Contents

List of Tables

List of Figures

1 Introduction

1.1 Context

In this paper we attempt to model, mathematically, carbon as it moves through the ecosystem. In a typical forest, carbon dioxide (CO_2) exists everywhere in the atmosphere. As trees and other plants grow, they absorb the carbon dioxide from the atmosphere and convert it into the plant matter that makes up their stems, leaves, branches, and roots. Eventually, the plants drop their leaves and other bits, which accumulate on the forest floor. We call this accumulated organic material *litter*. Over time, this litter begins to decay, a process which releases some of the stored carbon back into the atmosphere as CO_2. The rest of the stored carbon is converted into *humus* (the black substance that gives soil its characteristic colour and texture) as well as *stable humus charcoal* (pure carbon produced as humus decays).

Our goal is to be able to compute the density of carbon stored in each phase of the biosphere life-cycle at any given point in time. We begin with an admittedly naive model that examines only the carbon stored in the litter deposited on the forest floor. We then examine a more complicated model that addresses some of the shortcomings of the naive model, and allows us to compute the amount of carbon present in any part of the biosphere.

1.2 Overview of Results (Executive Summary)

TODO: write this!

2 Carbon in a Simple Litter Model

2.1 Context

The ecosystem of a forest is in a constant and perpetual cycle of growth and decay. As trees grow they drop leaves, branches, and other organic material to the forest floor. This material is called *litter*. Being organic debris from trees, the litter contains trapped carbon, and as the litter decays, the trapped carbon is released in two forms. First, upon consumption of litter by living organisms, carbon dioxide molecules are released into the atmosphere through respiration. Second, a certain proportion of the carbon in the litter is converted into the black substance, known as *humus*, which gives compost and soil their characteristic colour and texture. This conversion from litter to humus is known as *humification*.

By demarcating a boundary defining our system, we could set up an area of the forest floor where we monitor and measure the density of the carbon in the litter located within the boundary.

Carbon enters such a system continuously through the litter-fall from trees and plants living within the boundary and carbon exits the system via respiration and humification.

2.2 Modelling the Carbon in a Simple Litter Model

We assume that the rate of litter-fall is constant, as is the sum of the rates of respiration and humification, within the system. Our model, then relies on the following variables and parameters:

	Type	Description
$t \geq 0$	Variable	Time, represented continuously
$x(t) \geq 0$	Function	The amount of carbon in litter form at time t, in $g\ C/m^2$
$z \geq 0$	Parameter	The litter-fall rate, in $g\ C/m^2$
$r \geq 0$	Parameter	The respiration proportionality constant, in $/year$
$h \geq 0$	Parameter	The humification proportionality constant, in $/year$

Tab. 1: Parameters and variables in the Simple Litter Model

We assume that carbon enters the system at a constant rate through litter-fall, and that carbon exits the system by respiration at rate r and humification at rate h, such that $r + h$ is a constant. We also assume that initially there is a zero litter density (and thus zero carbon in the area under surveillance). This assumption corresponds to a simulation of forest rejuvenation after a ground fire.

Our objective, then, is to model the amount of carbon contained in the litter at any point in time t. In mathematical terms, we seek to compute $x(t)$ given the constant rate of litter-fall z, and the constant rate, $r + h$, of respiration plus humification within the system.

The model can be represented with the following differential equation:

$$x'(t) = z - (r + h) * x(t) \tag{1}$$

Since $r + h$ is a constant, we can combine the rate into a single rate $k = r + h$ which yields:

$$x'(t) = z - k * x(t) \tag{2}$$

2.3 Solution to the Simple Litter Model

The general solution to the first-order constant-coefficient differential equation (2) can be easily found using integration factor as shown below

$$\begin{aligned}
x'(t) &= z - kx(t) \\
x'(t) + kx(t) &= z \\
\frac{d}{dt}[e^{kt}x(t)] &= e^{kt}z \\
\int d[e^{kt}x(t)] &= z\int e^{kt}dt \\
e^{kt}x(t) &= z\frac{1}{k}e^{kt} + c_1
\end{aligned}$$

