))$ is a discrete dual framelet in $l_2(\mathbb{Z})$ if each of $\mathrm{DAS}(\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s)$ and $\mathrm{DAS}(a; b_1, \ldots, b_s)$ is a discrete framelet in $l_2(\mathbb{Z})$ and

$$v = \sum_{k\in\mathbb{Z}} \langle v, \tilde{a}_{J;k}\rangle a_{J;k} + \sum_{j=1}^{J}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle v, \tilde{b}_{\ell,j;k}\rangle b_{\ell,j;k}, \quad \forall\, v \in l_2(\mathbb{Z}), J \in \mathbb{N}$$

with the series converging unconditionally.

Refinable Functions

- Let $a \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = \sum_{k\in\mathbb{Z}} a(k) = 1$.
- The refinable function $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ is well defined for $\xi \in \mathbb{R}$ and satisfies

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k) \quad \text{i.e.,} \quad \widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi).$$

Indeed,

$$\widehat{\phi}(2\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{1-j}\xi) = \widehat{a}(\xi)\widehat{a}(2^{-1}\xi)\widehat{a}(2^{-2}\xi)\cdots = \widehat{a}(\xi)\prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi).$$

- Note that the Fourier transform of $\phi(2x-k)$ is

$$\phi(\widehat{2\cdot -}k)(\xi) = \int_{\mathbb{R}} \phi(2x-k)e^{-ix\xi}dx = \frac{1}{2}\int_{\mathbb{R}} \phi(y)e^{-i\frac{1}{2}(y+k)\xi}dy = \frac{1}{2}e^{-ik\xi}\widehat{\phi}(\xi/2).$$

Therefore, the Fourier transform of $2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k)$ is

$$2\sum_{k\in\mathbb{Z}} a(k)\frac{1}{2}e^{-ik\xi/2}\widehat{\phi}(\xi/2) = \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi/2}\widehat{\phi}(\xi/2) = \widehat{a}(\xi/2)\widehat{\phi}(\xi/2) = \widehat{\phi}(\xi).$$

This proves $2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k) = \phi(x)$.

Some Basics on Wavelets in $L_2(\mathbb{R})$

- For $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$, define an affine system as

$$\begin{aligned}\mathrm{AS}(\phi;\psi^1,\ldots,\psi^s) &:= \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \\ &\cup \{\psi^\ell_{2^j;k} := 2^{j/2}\psi^\ell(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}, \ell = 1, \ldots, s\}.\end{aligned}$$

- We say that $\{\phi;\psi^1,\ldots,\psi^s\}$ is a framelet in $L_2(\mathbb{R})$ if $\mathrm{AS}(\phi;\psi^1,\ldots,\psi^s)$ is a framelet in $L_2(\mathbb{R})$, that is, there exist positive constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_2^2 \leqslant \sum_{k\in\mathbb{Z}} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{\ell=1}^{s}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{2^j;k}\rangle|^2 \leqslant C_2\|f\|_2^2, \quad \forall\, f \in L_2(\mathbb{R}).$$

- In particular, $\{\phi;\psi^1,\ldots,\psi^s\}$ is called a tight framelet in $L_2(\mathbb{R})$ if

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{\ell=1}^{s}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{2^j;k}\rangle|^2 = \|f\|_2^2, \quad \forall\, f \in L_2(\mathbb{R}).$$

- Then $f = \sum_{k\in\mathbb{Z}}\langle f, \phi(\cdot - k)\rangle\phi(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}}\langle f, \psi^\ell_{2^j;k}\rangle\psi^\ell_{2^j;k}$.
- $\{\phi;\psi^1,\ldots,\psi^s\}$ is called an orthogonal wavelet in $L_2(\mathbb{R})$ if $\mathrm{AS}(\phi;\psi^1,\ldots,\psi^s)$ is an orthonormal basis in $L_2(\mathbb{R})$.
- $\{\phi;\psi^1,\ldots,\psi^s\}$ is an orthogonal wavelet in $L_2(\mathbb{R})$ if and only if it is a tight framelet in $L_2(\mathbb{R})$ and $\|\phi\|_2 = \|\psi_1\|_2 = \cdots = \|\psi^s\|_2 = 1$.

Dual Framelets in $L_2(\mathbb{R})$

For $\tilde{\phi}, \tilde{\psi}^1, \ldots, \tilde{\psi}^s \in L_2(\mathbb{R})$ and $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$, we say that $(\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\phi; \psi^1, \ldots, \psi^s\})$ is a dual framelet in $L_2(\mathbb{R})$ if

1. $\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}$ is a framelet in $L_2(\mathbb{R})$.
2. $\{\phi; \psi^1, \ldots, \psi^s\}$ is a framelet in $L_2(\mathbb{R})$.
3. The following identity holds:

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k)\rangle\langle \phi(\cdot - k), g\rangle + \sum_{\ell=1}^{s}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle\langle \psi^{\ell}_{2^j;k}, g\rangle, \ \forall\, f, g \in L_2(\mathbb{R})$$

   with series converging absolutely.

Consequently, we have the wavelet representation of functions in $L_2(\mathbb{R})$:

$$f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle \psi^{\ell}_{2^j;k}.$$

with the series converging unconditionally.

Characterization of Dual Framelets in $L_2(\mathbb{R})$

Theorem

Let $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_s, a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ *such that* $\widehat{a}(0) = \widehat{\tilde{a}}(0) = 1$. *Define* $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$, $\widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\tilde{a}}(2^{-j}\xi)$ *and*

$$\widehat{\psi^\ell}(\xi) := \widehat{b_\ell}(\xi/2)\widehat{\phi}(\xi/2), \qquad \widehat{\tilde{\psi}^\ell}(\xi) := \widehat{\tilde{b}_\ell}(\xi/2)\widehat{\tilde{\phi}}(\xi/2), \qquad \ell = 1, \ldots, s.$$

Then the following are equivalent to each other

1. $(\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\phi; \psi^1, \ldots, \psi^s\})$ *is a dual framelet in* $L_2(\mathbb{R})$.
2. $(\mathrm{DAS}(\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s), \mathrm{DAS}(a; b_1, \ldots, b_s))$ *is a discrete dual framelet in* $l_2(\mathbb{Z})$.
3. $\phi, \tilde{\phi} \in L_2(\mathbb{R})$, $\widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = 0$, $\widehat{\tilde{b}_1}(0) = \cdots = \widehat{\tilde{b}_s}(0) = 0$, *and* $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ *is a dual framelet filter bank, i.e.,*

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}_1}(\xi) & \cdots & \widehat{\tilde{b}_s}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}_1}(\xi+\pi) & \cdots & \widehat{\tilde{b}_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}^\star = I_2.$$

Wavelet Transform in $L_2(\mathbb{R})$

- Let $(\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\phi; \psi^1, \ldots, \psi^s\})$ is a dual framelet in $L_2(\mathbb{R})$ with a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$.
- For a given function $f \in L_2(\mathbb{R})$, we define

$$v^j(k) := \langle f, \tilde{\phi}_{2^j;k} \rangle, \qquad w^{\ell,j}(k) := \langle f, \tilde{\psi}^\ell_{2^j;k} \rangle, \qquad j, k \in \mathbb{Z}, \ell = 1, \ldots, s.$$

- They can be computed by fast wavelet transform:

$$v^{j-1} = \frac{\sqrt{2}}{2} \mathcal{T}_{\tilde{a}} v^j, \qquad w^{\ell,j-1} = \frac{\sqrt{2}}{2} \mathcal{T}_{\tilde{b}_\ell} v^j, \qquad \ell = 1, \ldots, s,$$

$$v^j = \frac{\sqrt{2}}{2} \mathcal{S}_a v^{j-1} + \sum_{\ell=1}^{s} \frac{\sqrt{2}}{2} \mathcal{S}_{\tilde{b}_\ell} w^{\ell,j-1}.$$

- For $J \in \mathbb{N}$, approximate $f \approx f_J := \sum_{k \in \mathbb{Z}} v_J(k) \phi_{2^J;k} = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{2^j;k} \rangle \phi_{2^j;k}$. Because $\int \tilde{\phi}(x) dx = \widehat{\tilde{\phi}}(0) = 1$, $\langle f, \tilde{\phi}_{2^j;k} \rangle \approx f(2^{-j}k) \langle 1, \tilde{\phi}_{2^j;k} \rangle = 2^{-J/2} f(2^{-j}k)$.
- $f_j = f_{j-1} + \sum_{\ell=1}^{s} \sum_{k \in \mathbb{Z}} w^{\ell,j-1} \psi^\ell_{2^{j-1};k} = f_{j-1} + \sum_{\ell=1}^{s} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}^\ell_{2^{1-j};k} \rangle \psi^\ell_{2^{j-1};k}$.

$$f_J = f_0 + \sum_{\ell=1}^{s} \sum_{j=0}^{J-1} \sum_{k \in \mathbb{Z}} w^{\ell,j}(k) \psi_{2^j;k} = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k) + \sum_{\ell=1}^{s} \sum_{j=0}^{J-1} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}^\ell_{2^j;k} \psi_{2^j;k}.$$

Why Wavelets?

A wavelet ψ often has

1. compact support $\Rightarrow$ good spatial localization.
2. high smoothness/regularity $\Rightarrow$ good frequency localization.
3. high vanishing moments $\Rightarrow$ multiscale sparse representation. That is, most wavelet coefficients are small for smooth functions/signals.
4. associated filter banks $\Rightarrow$ fast wavelet transform to compute coefficients $\langle f, \psi^{\ell}_{2^j;k} \rangle$ through filter banks.
5. singularities of signals and their locations can be captured in large wavelet coefficients.
6. function spaces (Sobolev and Besov spaces) can be characterized by wavelets. This is important in harmonic analysis and numerical PDEs.

Explanation for Sparse Representation

- A wavelet function ψ has m vanishing moments if

$$\int_{\mathbb{R}} x^n \psi(x) dx = 0, \qquad n = 0, \ldots, m-1.$$

That is, $\widehat{\psi}(0) = \widehat{\psi}'(0) = \cdots = \widehat{\psi}^{(m-1)}(0) = 0$. Define $\mathrm{vm}(\psi) := m$ largest.

- If $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ and $\widehat{\phi}(0) \neq 0$, then $\mathrm{vm}(\psi) = \mathrm{vm}(b)$.
- The multiscale wavelet representation of $f \in L_2(\mathbb{R})$ is

$$f = \sum_{k \in \mathbb{Z}} \langle f, \phi(\cdot - k) \rangle \tilde{\phi}(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{\ell=1}^{s} \langle f, \tilde{\psi}^{\ell}_{2^j;k} \rangle \psi^{\ell}_{2^j;k}$$

with $\psi^{\ell}_{2^j;k}(x) := 2^{j/2}\psi^{\ell}(2^j x - k)$.

