

# 1 Asymptotics

Limit	Function $f(x)$	Equivalent $g(x)$	Condition
$x \rightarrow 0$	$\log(1+x)$	$x$	$ x  \ll 1$
$x \rightarrow 0$	$\log(1-x)$	$-x$	$x \ll 1$
$x \rightarrow 0$	$e^x - 1$	$x$	$ x  \ll 1$
$x \rightarrow 0$	$\sin x$	$x$	$ x  \ll 1$
$x \rightarrow 0$	$1 - \cos x$	$\frac{x^2}{2}$	$ x  \ll 1$
$x \rightarrow 0$	$\tan x$	$x$	$ x  \ll 1$
$x \rightarrow 0$	$\arcsin x$	$x$	$ x  \ll 1$
$x \rightarrow 0$	$(1+x)^\alpha$	$1 + \alpha x$	Fixed $\alpha$ , $ x  \ll 1$
$x \rightarrow 0$	$\Gamma(1+x)$	$1 - \gamma x$	$\gamma = 0.577 \dots$
$n \rightarrow \infty$	$n!$	$\sqrt{2\pi n} (n/e)^n$	Stirlings approximation
$n \rightarrow \infty$	$H_n = \sum_{k=1}^n \frac{1}{k}$	$\log n + \gamma$	Harmonic numbers
$n \rightarrow \infty$	$\binom{2n}{n}$	$\frac{4^n}{\sqrt{\pi n}}$	Central binomial
$n \rightarrow \infty$	$(1 + \frac{1}{n})^n$	$e$	Definition of $e$
$n \rightarrow \infty$	$\zeta(n)$	1	Riemann zeta tail

Table 1: Common asymptotic equivalences:  $f(x) \sim g(x)$  means  $f(x)/g(x) \rightarrow 1$ .

Operation (as $n \rightarrow \infty$ or $x \rightarrow 0$ )	Safe to replace $f$ by $g$ ?	Remarks
$\lim f(n)$	Yes	If $f \sim g$ and $\lim g = L \in \mathbb{R} \cup \{\infty\}$ , then $\lim f = L$ .
$\frac{f(n)}{g(n)}$	Yes	By definition $\frac{f}{g} \rightarrow 1$ . Useful for verifying asymptotic equivalence itself.
$f(n)g(n)$ or $f(n) \cdot h(n)$	Yes (usually)	Multiplicative errors stay small: $(fg)/(gg) = f/g \rightarrow 1$ if $h \sim g$ . Be sure $h$ is bounded away from 0.
$f(n) - g(n)$	No	Only $f - g = o(g)$ is guaranteed. The difference need <u>not</u> vanish; e.g. $\log n + \gamma - \log n \rightarrow \gamma$ .
$\log f(n)$	Caution	If $f \sim g$ and both $\rightarrow \infty$ at comparable rates, $\log f - \log g = \log(1 + o(1)) = o(1)$ , so safe. If $f \rightarrow C > 0$ , extra care needed.
$e^{f(n)}$ or any non-linear analytic map	Caution / No	Small <u>relative</u> error in exponent can balloon: $e^f = e^{\overline{g(1+o(1))}} = e^g e^{o(g)}$ . Safe only when $g = o(1)$ .
$\lfloor f(n) \rfloor, \text{sign}(f(n))$	No	Discontinuous operations destroy the $f/g \rightarrow 1$ guarantee. Analyze separately.

Table 2: Rule of thumb for substituting  $f \sim g$  in various expressions. Here  $f \sim g$  means  $\frac{f(n)}{g(n)} \rightarrow 1$ .