# 310 Quals Strategy Compendium

July 5, 2025

## 1 Permutation and counting facts

**Fact 1** (Number derangements of $k$-element set)
Derangements: $D_n$ is the number of permutations with no fixed points.

Via inclusion exclusion.

$$D_k = k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}.$$

Eg, if $T$ is number of fixed points:

$$P(T = k) = \frac{1}{n!} \binom{n}{k} D_{n-k},$$

since the remaining $n - k$ must **not** be fixed.

Apply the above to "Distribution of number of fixed points"- type questions.

**Fact 2** (Catalan Numbers)
Catalan Numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad n \geqslant 0$$

Dyck Paths

**Definition 3** (Cycles)
Let $s \in S_n$ be a permutation. Then $L_i(s) = \inf\{j : s^j(i) = i\}$. Ie the first time we come back to $i$. See pg 101 Dembo remark. Durret Ex 2.2.4. Then the cycle is the collection $\{s^j(i) : 1 \leqslant j \leqslant L_i(s)\}$– ie the elements of $[n]$ before we come back to $i$.

If $T_n$ = #cycles, $(T_n - \log n)/\sqrt{\log n} \xrightarrow{d} \mathcal{N}(0, 1)$.

**Definition 4** (Descents)

Reference: check Persi and Susan's paper

# 2 Distribution Facts

## 2.1 Gaussian Facts

**Fact 5** (Max of Gaussians and fluctuations)
Max of Gaussians $\sqrt{2 \log n}$ with fluctuations $1/\sqrt{\log n}$

**Fact 6** (Max of sub-gaussian)
Expectation upper bound applies to correlated sub-gaussian reandom variables – see Vershynin.

**Fact 7** (Mill's Ratio)

**Fact 8** (Normal Conditional Distributions)

$$(X, Y) \sim MVN(\mu, \Sigma \implies X|Y \sim \ldots$$

## 2.2 Poisson, Exponential Distribution

Superposition and thinning.

**Fact 9** (Superposition)
See Poisson with integer mean? Try superposition:

$$Pois(n) = \sum_{i=1}^{n} Pois(1)$$

**Fact 10** (Renyi Representation of Exponential)

**Fact 11** (Maximum, minimum of Exponential)

# 3 Basic set theory and measure theory

**Definition 12** (Sigma Algebra)

**Definition 13** (Algebra/Field)

**Definition 14** (Outer measure)
Defined by

1. A non-negative set function
2. ......

The typical outer measure is WRite this out

**Definition 15** (Measurable sets)
$E$ is $\mu^*$ measurable if for all $B \subset \Omega$:

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Write about littlewood's principles here

**Example 16** (Non-measurable sets)
Todo

## 4 Pi-Lambda and Good Sets

**Definition 17** ($\pi$-system)
Collection of sets that is closed under **finite intersections**

**Definition 18** ($\lambda$-system)
$L$ lambda system if

1. $\Omega \in L$
2. closed under complements
3. closed under countable disjoint unions

Alternative definition:

1. $\Omega \in L$
2. $A, B \in L$ and $A \subset B$ then $B \setminus A \in L$
3. $A_1, A_2, \dots, \in L$ an increasing sequence of sets, then $\bigcup_i A_i \in L$

Note that a collection is a $\sigma$-algebra $\iff$ it is both a pi system and a lambda system.

**Theorem 19** ($\pi - \lambda$ Theorem)
If $P$ is a $\pi$ system and $L$ is a $\lambda$-system with $P \subset L$, then $\sigma(P) \subset L$.
Use to proof uniqueness of extension from an algebra to the sigma field.

**Example 20** (Quals 2017, Question 2)

**Theorem 21** (Monotone Class Theorem)
Use def:

**Definition 22** (Monotone Class)
$M$ is a monotone class if

1. Closed under increasing unions
2. Closed under decreasing intersections

If an algebra $A$ is contained in a monotone class $M$, then $\sigma(A) \subset M$.

