

HYPERUNIFORM POINT SETS ON PROJECTIVE SPACES

JOHANN S. BRAUCHART AND PETER J. GRABNER[‡]

ABSTRACT. We extend the notion of hyperuniformity to the projective spaces \mathbb{RP}^{d-1} , \mathbb{CP}^{d-1} , \mathbb{HP}^{d-1} , and \mathbb{OP}^2 . We show that hyperuniformity implies uniform distribution and present examples of deterministic point sets as well as point processes which exhibit hyperuniform behaviour.

1. INTRODUCTION

The quantitative characterisation of density fluctuation in many-particle systems led to the introduction of the concept of “hyperuniformity” (S. Torquato; cf. [37]) also called “superhomogeneity” (J. Lebowitz, cf. [18]).

The main feature of hyperuniformity of an infinite discrete point configuration X in \mathbb{R}^d is the fact that local density fluctuations are of smaller order than for a random (“Poissonian”) point configuration. In terms of the structure factor

$$S(\mathbf{k}) = \lim_{B \rightarrow \mathbb{R}^d} \frac{1}{\#(B \cap X)} \sum_{\mathbf{x}, \mathbf{y} \in B \cap X} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} \quad (\text{thermodynamic limit})$$

this means that $\lim_{\mathbf{k} \rightarrow 0} S(\mathbf{k}) = 0$; i.e., normalised density fluctuations are completely suppressed at very large length-scales (cf. [37]). This *thermodynamic limit* is understood in the sense that the volume B (for instance a ball of radius R) tends to the whole space \mathbb{R}^d while $\lim_{B \rightarrow \mathbb{R}^d} \frac{\#(B \cap X)}{\text{vol}(B)} = \rho$, where ρ is the density. Equivalently, the number variance $\mathbb{V}[N_R]$ (understood in the sense of thermodynamic limit) of particles within a spherical observation window of radius R can be used. When scaled by the window volume, this ratio tends to zero as $R \rightarrow \infty$; i.e., $\mathbb{V}[N_R]$ grows more slowly than the window volume of order R^d in the large- R limit. When

$$S(\mathbf{k}) \sim \|\mathbf{k}\|^\alpha \quad \text{as } \|\mathbf{k}\| \rightarrow \infty,$$

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where $\alpha > 0$, the number variance has the following large- R scaling:

$$\mathbb{V}[N_R] \sim \begin{cases} R^{d-1} & \alpha > 1, \\ R^{d-1} \log R & \alpha = 1, \\ R^{d-\alpha} & 0 < \alpha < 1. \end{cases}$$

This asymptotic behaviour provides a way to distinguish between different classes of hyperuniformity: Class I (strongest), II, and III (weakest). For a thorough discussion and survey regarding, in particular, hyperuniform states of matter, we refer to [36]. More recent contributions consider hyperuniformity in biology and geophysics [17, 26, 40], cryptography [20], and mathematics [7] to name a few. A nice layman's introduction to hyperuniformity can be found in [39].

In an answer to a question of S. Torquato, we together with W. Kusner [11] were able to extend the concept of hyperuniformity to the compact unit sphere \mathbb{S}^d in the Euclidean space \mathbb{R}^{d+1} by considering a fixed sequence of N -point distributions and analysing the asymptotic behaviour of the number variance with respect to certain spherical caps as $N \rightarrow \infty$. It turns out that there are three regimes of growth of the spherical caps as test sets that are of interest and only the one dubbed “hyperuniform with respect to threshold order” can be seen as a direct analogue to the classical notion of hyperuniformity; see Definition 2 below. In [11] it is shown that hyperuniformity in each of the three regimes implies uniform distribution. Furthermore, it is proven that so-called Quasi-Monte Carlo design sequences (cf. [13]) with strength at least $\frac{d+1}{2}$, certain energy minimising point set sequences (see, e.g., [9, 10]), and especially sequences of spherical designs of optimal growth order have hyperuniform behaviour. The paper [12] studies hyperuniformity on the sphere for samples of point processes on the sphere. The spherical ensemble (cf. [25]), the harmonic ensemble introduced in [6], and the jittered sampling process (shown to be a determinantal) exhibit hyperuniform behaviour.

An essential requirement for our study of the hyperuniformity phenomenon in the compact setting is that the space is homogeneous. Having considered the sphere, it seems natural to turn to doubly homogeneous spaces next, which is the aim of this paper. We point out that another compact setting, flat tori, is studied in [34].

The paper is organised as follows. Section 2 introduces compact doubly homogeneous spaces, briefly discusses their harmonic analysis, and defines hyperuniformity on these spaces. Like in the spherical

case [11, 12], there are three regimes. We arrive at the result that hyperuniformity, indeed, implies uniform distribution in each case. Section 3 discusses examples: maximisers of the sum of distances, jittered sampling, and the harmonic ensemble.

2. HYPERUNIFORMITY ON COMPACT DOUBLY HOMOGENEOUS SPACES

The main purpose of this paper is to extend the notion of hyperuniformity to doubly homogeneous manifolds. Thus we first give a concise description of these spaces and collect the relevant facts that will be used later.

