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2.1 Context

The ecosystem in a forest is in a constant and perpetual cycle of growth and decay. As trees grow they drop leaves, branches, and other organic material to the forest floor. This material is called *litter*. Being organic debris from trees, the litter contains trapped carbon. As the litter decays, the trapped carbon is released in two forms. First, the process of decay naturally respirates carbon dioxide molecules into the atmosphere. Second, a certain proportion of the carbon in the litter is converted into the black substance, known as *humus*, which gives compost and soil their characteristic colour and texture. This conversion from litter to humus is called *humification*.

By demarcating a boundary defining our system, we can set up an area of the forest floor where we can monitor and measure the density of the carbon in the litter located within the boundary.

Carbon enters such a system continuously through the litter-fall from trees and plants living within the boundary and carbon exits the system via respiration and humification.

2.2 Modelling the Carbon in a Simple Litter Model

We assume that the rate of litter-fall is constant, as are the rates of respiration and humification, within the system. Our model, then relies on the following variables and parameters:

	Type	Description
t	Variable	Time, represented continuously
$x(t)$	Function	The amount of carbon in the litter at time t. In $g\ C/m^2$
z	Parameter	The litter-fall rate, in $g\ C/m^2$.
r	Parameter	The respiration proportionality constant, in $/year$.
h	Parameter	The humification proportionality constant, in $/year$.

Tab. 1: Parameters and Variables in the Simple Litter Model

We assume that carbon enters the system at a constant rate through litter-fall, and that carbon exits the system by respiration and humification, also at constant rates. We also assume that the model starts out with zero litter (and thus zero carbon). This assumption seems reasonable, since it simulates forest rejuvenation after a ground fire.

Our objective, then, is to model the amount of carbon contained in the litter at any point in time. In mathematical terms, we seek to compute $x(t)$ at any point in time t, given the constant rate of litter-fall z, the constant rate of respiration r, and the constant rate of humification h, within the system.

We've chosen to represent the model as a differential equation,

$$x'(t) = z - x(t) * (r + h) \tag{1}$$

We now also assume that $r = h$, meaning that the rates of respiration and humifcation are equal. Thus, we can combine them into a single rate $k = r + h$. Updating (1), we have

$$x'(t) = z - k * x(t) \tag{2}$$

2.3 Solution to the Simple Litter Model

The general solution to the differential equation (2) is

$$x(t) = c_1 e^{-kt} + \frac{z}{k} \tag{3}$$

To find a specific solution to the differential equation, we rely on the assumption that there is initially zero carbon in the system: $x(0) = 0$. Thus, we can find c_1

$$\begin{aligned} x(0) &= c_1 e^{-k0} + \frac{z}{k} \\ \implies c_1 &= -\frac{z}{k} \end{aligned}$$

2.3.1 A Concrete Solution to the Simple Litter Model

From the existing literature, we can estimate parameter values to represent real-world measurements of a temperate forest. We let

$$\begin{aligned} z &= 240\ g\ C/m^2 \\ k = r + h &= 0.4/year \end{aligned}$$

Plugging these parameter values into the model equations, we find

$$\begin{aligned} c_1 &= -\frac{z}{k} \\ &= -\frac{240}{0.4} \\ &= -600 \end{aligned}$$

