Estimation of the effect of climate on infectious disesaes

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1 Motivating Question

If climate change (through some sort of variable such as temperature or precipitation) is affecting the number of cases of infectious diseasse, it is an outstanding question how strong this effect must be to identify it from some background autocorrelated value.

How strong does the signal of climate change need to be to detect it?

2 Data & model

Assuming we have some data on observed cases of a given infectious disease. The relationship between those observed cases and actual cases is a state process with some observation error, ϵ_o. Cases themselves are now given as a state space model where the number of cases at time $t+1$ are driven by the effect of both temperature variance (consistent through time) and mean temperature (increasing through time), as well as an unobserved driver that is correlated through time with the mean temperature.

Let: - Y_t represent the true number of cases at time t. - Y_t^{obs} represent the observed number of cases at time t, which includes observational error ϵ_o. - X_μ represent the mean temperature at time t, and let X_σ represent the temperature variance, assumed constant over time. - U_t represent the unobserved driver correlated with mean temperature X_μ.

2.1 State Process (Evolution) Equation

For the true cases Y_t, we specify that the cases at time $t+1$ are driven by the temperature effects and an unobserved driver U_t as follows:

$$Y_{t+1} \mid Y_t, X_\mu, U_t \sim \text{Poisson}\left(Y_t \exp\left(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t\right)\right) \tag{1}$$

where: - α is an intercept term. - β_μ is the effect of mean temperature X_μ on the cases. - β_σ is the effect of temperature variance X_σ. - γ captures the effect of the unobserved driver U_t on the cases.

2.2 Latent Driver Process

We assume that the unobserved driver U_t has temporal correlation and is influenced by the mean temperature:

$$U_{t+1} \mid U_t, X_\mu \sim \mathcal{N}(\phi U_t + \eta X_\mu, \sigma_U^2) \tag{2}$$

where: - ϕ is an autoregressive parameter governing the temporal correlation of U_t. - η is the strength of the correlation between U_t and X_μ. - σ_U^2 is the variance of U_t.

2.3 Observation Equation

The observed cases Y_t^{obs} are related to the true cases Y_t with observation error ϵ_o:

$$Y_t^{\text{obs}} \mid Y_t \sim \text{Poisson}(Y_t e^{\epsilon_o}) \tag{3}$$

with $\epsilon_o \sim \mathcal{N}(0, \sigma_o^2)$, where σ_o^2 is the variance of the observation error.

2.4 Priors

The priors for the parameters could be specified as follows:

$$\begin{aligned}
\alpha &\sim \mathcal{N}(0, 10) && \text{(4a)}\\
\beta_\mu &\sim \mathcal{N}(0, 1) && \text{(4b)}\\
\beta_\sigma &\sim \mathcal{N}(0, 1) && \text{(4c)}\\
\gamma &\sim \mathcal{N}(0, 1) && \text{(4d)}
\end{aligned}$$

$$\begin{aligned}
\phi &\sim \text{Uniform}(-1, 1) && \text{(5a)}\\
\eta &\sim \mathcal{N}(0, 1) && \text{(5b)}\\
\sigma_U^2 &\sim \text{Inverse-Gamma}(2, 1) && \text{(5c)}\\
\sigma_o^2 &\sim \text{Inverse-Gamma}(2, 1) && \text{(5d)}
\end{aligned}$$

2.5 Full Conditional Derivations

2.5.1 1. Full Conditional for Y_t

The likelihood for Y_t comes from the Poisson model and the observation model:

$$Y_{t+1} \mid Y_t, X_\mu, U_t \sim \text{Poisson}\left(Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right) \tag{6}$$

$$Y_t^{\text{obs}} \mid Y_t \sim \text{Poisson}(Y_t e^{\epsilon_o}) \tag{7}$$

The joint likelihood is:

$$
\begin{aligned}
p(Y_t \mid Y_{t+1}, Y_t^{\text{obs}}, \alpha, \beta_\mu, \beta_\sigma, \gamma, U_t, \epsilon_o) \propto \text{Poisson}(Y_{t+1} \mid Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t))\times \\
\text{Poisson}(Y_t^{\text{obs}} \mid Y_t e^{\epsilon_o})
\end{aligned}
\tag{8}
$$

