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## 1 Introduction

### 1.1 Historical Context

### 1.2 Overview of Results (Executive Summary)

## 2 Carbon in a Simple Litter Model

### 2.1 Context

The ecosystem in a forest is in a constant and perpetual cycle of growth and decay. As trees grow they drop leaves, branches, and other organic material to the forest floor. This material is called *litter*. Being organic debris from trees, the litter contains trapped carbon. As the litter decays, the trapped carbon is released in two forms. First, the process of decay naturally respirates carbon dioxide molecules into the atmosphere. Second, a certain proportion of the carbon in the litter is converted into the black substance, known as *humus*, which gives compost and soil their characteristic colour and texture. This conversion from litter to humus is called *humification*.

By demarcating a boundary defining our system, we can set up an area of the forest floor where we can monitor and measure the density of the carbon in the litter located within the boundary.

Carbon enters such a system continuously through the litter-fall from trees and plants living within the boundary and carbon exits the system via respiration and humification.

### 2.2 Modelling the Carbon in a Simple Litter Model

We assume that the rate of litter-fall is constant, as are the rates of respiration and humification, within the system. Our model, then relies on the following variables and parameters:

	Type	Description
$t$	Variable	Time, represented continuously
$x(t)$	Function	The amount of carbon in the litter at time $t$ , in $g\ C/m^2$
$z$	Parameter	The litter-fall rate, in $g\ C/m^2$
$r$	Parameter	The respiration proportionality constant, in $/year$
$h$	Parameter	The humification proportionality constant, in $/year$

Tab. 1: Parameters and Variables in the Simple Litter Model

We assume that carbon enters the system at a constant rate through litter-fall, and that carbon exits the system by respiration and humification, also at constant rates. We also assume that the model starts out with zero litter (and thus zero carbon). This assumption seems reasonable, since it simulates forest rejuvenation after a ground fire.

Our objective, then, is to model the amount of carbon contained in the litter at any point in time. In mathematical terms, we seek to compute  $x(t)$  at any point in time  $t$ , given the constant rate of litter-fall  $z$ , the constant rate of respiration  $r$ , and the constant rate of humification  $h$ , within the system.

We've chosen to represent the model with a differential equation

$$x'(t) = z - (r + h) * x(t) \quad (1)$$

We now also assume that  $r = h$ , meaning that the rates of respiration and humification are equal. Thus, we can combine them into a single rate  $k = r + h$ . Updating (1), we have

$$x'(t) = z - k * x(t) \quad (2)$$

## 2.3 Solution to the Simple Litter Model

The general solution to the differential equation (2) is

$$x(t) = c_1 e^{-kt} + \frac{z}{k} \quad (3)$$

To find a specific solution to the differential equation, we rely on the assumption that there is initially zero carbon in the system:  $x(0) = 0$ . Thus, we can find  $c_1$

$$\begin{aligned} x(0) &= c_1 e^{-k0} + \frac{z}{k} \\ \implies c_1 &= -\frac{z}{k} \end{aligned}$$

### 2.3.1 A Concrete Solution to the Simple Litter Model

From the existing literature, we can estimate parameter values to represent real-world measurements of a temperate forest. We let

$$\begin{aligned} z &= 240 \text{ g C/m}^2 \\ k = r + h &= 0.4/\text{year} \end{aligned}$$

Plugging these parameter values into the model equations, we find

$$\begin{aligned} c_1 &= -\frac{z}{k} \\ &= -\frac{240}{0.4} \\ &= -600 \end{aligned}$$

