

310 Quals Strategy Compendium

June 28, 2025

1 Permutation and counting facts

Fact 1 (Number derangements of k -element set)

Derangements: D_n is the number of permutations with no fixed points.

Via inclusion exclusion.

$$D_k = k! \sum_{j=0}^k \frac{(-1)^j}{j!}.$$

Eg, if T is number of fixed points:

$$P(T = k) = \frac{1}{n!} \binom{n}{k} D_{n-k},$$

since the remaining $n - k$ must **not** be fixed.

Apply the above to "Distribution of number of fixed points"- type questions.

Fact 2 (Catalan Numbers)

Catalan Numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad n \geq 0$$

Dyck Paths

Definition 3 (Cycles)

Cycle of a permutations

Definition 4 (Descents)

Reference: check Persi and Susan's paper

2 Distribution Facts

2.1 Gaussian Facts

Fact 5 (Max of Gaussians and fluctuations)

Max of Gaussians $\sqrt{2 \log n}$ with fluctuations $1/\sqrt{\log n}$

Fact 6 (Max of sub-gaussian)

Expectation upper bound applies to correlated sub-gaussian random variables – see Vershynin.

Fact 7 (Mill's Ratio)**Fact 8 (Normal Conditional Distributions)**

$$(X, Y) \sim MVN(\mu, \Sigma) \implies X|Y \sim \dots$$

2.2 Poisson, Exponential Distribution

Superposition and thinning.

Fact 9 (Superposition)

See Poisson with integer mean? Try superposition:

$$Pois(n) = \sum_{i=1}^n Pois(1)$$

Fact 10 (Renyi Representation of Exponential)**Fact 11 (Maximum, minimum of Exponential)****3 Basic set theory and measure theory****Definition 12 (Sigma Algebra)****Definition 13 (Algebra/Field)**

Definition 14 (Outer measure)

Defined by

1. A non-negative set function
2.

The typical outer measure is **WRite this out**

Definition 15 (Measurable sets)

E is μ^* measurable if for all $B \subset \Omega$:

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Littlewood's principles idea.

4 Pi-Lambda and Good Sets**Definition 16 (π -system)**

Collection of sets that is closed under **finite intersections**

Definition 17 (λ -system)

L lambda system if

1. $\Omega \in L$
2. closed under complements
3. closed under countable disjoint unions

Alternative definition:

1. $\Omega \in L$
2. $A, B \in L$ and $A \subset B$ then $B \setminus A \in L$
3. $A_1, A_2, \dots \in L$ an increasing sequence of sets, then $\bigcup_i A_i \in L$

Theorem 18 ($\pi - \lambda$ Theorem)

If P is a π system and L is a λ -system with $P \subset L$, then $\sigma(P) \subset L$.

Use to proof uniqueness of extension from an algebra to the sigma field.

Example 19 (Qals 2017, Question 2)

Theorem 20 (Monotone Class Theorem)

Use def:

Definition 21 (Monotone Class)

M is a monotone class if

1. Closed under increasing unions
2. Closed under decreasing intersections

If an algebra A is contained in a monotone class M , then $\sigma(A) \subset \sigma(M)$.

Theorem 22 (Monotone Class for functions)

Double check this

M be a vector space of measurable functions such that

1. $1 \in M$
2. M^+ (positive functions in M) is closed under increasing limits
3. $1_A \in M$ for all A in a pi system generating \mathcal{F} .

Then M contains all bounded measurable functions

Theorem 23 (Independent π systems generate independent sigma algebras)

$$\{C_i\}_{i \in I} \text{ independent} \implies \{\sigma(C_i)\} \text{ independent}$$

Example 24 (Prove two random variables independent)

Check that generating pi systems are independent, ie $P(X \leq b, Y \leq a) = \prod P(X \leq b)P(Y \leq a)$.

5 Borel Cantelli

Example 25 (Longest head runs)

See Durrett.

A helpful idea is splitting into recurrent events. Eg in head runs: $\log_2 n(1 - \varepsilon)$: if $L_n < \log_2 n(1 - \varepsilon)$, then none of these blocks can be all heads.

6 Integration

Definition 26 (Lebesgue Integral)

$$\int_{\Omega} f(\omega) d\mu(\omega) = \sup \left(\sum_{i=1}^n v_i \mu(A_i) \right)$$

where $v_i = \inf_{\omega \in A_i} f(\omega)$ and the sup is over all partitions of Ω .

