310 Quals Strategy Compendium

June 23, 2025

1 Permutation and counting facts

Fact 1 (Number derangements of k-element set)
Derangements: D_n is the number of permutations with no fixed points.

Via inclusion exclusion.

$$D_k = k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}.$$

Eg, if T is number of fixed points:

$$P(T = k) = \frac{1}{n!} \binom{n}{k} D_{n-k},$$

since the remaining $n - k$ must **not** be fixed.

Apply the above to "Distribution of number of fixed points"- type questions.

Fact 2 (Catalan Numbers)
Catalan Numbers:
Dyck Paths

Definition 3 (Cycles)
Cycle of a permutations

Definition 4 (Descents)

Reference: check Persi and Susan's paper

2 Distribution Facts

2.1 Gaussian Facts

Fact 5 (Max of Gaussians and fluctuations)
Max of Gaussians $\sqrt{2 \log n}$ with fluctuations $1/\sqrt{\log n}$

Fact 6 (Max of sub-gaussian)
Expectation upper bound applies to correlated sub-gaussian reandom variables – see Vershynin.

Fact 7 (Mill's Ratio)

Fact 8 (Normal Conditional Distributions)

$$(X, Y) \sim MVN(\mu, \Sigma \implies X|Y \sim \ldots$$

2.2 Poisson, Exponential Distribution

Superposition and thinning.

Fact 9 (Superposition)
See Poisson with integer mean? Try superposition:

$$Pois(n) = \sum_{i=1}^{n} Pois(1)$$

Fact 10 (Renyi Representation of Exponential)

Fact 11 (Maximum, minimum of Exponential)

3 Stein's Method (Poisson)

3.1 Method 1 - Dependency Graphs

3.2 Method 2 - when dependency graph doesn't work (ie complete)

Example 12 (Fixed Points - 310a HW8)

Let σ be a uniformly chosen permutation in the symmetric group S_n. Let $W = \#\{i : \sigma(i) = i\}$ (the number of fixed points in σ). Show that W has an approximate Poisson(1) distribution by using Stein's method to get an upper bound on $\|P_W - \text{Poisson}(1)\|$. (Hint: see section 4.5 of Arratia-Goldstein-Gordon.) Give details for this specific case.

Let $I = [n]$. We choose $B_\alpha = \{\alpha\}$ and use Theorem 1 from Arratia-Goldstein-Gordon.
For each $i \in I$, let

$$X_i = \begin{cases} 1 \text{ if } \sigma(i) = i \\ 0 \text{ otherwise} \end{cases}.$$

Naturally, $P(X_i = 1) = \frac{1}{n}$. We let $W = \sum_{i \in I} X_i$ and $\lambda = E[W] = 1$. We now use Stein's method as given in Arratia-Goldstein-Gordon Theorem 1 to get an upper bound on $\|P_W - Pois(1)\|$.

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta = \sum_{\alpha \in I} p_\alpha^2 = \frac{1}{n}.$$

Next, because we let $B_\alpha = \{\alpha\}$,

$$b_2 = 0.$$

Finally, for the third term, by Lemma 2 (p 418) in Arratia et al,

$$b_3 \leqslant \min_{1<k<n}\left(\frac{2k}{n-k} + 2n2^{-k}e^e\right) \sim 2\frac{(2log_2 n + e/ln2)}{n},$$

due to the fact that $\lambda = 1$ in our problem, so $\lambda = o(n)$
Now note that as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so, noting that the Arratia paper's definition of TV distance is twice our definition of TV distance:

$$\|P_W - Pois(1)\| \leqslant b_1 + b_2 + b_3 = \frac{1}{n} + \frac{4\log_2 n + 2\epsilon/\ln 2}{n} + o(1)$$

Now as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so $\|P_W - Pois(1)\| \longrightarrow 0$.

Example 13 (Near Fixed Points- 2004 Q2)

4 Approximations

$$1 - x \leqslant e^{-x} \quad 1 - x \geqslant e^{-2x} \quad \text{both for small } x?$$

5 Asymptotics

Limit	**Function** $f(x)$	**Equivalent** $g(x)$	**Condition**
$x \longrightarrow 0$	$\log(1+x)$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\log(1-x)$	$-x$	$x \ll 1$
$x \longrightarrow 0$	$e^x - 1$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\sin x$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$1 - \cos x$	$\frac{x^2}{2}$	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\tan x$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\arcsin x$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$(1+x)^\alpha$	$1 + \alpha x$	Fixed α, $\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\Gamma(1+x)$	$1 - \gamma x$	$\gamma = 0.577\ldots$
$n \longrightarrow \infty$	$n!$	$\sqrt{2\pi n}\,(n/e)^n$	Stirlings approximation
$n \longrightarrow \infty$	$H_n = \sum_{k=1}^{n} \frac{1}{k}$	$\log n + \gamma$	Harmonic numbers
$n \longrightarrow \infty$	$\binom{2n}{n}$	$\frac{4^n}{\sqrt{\pi n}}$	Central binomial
$n \longrightarrow \infty$	$\left(1 + \frac{1}{n}\right)^n$	e	Definition of e
$n \longrightarrow \infty$	$\zeta(n)$	1	Riemann zeta tail

Table 1: Common asymptotic equivalences: $f(x) \sim g(x)$ means $f(x)/g(x) \longrightarrow 1$.

Operation (as $n \longrightarrow \infty$ or $x \longrightarrow 0$)	**Safe to replace f by g?**	**Remarks**
$\lim f(n)$	Yes	If $f \sim g$ and $\lim g = L \in \mathbb{R} \cup \{\infty\}$, then $\lim f = L$.
$\frac{f(n)}{g(n)}$	Yes	By definition $\frac{f}{g} \longrightarrow 1$. Useful for verifying asymptotic equivalence itself.
$f(n)\,g(n)$ or $f(n) \cdot h(n)$	Yes (usually)	Multiplicative errors stay small: $(fg)/(gg) = f/g \longrightarrow 1$ if $h \sim g$. Be sure h is bounded away from 0.
$f(n) - g(n)$	No	Only $f - g = o(g)$ is guaranteed. The difference need <u>not</u> vanish; e.g. $\log n + \gamma - \log n \longrightarrow \gamma$.
$\log f(n)$	Caution	If $f \sim g$ and both $\longrightarrow \infty$ at comparable rates, $\log f - \log g = \log(1 + o(1)) = o(1)$, so safe. If $f \longrightarrow C > 0$, extra care needed.
$e^{f(n)}$ or any non-linear analytic map	Caution / No	Small <u>relative</u> error in exponent can balloon: $e^f = e^{g(1+o(1))} = e^g\, e^{o(g)}$. Safe only when $g = o(1)$.
$\lfloor f(n) \rfloor$, $\operatorname{sign}\big(f(n)\big)$	No	Discontinuous operations destroy the $f/g \longrightarrow 1$ guarantee. Analyze separately.

Table 2: Rule of thumb for substituting $f \sim g$ in various expressions. Here $f \sim g$ means $\frac{f(n)}{g(n)} \longrightarrow 1$.