1 Grammar

1.1 Untyped Expressions

$$
\begin{array}{rcll}
eu & ::= & \mathbf{1} & \text{Unit} \\
 & | & v & \text{Var} \\
 & | & \lambda x \,.\, eu & \text{Lambda} \\
 & | & eu_1 \; eu_2 & \text{App}
\end{array}
$$

1.2 Typed Expressions

$$
\begin{array}{rcll}
e & ::= & \mathbf{1} & \text{Unit} \\
 & | & v & \text{Var} \\
 & | & \lambda\,(x \,:\, t)\;.\,e & \text{Lambda} \\
 & | & e_1 \; e_2 & \text{App}
\end{array}
$$

1.3 Types

$$
\begin{array}{rcll}
t & ::= & \mathbf{1} & \text{Unit} \\
 & | & t_1 \to t_2 & \text{Arrow} \\
 & | & u & \text{UVar}
\end{array}
$$

1.4 Typing Evironments

$$
\begin{array}{rcll}
\Gamma & ::= & \cdot & \text{Empty} \\
 & | & \Gamma, v \,:\, t & \text{Var with Type}
\end{array}
$$

1.5 Type Stores

$$
\begin{array}{rcll}
\mathcal{S} & ::= & \cdot & \text{Empty} \\
 & | & \mathcal{S}, u = t & \text{UVar with Type}
\end{array}
$$

2 Typing

2.1 Declarative Typing

Judgements of the form:

$$\Gamma \vdash e \,:\, t$$

$$\frac{}{\Gamma \vdash \mathbf{1} \,:\, \mathbf{1}}\ (\text{DeclUnit}) \qquad \frac{\Gamma(v) = t}{\Gamma \vdash v \,:\, t}\ (\text{Var})$$

$$\frac{\Gamma, x \,:\, t_1 \vdash e \,:\, t_2}{\Gamma \vdash \lambda\,(x \,:\, t_2)\;.\,e \,:\, t_1 \to t_2}\ (\text{Lambda})$$

$$\frac{\Gamma \vdash e_1 \,:\, t_1 \to t_2 \qquad \Gamma \vdash e_2 \,:\, t_2}{\Gamma \vdash e_1 \; e_2 \,:\, t_2}\ (\text{App})$$

2.2 Algorithmic Typing

Jugements of the form:

$$\Gamma, \mathcal{S} \vdash eu \Rightarrow e \,:\, t \dashv \mathcal{S}'$$

$$\frac{}{\Gamma, \mathcal{S} \vdash \mathbf{1} \Rightarrow \mathbf{1} \,:\, \mathbf{1} \dashv \mathcal{S}} \text{ (Unit)} \qquad \frac{\Gamma(v) = t}{\Gamma, \mathcal{S} \vdash v \Rightarrow v \,:\, t \dashv \mathcal{S}} \text{ (Var)}$$

$$\frac{\begin{array}{cc} \text{Fresh } u & \Gamma, \mathcal{S}_0 \vdash eu_1 \Rightarrow e_1 \,:\, t_{\text{arr}} \dashv \mathcal{S}_1 \\ \Gamma, \mathcal{S}_1 \vdash eu_2 \Rightarrow e_2 \,:\, t_{\text{arg}} \dashv \mathcal{S}_2 & \mathcal{S}_2 \vdash t_{\text{arr}} == t_{\text{arg}} \to u \dashv \mathcal{S}_3 \end{array}}{\Gamma, \mathcal{S}_0 \vdash eu_1 \; eu_2 \Rightarrow e_1 \; e_2 \,:\, u \dashv \mathcal{S}_3} \text{ (App)}$$

$$\frac{\begin{array}{c} \text{Fresh } u_1, u_2 \qquad \Gamma, x \,:\, u_1, \mathcal{S}_0 \vdash eu \Rightarrow e \,:\, t_{\text{ret}} \dashv \mathcal{S}_1 \\ \mathcal{S}_1 \vdash t_{\text{ret}} == u_2 \dashv \mathcal{S}_2 \end{array}}{\Gamma, \mathcal{S}_0 \vdash \lambda x \,.\, eu \Rightarrow \lambda\,(x \,:\, u_1)\;.\, e \,:\, u_1 \to u_2 \dashv \mathcal{S}_2} \text{ (Arr)}$$

Judgements of the form:

$$\mathcal{S}_0 \vdash t_1 == t_2 \dashv \mathcal{S}_1$$

$$\frac{}{\mathcal{S} \vdash t == t \dashv \mathcal{S}} \text{ (Base Eq)} \qquad \frac{\mathcal{S}_0 \vdash t_1 == t_3 \dashv \mathcal{S}_1 \qquad \mathcal{S}_1 \vdash t_2 == t_4 \dashv \mathcal{S}_2}{\mathcal{S}_0 \vdash t_1 \to t_2 == t_3 \to t_4 \dashv \mathcal{S}_2} \text{ (Arr Eq)}$$

$$\frac{u \notin t \qquad u \notin \mathcal{S}}{\mathcal{S} \vdash u == t \dashv \mathcal{S}, u = t} \text{ (UVar Left Eq)} \qquad \frac{u \notin t \qquad u \notin \mathcal{S}}{\mathcal{S} \vdash t == u \dashv \mathcal{S}, u = t} \text{ (UVar Right Eq)}$$

3 Theorems

Lemma 1 (Equality)**.** *If there exists a judgement*

$$\mathcal{S} \vdash t_1 == t_2 \dashv \mathcal{S}',$$

then under context $\mathcal{S}'$, the types t_1 and t_2 are equal.

