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2.1 Context

The ecosystem of a forest is in a constant and perpetual cycle of growth and decay. As trees grow they drop leaves, branches, and other organic material to the forest floor. This material is called *litter*. Being organic debris from trees, the litter contains trapped carbon, and as the litter decays, the trapped carbon is released in two forms. First, upon consumption of litter by living organisms, carbon dioxide molecules are released into the atmosphere through respiration. Second, a certain proportion of the carbon in the litter is converted into the black substance, known as *humus*, which gives compost and soil their characteristic colour and texture. This conversion from litter to humus is known as *humification*.

By demarcating a boundary defining our system, we could set up an area of the forest floor where we monitor and measure the density of the carbon in the litter located within the boundary.

Carbon enters such a system continuously through the litter-fall from trees and plants living within the boundary and carbon exits the system via respiration and humification.

2.2 Modelling the Carbon in a Simple Litter Model

We assume that the rate of litter-fall is constant, as is the sum of the rates of respiration and humification, within the system. Our model, then relies on the following variables and parameters:

	Type	Description
$t \geq 0$	Variable	Time, represented continuously
$x(t) \geq 0$	Function	The amount of carbon in litter form at time t, in $g\ C/m^2$
$z \geq 0$	Parameter	The litter-fall rate, in $g\ C/m^2$
$r \geq 0$	Parameter	The respiration proportionality constant, in $/year$
$h \geq 0$	Parameter	The humification proportionality constant, in $/year$

Tab. 1: Parameters and Variables in the Simple Litter Model

We assume that carbon enters the system at a constant rate through litter-fall, and that carbon exits the system by respiration at rate r and humification at rate h, such that $r + h$ is a constant. We also assume that initially there ia a zero litter density (and thus zero carbon in the area under survailence). This assumption corresponds to a simulation of forest rejuvenation after a ground fire.

Our objective, then, is to model the amount of carbon contained in the litter at any point in time t. In mathematical terms, we seek to compute $x(t)$ given the constant rate of litter-fall z, the constant rate $r+h$ of respiration plus humification within the system.

The model can be represented with the following differential equation:

$$x'(t) = z - (r+h) * x(t) \tag{1}$$

Since $r+h$ is a constant, we can combine the rate into a single rate $k = r+h$ which yields:

$$x'(t) = z - k * x(t) \tag{2}$$

2.3 Solution to the Simple Litter Model

The general solution to the first-order constant-coefficient differential equation (2) can be easily found using integration factor as shown below

$$\begin{aligned} x'(t) &= z - kx(t) \\ x'(t) + kx(t) &= z \\ \frac{d}{dt}[e^{kt}x(t)] &= e^{kt}z \\ \int d[e^{kt}x(t)] &= z\int e^{kt}dt \\ e^{kt}x(t) &= z\frac{1}{k}e^{kt} + c_1 \end{aligned}$$

$$x(t) = c_1 e^{-kt} + \frac{z}{k} \tag{3}$$

To find a specific solution to the differential equation, we rely on the assumption that initially there is zero carbon in the system: $x(0) = 0$. Thus,

$$\begin{aligned} x(0) &= c_1 e^{-k0} + \frac{z}{k} \\ \implies c_1 &= -\frac{z}{k} \end{aligned}$$

2.3.1 A Concrete Solution to the Simple Litter Model

From the existing literature, we estimate parameter values to represent real-world measurements of a temperate forest. We let

$$\begin{aligned} z &= 240\ g\ C/m^2 \\ k = r + h &= 0.4/year \end{aligned}$$

Plugging these parameter values into the model equations, we find

$$\begin{aligned} c_1 &= -\frac{z}{k} \\ &= -\frac{240}{0.4} \\ &= -600 \end{aligned}$$

And equation (3) becomes

$$x(t) = 600 - 600e^{-0.4t} \tag{4}$$

Here we see a plot of the solution equation (4)