And equation (3) becomes

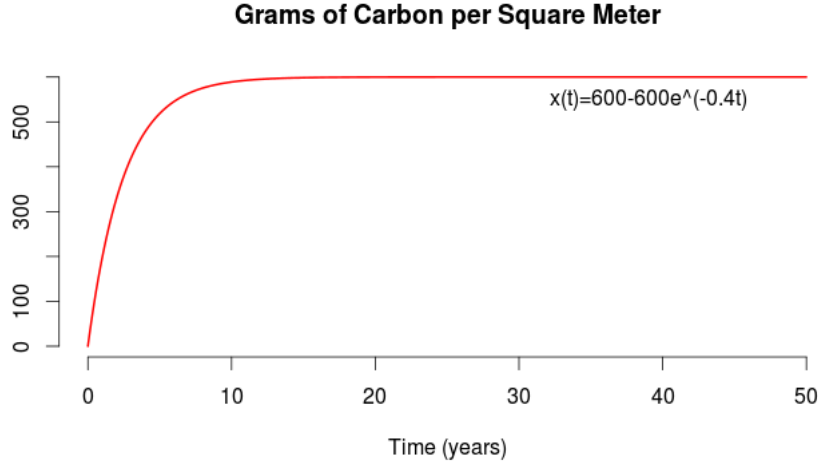


Fig. 1: Plot of the Solution to the Simple Litter Model

$$x(t) = 600 - 600e^{-0.4t} \quad (4)$$

Here we see a plot of the solution equation (4)

## 2.4 Analysis of the Simple Litter Model

From Figure 1, it appears that the model achieves equilibrium at  $x(t) = 600$ . We can validate this observation by finding the fixed points of the model.

By definition, the fixed point of our model exists when  $x'(t) = 0$ , which occurs only when  $-600 = -600e^{-0.4t}$ . Since this equation does not have any real roots, we can't analytically find a fixed point  $t^*$  for any sensible measure of time (time can't be measured with a complex number), but we can approximate it numerically. Indeed by  $t = 12$ ,  $x(t)$  is within 1% of the equilibrium value  $x(t) = 600$ .

Working with the assumption that there exists a fixed point  $t^*$  such that  $x(t^*) = 600$ , we can use graphical stability analysis to determine the stability of the point  $x(t^*)$ . Here we see a plot of  $x(t)$  versus  $x'(t)$ , annotated with the usual markings used in graphical stability analysis

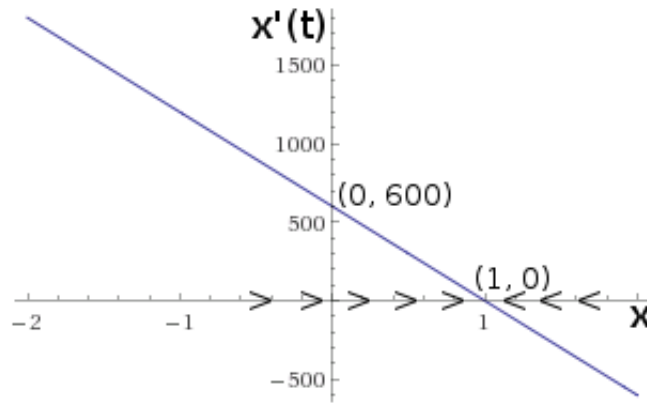


Fig. 2: Graphical Stability Analysis of the Simple Litter Model

The graphical stability analysis confirms that  $x(t^*) = 600$  is indeed a stable point, where we should expect the system to stabilize. This confirms the observations of Figure 1 and explains why the graph of the solution reaches equilibrium.

## 2.5 Observations and Conclusions of the Simple Litter Model

Our model suggests that the density of carbon sequestered in the litter found on the floor of a temperate forest should reach an equilibrium of about  $600 \text{ g C/m}^2$ , even if the initial state of the forest is free of litter due to a recent ground fire.

Intuitively, this makes sense based on our familiarity with nature. Biological systems generally must tend toward equilibrium since any environment containing finite resources cannot support infinite growth. Also, if systems tended toward unbounded decay (or decline), life would have been extinguished billions of years ago.

## 2.6 Criticisms of the Simple Litter Model

This model is obviously very simplistic. We measure carbon only within the litter, and only within an arbitrarily bounded region of the forest. We also assume that carbon exits the litter via respiration and humification at the same rate.