**Theorem 23** (Monotone Class for functions)
Double check this
$M$ be a vector space of $\mathbb{R}$-valued functions on $\Omega$ such that

1. $1 \in M$
2. $M$ is a vector space over $\mathbb{R}$. Ie closed under addition and scalar mult
3. If $h_n \geqslant 0 \in M$ and $h_n \uparrow h$, then $h \in M$.

Then if $P$ is a pi system such that $\mathbf{1}_A \in M$ for all $M \in P$, then $M$ contains all functions that are measurable with respect to $\sigma(P)$.

Ie want to prove something about some functions measurable on $\sigma(P)$...

**Theorem 24** (Independent $\pi$ systems generate independent sigma algebras)

$$\{C_i\}_{i \in I} \text{ independent} \implies \{\sigma(C_i)\} \text{ independent}$$

**Example 25** (Prove two random variables independent)
Check that generating pi systems are independent, ie $P(X \leqslant b, Y \leqslant a) = \prod P(X \leqslant b)P(Y \leqslant a)$.

**Definition 26** (Independence of (uncountable) collections of sets)
$\{A_\alpha\}_{\alpha \in I}$ each be a collection of sets. Independent if any **finite subcollection** are mutually independent.

## 5 Extension theorems

**Theorem 27** (Caratheodory)
Extend a measure on an algebra to a $\sigma$ algebra, uniquely if finite.

The important idea is make an outer measure, then prove the lemma is that the $\mu^*$ measurable sets in $\mathcal{F}$ for a sigma algebra on which $\mu^*$ is countably additive, ie a bonafide measure.

## 6 Random variables

**Definition 28** (Random variable)
$X : \Omega \longrightarrow \$$ is measurable if:

$$X^{-1}(B) := \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{S}$$

Fix an arbitrary set in the co-sigma algebra and check that its preimage is measurable-$\mathcal{F}$.

**Recipe 29** (Measurability)
If $\mathcal{S} = \sigma(\mathcal{A})$ and $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{A}$, then $X$ is measurable.

Note– nothing necessary about it being a pi system, just pick any convenient generators.

*Proof.* Such sets form a sigma algebra and $\mathcal{S} = \sigma(\mathcal{A})$. □

**Definition 30** (Characterize distribution functions of real-valued random variable)
Dembo Theorem 1.2.37.

1. Non decreasing
2. $\lim_{x \longrightarrow \infty} F(x) = 1$ and $\longrightarrow -\infty$ gives 0
3. $F$ right continuous

## 7 0-1 Laws

**Definition 31** (Tail field)
Defining $\mathcal{T}_n = \sigma(X_r, r > n)$ and the tail sigma algebra of the process is $\mathcal{T} = \cap_n \mathcal{T}_n$.

**Theorem 32** (Kolmogorov 0-1)
The tail sigma field is P-trivial. Dembo 1.4.10

## 8 Borel Cantelli

**Example 33** (Longest head runs)
See Durrett.

A helpful idea is splitting into blocks.

**Truncation** arguments. Want to show something about $\{X_n\}$. Consider:

$$Y_n = X_n \mathbf{1}[|X_n| \leqslant c_n],$$

which will then be integrable. Then prove something about $Y_n$, you can transfer this knowledge to knowledge about $X_n$ by the following idea:

$$P(X_n \neq Y_n \text{ io}) = 0 \text{ if } \sum_{i=1}^{n} P(|X_n| > c_n) < \infty$$

So if the above is zero, eventually $X_n = Y_n$, so the limiting behavior of $Y_n$ is the same as that of $X_n$. The difficulty is picking a $c_n$ such that the above holds.