2.1. Harmonic analysis on compact doubly homogeneous spaces. Let us recall the definition of doubly homogeneous spaces. A metric space (X, ϑ) is called doubly homogeneous, if there exists a group G acting isometrically on X so that

$$\forall x_1, x_2, y_1, y_2 \in X : \vartheta(x_1, x_2) = \vartheta(y_1, y_2) \Rightarrow \exists g \in G : y_1 = gx_1 \wedge y_2 = gx_2.$$

The compact doubly homogeneous Riemannian manifolds have been characterised in [38] (see also [22]):

$$\begin{aligned} \mathbb{S}^d &\cong \mathrm{SO}(d+1)/\mathrm{SO}(d) \\ \mathbb{RP}^{d-1} &\cong \mathrm{O}(d)/(\mathrm{O}(d-1) \times \mathrm{O}(1)) \\ \mathbb{CP}^{d-1} &\cong \mathrm{U}(d)/(\mathrm{U}(d-1) \times \mathrm{U}(1)) \\ \mathbb{HP}^{d-1} &\cong \mathrm{Sp}(d)/(\mathrm{Sp}(d-1) \times \mathrm{Sp}(1)) \\ \mathbb{OP}^2 &\cong \mathrm{F}_4/\mathrm{Spin}(9). \end{aligned}$$

We also refer to [1] for a concise introduction. The case of the sphere has been studied in [11, 12]; thus we mainly focus on the projective spaces \mathbb{FP}^{d-1} in this paper, where we use \mathbb{F} to represent the underlying scalar “field” ($\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$). The dimension of the space as a real manifold is then given by

$$D = (d-1) \dim_{\mathbb{R}}(\mathbb{F}),$$

where

$$\dim_{\mathbb{R}}(\mathbb{C}) = 2, \quad \dim_{\mathbb{R}}(\mathbb{H}) = 4, \quad \dim_{\mathbb{R}}(\mathbb{O}) = 8.$$

Furthermore, we associate the parameters

$$\alpha = \frac{d-1}{2} \dim_{\mathbb{R}}(\mathbb{F}) - 1 \quad \text{and} \quad \beta = \frac{1}{2} \dim_{\mathbb{R}}(\mathbb{F}) - 1$$

to the spaces \mathbb{FP}^{d-1} . We normalise the geodesic metric ϑ to take values in $[0, \frac{\pi}{2}]$, and equip the space with the normalised surface measure σ .

The measure of the geodesic ball $B(x, r) = \{y \in \mathbb{FP}^{d-1} \mid \vartheta(x, y) < r\}$ is then given by

$$\sigma(B(x, r)) = \int_0^r A(\vartheta) d\vartheta,$$

where

$$A(r) = \frac{2\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \sin(r)^{2\alpha+1} \cos(r)^{2\beta+1}$$

denotes the surface area of the geodesic sphere $\mathbb{S}(a, r) = \{x \in \mathbb{FP}^{d-1} \mid \vartheta(a, x) = r\}$. From now on, we use the notation

$$C_{\alpha, \beta} = \frac{2\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}$$

for the normalising constant.

The Laplace operator associated to the underlying metric tensor is then given by

$$\Delta f = -\frac{1}{A(r)} \frac{\partial}{\partial r} \left(A(r) \frac{\partial f}{\partial r} \right) + \Delta_{\mathbb{S}(a, r)} f,$$

where $\Delta_{\mathbb{S}(a, r)}$ denotes the Laplace operator on the geodesic sphere. In the following we will restrict our attention to zonal functions depending only on r ; for these functions $\Delta_{\mathbb{S}(a, r)} f$ vanishes. Furthermore, we notice that we use the geometers convention that the Laplacian has non-negative eigenvalues.

The zonal eigenfunctions of Δ are then given by the functions

$$P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, a))),$$

the Jacobi-polynomials with parameters α and β . The corresponding eigenvalues are given by

$$\lambda_n = 4n(n + \alpha + \beta + 1).$$

The dimension of the corresponding space of eigenfunctions V_n is then given by

$$m_n = \frac{2n + \alpha + \beta + 1}{\alpha + \beta + 1} \frac{(\alpha + \beta + 1)_n (\alpha + 1)_n}{n! (\beta + 1)_n},$$

where $(x)_n = x(x + 1) \cdots (x + n - 1)$ denotes the rising factorial (Pochhammer symbol).

The spaces of eigenfunctions V_n are invariant under the group G acting on \mathbb{FP}^{d-1} . The corresponding representation turns out to be irreducible. Let $Y_{n,k}(x)$ ($k = 1, \dots, m_n$) denote an orthonormal basis of V_n . Then the addition theorem

$$(1) \quad \sum_{k=1}^{m_n} Y_{n,k}(x) Y_{n,k}(y) = \frac{m_n}{P_n^{(\alpha, \beta)}(1)} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y)))$$

