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## 1 Introduction

The problem of *effectively* allocating finite resources has been known for centuries. Certainly, as long as there have been functional economies (even systems as simple as barter networks) producers have been pressured to utilize labour, capital, and raw materials in the most efficient manner possible. More generally, there exist a class of problems known as Assignment Problems.

In the most general case, the Assignment Problem boils down to “essentially [trying] to address the issue of getting the best from every [resource] allocation” [Pai, 2012, p. 8]. This could mean assigning workers of various skill levels to machines in a factory, scheduling specific types of products to be produced on assembly lines of varying capacity, reducing change-over and re-tooling costs during production, or many other problems. In this paper, we examine the problem of assigning a varied workload of tasks to a group of employees of varying skill levels in such a way that we minimize the elapsed time, commonly referred to as the MAKESPAN.

Our problem is important because it directly affects the throughput and, therefore, the profitability of the office we study. By finding an optimal assignment strategy that minimizes the amount of time it takes to complete a task, we can be sure that the office operates in an efficient, cost-effective, and sustainable manner over the long term.

### 1.1 Historical Context

While people have encountered Assignment Problems for at least several centuries, they lacked the tools to effectively tackle them, especially in the general sense, until fairly recently. Indeed, neither the Renaissance nor the Industrial Revolution produced the necessary confluence of applied and theoretical mathematics and advanced computational technology required to enable business owners to solve these types of problems unassisted.

By the mid-20<sup>th</sup> century, a number of critical breakthroughs began to bring powerful mathematical techniques to the masses. Alan Turing had worked to design high-performance code-breaking analog computers and had invented what we now call software while “attacking problems of calculability and Hilbert’s Entscheidungsproblem” [Petzold, 2008, p. 65]. John Von Neumann envisioned the architecture of the modern digital computer. Wassily Leontif had “opened the door to a new era in mathematical modelling [sic]” [Lay, 2006, p. 1], work for which he would be “awarded the 1973 Nobel Prize in Economic Science” [Lay, 2006, p. 1]. And a secret group colloquially referred to as “Blackett’s Circus” had created a new industry while using science and mathematics to defeat the Nazis. They called it OPERATIONS RESEARCH.

Within the British Admiralty, and reporting directly to Winston Churchill, a scientist and Navy officer named Patrick Blackett led the “operations research group, ...a subdepartment of the navy’s Scientific Research and Experiment Department.” [Budiansky, 2013, p. 462] While developing technologies like RADAR and pioneering the application of science to military strategy, Black-

ett’s team laid the foundation for the post-war surge in operations research work. Such work would eventually lead to minor business revolutions like lean manufacturing and its offshoot, agile software development.

It was during these decades of rapid technological advancement that the mathematical modelling techniques used in this paper were refined. Fortunately for the authors, we now live in a time where complex mathematical models can be effortlessly solved with even a mediocre personal computer.

## 2 The Employee Scheduling Problem

In this paper, we examine an insurance office that handles two types of work: creating new policies, and preparing claims. The office has three employees of varying skill levels, who each take different lengths of time (on average) to complete a given task. Here we see the average time (in minutes) required for each employee to complete each task.

	New Policy	Claim
Employee 1	10	28
Employee 2	15	22
Employee 3	13	18

Tab. 1: Average task processing time in minutes.

The insurance company has compiled the timings above by empirical measurement. We assume that the company has a healthy business and, therefore, that there is a steady stream of New Policy and Claim tasks that require processing by the office employees. Under these circumstances, the insurance company is interested in assigning some fraction of the incoming tasks to each employee so as to minimize the makespan, or the elapsed time required to finish all of the tasks, while maintain the ratio of New Policy and Claim constant.

### 2.1 Modelling Approach

To answer the question “how should the insurance company assign tasks to its employees to minimize the makespan?”, we rely on a linear programming technique known as Simplex Minimization. We consider  $p_i$  and  $q_i$ , the fraction of New Policy tasks and the fraction of Claim tasks, respectively, to be assigned to the  $i^{th}$  employee.

Our objective function is simply  $t$ , the elapsed time (in minutes) required to complete all of the tasks.

#### 2.1.1 Parameters

We consider all of the average processing times listed in Table 1 to be model parameters.

