Lecture 23

<u>Plan</u>: 1) finish stating ellipsoid
2) Analyze ellipsoid
3) Apply to LP.

Analysis of ellipsoid

Recall main lemma:

Volume Lemma: Let E' be ellipsoid after E in the algorithm.
<u>Then</u>:

$$\mathrm{vol}(E') \le e^{-\frac{1}{2(n+1)}} \mathrm{vol}(E).$$

Before proving, some preliminaries:

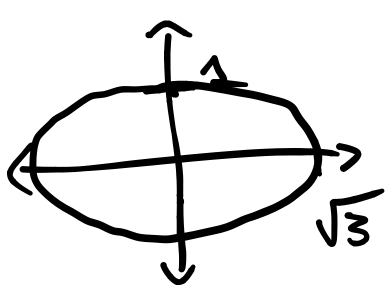
Def: Given a center $e \in \mathbb{R}^n$ & a positive-definite matrix $A \in \mathbb{R}^{n \times n}$ the **ellipsoid**

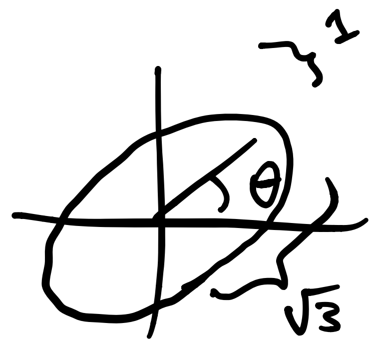
$$E(e, A) := \{ x \in \mathbb{R}^n : (x-e)^T A^{-1} (x-e) \leq 1 \}.$$

e.g. $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad A' = R(\theta) A R(\theta)^T$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$a = (0,0)$





Recall: Matrix $A \in \mathbb{R}^{n \times n}$ **positive-definite** if symmetric ($A^T = A$) and

$$x^T A x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}.$$

Equivalent conditions: Let A be a symmetric matrix. Then:

A P.D.

$\iff \exists\, B \in \mathbb{R}^{n \times n}$ s.t. $A = B^T B$

$\iff A^{-1}$ P.D. $\quad (A^{-1} = B^{-1}(B^{-1})^T)$

$\iff A$ has n orthonormal eigenvectors w/ positive eigenvalues.

Facts about ellipsoids:

- They are **Affine transformations** (linear map + translation) of unit spheres

$$E(0,I) = \{x \in \mathbb{R}^n : x^T x \leq 1\}.$$

Proof:

Let $A = B^T B$;

$$E(0,I) = T\, E(e,A)$$

where T is affine bijection

$$T : x \longmapsto y := (B^{-1})^T (x - e).$$

$$y^T y \le 1 \iff (x-e)^T B^{-1} (B^{-1})^T (x-e) \le 1$$

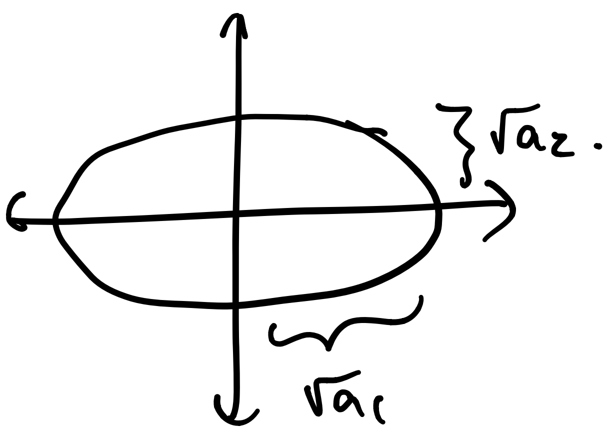
$$\| \quad (x-a)^T A^{-1} (x-e).$$

- **Volume:** if $A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$,

i.e. $E(e,A)$ "coordinate aligned",

then

$$\operatorname{vol} E(e,A) = \sqrt{a_1 \cdots a_n}\, \operatorname{vol} E(0,I)$$



