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1 Introduction

The problem of *effectively* allocating finite resources has been known for centuries. Certainly, as long as there have been functional economies (even systems as simple as barter networks) producers have been pressured to utilize labour, capital, and raw materials in the most efficient manner possible. More generally, there exist a class of problems known as Assignment Problems.

In the most general case, the Assignment Problem boils down to "essentially [trying] to address the issue of getting the best from every [resource] allocation" [Pai, 2012, p. 8]. This could mean assigning workers of various skill levels to machines in a factory, scheduling specific types of products to be produced on assembly lines of varying capacity, reducing change-over and re-tooling costs during production, or many other problems. In this paper, we examine the problem of assigning a varied workload of tasks to a group of employees of varying skill levels in such a way that we minimize the elapsed time, commonly referred to as the MAKESPAN.

Our problem is important because it directly affects the throughput and, therefore, the profitability of the office we study. By finding an optimal assignment strategy that minimizes the amount of time it takes to complete a task, we can be sure that the office operates in an efficient, cost-effective, and sustainable manner over the long term.

1.1 Historical Context

While people have encountered Assignment Problems for at least several centuries, they lacked the tools to effectively tackle them, especially in the general sense, until fairly recently. Indeed, neither the Renaissance nor the Industrial Revolution produced the necessary confluence of applied and theoretical mathematics and advanced computational technology required to enable business owners to solve these types of problems unassisted.

By the mid-20$^{\text{th}}$ century, however, a number of critical breakthroughs would begin to bring powerful mathematical techniques to the masses. Alan Turing had worked to design high-performance code-breaking analog computers and had invented what we now call software while "attacking problems of calculability and Hilbert's Entsheidungsproblem"[Petzold, 2008, p. 65]. John Von Neumann envisioned the architecture of the modern digital computer. Wassily Leontif had "opened the door to a new era in mathematical modeling [sic]"[Lay, 2006, p. 1], work for which he would be "awarded the 1973 Nobel Prize in Economic Science" [Lay, 2006, p. 1]. And a secret group colloquially referred to as "Blackett's Circus" had, while using science and mathematics to defeat the Nazis, created a new industry. They called it OPERATIONS RESEARCH.

Within the British Admiralty, and reporting directly to Winston Churchill, a scientist and Navy officer named Patrick Blackett led the "operations research group, ...a subdepartment of the navy's Scientific Research and Experiment Department." [Budiansky, 2013, p. 462] While developing technologies like RADAR and pioneering the application of science to military strategy, Blackett's team laid the foundation for the post-war surge in operations research work. Such work would eventually lead to minor business revolutions like lean manufacturing and its offshoot, agile software development.

It was during these decades of rapid technological advancement that the mathematical modelling techniques used in this paper were refined. Fortunately for the authors, we now live in a time where complex mathematical models can be effortlessly solved with even a mediocre personal computer.

2 The Employee Scheduling Problem

In this paper, we examine an insurance office that handles two types of work: creating new policies, and preparing claims. The office has three employees of varying skill levels, who each take different lengths of time (on average) to complete a given task. Here we see the average time (in minutes) required for each employee to complete each task type.

	New Policy	Claim
Employee 1	10	28
Employee 2	15	22
Employee 3	13	18

Tab. 1: Average Task Processing Time (in minutes)

The insurance company has compiled the timings above by empirical measurement. We assume that the company has a healthy business and, therefore, that there is a steady stream of New Policy and Claim tasks that require processing by the office employees. Under these circumstances, the insurance company is interested in assigning some fraction of the incoming tasks to each employee so as to minimize the MAKESPAN, or the elapsed time required to finish all of the tasks.

2.1 Modelling Approach

To answer the question "how should the insurance company assign tasks to its employees to minimize the makespan (or elapsed time)?", we rely on a linear programming technique known as Simplex Minimization. We consider p_i and q_i, the fraction of New Policy tasks and the fraction of Claim tasks (respectively) to be assigned to the i^{th} employee.

Our objective function is simply t, the elapsed time (in minutes) required to complete all of the tasks.

2.1.1 Parameters

We consider all of the average processing times listed in Table 1 to be model parameters.