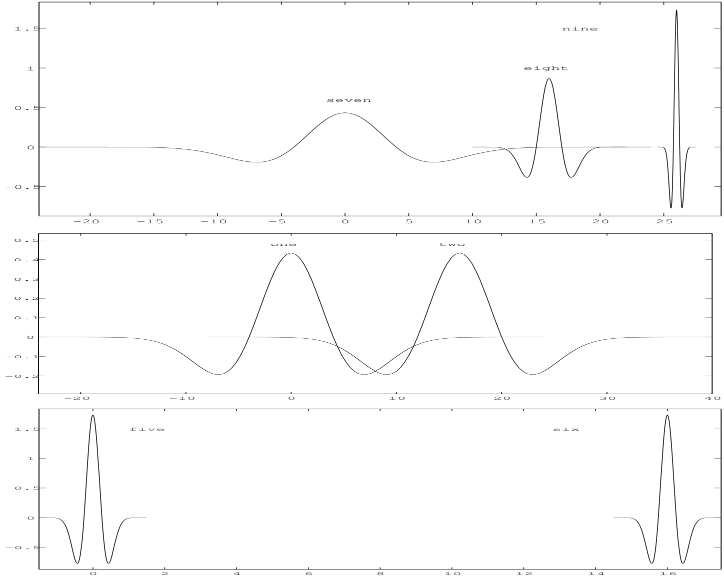
- $\mathrm{supp}\tilde{\psi}^{\ell}_{2^j;k} = 2^{-j}k + 2^{-j}\mathrm{supp}\tilde{\psi}^{\ell} \approx 2^{-j}k$ when $j \to \infty$.
- Wavelet coefficient $\langle f, \tilde{\psi}^{\ell}_{2^j;k} \rangle$ only depends f in the support of $\tilde{\psi}^{\ell}_{2^j;k}$. If f is smooth and can be well approximated by a polynomial P of degree $< m$, then

$$|\langle f, \tilde{\psi}^{\ell}_{2^j;k} \rangle| = |\langle f - P, \tilde{\psi}^{\ell}_{2^j;k}|\rangle = \|(f-P)\chi_{\mathrm{supp}(\tilde{\psi}^{\ell}_{2^j;k})}\|_2 \|\tilde{\psi}^{\ell}\|_2 \approx 0,$$

where $\langle P, \psi^{\ell}_{2^j;k} \rangle = 2^{j/2}\int_{\mathbb{R}} P(x)\psi^{\ell}(2^j x - k)dx = 2^{-j/2}\int_{\mathbb{R}} P(2^{-j}(x+k))\psi^{\ell}(y)dy = 0$.

- If $\langle f, \tilde{\psi}^{\ell}_{2^j;k} \rangle$ is large for large j, we know the position of singularity, since $\mathrm{supp}\tilde{\psi}^{\ell}_{2^j\ k} = 2^{-j}\mathrm{supp}\tilde{\psi}^{\ell} + 2^{-j}k \approx 2^{-j}k$.

Dilates and Shifts of Multiscale Affine Systems



Tensor Product (Separable) Wavelets and Framelets in $\mathbb{R}^d$

- Let $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})$ be a dual framelet filter bank.
- Tensor product filters: $[u_1 \otimes \cdots \otimes u_d](k_1, \dots, k_d) = u_1(k_1) \cdots u_d(k_d)$.
- Tensor product two-dimensional dual framelet filter bank:

$$\Big(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\} \otimes \{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\} \otimes \{a; b_1, \dots, b_s\}\Big).$$

That is,

$$\{a; b_1, \dots, b_s\} \otimes \{a; b_1, \dots, b_s\} = \{a \otimes a; b_1 \otimes a, \dots, b_s \otimes a,$$
$$b_1 \otimes b_1, \dots, b_s \otimes b_1, \dots, b_s \otimes b_1, \dots, b_s \otimes\}$$

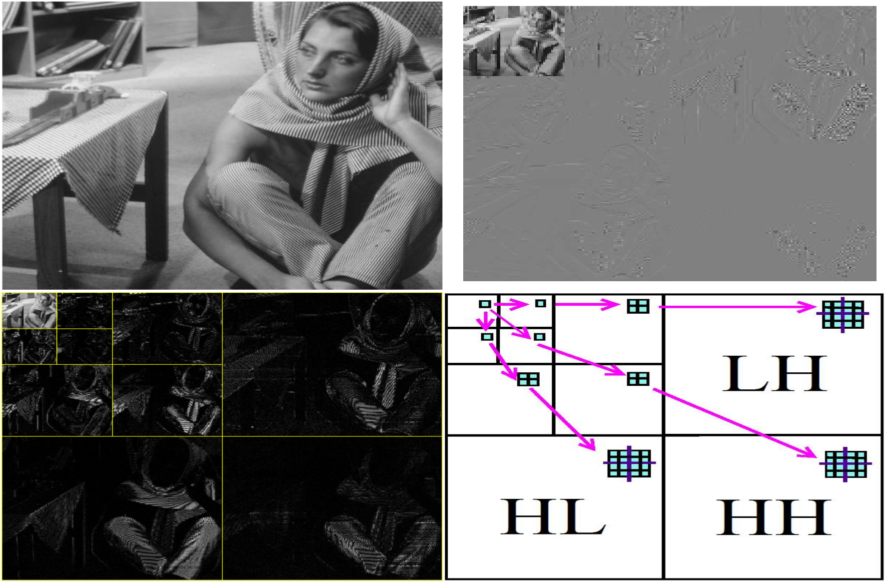
consists of one low-pass tensor product filter $a \otimes a$ and total $(s+1)^2 - 1 = s^2 + 2s$ high-pass tensor product filters.

- Tensor product functions: $[f_1 \otimes \cdots \otimes f_d](x_1, \dots, x_d) = f_1(x_1) \cdots f_d(x_d)$.
- Let $(\{\tilde{\phi}; \tilde{\psi}_1, \dots, \tilde{\psi}_s\}, \{\phi; \psi_1, \dots, \psi_s\})$ be a dual framelet in $L_2(\mathbb{R})$.
- Tensor product two-dimensional dual framelet in $L_2(\mathbb{R}^2)$:

$$\Big(\{\tilde{\phi}; \tilde{\psi}_1, \dots, \tilde{\psi}_s\} \otimes \{\tilde{\phi}; \tilde{\psi}_1, \dots, \tilde{\psi}_s\}, \{\phi; \psi_1, \dots, \psi_s\} \otimes \{\phi; \psi_1, \dots, \psi_s\}\Big).$$

- Advantages: fast and simple algorithm.

Sparsity and Multiscale Structure for Images



Subdivision Curves in Computer Graphics

- Let $v : \mathbb{Z} \to \mathbb{R}^2$ or $\mathbb{R}^3$ be given initial 2D or 3D curves outlining the rough shape of the curve. Write $v = (v^1, v^2, v^3)$ with sequences $v^1, v^2, v^3 : \mathbb{Z} \to \mathbb{R}$.
- Apply the subdivision operator to each entry of v to obtain $\mathcal{S}_a^n v := (\mathcal{S}_a^n v^1, \mathcal{S}_a^n v^2, \mathcal{S}_a^n v^3)$. Then plot the curve $\mathcal{S}_a^n v$.
- Different choices of filters (called masks in computer aided geometric design) affect the shapes of subdivision curves.
- A subdivision scheme with mask a is often used to compute a refinable function ϕ, where $\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$:

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |(\mathcal{S}_a^n \boldsymbol{\delta})(k) - \phi(2^{-j}k)| = 0.$$

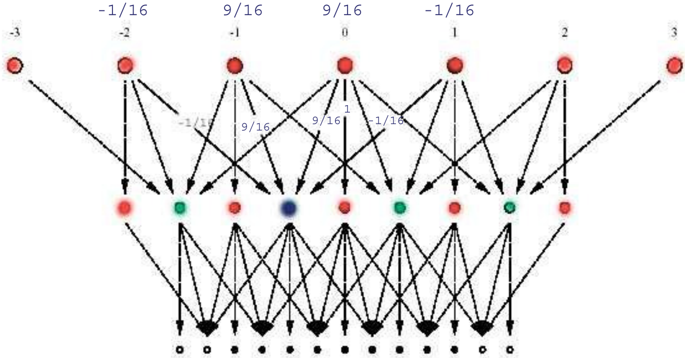
- $a^{[0]}(k) = a(2k)$ and $a^{[1]}(k) := a(2k+1)$ for $k \in \mathbb{Z}$. Then

$$[\mathcal{S}_a v](2n) = 2\sum_{k\in\mathbb{Z}} v(k)a(2n-2k) = 2[a^{[0]} * v](n),$$

$$[\mathcal{S}_a v](2n+1) = 2\sum_{k\in\mathbb{Z}} v(k)a(2n+1-2k) = 2[a^{[1]} * v](n).$$

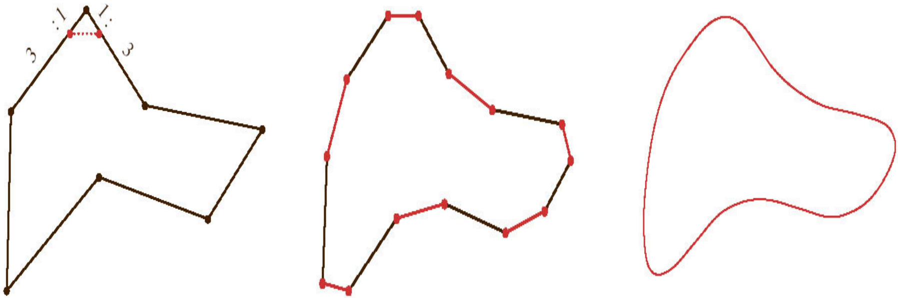
- Even stencil $\{2a^{[0]}(-k) = 2a(-2k)\}_{k\in\mathbb{Z}}$ and odd stencil $\{2a^{[1]}(-k) = 2a(1-2k)\}_{k\in\mathbb{Z}}$.
- A mask a is interpolatory if $a(2k) = \frac{1}{2}\boldsymbol{\delta}(k)$ for all $k \in \mathbb{Z}$.

The 4-point Interpolatory Subdivision Scheme



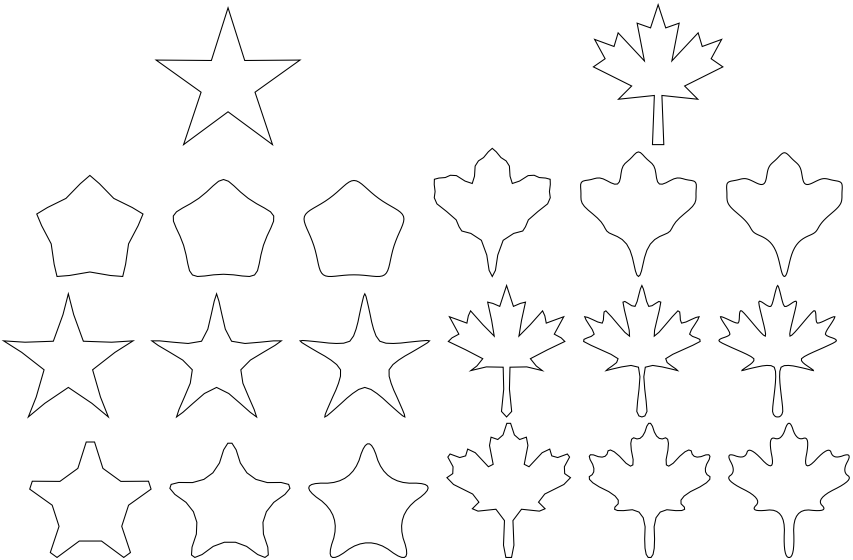
• The mask is the Deslauriers-Dubuc interpolatory mask
$a_4^I = [-\frac{1}{32}, 0, \frac{9}{32}, \frac{1}{2}, \frac{9}{32}, 0, -\frac{1}{32}]$.
Even stencil $\{1\}_{[0,0]}$ and odd stencil $\{-\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, -\frac{1}{16}\}_{[-1,2]}$.