$$x(t) = c_1 e^{-kt} + \frac{z}{k} \tag{3}$$

To find a specific solution to the differential equation, we rely on the assumption that initially there is zero carbon in the system: $x(0) = 0$. Thus,

$$\begin{aligned} x(0) &= c_1 e^{-k0} + \frac{z}{k} \\ \implies c_1 &= -\frac{z}{k} \end{aligned}$$

2.3.1 A Concrete Solution to the Simple Litter Model

From the existing literature, we estimate parameter values to represent real-world empirical measurements of a temperate forest. We let

$$\begin{aligned} z &= 240\ g\ C/m^2 \\ k = r + h &= 0.4/year \end{aligned}$$

Plugging these parameter values into the model equations, we find

$$\begin{aligned} c_1 &= -\frac{z}{k} \\ &= -\frac{240}{0.4} \\ &= -600 \end{aligned}$$

And equation (3) becomes

$$x(t) = 600 - 600e^{-0.4t} \tag{4}$$

Here we see a plot of the solution equation (4)

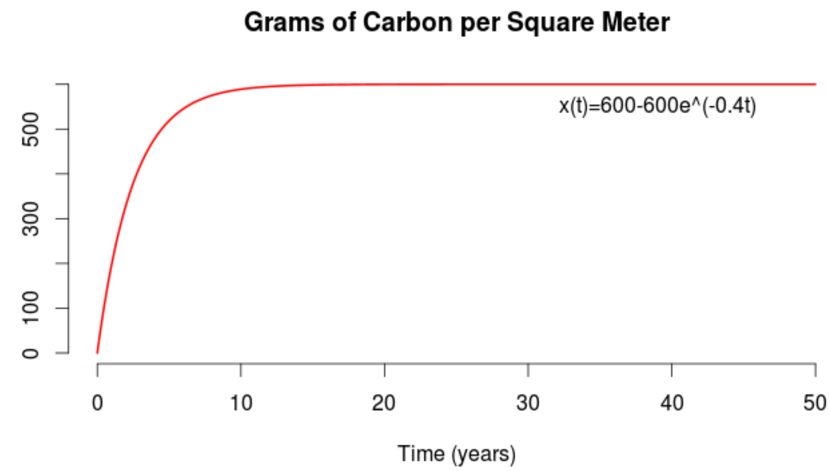

Fig. 1: Plot of the solution to the Simple Litter Model

2.3.2 Numerical Solution to the Simple Litter Model

Oftentimes it is not possible to obtain a closed form or easily calculable analytical solution to differential equations. To investigate farther into this topic, we stipulated a numerical solution to equation (2). Using initial condition $x(0) = 0$ and Euler's method for solving oridnary differential equations, we preceeded as shown in Algorithm 1. For the first 50 years, a plot of the numerical and analytical solutions were overlayed on the top of their difference $|\Delta x(t)|$ in figure (2). The two solutions agree within 2.43 $g\ C/m^2$ 89% of the times and they reach maximum difference of 4.49 $g\ C/m^2$after 2.5 years.

Algorithm 1 Euler method for solving simple litter model ODE, eqn.(2)

1. Initial Conditions: $t = 0$ & $x(0) = 0$
2. $\delta t = 0.1$Year
3. $x'(t + \delta t) = 240 - 0.4x(t)$
4. $x(t + \delta t) = x(t) + \delta t\ x'(t)$
5. $t = t + 0.1$
6. While $t < 50$ Years, goto 3