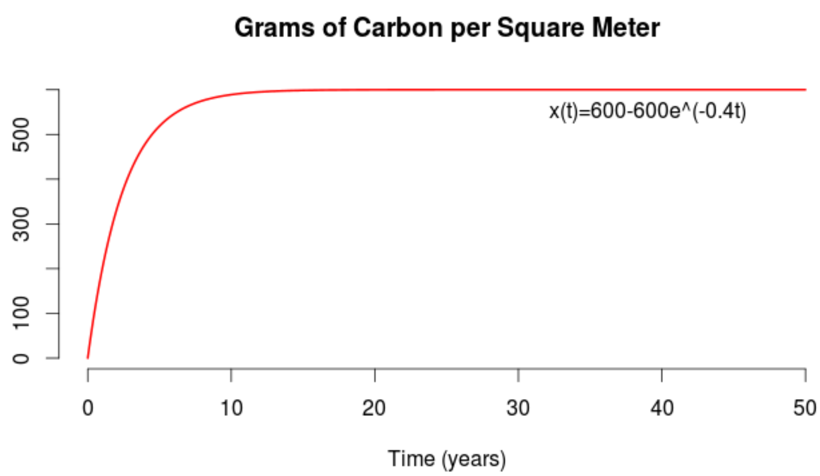

Fig. 1: Plot of the Solution to the Simple Litter Model

2.3.2 Numerical Solution to the Simple Litter Model

Oftentimes it is not possible to obtain a closed form or easily calculable analytical solution to differential equations. To investigate farther into this topic, we stipulated a numerical solution to equation 2. Using initial condition $x(0) = 0$ and Eauler's method for solving ODEs, we preceeded as shown in Algorithm 1.

Plot of the numerical solution which is overlayed on the top of the analytical solution is depicted in figure 2. Additionally

Algorithm 1 Euler method for solving simple litter model ODE, 2

1. Initial Conditions: $t = 0$ & $x(0) = 0$
2. $\delta t = 0.1\text{Year}$
3. $x'(t + \delta t) = 240 - 0.4x(t)$
4. $x(t + \delta t) = x(t) + \delta t\ x'(t)$
5. $t = t + 0.1$
6. While $t < 50$ Years, goto 3

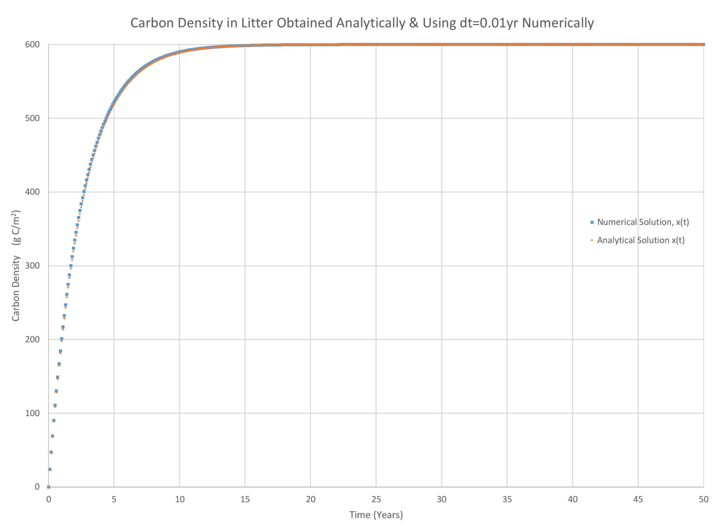

Fig. 2: Plot of the Numerical Solution Overlayed on Top of Analytical Solution to the Simple Litter Model

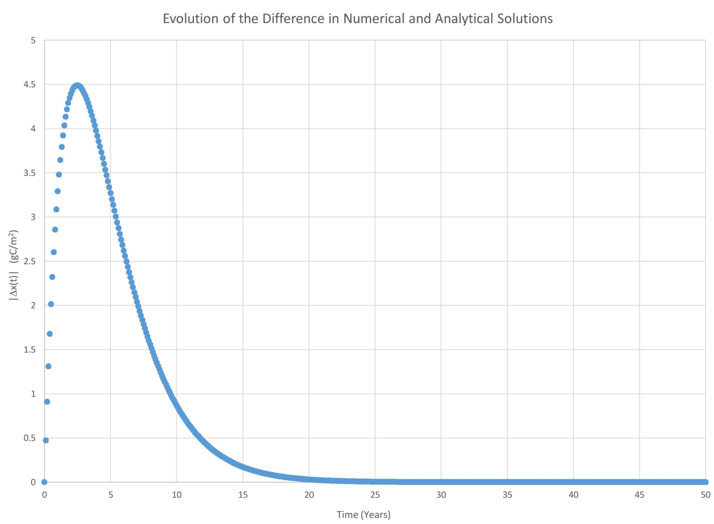

Fig. 3: Plot of the Absolute Value of the Difference Between Numerical and Analytical Solutions to Simple Litter Problem

2.4 Analysis of the Simple Litter Model

From Figure 1, it appears that the model achieves equilibrium at $x(t) = 600$. We can validate this observation by finding the fixed points of the model.

By definition, the fixed point of our model exists when $x'(t) = 0$, which occurs only when $-600 = -600e^{-0.4t}$. Since this equation does not have any real roots, we can't analytically find a fixed point t^* for any sensible measure of time (time can't be measured with a complex number), but we can approximate it numerically. Indeed by $t = 12$, $x(t)$ is within 1% of the equilibrium value $x(t) = 600$.