The Corner Cutting Scheme



• The mask is the cubic B-spline filter of order 4 $a_4^B = \{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\}_{[-1,2]}$.
Even stencil $\{\frac{1}{4}, \frac{3}{4}\}_{[0,1]}$ and odd stencil $\{\frac{3}{4}, \frac{1}{4}\}_{[0,1]}$.

Example of Subdivision Curves Using Different Masks



Subdivision Schemes and Cascade Algorithm

- Let $a \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = \sum_{k\in\mathbb{Z}} a(k) = 1$.
- Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ for $\xi \in \mathbb{R}$.
- **Definition:** We say that the subdivision scheme with mask a is convergent in $C(\mathbb{R})$ if for every $v \in l_\infty(\mathbb{Z})$, there exists a continuous function η_v such that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |[\mathcal{S}_a^n v](k) - \eta_v(2^{-n}k)| = 0.$$

  If the subdivision scheme is convergent, then $\eta_v = \sum_{k\in\mathbb{Z}} v(k)\phi(\cdot - k)$.
- A function $f \in L_2(\mathbb{R})$ is called an admissible initial function if

$$\widehat{f}(0) = 1, \quad \text{and} \quad \widehat{f}(2\pi k) = 0, \qquad k \in \mathbb{Z}\backslash\{0\}.$$

- The cascade operator $\mathcal{R}_a$ is defined to be

$$[\mathcal{R}_a f](x) := 2\sum_{k\in\mathbb{Z}} a(k) f(2x-k), \quad \text{that is,} \quad \widehat{\mathcal{R}_a f}(2\xi) = \widehat{a}(\xi)\widehat{f}(\xi).$$

- Note that $\mathcal{R}_a\phi = \phi$. That is, ϕ is a fixed point of $\mathcal{R}_a$.
- We say that the cascade algorithm with mask a is convergent in $L_2(\mathbb{R})$ if $\{\mathcal{R}_a^n f\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_2(\mathbb{R})$ for every initial admissible initial function f. If the cascade algorithm is convergent in $L_2(\mathbb{R})$, then $\phi \in L_2(\mathbb{R})$ and $\lim_{n\to\infty} \|\mathcal{R}_a^n f - \phi\|_2 = 0$.
- By induction, we have $\mathcal{R}_a^n f = \sum_{k\in\mathbb{Z}} [\mathcal{S}_a^n \boldsymbol{\delta}](k) f(2^{-n}\cdot -k)$.

Convergence of Subdivision Schemes and Cascade Algorithm

- Let $a \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = 1$ and $m := \mathrm{sr}(a)$. Write

$$\widehat{a}(\xi) = (1 + e^{-i\xi})^m \widehat{b}(\xi) \quad \text{with} \quad b \in l_0(\mathbb{Z}),\ \widehat{b}(0) \neq 0.$$

- Define $c \in l_0(\mathbb{Z})$ by $\widehat{c}(\xi) := |\widehat{b}(\xi)|^2$ and $\mathrm{supp}(c) = [-N, N]$, $N \in \mathbb{N} \cup \{0\}$.
- Define a smoothness exponent

$$\mathrm{sm}(a) := -\frac{1}{2} \log_2 \lambda_c \ \text{ with } \ \lambda_c := \max\{|\lambda| \ : \ \lambda \in \mathrm{spec}((2c(2k-j))_{-N \leqslant j,k \leqslant N})\},$$

where $\mathrm{spec}(A)$ is the set of all eigenvalues of a matrix A.

Theorem

Let $a \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = 1$ and $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$. Then the cascade algorithm with mask a is convergent in $L_2(\mathbb{R})$ if and only if $\mathrm{sm}(a) > 0$. *Moreover,* $\mathrm{sm}(a) > 0$ *implies $\phi \in L_2(\mathbb{R})$ and* $\lim_{n\to\infty} \|\mathcal{R}_a^n f - \phi\|_2 = 0$. *Moreover, $\phi \in C^m(\mathbb{R})$ if* $\mathrm{sm}(a) > m + 1/2$.

Corollary

If $\mathrm{sm}(a) > \frac{1}{2}$, *then the subdivision scheme with mask a is convergent in $C(\mathbb{R})$ and*

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |[\mathcal{S}_a^n v](k) - \eta_v(2^{-n}k)| = 0 \quad \text{with} \quad \eta_v := \sum_{k\in\mathbb{Z}} v(k)\phi(\cdot - k).$$

B-spline Functions

- For $m \in \mathbb{N}$, the B-spline function B_m of order m is defined to be

$$B_1 := \chi_{(0,1]} \quad \text{and} \quad B_m := B_{m-1} * B_1 = \int_0^1 B_{m-1}(\cdot - t)dt.$$

- $\operatorname{supp}(B_m) = [0, m]$ and $B_m(x) > 0$ for all $x \in (0, m)$.
- $B_m = B_m(m - \cdot)$, $B_m \in \mathscr{C}^{m-2}(\mathbb{R})$, $B_m|_{(k,k+1)} \in \mathbb{P}_{m-1}$ for all $k \in \mathbb{Z}$.
- $\widehat{B_m}(\xi) = (\frac{1-e^{-i\xi}}{i\xi})^m$ and B_m is refinable:

$$B_m = 2 \sum_{k \in \mathbb{Z}} a_m^B(k) B_m(2 \cdot -k),$$

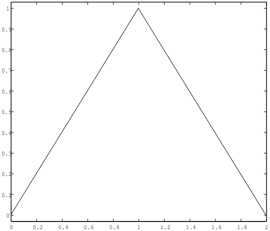
  where a_m^B is the B-spline filter of order m:

$$\widehat{a_m^B}(\xi) := 2^{-m}(1 + e^{-i\xi})^m.$$

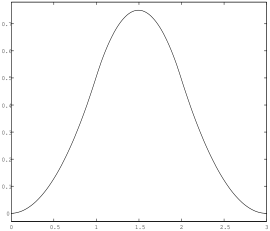
- $\operatorname{sr}(a_m^B) = m$, that is, a_m^B has m sum rules.

Graphs of B-spline Functions

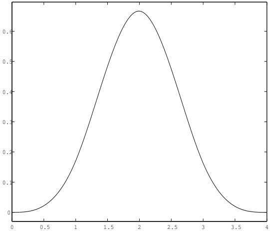
(a) B_1



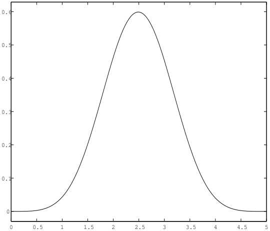
(b) B_2



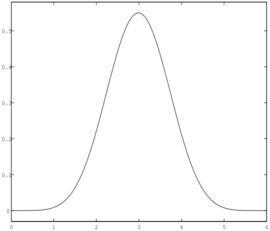
(c) B_3



(d) B_4



(e) B_5



(f) B_6

B-spline Filters a_m^B

$$\begin{aligned}
a_1^B &= \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]},\\
a_2^B &= \{\underline{\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},\\
a_3^B &= \{\underline{\mathbf{\tfrac{1}{8}}}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]},\\
a_4^B &= \{\underline{\mathbf{\tfrac{1}{16}}}, \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{4}, \tfrac{1}{16}\}_{[0,4]},\\
a_5^B &= \{\underline{\mathbf{\tfrac{1}{32}}}, \tfrac{5}{32}, \tfrac{5}{16}, \tfrac{5}{16}, \tfrac{15}{32}, \tfrac{1}{32}\}_{[0,5]},\\
a_6^B &= \{\underline{\mathbf{\tfrac{1}{64}}}, \tfrac{3}{32}, \tfrac{15}{64}, \tfrac{5}{16}, \tfrac{15}{64}, \tfrac{3}{32}, \tfrac{1}{64}\}_{[0,6]}.
\end{aligned}$$

- Note $\widehat{a_m^B}(\xi) = (1+e^{-i\xi})^m \widehat{b}(\xi)$ with $\widehat{b}(\xi) := 2^{-m}$. Hence, $\widehat{c}(\xi) := |\widehat{b}(\xi)|^2 = 2^{-2m}$. Therefore, $c = 2^{-2m}\boldsymbol{\delta}$ and $\mathrm{spec}(\{2c(2k-j)\}_{-Nj,k\leqslant N}) = 2^{1-2m}$ with $N = 0$. Therefore,
$$\mathrm{sm}(a_m^B) = -\frac{1}{2}\log_2 2^{1-2m} = m - 1/2.$$
- Note that $a_1^B = [\frac{1}{2}, \frac{1}{2}]_{[0,1]}$ is the Haar low-pass filter and $\mathrm{sm}(a_1^B) = 1/2$. Hence, its refinable function $\phi = \chi_{[0,1]} \in L_2(\mathbb{R})$.
- Note that $\mathrm{sm}(a_4^B) = 3.5$. Hence, its refinable function $\phi = B_4$ belongs to C^2 by $\mathrm{sm}(a_4^B) = 3.5 > 2 + 1/2$.