And equation (2) becomes

$$x(t) = 600 - 600e^{-0.4t} \tag{4}$$

Here we see a plot of the solution

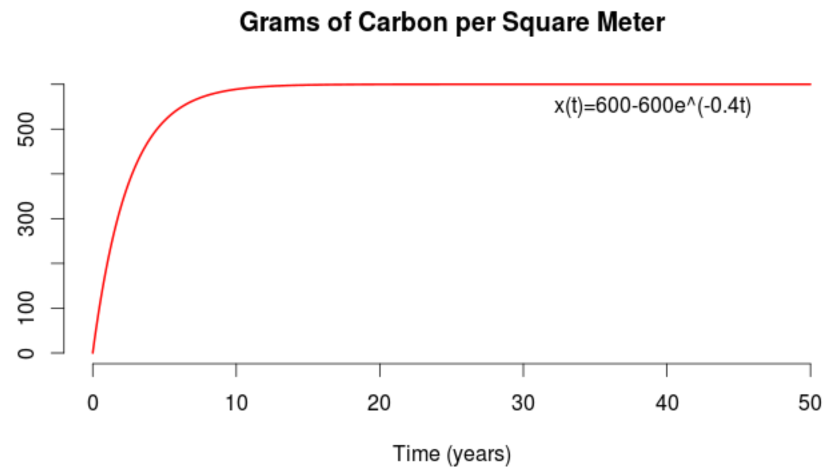

Fig. 1: A Plot of the Solution to the Simple Litter Model

2.4 Analysis of the Simple Litter Model

From Figure 1, it appears that the model has acheives equilibrium at $x(t) = 600$. We can validate this observation by finding the fixed points of the model.

By definition, the fixed point of our model exists when $x'(t) = 0$, which occurs only when $-600 = -600e^{-0.4t}$. Since this equation does not have any real roots, we can't analytically find a fixed point t^* for any sensible measure of time (time can't be measured with a complex number), but we can approximate it numerically. Indeed by $t = 12$, $x(t)$ is within 1% of the equilibrium value $x(t) = 600$.

Working with the assumption that there exists a fixed point t^* such that $x(t^*) = 600$, we can use graphical stability analysis to determine the stability of the point $x(t^*)$. Here we see a plot of $x(t)$ versus $x'(t)$, annotated with the usual markings used in graphical stability analysis

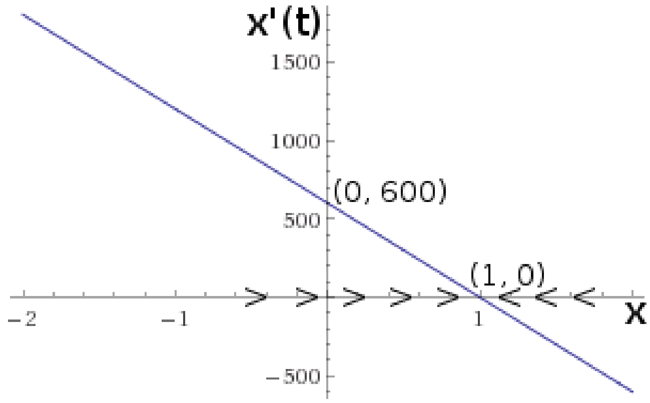

Fig. 2: Graphical Stability Analysis of the Simple Litter Model

The graphical stability analysis confirms that $x(t^*) = 600$ is indeed a stable point, where we should expect the system to stabilize. This confirms the observations of Figure 1.

3 The Carbon Cycle Problem

3.1 Modelling Approach

3.1.1 Variables

p_i	Flow portion from atmosphere to i
k_{ij}	Flow portion from i to j

Tab. 2: The Model Variables

3.1.2 Differential Equations

$$
\begin{aligned}
\dot{x}_1 &= p_1 z - k_{15} x_1 \\
\dot{x}_2 &= p_2 z - k_{25} x_2 \\
\dot{x}_3 &= p_3 z - k_{35} x_3 \\
\dot{x}_4 &= p_4 z - k_{46} x_4 \\
\dot{x}_5 &= k_{15} x_1 + k_{25} x_2 + k_{35} x_3 - k_{50} x_5 - k_{56} x_5 \\
\dot{x}_6 &= k_{46} x_4 + k_{56} x_5 - k_{60} x_6 - k_{67} x_6 \\
\dot{x}_7 &= k_{67} x_6 - k_{70} x_7
\end{aligned}
$$

Tab. 3: The Model Variables

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{pmatrix} =$$

$$\begin{bmatrix} -k_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_{25} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_{46} & 0 & 0 & 0 \\ k_{15} & k_{25} & k_{35} & 0 & -k_{50}-k_{56} & 0 & 0 \\ 0 & 0 & 0 & k_{46} & k_{56} & -k_{60}-k_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{67} & -k_{70} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} +$$

$$\begin{bmatrix} p_1 z \\ p_2 z \\ p_3 z \\ p_4 z \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3.2 Solving the Model

3.3 Sensitivity Analysis

4 Conclusion

5 Appendix A