310 Quals Strategy Compendium

July 5, 2025

1 Permutation and counting facts

Fact 1 (Number derangements of k-element set)
Derangements: D_n is the number of permutations with no fixed points.

Via inclusion exclusion.

$$D_k = k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}.$$

Eg, if T is number of fixed points:

$$P(T = k) = \frac{1}{n!} \binom{n}{k} D_{n-k},$$

since the remaining $n - k$ must **not** be fixed.

Apply the above to "Distribution of number of fixed points"- type questions.

Fact 2 (Catalan Numbers)
Catalan Numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad n \geqslant 0$$

Dyck Paths

Definition 3 (Cycles)
Cycle of a permutations

Definition 4 (Descents)

Reference: check Persi and Susan's paper

2 Distribution Facts

2.1 Gaussian Facts

Fact 5 (Max of Gaussians and fluctuations)
Max of Gaussians $\sqrt{2\log n}$ with fluctuations $1/\sqrt{\log n}$

Fact 6 (Max of sub-gaussian)
Expectation upper bound applies to correlated sub-gaussian reandom variables – see Vershynin.

Fact 7 (Mill's Ratio)

Fact 8 (Normal Conditional Distributions)

$$(X, Y) \sim MVN(\mu, \Sigma \implies X|Y \sim \ldots$$

2.2 Poisson, Exponential Distribution

Superposition and thinning.

Fact 9 (Superposition)
See Poisson with integer mean? Try superposition:

$$Pois(n) = \sum_{i=1}^{n} Pois(1)$$

Fact 10 (Renyi Representation of Exponential)

Fact 11 (Maximum, minimum of Exponential)

3 Basic set theory and measure theory

Definition 12 (Sigma Algebra)

Definition 13 (Algebra/Field)

Definition 14 (Outer measure)
Defined by

1. A non-negative set function
2. ......

The typical outer measure is WRite this out

Definition 15 (Measurable sets)
E is μ^* measurable if for all $B \subset \Omega$:

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Write about littlewood's principles here

Example 16 (Non-measurable sets)
Todo

4 Pi-Lambda and Good Sets

Definition 17 (π-system)
Collection of sets that is closed under **finite intersections**

Definition 18 (λ-system)
L lambda system if

1. $\Omega \in L$
2. closed under complements
3. closed under countable disjoint unions

Alternative definition:

1. $\Omega \in L$
2. $A, B \in L$ and $A \subset B$ then $B \setminus A \in L$
3. $A_1, A_2, \dots, \in L$ an increasing sequence of sets, then $\bigcup_i A_i \in L$

Note that a collection is a σ-algebra $\iff$ it is both a pi system and a lambda system.

Theorem 19 ($\pi - \lambda$ Theorem)
If P is a π system and L is a λ-system with $P \subset L$, then $\sigma(P) \subset L$.
Use to proof uniqueness of extension from an algebra to the sigma field.

Example 20 (Quals 2017, Question 2)

Theorem 21 (Monotone Class Theorem)
Use def:

Definition 22 (Monotone Class)
M is a monotone class if

1. Closed under increasing unions
2. Closed under decreasing intersections

If an algebra A is contained in a monotone class M, then $\sigma(A) \subset M$.

Theorem 23 (Monotone Class for functions)
Double check this
M be a vector space of $\mathbb{R}$-valued functions on Ω such that

1. $1 \in M$
2. M is a vector space over $\mathbb{R}$. Ie closed under addition and scalar mult
3. If $h_n \geqslant 0 \in M$ and $h_n \uparrow h$, then $h \in M$.

Then if P is a pi system such that $\mathbf{1}_A \in M$ for all $M \in P$, then M contains all functions that are measurable with respect to $\sigma(P)$.

Ie want to prove something about some functions measurable on $\sigma(P)$...