The Bracket Product

For $f, g \in L_2(\mathbb{R})$, we define the bracket product to be

$$[f,g](\xi) := \sum_{k\in\mathbb{Z}} f(\xi+2\pi k)\overline{g(\xi+2\pi k)} = \left\langle \{f(\xi+2\pi k)\}_{k\in\mathbb{Z}}, \{g(\xi+2\pi k)\}_{k\in\mathbb{Z}} \right\rangle_{l_2(\mathbb{Z})}.$$

Note that $[f,f] \in L_1(\mathbb{T})$ by

$$2\pi\|[f,f]\|_1 = \int_{-\pi}^{\pi} [f,f](\xi)d\xi = \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} |f(\xi+2\pi k)|^2 d\xi = \int_{\mathbb{R}} |f(\xi)|^2 d\xi = \|f\|_2^2.$$

By the Cauchy-Schwarz inequality, $|[f,g](\xi)| \leqslant$
$\|\{f(\xi+2\pi k)\}_{k\in\mathbb{Z}}\|_{l_2(\mathbb{Z})}\|\{g(\xi+2\pi k)\}_{k\in\mathbb{Z}}\|_{l_2(\mathbb{Z})} = \sqrt{[f,f](\xi)}\sqrt{[g,g](\xi)}$.

$$\begin{aligned} |[f,g](\xi)| &\leqslant \sum_{k\in\mathbb{Z}} |f(\xi+2\pi k)g(\xi+2\pi k)| \\ &\leqslant \Big(\sum_{k\in\mathbb{Z}} |f(\xi+2\pi k)|^2\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}} |g(\xi+2\pi k)|^2\Big)^{1/2} = \sqrt{[f,f](\xi)}\sqrt{[g,g](\xi)}. \end{aligned}$$

Hence, $|[f,g](\xi)|^2 \leqslant [f,f](\xi)[g,g](\xi)$ and $\|[f,g]\|_1 \leqslant \|[f,f]\|_1\|[g,g]\|_1$ by

$$2\pi\|[f,g]\|_1 = \int_{\mathbb{T}} |[f,g](\xi)|d\xi \leqslant \left(\int_{\mathbb{T}} [f,f](\xi)d\xi\right)^{1/2} \left(\int_{\mathbb{T}} [f,f](\xi)d\xi\right)^{1/2} = \|f\|_2\|g\|_2.$$

Some Properties of the Bracket Product

Some properties of the bracket product are as follows.

Lemma

For $f, g \in L_2(\mathbb{R})$,

(1) *The bracket product* $[\widehat{f}, \widehat{g}] \in L_1(\mathbb{T})$.

(2) *The Fourier series of* $[\widehat{f}, \widehat{g}]$ *is* $\sum_{k\in\mathbb{Z}} \langle f, g(\cdot + k)\rangle e^{ik\xi}$.

(3) $\langle f, g(\cdot + k)\rangle = 0$ *for all* $k \in \mathbb{Z}$ *if and only if* $[\widehat{f}, \widehat{g}](\xi) = 0$ *for a.e.* $\xi \in \mathbb{R}$.

(4) $\langle f, g(\cdot + k)\rangle = \delta(k)$ *for all* $k \in \mathbb{Z}$ *if and only if* $[\widehat{f}, \widehat{g}](\xi) = 1$ *for a.e.* $\xi \in \mathbb{R}$.

Proof

Proof. We already proved item (1) because $\widehat{f}, \widehat{g} \in L_2(\mathbb{R})$ by Plancherel's Theorem. We now calculate its kth Fourier coefficient:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}[\widehat{f},\widehat{g}](\xi)e^{-ik\xi}d\xi &= \frac{1}{2\pi}\int_{\pi}^{\pi}\sum_{k\in\mathbb{Z}}\widehat{f}(\xi+2\pi k)\overline{\widehat{g}(\xi+2\pi k)}e^{-ik\xi}d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}e^{-ik\xi}d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\xi)\overline{\widehat{g}(\xi)e^{ik\xi}}d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\xi)\overline{\widehat{g(\cdot+k)}(\xi)}d\xi \\
&= \frac{1}{2\pi}\langle\widehat{f},\widehat{g(\cdot+k)}\rangle \\
&= \langle f, g(\cdot+k)\rangle,
\end{aligned}$$

where we used $\widehat{g(\cdot+k)}(\xi) = \widehat{g}(\xi)e^{ik\xi}$ and the Plancherel's Theorem. This proves item (2).

Items (3) and (4) follow trivially from item (2).

Connections of Tight Framelets and Tight Framelet Filter Banks

Theorem

Let $a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ *with* $\widehat{a}(0) = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \qquad \widehat{\psi^{\ell}}(\xi) := \widehat{b_\ell}(\xi/2)\widehat{\phi}(\xi/2), \qquad \ell = 1, \ldots, s.$$

Then $\{\phi; \psi^1, \ldots, \psi^s\}$ *is a tight framelet in* $L_2(\mathbb{R})$, *that is,*

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle f, \psi^{\ell}_{2^j;k}\rangle \psi^{\ell}_{2^j;k}, \qquad \forall\, f \in L_2(\mathbb{R})$$

if and only if $\{a; b_1, \ldots, b_s\}$ *is a tight framelet filter bank:*

$$|\widehat{a}(\xi)|^2 + \sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi)|^2 = 1, \quad \widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} + \sum_{\ell=1}^{s} \widehat{b_\ell}(\xi)\overline{\widehat{b_\ell}(\xi+\pi)} = 0.$$

The key: The tight framelet filter bank forces $\phi \in L_2(\mathbb{R})$ and $\widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = 0$.

Existence of Refinable Functions $\phi \in L_2(\mathbb{R})$

Theorem

Let $\widehat{a}$ be a 2π-periodic continuous function such that $|\widehat{a}(\xi) - 1| \leqslant C|\xi|^\tau$ for all $\xi \in \mathbb{R}$ for some positive constants τ and C (this condition is satisfied for $a \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = 1$). Then

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \quad \xi \in \mathbb{R}$$

is a well-defined continuous function on $\mathbb{R}$ satisfying $\widehat{\phi}(0) = 1$ and $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. If in addition

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + \pi)|^2 \leqslant 1 \qquad \forall\, \xi \in \mathbb{R},$$

then $[\widehat{\phi}, \widehat{\phi}](\xi) \leqslant 1$, $\|\widehat{\phi}\|_2^2 \leqslant 2\pi$, and $\|\phi\|_2 \leqslant 1$.

Proof

Proof. Since $\sum_{j=1}^{\infty} |\widehat{a}(2^{-j}\xi) - 1| \leqslant C \sum_{j=1}^{\infty} 2^{-\tau j}|\xi|^{\tau} < \infty$, the infinite product $\prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ is convergent and $\widehat{\phi}$ is a well-defined continuous function. Define $f_n(\xi) := \chi_{(-2^n\pi, 2^n\pi]}(\xi) \prod_{j=1}^{n} \widehat{a}(2^{-j}\xi)$ for $n \in \mathbb{N} \cup \{0\}$. Then $\lim_{n\to\infty} f_n(\xi) = \widehat{\phi}(\xi)$. We now claim that $[\widehat{f_n}, \widehat{f_n}](\xi) \leqslant 1$ for $\xi \in \mathbb{R}$. Clearly, by $f_0 = \chi_{(-\pi,\pi]}$, we trivially have $[f_0, f_0] = 1$. Suppose that $[f_{n-1}, f_{n-1}] \leqslant 1$. Observing $f_n(2\xi) = \widehat{a}(\xi)\widehat{f_{n-1}}(\xi)$, we have

$$\begin{aligned}[f_n, f_n](\xi) &= |\widehat{a}(\xi/2)|^2[f_{n-1}, f_{n-1}](\xi/2) + |\widehat{a}(\xi/2 + \pi)|^2[f_{n-1}, f_{n-1}](\xi/2 + \pi)\\ &\leqslant |\widehat{a}(\xi/2)|^2 + |\widehat{a}(\xi/2 + \pi)|^2 \leqslant 1.\end{aligned}$$

By induction, the claim $[f_n, f_n] \leqslant 1$ holds for all $n \in \mathbb{N} \cup \{0\}$. Because $|\widehat{a}(\xi)| \leqslant 1$ for all $\xi \in \mathbb{R}$, we have $0 \leqslant |\widehat{\phi}|^2 \leqslant 1$. Therefore, $0 \leqslant |\widehat{\phi}|^2\chi_{(-2^n\pi,2^n\pi]} \leqslant |f_n|^2$. Hence,

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + 2\pi k)|^2\chi_{(-2^n\pi,2^n\pi]}(\xi + 2\pi k) \leqslant [f_n, f_n](\xi) \leqslant 1.$$

Taking $n \to \infty$, we conclude from the above inequality that $[\widehat{\phi}, \widehat{\phi}] \leqslant 1$. Hence,

$$\|\widehat{\phi}\|_2^2 = \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 d\xi = \int_{-\pi}^{\pi} [\widehat{\phi}, \widehat{\phi}](\xi) d\xi \leqslant 2\pi.$$

By Plancherel's Theorem, we conclude that $\|\phi\|_2^2 = (2\pi)^{-1}\|\widehat{\phi}\|_2^2 \leqslant 1$. $\square$

Tight Framelet Filter Banks

Theorem

If $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank, that is,

$$|\widehat{a}(\xi)|^2 + \sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi)|^2 = 1, \quad \widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} + \sum_{\ell=1}^{s} \widehat{b_\ell}(\xi)\overline{\widehat{b_\ell}(\xi+\pi)} = 0,$$

then we must have

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \leqslant 1, \qquad \forall\, \xi \in \mathbb{R}.$$

Tight Framelet Filter Banks

Theorem

If $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank, that is,

$$|\widehat{a}(\xi)|^2 + \sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi)|^2 = 1, \quad \widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} + \sum_{\ell=1}^{s} \widehat{b_\ell}(\xi)\overline{\widehat{b_\ell}(\xi+\pi)} = 0,$$

then we must have

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \leqslant 1, \qquad \forall\, \xi \in \mathbb{R}.$$

Proof. Note that $\sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi)|^2 = 1 - |\widehat{a}(\xi)|^2$ and

$$\begin{aligned}
|\widehat{a}(\xi)\widehat{a}(\xi+\pi)|^2 &= \left|\sum_{\ell=1}^{s} \widehat{b_\ell}(\xi)\overline{\widehat{b_\ell}(\xi+\pi)}\right|^2 \leqslant \left(\sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi)|^2\right)\left(\sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi+\pi)|^2\right) \\
&= (1 - |\widehat{a}(\xi)|^2)(1 - |\widehat{a}(\xi+\pi)|^2) \\
&= 1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2 + |\widehat{a}(\xi)\widehat{a}(\xi+\pi)|^2,
\end{aligned}$$

from which we have $1 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2 \geqslant 0$. $\square$

Orthogonal Wavelets vs Orthogonal Wavelet Filter Banks

Theorem

Let $a, b \in l_0(\mathbb{Z})$ *with* $\widehat{a}(0) = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \qquad \widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2).$$

Then the following are equivalent to each other:

1. $\{\phi; \psi\}$ *is an orthogonal wavelet in* $L_2(\mathbb{R})$*, that is,*

   $$\mathrm{AS}(\phi;\psi) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi_{2^j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}\}$$

   is an orthonormal basis of $L_2(\mathbb{R})$.
2. $\{a; b\}$ *is an orthogonal wavelet filter bank and* $[\widehat{\phi}, \widehat{\phi}](\xi) = 1$ *almost everywhere (Note that* $[\widehat{\phi}, \widehat{\phi}] = 1 \iff \langle \phi, \phi(\cdot - k)\rangle = \boldsymbol{\delta}(k)$ *for* $k \in \mathbb{Z}$*)*
3. $\{a; b\}$ *is an orthogonal wavelet filter bank and* $\mathrm{sm}(a) > 0$.