**Theorem 34** (Borel Cantelli 3)
Not as useful but still check these
If you have a filtration $\{F_n\}$ and $A_n \in \mathcal{F}_n$, then:

1. ( analog of Borel Cantelli 1)

$$\sum_{k=1}^{\infty} P(A_k|\mathcal{F}_{k-1})(\omega) < \infty \implies \sum \mathbf{1}[\omega \in A_k\} < \infty$$

2. Analog of BC 2

$$\sum_k P(A_k|\mathcal{F}_{k-1})(\omega) = \infty \implies \frac{\sum \mathbf{1}[A_k(\omega)]}{\sum P(A_k|\mathcal{F}_{k-1})(\omega)|} \to 1$$

## 9 Modes of Convergence

**Recipe 35** (Proving as convergence)
Some ideas:

1. BC 1: check that $P(|X_n - X| > \epsilon \text{ i.o}) = 0$. Consider *subsequence trick* below.
2. If $X_n \xrightarrow{p} X$, then $X_{n_k} \xrightarrow{\text{a.s.}} X$ for a subsequence. (Good for counter examples)
3. $X_n \xrightarrow{p} X$ and $X_n$ is monotone (ie for all $\omega$, $X_n(\omega)$ is increasing in $n$), then $X_n \xrightarrow{\text{a.s.}} X$.
4. Continuous mapping theorem
5. Skorohod's Representation. If $X_n \implies X$ weakly, then there exists a probability space and random variables $Z_n =^d X_n$ and $Z =^d X$ such that $Z_n \xrightarrow{\text{a.s.}} Z$. (Good for counter examples)

**Theorem 36** (BC Subsequence trick)
Sometimes $\sum P(A_n) = \infty$.

1. Pick a subsequence such that it's finite.
2. Shows that $A_{n_k}$ infinitely often occurs with probability 0.
3. Apply interpolation or some other argument for everything in between, ie to conclude that $A_n$ infinitely often wp 0 also.

Eg for monotone $X_n$:

$$\frac{X_n}{\mathbf{E}X_n} \leqslant \frac{X_{n_{k+1}-1}}{\mathbf{E}X_{n_k}}$$

So $\limsup_k \frac{X_{n_{k+1}-1}}{\mathbf{E}X_{n_k}} \leqslant 1 \implies \limsup_n \frac{X_n}{\mathbf{E}X_n} \leqslant 1$. We can control the left hand side by controlling the right hand side along our choice of subsequence.

See Dembo notes subsequence section Examples– SLLN, Brownian LLN, LIL.

SLLN in Persi notes and Renewal theorem in Dembo

**Recipe 37** (Proving almost sure limit is finite)
Try:

1. Take expectation - use MCT or something and show that finite, so random variable is finite wp 1
2. If $\sum_n \mathbf{Var}(X_n) < \infty$ and $\mathbf{E}X_n = 0$ (2.3.17 Dembo) (Kolmogorov 2 series theorem)
3. Kolmogorov 3 series

**Recipe 38** (Proving conv in prob)
Some ideas:

1. Show almost sure convergence
2. Show convergence in $L^p$ for $p \geqslant 1$ (in $L^2$ in particular useful):
    - Show that $\mathbf{E}X_n \longrightarrow \mu$, $\mathbf{Var}X_n \longrightarrow 0$, then $X_n \longrightarrow \mu$ in $L^2$
3. Show $\mathbf{Var}X_n/b_n^2 \longrightarrow 0$ and use Markov to get $b_n^{-1}(X_n - \mathbf{E}X_n) \xrightarrow{p} 0$ (and also in $L^2$).
4. CMT
5. $X_n \xrightarrow{d} c$ constant
6. Truncation tool: **Weak law for Triangular Array**. For when we don't have second moments.

**Theorem 39** (Coupon Collector)
If $T_n$ is time to get all $n$ possible coupons. Then

$$T_n/(n \log n) \xrightarrow{p} 1$$

(and in $L^2$)– see Dembo 2.1.8.