### 2.1.2 Variables

We formally define the model variables as follows:

$p_1, p_2, p_3$	The fraction of New Policy work assigned to each employee
$q_1, q_2, q_3$	The fraction of Claim work assigned to each employee

Tab. 2: Basic Model Variables

### 2.1.3 Constraints

Since we are dealing with human employees and real-world tasks, we assume all trivial and non-negative constraints:  $q_i, t, p_i \geq 0$ . For example, it would be rather foolish to assign a negative number of tasks to an employee for completion. Here we also define the non trivial constraints.

$p_1 + p_2 + p_3 = 1$	We must assign exactly all New Policy tasks
$q_1 + q_2 + q_3 = 1$	We must assign exactly all Claim tasks
$10p_1 + 28q_1 \leq t$	Ensure minimal makespan
$15p_2 + 22q_2 \leq t$	Ensure minimal makespan
$13p_3 + 18q_3 \leq t$	Ensure minimal makespan

Tab. 3: Basic Model Constraints

The first two constraints ensure that all of our tasks get assigned. For obvious reasons, it would not be an optimal schedule to only assign a portion of, say, the New Policy tasks since by definition the unassigned portion would not be completed. The final three constraints ensure that when we allocate tasks to each employee, each allocation falls below the same bounding value for elapsed time.

By constraining the model in this manner, we ensure that no employee exceeds the elapsed time of his peers or the optimal elapsed time,  $t$  and we disqualify the trivial solution  $p_i = q_i = t = 0$ .

### 2.1.4 The Objective Function

Stated more precisely in mathematical terms, our objective is to “choose values of  $p_i$  and  $q_i$  (listed in Table 2) that will minimize the objective function  $t$ , subject to all trivial constraints,  $q_i, t, p_i \geq 0$ , and all non-trivial constraints listed in Table 3.”

## 3 Solving the Basic Employee Scheduling Model

The basic model described above is well-suited for the Simplex Minimization algorithm, and requires little modification. We set up and solved the model in a spreadsheet, seen here.

Efficiency of Workers				
	Processing Time per Policy (min)	Processing Time per Claim (min)		
Employee 1	10	28		
Employee 2	15	22		
Employee 3	13	18		
Decision Variables			Constraints	
	Percent of Policies Processed	Percent of Claims Processed	Employee's Processing "Time"	Total Processing "Total Time"
Employee 1	99.402985%	0.000000%	9.940298507	9.940298507
Employee 2	0.597015%	44.776119%	9.940298507	9.940298507
Employee 3	0.000000%	55.223881%	9.940298507	9.940298507
Total	100%	100%		
Objective function t			9.940298507	

Fig. 1: Basic Model Solution (Excel)

To download the spreadsheet for closer inspection, see Appendix A. From the solution, we find that the optimal assignment strategy is

New Policy Assignments	Claim Assignments
$p_1 = 333/335 \approx 0.994$	$q_1 = 0$
$p_2 = 2/335 \approx 0.006$	$q_2 = 30/67 \approx 0.448$
$p_3 = 0$	$q_3 = 37/67 \approx 0.552$

Tab. 4: Optimal solution to the basic model

Put plainly, over the long term management should assign about 99.4% of New Policy tasks to Employee 1, about 0.6% of New Policy tasks and about 44.8% of Claim tasks to Employee 2, and about 55.2% of Claim tasks to Employee 3.

With this optimal solution, we find resulting optimal makespan is  $t = \frac{666}{67} \approx 9.94$  minutes or about 9:56.

### 3.1 Sensitivity Analysis

From a sensitivity analysis obtained using Excel's simplex Solver, we see here the shadow prices for our two most important constraints.

Constraint	Shadow Price
$p_1 + p_2 + p_3 = 1$	4.0299
$q_1 + q_2 + q_3 = 1$	5.9104

Tab. 5: Constraint shadow prices

We see that the basic model is slightly more sensitive to the constraint  $q_1 + q_2 + q_3 = 1$ , the fraction of Claim tasks assigned, since that constraint has the higher shadow price. We believe that this is caused because the average

time it takes for any employee to handle a Claim task is higher than the average time to process a New Policy task. Thus, the insurance company may be able to streamline its operation by finding ways to reduce the time it takes to process a Claim task. Such strategies might include more employee training, process automation, or improvements in electronic record keeping systems

## 3.2 Observations of the Basic Solution

Our solution is only useful in the long-term where there is a long sequence of tasks to be processed. For example, it wouldn't make sense to assign 0.6% of a single task to one employee. In addition to being impractical, the cost of context-switching as the employees handed off the partially completed task would outweigh any potential time savings. Rather, our scheduling solution would be better applied to a large list of tasks. For example, if the office had a long backlog of work (say, several hundred tasks) or if the insurance company is busy (and healthy), receiving a constant and fairly steady stream of work every day for the foreseeable future.

