Supplementary Material

Proof Lemma 4

We derive two inequalities in terms of NE-seeking error and compression error, respectively.
NE-seeking error:
Based on Proposition 1 in [1], we conclude that $\mathbf{X}^\star = \mathbf{X}^\star - \gamma \mathbf{F}_a(\mathbf{X}^\star), \forall \gamma > 0$. Thus, we can obtain

$$\begin{aligned}
&||\mathbf{X}^{k+1} - \mathbf{X}^\star||_{\mathrm{F}}^2 \\
&= ||\mathbf{X}^k - \gamma \mathbf{F}_a(\mathbf{X}^k) + \gamma (I - W)\mathbf{E}^k + \boldsymbol{\xi}^k - \mathbf{X}^\star + \gamma \mathbf{F}_a(\mathbf{X}^\star)||_{\mathrm{F}}^2 \\
&\leq c_1 ||\mathbf{X}^k - \gamma \mathbf{F}_a(\mathbf{X}^k) - \mathbf{X}^\star + \gamma \mathbf{F}_a(\mathbf{X}^\star)||_{\mathrm{F}}^2 \\
&+ \frac{c_1}{c_1 - 1}(2\gamma^2 ||I - W||_{\mathrm{F}}^2 ||\mathbf{E}^k||_{\mathrm{F}}^2 + 2||\boldsymbol{\xi}||_{\mathrm{F}}^2),
\end{aligned} \tag{1}$$

where the last equality comes from Lemma 5 in [2] with $c_1 > 1$.
Next, we bound

$$\begin{aligned}
&||\mathbf{X}^k - \gamma \mathbf{F}_a(\mathbf{X}^k) - \mathbf{X}^\star + \gamma \mathbf{F}_a(\mathbf{X}^\star)||_{\mathrm{F}}^2 \\
&= ||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2 + \gamma^2 ||\mathbf{F}_a(\mathbf{X}^k) - \mathbf{F}_a(\mathbf{X}^\star)||_{\mathrm{F}}^2 \\
&- 2\gamma \langle \mathbf{F}_a(\mathbf{X}^k) - \mathbf{F}_a(\mathbf{X}^\star), \mathbf{X}^k - \mathbf{X}^\star \rangle \\
&\leq (1 + \gamma^2 L_F^2 - 2\gamma \mu_F) ||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2,
\end{aligned} \tag{2}$$

where the last inequality is based on Lemma 1 and Lemma 2 in the manuscript.
From (2), we have $\min\{1 + \gamma^2 L_F^2 - 2\gamma \mu_F\} = (1 - \mu_F^2 / L_F^2) > 0$. Combining (1) and (2) after taking taking $c_1 = \frac{2L_F^2 - \mu_F^2}{2L_F^2 - 2\mu_F^2}$, we have

$$\begin{aligned}
||\mathbf{X}^{k+1} - \mathbf{X}^\star||_{\mathrm{F}}^2 \leq\ & c_1 (1 + \gamma^2 L_F^2 - 2\gamma \mu_F) ||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2 \\
& + \frac{2c_1 \gamma^2 ||I - W||_{\mathrm{F}}^2}{c_1 - 1} ||\mathbf{E}^k||_{\mathrm{F}}^2 + \frac{2c_1}{c_1 - 1} ||\boldsymbol{\xi}^k||_{\mathrm{F}}^2,
\end{aligned} \tag{3}$$

Then, from Lemma 3, we can obtain

$$\begin{aligned}
&\mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{X}^\star||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq c_1 (1 + \gamma^2 L_F^2 - 2\gamma \mu_F) \mathbb{E}[||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&+ c_2 \gamma^2 \mathbb{E}[||\tilde{\mathbf{X}}^k - \mathbf{H}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] + c_3 \bar{\theta}^2,
\end{aligned} \tag{4}$$

where $c_2 = \frac{4c_1 ||I - W||_{\mathrm{F}}^2 C}{c_1 - 1}$, $c_3 = \frac{8c_1 ||I - W||_{\mathrm{F}}^2 n^2}{c_1 - 1} + \frac{4c_1 n^2}{c_1 - 1}$.
Compression error:

Denote $\mathcal{C}_r(\mathbf{X}^k) = \mathcal{C}(\mathbf{X}^k)/r$, according to (7e) in the manuscript, for $0 < \alpha \leq \frac{1}{r}$,