X	α	β	λ_k	$m_k = \dim(V_k)$
\mathbb{RP}^{d-1}	$\frac{d-3}{2}$	$-\frac{1}{2}$	$2k(2k+d-2)$	$\frac{4k+d-2}{d-2} \binom{2k+d-3}{d-3}$
\mathbb{CP}^{d-1}	$d-2$	0	$4k(k+d-1)$	$\frac{2k+d-1}{d-1} \binom{d+k-2}{d-2}^2$
\mathbb{HP}^{d-1}	$2d-3$	1	$4k(k+2d-1)$	$\frac{2k+2d-1}{(2d-1)(2d-2)} \binom{k+2d-2}{2d-2} \binom{k+2d-3}{2d-3}$
\mathbb{OP}^2	7	3	$4k(k+11)$	$\frac{2k+11}{1320} \binom{k+10}{7} \binom{k+7}{7}$

TABLE 1. The eigenvalues and dimensions of the eigenspaces of the Laplace operator on the projective spaces \mathbb{FP}^{d-1}

is an immediate consequence of the irreducibility of V_n . Furthermore, we recall the formula for integrals of zonal functions

$$(2) \quad \begin{aligned} & \int_{\mathbb{FP}^{d-1}} f(\cos(2\vartheta(x, y))) d\sigma(x) \\ &= C_{\alpha, \beta} \int_0^{\frac{\pi}{2}} f(\cos(2\vartheta)) \sin(\vartheta)^{2\alpha+1} \cos(\vartheta)^{2\beta+1} d\vartheta, \end{aligned}$$

see for instance [21].

Every function $f(\cos(2t)) \in L^2([0, \frac{\pi}{2}], \sin(t)^{2\alpha+1} \cos(t)^{2\beta+1} dt)$ can be expanded in terms of the orthogonal system $P_n^{(\alpha, \beta)}(\cos(2t))$:

$$(3) \quad f(\cos(2t)) = \sum_{n=0}^{\infty} \widehat{f}(n) P_n^{(\alpha, \beta)}(\cos(2t)),$$

where the coefficients are given by

$$\widehat{f}(n) = \frac{C_{\alpha, \beta} m_n}{\left(P_n^{(\alpha, \beta)}(1)\right)^2} \int_0^{\frac{\pi}{2}} f(\cos(2t)) P_n^{(\alpha, \beta)}(\cos(2t)) \sin(t)^{2\alpha+1} \cos(t)^{2\beta+1} dt.$$

The convergence in (3) is *a priori* only in the L^2 -sense; for continuous functions f satisfying

$$(4) \quad \sum_{j, k=1}^N c_j \overline{c_k} f(\cos(2\vartheta(x_j, x_k))) \geq 0$$

for all choices of N , $x_1, \dots, x_N \in X$, and $c_1, \dots, c_N \in \mathbb{C}$, Mercer's theorem ensures absolute and uniform convergence of the series (3) (see

for instance [16]). Functions satisfying (4) are called *positive definite*. It follows from [8] that positive definite functions are characterised by the non-negativity of the coefficients $\widehat{f}(n)$.

2.2. Hyperuniformity on \mathbb{FP}^{d-1} . We use similar ideas as in [11, 12] to define hyperuniform sequences of point sets in the setting of the spaces \mathbb{FP}^{d-1} . As in the original euclidean setting the concept is based on the *number variance*.

Definition 1. Let $(X_N)_{N \in \mathbb{N}}$ a sequence of point sets in \mathbb{FP}^{d-1} with $\lim_{N \rightarrow \infty} \#X_N = \infty$. The number variance of this sequence is then given by

$$(5) \quad \begin{aligned} \mathbb{V}(X_N, r) &= \mathbb{V}_x \#(X_N \cap B(x, r)) \\ &= \int_{\mathbb{FP}^{d-1}} \left(\sum_{y \in X_N} \mathbb{1}_{B(x, r)}(y) - \#X_N \sigma(B(x, r)) \right)^2 d\sigma(x). \end{aligned}$$

In a probabilistic setting the number variance is the variance of the number of points in a geodesic ball of radius r . The interest in this quantity stems from the fact that it can be used as a measure for the quality of a point distribution in the deterministic as well as the probabilistic setting.

Remark 1. The number variance is intimately related to the L^2 -discrepancy

$$D_{L^2}(X_N) = \frac{1}{\#X_N} \left(\int_0^{\frac{\pi}{2}} \mathbb{V}(X_N, r) dr \right)^{\frac{1}{2}}.$$

This notion is a classical measure for the deviation of the discrete distribution $\frac{1}{\#X_N} \sum_{x \in X_N} \delta_x$ from uniform distribution. L^2 -discrepancy has been studied to find lower bounds for this deviation; the theory of *irregularities of distribution* has developed techniques to derive lower bounds for various notions of discrepancy. For a comprehensive introduction and a very good overview over results in this context we refer to [3].

Remark 2. Stolarsky's celebrated invariance principle [35] relates the L^2 -discrepancy of a point set X_N on the sphere \mathbb{S}^d to the sum of mutual chordal distances

$$(6) \quad D_{L^2}(X_N)^2 + c_d \frac{1}{(\#X_N)^2} \sum_{x, y \in X_N} \|x - y\| = c_d \iint_{\mathbb{S}^d \times \mathbb{S}^d} \|x - y\| d\sigma(x) d\sigma(y)$$

for a suitable constant c_d . This shows that the L^2 -discrepancy is minimised for sets maximising the sum of mutual distances.