It is interesting to note that, while no *individual* employee is able to finish any single task in less than ten minutes, our optimal scheduling strategy has all tasks being completed in slightly less than 10 minutes per task, over the long term.

### 3.2.1 Criticism

This basic employee task assignment strategy has multiple flaws. First, there is no information provided regarding the accuracy and methodology of the measured timings listed in Table 1. Additionally, a linear approximation of the employees' average processing time (Table 1) cannot be an accurate representation of real-world worker productivity over the long term. For example, the authors believe that it is unlikely that an employee's productivity would remain constant throughout the day (let alone over weeks, months, or years). Our model does not take into account any aspect of productivity fluctuations caused by factors such as fatigue, boredom, enthusiasm, learning over course of time, etc. The authors also wonder whether such a scheduling strategy could foster animosity and politics between the employees. We leave this as an open question for the reader.

Additionally, the model does not take into account the ratio of policies and claims that need to be processed and assumes that there are an infinite number of policies and claims waiting at any given time. It is very likely that the average number of policies and claims that need to be processed annually are not the same, for example, there could be more policies that need to be processed and fewer claims. This means that the task division may not truly lead to the best strategy for the given insurance company. To further dig into the probable complexity of this problem, we investigate some variation of this problem by introducing further or different constraints and parameters.



## 4 Enhanced Employee Scheduling Problems

The insurance company may be interested to learn how the Basic Solution reacts to other real-world conditions. Here, we present some augmented models to address other scenarios.

### 4.1 Finite Number of Policies and Claims

Consider the scenario in which the insurance company signs a contract to process 500 new policies and 100 claims. To solve this model, in addition to the constraints in Table 3 and Table 1 we have included the constraints  $p_i, q_i \in \mathbb{N}$  and replaced the first two conditions in Table 2 with  $p_1 + p_2 + p_3 = 500$  and  $q_1 + q_2 + q_3 = 100$ .

Employee	Number of Policies	Number of Claims	Processing Time (min)
1	261	1	2638
2	170	4	2638
3	69	95	2607
Project Completion Time:			2638 Minutes

Tab. 6: Finite number of policies and claims solution given 500 and 100, respectively, policies and claims need processing

We also found that the total number of policies and claims that need processing affects the proportion of policies and claims assigned to a given employee. The table below shows this effect, where we assumed that there are 50,000 policies and 10,000 claims that need to be processed.

Employee	Number of Policies	Number of Claims	Processing Time (min)
1	26210	1	262128
2	17462	9	262128
3	6328	9990	262084
Project Completion Time:			262128 Minutes

Tab. 7: Finite number of policies and claims solution given 50,000 and 10,000, respectively, policies and claims needs processing

### 4.2 Employee Sick Time

Suppose that one of the employees becomes ill and cannot work. How would that affect the Basic Model? Does it matter which employee becomes ill? The answers to these questions could help management determine which employees to target with extra preventative health and wellness benefits (perhaps free flu vaccinations, or similar).

We can examine this question by removing one employee at a time from the scheduling model, then running the Simplex Minimization algorithm again on the augmented model. This results in three new solutions (one for each ill employee) which can be compared to determine which ill worker has the greatest impact on overall office productivity. Here we see the solutions to each augmented model.

