Distribution stuff comnpendium

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Distribution	**Support**	**p.m.f.** $P(X = x)$	**c.d.f.** $F(x)$
Bernoulli(p)	$\{0, 1\}$	$p^x(1-p)^{1-x}$	$\mathbf{1}_{x \geqslant 1} p + \mathbf{1}_{0 \leqslant x < 1}(1-p)$
Binomial(n, p)	$\{0, \dots, n\}$	$\binom{n}{x} p^x (1-p)^{n-x}$	$\sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k (1-p)^{n-k}$
Geometric (p) (shift-1)	$\{1, 2, \dots\}$	$(1-p)^{x-1} p$	$1 - (1-p)^{\lfloor x \rfloor}$
Negative Binomial(r, p)	$\{0, 1, \dots\}$	$\binom{r+x-1}{x} (1-p)^r p^x$	$1 - \mathrm{B}_p\big(\lfloor x \rfloor + 1, r\big)$ (B_p = regularized Beta)
Poisson(λ)	$\{0, 1, \dots\}$	$e^{-\lambda} \frac{\lambda^x}{x!}$	$e^{-\lambda} \sum_{k=0}^{\lfloor x \rfloor} \frac{\lambda^k}{k!}$
Hypergeometric(N, K, n)	$\{0, \dots, n\}$	$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$	$\sum_{k=0}^{\lfloor x \rfloor} \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$
Discrete Uniform($a : b$)	$\{a, a+1, \dots, b\}$	$\frac{1}{b-a+1}$	$\frac{\lfloor x \rfloor - a + 1}{b - a + 1} \mathbf{1}_{x \geqslant a}$

Table 1: Common discrete distributions. Here $\mathbf{1}_A$ is the indicator of event A.

1 Useful Identities

1.1 Normal

Conditional distributions for MVN. Relationship between conditional distributions (mean) and OLS. If mean 0, then $E[Y|Z] = \frac{EYZ}{EZ^2} Z$, ala $Z(Z^T Z)^{-1} Z^T Y$ add this

1.2 Cauchy

"Sample median is a better estimator than sample mean". Sample mean is consistent, sample mean is not.

1.3 Exponential

"Sum of independent exponential with the same rate parameter is Gamma".
If $X_i \sim Expo(\lambda)$, then:

$$\sum_{i=1}^{n} X_i \sim \Gamma(n, \lambda).$$

If $X_i \sim Laplace(\mu, \theta)$, then $Y_i = \frac{|X_i - \mu|}{\theta} \sim Expo(1)$.