This invariance principle was generalised to the projective spaces \mathbb{FP}^{d-1} by Skriganov [33]. Furthermore, it is observed there that the invariance principle is a special case of a much more general relation.

As in the classical notion of hyperuniformity we will be interested in the dependence of the number variance on the radius r . Three regimes turn out to be of interest.

Definition 2. A sequence of point sets $(X_N)_{N \in \mathbb{N}}$ is called

- (1) *hyperuniform for large balls*, if for every $r \in (0, \frac{\pi}{2})$

$$\mathbb{V}(X_N, r) = o(\#X_N),$$

- (2) *hyperuniform for small balls*, if for every sequence of $(r_N)_{N \in \mathbb{N}}$ of radii satisfying

$$\begin{aligned} \lim_{N \rightarrow \infty} r_N &= 0 \\ \lim_{N \rightarrow \infty} \sigma(B(\cdot, r_N)) \# X_N &= \infty \end{aligned}$$

the following holds

$$\mathbb{V}(X_N, r) = o(\#X_N \sigma(B(\cdot, r_N))),$$

- (3) *hyperuniform for balls at threshold order*, if

$$\limsup_{N \rightarrow \infty} \mathbb{V}(X_N, r(\#X_N)^{-\frac{1}{D}}) = \mathcal{O}(r^{D-1})$$

for $r \rightarrow \infty$.

We notice that hyperuniformity at threshold order is obtained by rescaling and adapting the classical notion to the compact situation. The motivation for the order r^{D-1} on the right hand side comes from the order of the surface area of the geodesic sphere; i.i.d. random points would achieve the order r^D . Furthermore, it is clear that a smaller order of magnitude than $(\#X_N)^{-\frac{1}{D}}$ for the radii would not be sensible, because then the expected number of points in a ball would tend to 0. The other two notions of hyperuniformity have no obvious counterpart in the classical setting. All three notions have in common that the number variance is required to have smaller order of magnitude than the variance of i.i.d. random point sets of the same cardinality.

2.3. Uniform distribution. Uniform distribution is a notion that formalises the discretisation of measures.

Definition 3. A sequence of point sets $(X_N)_{N \in \mathbb{N}}$ in \mathbb{FP}^{d-1} with $\lim_{N \rightarrow \infty} \#X_N = \infty$ is called uniformly distributed, if

$$\lim_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{x \in X_N} \mathbb{1}_{B(x,r)} = \sigma(B(x,r))$$

for all $x \in \mathbb{FP}^{d-1}$ and $r \in [0, \frac{\pi}{2}]$.

For a comprehensive introduction to uniform distribution we refer to [15, 28].

From the general theory (see [28]) it is known that a sequence of point sets is uniformly distributed, if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{x \in X_N} f(x) = \int_{\mathbb{FP}^{d-1}} f(x) d\sigma(x)$$

for all continuous functions $f : \mathbb{FP}^{d-1} \rightarrow \mathbb{C}$. From this it follows immediately (using the denseness of the Laplace eigenfunctions and the addition theorem (1)) that $(X_N)_{N \in \mathbb{N}}$ is uniformly distributed, if and only if

$$(7) \quad \lim_{N \rightarrow \infty} \frac{1}{(\#X_N)^2} \sum_{x,y \in X_N} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))) = 0$$

for all $n \geq 1$.

We now want to relate hyperuniformity to uniform distribution. Since uniform distribution is a property modelled after the law of large numbers, which holds especially for i.i.d. random points, and hyperuniformity is modelled as a property “better than i.i.d. random points”, we can only expect that hyperuniformity implies uniform distribution.

In order to show that in all three regimes, we first derive an expression for the number variance of X_N in terms of Jacobi polynomials. We observe that $\mathbb{1}_{B(x,r)}(y) = \mathbb{1}_{[0,r]}(\vartheta(x,y))$ is a zonal function. Thus we can expand it as a series in terms of Jacobi polynomials in $\cos(2\vartheta(x,y))$. Using the formula

$$\begin{aligned} & \int_0^r P_n^{(\alpha,\beta)}(\cos(2t)) \sin(t)^{2\alpha+1} \cos(t)^{2\beta+1} dt \\ &= \frac{1}{2n} \sin(r)^{2\alpha+2} \cos(r)^{2\beta+2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos(2r)) \end{aligned}$$

for $n \geq 1$ (see [29]), we obtain

$$\mathbb{1}_{B(x,r)}(y) = \sigma(B(x,r)) + \sum_{n=1}^{\infty} \frac{C_{\alpha,\beta} m_n}{P_n^{(\alpha,\beta)}(1)} a_n(r) P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))).$$

with

$$a_n(r) = \frac{1}{2nP_n^{\alpha,\beta}(1)} \sin(r)^{2\alpha+2} \cos(r)^{2\beta+2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos(2r)).$$

This equation is valid in the L^2 -sense. Applying Parseval's identity we obtain

$$(8) \quad \mathbb{V}(X_n, r) = \sum_{n=1}^{\infty} \frac{C_{\alpha,\beta}^2 m_n}{P_n^{\alpha,\beta}(1)} a_n(r)^2 \sum_{x,y \in X_N} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))).$$

