310 Quals Strategy Compendium

June 26, 2025

1 Permutation and counting facts

Fact 1 (Number derangements of k-element set)
Derangements: D_n is the number of permutations with no fixed points.

Via inclusion exclusion.

$$D_k = k! \sum_{j=0}^{k} \frac{(-1)^j}{j!}.$$

Eg, if T is number of fixed points:

$$P(T = k) = \frac{1}{n!} \binom{n}{k} D_{n-k},$$

since the remaining $n - k$ must **not** be fixed.

Apply the above to "Distribution of number of fixed points"- type questions.

Fact 2 (Catalan Numbers)
Catalan Numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad n \geqslant 0$$

Dyck Paths

Definition 3 (Cycles)
Cycle of a permutations

Definition 4 (Descents)

Reference: check Persi and Susan's paper

2 Distribution Facts

2.1 Gaussian Facts

Fact 5 (Max of Gaussians and fluctuations)
Max of Gaussians $\sqrt{2 \log n}$ with fluctuations $1/\sqrt{\log n}$

Fact 6 (Max of sub-gaussian)
Expectation upper bound applies to correlated sub-gaussian reandom variables – see Vershynin.

Fact 7 (Mill's Ratio)

Fact 8 (Normal Conditional Distributions)

$$(X, Y) \sim MVN(\mu, \Sigma \implies X|Y \sim \ldots$$

2.2 Poisson, Exponential Distribution

Superposition and thinning.

Fact 9 (Superposition)
See Poisson with integer mean? Try superposition:

$$Pois(n) = \sum_{i=1}^{n} Pois(1)$$

Fact 10 (Renyi Representation of Exponential)

Fact 11 (Maximum, minimum of Exponential)

3 Basic set theory and measure theory

Definition 12 (Sigma Algebra)

Definition 13 (Algebra/Field)

Definition 14 (Outer measure)
Defined by

1. A non-negative set function
2. ......

The typical outer measure is WRite this out

Definition 15 (Measurable sets)
E is μ^* measurable if for all $B \subset \Omega$:

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Littlewood's principles idea.

4 Pi-Lambda and Good Sets

Definition 16 (π-system)
Collection of sets that is closed under **finite intersections**

Definition 17 (λ-system)
L lambda system if

1. $\Omega \in L$
2. closed under complements
3. closed under countable disjoint unions

Alternative definition:

1. $\Omega \in L$
2. $A, B \in L$ and $A \subset B$ then $B \setminus A \in L$
3. $A_1, A_2, \dots, \in L$ an increasing sequence of sets, then $\bigcup_i A_i \in L$

Theorem 18 ($\pi - \lambda$ Theorem)
If P is a π system and L is a λ-system with $P \subset L$, then $\sigma(P) \subset L$.
Use to proof uniqueness of extension from an algebra to the sigma field.

Example 19 (Quals 2017, Question 2)

Theorem 20 (Monotone Class Theorem)
Use def:

Definition 21 (Monotone Class)
M is a monotone class if

1. Closed under increasing unions
2. Closed under decreasing intersections

If an algebra A is contained in a monotone class M, then $\sigma(A) \subset \sigma(M)$.

Theorem 22 (Monotone Class for functions)
Double check this
M be a vector space of measurable functions such that

1. $1 \in M$
2. M^+ (positive functions in M) is closed under inreasing limits
3. $\mathbf{1}_A \in M$ for all A in a pi system generating $\mathcal{F}$.

Then M contains all bounded measurable functions

Theorem 23 (Independent π systems generate independent sigma algebras)

$$\{C_i\}_{i \in I} \text{ independent } \implies \{\sigma(C_i)\} \text{ independent}$$

Example 24 (Prove two random variables independent)
Check that generating pi systems are independent, ie $P(X \leqslant b, Y \leqslant a) = \prod P(X \leqslant b) P(Y \leqslant a)$.

5 Borel Cantelli

6 Integration

Definition 25 (Lebesgue Integral)

$$\int_\Omega f(\omega) d\mu(\omega) = \sup \left(\sum_{i=1}^{n} v_i \mu(A_i) \right)$$

where $v_i = \inf_{\omega \in A_i} f(\omega)$ and the sup is over all partitions of Ω.