Working with the assumption that there exists a fixed point t^* such that $x(t^*) = 600$, we can use graphical stability analysis to determine the stability of the point $x(t^*)$. Here we see a plot of $x(t)$ versus $x'(t)$, annotated with the usual markings used in graphical stability analysis

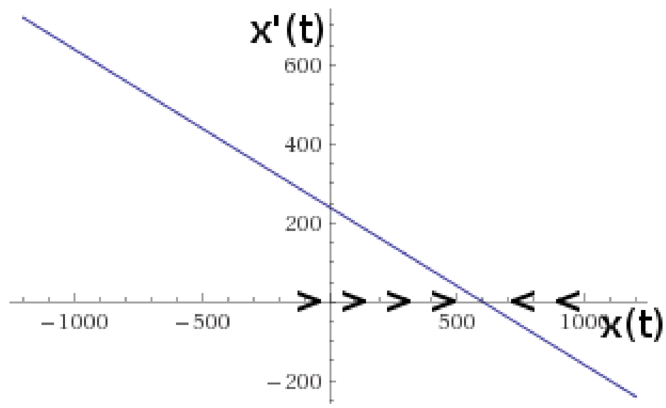

Fig. 4: Graphical Stability Analysis of the Simple Litter Model

The graphical stability analysis confirms that $x(t^*) = 600$ is indeed a stable point, where we should expect the system to stabilize. This confirms the observations of Figure 1 and explains why the graph of the solution reaches equilibrium.

2.5 Observations and Conclusions of the Simple Litter Model

Our model suggests that the density of carbon sequestered in the litter found on the floor of a temperate forest should reach an equilibrium of about 600 $g\ C/m^2$, even if the initial state of the forest is free of litter due to a recent ground fire.

Intuitively, this makes sense based on our familiarity with nature. Biological systems generally must tend toward equilibrium since any environment containing finite resources cannot support infinite growth. Also, if systems tended toward unbounded decay (or decline), life would have been extinguished billions of years ago.

2.6 Criticisms of the Simple Litter Model

This model is obviously very simplistic. We measure carbon only within the litter, and only within an arbitrarily bounded region of the forest. We also assume that carbon exits the litter via respiration and humification at the same rate.

Assuming that all of the rates in the model are constant is probably not a good representation of a real biological system. In a temperate climate, trees drop their leaves in the autumn. Surely this type of seasonal fluctuation would affect the rate of litter-fall, z. Since we have assumed that z is constant, this model would be more suited to a discreet modelling approach where each discrete time step, t, represents one year. However, since we've viewed time continuously, our rate of litter-fall should really fluctuate seasonally.

Finally, we don't have a good understanding of the constants provided in the problem, nor a clear picture of how they were measured and the accuracy of those measurements.

In the next section, we look at a more complicated problem, The Carbon Cycle, which addresses some of these shortcomings.

3 An Enhanced Model: The Carbon Cycle

3.1 Context

We now focus on a more complicated model that addresses some of the concerns outlined above. It is called the Carbon Cycle Model. This model extends the previous Simple Litter Model to examine carbon as it flows throughout an ecosystem. In addition to litter, we also consider leaves, branches, stems, roots, humus, and stable humus charcoal.

Here we see a conceptual diagram of the Carbon Cycle Model

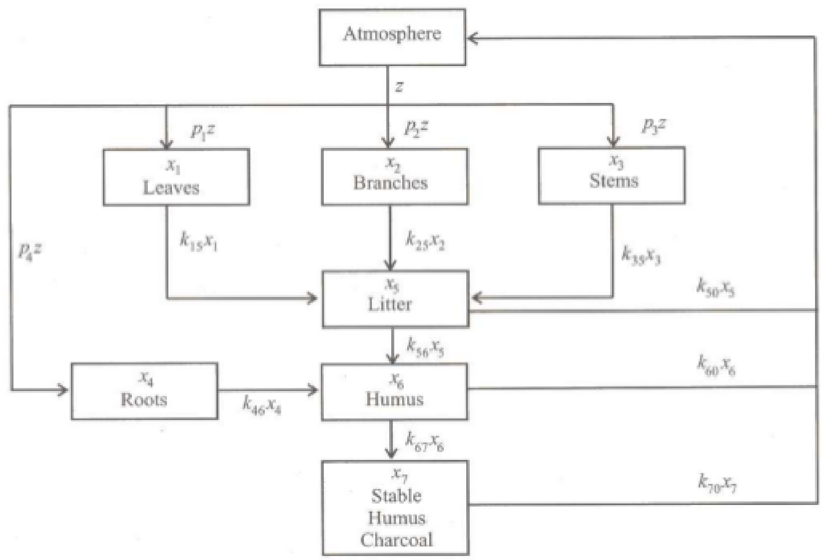

Fig. 5: Conceptual Diagram of the Carbon Cycle Model

Although noted in the diagram, we consider the atmosphere to be external to the model. This is mainly due to the enormity of the atmosphere as compared with the finite region being modelled. Carbon that is absorbed by plants from the atmosphere is considered to be entering the model, while carbon respirated to the atmosphere during the decay process is considered to be leaving the model.