This series is absolutely and uniformly (as a function of x and y) convergent by Mercer's theorem. Notice that the function

$$\begin{aligned} (9) \quad g_r(x, y) &= \int_{\mathbb{FP}^{d-1}} (\mathbb{1}_{B(x,r)}(z) - \sigma(B(x,r))) (\mathbb{1}_{B(y,r)}(z) - \sigma(B(y,r))) d\sigma(z) \\ &= \sigma(B(x,r) \cap B(y,r)) - \sigma(B(\cdot, r))^2 \end{aligned}$$

is positive definite and that

$$(10) \quad \mathbb{V}(X_N, r) = \sum_{x,y \in X_N} g_r(x, y).$$

2.3.1. Hyperuniformity for large balls implies uniform distribution. Assume now that $(X_N)_{N \in \mathbb{N}}$ is hyperuniform for large balls. For fixed n choose a distance $0 < r < \frac{\pi}{2}$ such that $P_n^{(\alpha,\beta)}(\cos(2r)) \neq 0$. Then equation (8) yields

$$\begin{aligned} 0 &= \lim_{N \rightarrow \infty} \frac{\mathbb{V}(X_N, r)}{\#X_N} \geq \\ &\quad \frac{C_{\alpha,\beta}^2 m_n}{P_n^{\alpha,\beta}(1)} a_n(r)^2 \lim_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{x,y \in X_N} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))). \end{aligned}$$

This proves

Theorem 1. *Let $(X_N)_{N \in \mathbb{N}}$ be hyperuniform for large balls. Then $(X_N)_{N \in \mathbb{N}}$ is uniformly distributed. More precisely,*

$$(11) \quad \lim_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{x,y \in X_N} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))) = 0$$

holds for all $n \geq 1$.

Remark 3. Notice that (11) gives a better rate of convergence as compared to the criterion (7) for uniform distribution. This reflects the heuristic expectation that hyperuniformity should give an improved quality of uniform distribution.

2.3.2. Hyperuniformity for small balls implies uniform distribution. Assume that $(X_N)_{N \in \mathbb{N}}$ is hyperuniform for small balls. Furthermore, we observe that for fixed $n \geq 1$ the coefficient of $P_n^{(\alpha, \beta)}$ in (8) is of order r_N^{2D} for $N \rightarrow \infty$ (notice that $4\alpha + 4 = 2D$). Arguing as above we obtain that

$$\lim_{N \rightarrow \infty} \frac{\mathbb{V}(X_N, r_N)}{\#X_N \sigma(B(\cdot, r_N))} = 0$$

implies

$$\lim_{N \rightarrow \infty} \frac{r_N^D}{\#X_N} \sum_{x, y \in X_N} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y))) = 0.$$

Since r_N can be chosen to tend to 0 arbitrarily slowly, this implies

$$\limsup_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{x, y \in X_N} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y))) < \infty.$$

Thus we have shown

Theorem 2. *Let $(X_N)_{N \in \mathbb{N}}$ be hyperuniform for small balls. Then $(X_N)_{N \in \mathbb{N}}$ is uniformly distributed. More precisely,*

$$(12) \quad \limsup_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{x, y \in X_N} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y))) < \infty$$

holds for all $n \geq 1$.

Remark 4. Notice that we have an improvement in the quality of uniform distribution in comparison with (7) by a power $(\#X_N)^{1-\varepsilon}$ for $\varepsilon > 0$.

2.3.3. Hyperuniformity for balls at threshold order implies uniform distribution. Assume that $(X_N)_{N \in \mathbb{N}}$ is hyperuniform for balls at threshold order. For fixed $n \geq 1$ we notice that

$$a_n(r(\#X_N)^{-\frac{1}{D}})^2 \sim \frac{P_{n-1}^{(\alpha+1, \beta+1)}(1)^2}{(2n P_n^{(\alpha, \beta)}(1))^2} \frac{r^{2D}}{(\#X_N)^2}.$$

Combining this with (8) gives

$$\begin{aligned} r^{2D} \frac{C_{\alpha, \beta}^2 m_n P_{n-1}^{(\alpha+1, \beta+1)}(1)^2}{(2n)^2 (P_n^{(\alpha, \beta)}(1))^3} \limsup_{N \rightarrow \infty} \frac{1}{(\#X_N)^2} \sum_{x, y \in X_N} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y))) \\ \leq \limsup_{N \rightarrow \infty} \mathbb{V}(X_N, r(\#X_N)^{-\frac{1}{D}}) = \mathcal{O}(r^{D-1}). \end{aligned}$$

This can only hold, if

$$\limsup_{N \rightarrow \infty} \frac{1}{(\#X_N)^2} \sum_{x, y \in X_N} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y))) = 0.$$

Summing up, we have proved

Theorem 3. *Let $(X_N)_{N \in \mathbb{N}}$ be hyperuniform for balls at threshold order. Then $(X_N)_{N \in \mathbb{N}}$ is uniformly distributed.*

Remark 5. In this case there seems to be no improvement in the order of convergence of the sum (7).