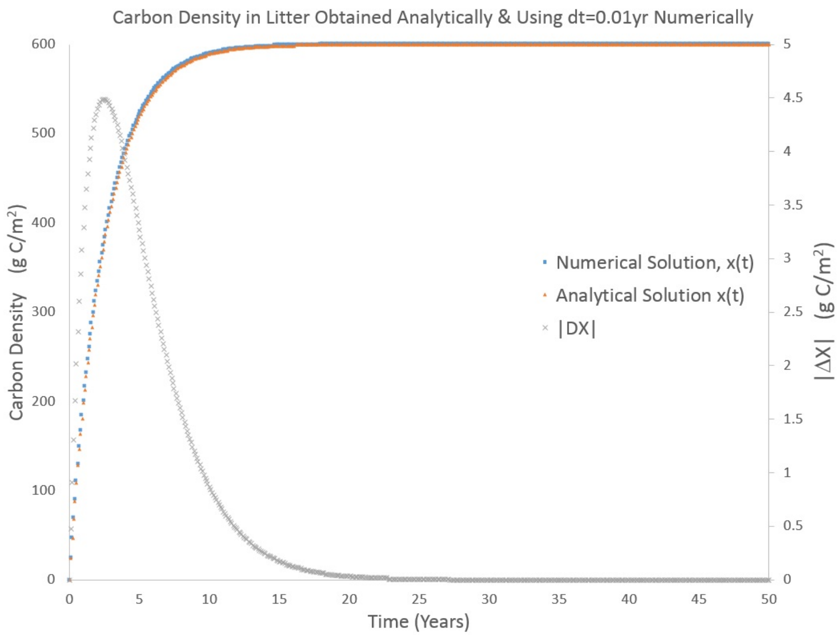

Fig. 2: Plot of numerical and analytical solutions to the Simple Litter Model (left axis) as well as a plot of the absolute value of the difference between analytical and numerical solution $|\Delta x(t)|$ (right axis). The two solutions reach maximum disagreement of 4.49 $g\ C\ /\ m^2$at 2.5 years.

2.4 Analysis of the Simple Litter Model

From Figure 1, it appears that the model achieves equilibrium at $x(t) = 600$. We can validate this observation by finding the fixed points of the model.

By definition, the fixed point of our model exists when $x'(t) = 0$. By rearranging the equation as $x'(t) = -0.4(x(t) - 600)$, it is easy to see that $x'(t) = 0$ only when $x(t) = 600$. Since equation (4) does not have any real roots, we can't analytically find a fixed point t^* for any sensible measure of time (time can't be measured with a complex number), but we can approximate it numerically. Indeed by $t = 12$, $x(t)$ is within 1% of the equilibrium value $x(t) = 600$.

Working with the assumption that there exists a fixed point t^* such that $x(t^*) = 600$, we can use graphical stability analysis to determine the stability of the point $x(t^*)$. Here we see a plot of $x(t)$ versus $x'(t)$, annotated with the usual markings used in graphical stability analysis

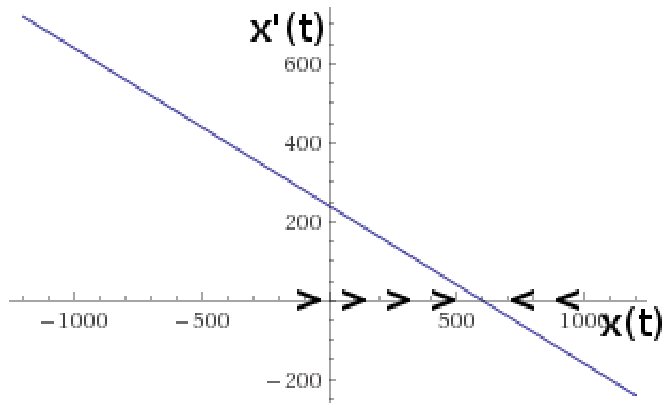

Fig. 3: Graphical Stability Analysis of the Simple Litter Model

The graphical stability analysis confirms that $x(t^*) = 600$ is indeed a stable point, where we should expect the system to equalize. This confirms the observations of Figure 1 and explains why the graph of the solution reaches equilibrium.