(because $\operatorname{vol} E(e,A) = \operatorname{vol} E(0,A)$,

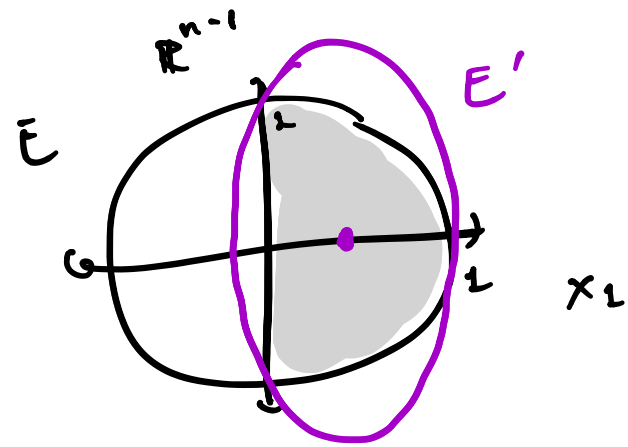
$$\& \quad B^{-1} E(0,A) = E(0,I)$$

$$\Rightarrow \text{vol } E(a,A) = \det B \text{ vol } E(0,I)$$

$$= \sqrt{\det A} \text{ vol } E(0,I) = \sqrt{a_1 \cdots a_n} \text{ vol } E(0,I).$$

Proof of Volume Lemma:

- Begin with case where $E = E(0,I)$ (unit sphere) and inequality is $x_1 \geq 0$.



- **Claim**: We can take

$$E' = \left\{ x : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\}.$$

i.e. $E' = E(a, A)$ where $a = \left(\frac{1}{n+1}, 0, \dots, 0\right)$

and $A = \mathrm{diag}\left(\left(\frac{n}{n+1}\right)^2, \frac{n}{n^2-1}, \dots, \frac{n}{n^2-1}\right)$.

Proof of claim:

- Need to show $E \cap \{x : x_1 \geq 0\} \subseteq E'$.
- Let $x \in E \cap \{x : x_1 \geq 0\}$. Then

want to show ≤ 1.

$$\left(\frac{n+1}{n}\right)^2 \underbrace{\left(x_1 - \frac{1}{n+1}\right)^2}_{\text{expand}} + \frac{n^2-1}{n} \sum_{i=2}^{n} x_i^2$$

$$= \frac{n^2+2n+1}{n^2} x_1^2 - \left(\frac{n+1}{n}\right)^2 \frac{2x_1}{n+1} + \frac{1}{n^2} + \frac{n^2-1}{n} \sum_{i=2}^{n} x_i^2$$

move $\frac{n^2-1}{n} x_1$

cancel

$$= \underbrace{\frac{2n+2}{n^2} x_1^2 - \frac{2n+2}{n^2} x_1}_{\text{collect}} + \frac{1}{n^2} + \frac{n^2-1}{n} \sum_{i=1}^{n} x_i^2$$

$$= \frac{2n+2}{n^2} \underbrace{x_1(x_1-1)}_{\leq 0} + \frac{1}{n^2} + \frac{n^2-1}{n} \underbrace{\sum_{i=1}^{n} x_i^2}_{\leq 1}$$

$$\leq \frac{1}{n^2} + \frac{n^2-1}{n} \leq 1. \qquad \square$$

- Proof of volume lemma in this case:

$$\text{vol } E(e_1 A) = \sqrt{a_1 \cdots a_n}\, E(0, I)$$

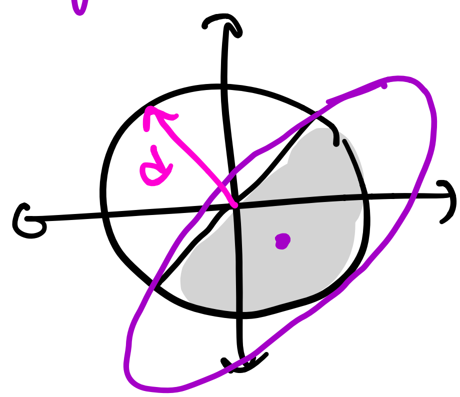
$$= \sqrt{\left(\frac{n}{n+1}\right)^2 \cdot \frac{n}{n^2-1} \cdot \ldots \cdot \frac{n}{n^2-1}}$$

$$= \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} = \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}}$$

$$\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n^2-1)}}$$

$$= e^{-\frac{1}{n+1}} e^{+\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}. \quad \square$$

- what if we have some other inequality $d^T x \le 0$?



▷ can assume $\|d\| = 1$ by $d \leftarrow \frac{d}{\|d\|}$

▷ Figure out E' by rotating so $d = e_1$, then rotating back; shows ratio still $\le \exp\left(-\frac{1}{2(n+1)}\right)$.

