Determinism

- im not in college im js a junior in HS anyway were idk

We wanna show
if some X can go to X_1 and X_2 ($X \to X_1$ & $X \to X_2$) then $X_1 = X_2$.

Let's define 3 rules here.

E-Beta:

$(\lambda x.\cancel{X})V \to [V/x]X$ (for some value V)

E-App1

$$\frac{X_1 \to X_1'}{X_1 X_2 \to X_1' X_2}$$

E-App2

$$\frac{X_2 \to X_2'}{V X_2 \to V X_2'}$$ for some V

So now we do some induction ~~stuff~~ stuff

Assume [redacted] $X \to X_1$ and $X \to X_2$

Case 1: X is a variable or a val that's not an application

(if X is a variable, no rule holds so vacous truth)

(if X is a value like a lambda abstraction for example, no rule holds since values can't reduce. No $M \to M_i$ for $i \in 1,2$)

Case 2: X is an application, we split into 3 sub cases

$X = X_1' X_2'$

SC 1: Reduction by E-App 1
in case 2

Reduction must have come from reducing left part of app

$$\frac{X_1' \to X_1''}{X_1' X_2' \to X_1'' X_2'}$$

So if one derivation is $X \to X_1$, by E-App1 we get $X_1 = X_1'' X_2'$ with derivation

$X_1' \to X_1''$

Now suppose another one deriv: $X \to X_2$. There are 2 possibilities

a) it uses E App 1 again. Then $\exists X_1'''$ st $X_1' \to X_1'''$ and $X_2 = X_1''' X_2'$

By our inductive hyp., reduction on left is deterministic, $X_1'' = X_1'''$ and thus $X_1 = X_2$.

b) maybe a diff rule. Note for any app $X_1' X_2'$, if X_1' isn't a value then only rule that holds is E-App1. If the second derivation uses a rule not E-App 1, then X_1' is a value. But if X_1' is a value then E-App 1 can't work (only E-App 2 or E-Beta). This contradicts the assumption that one deriv used E-App 1. Thus both derivations use E-App 1.

we say $X_1 = X_2$.

$\boxtimes$

SC 2: Reduction by rule E-App 2

in Case 2

Now, our term is $X = V X_2'$ where X is a value and step comes from reduction.

$$\frac{X_2' \to X_2''}{V X_2' \to V X_2''} \quad \text{(via E-App 2) so that } X_1 = V X_2''.$$

Now let's suppose ANOTHER derivation $X \to X_2$. We consider possibilities

a) if other deriv uses E-App 2, then $\exists X_2'''$ st $X_2 = V X_2'''$ with $X_2' \to X_2'''$. By inductive hyp (used for deriv $X_2' \to X_2''$ and $X_2' \to X_2'''$), we get $X_2'' = X_2'''$ and thus $X_1 = X_2$.

b) maybe a diff rule, like Beta. but E-Beta only works when left part of application is lambda abstraction. V is a value and in all values the only values are lambda abstractions, our only possibility is

$V = \lambda x.\ X'$ and our "redex" here is $V X_2'$.

However, we said E-Beta should be used instead of E-App 2. But, by our conditions, when function part is a lambda abstraction and our argument is a value, only one rule, E-beta, can apply. If one deriv used E-App 2, then it has to be because x_2' was not yet a value. If x_2' were a value, then only rule that works is E-Beta. Thus two deriv can't disagree on what rule applies, so both derivations must use E-App 2.

we can say $x_1 = x_2$ ☒

SC 3: Reduction by rule E-Beta.
in case 2

Now, our term is in form $x = (\lambda x . x')v$, where v is some value.
from E-Beta, the reduction is

$$x \to [v/x]\, x'. \quad \text{Thus, } x_1 = [v/x]x'.$$

Now let's assume there is an alt. derivation $x \to x_2$. Since the term is exactly $(\lambda x . x')v$ with v as a value, our only rule that works is E-Beta since "redex" is fully formed. Alt derivation must also use E-Beta, so $x_2 = [v/x]\, x'$.

Thus $x_1 = x_2$ ☒

All cases show if $\exists\ x \to x_1$ and $x \to x_2$, by casework on X form and using ind. hyp. on sub-derivations, we get $x_1 = x_2$. Thus, whatever I js proved is deterministic -

small-step operational semantic for simply-typed lambda calc.

Note: My hand hurts pls let this work bro my head hurts too this hurts more than ISLs.