Theorem 24 (Independent π systems generate independent sigma algebras)

$$\{C_i\}_{i \in I} \text{ independent } \implies \{\sigma(C_i)\} \text{ independent}$$

Example 25 (Prove two random variables independent)
Check that generating pi systems are independent, ie $P(X \leqslant b, Y \leqslant a) = \prod P(X \leqslant b)P(Y \leqslant a)$.

Definition 26 (Independence of (uncountable) collections of sets)
$\{A_\alpha\}_{\alpha \in I}$ each be a collection of sets. Independent if any **finite subcollection** are mutually independent.

5 Extension theorems

Theorem 27 (Caratheodory)
Extend a measure on an algebra to a σ algebra, uniquely if finite.

The important idea is make an outer measure, then prove the lemma is that the μ^* measurable sets in $\mathcal{F}$ for a sigma algebra on which μ^* is countably additive, ie a bonafide measure.

6 Random variables

Definition 28 (Random variable)
$X : \Omega \longrightarrow \mathbb{S}$ is measurable if:

$$X^{-1}(B) := \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{S}$$

Fix an arbitrary set in the co-sigma algebra and check that its preimage is measurable-$\mathcal{F}$.

Recipe 29 (Measurability)
If $\mathcal{S} = \sigma(\mathcal{A})$ and $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{A}$, then X is measurable.

Note– nothing necessary about it being a pi system, just pick any convenient generators.

Proof. Such sets form a sigma algebra and $\mathcal{S} = \sigma(\mathcal{A})$. □

Definition 30 (Characterize distribution functions of real-valued random variable)
Dembo Theorem 1.2.37.

1. Non decreasing
2. $\lim_{x \longrightarrow \infty} F(x) = 1$ and $\longrightarrow -\infty$ gives 0
3. F right continuous

7 0-1 Laws

Definition 31 (Tail field)
Defining $\mathcal{T}_n = \sigma(X_r, r > n)$ and the tail sigma algebra of the process is $\mathcal{T} = \cap_n \mathcal{T}_n$.

Theorem 32 (Kolmogorov 0-1)
The tail sigma field is P-trivial. Dembo 1.4.10

8 Borel Cantelli

Example 33 (Longest head runs)
See Durrett.

A helpful idea is splitting into blocks.

Truncation arguments. Want to show something about $\{X_n\}$. Consider:

$$Y_n = X_n \mathbf{1}[|X_n| \leqslant c_n],$$

which will then be integrable. Then prove something about Y_n, you can transfer this knowledge to knowledge about X_n by the following idea:

$$P(X_n \neq Y_n \text{ io}) = 0 \text{ if } \sum_{i=1}^{n} P(|X_n| > c_n) < \infty$$

So if the above is zero, eventually $X_n = Y_n$, so the limiting behavior of Y_n is the same as that of X_n. The difficulty is picking a c_n such that the above holds.

Theorem 34 (Borel Cantelli 3)
Not as useful but still check these
If you have a filtration $\{F_n\}$ and $A_n \in \mathcal{F}_n$, then:

1. (analog of Borel Cantelli 1)

$$\sum_{k=1}^{\infty} P(A_k|\mathcal{F}_{k-1})(\omega) < \infty \implies \sum \mathbf{1}[\omega \in A_k\} < \infty$$

2. Analog of BC 2

$$\sum_k P(A_k|\mathcal{F}_{k-1})(\omega) = \infty \implies \frac{\sum \mathbf{1}[A_k(\omega)]}{\sum P(A_k|\mathcal{F}_{k-1})(\omega)|} \to 1$$

9 Modes of Convergence

Recipe 35 (Proving as convergence)
Some ideas:

1. BC 1: check that $P(|X_n - X| > \epsilon \text{ i.o}) = 0$. Consider *subsequence trick* below.
2. If $X_n \xrightarrow{p} X$, then $X_{n_k} \xrightarrow{\text{a.s.}} X$ for a subsequence. (Good for counter examples)
3. $X_n \xrightarrow{p} X$ and X_n is monotone (ie for all ω, $X_n(\omega)$ is increasing in n), then $X_n \xrightarrow{\text{a.s.}} X$.
4. Continuous mapping theorem
5. Skorohod's Representation. If $X_n \implies X$ weakly, then there exists a probability space and random variables $Z_n =^d X_n$ and $Z =^d X$ such that $Z_n \xrightarrow{\text{a.s.}} Z$. (Good for counter examples)

Theorem 36 (BC Subsequence trick)
Sometimes $\sum P(A_n) = \infty$.

