Topological susceptibility and the η' meson mass

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Abstract

We computed the mass of the η' meson using the Witten Veneziano formula accounting for 3 quark flavors. We simulated an $SU(3)$ Yang-Mills theory with Wilsonian discretization on three lattices with fixed physical volume and progressively finer spacing, from which we estimated the continuum limit of the topological susceptibility χ. We considered a naive discretization to define the topological charge density q^t, which we evolved with the Wilson flow, to compute its cumulants Q^t and $t_0^2\chi^t$ at a reference time t_0. Due to the lack of statistics, our best results were only qualitatively correct.

Contents

1 Introduction

The classical QCD action, in the massless fermions limit, has an accidental global symmetry called chiral simmetry. The $U(1)_L \times U(1)_R$ subgroup is broken to $U(1)_B$[1] as a consequence of its quantization via the path integral. This is called the *chiral anomaly*: it happens because the integration measure is not invariant under the action of the axial group $U(1)_A$ and, ultimately, it is the reason for the unexpectedly large mass of the η' meson $m_{\eta'} = 0.958\,\text{GeV}$, which is 0 in the classical theory.

The Witten-Veneziano formula aims at providing an explaination for the mass of the η' meson by connecting it to non trivial topological fluctuations of the guage fields, encoded by the topological susceptibility χ in pure Yang-Mills theories. The formula takes its simplest form in the chiral limit

$$m_{\eta'}^{chiral} = \sqrt{\frac{2N_f\chi}{F_{\eta'}^2}} \tag{1}$$

where N_f is the flavor number and $F_{\eta'}$ is the leptonic decay constant of the η'. The chiral contribution proves to be the dominant one even in the non chiral limit

$$m_{\eta'} = \sqrt{\frac{2N_f\chi}{F_{\eta'}^2} + 2m_K^2 - m_\eta^2} \tag{2}$$

where m_K and m_η are the masses of the K^0 and η mesons.

The nature of topological susceptibility lies in the fact that in SU invariant theories the allowed field configurations form disjoint clusters characterized by the topological charge Q, which takes integer values and is non zero on average[2]. The topological susceptibility χ is the variance of Q.

Moreover, since $\chi = 0$ both in the classical theory and in perturbative quantization, measurements of $m_{\eta'}$ are a strong confirmation of non perturbative QCD quantization.

This work is heavily based on [Cè+15], both for the theoretical formulation and for the computational implementation. We will consider Wilson discretization to define lattice QCD and, through the Wilson flow definition of the topological charge, we will compute the leading contribution to the topological susceptibility as the continuum limit of the values obtained on three different lattices; then we will use it to compute the η' mass.

[1] L and R stand for left and right, B stands for baryonic.

[2] This is the reason behind the integration measure not being $U(1)_A$ invariant.

2 Elements of theory

2.1 Continuum theory

In a $4D$ pure $SU(N)$ gauge theory, the fundamental gauge fields $A_\mu = A_\mu^a T^a$ evolves via Yang-Mills gradient flow, obeying the equation

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu \tag{3}$$

where $G_{\mu\nu} = \partial_{[\mu} B_{\nu]} - i\left[B_\mu, B_\nu\right]$ is the field strength tensor, $D_\mu = \partial_\mu - i[B_\mu, \cdot]$ is the covariant derivative, $B_\mu \equiv A_\mu\big|_{t=0}$ and α_0 is a gauge fixing parameter. This allows for the definition of the topological charge density at flow time t

$$q^t \equiv \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \operatorname{Tr}\left[G_{\mu\nu} G_{\rho\sigma}\right] \tag{4}$$

whose integral is the topological charge

$$Q^t = \int \mathrm{d}^4x\, q^t(x). \tag{5}$$

For a fixed gauge field configuration, the purely topological nature of Q^t is proven by the relation

$$\partial_t Q^t = 0. \tag{6}$$

It can also be shown that, given a finite local operator $O(y)$ with $x \neq y$, the correlation function

$$\left\langle q^{t=0}(x) O(y) \right\rangle = \lim_{t\to 0} \left\langle q^t(x) O(y) \right\rangle \tag{7}$$

is finite and can therefore be used as a definition of $q^0(x)$ for the renormalized theory.