Distribution	Support	p.d.f. $f(x)$	c.d.f. $F(x)$
Uniform(a, b)	(a, b)	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$, $a < x < b$
Exponential(λ)	$(0, \infty)$	$\lambda e^{-\lambda x}$	$1 - e^{-\lambda x}$
Gamma(α, θ)	$(0, \infty)$	$\frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}$	$\frac{\gamma(\alpha, x/\theta)}{\Gamma(\alpha)}$ (γ = lower incomplete Γ)
χ^2_k	$(0, \infty)$	$\frac{x^{k/2-1} e^{-x/2}}{2^{k/2}\Gamma(k/2)}$	$P\big(k/2, x/2\big)$ (regularized Γ)
Normal(μ, σ^2)	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\Big]$	$\Phi\Big(\frac{x-\mu}{\sigma}\Big)$ (Φ = standard normal c.d.f.)
Lognormal(μ, σ)	$(0, \infty)$	$\frac{1}{x\sigma\sqrt{2\pi}} \exp\Big[-\frac{(\ln x-\mu)^2}{2\sigma^2}\Big]$	$\Phi\Big(\frac{\ln x-\mu}{\sigma}\Big)$
Student-$t(\nu)$	$(-\infty, \infty)$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$	$\frac{1}{2} + x\,\frac{{}_2F_1\left(\frac{1}{2}, \frac{\nu+1}{2}; \frac{3}{2}; -\frac{x^2}{\nu}\right)}{\sqrt{\nu\pi}\,B\left(\frac{\nu}{2}, \frac{1}{2}\right)}$ (symmetric)
Cauchy(x_0, γ)	$(-\infty, \infty)$	$\frac{1}{\pi\gamma\,[1 + ((x-x_0)/\gamma)^2]}$	$\frac{1}{\pi}\arctan\Big(\frac{x-x_0}{\gamma}\Big) + \frac{1}{2}$
Laplace(μ, b)	$(-\infty, \infty)$	$\frac{1}{2b}\exp\left(-\frac{\lvert x-\mu\rvert}{b}\right)$	$\begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right), & x < \mu, \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right), & x \geqslant \mu, \end{cases}$
Weibull(k, λ)	$(0, \infty)$	$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$	$1 - e^{-(x/\lambda)^k}$
Pareto(x_m, α)	(x_m, ∞)	$\frac{\alpha x_m^\alpha}{x^{\alpha+1}}$	$1 - \left(\frac{x_m}{x}\right)^\alpha$
Beta(α, β)	$(0, 1)$	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha, \beta)}$	$I_x(\alpha, \beta)$ (I_x = regularized Beta)
Rayleigh(σ)	$(0, \infty)$	$\frac{x}{\sigma^2}\exp\big(-x^2/(2\sigma^2)\big)$	$1 - \exp\big(-x^2/(2\sigma^2)\big)$
Triangular(a, c, b)	(a, b)	$\frac{2(x-a)}{(b-a)(c-a)}\mathbf{1}_{a\leqslant x<c} + \frac{2(b-x)}{(b-a)(b-c)}\mathbf{1}_{c\leqslant x<b}$	piecewise quadratic (integral of pdf)

Table 2: Common continuous distributions. Special-function notation follows standard texts.

1.4 Uniform

$$U_{(k)} \sim Beta(k, n+1-k)$$

$$-\log U \sim Exp(1)$$

1.5 Cauchy

$$X \sim Cauchy \implies 1/X \sim Cauchy.$$

2 Conjugate Priors

3 Identities

3.1 Convolution

Discrete:

$$P(Z = z) = \sum_{k=-\infty}^{\infty} P(X = k)P(Y = z - k)$$

Table 3: Effect of multiplying a random variable by a positive constant $c > 0$

✓/×	Distribution X	Shape-only pdf/pmf	Law of cX	Notes
✓	Exponential $\mathrm{Exp}(\lambda)$	$\lambda e^{-\lambda x},\ x > 0$	$cX \sim \mathrm{Exp}(\lambda/c)$	Pure scale family
✓	Gamma $\Gamma(k, \theta)$	$\dfrac{x^{k-1}e^{-x/\theta}}{\Gamma(k)\theta^k}$	$cX \sim \Gamma(k,\ c\theta)$	Includes χ^2, Erlang
✓	Weibull $\mathrm{Wei}(k, \lambda)$	$\dfrac{k}{\lambda}(x/\lambda)^{k-1}e^{-(x/\lambda)^k}$	$cX \sim \mathrm{Wei}(k,\ c\lambda)$	Shape k unchanged
✓	Log-normal $\mathcal{LN}(\mu, \sigma^2)$		$cX \sim \mathcal{LN}(\mu + \ln c, \sigma^2)$	Scaling shifts log-mean
✓	Pareto $\mathrm{Par}(\alpha, x_m)$	$\dfrac{\alpha x_m^\alpha}{x^{\alpha+1}},\ x > x_m$	$cX \sim \mathrm{Par}(\alpha,\ cx_m)$	Heavy-tailed example
✓	Normal $\mathcal{N}(\mu, \sigma^2)$		$cX \sim \mathcal{N}(c\mu,\ c^2\sigma^2)$	Locationscale family
✓	Student-$t_\nu(0, \sigma)$		$cX \sim t_\nu(0,\ c\sigma)$	Same d.f. ν
✓	Cauchy $\mathrm{Cauchy}(\mu, \gamma)$		$cX \sim \mathrm{Cauchy}(c\mu,\ c\gamma)$	Stable, heavy tail
✓	Uniform $\mathcal{U}(0, \theta)$	$1/\theta$	$cX \sim \mathcal{U}(0,\ c\theta)$	Classic scale
✓	Inverse-Gamma		Multiply the scale parameter by c	Same pattern as Gamma family
×	Poisson $\mathrm{Poi}(\lambda)$	$\dfrac{e^{-\lambda}\lambda^k}{k!}$	cX not integer-valued $\implies$ not Poisson	Closed under addition, not scaling
×	Binomial / Bernoulli		Support $\{0, \ldots, n\}$ broken by c	Sums, not scaling, stay in family
×	Geometric / Negative-Binomial		Same integer-support issue	
×	Beta $\mathrm{Beta}(\alpha, \beta)$		Lives on $(0, 1)$; cX usually leaves interval	
×	Discrete uniform on $\{1, \ldots, n\}$		Breaks discreteness unless c integer	