2.5 Observations and Conclusions of the Simple Litter Model

Our model suggests that the density of carbon sequestered in the litter found on the floor of a temperate forest should reach an equilibrium of about 600 $g\ C/m^2$, even if the initial state of the forest is free of litter due to a recent ground fire.

Intuitively, this makes sense based on our familiarity with nature. Biological systems generally must tend toward equilibrium since any environment containing finite resources cannot support infinite growth. Also, if systems tended toward unbounded decay (or decline), life would have been extinguished billions of years ago.

2.6 Criticisms of the Simple Litter Model

This model is obviously very simplistic. We measure carbon only within the litter, and only within an arbitrarily bounded region of the forest. We also assume that carbon exits the litter via respiration and humification at the same rate.

Assuming that all of the rates in the model are constant is probably not a good representation of a real biological system. In a temperate climate, trees drop their leaves in the autumn. Surely this type of seasonal fluctuation would affect the rate of litter-fall, z. Since we have assumed that z is constant, this model would be more suited to a discrete modelling approach where each discrete time step, t, represents one year. However, since we've viewed time continuously, our rate of litter-fall should really fluctuate seasonally.

Finally, we don't have a good understanding of the constants provided in the problem, nor a clear picture of how they were measured and the accuracy of those measurements.

In the next section, we look at a more complicated problem, The Carbon Cycle, which addresses some of these shortcomings.

3 An Enhanced Model: The Carbon Cycle

3.1 Context

We now focus on a more complicated model of the Carbon Cycle which addresses some of the concerns outlined above. This model extends the previous Simple Litter Model to examine carbon as it flows throughout an ecosystem. In addition to litter, we also consider the flow of Carbon between leaves, branches, stems, roots, humus, and stable humus charcoal. The flow diagram of this Carbon Cycle Model is depicted in figure 4.

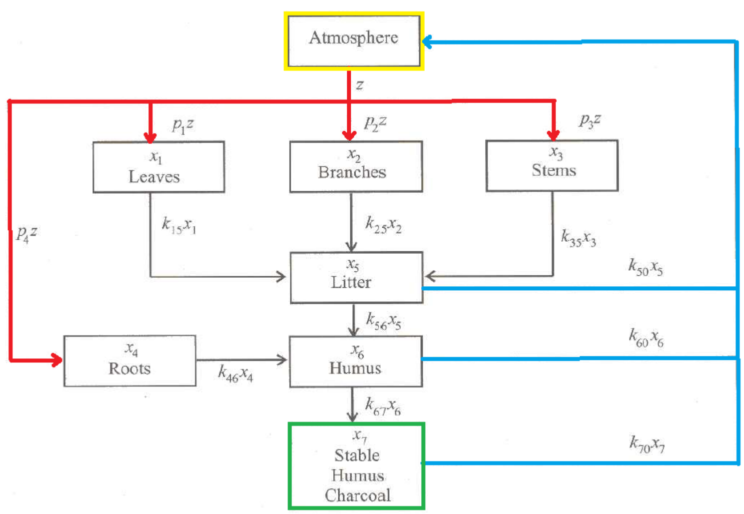

Fig. 4: Conceptual Diagram of the Carbon Cycle Model. Carbon flow into our system from Carbon reserrgvoir, Atmosphere. The arrows represent the Carbon flow directions and their values represent flow rates while x_i represent the amount of carbon in compartment i.

In this system, we consider the atmosphere to be external to the model. This is mainly due to the enormity of the atmosphere as compared with the finite region being modelled. Carbon that is absorbed by plants from the atmosphere is considered to be entering the model, while carbon respirated to the atmosphere during the decay process is considered to be leaving the model.

Carbon enters the cycle when growing plants absorb carbon dioxide, CO_2, from the atmosphere. This CO_2 is converted into organic plant matter and sequestered within the leaves, branches, stems, and roots of each plant. Over time, as plants drop leaves and stems, litter accumulates on the forest floor, as in the Simple Litter Model.