1. Pick a subsequence such that it's finite.
2. Shows that A_{n_k} infinitely often occurs with probability 0.
3. Apply interpolation or some other argument for everything in between, ie to conclude that A_n infinitely often wp 0 also.

Eg for monotone X_n:

$$\frac{X_n}{\mathbf{E}X_n} \leqslant \frac{X_{n_{k+1}-1}}{\mathbf{E}X_{n_k}}$$

So $\limsup_k \frac{X_{n_{k+1}-1}}{\mathbf{E}X_{n_k}} \leqslant 1 \implies \limsup_n \frac{X_n}{\mathbf{E}X_n} \leqslant 1$. We can control the left hand side by controlling the right hand side along our choice of subsequence.

See Dembo notes subsequence section Examples– SLLN, Brownian LLN, LIL.

SLLN in Persi notes and Renewal theorem in Dembo

Recipe 37 (Proving conv in prob)
Some ideas:

1. Show almost sure convergence
2. Show convergence in L^p for $p \geqslant 1$ (in L^2 in particular useful):
    - Show that $\mathbf{E}X_n \longrightarrow \mu$, $\mathbf{Var}X_n \longrightarrow 0$, then $X_n \longrightarrow \mu$ in L^2
3. CMT
4. $X_n \xrightarrow{d} c$ constant
5. Truncation tool: **Weak law for Triangular Array**. For when we don't have second moments.

Theorem 38 (Weak law triangular array - 2.1.11 Dembo)
Check this Suppose triangular array $\{X_{n,k}\}_{n\in\mathbb{N},k\leqslant n}$ of pairwise independent rv. Define a truncated array with the same *within-row* truncation:

$$\overline{X}_{n,k} = X_{n,k}\mathbf{1}[|X_{n,k}| < b_n]$$

. If we have two conditions:

$$\sum_{k=1}^{n} P(|X_{n,k}| > b_n) \longrightarrow^{n} 0,$$

and,

$$b_n^{-2}\sum_{k=1}^{n} \mathbf{Var}(\overline{X}_{n,k}) \longrightarrow 0,$$

Then:

$$b_n^{-1}(\sum_{k=1}^{n} X_{n,k} - a_n)\xrightarrow{p} 0 \quad \text{where } a_n = \sum_{k=1}^{n} \mathbf{E}\overline{X}_{n,k}.$$

In case of St Petersburg paradox, then show that $a_n/b_n \longrightarrow 1/2$ so that we get $b_n^{-1}\sum_{k=1}^{n} X_k = (b_n^{-1}\sum_{k=1}^{n} X_{n,k})\xrightarrow{p} 1/2$.

Recipe 39 (Prove L^p convergence)
Ideas - see session notes

1. $X_n \xrightarrow{\text{a.s.}} X$ and X_n uniformly bounded then $X_n \longrightarrow^{L1} X$.
2. If X_n uniformly bounded then $X_n \xrightarrow{d} 0 \implies X_n \longrightarrow^{L1} 0$.
3. If $|X_n|^p$ is UI then $X_n \xrightarrow{d} X \implies \mathbf{E}|X_n|^p \longrightarrow E|X|^p$
4. Scheffe's Lemma