Simplifying this:

$$
\begin{aligned}
p(Y_t \mid Y_{t+1}, Y_t^{\text{obs}}, \alpha, \beta_\mu, \beta_\sigma, \gamma, U_t, \epsilon_o) \propto{} & \exp\left(-Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right) \\
& \left(Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right)^{Y_{t+1}} \times \\
& \exp\left(-Y_t e^{\epsilon_o}\right) \left(Y_t e^{\epsilon_o}\right)^{Y_t^{\text{obs}}}
\end{aligned}
\tag{9}
$$

This simplifies to:

$$
\begin{aligned}
p(Y_t \mid Y_{t+1}, Y_t^{\text{obs}}, \alpha, \beta_\mu, \beta_\sigma, \gamma, U_t, \epsilon_o) \propto{} & Y_t^{Y_{t+1} + Y_t^{\text{obs}}} \times \\
& \exp\left(-Y_t \left(\exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t) + e^{\epsilon_o}\right)\right)
\end{aligned}
\tag{10}
$$

This is proportional to a Gamma distribution with shape $Y_{t+1} + Y_t^{\text{obs}}$ and rate $\exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t) + e^{\epsilon_o}$. Therefore, the full conditional for Y_t is Gamma-distributed, and there is a closed-form solution.

2.5.2 2. Full Conditional for U_t

The likelihood for U_t comes from the Normal distribution and the Poisson likelihood for Y_{t+1}:

$$
U_{t+1} \mid U_t, X_\mu \sim \mathcal{N}(\phi U_t + \eta X_\mu, \sigma_U^2) \tag{11}
$$

$$
Y_{t+1} \mid Y_t, U_t \sim \text{Poisson}(Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)) \tag{12}
$$

The joint likelihood is:

$$
\begin{aligned}
p(U_t \mid U_{t+1}, Y_{t+1}, Y_t, \alpha, \beta_\mu, \beta_\sigma, \gamma, X_\mu, \sigma_U^2) \propto & \\
& \mathcal{N}(U_{t+1} \mid \phi U_t + \eta X_\mu, \sigma_U^2)\times \\
& \text{Poisson}(Y_{t+1} \mid Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t))
\end{aligned}
\tag{13}
$$

Substituting the likelihoods:

$$
\begin{aligned}
p(U_t \mid U_{t+1}, Y_{t+1}, Y_t, \alpha, \beta_\mu, \beta_\sigma, \gamma, X_\mu, \sigma_U^2) \propto \\
\exp\left(-\frac{(U_{t+1} - (\phi U_t + \eta X_\mu))^2}{2\sigma_U^2}\right) \times \\
\exp\left(-Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right)
\end{aligned} \tag{14}
$$

This is a mixture of Gaussian and Poisson terms, which does not have a closed-form solution. Numerical methods such as Metropolis-Hastings would be used for approximation.

2.5.3 3. Full Conditional for α

The prior for α is $\mathcal{N}(0, 10)$, and the full conditional is:

$$
p(\alpha \mid Y, X_\mu, X_\sigma, U) \propto \mathcal{N}(0, 10) \times \prod_{t=1}^{T} \text{Poisson}(Y_{t+1} \mid Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)) \tag{15}
$$

Substituting the Poisson likelihood:

$$
p(\alpha \mid Y, X_\mu, X_\sigma, U) \propto \exp\left(-\frac{\alpha^2}{2 \cdot 10^2}\right) \times \prod_{t=1}^{T} \exp\left(-Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right) \tag{16}
$$

This simplifies to:

$$
p(\alpha \mid Y, X_\mu, X_\sigma, U) \propto \exp\left(-\frac{\alpha^2}{2 \cdot 10^2} - \sum_{t=1}^{T} Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right) \tag{17}
$$

This is a Gaussian-exponential mixture, and there is no closed-form solution. Approximation methods (e.g., numerical optimization) would be needed.