The plant roots and the litter then undergo humification which produces humus, as in the Simple Litter Model, and CO_2. As the humus decays, it produces stable humus charcoal, and more CO_2. As charcoal breaks down, it also releases some CO_2. All of the carbon dioxide produced during this cycle ends up back in the global atmosphere, ready to be reabsorbed by growing plants.

3.2 Modelling the Carbon in The Carbon Cycle

Our Carbon Cycle Model relies on a system of differential equations, each fairly similar to equation (2) from the Simple Litter Model.

3.2.1 Parameters

We assume that carbon enters the system from the atmosphere at a constant rate, z, measured in metric grams of carbon per year per squared meter ($g\ C/year/m^2$). We further partition z into the portion which goes into the leaves, branches, stems, and roots by the proportion parameters p_1, p_2, p_3, p_4. Note that p_i essentially represent a probability distribution for the dispersion of z into the components of the model. The parameters k_{ij} are proportions that indicate the rate of carbon transferred from x_i to x_j.

z	The rate that carbon enters the system, in $g\ C/year/m^2$
p_i	The proportion of z absorbed into x_i
k_{ij}	The rate of carbon transfer from x_i to x_j

Tab. 2: Parameters of the Carbon Cycle Model

3.2.2 Variables

We represent each of the seven model components [leaves, branches, stems, roots, litter, humus, stable humus charcoal) by a variable x_i, where $i = 1, 2, ...7$. Thus, x_1 represents leaves, x_4 represents roots, etc. These variables can be seen in Figure 4 and in the following table.

x_1	The amount of carbon stored in leaves
x_2	The amount of carbon stored in branches
x_3	The amount of carbon stored in stems
x_4	The amount of carbon stored in roots
x_5	The amount of carbon stored in litter
x_6	The amount of carbon stored in humus
x_7	The amount of carbon stored in charcoal

Tab. 3: Variables of the Carbon Cycle Model

We assume that carbon flowing between any two connected components in the diagram can be represented by a differential equation, as we did in the Simple Litter Model. Our model, then, consists of a system of seven differential equations. In contrast with the simple litter model, our goal is no longer to compute the amount of carbon in any given component, but rather to examine the system as a whole. Of particular interest are the steady-states, where the system achieves an equilibrium.

3.2.3 Differential Equations

A set of seven differential equations, one for each component of the model, was obtained by allowing the rate of change of carbon for a given component of the system to be the difference between the rate of inflow and outflow of carbon in that component of the system. This process is similar to that of the Simple Litter Model. Table 4 of the differential equations is shown below.

$$
\begin{aligned}
\dot{x}_1 &= [p_1 z] - [k_{15} x_1] \\
\dot{x}_2 &= [p_2 z] - [k_{25} x_2] \\
\dot{x}_3 &= [p_3 z] - [k_{35} x_3] \\
\dot{x}_4 &= [p_4 z] - [k_{46} x_4] \\
\dot{x}_5 &= [k_{15} x_1 + k_{25} x_2 + k_{35} x_3] - [k_{50} x_5 + k_{56} x_5] \\
\dot{x}_6 &= [k_{46} x_4 + k_{56} x_5] - [k_{60} x_6 + k_{67} x_6] \\
\dot{x}_7 &= [k_{67} x_6] - [k_{70} x_7]
\end{aligned}
$$

Tab. 4: Differential Equations of the Carbon Cycle Model. The rate of change of $x_i$was set to be the difference between the rate of inflow and outflow of the carbon into component i of the system.