Assuming that all of the rates in the model are constant is probably not a good representation of a real biological system. In a temperate climate, trees drop their leaves in the autumn. Surely this type of seasonal fluctuation would affect the rate of litter-fall,  $z$ . Since we have assumed that  $z$  is constant, this model would be more suited to a discrete modelling approach where each discrete time step,  $t$ , represents one year. However, since we've viewed time continuously, our rate of litter-fall should really fluctuate seasonally.

Finally, we don't have a good understanding of the constants provided in the problem, nor a clear picture of how they were measured and the accuracy of those measurements.

In the next section, we look at a more complicated problem, The Carbon Cycle, which addresses some of these shortcomings.

### 3 An Enhanced Model: The Carbon Cycle

#### 3.1 Context

We now focus on a more complicated model that addresses some of the concerns outlined above. It is called the Carbon Cycle Model. This model extends the previous Simple Litter Model to examine carbon as it flows throughout an ecosystem. In addition to litter, we also consider leaves, branches, stems, roots, humus, and stable humus charcoal.

Here we see a conceptual diagram of the Carbon Cycle Model

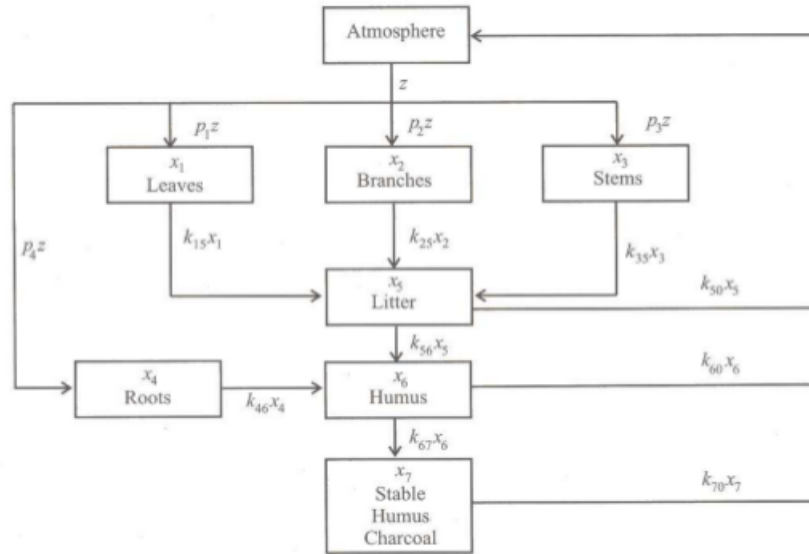


Fig. 3: Conceptual Diagram of the Carbon Cycle Model

Although noted in the diagram, we consider the atmosphere to be external to the model. This is mainly due to the enormity of the atmosphere as compared with the finite region being modelled. Carbon that is absorbed by plants from the atmosphere is considered to be entering the model, while carbon respired to the atmosphere during the decay process is considered to be leaving the model.

Carbon enters the cycle when growing plants absorb carbon dioxide ( $CO_2$ ) from the atmosphere. The  $CO_2$  is converted into organic plant matter and

sequestered within the leaves, branches, stems, and roots of each plant. Over time, as plants drop leaves and stems, litter accumulates on the forest floor, as in the Simple Litter Model.

The plant roots and the litter then undergo humification which produces humus, as in the Simple Litter Model, and  $CO_2$ . As the humus decays, it produces stable humus charcoal, and more  $CO_2$ . As charcoal breaks down, it also releases some  $CO_2$ . All of the carbon dioxide produced during this cycle ends up back in the global atmosphere, ready to be reabsorbed by growing plants.

## 3.2 Modelling the Carbon in The Carbon Cycle

Our Carbon Cycle Model relies on a system of differential equations, each fairly similar to equation 2 from the Simple Litter Model.