$$
\begin{aligned}
&||\tilde{\mathbf{X}}^{k+1} - \mathbf{H}^{k+1}||_{\mathrm{F}}^2 \\
=&||\tilde{\mathbf{X}}^{k+1} - \tilde{\mathbf{X}}^k + \tilde{\mathbf{X}}^k - \mathbf{H}^k - \alpha r \frac{\mathbf{Q}^k}{r}||_{\mathrm{F}}^2 \\
=&||\tilde{\mathbf{X}}^{k+1} - \tilde{\mathbf{X}}^k + \alpha r(\tilde{\mathbf{X}}^k - \mathbf{H}^k - \mathcal{C}_r(\tilde{\mathbf{X}}^k - \mathbf{H}^k))||_{\mathrm{F}}^2 \\
\leq&\tau_1 \Big[\alpha r||\tilde{\mathbf{X}}^k - \mathbf{H}^k - \mathcal{C}_r(\tilde{\mathbf{X}}^k - \mathbf{H}^k)||_{\mathrm{F}}^2 \\
&+ (1-\alpha r)||\tilde{\mathbf{X}}^k - \mathbf{H}^k||_{\mathrm{F}}^2\Big] + \frac{\tau_1}{\tau_1 - 1}||\tilde{\mathbf{X}}^{k+1} - \tilde{\mathbf{X}}^k||_{\mathrm{F}}^2,
\end{aligned} \tag{5}
$$

where in the first inequality we use the result of Lemma 5 in [2] with $\tau_1 = 1 + \alpha r \delta$.
Taking conditional expectation on both sides of (5), we obtain

$$
\begin{aligned}
&\mathbb{E}[||\tilde{\mathbf{X}}^{k+1} - \mathbf{H}^{k+1}||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq \tau_1[\alpha r(1-\delta) + (1-\alpha r)]\mathbb{E}[||\tilde{\mathbf{X}}^k - \mathbf{H}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&+ \frac{\tau_1}{\tau_1 - 1}\mathbb{E}[||\tilde{\mathbf{X}}^{k+1} - \tilde{\mathbf{X}}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k],
\end{aligned} \tag{6}
$$

where the inequality holds based on Assumption 2.
Moreover, we have

$$
\begin{aligned}
&\mathbb{E}[||\tilde{\mathbf{X}}^{k+1} - \tilde{\mathbf{X}}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&= \mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{X}^k - (\boldsymbol{\xi}^{k+1} - \boldsymbol{\xi}^k)||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq 2\mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{X}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] + 2\mathbb{E}[||\boldsymbol{\xi}^{k+1} - \boldsymbol{\xi}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq 2\mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{X}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] + 8n^2\bar{\theta}^2.
\end{aligned} \tag{7}
$$

Next, we bound $\mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{X}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k]$.

$$
\begin{aligned}
&\mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{X}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&= \mathbb{E}[||\gamma(I-W)\mathbf{E}^k - \gamma(\mathbf{F}_a(\mathbf{X}^k) - \mathbf{F}_a(\mathbf{X}^\star)) + \boldsymbol{\xi}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq 3\gamma^2||(I-W)||_{\mathrm{F}}^2\mathbb{E}[||\mathbf{E}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&+ 3\gamma^2 L_F^2\mathbb{E}[||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2 \mid \mathcal{F}^k] + 3\mathbb{E}[||\boldsymbol{\xi}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq 6C\gamma^2||(I-W)||_{\mathrm{F}}^2\mathbb{E}[||\tilde{\mathbf{X}}^k - \mathbf{H}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&+ 3\gamma^2 L_F^2\mathbb{E}[||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2 \mid \mathcal{F}^k] + (12\gamma^2||(I-W)||_{\mathrm{F}}^2 + 6)n^2\bar{\theta}^2.
\end{aligned} \tag{8}
$$

Bringing (7) and (8) into (6) and denoting $c_4 = \frac{6(1+\alpha r\delta)}{\alpha r\delta} > 1, c_x = \tau_1[\alpha r(1-\delta) + (1-\alpha r)] = 1 - \alpha^2 r^2 \delta^2 < 1$, we have

$$
\begin{aligned}
&\mathbb{E}[||\mathbf{X}^{k+1} - \mathbf{H}^{k+1}||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&\leq c_4\gamma^2 L_F^2\mathbb{E}[||\mathbf{X}^k - \mathbf{X}^\star||_{\mathrm{F}}^2 \mid \mathcal{F}^k] \\
&+ (c_x + c_5\gamma^2)\mathbb{E}[||\tilde{\mathbf{X}}^k - \mathbf{H}^k||_{\mathrm{F}}^2 \mid \mathcal{F}^k] + c_6\bar{\theta}^2,
\end{aligned} \tag{9}
$$

where $c_5 = 2c_4C||(I-W)||_{\mathrm{F}}^2$ and $c_6 = (2c_4 + 6)n^2$.

References

[1] T. Tatarenko, W. Shi, and A. Nedić, "Geometric convergence of gradient play algorithms for distributed Nash equilibrium seeking," *IEEE Transactions on Automatic Control*, vol. 66, no. 11, pp. 5342–5353, 2020.

[2] Y. Liao, Z. Li, K. Huang, and S. Pu, "A compressed gradient tracking method for decentralized optimization with linear convergence," *IEEE Transactions on Automatic Control*, vol. 67, no. 10, pp. 5622–5629, 2022.