**Theorem 40** (Weak law triangular array - 2.1.11 Dembo)
Check this Suppose triangular array $\{X_{n,k}\}_{n \in \mathbb{N}, k \leqslant n}$ of pairwise independent rv. Define a truncated array with the same *within-row* truncation:

$$\overline{X}_{n,k} = X_{n,k}\mathbf{1}[|X_{n,k}| < b_n]$$

. If we have two conditions:

$$\sum_{k=1}^{n} P(|X_{n,k}| > b_n) \longrightarrow^{n} 0,$$

and,

$$b_n^{-2} \sum_{k=1}^{n} \mathbf{Var}(\overline{X}_{n,k}) \longrightarrow 0,$$

Then:

$$b_n^{-1}(\sum_{k=1}^{n} X_{n,k} - a_n) \xrightarrow{p} 0 \quad \text{where } a_n = \sum_{k=1}^{n} \mathbf{E}\overline{X}_{n,k}.$$

In case of St Petersburg paradox, then show that $a_n/b_n \longrightarrow 1/2$ so that we get $b_n^{-1} \sum_{k=1}^{n} X_k = (b_n^{-1} \sum_{k=1}^{n} X_{n,k}) \xrightarrow{p} 1/2$.

**Recipe 41** (Prove $L^p$ convergence)
Ideas - see session notes

1. $X_n \xrightarrow{\text{a.s.}} X$ and $X_n$ uniformly bounded then $X_n \longrightarrow^{L1} X$.
2. If $X_n$ uniformly bounded then $X_n \xrightarrow{d} 0 \implies X_n \longrightarrow^{L1} 0$.
3. If $|X_n|^p$ is UI then $X_n \xrightarrow{d} X \implies \mathbf{E}|X_n|^p \longrightarrow E|X|^p$
4. Scheffe's Lemma

**Recipe 42** (Proving UI)
Try

1. Sums of UI sequence of rvs are UI
2. Domination: If $E \sup |X_i| < \infty$ then $\{X_i\}$ is UI
3. If $\{X_i\}$ is UI, then $\sup E|X_i| < \infty$ but not the converse.
4. Prop 8.3.3: $X_n$ UI $\iff$ $\sup E|X_i| < \infty$ and for all $\varepsilon$, there exists $\delta$ for all A such that $P(A) < delta$ then $\mathbf{E}[|X_n|\mathbf{1}[A]] < \varepsilon$
5. If $\sup \mathbf{E}|X_i|^r < \infty$ if $r > 1$, then UI
6. $X \in L^1$, then $\{\mathbf{E}[X|\mathcal{G}] : \mathcal{G} \subset \mathcal{F}\}$ is UI
7. $L^1$ convergence implies UI

## 10 Integration

**Definition 43** (Lebesgue Integral)

$$\int_\Omega f(\omega) d\mu(\omega) = \sup \left( \sum_{i=1}^{n} v_i \mu(A_i) \right)$$

where $v_i = \inf_{\omega \in A_i} f(\omega)$ and the sup is over all partitions of $\Omega$.

**Theorem 44** (MCT)
Note that can also do for general functions (not necessarily non negative) so long as $f_n \geqslant g \in L^1$.

**Theorem 45** (Fatou's Lemma)

$$\int \liminf_n f_n d\mu \leqslant \liminf_n \int f_n d\mu$$

**Theorem 46** (DCT)
If $f_n \longrightarrow f$ a.e $\omega$, and there exists $g$ such that $|f_n(\omega)| \leqslant g(\omega)$ a.e $\omega$ and $\int g d\mu < \infty$ then exchange integral and limit. Must dominate the sequence.

**Theorem 47** (Scheffe's Lemma)
Is a statement about combining $L^1$ and as convergence. If $f_n \xrightarrow{\text{a.s.}} f \in L^1$, then:

$$\|f_n - f\|_{L^1} \longrightarrow 0 \iff \int |f_n| d\mu \longrightarrow \int |f| d\mu$$

We always have that convergence in $L^1$ implies convergence of the expectations (without any a.s. convergence assumption) , but the other direction is Scheffe's contribution.

**Theorem 48** (Reverse Fatou)
If $f_n \leqslant g \in L^1$ then :

$$\limsup_n \int f_n d\mu \leqslant \int \limsup f_n d\mu$$

**Fact 49** ($L^p$ spaces nested)

$$\|Y\|_r \leqslant \|Y\|_q$$

**Fact 50** ($L^q$ convergence fact - Dembo 1.3.28)
$X_n \rightarrow^{L^q} X \implies \mathbf{E}|X_n|^q \longrightarrow \mathbf{E}|X_\infty|^q$ for any $q$. (Minkowski)

Also for only $q \in \mathbb{N}$, $\mathbf{E}X_n^q \longrightarrow \mathbf{E}X_\infty^q$. (some wild algebraic shit for odd $q$).