Continuous. CDF:

$$H(z) \int_{-\infty}^{\infty} F(z-t)g(t)dt = \int_{-\infty}^{\infty} G(t)f(z-t)dt$$

Density:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)dx.$$

3.2 Characteristic functions

Table 4: Conjugate priorposterior relationships after n observations $\mathbf{x} = (x_1, \dots, x_n)$

Likelihood (parameter)	$f(x \mid \theta)$	**Conjugate prior** $\pi(\theta)$	**Posterior hyper-parameters**
Bernoulli/Binomial (m)	$\binom{m}{x} p^x (1-p)^{m-x}$	Beta(α, β) $\pi(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}$	$\alpha' = \alpha + \sum_i x_i,\ \beta' = \beta + nm - \sum_i x_i$
Negative-Binomial (r)	$\binom{x+r-1}{x} (1-p)^r p^x$	same Beta(α, β)	$\alpha' = \alpha + rn,\ \beta' = \beta + \sum_i x_i$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$	Gamma(α, β) $\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$	$\alpha' = \alpha + \sum_i x_i,\ \beta' = \beta + n$
Exponential (rate)	$\lambda e^{-\lambda x}$	same Gamma(α, β)	$\alpha' = \alpha + n,\ \beta' = \beta + \sum_i x_i$
Exponential (scale) θ	$\theta^{-1} e^{-x/\theta}$	Inv–$\Gamma(\alpha, \beta)$ $\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-\alpha-1} e^{-\beta/\theta}$	$\alpha' = \alpha + n,\ \beta' = \beta + \sum_i x_i$
Gamma (k **known, scale)** θ	$\frac{x^{k-1} e^{-x/\theta}}{\Gamma(k)\theta^k}$	same Inv–$\Gamma(\alpha, \beta)$	$\alpha' = \alpha + nk,\ \beta' = \beta + \sum_i x_i$
Gamma (k **known, rate)** β	$\frac{\beta^k x^{k-1} e^{-\beta x}}{\Gamma(k)}$	Gamma(α, η) (hyper-rate η)	$\alpha' = \alpha + nk,\ \eta' = \eta + \sum_i x_i$
Gamma (β **fixed, shape)** α	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$\pi(\alpha) \propto a^{\alpha-1} [\Gamma(\alpha)]^{-b}$	$a' = a \prod_i x_i,\ b' = b + n$
Multinomial (K)	$\frac{n!}{\prod_k x_k!} \prod_{k=1}^{K} p_k^{x_k}$	Dirichlet$(\boldsymbol{\alpha})$ $\pi(\mathbf{p}) = \frac{\Gamma(\alpha_0)}{\prod_k \Gamma(\alpha_k)} \prod_{k=1}^{K} p_k^{\alpha_k - 1}$	$\alpha'_k = \alpha_k + x_k,\ k = 1{:}K$
Normal (σ^2 **known)**	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathcal{N}(\mu_0, \sigma_0^2)$ $\pi(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}}$	$\sigma_n^2 = (\sigma_0^{-2} + n\sigma^{-2})^{-1},\ \mu' = \sigma_n^2(\mu_0\sigma_0^{-2} + n\overline{x}\,\sigma^{-2})$
Normal (μ **known)**		same Inv–$\Gamma(\alpha, \beta)$	$\alpha' = \alpha + \frac{n}{2},\ \beta' = \beta + \frac{1}{2} \sum_i (x_i - \mu)^2$
Normal (both unknown)		$\mathcal{N}$–Inv–Γ $(\mu_0, \lambda, \alpha, \beta)$	$\lambda' = \lambda + n,\ \mu' = \frac{\lambda\mu_0 + n\overline{x}}{\lambda + n},\ \alpha' = \alpha + \frac{n}{2}$ $\beta' = \beta + \frac{1}{2} \sum_i (x_i - \overline{x})^2 + \frac{\lambda n(\overline{x} - \mu_0)^2}{2(\lambda + n)}$