Proof. TBD □

Theorem 1 (Soundness)**.** *If there exists an algorithmic judgement*

$$\Gamma, \mathcal{S} \vdash eu \Rightarrow e \,:\, t \dashv \mathcal{S}'$$

then we may derive a dedclarative judgement

$$\mathcal{S}(\Gamma) \vdash \mathcal{S}(e) \,:\, \mathcal{S}(t).$$

Proof. The proof is by induction over the expressions.

The base cases for the rules Unit and Var are trivial.

For App, we start with the given judgement:

$$\frac{\begin{array}{cc}\text{Fresh } u & \Gamma, \mathcal{S}_0 \vdash eu_1 \Rightarrow e_1 : t_{\text{arr}} \dashv \mathcal{S}_1 \\ \Gamma, \mathcal{S}_1 \vdash eu_2 \Rightarrow e_2 : t_{\text{arg}} \dashv \mathcal{S}_2 & \mathcal{S}_2 \vdash t_{\text{arr}} == t_{\text{arg}} \to u \dashv \mathcal{S}_3\end{array}}{\Gamma, \mathcal{S}_0 \vdash eu_1\ eu_2 \Rightarrow e_1\ e_2 : u \dashv \mathcal{S}_3}\ \text{(App)}$$

Let $\mathcal{S}$ be $\mathcal{S}_3$ for brevity.

By induction, we know that there are declarative judgements

$$\mathcal{S}(\Gamma) \vdash \mathcal{S}(e_1) : \mathcal{S}(t_{\text{arr}})$$

and

$$\mathcal{S}(\Gamma) \vdash \mathcal{S}(e_2) : \mathcal{S}(t_{\text{arg}}).$$

By the equality lemma, we know that $t_{\text{arr}} = t_{\text{arg}} \to u$, so we may substitute for it:

$$\mathcal{S}(\Gamma) \vdash \mathcal{S}(e_1) : \mathcal{S}(t_{\text{arg}} \to u).$$

By property of substitution, we get

$$\mathcal{S}(\Gamma) \vdash \mathcal{S}(e_1) : \mathcal{S}(t_{\text{arg}}) \to \mathcal{S}(u).$$

Thus we may construct the declarative judgement

$$\frac{\mathcal{S}(\Gamma) \vdash \mathcal{S}(e_1) : \mathcal{S}(t_{\text{arg}}) \to \mathcal{S}(u) \qquad \mathcal{S}(\Gamma) \vdash \mathcal{S}(e_2) : \mathcal{S}(t_{\text{arg}})}{\mathcal{S}(\Gamma) \vdash \mathcal{S}(e_1)\ \mathcal{S}(e_2) : \mathcal{S}(u)}$$

By property of substitution, we know that $\mathcal{S}(e_1)\ \mathcal{S}(e_2)$ is the same as $\mathcal{S}(e_1\ e_2)$, which gives the desired.

For Arr, we start with the given judgement:

$$\frac{\begin{array}{cc}\text{Fresh } u_1, u_2 & \Gamma, x : u_1, \mathcal{S}_0 \vdash eu \Rightarrow e : t_{\text{ret}} \dashv \mathcal{S}_1 \\ \multicolumn{2}{c}{\mathcal{S}_1 \vdash t_{\text{ret}} == u_2 \dashv \mathcal{S}_2}\end{array}}{\Gamma, \mathcal{S}_0 \vdash \lambda x \,.\, eu \Rightarrow \lambda\,(x : u_1)\ .\, e : u_1 \to u_2 \dashv \mathcal{S}_2}\ \text{(Arr)}$$

Let $\mathcal{S}$ be $\mathcal{S}_2$ for brevity.

By induction, we can conclude that there exists the declarative judgement

$$\mathcal{S}(\Gamma), x : \mathcal{S}(u_1) \vdash \mathcal{S}(e) : \mathcal{S}(t_{\text{ret}}).$$

By the equality lemma, we may substitute u_2 for t_{ret}, giving

$$\mathcal{S}(\Gamma), x : \mathcal{S}(u_1) \vdash \mathcal{S}(e) : \mathcal{S}(u_2).$$

From this, we may derive the declarative judgement

$$\frac{\mathcal{S}(\Gamma), x : \mathcal{S}(u_1) \vdash \mathcal{S}(e) : \mathcal{S}(u_2)}{\mathcal{S}(\Gamma) \vdash \lambda x \,.\, \mathcal{S}(e) : \mathcal{S}(u_1) \to \mathcal{S}(u_2)}$$

By the property of substitution, we know that $\mathcal{S}(u_1) \to \mathcal{S}(u_2$ is the same as $\mathcal{S}(u_1 \to u_2)$, which gives the desired.

$\square$