Symmetry Breaking: The social golfers problem as a case study

Javier Larrosa

LSI-UPC

1 Description of the Problem

There is a group of n people that want to play golf once a week during w weeks. Each week, players are divided into g groups. Each group has s people. Thus, $n = g \cdot s$. The rules of the tournament require that two people cannot play in the same group more than once.

2 Basic Modelling

We will use the following indexes:

- i will refer to a group. Thus, it will range between 0 and $g-1$.
- j will refer to a member of a group. Thus, it will range between 0 and $s-1$.
- k will refer to a week. Thus, it will range between 0 and $w-1$.

We define integer variables x_{ijk} taking values in the range $[0..n-1]$. If $x_{ijk} \leftarrow v$, it means that person v is the j-th member of group i at week k

Consider the following 2-dimensional arrangement of the variables:

x_{000} x_{010} ... $x_{0(s-1)0}$	x_{001} x_{011} ... $x_{0(s-1)1}$	... ...	$x_{00(w-1)}$ $x_{01(w-1)}$ ... $x_{0(s-1)(w-1)}$
x_{100} x_{110} ... $x_{1(s-1)0}$	x_{101} x_{111} ... $x_{1(s-1)1}$	... ...	$x_{10(w-1)}$ $x_{11(w-1)}$ ... $x_{1(s-1)(w-1)}$
...	...	...	...
$x_{(g-1)00}$ $x_{(g-1)10}$ ... $x_{(g-1)(s-1)0}$	$x_{(g-1)01}$ $x_{(g-1)11}$ ... $x_{(g-1)(s-1)1}$	... ...	$x_{(g-1)0(w-1)}$ $x_{(g-1)1(w-1)}$ ... $x_{(g-1)(s-1)(w-1)}$

where each column is a week, and columns are divided by groups.

We define the following constraints:

- Every player plays every week: for each $k = 0..w-1$ we add a constraint
$$alldiff(x_{**k})$$
- Nobody meets more than once: For every pair of different weeks $k, k' = 1..w-1$, for every pair of groups $i, i' = 0..g-1$, for every pair of different players of group i $j_i, j'_i = 0..s-1$, for every pair of different players of group i' $j_{i'}, j'_{i'} = 0..s-1$ we add a constraint
$$x_{ij_ik} \neq x_{i'j_{i'}k'} \quad \vee x_{ij'_ik} \neq x_{i'j'_{i'}k'}$$

A solution example for the $s = 3, g = 3, w = 3$ case is

```
0, 0, 0, 0,
1, 3, 4, 5,
2, 6, 7, 8,
3, 1, 1, 1,
4, 4, 5, 3,
5, 8, 6, 7,
6, 2, 2, 2,
7, 5, 3, 4,
8, 7, 8, 6,
```

3 Symmetries

From any solution, we can obtain a number of *symmetrical* solutions by:

- Player Permutations: players are indistinguishable, thus we can permut them in the solution. $(n!)$
- Week permutations: the order of the weeks is irrelevant, thus we can permut them in the solution. $(w!)$
- Group permutations: For a given week, the order of the groups is irrelevant. Thus we can permut them in the solution. $(w \cdot (g!))$
- In-group permutatations: for a given group of a given week, the order of the people is irrelevant. Thus we can permut them in the solution. $(w \cdot g \cdot (s!))$

Therefore, from one solution we can obtain $n! \cdot w! \cdot w \cdot (g!) \cdot w \cdot g \cdot (s!)$ equivalent ones.

As an example, for the tiny $s = 3, g = 3, w = 3$ case the number of simmetries is more than 264 millions.

4 Breaking Symmetries

- Player Permutations: We only need to assign consecutive values $0, 1, .., n - 1$ to the variables of the first column.
- In-group permutatations: We only need to force the members to be in increasing order: $x_{ijk} < x_{i(j+1)k}$
- Group permutations: Taking into account that the first elements of the groups in the same week must be different and smaller than the other elements of the corresponding groups, we can order the group in increasing order of their first member: $x_{i0k} < x_{(i+1)0k}$
- Week permutations: The previous constraints force, for all weeks, the first element of the first group to be 1. Then the second elements must be diferent at each week. We can order the weeks in increasing order of the second member of the first group: $x_{01k} < x_{01(k+1)}$

Exercise: model this problem with Gecode and count the number of solution with and without symmetry-breaking for different values of g, s, w.