Sick Employee	$p_i$ values	$q_i$ values	$t$ value
1	$p_2 = 1, p_3 = 0$	$q_2 = 3/40, q_3 = 37/40$	$333/20 \approx 16.65$
2	$p_1 = 1, p_3 = 0$	$q_1 = 4/23, q_3 = 19/23$	$342/23 \approx 14.870$
3	$p_1 = 1, p_2 = 0$	$q_1 = 6/25, q_2 = 19/25$	$418/25 \approx 16.72$

Tab. 8: Solutions to the Employee Sick Time Model

While sick time for any of the employees has a clear negative impact on the processing time (that is, the elapsed time,  $t$ , to process a task increases) we can spot some differences. Sick time by Employees 1 and 3 has roughly the same affect on the makespan,  $t$ , whereas when Employee 2 is ill we see that the overall impact is comparatively low. Thus, from the insurance company's point of view, if any one employee should get sick, it would be preferable that Employee 2 becomes ill. Therefore, management could mitigate the risk, at the least cost, by offering only Employees 1 and 3 extra preventative health care coverage such as seasonal flu vaccinations.

These results also suggest that Employee 2 may be the least productive member of the team.

With these findings on hand, management would be able to confidently predict the adjusted company-wide productivity and adjust deadlines, if necessary, as soon as they get news that an employee has taken ill.

### 4.3 Employee Training

Companies often train their employees to increase efficiency and productivity. Moreover, since professional development budgets are typically limited, the company may wish to offer training to only a subset of its employees. To model a similar problem for our insurance company, we assume that the company has a budget for training only one of its employees with the aim still to reduce the total time needed to complete New Policy and Claim tasks.

#### 4.3.1 Employee Selection Criteria

Percent increased efficiency of a given employee depends on the learning curve of that individual and, certainly, on the suitability of the training curriculum and the length of the training. The simplest scenario would be to assume that all the employees have the same linear learning curve and that the curriculum would be the same for all employees. This assumption seems reasonable since

there are few quantitative metrics available that would measure an individual's potential suitability to a training program.

As a result, it is reasonable to conclude that no matter which employee is chosen, the employee would become more efficient by the same percentage. We assume that an employee would become 10% more efficient upon completion of the training.

### 4.3.2 Methodology and Outcome

Mathematically, the solution to this problem is the same as solving the basic model after replacing for the enhanced task completion time of the trained employee. Hence, this methodology requires solving the Basic Model 3 times, once per employee, and selecting the employee who yields a larger effect on the predicted company-wide makespan. Here we see the revised task completion times for each employee.

Employee Selected for Training	New Policy	Claims
1	$10/1.1 \approx 9.09$	$28/1.1 \approx 25.45$
2	$15/1.1 \approx 13.64$	$22/1.1 \approx 20$
3	$13/1.1 \approx 11.82$	$18/1.1 \approx 16.36$

Tab. 9: Revised Task Processing Time in Minutes for Each Employee

Upon using Maple to compute the revised solutions, we found that training Employee 3 would yield the shortest makespan,  $t$ . In addition, upon the training completion, Employee 1 would be assigned 3.2% fewer New Policy tasks while Employee 2 would be assigned 5.3 times as many New Policy tasks, when compared with the Basic Model results in Table 4. Here we see the details of the revised solutions.

Employee	$p_i$ values	$q_i$ values	$t$ value
1	$p_1 = 1, p_2 = 0, p_3 = 0$	$q_1 = \frac{89}{3889}, q_2 = \frac{1710}{3889}, q_3 = \frac{2090}{3889}$	$\frac{37620}{3889} \approx 9.67$
2	$p_1 = \frac{333}{344}, p_2 = \frac{11}{344}, p_3 = 0$	$q_1 = 0, q_2 = \frac{159}{344}, q_3 = \frac{185}{344}$	$\frac{1665}{172} \approx 9.68$
3	$p_1 = \frac{333}{346}, p_2 = \frac{13}{346}, p_3 = 0$	$q_1 = 0, q_2 = \frac{285}{692}, q_3 = \frac{407}{692}$	$\frac{1665}{173} \approx 9.62$

Tab. 10: Revised assignment schedule, assuming a 10% productivity boost for the trained employee

One could wonder how the makespan would be different if the percent increase in efficiency of the trained employees is not 10%. To investigate this scenario, we computed the makespan for decrease in average processing time of 0% to 20% for each employee, see Figure 2. From the plot, the insurance company would be better off to select Employee 1 for training if the decrease in processing time is just over 2%. Otherwise, the company would be better off to train Employee 3.

