310 Quals Strategy Compendium

June 23, 2025

1 Permutation and counting facts

Fact 1 (Number derangements of k-element set)
Derangements: D_n is the number of permutations with no fixed points.

Via inclusion exclusion.

$$D_k = k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}.$$

Eg, if T is number of fixed points:

$$P(T = k) = \frac{1}{n!}\binom{n}{k} D_{n-k},$$

since the remaining $n - k$ must **not** be fixed.

Apply the above to "Distribution of number of fixed points"- type questions.

Fact 2 (Catalan Numbers)
Catalan Numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad n \geqslant 0$$

Dyck Paths

Definition 3 (Cycles)
Cycle of a permutations

Definition 4 (Descents)

Reference: check Persi and Susan's paper

2 Distribution Facts

2.1 Gaussian Facts

Fact 5 (Max of Gaussians and fluctuations)
Max of Gaussians $\sqrt{2 \log n}$ with fluctuations $1/\sqrt{\log n}$

Fact 6 (Max of sub-gaussian)
Expectation upper bound applies to correlated sub-gaussian reandom variables – see Vershynin.

Fact 7 (Mill's Ratio)

Fact 8 (Normal Conditional Distributions)

$$(X, Y) \sim MVN(\mu, \Sigma \implies X|Y \sim \ldots$$

2.2 Poisson, Exponential Distribution

Superposition and thinning.

Fact 9 (Superposition)
See Poisson with integer mean? Try superposition:

$$Pois(n) = \sum_{i=1}^{n} Pois(1)$$

Fact 10 (Renyi Representation of Exponential)

Fact 11 (Maximum, minimum of Exponential)

3 Stein's Method (Poisson)

3.1 Method 1 - Dependency Graphs

3.2 Method 2 - when dependency graph doesn't work (ie complete)

Example 12 (Fixed Points - 310a HW8)

Let σ be a uniformly chosen permutation in the symmetric group S_n. Let $W = \#\{i : \sigma(i) = i\}$ (the number of fixed points in σ). Show that W has an approximate Poisson(1) distribution by using Stein's method to get an upper bound on $\|P_W - \text{Poisson}(1)\|$. (Hint: see section 4.5 of Arratia-Goldstein-Gordon.) Give details for this specific case.

Let $I = [n]$. We choose $B_\alpha = \{\alpha\}$ and use Theorem 1 from Arratia-Goldstein-Gordon.
For each $i \in I$, let

$$X_i = \begin{cases} 1 \text{ if } \sigma(i) = i \\ 0 \text{ otherwise} \end{cases}.$$

Naturally, $P(X_i = 1) = \frac{1}{n}$. We let $W = \sum_{i \in I} X_i$ and $\lambda = E[W] = 1$. We now use Stein's method as given in Arratia-Goldstein-Gordon Theorem 1 to get an upper bound on $\|P_W - Pois(1)\|$.

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta = \sum_{\alpha \in I} p_\alpha^2 = \frac{1}{n}.$$

Next, because we let $B_\alpha = \{\alpha\}$,

$$b_2 = 0.$$

Finally, for the third term, by Lemma 2 (p 418) in Arratia et al,

$$b_3 \leqslant \min_{1<k<n}\left(\frac{2k}{n-k} + 2n2^{-k}e^e\right) \sim 2\frac{(2log_2 n + e/ln2)}{n},$$

due to the fact that $\lambda = 1$ in our problem, so $\lambda = o(n)$
Now note that as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so, noting that the Arratia paper's definition of TV distance is twice our definition of TV distance:

$$\|P_W - Pois(1)\| \leqslant b_1 + b_2 + b_3 = \frac{1}{n} + \frac{4\log_2 n + 2\epsilon/\ln 2}{n} + o(1)$$

Now as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so $\|P_W - Pois(1)\| \longrightarrow 0$.

Example 13 (Near Fixed Points- 2004 Q2)

4 Approximations

$$1 - x \leqslant e^{-x} \quad 1 - x \geqslant e^{-2x} \quad \text{both for small } x?$$