2.1.2 Variables

We now formally define the model variables

p_1, p_2, p_3	The fraction of New Policy work assigned to each employee
q_1, q_2, q_3	The fraction of Claim work assigned to each employee

Tab. 2: Basic Model Variables

2.1.3 Constraints

Since we are dealing with human employees and real-world tasks, we assume all trivial and non-negative constraints. For example, it would be rather foolish to assign a negative number of tasks to an employee for completion. Here we also define the non trivial constraints.

$p_1 + p_2 + p_3 = 1$	We must assign exactly all New Policy tasks
$q_1 + q_2 + q_3 = 1$	We must assign exactly all Claim tasks
$10p_1 + 28q_1 \leq t$	Ensure minimal makespan
$15p_2 + 22q_2 \leq t$	Ensure minimal makespan
$13p_3 + 18q_3 \leq t$	Ensure minimal makespan

Tab. 3: Basic Model Constraints

The first two constraints ensure that all of our tasks get assigned. For obvious reasons, it would not be an optimal schedule to only assign a portion of, say, the New Policy tasks, since by definition the unassigned portion would not be completed. The final three constraints ensure that when we allocate tasks to each employee, each allocation falls below the same bounding value for elapsed time. By constraining the model in this manner, we ensure that no employee exceeds the elapsed time of his peers or the optimal elapsed time, t.

2.1.4 The Objective Function

Stated more precisely in mathematical terms, our question is “what values of p_i and q_i (listed in Table 2) will minimize the objective function t, subject to all trivial constraints and all non-trivial constraints listed in Table 3 ?”

3 Solving the Basic Employee Scheduling Model

The basic model described above is well-suited to the Simplex Minimization algorithm, and requires little modification. We set up and solved the model in a spreadsheet, seen here.

Efficiency of Workers		
	Processing Time per Policy (min)	Processing Time per Claim (min)
Worker 1	10	28
Worker 2	15	22
Worker 3	13	18

	Decision Variables		Constraints	
	Fraction of Policies Processed	Fraction of Claims Processed	Processing Time LHS (min)	Processing Time RHS (min)
Worker 1	0.994029851	0	9.940298507	9.940298507
Worker 2	0.005970149	0.447761194	9.940298507	9.940298507
Worker 3	0	0.552238806	9.940298507	9.940298507
Constraints LHS	1	1		
Constraints RHS	1	1		
Objective	9.940298507			

Fig. 1: Basic Model Solution (Excel)

To download the spreadsheet for closer inspection, see Appendix A. From the solution, we find that the optimal assignment strategy is

New Policy Assignments	Claim Assignments
$p_1 = 333/335 \approx 0.994$	$q_1 = 0$
$p_2 = 2/335 \approx 0.006$	$q_2 = 30/67 \approx 0.448$
$p_3 = 0$	$q_3 = 37/67 \approx 0.552$

Tab. 4: Optimal Solution to the Basic Model

Put plainly, over the long term management should assign about 99.4% of New Policy tasks to Employee 1, about 0.6% of New Policy tasks and about 44.8% of Claim tasks to Employee 2, and about 55.2% of Claim tasks to Employee 3.

With this optimal solution, we find resulting optimal makespan is $t = \frac{666}{67} \approx$ 9.94 minutes or about 9:56.

3.1 Observations of the Basic Solution

First, note that our solution is only useful in the long-term. It wouldn't make sense to assign 0.6% of a single task to one employee. In addition to being impractical, the cost of context-switching as the employees handed off the partially completed task would outweigh any potential time savings. Rather, our scheduling solution would be better applied to a large list of tasks. For example, if the office had a long backlog of work (say, several hundred tasks) or if the insurance company is busy (and healthy), receiving a constant and fairly steady stream of work every day for the foreseeable future.

It is interesting to note that, while no *individual* employee is able to finish any single task in less than ten minutes, our optimal scheduling strategy has all tasks being completed in slightly less than 10 minutes per task, over the long term.

4 Enhanced Employee Scheduling Problems

5 Appendix A

Rather than attach printed spreadsheets as an appendix, all of our project documents including spreadsheets, Maple source code, charts, and LaTeX files are available from our public repository, located at https://github.com/colefichter/math372_project_1.

References

[Budiansky, 2013] Budiansky, S. (2013). *Blackett's War: The Men Who Defeated the Nazi U-Boats and Brought Science to the Art of Warfare*. Knopf, Toronto.

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