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Project 5: Relative State Estimation for Two Spacecraft in LEO using Kalman Filtering

1. Introduction and Problem Statement

This project investigates relative navigation for a deputy spacecraft in low Earth orbit as a benchmark for comparing truth model propagation with state estimation using a discrete Kalman filter. The chief follows a near circular reference orbit at a prescribed altitude, while the deputy starts from a known relative offset in the local vertical local horizontal frame and evolves according to the linearized Hill Clohessy Wiltshire (HCW) equations. The continuous dynamics are discretized with a one second sample time to construct the state transition and process noise models used in the filter. At each step a synthetic relative position measurement in the local frame, corrupted by zero mean Gaussian noise, is supplied to the estimator, which performs the standard predict and update cycle. The objective is to quantify estimation accuracy and covariance consistency over multiple orbits using position and velocity error histories and innovation statistics as primary performance metrics.

2. Governing Equations

Relative motion is modeled in the LVHF using the HCW equations. The chief spacecraft is assumed to follow a circular orbit with semimajor axis (a) and gravitational parameter (μ), which defines mean motion:

$$n = \sqrt{\frac{\mu}{a^3}}$$

The deputy state is written as:

$$\boldsymbol{x} = [x\ y\ z\ \dot{x}\ \dot{y}\ \dot{z}]^T$$

Where x, y, and x are the radial, along track, and cross track offsets from the chief in the LVLH frame, and $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the corresponding velocities.
The continuous dynamics satisfy:

$$\ddot{x} = 3n^2x + 2n\dot{y}, \qquad \ddot{y} = -2n\dot{x}, \qquad \ddot{z} = -n^2 z$$

These equations capture the coupling between radial and the along track motion created by orbital curvature, while the cross-track motion behaves as a simple harmonic oscillator with natural frequency *n*.
For Kalman filtering it is convenient to express the system the system in first order state space form:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{w}(t)$$

With:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^3 & 0 & 0 & 0 \end{bmatrix}$$

Where w(t) represents the unmodeled accelerations and processes disturbances. The process noise term accounts for effects such as small mismatches in the chief orbit and neglected perturbations that are not included in the linear model.
Measurements are modeled as noisy observations of the deputy position in LVLH frame:

$$\boldsymbol{y}(t) = \mathrm{H}\boldsymbol{x}(t) + \boldsymbol{v}(t)$$

with,

$$\mathrm{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

And ***v(t)*** a zero mean Gaussian measurement noise process with covariance ***R***. This structure isolates the physics of relative motion and sensing, which are later discretized and embedded in the Kalman filter implementation.

3. Numerical Method
The continuous LVLH model and measurement equations from the governing equation section were implemented in discrete time on a uniform grid $t_k = k\Delta t$ with $\Delta t = 1\mathrm{s}$. The discrete transition matrix was computed in MATLAB as: $\phi = \exp(A\Delta t)$ and the same ϕ was used to propagate both the truth model and Kalman prediction so that discretization does not introduce additional mismatch. Process noise was modeled as small white accelerations in each axis with standard deviation $\sigma_a = 2.0 \times 10^{-6}\ \mathrm{km/s^2}$; the corresponding discrete covariance was taken diagonal:

$$Q = \mathrm{diag}(q_x, q_y, q_z, q_{\dot{x}}, q_{\dot{y}}, q_{\dot{z}})$$

with larger entries in the velocity rows so that uncertainty growth is mainly driven by the unmodeled accelerations rather than artificial jumps in position.
Synthetic measurements were generated by sampling the true LVLH position each step and adding Gaussian noise:

$$y_k = \mathrm{H}x_k^{\mathrm{true}} + v_k$$

chosen to mimic flash LIDAR with 7cm range accuracy at 1hz. The noise covariance was modeled as: $R = \mathrm{diag}(\sigma_x^2, \sigma_y^2, \sigma_z^2)$. With σ equal to the range standard deviation. The discrete Kalman filter was coded explicitly rather than using a built-in routine: each iteration applied the standard predict equations for $\hat{x}$ and P, formed the innovation and its covariance, computed the Kalman gain, and then applied the correction step. The initial $\hat{x}_0$ and P_0 were chosen to contain noticeable errors in both position and velocity, which makes the convergence phase visible in the results while keeping the filter numerically stable over the full simulation.

4. Results and Discussion

The figure below shows the true and estimated relative position and velocity in each LVLH axis together with the three sigma bounds over two chief orbits. The deputy starts tens of kilometers from the chief with a small position error but a large initial velocity error in the estimate. Given the measurement and process noise specified above, the filter quickly removes the position bias and locks onto the periodic Hill motion. After the initial transient the estimated position curves lie essentially on top of the truth, and the reported three-dimensional position RMSE of a few centimeters is consistent with the narrow three sigma envelopes.

Velocity converges more slowly and exhibits larger residual error. Since the sensor only provides position information, the filter must infer velocity through the dynamics and the process noise model, so the large initial velocity error persists over a longer portion of the run. The acceleration noise level continually injects small disturbances that keep the covariance from collapsing and produce the small oscillations visible in the velocity error. As a result, the final three-dimensional velocity RMSE remains on the order of meters per second, while the three sigma bounds shrink to values that still cover the residual errors. Together, the plots and RMSE values indicate that the chosen Q and R yield an estimator that is accurate, statistically consistent, and robust to the modest unmodeled perturbations represented in the process noise.