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EML 6415 Project 2: Discrete and Continuous Optimal Control of Satellite Relative Motion

1. Introduction and Problem Statement

This project investigates optimal control of relative satellite motion using Clohessy-Wiltshire (CW) equations, which describe linearized dynamics of a deputy satellite orbiting near a chief in circular motion. Two methods considered are fuel-optimal impulsive maneuvers – four discrete burns equally spaced over one orbital period and continuous feedback control – a Linear Quadratic Regular (LQR) formulation. The objective is to minimize the total control effort required to transfer the deputy from its initial state to a desired terminal configuration over one full revolution.

2. Governing Equations

The **Clohessy-Wiltshire (CW)** in a nondimensionalized form where 2π time units correspond to one orbital period; the linearized equations of motion are:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 3\mathrm{x} + 2\dot{y} \\ -2\dot{x} \\ -z \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

Where x, y, z represents the relative position components of the deputy spacecraft, and u_x, u_{y}, u_{z} are the applied control accelerations.

To express this as a first-order state-space system, the state vector is defined as:

$$\boldsymbol{X} = [\,\boldsymbol{x}\ \boldsymbol{y}\ \boldsymbol{z}\ \dot{\boldsymbol{x}}\ \dot{\boldsymbol{y}}\ \dot{\boldsymbol{z}}\,]^{\boldsymbol{T}}$$

yielding the linear system:

$$\boldsymbol{X} = \boldsymbol{A}\dot{\boldsymbol{X}} + \boldsymbol{B}\boldsymbol{U},\ \ \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},\quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The solution of this linear time-invariant system over an interval $\Delta\theta$ can be written using the state transition matrix:

$$\mathbf{X}(\boldsymbol{\theta} + \boldsymbol{\Delta\theta}) = \boldsymbol{\Phi}(\boldsymbol{\Delta\theta})\mathbf{X}(\boldsymbol{\theta})$$

where:

$$\boldsymbol{\Phi}(\theta) = \begin{pmatrix} 4-3\cos\theta & 0 & 0 & \sin\theta & 2(1-\cos\theta) & 0 \\ -6(\theta-\sin\theta) & 1 & 0 & -2(1-\cos\theta) & 4\sin\theta-3\theta & 0 \\ 0 & 0 & \cos\theta & 0 & 0 & \sin\theta \\ 3\sin\theta & 0 & 0 & \cos\theta & 2\sin\theta & 0 \\ -6(1-\cos\theta) & 0 & 0 & -2\sin\theta & -3+4\cos\theta & 0 \\ 0 & 0 & -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$$

3. Numerical Method

The system was implemented in MATLAB, solving a nonlinear optimization problem using *fmincon* function to minimize total fuel consumption. The control sequence consisted of four impulsive burns equally spaced at intervals of $\Delta\theta = \pi/2$.

The optimization variable vector included the thrust components for each impulse along with an undetermined phase angle α, defining the terminal configuration. The cost function was defined as:

$$J = \sum_{i=1}^{n_{imp}} ||\Delta V_i||$$

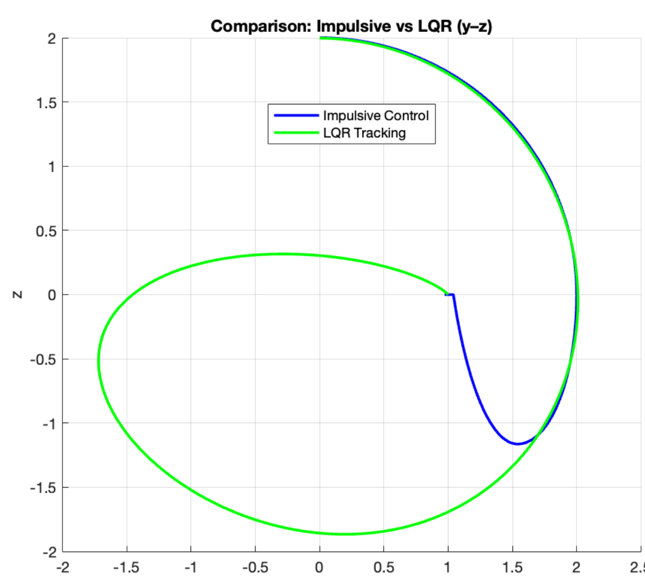
For continuous case, the LQR tracking formulation was applied:

$$J = \int_0^{2\pi} (e^T Q e + u^T R u) d\theta$$

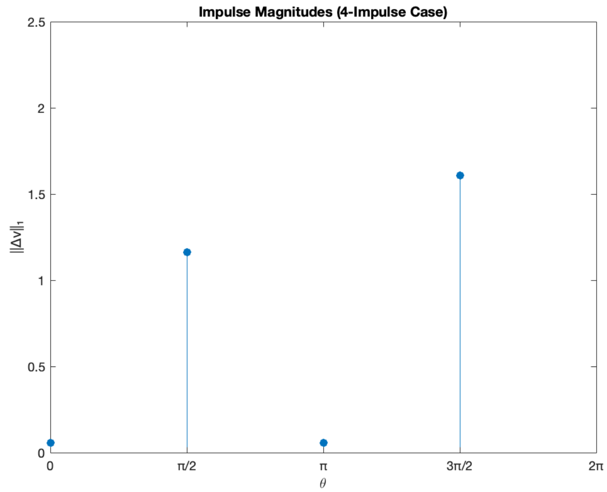
Where $e = X - X_{ref}$ is the tracking error. This cost penalizes both deviations from the desired CW trajectory and excessive control effort, balancing accuracy, and energy use. The optimal gain matrix K was obtained using MATLAB's *lqr* command, and the closed-loop error dynamics was integrated with *ode45*.

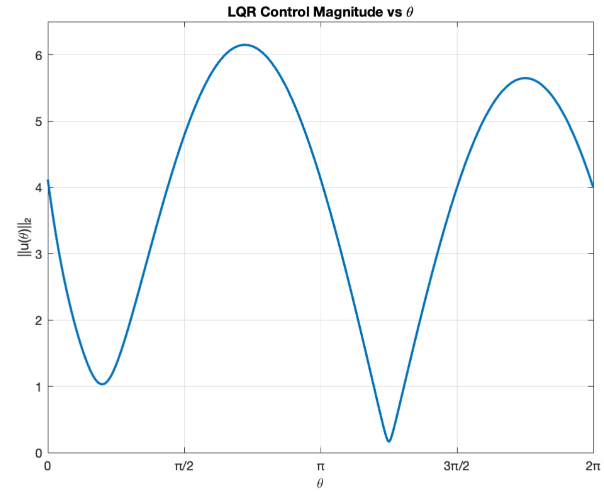
4. Results and Discussion

The comparison of orbits demonstrates that the LQR formulation yields a dynamically smoother and more stable transfer compared to the segmented impulsive trajectory. The weighting matrices Q and R were selected to balance state-tracking accuracy with control energy, producing a continuous thrust profile that regulates deviations throughout the orbit. Unlike the impulsive solution, which concentrates control authority into a few high-efficiency burns, the LQR controller distributes smaller accelerations over the entire period to maintain phase and minimize transient error. Both methods converge to the same final state, but the continuous LQR path exhibits superior transient smoothness and



robustness to modeling perturbations, consistent with the energy-optimal nature of feedback control.





Parameter	Impulsive (4 burns)	LQR Continuous
Total Impulse ($\sum\|\|\Delta v\|\|$)	2.5882	-
Continuous Thrust ($\int\|\|u(\theta)\|\|$	-	27.61
Quadratic Cost (J)	-	279.12

The performance comparison highlights the difference between impulsive and continuous optimal control. The impulsive solution minimizes fuel directly through high efficiency thrust events, where each impulse represents an instantaneous velocity change $||\Delta v_i||$. In contrast, the LQR controller minimizes a quadratic cost based on control energy rather than fuel, producing a continuous acceleration profile $||u(\theta)||$ that remains active throughout the orbit. Although each instantaneous thrust is small, integrating the control magnitude over time yields a larger accumulated control effort or the continuous analogue of total Δv. The higher total impulse therefore reflects the LQR controller's energy-optimal nature favoring stability and smooth regulation over strict fuel minimization whereas the impulsive approach is explicitly fuel-optimal but dynamically less adaptive.

5. Conclusion

The results demonstrate the inherent tradeoffs between impulsive and continuous optimal control strategies for relative orbital motion. The impulsive solution is fuel-optimal and achieves the transfer efficiently through discrete maneuvers, while the continuous LQR controller provides a smoother, dynamically stable trajectory with improved regulation characteristics. Although the LQR approach incurs higher integrated control effort, its continuous feedback structure offers greater robustness to disturbances and modeling errors qualities desirable in autonomous spacecraft guidance and formation control applications.