2.5.4 4. Full Conditional for β_μ

The prior for β_μ is $\mathcal{N}(0,1)$, and the full conditional is:

$$p(\beta_\mu \mid Y, X_\mu, X_\sigma, U) \propto \mathcal{N}(0,1) \times \prod_{t=1}^{T} \text{Poisson}(Y_{t+1} \mid Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)) \quad (18)$$

Substituting the Poisson likelihood:

$$p(\beta_\mu \mid Y, X_\mu, X_\sigma, U) \propto \exp\left(-\frac{\beta_\mu^2}{2}\right) \times \prod_{t=1}^{T} \exp\left(-Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)\right) \quad (19)$$

This leads to a Gaussian-exponential mixture, which does not have a closed-form solution.

2.5.5 5. Full Conditional for β_σ

The full conditional for β_σ follows the same structure as for β_μ:

$$p(\beta_\sigma \mid Y, X_\mu, X_\sigma, U) \propto \mathcal{N}(0,1) \times \prod_{t=1}^{T} \text{Poisson}(Y_{t+1} \mid Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)) \quad (20)$$

This does not have a closed-form solution either.

2.5.6 6. Full Conditional for γ

The full conditional for γ is:

$$p(\gamma \mid Y, X_\mu, X_\sigma, U) \propto \mathcal{N}(0,1) \times \prod_{t=1}^{T} \text{Poisson}(Y_{t+1} \mid Y_t \exp(\alpha + \beta_\mu X_\mu + \beta_\sigma X_\sigma + \gamma U_t)) \quad (21)$$

This does not have a closed-form solution.

2.5.7 7. Full Conditional for ϕ

The prior for ϕ is Uniform$(-1, 1)$, and the full conditional is:

$$p(\phi \mid U, X_\mu) \propto \text{Uniform}(-1, 1) \times \prod_{t=1}^{T} \mathcal{N}(U_{t+1} \mid \phi U_t + \eta X_\mu, \sigma_U^2) \tag{22}$$

This is a product of Gaussian distributions. The full conditional does not have a closed-form solution but can be approximated numerically.

2.5.8 8. Full Conditional for η

Similarly, for η:

$$p(\eta \mid U, X_\mu) \propto \mathcal{N}(0, 1) \times \prod_{t=1}^{T} \mathcal{N}(U_{t+1} \mid \phi U_t + \eta X_\mu, \sigma_U^2) \tag{23}$$

This is a Gaussian-exponential mixture, and no closed-form solution exists.

2.5.9 9. Full Conditional for σ_U^2

For σ_U^2, the prior is Inverse-Gamma$(2, 1)$, and the full conditional is:

$$p(\sigma_U^2 \mid U, X_\mu, \phi, \eta) \propto \text{Inverse-Gamma}(2, 1) \times \prod_{t=1}^{T} \mathcal{N}(U_{t+1} \mid \phi U_t + \eta X_\mu, \sigma_U^2) \tag{24}$$

The full conditional is an **Inverse-Gamma distribution** with updated shape and rate parameters. This has a **closed-form solution**.

2.5.10 10. Full Conditional for σ_o^2

For σ_o^2, the prior is Inverse-Gamma$(2, 1)$, and the full conditional is:

$$p(\sigma_o^2 \mid Y^{\text{obs}}, Y) \propto \text{Inverse-Gamma}(2, 1) \times \prod_{t=1}^{T} \mathcal{N}(\log(Y_t^{\text{obs}}/Y_t) \mid 0, \sigma_o^2) \tag{25}$$

This conditional is also **Inverse-Gamma** with parameters updated based on the data. Hence, it has a **closed-form solution**.