# Tree HMM Derivation

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Nov 2025

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## 1 Setup

Suppose (for illustration only- method will be fully general) that cell $r$ has daughters $n, m$, with division at time $\tilde{t}$. Assume:

$$z_{r,1} \sim \mathrm{Cat}(\pi_0) \quad \text{initial distribution of root}$$

$$z_{r,t+1} | z_{r,t} = k \sim \mathrm{Cat}(\pi_k) \quad \text{Markov transitions}$$

Identically

$$z_{r,p+1} | z_{r,p} = k \sim \mathrm{Cat}(\pi_k) \quad \text{for } p > \tilde{t} + 1$$

Assume likelihood identical for parents and children, conditional on latent $z$:

$$p(x_{r,t} | z_{r,t}, x_{r,t-1}) := \ell_t^{(r)},$$

above is probably just typical AR(1) Gaussian emission.

Finally, assume that:

$$(z_{n,\tilde{t}+1} | z_{r,\tilde{t}}) = k \sim \mathrm{Cat}(\tilde{\pi}_k)$$

### 1.1 Backwards Messages

Initialize for all cells at their "endpoint"- $\beta_t^{(c)} = \mathbf{1}_k$ For a cell $r$ at time $t$, backwards message is given by:

$$\beta_t^{(r)} = \bigodot_{c \in C(r)} \left[ P_c^T (\beta_{t+1}^{(c)} \odot \ell_{t+1}^{(c)} \right].$$

Here, if $C(r) = \{r\}$, then $P_c = P$. If $C(r) = \{n, m\}$, then $P_c = \tilde{P}$.

That is, the backwards message is from the immediate children, and propagates up the tree so that $\beta_1^{(r)}$ includes the messages from all children.

## 2 Forward-Backward (E-step) Derivation

### 2.1 Forward Messages

Initialize $\alpha_1^{(root})=\pi\odot l_0^{(root)}$.

$$\alpha_{t+1}^{(c)}=\ell_{t+1}^{(c)}\odot\left[P_c^T\alpha_t^{(p)}\right].$$

### 2.2 Smoother Posteriors

$$p(z_t^{(n)}|\text{All observations})\propto\alpha_t^{(n)}\odot\beta_t^{(n)}.$$

### 2.3 Jax implementation

## 3 M Step