To simplify the notation, let $X' = \langle \dot{x}_1, \dot{x}_2, ..., \dot{x}_7 \rangle^t$. That is, we place the left hand side of each differential equation in Table 4 into a column vector and denote it X'. Next, from the right hand side of the differential equations in Table 4, we place the k_{ij} coefficients into a matrix A, as follows

$$A = \begin{bmatrix} -k_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_{25} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_{46} & 0 & 0 & 0 \\ k_{15} & k_{25} & k_{35} & 0 & -(k_{50}+k_{56}) & 0 & 0 \\ 0 & 0 & 0 & k_{46} & k_{56} & -(k_{60}+k_{67}) & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{67} & -k_{70} \end{bmatrix}$$

We also let $\vec{x} = \langle\, x_1,\ x_2,\ \ldots,\ x_7 \,\rangle^t$ (a column vector of variables from the right hand side of Table 4). Finally, we add the remaining constant terms from the right hand side of Table 4 to a constant vector, $\vec{b}$

$$\overrightarrow{b} \;=\; \langle\, p_1 z,\ p_2 z,\ p_3 z,\ p_4 z,\ 0\ ,\ 0\ ,\ 0 \,\rangle^t$$

Thus, we can now represent our system of differential equations with the vector equation

$$X' = A\vec{x} + \vec{b} \tag{5}$$

3.3 Solution to the Carbon Cycle Model

From literature, we were able to calculate the rate of carbon entery into our system z, partition coefficient p_i, and flow rate coeffients k_{ij} and therefore, the matrix A and vector $\overrightarrow{b}$ were know quantities in equation 5. The data available were for seven ecosystems of which we chose to present our findings for tropical forests, grasslands, and agricultural lands.

Variables	Tropical Forests	Grassland	Agricultural Land
Z in $(Gt\ C/yr)$	27.8	10.7	7.5
Area a $(10^{12}\ m^2)$	36.1	18.8	17.4
p_1	0.30	0.60	0.80
p_2	0.20	0.00	0.00
p_3	0.30	0.00	0.00
p_4	0.20	0.40	0.20
k_{15}	1.00	1.00	1.00
k_{25}	0.10	0.10	0.10
k_{35}	0.033	0.02	0.02
k_{46}	0.10	1.00	1.00
$k_{56} + k_{50}$	1.00	0.50	1.00
$k_{67} + k_{60}$	0.10	0.025	0.04
k_{70}	0.002	0.002	0.002
Humification $h = k_{56}$	0.40	0.60	0.20
Carbonization $c = k_{67}$	0.05	0.05	0.05

Tab. 5: Data as was presented by literature for three ecosystems: Tropical Forests, Grassland, and Agricultural Land. The anual amount of carbon entering into the system, Z in gigatons of carbon per year, in an area of a was used to calculate grams of carbon z entering into the sysstem per year per squared meters.

3.3.1 Steady States of the Ecosystems

To find the fixed point of the system, we set $X' = 0$, as the steady states describe non-evolving carbon distribusion within our system. This process simplified the equation5 to $A\overrightarrow{x} = -\overrightarrow{b}$ which has a unique non-trivial solution $\overrightarrow{x} = A^{-1}\overrightarrow{b}$ for every invertible matrix A, and a least square fit solution otherwise.

3.3.2 Time to Attaine 95% of Stable Values

3.3.3 Analytical Solutions

3.3.4 Solution Within Tropical Forest

3.3.5 Solution Within Grassland

3.3.6 Solution Within Agricultural Land

3.4 Analysis of the Carbon Cycle Model

The analysis of the Carbon Cycle Model consists of stability analysis and its corresponding sensitivity analysis.