Theorem 27 (MCT)

Note that can also do for general functions (not necessarily non negative) so long as $f_n \geq g \in L^1$.

Theorem 28 (Fatou's Lemma)

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu$$

Theorem 29 (DCT)

If $f_n \rightarrow f$ a.e ω , and there exists g such that $|f_n(\omega)| \leq g(\omega)$ a.e ω and $\int g d\mu < \infty$ then exchange integral and limit. Must dominate the sequence.

Theorem 30 (Scheffe's Lemma)

Is a statement about combining L^1 and as convergence. If $f_n \xrightarrow{\text{a.s.}} f \in L^1$, then:

$$\|f_n - f\|_{L^1} \rightarrow 0 \iff \int |f_n| d\mu \rightarrow \int |f| d\mu$$

Theorem 31 (Vitali Convergence Theorem)

If f_n is UI and $f_n \xrightarrow{p} f$, then $f \in L^1$ and $\|f_n - f\|_{L^1} \rightarrow 0$.

Theorem 32 (Generalized DCT)

???? If $|f_n| \leq g_n$ such that $g_n \rightarrow g \in L^1$ (convergence in L^1) then $\int f_n d\mu \rightarrow \int f d\mu$

Theorem 33 (Reverse Fatou)

If $f_n \leq g \in L^1$ then :

$$\limsup_n \int f_n d\mu \leq \int \limsup_n f_n d\mu$$

7 Product σ -algebras

8 Stein's Method (Poisson)

8.1 Method 1 - Dependency Graphs

8.2 Method 2 - when dependency graph doesn't work (ie complete)

Example 34 (Fixed Points - 310a HW8)

Let σ be a uniformly chosen permutation in the symmetric group S_n . Let $W = \#\{i : \sigma(i) = i\}$ (the number of fixed points in σ). Show that W has an approximate Poisson(1) distribution by using Stein's method to get an upper bound on $\|P_W - \text{Poisson}(1)\|$. (Hint: see section 4.5 of Arratia-Goldstein-Gordon.) Give details for this specific case.

Let $I = [n]$. We choose $B_\alpha = \{\alpha\}$ and use Theorem 1 from Arratia-Goldstein-Gordon.
For each $i \in I$, let

$$X_i = \begin{cases} 1 & \text{if } \sigma(i) = i \\ 0 & \text{otherwise} \end{cases}.$$

Naturally, $P(X_i = 1) = \frac{1}{n}$. We let $W = \sum_{i \in I} X_i$ and $\lambda = E[W] = 1$. We now use Stein's method as given in Arratia-Goldstein-Gordon Theorem 1 to get an upper bound on $\|P_W - \text{Pois}(1)\|$.

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta = \sum_{\alpha \in I} p_\alpha^2 = \frac{1}{n}.$$

Next, because we let $B_\alpha = \{\alpha\}$,

$$b_2 = 0.$$

Finally, for the third term, by Lemma 2 (p 418) in Arratia et al,

$$b_3 \leq \min_{1 \leq k \leq n} \left(\frac{2k}{n-k} + 2n2^{-k} e^e \right) \sim 2 \frac{(2 \log_2 n + e/\ln 2)}{n},$$

due to the fact that $\lambda = 1$ in our problem, so $\lambda = o(n)$

Now note that as $n \rightarrow \infty$, $b_1 \rightarrow 0$ and $b_3 \rightarrow 0$, so, noting that the Arratia paper's definition of TV distance is twice our definition of TV distance:

$$\|P_W - \text{Pois}(1)\| \leq b_1 + b_2 + b_3 = \frac{1}{n} + \frac{4 \log_2 n + 2e/\ln 2}{n} + o(1)$$

Now as $n \rightarrow \infty$, $b_1 \rightarrow 0$ and $b_3 \rightarrow 0$, so $\|P_W - \text{Pois}(1)\| \rightarrow 0$.

Example 35 (Near Fixed Points- 2004 Q2)

9 Approximations

$$1 - x \leq e^{-x} \quad 1 - x \geq e^{-2x} \quad \text{both for small } x?$$

9.1 Binomial Coeffs and Stirlings

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

Stirlings