Carbon enters the cycle when growing plants absorb carbon dioxide (CO_2) from the atmosphere. The CO_2 is converted into organic plant matter and sequestered within the leaves, branches, stems, and roots of each plant. Over time, as plants drop leaves and stems, litter accumulates on the forest floor, as in the Simple Litter Model.

The plant roots and the litter then undergo humification which produces humus, as in the Simple Litter Model, and CO_2. As the humus decays, it produces stable humus charcoal, and more CO_2. As charcoal breaks down, it also releases some CO_2. All of the carbon dioxide produced during this cycle ends up back in the global atmosphere, ready to be reabsorbed by growing plants.

3.2 Modelling the Carbon in The Carbon Cycle

Our Carbon Cycle Model relies on a system of differential equations, each fairly similar to equation 2 from the Simple Litter Model.

3.2.1 Parameters

We assume that carbon enters the system from the atmosphere at a constant rate, z, measured in metric Gigatonnes of carbon per year ($Gt\ C/year$). We further partition z into the portion which goes into the leaves, branches, stems, and roots by the proportion parameters p_1, p_2, p_3, p_4. Note that the p_i essentially represent a probability distribution for the dispersion of z into the components of the model. The parameters k_{ij} are proportions that indicate the rate of carbon transferred from x_i to x_j.

z	The rate that carbon enters the system, in $Gt\ C/year$
p_i	The proportion of z absorbed into x_i
k_{ij}	The rate of carbon transfer from x_i to x_j

Tab. 2: Parameters of the Carbon Cycle Model

3.2.2 Variables

We represent each of the seven model components (leaves, branches, stems, roots, litter, humus, stable humus charcoal) by a variable x_i, where $i = 1, 2, ...7$. Thus, x_1 represents leaves, x_4 represents roots, etc. These variables can be seen in Figure 5 and in the following table.

x_i	The amount of carbon stored in each of the model components, $i = 1, 2, ...7$

Tab. 3: Variables of the Carbon Cycle Model

We assume that carbon flowing between any two connected components in the diagram can be represented by a differential equation, as we did in the Simple Litter Model. Our model, then, consists of a system of seven differential equations. In contrast with the simple litter model, our goal is no longer to compute the amount of carbon in any given component, but rather to examine the system as a whole. Of particular interest are so called steady-states, where the system may achieve an equilibrium.

3.2.3 Differential Equations

$$
\begin{aligned}
\dot{x}_1 &= p_1 z - k_{15} x_1 \\
\dot{x}_2 &= p_2 z - k_{25} x_2 \\
\dot{x}_3 &= p_3 z - k_{35} x_3 \\
\dot{x}_4 &= p_4 z - k_{46} x_4 \\
\dot{x}_5 &= k_{15} x_1 + k_{25} x_2 + k_{35} x_3 - k_{50} x_5 - k_{56} x_5 \\
\dot{x}_6 &= k_{46} x_4 + k_{56} x_5 - k_{60} x_6 - k_{67} x_6 \\
\dot{x}_7 &= k_{67} x_6 - k_{70} x_7
\end{aligned}
$$

Tab. 4: Differential Equations of the Carbon Cycle Model

To make the notation a little less cumbersome, let $X' = \langle \dot{x}_1, \dot{x}_2, ..., \dot{x}_7 \rangle$. That is, we place the left hand side of each differential equation in Table 4 into a column vector and denote it X'.

Next, from the right hand side of the differential equations in Table 4, we place the k_{ij} coefficients into a matrix A, as follows

$$A = \begin{bmatrix} -k_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_{25} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_{46} & 0 & 0 & 0 \\ k_{15} & k_{25} & k_{35} & 0 & -k_{50} - k_{56} & 0 & 0 \\ 0 & 0 & 0 & k_{46} & k_{56} & -k_{60} - k_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{67} & -k_{70} \end{bmatrix}$$

We also let $\vec{x} = \langle x_1, x_2, ..., x_7 \rangle$ (a column vector of variables from the right hand side of Table 4). Finally, we add the remaining constant terms from the right hand side of Table 4 to a constant vector, $\vec{b}$

$$\vec{b} = \begin{bmatrix} p_1 z \\ p_2 z \\ p_3 z \\ p_4 z \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we can now represent our system of differential equations with the vector equation

$$X' = A\vec{x} + \vec{b} \tag{5}$$

3.3 Solution to the Carbon Cycle Model

3.4 Analysis of the Carbon Cycle Model

3.5 Observations and Conclusions of the Carbon Cycle Model

3.6 Criticisms of the Carbon Cycle Model

4 Conclusion

5 Appendix A