Tight Framelets vs Orthogonal Wavelets in $L_2(\mathbb{R})$

- Consider the Haar orthogonal filter bank $\{a; b\}$ with

$$a = \{\tfrac{1}{2}, \tfrac{1}{2}\}_{[0,1]}, \qquad b := \{-\tfrac{1}{2}, \tfrac{1}{2}\}_{[0,1]}.$$

- Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ and $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ (i.e., $\psi(x) = 2\sum_{k\in\mathbb{Z}} b(k)\phi(2x-k)$. In fact,

$$\phi = \chi_{[0,1]}, \quad \psi = \chi_{[1/2,1]} - \chi_{[0,1/2]}.$$

- Because $\mathrm{sm}(a) = 1/2 > 0$, $\{\phi; \psi\}$ is an orthogonal wavelet in $L_2(\mathbb{R})$.
- Now consider a dilated version of Haar wavelet filter bank:

$$a_1 := \{\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\}_{[0,3]}, \qquad b_1 := \{-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\}_{[0,3]}.$$

  Then $\{a_1, b_1\}$ is an still orthogonal wavelet filter bank.
- Define $\widehat{\phi_1}(\xi) := \prod_{j=1}^{\infty} \widehat{a_1}(2^{-j}\xi)$ and $\widehat{\psi_1}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi_1}(\xi/2)$. In fact,

$$\phi_1 = \phi(\cdot/3) = \chi_{[0,3]}, \qquad \psi_1 = \psi(\cdot/3) = \chi_{[3/2,3]} - \chi_{[0,3/2]}.$$

- Then $\{\phi_1; \psi_1\}$ is a tight framelet in $L_2(\mathbb{R})$.
- But $\mathrm{sm}(a) \leqslant 0$ and $\{\phi_1; \psi_1\}$ is not an orthogonal wavelet in $L_2(\mathbb{R})$. In particular,

$$\langle \phi_1(\cdot - 1), \phi_1 \rangle = 2 \neq 0.$$

Construction of Orthogonal Wavelet Filter Bank

- An orthogonal wavelet filter bank $\{a; b\}$ satisfies

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)} & \overline{\widehat{a}(\xi+\pi)} \\ \overline{\widehat{b}(\xi)} & \overline{\widehat{b}(\xi+\pi)} \end{bmatrix} = I_2.$$

$$\text{i.e.,} \quad \begin{bmatrix} \overline{\widehat{a}(\xi)} & \overline{\widehat{a}(\xi+\pi)} \\ \overline{\widehat{b}(\xi)} & \overline{\widehat{b}(\xi+\pi)} \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix} = I_2,$$

- which is further equivalent to

$$\begin{aligned} |\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 &= 1, \\ |\widehat{b}(\xi)|^2 + |\widehat{b}(\xi+\pi)|^2 &= 1, \\ \overline{\widehat{a}(\xi)}\widehat{b}(\xi) + \overline{\widehat{a}(\xi+\pi)}\widehat{b}(\xi+\pi) &= 0. \end{aligned}$$

- The second and third identities hold if $\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}$. Hence, we only need an orthogonal low-pass filter $a \in l_0(\mathbb{Z})$ satisfying

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 = 1.$$

- A filter c is interpolatory if $\widehat{c}(\xi) + \widehat{c}(\xi+\pi) = 1$.
- Define $\widehat{c}(\xi) := |\widehat{a}(\xi)|^2 = \overline{\widehat{a}(\xi)}\widehat{a}(\xi)$, that is, $c = a^\star * a$. Then a is an orthogonal low-pass filter if and only if c is an interpolatory filter.

A Basic Identity

For $m, n \in \mathbb{N}$, $\mathrm{P}_{m,n}$ is the unique polynomial of degree at most $n-1$ satisfying

$$\mathrm{P}_{m,n}(x) := (1-x)^{-m} + \mathscr{O}(x^n), \quad x \to 0, \quad \text{that is,} \quad \mathrm{P}_{m,n}(x) = \sum_{j=0}^{n-1} \binom{m+j-1}{j} x^j.$$

Theorem

$(1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1$ *for all* $x \in \mathbb{R}, m \in \mathbb{N}$.

A Basic Identity

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Theorem

$(1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1$ *for all* $x \in \mathbb{R}, m \in \mathbb{N}$.

Proof. Define $\mathrm{P}(y,x) := \sum_{j=0}^{m-1} \binom{2m-1}{j} x^j y^{m-j-1}$. Then

$$(x+y)^{2m-1} = \sum_{j=0}^{2m-1} \binom{2m-1}{j} x^j y^{2m-1-j} = x^m \mathrm{P}(x,y) + y^m \mathrm{P}(y,x).$$

Note $\deg(\mathrm{P}(1-x,x)) < m$ and $x^m \mathrm{P}(x, 1-x) + (1-x)^m \mathrm{P}(1-x, x) = 1$, from which we have

$$\begin{aligned} \mathrm{P}(1-x,x) &= (1-x)^{-m}[(1-x)^m \mathrm{P}(1-x,x)] = (1-x)^{-m}[1 - x^m \mathrm{P}(x, 1-x)] \\ &= (1-x)^{-m} + \mathscr{O}(x^m), \quad x \to 0. \end{aligned}$$

By the uniqueness of $\mathrm{P}_{m,m}$, we must have $\mathrm{P}(x, 1-x) = \mathrm{P}_{m,m}$. Hence, we proved

$$(1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1.$$

Construction of Interpolatory Filters

- A filter $a \in l_0(\mathbb{Z})$ is interpolatory if $\widehat{a}(\xi) + \widehat{a}(\xi+\pi) = 1$, i.e.,
$$a(0) = \frac{1}{2} \quad \text{and} \quad a(2k) = 0, \quad \forall\, k \in \mathbb{Z}\backslash\{0\}.$$
- For $m \in \mathbb{N}$, a family of interpolatory filters a^I_{2m} is given by
$$\widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathrm{P}_{m,m}(\sin^2(\xi/2)).$$
Set $x = \sin^2(\xi/2)$. Then $\sin^2((\xi+\pi)/2) = \cos^2(\xi/2)$. Hence,
$$\widehat{a^I_{2m}}(\xi) = (1-x)^m \mathrm{P}_{m,m}(x) \quad \text{and} \quad \widehat{a^I_{2m}}(\xi+\pi) = x^m \mathrm{P}_{m,m}(1-x).$$
Therefore, $\widehat{a^I_{2m}}(\xi) + \widehat{a^I_{2m}}(\xi+\pi) = (1-x)^m\mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1.$
- Note that $|1+e^{-i\xi}|^2 = 2^{-m}\cos^{2m}(\xi/2)$ and $\mathrm{P}(0) = 1$. The mask a^I_{2m} has $2m$ sum rules satisfying $(1-e^{-i\xi})^{2m} \mid \widehat{a^I_{2m}}(\xi)$.
- Hence, $\mathrm{sr}(a^I_{2m}) = 2m$, $\widehat{a^I_{2m}}(0) = 1$, and a^I_{2m} is supported inside $[1-2m, 2m-1]$.
- $\widehat{a^I_{2m}}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$.
- The filters are called Deslauriers-Dubuc interpolatory filters.

Interpolatory Filters a^I_{2m}

$$
\begin{aligned}
a^I_2 &= \{\tfrac{1}{4}, \underline{\tfrac{1}{2}}, \tfrac{1}{4}\}_{[-1,1]},\\
a^I_4 &= \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\tfrac{1}{2}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]},\\
a^I_6 &= \{\tfrac{3}{512}, 0, -\tfrac{25}{512}, 0, \tfrac{75}{256}, \underline{\tfrac{1}{2}}, \tfrac{75}{256}, 0, -\tfrac{25}{512}, 0, \tfrac{3}{512}\}_{[-5,5]},\\
a^I_8 &= \{-\tfrac{5}{4096}, 0, \tfrac{49}{4096}, 0, -\tfrac{245}{4096}, 0, \tfrac{1225}{4096}, \underline{\tfrac{1}{2}}, \tfrac{1225}{4096}, 0, -\tfrac{245}{4096},\\
&\qquad 0, \tfrac{49}{4096}, 0, -\tfrac{5}{4096}\}_{[-7,7]}.
\end{aligned}
$$

m	1	2	3	4	5
$\mathrm{sm}(a^I_{2m})$	1.5	2.440765	3.175132	3.793134	4.344084

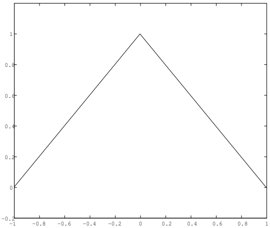
Theorem

Let $a \in l_0(\mathbb{Z})$ be interpolatory: $a(2k) = \frac{1}{2}\boldsymbol{\delta}(k)$ for $k \in \mathbb{Z}$. Define a refinable function by $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ for $\xi \in \mathbb{R}$. If $\mathrm{sm}(a) > 1/2$, then ϕ is a compactly supported continuous function and is interpolating:

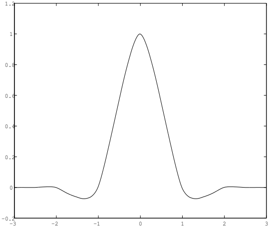
$$\phi(k) = \boldsymbol{\delta}(k), \qquad k \in \mathbb{Z}.$$

In particular, if $a = a^I_{2m}$ with $m \in \mathbb{N}$, then $\phi(k) = \boldsymbol{\delta}(k)$ for all $k \in \mathbb{Z}$.

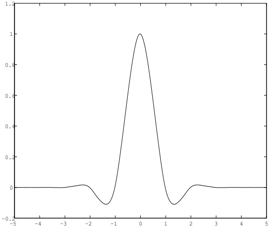
Compactly Supported Interpolating Function



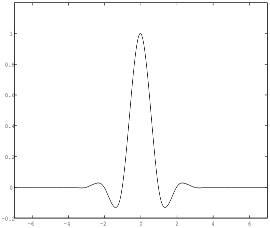
(a) $\phi^{a_2^l}$



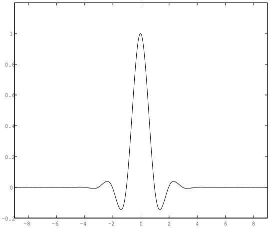
(b) $\phi^{a_4^l}$



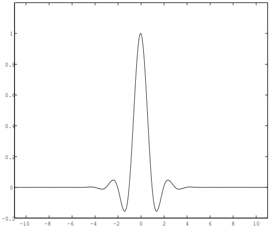
(c) $\phi^{a_6^l}$



(d) $\phi^{a_8^l}$



(e) $\phi^{a_{10}^l}$



(f) $\phi^{a_{12}^l}$

Fejér-Riesz Lemma

Lemma

Let $\boldsymbol{\Theta}$ *be a* 2π*-periodic trigonometric polynomial with real coefficients (or with complex coefficients) such that* $\boldsymbol{\Theta}(\xi) \geqslant 0$ *for all* $\xi \in \mathbb{R}$*. Then there exists a* 2π*-periodic trigonometric polynomial* $\boldsymbol{\theta}$ *with real coefficients (or with complex coefficients) such that* $|\boldsymbol{\theta}(\xi)|^2 = \boldsymbol{\Theta}(\xi)$ *for all* $\xi \in \mathbb{R}$*. Moreover, if* $\boldsymbol{\Theta}(0) \neq 0$*, then we can further require* $\boldsymbol{\theta}(0) = \sqrt{\boldsymbol{\Theta}(0)}$*.*