### 3.2.1 Parameters

We assume that carbon enters the system from the atmosphere at a constant rate,  $z$ , measured in metric Gigatonnes of carbon per year ( $Gt\ C/year$ ). We further partition  $z$  into the portion which goes into the leaves, branches, stems, and roots by the proportion parameters  $p_1, p_2, p_3, p_4$ . Note that the  $p_i$  essentially represent a probability distribution for the dispersion of  $z$  into the components of the model. The parameters  $k_{ij}$  are proportions that indicate the rate of carbon transferred from  $x_i$  to  $x_j$ .

$z$	The rate that carbon enters the system, in $Gt\ C/year$
$p_i$	The proportion of $z$ absorbed into $x_i$
$k_{ij}$	The rate of carbon transfer from $x_i$ to $x_j$

Tab. 2: Parameters of the Carbon Cycle Model

### 3.2.2 Variables

We represent each of the seven model components (leaves, branches, stems, roots, litter, humus, stable humus charcoal) by a variable  $x_i$ , where  $i = 1, 2, \dots, 7$ . Thus,  $x_1$  represents leaves,  $x_4$  represents roots, etc. These variables can be seen in Figure 3 and in the following table.

$x_i$	The amount of carbon stored in each of the model components, $i = 1, 2, \dots, 7$
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Tab. 3: Variables of the Carbon Cycle Model

We assume that carbon flowing between any two connected components in the diagram can be represented by a differential equation, as we did in the Simple Litter Model. Our model, then, consists of a system of seven differential equations. In contrast with the simple litter model, our goal is no longer to



compute the amount of carbon in any given component, but rather to examine the system as a whole. Of particular interest are so called steady-states, where the system may achieve an equilibrium.

### 3.2.3 Differential Equations

$$\begin{aligned}
 \dot{x}_1 &= p_1 z - k_{15} x_1 \\
 \dot{x}_2 &= p_2 z - k_{25} x_2 \\
 \dot{x}_3 &= p_3 z - k_{35} x_3 \\
 \dot{x}_4 &= p_4 z - k_{46} x_4 \\
 \dot{x}_5 &= k_{15} x_1 + k_{25} x_2 + k_{35} x_3 - k_{50} x_5 - k_{56} x_5 \\
 \dot{x}_6 &= k_{46} x_4 + k_{56} x_5 - k_{60} x_6 - k_{67} x_6 \\
 \dot{x}_7 &= k_{67} x_6 - k_{70} x_7
 \end{aligned}$$

Tab. 4: Differential Equations of the Carbon Cycle Model

To make the notation a little less cumbersome, let  $X' = \langle \dot{x}_1, \dot{x}_2, \dots, \dot{x}_7 \rangle$ . That is, we place the left hand side of each differential equation in Table 4 into a column vector and denote it  $X'$ .

Next, from the right hand side of the differential equations in Table 4, we place the  $k_{ij}$  coefficients into a matrix  $A$ , as follows

$$A = \begin{bmatrix} -k_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_{25} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_{46} & 0 & 0 & 0 \\ k_{15} & k_{25} & k_{35} & 0 & -k_{50} - k_{56} & 0 & 0 \\ 0 & 0 & 0 & k_{46} & k_{56} & -k_{60} - k_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{67} & -k_{70} \end{bmatrix}$$

We also let  $\vec{x} = \langle x_1, x_2, \dots, x_7 \rangle$  (a column vector of variables from the right hand side of Table 4). Finally, we add the remaining constant terms from the right hand side of Table 4 to a constant vector,  $\vec{b}$

$$\vec{b} = \begin{bmatrix} p_1 z \\ p_2 z \\ p_3 z \\ p_4 z \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we can now represent our system of differential equations with the vector equation

$$X' = A\vec{x} + \vec{b} \quad (5)$$

- 3.3 Solution to the Carbon Cycle Model
- 3.4 Analysis of the Carbon Cycle Model
- 3.5 Observations and Conclusions of the Carbon Cycle Model
- 3.6 Criticisms of the Carbon Cycle Model
- 4 Conclusion

## 5 Appendix A