The cumulants of q^t are defined

$$C_n^t \equiv \int \mathrm{d}^4x_1\, \mathrm{d}^4x_{2n-1} \left\langle q^t(x_1) \dots q^t(x_{2n}) \right\rangle_c \tag{8}$$

and, because of equation 6, we can prove that they are independent of the flow time $t \geq 0$. In particular, we are going to focus on

$$\begin{aligned} \chi^t \equiv C_2^t &= \frac{1}{V} \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \left\langle q^t(x_1) q^t(_2) \right\rangle_c \\ &= \frac{\left\langle \left(Q^t\right)^2 \right\rangle}{V} \end{aligned} \tag{9}$$

where V is the four dimensional volume on which we consider the theory. It can also be shown that χ^t is finite both when $t \to 0$ and when $x \to 0$.

2.2 Lattice theory

An analogous of the YM gradient flow equation 3 can be defined on the lattice

$$\partial_t V_\mu = -g_0^2 \left(\partial_{x,\mu} S[V]\right) V_\mu(x),$$
$$V_\mu(x)\Big|_{t=0} \equiv U_\mu(x) \tag{10}$$

where S is the Wilson gauge action, U_μ are the gauge link operators and $\partial_{x,\mu}$ are the appropriate differential operators. On the lattice there are different definitions of topological charge density, depending on the discretization choice. It is proven that the continuum limit of the topological charge defined from the Neuberger operator[3], at positive flow time, obeys the proper singlet chiral Ward identities to be inserted in the Witten Veneziano relation.

We started with a naive discretization of a lattice with spacing a. The charge density is defined

$$q^t = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[G_{\mu\nu}G_{\rho\sigma}\right](x) \tag{11}$$

where the field strength tensor $G_{\mu\nu}$ is defined in terms of the plaquette $Q_{\mu\nu}$

$$G_{\mu\nu}(x) = -\frac{i}{4a^2}T^a\,\mathrm{tr}_a\left[Q_{[\mu\nu]}(x)T^a\right], \tag{12}$$

$$\begin{aligned} Q_{\mu\nu}(x) &= V_\mu(x)V_\nu(x+a\hat{\mu})V^\dagger(x+a\hat{\nu})V_\nu^\dagger(x)+\\ &V_\nu(x)V_\mu^\dagger(x-a\hat{\mu}+a\hat{\nu})V_\nu^\dagger(x-a\hat{\mu})V_\mu(x-a\hat{\nu})+\\ &V_\mu^\dagger(x-a\hat{\mu})V_\nu^\dagger(x-a\hat{\nu}-a\hat{\mu})V_\mu(x-a\hat{\mu}-a\hat{\nu})\times\\ &\times V_\nu(x-a\hat{\nu})+\\ &V_\nu^\dagger(x-a\hat{\nu})V_\mu(x-a\hat{\nu})V_\nu(x+a\hat{\mu}-a\hat{\nu})V_\mu^\dagger(x); \end{aligned} \tag{13}$$

thus the topological charge is

$$Q^t \equiv a^4\sum_x q^t(x). \tag{14}$$

The topological charge density defined on a naively discretized lattice, when we consider evolved gauge fields, has a well defined finite continuum limit which coincides with the continuum limit of the definition that comes from Neuberger's approach. Its cumulants are

$$C_n^t = a^{8n-4}\sum_{x_1\dots x_{2n-1}}\left\langle q^t(x_1)\dots q^t(x_{2n-1})q^t(0)\right\rangle \tag{15}$$

and, since at positive flow time short-distance singularities cannot arise, the continuum limit[4] equivalence also holds true for the cumulants. Particularly, the continuum limit of the topological susceptibility is

$$\chi^t = a^4\sum_x\left\langle q^t(x)q^t(0)\right\rangle = \frac{\left\langle (Q^t)^2\right\rangle}{V} \tag{16}$$

where V is the volume of the lattice and Q is computed like in equation 14, provides us with the right quantity to compute the η' mass from the Witten-Veneziano relation.

3 Numerical setup

We ran three simulations on different lattices of physical volume $L^4 \approx (1.2\,\mathrm{fm})^4$ and varibale spacing a, imposing periodic boundary conditions on the gauge fields; the different spacings influence the values of the β function and the adimensional reference scale-spacing ratio t_0/a^2 in each run of the simulation. To avoid autocorrelation, the data were saved once every N_{it} configurations. The parameters are summarized in table 1.

	β	t_0/a^2	L/a	a[fm]	N_{conf}	N_{it}
$B1$	5.96	2.7984(9)	12	0.102	1001	20
$B2$	6.05	3.7960(12)	14	0.087	1040	20
$B3$	6.13	4.8855(15)	16	0.068	1001	30

Table 1: Simulation parameters. The β function depends on the running coupling constant $6/g_0^2$; N_{conf} is the number of configurations we used and N_{it} is how many configurations were discarded to avoid autocorrelation.

