1 Asymptotics

Limit	**Function** $f(x)$	**Equivalent** $g(x)$	**Condition**
$x \longrightarrow 0$	$\log(1+x)$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\log(1-x)$	$-x$	$x \ll 1$
$x \longrightarrow 0$	$e^x - 1$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\sin x$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$1 - \cos x$	$\frac{x^2}{2}$	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\tan x$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\arcsin x$	x	$\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$(1+x)^\alpha$	$1 + \alpha x$	Fixed α, $\lvert x\rvert \ll 1$
$x \longrightarrow 0$	$\Gamma(1+x)$	$1 - \gamma x$	$\gamma = 0.577\ldots$
$n \longrightarrow \infty$	$n!$	$\sqrt{2\pi n}\,(n/e)^n$	StirlingâĂŹs approximation
$n \longrightarrow \infty$	$H_n = \sum_{k=1}^{n} \frac{1}{k}$	$\log n + \gamma$	Harmonic numbers
$n \longrightarrow \infty$	$\binom{2n}{n}$	$\frac{4^n}{\sqrt{\pi n}}$	Central binomial
$n \longrightarrow \infty$	$\left(1 + \frac{1}{n}\right)^n$	e	Definition of e
$n \longrightarrow \infty$	$\zeta(n)$	1	Riemann zeta tail

Table 1: Common asymptotic equivalences: $f(x) \sim g(x)$ means $f(x)/g(x) \longrightarrow 1$.

Operation (as $n \longrightarrow \infty$ or $x \longrightarrow 0$)	**Safe to replace f by g?**	**Remarks**
$\lim f(n)$	Yes	If $f \sim g$ and $\lim g = L \in \mathbb{R} \cup \{\infty\}$, then $\lim f = L$.
$\dfrac{f(n)}{g(n)}$	Yes	By definition $\dfrac{f}{g} \longrightarrow 1$. Useful for verifying asymptotic equivalence itself.
$f(n)\,g(n)$ or $f(n) \cdot h(n)$	Yes (usually)	Multiplicative errors stay small: $(fg)/(gg) = f/g \longrightarrow 1$ if $h \sim g$. Be sure h is bounded away from 0.
$f(n) - g(n)$	No	Only $f - g = o(g)$ is guaranteed. The difference need <u>not</u> vanish; e.g. $\log n + \gamma - \log n \longrightarrow \gamma$.
$\log f(n)$	Caution	If $f \sim g$ and both $\longrightarrow \infty$ at comparable rates, $\log f - \log g = \log(1 + o(1)) = o(1)$, so safe. If $f \longrightarrow C > 0$, extra care needed.
$e^{f(n)}$ or any non-linear analytic map	Caution / No	Small <u>relative</u> error in exponent can balloon: $e^f = e^{g(1+o(1))} = e^g\, e^{o(g)}$. Safe only when $g = o(1)$.
$\lfloor f(n) \rfloor$, $\operatorname{sign}(f(n))$	No	Discontinuous operations destroy the $f/g \longrightarrow 1$ guarantee. Analyze separately.

Table 2: Rule of thumb for substituting $f \sim g$ in various expressions. Here $f \sim g$ means $\dfrac{f(n)}{g(n)} \longrightarrow 1$.