Daubechies Orthogonal Wavelets

Let a^I_{2m} be the interpolatory filter. Since $\widehat{a^I_{2m}}(\xi) \geqslant 0$, by Fejér-Riesz lemma, there exists $a^D_m \in l_0(\mathbb{Z})$ such that $\widehat{a^D_m}(0) = 1$ and

$$|\widehat{a^D_m}(\xi)|^2 = \widehat{a^I_{2m}}(\xi) := \widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathsf{P}_{m,m}(\sin^2(\xi/2)).$$

Then $\mathrm{sr}(a^D_m) = m$ (i.e., a^D_m has m sum rules) and

$$|\widehat{a^D_m}(\xi)|^2 + |\widehat{a^D_m}(\xi+\pi)|^2 = \widehat{a^I_{2m}}(\xi) + \widehat{a^I_{2m}}(\xi+\pi) = 1.$$

Define ϕ through $\widehat{\phi}(\xi) := \prod_{j=1}^\infty \widehat{a^D_m}(2^{-j}\xi)$. Then

$$[\widehat{\phi}, \widehat{\phi}] := \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + 2\pi k)|^2 = 1$$

and $\{a^D_m; b^D_m\}$ is an orthogonal wavelet filter bank with

$$\widehat{b^D_m}(\xi) := e^{-i\xi}\overline{\widehat{a^D_m}(\xi+\pi)}.$$

Then $\mathrm{vm}(b^D_m) = m$ and $\{\phi; \psi\}$ is a compactly supported orthogonal wavelet, where

$$\widehat{\psi}(\xi) := \widehat{b^D_m}(\xi/2)\widehat{\phi}(\xi/2)$$

such that the low-pass filter a^D_m has order m sum rules and the high-pass filter b^D_m has m vanishing moments, called the Daubechies orthogonal wavelet of order m.

Daubechies Orthogonal Refinable Functions

Theorem

Let $a = a_m^D$ *be the orthogonal filter and* $\widehat{\varphi}(\xi) := \prod_{j=1}^{\infty} \widehat{a_m^D}(2^{-j}\xi)$. *Then* $\varphi \in L_2(\mathbb{R})$, *the integer shifts of* φ *are orthonormal:* $\langle \varphi(\cdot - k), \varphi \rangle = \boldsymbol{\delta}(k)$ *for all* $k \in \mathbb{Z}$, *and*

$$[\widehat{\varphi}, \widehat{\varphi}] := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2\pi k)|^2 = 1.$$

Proof

Proof. Because $\widehat{a_m^D}(0) = 1$ and $|\widehat{a_m^D}(\xi)|^2 + |\widehat{a_m^D}(\xi+\pi)|^2 = 1$, we proved $\varphi \in L_2(\mathbb{R})$ and $[\widehat{\varphi}, \widehat{\varphi}] \leqslant 1$.

We now prove that $[\widehat{\varphi}, \widehat{\varphi}] = 1$. Define $f_n(\xi) := \chi_{(-2^n\pi, 2^n\pi]}(\xi) \prod_{j=1}^n \widehat{a_m^D}(2^{-j}\xi)$ for $n \in \mathbb{N} \cup \{0\}$. Then $\lim_{n\to\infty} f_n(\xi) = \widehat{\varphi}(\xi)$ for every $\xi \in \mathbb{R}$. Since $f_0 = \chi_{(-\pi,\pi]}$, we trivially have $[f_0, f_0](\xi) = \sum_{k\in\mathbb{Z}} |f_0(\xi + 2\pi k)|^2 = 1$. Suppose that $[f_{n-1}, f_{n-1}] = 1$. Then by $f_n(\xi) = \widehat{a_m^D}(\xi/2) f_{n-1}(\xi/2)$, we have

$$
\begin{aligned}
[f_n, f_n](\xi) &= |\widehat{a_m^D}(\xi/2)|^2 [f_{n-1}, f_{n-1}](\xi/2) + |\widehat{a_m^D}(\xi/2+\pi)|^2 [f_{n-1}, f_{n-1}](\xi/2+\pi)| \\
&= |\widehat{a_m^D}(\xi/2)|^2 + |\widehat{a_m^D}(\xi/2+\pi)|^2 = 1.
\end{aligned}
$$

By induction, we have $[f_n, f_n] = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, $\int_{\mathbb{R}} |f_n(\xi)|^2 d\xi = \int_{-\pi}^{\pi} [f_n, f_n](\xi) d\xi = 2\pi$.

By $\widehat{a_{2m}^I}(\xi) > 0$ for all $\xi \in (-\pi, \pi)$, since $\widehat{\varphi}$ is continuous, we have $c := \inf_{\xi\in[-\pi,\pi]} |\widehat{\varphi}(\xi)|^2 > 0$ and hence $0 \leqslant |f_n(\xi)|^2 \leqslant c^{-1} |\widehat{\varphi}(\xi)|^2 \in L_1(\mathbb{R})$. By the Dominated Convergence Theorem, $\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 d\xi = \lim_{n\to\infty} \int_{\mathbb{R}} |f_n(\xi)|^2 d\xi = 2\pi$.

Since $\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 d\xi = 2\pi$ and $[\widehat{\varphi}, \widehat{\varphi}] \leqslant 1$, we have $2\pi = \int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 d\xi = \int_{-\pi}^{\pi} [\widehat{\varphi}, \widehat{\varphi}](\xi) d\xi \leqslant 2\pi$, from which we have $[\widehat{\varphi}, \widehat{\varphi}](\xi) = 1$ a.e. $\xi \in \mathbb{R}$. $\square$

Daubechies Orthogonal Filters

$$a_1^D = \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]},$$

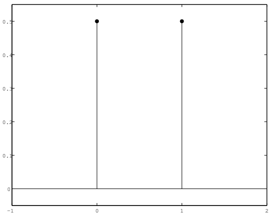
$$a_2^D = \{\tfrac{1+\sqrt{3}}{8}, \underline{\mathbf{\tfrac{3+\sqrt{3}}{8}}}, \tfrac{3-\sqrt{3}}{8}, \tfrac{1-\sqrt{3}}{8}\}_{[-1,2]}$$

$$a_3^D = \{\tfrac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{32}, \tfrac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{32}, \underline{\mathbf{\tfrac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}}}, \tfrac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{32}, \tfrac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{32}\}_{[-2,3]},$$

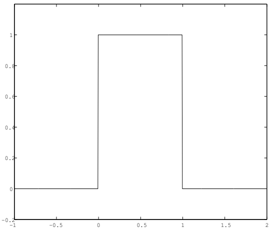
$$a_4^D = \{-0.0535744507091, -0.0209554825625, 0.351869534328, \underline{\mathbf{0.568329121704}}, 0.210617267102, -0.0701588120893, -0.00891235072084, 0.0227851729480\}_{[-3,4]}.$$

m	1	2	3	4	5	6
$\mathrm{sm}(a_m^D)$	0.5	1.0	1.415037	1.775565	2.096787	2.388374

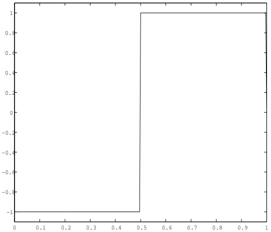
Daubechies Orthogonal Wavelets



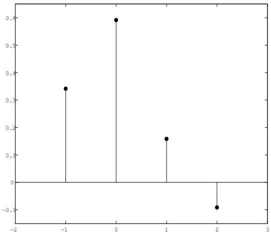
(a) Filter a_1^D



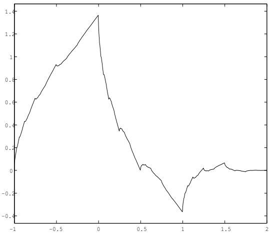
(b) $\phi^{a_1^D}$



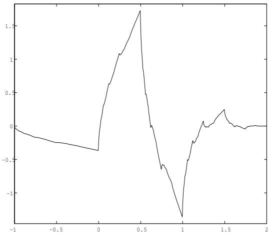
(c) $\psi^{a_1^D}$



(d) Filter a_2^D



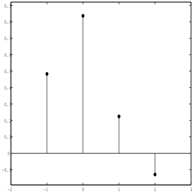
(e) $\phi^{a_2^D}$



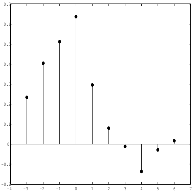
(f) $\psi^{a_2^D}$

An Example: Daubechies Orthogonal Wavelets

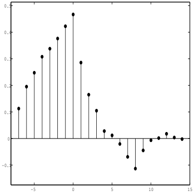
$$a = \left\{\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\right\}, \quad b = \left\{-\frac{1-\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, -\frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8}\right\}.$$



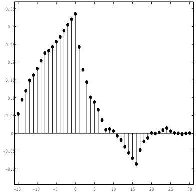
(g) a_1



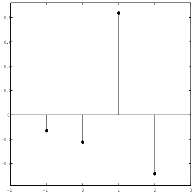
(h) a_2



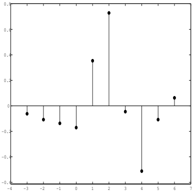
(i) a_3



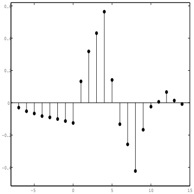
(j) a_4



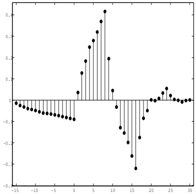
(k) b_1



(l) b_2



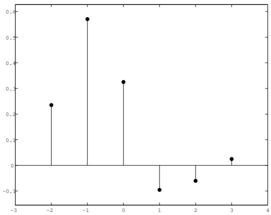
(m) b_3



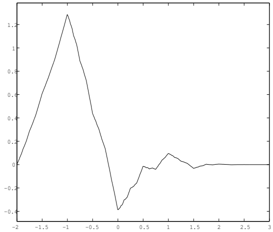
(n) b_4

Figure: $\mathrm{DAS}_J(\{a; b\})$ is an orthonormal basis of $l_2(\mathbb{Z})$ for all $J \in \mathbb{N}$