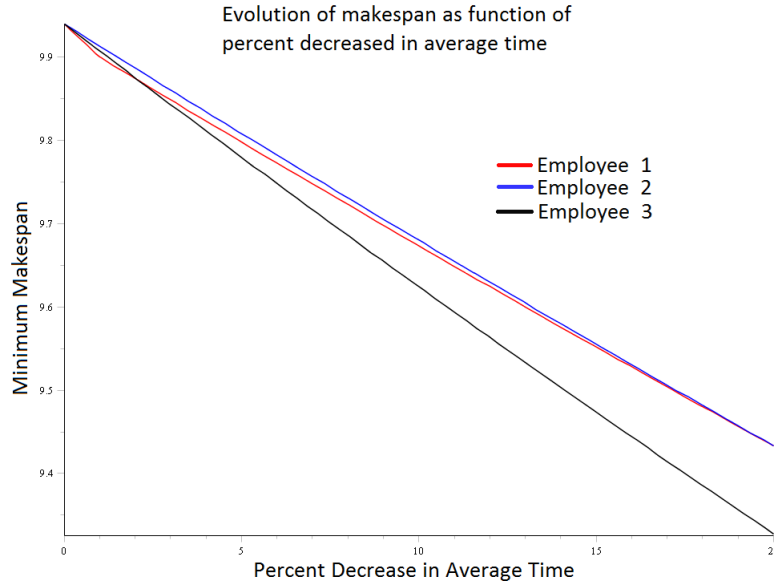


Fig. 2: Minimum Makespan as a Function of Percent Decrease in Average Processing Time for Each Trained Employee

From our analysis, the insurance company would achieve the best outcome, in the face of uncertain metrics, by allocating their limited professional development budget towards Employee 3, whose improvement would likely have the greatest positive impact on task processing times.

#### 4.4 Worker Happiness

Another approach that we looked at is the change in productivity due to an increase in worker happiness. This could be achieved by ensuring every worker has the chance of sharing their ideas during meetings that allow workers to offer their input on changes and policies. Research has shown that engaged employees tend to perform better on the job as compared to workers that have no feeling of contribution to the company. For example, Google considers worker satisfaction to be crucial to its success. As a result, the company provides many benefits and perks such as free meals, health coverage, daycare, gym access, on-site physician, and many more. “Obviously, all these perks come at a cost for Google. But so does employee dissatisfaction and high turnover”[Bulygo, 2013]. Looking at this perspective, we can see that it may also be the right approach to spend a little additional money to ensure that workers are satisfied and motivated to work hard and remain productive.

This problem could be modelled by first gathering preliminary statistics such as by how much an average employee’s productivity would increase as a result of implementation of some employee incentive programs, such as

offering free lunch. As we don't have access to such information, let's assume that average processing time for each application by each employee decreases by 10%. Since the Basic Model that we solved would yield the same result if the relative average processing time of employees stay unchanged, we assume that the increased productivity affects only the average time to process New Policy tasks. Perhaps we found that it is easier to improve in this area as policies result in revenue while claims cost the insurance company money. Here we see the adjusted processing times.

	New Policy	Claim
Employee 1	$10/1.1 \approx 9.09$	28
Employee 2	$15/1.1 \approx 13.64$	22
Employee 3	$13/1.1 \approx 11.82$	18

Tab. 11: Average Task Processing Time in Minutes After Implementation of Employee Incentive Programs

The constraints for this model would be set as follows, while the objective function and methodology are the same as in the Basic Model.

$p_1 + p_2 + p_3 = 1$	We must assign exactly all New Policy tasks
$q_1 + q_2 + q_3 = 1$	We must assign exactly all Claim tasks
$\frac{10}{1.1}p_1 + 28q_1 \leq t$	Ensure minimal makespan
$\frac{15}{1.1}p_2 + 22q_2 \leq t$	Ensure minimal makespan
$\frac{13}{1.1}p_3 + 18q_3 \leq t$	Ensure minimal makespan

Tab. 12: Constraints for the Worker Happiness Model

The optimized assignment strategy would be as follows, resulting in the makespan  $t = \frac{3672}{379} \approx 9.689$ .

New Policy Assignments	Claim Assignments
$p_1 = 1$	$q_1 = \frac{89}{4169} \approx 0.021$
$p_2 = 0$	$q_2 = \frac{1836}{4169} \approx 0.440$
$p_3 = 0$	$q_3 = \frac{204}{379} \approx 0.538$

Tab. 13: Optimal solution to the Worker Happiness Model

As expected, initiatives that reduce the makespan of the optimal task assignment schedule would be beneficial to the insurance company.