Recipe 40 (Proving UI)
Try

1. Sums of UI sequence of rvs are UI
2. Domination: If $E\sup|X_i| < \infty$ then $\{X_i\}$ is UI
3. If $\{X_i\}$ is UI, then $\sup E|X_i| < \infty$ but not the converse.
4. Prop 8.3.3: X_n UI $\iff$ $\sup E|X_i| < \infty$ and for all ε, there exists δ for all A such that $P(A) < delta$ then $\mathbf{E}[|X_n|\mathbf{1}[A]] < \varepsilon$
5. If $\sup \mathbf{E}|X_i|^r < \infty$ if $r > 1$, then UI
6. $X \in L^1$, then $\{\mathbf{E}[X|\mathcal{G}] : \mathcal{G} \subset \mathcal{F}\}$ is UI
7. L^1 convergence implies UI

10 Integration

Definition 41 (Lebesgue Integral)

$$\int_\Omega f(\omega)d\mu(\omega) = \sup\left(\sum_{i=1}^{n} v_i \mu(A_i)\right)$$

where $v_i = \inf_{\omega\in A_i} f(\omega)$ and the sup is over all partitions of Ω.

Theorem 42 (MCT)
Note that can also do for general functions (not necessarily non negative) so long as $f_n \geqslant g \in L^1$.

Theorem 43 (Fatou's Lemma)

$$\int \liminf_n f_n d\mu \leqslant \liminf_n \int f_n d\mu$$

Theorem 44 (DCT)
If $f_n \longrightarrow f$ a.e ω, and there exists g such that $|f_n(\omega)| \leqslant g(\omega)$ a.e ω and $\int g d\mu < \infty$ then exchange integral and limit. Must dominate the sequence.

Theorem 45 (Scheffe's Lemma)
Is a statement about combining L^1 and as convergence. If $f_n \xrightarrow{\text{a.s.}} f \in L^1$, then:

$$\|f_n - f\|_{L^1} \longrightarrow 0 \iff \int |f_n| d\mu \longrightarrow \int |f| d\mu$$

We always have that convergence in L^1 implies convergence of the expectations (without any a.s. convergence assumption) , but the other direction is Scheffe's contribution.

Theorem 46 (Reverse Fatou)
If $f_n \leqslant g \in L^1$ then :

$$\limsup_n \int f_n d\mu \leqslant \int \limsup f_n d\mu$$

Fact 47 (L^p spaces nested)

$$\|Y\|_r \leqslant \|Y\|_q$$

Fact 48 (L^q convergence fact - Dembo 1.3.28)
$X_n \longrightarrow^{L^q} X \implies \mathbf{E}|X_n|^q \longrightarrow \mathbf{E}|X_\infty|^q$ for any q. (Minkowski)

Also for only $q \in \mathbb{N}$, $\mathbf{E}X_n^q \longrightarrow \mathbf{E}X_\infty^q$. (some wild algebraic shit for odd q).

Theorem 49 (Holder's)
If $p, q > 1$ with $1/p + 1/q = 1$ then

$$E|XY| \leqslant \|X\|_p \|Y\|_q$$

Cauchy Schwarz is special case.

Theorem 50 (Minkowski)
Triangle inequality for the $\|\cdot\|_p$ norm

Definition 51 (Uniform integrability (UI))
Possibly uncountable collection $\{X_\alpha : \alpha \in I\}$ is called UI if

$$\lim_{M \to \infty} \sup_{\alpha \in I} \mathbf{E}[|X_\alpha| \mathbf{1}[|X_\alpha| > M] = 0$$

Fact 52 (Dominated implies UI)
If $|X_\alpha| \leqslant Y$ for integrable Y, then collection is UI.

As a corollary, any finite collection of integrable rv is UI.

Theorem 53 (Vitali Convergence Theorem)
Supposing that $X_n \xrightarrow{p} X$, then:

$$\{X_n\} \text{ is UI} \iff X_n \to^{L1} X \iff X_n \text{ is integrable for all } n \leqslant \infty \text{ and } E|X_n| \to E|X_\infty|.$$

11 Product σ-algebras

Existence of unique product measure of n $\sigma-$ finite measures.

