Proof of the Uniform Boundedness Principle

Let X be a Banach space, Y a normed space, and {T_n: X→Y} a family of bounded linear operators. Suppose:

∀x ∈ X, sup_n ■T_n x■ < ∞.

We show sup_n ■T_n■ < ∞.

1. For each m ∈ ■, define:
   E_m = { x ∈ X : sup_n ■T_n x■ ≤ m }.
2. Each E_m is closed (by continuity of T_n and taking supremum).
3. ■_{m=1}^∞ E_m = X, so by Baire Category, some E_M has nonempty interior.
4. ∃ ball B(x■, r) ⊆ E_M ⇒ ∀■h■ ≤ r, sup_n ■T_n (x■ + h)■ ≤ M.
5. Then for any y ∈ X, write y = x■ + h scaled: T_n y = T_n(x■ + h) – T_n(x■).
   ⇒ ■T_n y■ ≤ 2M·(■y■/r + 1).
6. Hence sup_n ■T_n■ < ∞.

✎ See Rudin, Functional Analysis, Theorem 1.14.