## 4.5 Generalized Parameters

Parameters of a given problem, such as parameters in 1, may change over time or may be subject to uncertainty. To investigate this scenario, we considered individual effects of each parameter, one parameter at a time. We varied each  $p_i$

and  $q_i$  between 0.9 to 1.1 times the provided value and computed the optimum task division strategy. The result of this long process was a series of plots of which 1 is illustrated below.

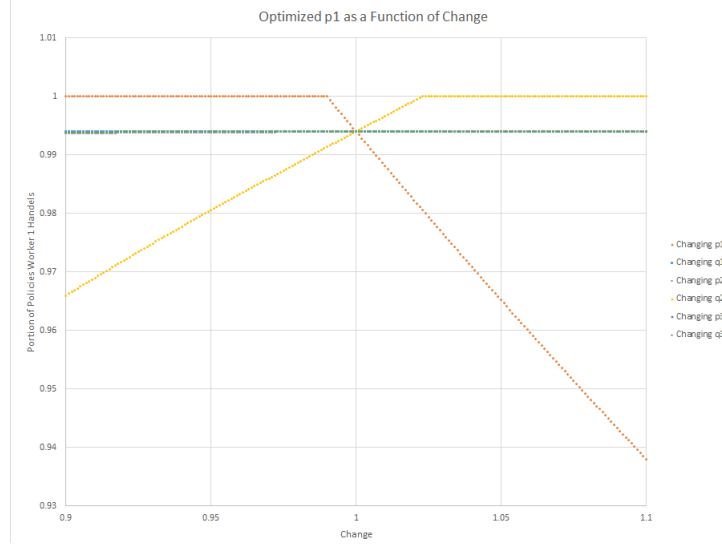


Fig. 3: Portion of Policies That Employee 1 Processes Due to Changes in  $p_i$  and  $q_i$

The plot in Figure 3 can be interpreted as follows: Employee 1 would process 99.4 % of the policies regardless of  $\pm 10\%$  changes in  $p_2$ ,  $p_3$ ,  $q_1$ , and  $q_3$ . If  $q_2$  (Employee 2's claim processing average time) increases by more than 2.3%, or  $p_1$  decreases by more than 1%, then Employee 1 would be processing all of the policies.

## 5 Conclusion

In this paper, we derived a solution to the basic employee task scheduling problem, a specialized case from the class of resource allocation problems. We showed that the insurance company can achieve a substantial improvement in productivity by thoughtfully assigning tasks to each employee. We also cautioned that our model may have flaws, and suggested that the insurance company perform further research before implementing a strategy.

We then investigated more specialized cases of the assignment problem, taking into account factors such as fluctuations in the ratio of task types in the workload, employee sick time, training and professional development, and work happiness. Here, again, we cautioned that a truly complete scheduling model must take into account complicating factors that are likely to arise in the real world.

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Regardless of the extra complexity and uncertainty, the insurance office has stumbled upon a problem whose solution has both interesting and practical consequences. We recommend that they continue to investigate the problem, since we believe that it is highly likely that an effective and efficient solution can be found and implemented.

## 6 Appendix A

Rather than attach printed spreadsheets as an appendix, all of our project documents including spreadsheets, Maple source code, charts, and  $\text{\LaTeX}$  files are available at [https://github.com/colefichter/math372\\_project\\_1](https://github.com/colefichter/math372_project_1).



## References

- [Budiansky, 2013] Budiansky, S. (2013). *Blackett's War: The Men Who Defeated the Nazi U-Boats and Brought Science to the Art of Warfare*. Knopf, Toronto.
- [Bulygo, 2013] Bulygo, Z. (2013). *Inside Google's Culture of Success and Employee Happiness*. <http://blog.kissmetrics.com/googles-culture-of-success/>.
- [Lay, 2006] Lay, D. C. (2006). *Linear Algebra and Its Applications*. Pearson Education, Indianapolis, 3rd edition.
- [Pai, 2012] Pai, P. P. (2012). *Operations Research: Principles and Practice*. Oxford University Press, New Delhi.
- [Petzold, 2008] Petzold, C. (2008). *The Annotated Turing*. Wiley Publishing, Indianapolis.