**Theorem 51** (Holder's)
If $p, q > 1$ with $1/p + 1/q = 1$ then

$$E|XY| \leqslant \|X\|_p \|Y\|_q$$

Cauchy Schwarz is special case.

**Theorem 52** (Minkowski)
Triangle inequality for the $\|\cdot\|_p$ norm

**Definition 53** (Uniform integrability (UI))
Possibly uncountable collection $\{X_\alpha : \alpha \in I\}$ is called UI if

$$\lim_{M \to \infty} \sup_{\alpha \in I} \mathbf{E}[|X_\alpha| \mathbf{1}[|X_\alpha| > M] = 0$$

**Fact 54** (Dominated implies UI)
If $|X_\alpha| \leqslant Y$ for integrable Y, then collection is UI.

As a corollary, any finite collection of integrable rv is UI.

**Theorem 55** (Vitali Convergence Theorem)
Supposing that $X_n \xrightarrow{p} X$, then:

$$\{X_n\} \text{ is UI} \iff X_n \rightarrow^{L1} X \iff X_n \text{ is integrable for all } n \leqslant \infty \text{ and } E|X_n| \longrightarrow E|X_\infty|.$$

# 11 Product $\sigma$-algebras

Existence of unique product measure of $n$ $\sigma-$ finite measures.

**Theorem 56** (Kolmogorov Extension)
Unique probability measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_c)$ with correct FDDs.

**Theorem 57** (Fubini's)
Conditions: $h \geqslant 0$ or $\int |h| d\mu < \infty$ where $\mu = \mu_1 \times \mu_2$.

# 12 Weak Convergence

## 12.1 Methods

1. **Direct**. Show that $F_n(x) \longrightarrow F(x)$ for all continuity points.
2. If there's a density, try to show that $f_n(x) \longrightarrow f_\infty$ and check that $f_\infty$ a valid pdf.
3. If $X_n \geqslant 0$, show that $\int_0^\infty \exp(-\lambda x) d\mu_n(x) \longrightarrow L(\lambda) = \int_0^\infty \exp(-\lambda x) d\mu_\infty(x)$. Note that $L(\lambda)$ is a Laplace transform of some $\mu$ if $L(\lambda) \downarrow 1$ as $\lambda \downarrow 0$. (Just need for positive $\lambda$.
4. MGFs
5. Characteristic functions- show that $\phi_n(t) \longrightarrow \phi(t)$ for all $t \in \mathbb{R}$. ($\phi$ is a characteristic function of a probability measure if $\psi(t) \longrightarrow 1$ as $t \downarrow 0$.
6. CLT

**Example 58** (Cycling of Random Number Generators (2007 Q2))
Similar to birthday problem.

$$\begin{aligned} P(T > k) &= \prod_{i=1}^{k} (1 - \frac{i}{n}) && (1) \\ &\approx \prod \exp(-1/n) && (2) \\ &\approx \exp(-k^2/n). && (3) \end{aligned}$$

So $P(T > x\sqrt{n}) \approx \exp(-x^2/2)$ should work.

Need to justify this rigorously. To do so, use $|\log(1+x) - x| < Cx^2$ when $|x| < 1/2$. Ie, $\log(1+x) = x + O(x^2)$.

See lecture 1 in 310a.

## 13 CLT

Heuristic: "not too dependent", "no few terms dominate".