In order to recover the continuum results, we used

$$t_0 = (0.176(4)\mathrm{fm})^2\,. \tag{17}$$

Simulations for lattices $B1$ and $B3$ were run on parallel using every processor available, while for lattice $B2$ we run 14 different simulations and averaged the results.

4 Results

For each lattice we computed N_{conf} values of the topological charge Q^t at different flow times.

We simulated flow time evolution up a reference scale t_0 corresponding to the closest integer to $100 t_0/a^2$ that also belongs to a *saved* configuration (see table 1). For the $B2$ lattice, the sensible choice would be 380 but the corresponding configuration was not saved by the algorithm, therefore we computed $\langle Q^{t_0}\rangle_{B2}$ as the weigthed average of the values at $t = 360$ and $t = 400$.

[3] $a^4 q_N \equiv -\frac{a}{2(1+s)}\,\mathrm{tr}[\gamma_5 D(x,x)]$, where $s \in (-1,1)$ and D is the Neuberger operator, which satisfies the Ginsparg-Wilson relation that leads to the chiral anomaly.

[4] This is not generally true on the lattice.

The more the Wilson flow approaches the reference scale t_0, the more the field configurations with topological charge close to integer values become probable. Effectively, the Wilson flow splits the phase space in topologically distinct sectors.

Figures 1 to 4 show this tendency for the lattice $B3$.

For each lattice we then evaluated the average value of the topological charge squared at reference time $\langle (Q^{t_0})^2 \rangle \equiv \langle Q^2 \rangle$ and used it to recover the topological susceptibility $\chi^{t_0} \equiv \chi$ through

$$t_0^2 \chi = \frac{\langle Q^2 \rangle}{(L/a)^4} \left(\frac{t_0}{a^2} \right)^2 . \tag{18}$$

The results are summarized in table 2.

	$\langle Q^2 \rangle$	$t_0^2 \chi [\times 10^{-4}]$
$B1$	1.70(7)	6.43(28)
$B2$	1.70(4)	6.37(20)
$B3$	1.76(8)	6.40(30)

Table 2: Results of the simulations.

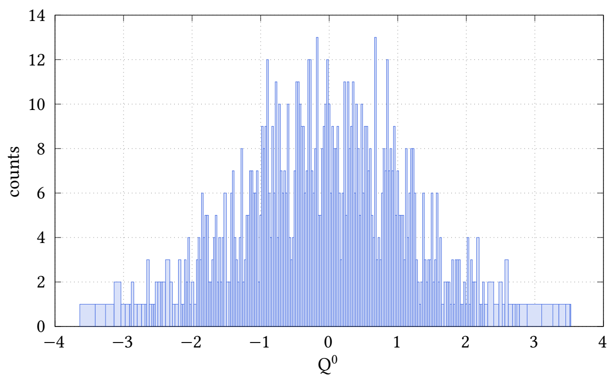

Figure 1: Q at flow time 0.