Theorem 54 (Kolmogorov Extension)
Unique probability measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_c)$ with correct FDDs.

Theorem 55 (Fubini's)
Conditions: $h \geqslant 0$ or $\int |h| d\mu < \infty$ where $\mu = \mu_1 \times \mu_2$.

12 Weak Convergence

12.1 Methods

1. **Direct**. Show that $F_n(x) \to F(x)$ for all continuity points.

2. If there's a density, try to show that $f_n(x) \longrightarrow f_\infty$ and check that f_∞ a valid pdf.

3. If $X_n \geqslant 0$, show that $\int_0^\infty \exp(-\lambda x) d\mu_n(x) \longrightarrow L(\lambda) = \int_0^\infty \exp(-\lambda x) d\mu_\infty(x)$. Note that $L(\lambda)$ is a Laplace transform of some μ if $L(\lambda) \downarrow 1$ as $\lambda \downarrow 0$. (Just need for positive λ.

4. MGFs

5. Characteristic functions- show that $\phi_n(t) \longrightarrow \phi(t)$ for all $t \in \mathbb{R}$. (ϕ is a characteristic function of a probability measure if $\psi(t) \longrightarrow 1$ as $t \downarrow 0$.

6. CLT

Example 56 (Cycling of Random Number Generators (2007 Q2))
Similar to birthday problem.

$$\begin{aligned} P(T > k) &= \prod_{i=1}^{k}(1 - \frac{i}{n}) && (1) \\ &\approx \prod \exp(-1/n) && (2) \\ &\approx \exp(-k^2/n). && (3) \end{aligned}$$

So $P(T > x\sqrt{n}) \approx \exp(-x^2/2)$ should work.

Need to justify this rigorously. To do so, use $|\log(1+x) - x| < Cx^2$ when $|x| < 1/2$. Ie, $\log(1+x) = x + O(x^2)$.

See lecture 1 in 310a.

13 CLT

Heuristic: "not too dependent", "no few terms dominate".

Theorem 57 (Lindeberg CLT)
Suppose we have a triangular array such that:

1. for fixed n, $\{X_{ni}\}_{i=1}^{k_n}$ are independent (ie independence within row).
2. Suppose also $\mathbf{E}(X_{ni}) = 0$ for all, and $\mathbf{Var}X_{ni} = \sigma_{ni}^2 < \infty$.
3. Define $S_n = \sum_{i=1}^{k_n} X_{ni}$ and $s_n^2 = \sum_{i=1}^{k_n} \sigma_{ni}^2$ (sum of rows)
4. **Lindeberg condition** holds ie for all $\varepsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{k_n} \int X_{ni}^2 \mathbf{1}[|X_{ni}| > \epsilon s_n] dP = 0$$

Then:

$$\frac{S_n}{s_n} \xrightarrow{d} \mathcal{N}(0,1).$$

Note that if not mean zero, subtract off the means:

$$\frac{S_n - \sum_{i=1}^{k_n} \mu_{kn}}{s_n} \xrightarrow{d} N(0,1)$$

Example 58 (CLT failures: too wild)

$$X_i = \begin{cases} 0 \text{ wp } 1 - 1/i \\ 1 \text{ wp } 1/i \end{cases}$$

The issue is that some X_i's dominate– ie the big ones.

Example 59 (CLT Failures: too dependent)

Recipe 60 (CLT for non-square integrable)
Session 4 notes. Similar to convergence in probability strategy.

Assume $X_{n,k}\ inL^1$ and exist c_n such that

1. $\sum_{k=1}^{\ell_n} P(|X_{n,k}| > c_n) = o(1)$
2. Lindeberg condition satisfied for truncated $Y_{n,k} = X_{n,k}\mathbf{1}[X_{n,k} \leqslant c_n]$
3. $\sum_{k=1}^{\ell_n} (\mathbf{E}X_{n,k} - \mathbf{E}Y_{n,k}) = o(s_n)$ where s_n^2 sum of variances of truncated in the n-th row.