**Theorem 59** (Lindeberg CLT)
Suppose we have a triangular array such that:

1. for fixed $n$, $\{X_{ni}\}_{i=1}^{k_n}$ are independent (ie independence within row).
2. Suppose also $\mathbf{E}(X_{ni}) = 0$ for all, and $\mathbf{Var}X_{ni} = \sigma_{ni}^2 < \infty$.
3. Define $S_n = \sum_{i=1}^{k_n} X_{ni}$ and $s_n^2 = \sum_{i=1}^{k_n} \sigma_{ni}^2$ (sum of rows)
4. **Lindeberg condition** holds ie for all $\varepsilon > 0$:

$$\lim_{n \longrightarrow \infty} \frac{1}{s_n^2} \sum_{i=1}^{k_n} \int X_{ni}^2 \mathbf{1}[|X_{ni}| > \epsilon s_n] dP = 0$$

Then:

$$\frac{S_n}{s_n} \xrightarrow{d} \mathcal{N}(0, 1).$$

Note that if not mean zero, subtract off the means:

$$\frac{S_n - \sum_{i=1}^{k_n} \mu_{kn}}{s_n} \xrightarrow{d} N(0, 1)$$

**Example 60** (CLT failures: too wild)

$$X_i = \begin{cases} 0 \text{ wp } 1 - 1/i \\ 1 \text{ wp } 1/i \end{cases}$$

The issue is that some $X_i$'s dominate– ie the big ones.

**Example 61** (CLT Failures: too dependent)

**Recipe 62** (CLT for non-square integrable)
Session 4 notes. Similar to convergence in probability strategy.

Assume $X_{n,k} \in L^1$ and exist $c_n$ such that

1. $\sum_{k=1}^{\ell_n} P(|X_{n,k}| > c_n) = o(1)$
2. Lindeberg condition satisfied for truncated $Y_{n,k} = X_{n,k}\mathbf{1}[X_{n,k} \leqslant c_n]$
3. $\sum_{k=1}^{\ell_n}(\mathbf{E}X_{n,k} - \mathbf{E}Y_{n,k}) = o(s_n)$ where $s_n^2$ sum of variances of truncated in the $n$-th row.

Then

$$\frac{\sum_{k=1}^{\ell_n} X_{n,k} - \mathbf{E}X_{n,k}}{s_n} \xrightarrow{d} N(0,1)$$

For example, see Dembo 3.1.12.

**Theorem 63** (Lyapunov CLT)
Lyapunov condition is sufficient for Lindeberg's condition. Same setup, check that:

$$s_n^{-2-\delta} \sum_{i=1}^{k_n} \mathbf{E}|X_{n,k} - \mathbf{E}(X_{n,k})|^{2+\delta} \longrightarrow 0 \quad \text{for some } \delta > 0$$

**Theorem 64** (Kolmogorov 3 Series Theorem)
....

Some helpful CLT references:

1. Exponential approximation for the geometric - pg 105 dembo
2. Normal approx to Poisson - Dembo ch 3
3. Normal approx to Binomial

Some limit theorems for max of random variables (Ex 3.2.13 Dembo). The below three are the only possible type of limits for max of iid random variables.

1. Max of exponentials or normals is is Gumbel-type $F_\infty(y) = \exp(-e^{-y})$. (see also 3.2.14)
2. Frechet type $F_\infty(y) = \exp(-y^{-\alpha})$
3. Weibull type $F_\infty(y) = \exp(-|y|^\alpha)$

A few more examples of max of rvs and limiting distributions

**Example 65** (Birthday problem limiting law)
$T_n$ is the number needed to get a match with $n$ possible birthdays (eg $n = 365$), then can show that $P(n^{-1/2}T_n > s) \longrightarrow \exp(-s^2/2)$ - Ex 3.2.15. See also the RNG 2007 Q2 question below.

## 14 Characteristic Functions

**Definition 66** (Characteristic Function)
Characteristic function is fourier transform of $\mu$.

$$\phi(t) = \mathbf{E}[\exp(itX)].$$

**Recipe 67** (Characteristic Function Chaos)
Try to get $\phi(t) \approx (1 + \frac{f(t)}{n})^n$ form so that we can use exp limit.

**Theorem 68** (Levy's continuity Theorem)
If $\phi_{\mu_n}(t) \longrightarrow \phi(t)$ pointwise and $\phi$ is continuous at $t = 0$, then $\mu_n$ is uniformly tight sequence and $\mu_n \implies \mu$ weakly where characteristic function of $\mu$ given by $\phi$.