3.4.1 Stability of the Fixed Points

A system is said to be stable around a fixed point if for any initial data, the solution tends toward to the neighbourhood of the fixed point. If given a system

$\vec{x}' = f(x)$ such that $x(0) = x_0$ and a fixed point $f(c) = 0$ (that is, $\vec{x}' = 0$), the system can be expanded using Taylor's Theorem around the fixed point:

$$\vec{x}' = \vec{f}(\vec{x}) = \vec{f}(\vec{c}) + D\vec{f}(\vec{c})(\vec{x}_0 - \vec{c}) + R(\vec{y}) = D\left(\vec{f}(\vec{c})\right)(\vec{x}_0 - \vec{c}) + R(\vec{y}) \tag{6}$$

If we let $R(\vec{y})$ be the neighbourhood and define $J = D\left(\vec{f}(\vec{c})\right)$ to be the Jacobian of the system, then the stability for the fixed point is found by the eigenvalues of the Jacobian. If every eigenvalue of the Jacobian at the point $\vec{c}$ is real and negative, then the solution is real.

For the Carbon Cycle Model in Table 4, the Jacobian is given by the matrix A in section **??**. Since this matrix is Lower Triangular, its eigenvalues are given by the diagonal entries.

$$\{\lambda_i\} = \{-k_{15}, -k_{25}, -k_{35}, -k_{46}, -(k_{50} + k_{56}), -(k_{60} + k_{67}), -k_{70}\} \tag{7}$$

Since all parameters k_i are positive and constant with respect to $\vec{x}_0$ and $\vec{c}$, all eigenvalues are negative for all $\vec{x}_0$ and $\vec{c}$. Thus, for the system in Table 4, the fixed points are stable.

3.4.2 Sensitivity of the Stability of the Fixed Points

The general sensitivity for an equation f to the parameter x is given by

$$s(f, x) = \left.\frac{d(f)}{dx} * \frac{x}{f}\right|_{x_0} \tag{8}$$

For the model in Table 4, the equations for the stability are given by equation (7).In terms of the general sensitivity equation, the acting function f is each λ_i and the acting parameter x is the corresponding entry. Since the equation for each λ_i is linear with respect to its acting x, the sensitivity for each is 1. This solution does not give much useful information about the sensitivity of the model to its parameter values.

A second method to examine the sensitivity of the model is to look at values of λ_i that would change the stability of the system. Clearly, these values occur when the corresponding entries in equation (7) are greater than or equal to zero. Consequently, for instability to occur, one of the given parameters k_j must be less than or equal to zero.

3.5 Observations and Conclusions of the Carbon Cycle Model

The Carbon Cycle Model has stable solutions that are affected strictly by the parameters given by the system. When Figure (4) is examined in perspective, the flows from each component must go in one direction; otherwise the system does not match physical intuition. More concretely, how could, for example, decaying leaves "reverse" and become living leaves again? This implies that all parameters must be greater than zero. With this in mind, it is clear that any system equivalent to the model derived from Figure (4) must have stable fixed points. Our confidence in the Carbon Cycle Model is boosted by the fact that our stability analysis is consistent with real constraints inherent in a biological system.

3.6 Criticisms of the Carbon Cycle Model

Like any mathematical model, there are errors made when creating a model for a system. The main contributor for these errors in general is a lack of model variables. Real systems have a large number of factors and to create a model that has a solution one must use a relatively small number of variables. More specific to this model, the equations had not taken into account that the number of trees change with respect to how much atmospheric carbon dioxide there is. This brings up the fact that the amount of atmospheric carbon dioxide changes with respect to time. Furthermore, this system is considered in isolation. There are different types of trees in a given ecosystem and thus the values for the parameters for each part of the tree are actually functions of how many of each tree there are. Note however, some of these issues do not affect the stability of the system since there parameters would still be positive; the issues affect the actual fixed point solutions.

This Carbon Cycle Model also suffers from the assumption of constant paramter values, as we saw in the Simple Litter Model. Again, it doesn't really make sense to assume that trees will drop their leaves at a constant rate through a calendar year, but such assumptions simplify the calculations and make the modelling process feasible.

4 Conclusion

5 Appendix A

All of the resources used to prepare this project are available in our public GitHub repository at https://github.com/colefichter/math372_project_2.

On the linked page you will also find a short menu describing where to find files of interest.