Then

$$\frac{\sum_{k=1}^{\ell_n} X_{n,k} - \mathbf{E}X_{n,k}}{s_n} \xrightarrow{d} N(0,1)$$

Theorem 61 (Lyapunov CLT)
Lyapunov condition is sufficient for Lindeberg's condition. Same setup, check that:

$$s_n^{-2-\delta} \sum_{i=1}^{k_n} \mathbf{E}|X_{n,k} - \mathbf{E}(X_{n,k})|^{2+\delta} \longrightarrow 0 \quad \text{for some } \delta > 0$$

14 Characteristic Functions

Definition 62 (Characteristic Function)
Characteristic function is fourier transform of μ.

$$\phi(t) = \mathbf{E}[\exp(itX)].$$

Recipe 63 (Characteristic Function Chaos)
Try to get $\phi(t) \approx (1 + \frac{f(t)}{n})^n$ form so that we can use exp limit.

15 Stein's Method (Poisson)

15.1 Method 1 - Dependency Graphs

15.2 Method 2 - when dependency graph doesn't work (ie complete)

Example 64 (Fixed Points - 310a HW8)

Let σ be a uniformly chosen permutation in the symmetric group S_n. Let $W = \#\{i : \sigma(i) = i\}$ (the number of fixed points in σ). Show that W has an approximate Poisson(1) distribution by using Stein's method to get an upper bound on $\|P_W - \text{Poisson}(1)\|$. (Hint: see section 4.5 of Arratia-Goldstein-Gordon.) Give details for this specific case.

Let $I = [n]$. We choose $B_\alpha = \{\alpha\}$ and use Theorem 1 from Arratia-Goldstein-Gordon.
For each $i \in I$, let

$$X_i = \begin{cases} 1 \text{ if } \sigma(i) = i \\ 0 \text{ otherwise} \end{cases}.$$

Naturally, $P(X_i = 1) = \frac{1}{n}$. We let $W = \sum_{i \in I} X_i$ and $\lambda = E[W] = 1$. We now use Stein's method as given in Arratia-Goldstein-Gordon Theorem 1 to get an upper bound on $\|P_W - Pois(1)\|$.

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta = \sum_{\alpha \in I} p_\alpha^2 = \frac{1}{n}.$$

Next, because we let $B_\alpha = \{\alpha\}$,

$$b_2 = 0.$$

Finally, for the third term, by Lemma 2 (p 418) in Arratia et al,

$$b_3 \leqslant \min_{1<k<n}\left(\frac{2k}{n-k} + 2n2^{-k}e^e\right) \sim 2\frac{(2log_2 n + e/ln2)}{n},$$

due to the fact that $\lambda = 1$ in our problem, so $\lambda = o(n)$
Now note that as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so, noting that the Arratia paper's definition of TV distance is twice our definition of TV distance:

$$\|P_W - Pois(1)\| \leqslant b_1 + b_2 + b_3 = \frac{1}{n} + \frac{4\log_2 n + 2\epsilon/\ln 2}{n} + o(1)$$

Now as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so $\|P_W - Pois(1)\| \longrightarrow 0$.

Example 65 (Near Fixed Points- 2004 Q2)

16 Approximations

$$1 - x \leqslant e^{-x} \quad 1 - x \geqslant e^{-2x} \quad \text{both for small } x?$$
$$\log(1 + x) = x + O(x^2)) \quad \text{for small } x$$

16.1 Binomial Coeffs and Stirlings

$$(\frac{n}{k})^k \leqslant \binom{n}{k} \leqslant (\frac{ne}{k})^k$$

Stirlings

17 Misc

Definition 66 (Metric)