Conversely, weak convergence implies convergence of characteristic functions.

## 15 Stein's Method (Poisson)

### 15.1 Method 1 - Dependency Graphs

### 15.2 Method 2 - when dependency graph doesn't work (ie complete)

**Example 69** (Fixed Points - 310a HW8)

Let $\sigma$ be a uniformly chosen permutation in the symmetric group $S_n$. Let $W = \#\{i : \sigma(i) = i\}$ (the number of fixed points in $\sigma$). Show that $W$ has an approximate Poisson(1) distribution by using Stein's method to get an upper bound on $\|P_W - \text{Poisson}(1)\|$. (Hint: see section 4.5 of Arratia-Goldstein-Gordon.) Give details for this specific case.

Let $I = [n]$. We choose $B_\alpha = \{\alpha\}$ and use Theorem 1 from Arratia-Goldstein-Gordon.
For each $i \in I$, let

$$X_i = \begin{cases} 1 \text{ if } \sigma(i) = i \\ 0 \text{ otherwise} \end{cases} .$$

Naturally, $P(X_i = 1) = \frac{1}{n}$. We let $W = \sum_{i \in I} X_i$ and $\lambda = E[W] = 1$. We now use Stein's method as given in Arratia-Goldstein-Gordon Theorem 1 to get an upper bound on $\|P_W - Pois(1)\|$.

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta = \sum_{\alpha \in I} p_\alpha^2 = \frac{1}{n}.$$

Next, because we let $B_\alpha = \{\alpha\}$,

$$b_2 = 0.$$

Finally, for the third term, by Lemma 2 (p 418) in Arratia et al,

$$b_3 \leqslant \min_{1<k<n}\left(\frac{2k}{n-k} + 2n2^{-k}e^e\right) \sim 2\frac{(2log_2 n + e/ln2)}{n},$$

due to the fact that $\lambda = 1$ in our problem, so $\lambda = o(n)$
Now note that as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so, noting that the Arratia paper's definition of TV distance is twice our definition of TV distance:

$$\|P_W - Pois(1)\| \leqslant b_1 + b_2 + b_3 = \frac{1}{n} + \frac{4\log_2 n + 2\epsilon/\ln 2}{n} + o(1)$$

Now as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so $\|P_W - Pois(1)\| \longrightarrow 0$.

**Example 70** (Near Fixed Points- 2004 Q2)

# 16 Approximations

$$1 - x \leqslant e^{-x} \quad 1 - x \geqslant e^{-2x} \quad \text{both for small } x?$$
$$\log(1 + x) = x + O(x^2)) \quad \text{for small } x$$

$$-x - x^2 \leqslant \log(1 - x) \leqslant -x \quad \text{for } x \in [0, 1/2].$$

## 16.1 Binomial Coeffs and Stirlings

$$(\frac{n}{k})^k \leqslant \binom{n}{k} \leqslant (\frac{ne}{k})^k$$

Stirlings

**Theorem 71** (Stirlings)

$$(1 - \epsilon)\sqrt{2\pi}k^{k+1/2}e^{-k} \leqslant k! \leqslant (1 + \epsilon)\sqrt{2\pi}k^{k+1/2}e^{-k}$$

# 17 Misc

**Definition 72** (Metric)

## 17.1 Series convergence

**Theorem 73** (Root Test)
If $S_n = \sum_{k=1}^{n} a_n$ then $S_n$ converges if $L = \limsup_n |a_n|^{1/n} < 1$. If $= 1$, no info. $> 1$, diverges.

**Theorem 74** (Kolmogorov's Maximal inequality)
$Y_i$ mutually independent with $\mathbf{E}Y_i^2 < \infty$ and $\mathbf{E}Y_i = 0$, then for any $z > 0$:

$$z^2 P(\max_{k \leqslant n} |S_k| \geqslant z) \leqslant \mathbf{Var}(Z_n)$$

where $S_n = \sum_{k=1}^{n} Y_k$.