Theorem 26 (MCT)
Note that can also do for general functions (not necessarily non negative) so long as $f_n \geqslant g \in L^1$.

Theorem 27 (Fatou's Lemma)

$$\int \liminf_n f_n d\mu \leqslant \liminf_n \int f_n d\mu$$

Theorem 28 (DCT)
If $f_n \longrightarrow f$ a.e ω, and there exists g such that $|f_n(\omega)| \leqslant g(\omega)$ a.e ω and $\int g d\mu < \infty$ then exchange integral and limit. Must dominate the sequence.

Theorem 29 (Scheffe's Lemma)
Is a statement about combining L^1 and as convergence. If $f_n \xrightarrow{\text{a.s.}} f \in L^1$, then:

$$\|f_n - f\|_{L^1} \longrightarrow 0 \iff \int |f_n| d\mu \longrightarrow \int |f| d\mu$$

Theorem 30 (Vitali Convergence Theorem)
If f_n is UI and $f_n \xrightarrow{p} f$, then $f \in L^1$ and $\|f_n - f\|_{L^1} \longrightarrow 0$.

Theorem 31 (Generalized DCT)
???? If $|f_n| \leqslant g_n$ such that $g_n \longrightarrow g \in L^1$ (convergence in L^1) then $\int f_n d\mu \longrightarrow \int f d\mu$

Theorem 32 (Reverse Fatou)
If $f_n \leqslant g \in L^1$ then :

$$\limsup_n \int f_n d\mu \leqslant \int \limsup f_n d\mu$$

7 Product σ-algebras

8 Stein's Method (Poisson)

8.1 Method 1 - Dependency Graphs

8.2 Method 2 - when dependency graph doesn't work (ie complete)

Example 33 (Fixed Points - 310a HW8)

Let σ be a uniformly chosen permutation in the symmetric group S_n. Let $W = \#\{i : \sigma(i) = i\}$ (the number of fixed points in σ). Show that W has an approximate Poisson(1) distribution by using Stein's method to get an upper bound on $\|P_W - \text{Poisson}(1)\|$. (Hint: see section 4.5 of Arratia-Goldstein-Gordon.) Give details for this specific case.

Let $I = [n]$. We choose $B_\alpha = \{\alpha\}$ and use Theorem 1 from Arratia-Goldstein-Gordon.
For each $i \in I$, let

$$X_i = \begin{cases} 1 \text{ if } \sigma(i) = i \\ 0 \text{ otherwise} \end{cases}.$$

Naturally, $P(X_i = 1) = \frac{1}{n}$. We let $W = \sum_{i \in I} X_i$ and $\lambda = E[W] = 1$. We now use Stein's method as given in Arratia-Goldstein-Gordon Theorem 1 to get an upper bound on $\|P_W - Pois(1)\|$.

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta = \sum_{\alpha \in I} p_\alpha^2 = \frac{1}{n}.$$

Next, because we let $B_\alpha = \{\alpha\}$,

$$b_2 = 0.$$

Finally, for the third term, by Lemma 2 (p 418) in Arratia et al,

$$b_3 \leqslant \min_{1<k<n} \left(\frac{2k}{n-k} + 2n2^{-k} e^e \right) \sim 2\frac{(2log_2 n + e/ln2)}{n},$$

due to the fact that $\lambda = 1$ in our problem, so $\lambda = o(n)$
Now note that as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so, noting that the Arratia paper's definition of TV distance is twice our definition of TV distance:

$$\|P_W - Pois(1)\| \leqslant b_1 + b_2 + b_3 = \frac{1}{n} + \frac{4\log_2 n + 2\epsilon/\ln 2}{n} + o(1)$$

Now as $n \longrightarrow \infty$, $b_1 \longrightarrow 0$ and $b_3 \longrightarrow 0$, so $\|P_W - Pois(1)\| \longrightarrow 0$.

Example 34 (Near Fixed Points- 2004 Q2)

9 Approximations

$$1 - x \leqslant e^{-x} \quad 1 - x \geqslant e^{-2x} \quad \text{both for small } x?$$