3. EXAMPLES

3.1. Maximisers of the sum of distances. As pointed out in Remark 2 Skriganov [33] generalised Stolarsky's invariance principle [35] to the projective spaces \mathbb{FP}^{d-1} . As a consequence the L^2 -discrepancy is minimised amongst all sets of the same cardinality by configurations maximising the sum of mutual "chordal" distances

$$(13) \quad \sum_{x,y \in X_N} \sin(\vartheta(x,y)).$$

It was shown in [32] that there exist positive constants K_d and k_d such that

$$(14) \quad \begin{aligned} -K_d(\#X_N)^{1-\frac{1}{D}} &\leq \\ \sum_{x,y \in X_N} \sin(\vartheta(x,y)) - (\#X_N)^2 \iint_{(\mathbb{FP}^{d-1})^2} &\sin(\vartheta(x,y)) d\sigma(x) d\sigma(y) \\ &\leq -k_d(\#X_N)^{1-\frac{1}{D}} \end{aligned}$$

for a point set X_N maximising (13).

Integrating (8) against $\sin(2r) dr$ and using

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin(r)^{4\alpha+4} \cos(r)^{4\beta+4} P_{n-1}^{(\alpha+1,\beta+1)}(\cos(2r))^2 \sin(2r) dr = \\ \frac{\Gamma(2\alpha+3)\Gamma(2\beta+3)}{\alpha+\beta+2} 2^{2n-2} (n+\alpha+\beta+1) \times \\ \frac{(\frac{1}{2})_{n-1}(\alpha+2)_{n-1}(\beta+2)_{n-1}(\alpha+\beta+2)_{n-1}}{((n-1)!)^2 \Gamma(2(n+\alpha+\beta+3))} \end{aligned}$$

(see [33]) we obtain

$$(15) \quad \begin{aligned} D_{L^2}(X_N)^2 &= C_{\alpha,\beta}^2 \Gamma(2\alpha+3)\Gamma(2\beta+3) \times \\ &\sum_{n=1}^{\infty} 2^{2n-2} (2n+\alpha+\beta+1) \frac{(\frac{1}{2})_{n-1}(\alpha+\beta+1)_n(\alpha+\beta+2)_n}{(\alpha+1)_n \Gamma(2(n+\alpha+\beta+3))} \times \\ &\sum_{x,y \in X_N} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))) \end{aligned}$$

(see also [31, 33]). We notice that the coefficients in the sum are of the order $n^{-\alpha-2}$.

We will use the inequality

$$(16) \quad \begin{aligned} & \sin(\vartheta)^{a+\frac{1}{2}} \cos(\vartheta)^{b+\frac{1}{2}} |P_n^{(a,b)}(\cos(2\vartheta))| \\ & \leq \frac{\Gamma(a+1)}{\sqrt{\pi}} \binom{n+a}{n} \left(n + \frac{1}{2}(a+b+1)\right)^{-a-\frac{1}{2}} = \mathcal{O}\left(n^{-\frac{1}{2}}\right) \end{aligned}$$

valid for $a \geq b > -1$ (see [14]). Inserting $a = \alpha + 1$ and $b = \beta + 1$ in (16) we estimate the number variance by

$$(17) \quad \mathbb{V}(X_N, r) \leq C \sin(r)^{2\alpha+1} \cos(r)^{2\beta+1} \sum_{n=1}^{\infty} n^{-\alpha-2} \sum_{x,y \in X_N} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))).$$

Putting (15) and (17) together and observing that $2\alpha + 1 = D - 1$ we obtain

$$(18) \quad \mathbb{V}(X_N, r) = \mathcal{O}\left(\sin(r)^{D-1} \mathrm{D}_{L^2}(X_N)\right).$$

For point sets with optimal L^2 -discrepancy (or equivalently, maximal sum of distances) we obtain

$$\mathbb{V}(X_N, r) = \mathcal{O}\left(\sin(r)^{D-1} (\#X_N)^{1-\frac{1}{D}}\right).$$

This inequality immediately implies the theorem.

Theorem 4. *Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of maximisers of the sum of mutual distances (13). Then this sequence is hyperuniform in all three regimes.*

3.2. Jittered sampling. A very simple and obvious probabilistic model for a point set X_N is given by jittered sampling. We start with a partition of the space \mathbb{FP}^{d-1} into N sets A_j ($j = 1, \dots, N$) of equal area and diameter $\leq CN^{-\frac{1}{D}}$, where the constant only depends on the space. It is known from [19] that such a partition exists.

We define a point process \mathcal{X}_N^J , called the *jittered sampling process*, by choosing N points $x_j \in A_j$ ($j = 1, \dots, N$), where each point is chosen with respect to the surface measure restricted to A_j . The number variance of this process is then given by

$$(19) \quad \begin{aligned} V_N &= \mathbb{V}(\mathcal{X}_N^J, r) \\ &= \int_{\mathbb{FP}^{d-1}} \int_{A_1} \cdots \int_{A_N} \left(\sum_{j=1}^N \mathbb{1}_{B(x,r)}(x_j) - N\sigma(B(x,r)) \right)^2 d\sigma_1(x_1) \cdots d\sigma_N(x_N) d\sigma(x), \end{aligned}$$

where $\sigma_j(A) = N\sigma(A_j \cap A)$ denotes the surface measure restricted to A_j .