Table 5: Characteristic functions $\varphi_X(t) = \mathrm{E}\left[e^{itX}\right]$ of selected distributions

Distribution (common parameterizations)	Characteristic function $\varphi_X(t)$
Bernoulli(p)	$1 - p + pe^{it}$
Binomial(n, p)	$\left(1 - p + pe^{it}\right)^n$
Poisson(λ)	$\exp\left(\lambda(e^{it} - 1)\right)$
Geometric(p), support $\{0, 1, \dots\}$	$\frac{p\,e^{it}}{1-(1-p)e^{it}}, \quad \lvert(1-p)e^{it}\rvert < 1$
Negative Binomial(r, p), failures r before r-th success	$\left(\frac{p}{1-(1-p)e^{it}}\right)^r$
Discrete Uniform$\{a, \dots, b\}$	$\frac{e^{ita}\left(1 - e^{it(b-a+1)}\right)}{(b-a+1)\left(1-e^{it}\right)}$
Normal(μ, σ^2)	$\exp\left(it\mu - \frac{1}{2}\sigma^2 t^2\right)$
Exponential(λ) (rate)	$\frac{\lambda}{\lambda - it}$
Gamma(k, θ) (shapescale) or Gamma(α, β) (shaperate)	$(1 - it\theta)^{-k}$ $(1 - it/\beta)^{-\alpha}$
Chisquare(ν)	$(1 - 2it)^{-\nu/2}$
Uniform(a, b)	$\frac{e^{itb} - e^{ita}}{it\,(b-a)}$
Laplace(μ, b)	$e^{it\mu}\left(1 + b^2 t^2\right)^{-1}$
Cauchy(μ, γ)	$\exp\left(it\mu - \gamma\lvert t\rvert\right)$
Logistic(μ, s)	$e^{it\mu}\,\frac{\pi st}{\sinh(\pi st)}$
Student $t(\nu)$	$\frac{\left(\sqrt{\nu}\lvert t\rvert\right)^{\nu/2} K_{\nu/2}\left(\sqrt{\nu}\lvert t\rvert\right)}{2^{\nu/2-1}\Gamma(\nu/2)} \quad (K_{\nu/2} = \text{modified Bessel})$
Beta(α, β)	${}_1F_1\left(\alpha;\ \alpha+\beta;\ it\right) \quad$ (confluent hypergeometric)
LogNormal(μ, σ^2)	$\exp\left(it\mu - \frac{1}{2}\sigma^2 t^2\right)\,{}_1F_1\left(\frac{it}{2}; \frac{1}{2}; \frac{\sigma^2 t^2}{2}\right)$
Cauchy (x_0 location, $\gamma > 0$ scale)	$\exp(x_0 it - \gamma\lvert t\rvert)$