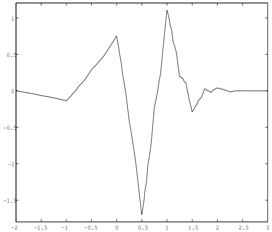
Daubechies Orthogonal Wavelets



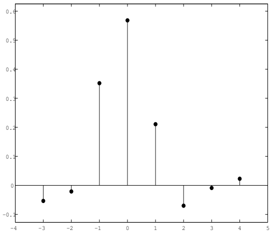
(a) Filter a_3^D



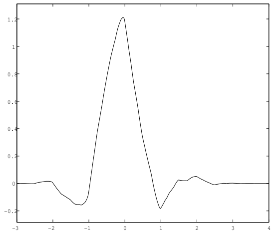
(b) $\phi^{a_3^D}$



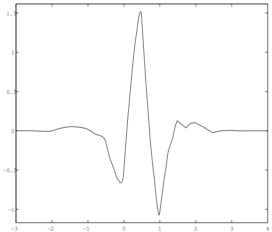
(c) $\psi^{a_3^D}$



(d) Filter a_4^D



(e) $\phi^{a_4^D}$



(f) $\psi^{a_4^D}$

Interpolating Refinable Functions

Theorem

Let $a = a^I_{2m}$ *and define a refinable function by* $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \qquad \xi \in \mathbb{R}.$
Then ϕ *is a compactly supported continuous function and is interpolating:*

$$\phi(k) = \boldsymbol{\delta}(k), \qquad k \in \mathbb{Z}.$$

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Proof. By definition, $\widehat{a^I_{2m}}(\xi) = \widehat{a^D_m}(\xi)\overline{\widehat{a^D_m}(\xi)}$. Define $\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} \widehat{a^D_m}(2^{-j}\xi)$. Then $\widehat{\phi}(\xi) = |\widehat{\varphi}(\xi)|^2$ and we proved $[\widehat{\varphi}, \widehat{\varphi}](\xi) = 1$. Then

$$\int_{\mathbb{R}} |\widehat{\phi}(\xi)| d\xi = \int_{\mathbb{R}} |\widehat{\varphi}(\xi)|^2 d\xi < \infty.$$

This proves that $\widehat{\phi} \in L_1(\mathbb{R})$ and hence ϕ must be continuous. On the other hand, for $k \in \mathbb{Z}$,

$$\phi(k) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}(\xi) e^{ik\xi} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi) e^{-ik\xi}} d\xi = \frac{1}{2\pi} \langle \widehat{\varphi}, \widehat{\varphi(\cdot - k)} \rangle$$
$$= \langle \varphi, \varphi(\cdot - k) \rangle = \boldsymbol{\delta}(k),$$

where we used the Plancherel's identity in the fourth identity. $\square$

Biorthogonal Wavelets in $L_2(\mathbb{R})$

- Let $\phi, \psi \in L_2(\mathbb{R})$ and $\tilde{\phi}, \tilde{\psi} \in L_2(\mathbb{R})$.
- $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$ if
  1. Both $\{\tilde{\phi}; \tilde{\psi}\}$ and $\{\phi; \psi\}$ are Riesz wavelets in $L_2(\mathbb{R})$, i.e.,

     $$C_1 \sum_{h \in \mathrm{AS}(\phi;\psi)} |c_h|^2 \leqslant \Big\| \sum_{h \in \mathrm{AS}(\phi;\psi)} c_h h \Big\|^2_{L_2(\mathbb{R})} \leqslant C_2 \sum_{h \in \mathrm{AS}(\phi;\psi)} |c_h|^2,$$

     where

     $$\mathrm{AS}(\phi;\psi) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi_{2^j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}\}.$$

  2. $\mathrm{AS}(\tilde{\phi}; \tilde{\psi})$ and $\mathrm{AS}(\phi; \psi)$ are biorthogonal to each other:

     $$\langle h, \tilde{h} \rangle = 1 \quad \text{and} \quad \langle h, g \rangle = 0, \quad \forall\, g \in \mathrm{AS}(\phi;\psi)\backslash\{h\}.$$

  3. The linear span of $\mathrm{AS}(\tilde{\phi}; \tilde{\psi})$ is dense in $L_2(\mathbb{R})$. The linear span of $\mathrm{AS}(\phi;\psi)$ is dense in $L_2(\mathbb{R})$.

Characterization of Biorthogonal Wavelets

Theorem

Let $a, b, \tilde{a}, \tilde{b} \in l_0(\mathbb{Z})$ *with* $\widehat{a}(0) = \widehat{\tilde{a}}(0) = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \qquad \widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2),$$

$$\widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\tilde{a}}(2^{-j}\xi), \qquad \widehat{\tilde{\psi}}(\xi) := \widehat{\tilde{b}}(\xi/2)\widehat{\tilde{\phi}}(\xi/2).$$

Then $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ *is a biorthogonal wavelet in* $L_2(\mathbb{R})$ *if and only if* $\mathrm{sm}_2(a) > 0$, $\mathrm{sm}_2(\tilde{\phi}) > 0$, *and* $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ *is a biorthogonal wavelet filter bank:*

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2.$$

Construction of Biorthogonal Wavelet Filter Bank

Proposition

Let $a, b, \tilde{a}, \tilde{b} \in l_0(\mathbb{Z})$. Then $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ is a biorthogonal wavelet filter bank:

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2$$

if and only if $(\tilde{a}, a)$ is a biorthogonal low-pass filter:

$$\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi)} + \widehat{\tilde{a}}(\xi+\pi)\overline{\widehat{a}(\xi+\pi)} = 1$$

and there exist $c \neq 0$ and $n, \tilde{n} \in \mathbb{Z}$ such that

$$\widehat{\tilde{b}}(\xi) = c e^{i(2n-1)\xi} \overline{\widehat{a}(\xi+\pi)}, \qquad \widehat{b}(\xi) = \overline{c^{-1}} e^{i(2\tilde{n}-1)\xi} \overline{\widehat{\tilde{a}}(\xi+\pi)}.$$

- We often take

$$\widehat{\tilde{b}}(\xi) = e^{-i\xi} \overline{\widehat{a}(\xi+\pi)}, \qquad \widehat{b}(\xi) = e^{-i\xi} \overline{\widehat{\tilde{a}}(\xi+\pi)}.$$

- $(\tilde{a}, a)$ is a biorthogonal low-pass filter if and only if $c := a^\star * \tilde{a}$ is an interpolatory mask.

Example of Biorthogonal Wavelets

We can obtain a pair of biorthogonal wavelet filters by splitting the interplatory filter

$$\overline{\widehat{\tilde{a}_m}(\xi)}\widehat{a_m}(\xi) := \widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathsf{P}_{m,m}(\sin^2(\xi/2))$$

as follows: $\mathsf{P}(x)\tilde{\mathsf{P}}(x) = \mathsf{P}_{m,m}(x)$ and

$$\widehat{a_m}(\xi) = 2^{-m}(1+e^{-i\xi})^m\mathsf{P}(\sin^2(\xi/2)), \qquad \widehat{b_m}(\xi) := e^{-i\xi}\overline{\widehat{\tilde{a}_m}(\xi+\pi)},$$

$$\widehat{\tilde{a}_m}(\xi) = 2^{-m}(1+e^{-i\xi})^m\tilde{\mathsf{P}}(\sin^2(\xi/2)), \qquad \widehat{\tilde{b}_m}(\xi) := e^{-i\xi}\overline{\widehat{a_m}(\xi+\pi)}.$$

For $m = 2$, we have the LeGall biorthogonal wavelet filter bank:

$$a_2 = \{\tfrac{1}{4}, \underline{\boldsymbol{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}$$

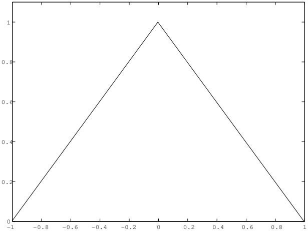
and

$$\tilde{a}_2 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\boldsymbol{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}.$$

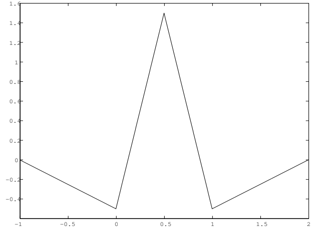
Note that

$$\mathrm{sm}(a_2) = 1.5, \qquad \mathrm{sm}(\tilde{a}_2) = 0.440765.$$

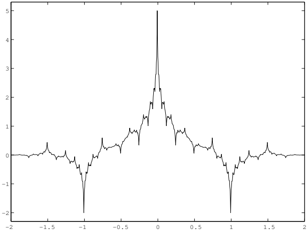
Examples: LeGall Biorthogonal Wavelet



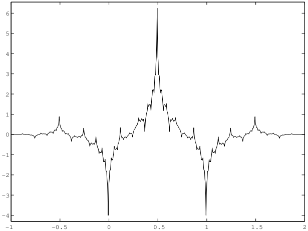
(g) ϕ^{a_2}



(h) ψ^{a_2,b_2}



(i) $\phi^{\tilde{a}_2}$



(j) $\psi^{\tilde{a}_2,\tilde{b}_2}$

The Most Famous Biorthogonal Wavelet

For $m = 4$,

$$a_4 = \left\{-\frac{t}{64}, \frac{2-t}{32}, \frac{16+t}{64}, \underline{\mathbf{\frac{6+t}{16}}}, \frac{16+t}{64}, \frac{2-t}{32}, -\frac{t}{64}\right\}_{[-3,3]},$$

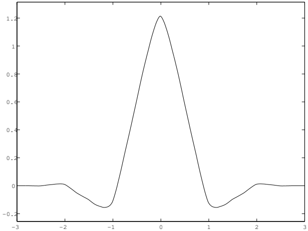
$$\tilde{a}_4 = \left\{\frac{t^2-4t+10}{256}, \frac{t-4}{64}, \frac{-t^2+6t-14}{64}, \frac{20-t}{64}, \underline{\mathbf{\frac{3t^2-20t+110}{128}}}, \frac{20-t}{64},\right.$$
$$\left.\frac{-t^2+6t-14}{64}, \frac{t-4}{64}, \frac{t^2-4t+10}{256}\right\}_{[-4,4]},$$

where $t \approx 2.92069$. The derived biorthogonal wavelet is called Daubechies 7/9 filter and has very impressive performance in many applications.

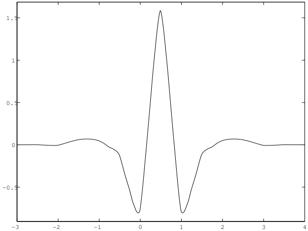
Note that

$$\mathrm{sm}(a) \approx 2.122644, \qquad \mathrm{sm}(\tilde{a}) \approx 1.409968.$$

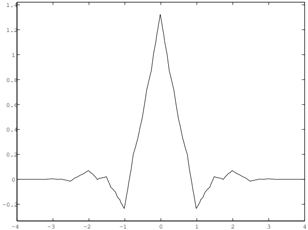
Example: Daubechies 7/9 Biorthogonal Wavelets



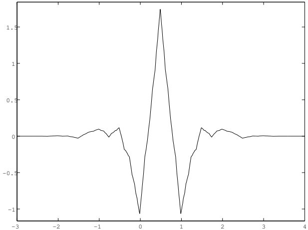
(k) ϕ^{a_4}



(l) ψ^{a_4,b_4}



(m) $\phi^{\tilde{a}_4}$



(n) $\psi^{\tilde{a}_4,\tilde{b}_4}$

Tight Framelet Filter Bank

- The definition of a tight framelet filter bank $\{a; b_1, \ldots, b_s\}$ can be given in the matrix form: $A(\xi)\overline{A(\xi)}^{\mathsf{T}} = I_2$, where
$$A(\xi) := \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}$$
- If $s = 1$, $\{a; b_1\}$ is called an orthogonal wavelet filter bank:
$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2$$
and
$$\overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix} = I_2.$$
- A tight framelet filter bank can have $s \geqslant 1$ high-pass filters.