Expanding the square in (19) and using the fact that $\mathbb{1}_{B(y,r)}(x) = \mathbb{1}_{B(x,r)}(y)$ we obtain

$$\begin{aligned} V_N &= \sum_{i \neq j} \int_{A_i} \int_{A_j} \sigma(B(x_i, r) \cap B(x_j, r)) d\sigma_i(x_i) d\sigma_j(x_j) \\ &\quad + N\sigma(B(\cdot, r)) - N^2\sigma(B(\cdot, r))^2 \\ &= N^2 \int_{\mathbb{FP}^{d-1}} \int_{\mathbb{FP}^{d-1}} \sigma(B(x, r) \cap B(y, r)) d\sigma(x) d\sigma(y) \\ &\quad - \sum_{i=1}^N \int_{A_i} \int_{A_i} \sigma(B(x_i, r) \cap B(x_j, r)) d\sigma_i(x_i) d\sigma_j(x_j) \\ &\quad + N\sigma(B(\cdot, r)) - N^2\sigma(B(\cdot, r))^2 \\ &= \frac{1}{2} \sum_{i=1}^N \int_{A_i} \int_{A_i} \sigma(B(x_i, r) \Delta B(y_i, r)) d\sigma_i(x_i) d\sigma_i(y_i), \end{aligned}$$

where we have used

$$\int_{\mathbb{FP}^{d-1}} \int_{\mathbb{FP}^{d-1}} \sigma(B(x, r) \cap B(y, r)) d\sigma(x) d\sigma(y) = \sigma(B(\cdot, r))^2$$

and used the notation $A \Delta B = A \cup B \setminus (A \cap B)$ for the symmetric difference. A similar computation has been used in [12].

The measure of the symmetric difference of two balls can be bounded

$$(20) \quad \sigma(B(x, r) \Delta B(y, r)) \leq C' \vartheta(x, y) \text{surface}(\partial B(\cdot, r))$$

for some constant $C' > 0$. Since the diameter of each part A_j of the partition is bounded by $CN^{-\frac{1}{D}}$ we obtain the bound

$$V_N \leq CC'N^{1-\frac{1}{D}}\text{surface}(\partial B(\cdot, r)) = \mathcal{O}(r^{D-1}N^{1-\frac{1}{D}}),$$

from which we derive the theorem.

Theorem 5. *The jittered sampling process \mathcal{X}_N^J is hyperuniform in all three regimes.*

3.3. The harmonic ensemble. Determinantal point processes (see [25]) have become a very useful tool to provide probabilistic models, which behave better than i.i.d. models due to built mutual repulsion of the points. Especially, in the context of finding well distributed point sets such processes have proven to be a very useful tool to establish good bounds for various quality measures of point distributions (see, for instance [1, 4–6, 23, 24, 30]). Since the notion of hyperuniformity has been motivated as a property of point sets, which behave “better than

i.i.d. random”, it is natural to test samples of determinantal processes for this property.

The rotation invariant processes on the spaces \mathbb{FP}^{d-1} have been characterised in [1]. The most natural amongst them is the process induced by the projection kernel on the spaces of eigenfunctions for the $N + 1$ smallest eigenvalues of the Laplace operator. This kernel is given by

(21)

$$\begin{aligned} K_N(x, y) &= \sum_{n=0}^N \sum_{k=1}^{m_n} Y_{n,k}(x) Y_{n,k}(y) = \sum_{n=0}^N \frac{m_n}{P_n^{(\alpha, \beta)}(1)} P_n^{(\alpha, \beta)}(\cos(2\vartheta(x, y))) \\ &= \frac{(\alpha + \beta + 2)_N}{(\beta + 1)_N} P_N^{(\alpha+1, \beta)}(\cos(2\vartheta(x, y))) \end{aligned}$$

(see, for instance [2]). This process produces

$$K_N(x, x) = \frac{(\alpha + \beta + 2)_N (\alpha + 2)_N}{N! (\beta + 1)_N} \sim \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2) \Gamma(\alpha + 2)} N^{2\alpha+2}$$

points; we use the notation \mathcal{X}_N^H for this process and call it the *harmonic process* like the corresponding process on the sphere introduced in [6]. For more details on determinantal point processes we refer to [25] again.