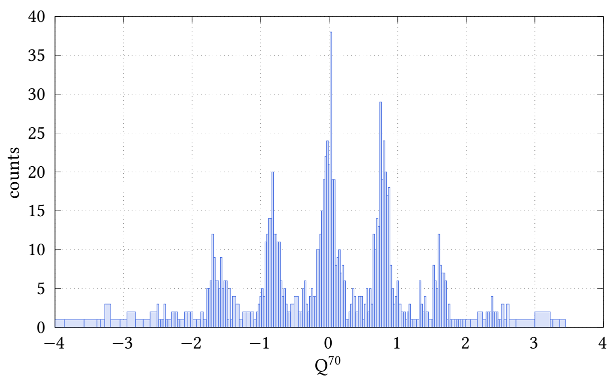

Figure 2: Q at flow time $70 = \frac{t_0}{7}$

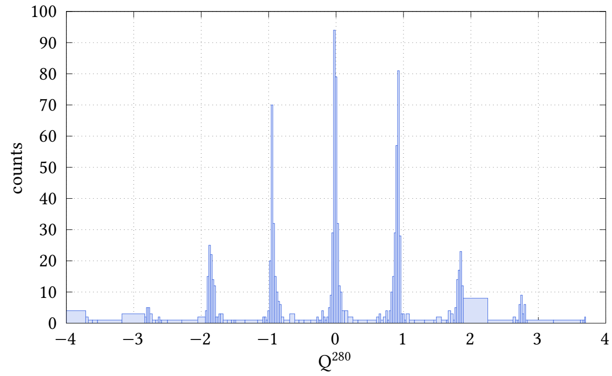

Figure 3: Q at flow time $280 = 4\frac{t_0}{7}$

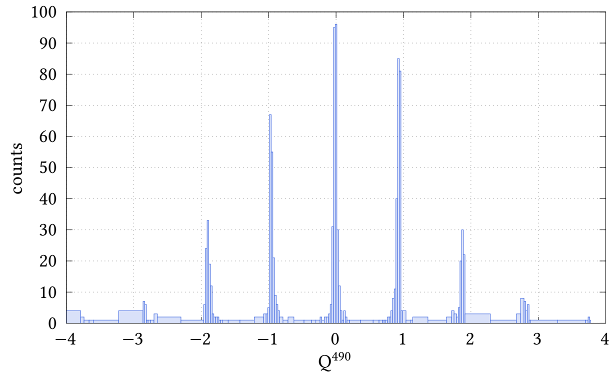

Figure 4: Q at flow time $490 = t_0$

The values collected in table 2 were then plotted against ${}^{a^2}/_{t_0}$ to extrapolate the continuum limit via a linear fit, shown in figure 5. The reduced χ^2 of the fit was $\sim 10^{-2}$ and the continuum limit we obtained is

$$t_0^2 \chi = 6.30(9) \times 10^{-4}. \tag{19}$$

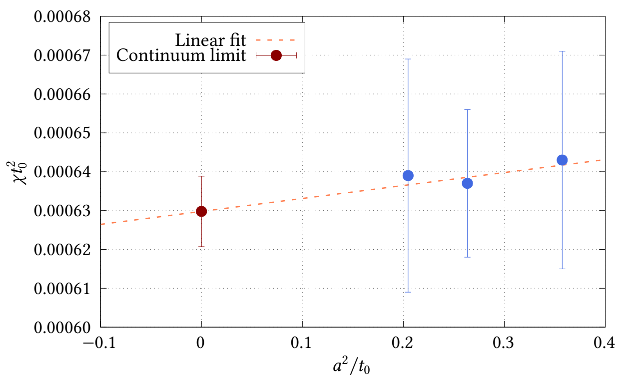

Figure 5: Continuum limit of $t_0^2 \chi$. Fit performed by Gnuplot, reduced $\chi^2 \sim 10^{-2}$.

To obtain the physical value of χ we divided the fit extrapolation by t_0^2 and then converted its value from fm^{-4} to MeV^4, remembering the conversion rate $1\mathrm{fm}^{-1} = 197.3\,\mathrm{MeV}$. Thus, the physical value of the topological susceptibility is

$$\chi = (178(8)\,\mathrm{MeV})^4 . \tag{20}$$

Because of the statistical error of our measurements, we can safely input the pion leptonic decay constant F_π instead of $F_{\eta'}$ inside the Witten Veneziano formula and discard any considerations about the errors of F_π, m_K and m_η.

Finally, using equation 2 with the 3 lightest quarks, we obtain the mass of the η' meson

$$m_{\eta'} = 944(69)\,\text{MeV}. \tag{21}$$

This result deviates less than 2σ from the measured value $m_{\eta'} = 958.66(24)$MeV but comes with a 7% relative error, whose magnitude overrules any significance of the formal compatibility between the results. Specifically, the fact that we found a smaller value for the η' mass suggests that we underestimated the value $t_0^2\chi$. This claim is supported by the results in [Cè+15] where $t_0^2\chi = 6.67(7) \times 10^{-4}$, more than 3 standard deviations away from our extimate (equation 19).

5 Conclusions

The results of this work are highly impacted by the lack of statistics. We ran three simulations, each 2 orders of magnitude smaller than the one in the reference article, on the lattices with smaller physical volume and larger spacing. This particularly afflicted the distribution of Q, which in this work were not normally distributed on the integers around 0, and ultimately the wellness of the continuum limit fit of $t_0\chi^2$. Still, given the relative novelty of the techniques and definitions that were used, a qualitatively correct result is a strong statement about the potential of this line of research.

Bibliography

[Cè+15] Marco Cè, Cristian Consonni, Georg P. Engel, and Leonardo Giusti. "Non-Gaussianities in the topological charge distribution of the SU(3) Yang-Mills theory". In: *Physical Review D* 92.7 (Oct. 2015). DOI: 10.1103/physrevd.92.074502. URL: https://doi.org/10.1103%2Fphysrevd.92.074502.