Matrix Form of Fejér-Riesz Lemma

Theorem

Let $U(\xi)$ be an $r \times r$ matrix of 2π-periodic trigonometric polynomials such that $U(\xi) \geqslant 0$, i.e., $\overline{U(\xi)}^{\mathsf{T}} = U(\xi)$ and $\overline{x}^{\mathsf{T}} U(\xi) x \geqslant 0$ for all $x \in \mathbb{C}^r$ and $\xi \in \mathbb{R}$. Then there exists an $r \times r$ matrix $V(\xi)$ of 2π-periodic trigonometric polynomials such that

$$V(\xi)\overline{V(\xi)}^{\mathsf{T}} = U(\xi).$$

- Note that $U(\xi) \geqslant 0$ particularly implies $\det(U(\xi)) \geqslant 0$.
- The choice of $V(\xi)$ is not unique.
- Effective algorithms exist and are still under development.

Construction of Tight Framelet Filter Banks

- If $\{a; b_1, \dots, b_s\}$ is a tight framelet filter bank, then
$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \leqslant 1, \quad \forall \xi \in \mathbb{R}.$$
- Conversely, if a satisfies the above inequality, one can obtain through Matrix Form of Fejér-Riesz lemma a tight framelet filter bank $\{a; b_1, b_2\}$.
- Recall that $a \in l_0(\mathbb{Z})$ is called an orthogonal low-pass filter if $|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 = 1$. Hence, the requirement for constructing a tight framelet filter bank is much weaker than for constructing an orthogonal wavelet filter bank.
- For example, all B-spline filters a_m^B and all interpolatory filters a_{2m}^I satisfy this condition.

Example from a_2^B

Let

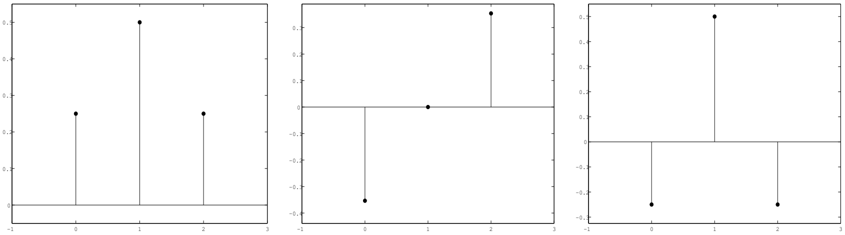
$$a_2^B = \{\underline{\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]}$$

be the B-spline filter of order 2. Let

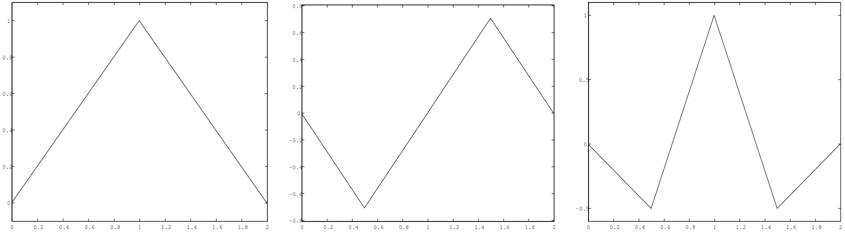
$$b_1 = \{\underline{-\mathbf{\tfrac{\sqrt{2}}{4}}}, 0, \tfrac{\sqrt{2}}{4}\}_{[0,2]},$$
$$b_2 = \{\underline{-\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, -\tfrac{1}{4}\}_{[0,2]}.$$

Then $\{a_2^B; b_1, b_2\}$ is a tight framelet filter bank such that a_2^B has order 2 sum rules and both b_1, b_2 have 1 vanishing moments.

Tight Framelet from B_2



(a) Filter a_2^B (b) Filter b_1 (c) Filter b_2



(d) B_2 (e) ψ^1 (f) ψ^2

Example from B_3

Let

$$a_3^B = \{\underline{\mathbf{\tfrac{1}{8}}}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]}$$

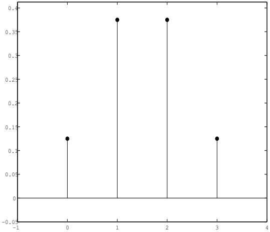
be the B-spline filter of order 3. Let

$$b_1 = \tfrac{\sqrt{3}}{4}\{\underline{-\mathbf{1}}, 1\}_{[0,1]},$$

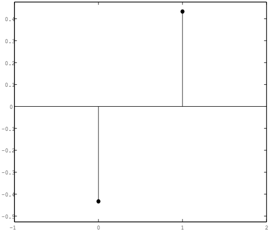
$$b_2 = \{\underline{-\mathbf{\tfrac{1}{8}}}, -\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]}$$

Then $\{a; b_1, b_2\}$ is a tight framelet filter bank such that a_2^B has order 3 sum rules and both b_1, b_2 have 1 vanishing moments.

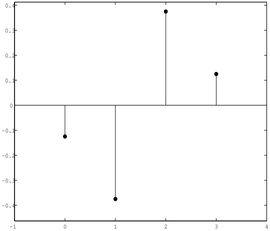
Tight Framelet from B_3



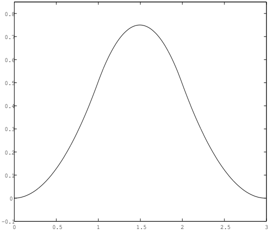
(a) Filter a_3^B



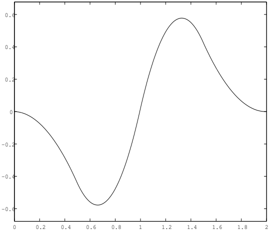
(b) Filter b_1



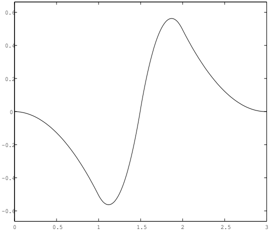
(c) Filter b_2



(d) B_3



(e) ψ^1



(f) ψ^2

Example from a_4^I

For a filter $a \in l_0(\mathbb{Z})$, its z-transform is defined to be

$$\mathrm{a}(z) := \sum_{k\in\mathbb{Z}} a(k) z^k, \qquad z \in \mathbb{C}\backslash\{0\}.$$

Let

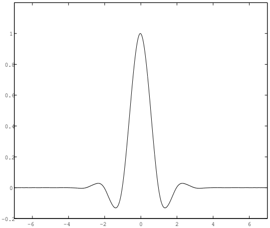
$$a_4^I = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\boldsymbol{\tfrac{1}{2}}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]}$$

be the interpolatory filter. Let

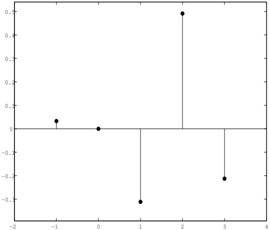
$$\begin{aligned} \mathrm{b}_1(z) &= \tfrac{\sqrt{2}}{8\sqrt{9-4\sqrt{3}}} z^2(1-z^{-1})^2(z^{-1}-\sqrt{3})(z+2-\sqrt{3}), \\ \mathrm{b}_2(z) &= \tfrac{2\sqrt{3}+1}{352\sqrt{9-4\sqrt{3}}}(1-z^{-1})^2(x+2-\sqrt{3})[(1-2\sqrt{3})z^{-1} \\ &\qquad + (6-\sqrt{3}) + 33\,z + 11\sqrt{3}z^2], \end{aligned}$$

where $\mathrm{b}(z) := \sum_{k\in\mathbb{Z}} b(k) z^k$. Then $\{a; b_1, b_2\}$ is a tight framelet filter bank such that a_4^I has order 4 sum rules and both b_1, b_2 have 2 vanishing moments.

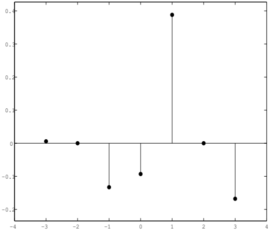
Tight Framelet from a_4^I



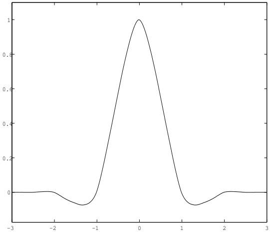
(a) Filter a_4^I



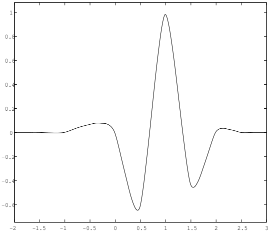
(b) Filter b_1



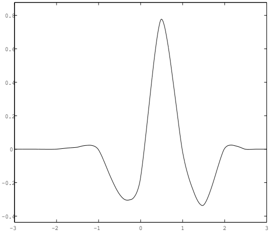
(c) Filter b_2



(d) $\phi^{a_4^I}$



(e) ψ^1



(f) ψ^2

Example from a_4^I

Let

$$a_4^I = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\boldsymbol{\tfrac{1}{2}}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]}$$

be the interpolatory filter. Let

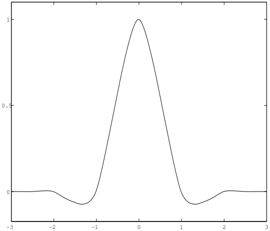
$$b_1 = \tfrac{1}{32}\{1, 0, -9, \underline{\mathbf{16}}, -9, 0, 1\}_{[-3,3]},$$

$$b_2 = \tfrac{\sqrt{6}}{32}\{-1, 0, 1, \underline{\mathbf{0}}, 1, 0, -1\}_{[-3,3]},$$

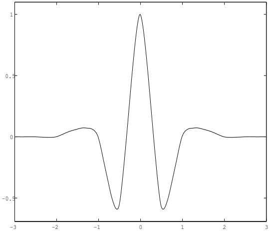
$$b_3 = \tfrac{\sqrt{2}}{16}\{-1, 0, 3, \underline{\mathbf{0}}, -3, 0, 1\}_{[-3,3]}.$$

Then $\{a; b_1, b_2, b_3\}$ is a real-valued interpolatory tight framelet filter bank such that a_4^I has order 4 sum rules and both b_1, b_2, b_3 have vanishing moments $4, 2, 3$, respectively.

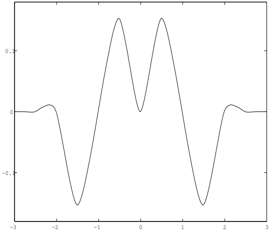
Tight Framelet from a_4^I



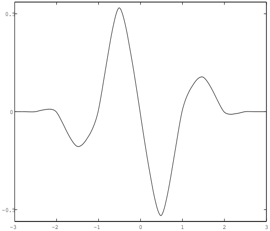
(a) ϕ



(b) ψ^1



(c) ψ^2



(d) ψ^3