We notice that the number variance of \mathcal{X}_N^H equals the expectation of the sum (10) under law of the process \mathcal{X}_N^H ,

$$(22) \quad \mathbb{V}(\mathcal{X}_N^H, r) = \mathbb{E} \sum_{x,y} g_r(x, y).$$

Such expectations can be expressed easily by the general theory of determinantal processes

$$\begin{aligned} \mathbb{E} \sum_{x,y} g_r(x, y) &= \mathbb{E} \sum_{x \neq y} g_r(x, y) + \mathbb{E} \sum_x g_r(x, x) \\ &= \iint_{(\mathbb{FP}^{d-1})^2} g_r(x, y) (K_N(x, x) K_N(y, y) - K_N(x, y) K_N(y, x)) d\sigma(x) d\sigma(y) \\ &\quad + K_N(\cdot, \cdot) g_r(\cdot, \cdot). \end{aligned}$$

Since all functions occurring in this last equation are zonal, we can use (2) to further simplify this equation

$$\begin{aligned} \mathbb{V}(\mathcal{X}_N^H, r) &= \\ C_{\alpha, \beta} \int_0^{\frac{\pi}{2}} g_r(\cos(2\vartheta)) & \left(K_N(1)^2 - K_N(\cos(2\vartheta))^2 \right) \sin(\vartheta)^{2\alpha+1} \cos(\vartheta)^{2\beta+1} d\vartheta \\ &\quad + K_N(1) g_r(1); \end{aligned}$$

here we have made use of the convention to write $f(\cos(2\vartheta(x, y))) = f(x, y)$ for zonal functions. Using the facts that

$$\begin{aligned} C_{\alpha, \beta} \int_0^{\frac{\pi}{2}} K_N(\cos(2\vartheta))^2 \sin(\vartheta)^{2\alpha+1} \cos(\vartheta)^{2\beta+1} d\vartheta &= K_N(1) \\ C_{\alpha, \beta} \int_0^{\frac{\pi}{2}} g_r(\cos(2\vartheta)) \sin(\vartheta)^{2\alpha+1} \cos(\vartheta)^{2\beta+1} d\vartheta &= 0 \end{aligned}$$

we obtain

$$\begin{aligned} \mathbb{V}(\mathcal{X}_N^H, r) &= \\ C_{\alpha, \beta} \int_0^{\frac{\pi}{2}} (g_r(1) - g_r(\cos(2\vartheta))) K_N(\cos(2\vartheta))^2 \sin(\vartheta)^{2\alpha+1} \cos(\vartheta)^{2\beta+1} d\vartheta. \end{aligned}$$

We now remark that

$$\begin{aligned} g_r(1) - g_r(\cos(2\vartheta(x, y))) &= \frac{1}{2}\sigma(B(x, r)\Delta B(y, r)) \\ &\leq C'\vartheta(x, y)\text{surface}(\partial B(\cdot, r)) \leq C''\sin(\vartheta)r^{D-1} \end{aligned}$$

using (20). Thus we have

$$\mathbb{V}(\mathcal{X}_N^H, r) = \mathcal{O}\left(r^{D-1} \int_0^{\frac{\pi}{2}} K_N(\cos(2\vartheta))^2 \sin(\vartheta)^{2\alpha+2} \cos(\vartheta)^{2\beta+1} d\vartheta\right),$$

and it remains to estimate the integral

$$\frac{(\alpha + \beta + 2)_N^2}{(\beta + 1)_N^2} \int_0^{\frac{\pi}{2}} P_N^{(\alpha+1, \beta)}(\cos(2\vartheta))^2 \sin(\vartheta)^{2\alpha+2} \cos(\vartheta)^{2\beta+1} d\vartheta;$$

notice that this is an integral over the square of a Jacobi polynomial against the “wrong” weight function. We use the estimate (16) for $a = \alpha + 1$ and $b = \beta$ in the interval $[\frac{c}{N}, \frac{\pi}{2}]$ and the trivial bound $|P_N^{(\alpha+1, \beta)}(\cos(2\vartheta))| \leq P_N^{(\alpha+1, \beta)}(1)$ in the interval $[0, \frac{c}{N}]$ to obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} P_N^{(\alpha+1, \beta)}(\cos(2\vartheta))^2 \sin(\vartheta)^{2\alpha+2} \cos(\vartheta)^{2\beta+1} d\vartheta \\ = \mathcal{O}\left(\int_0^{\frac{c}{N}} N^{2\alpha+2} \vartheta^{2\alpha+2} d\vartheta\right) + \mathcal{O}\left(\int_{\frac{c}{N}}^{\frac{\pi}{2}} \frac{1}{N \sin(\vartheta)} d\vartheta\right) \\ = \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{\log(N)}{N}\right) = \mathcal{O}\left(\frac{\log(N)}{N}\right). \end{aligned}$$

Summing up, we have obtained

$$\mathbb{V}(\mathcal{X}_N^H, r) = \mathcal{O}\left(r^{D-1} \frac{(\alpha + \beta + 2)_N^2}{(\beta + 1)_N^2} \frac{\log(N)}{N}\right) = \mathcal{O}\left(r^{D-1} \frac{K_N(1) \log(N)}{N}\right).$$

Thus we have shown the following theorem.

Theorem 6. *The harmonic process \mathcal{X}_N^H is hyperuniform for large and small balls. For balls at threshold order the weaker relation*

$$\mathbb{V}(\mathcal{X}_N^H, rK_N(1)^{-\frac{1}{D}}) = \mathcal{O}(r^{D-1} \log(N))$$

holds.

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INSTITUTE OF ANALYSIS AND NUMBER THEORY, GRAZ UNIVERSITY OF TECHNOLOGY, KOPERNIKUSGASSE 24. 8010 GRAZ, AUSTRIA

Email address: j.brauchart@tugraz.at

Email address: peter.grabner@tugraz.at