▷ End up with $E' = E\left(-\frac{1}{n+1} d, F\right)$,

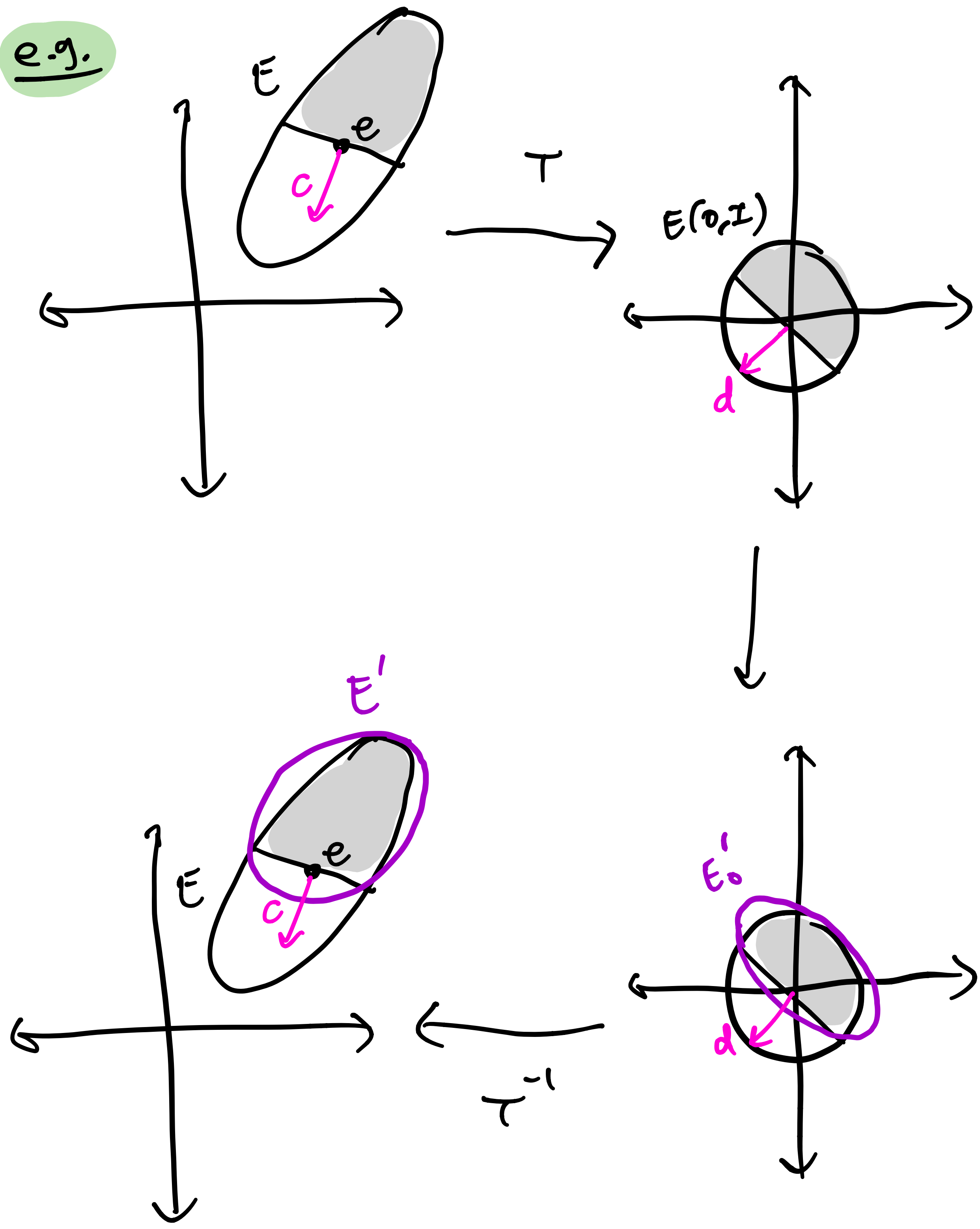
$$F = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} dd^T\right).$$

- What if E <u>not</u> unit sphere? Use affine transformation T; preserves ratios of volumes.

$$\begin{tikzcd} E \arrow[r, "T"] & E(0, I) \arrow[d] \\ E' & E_0' \arrow[l, "T^{-1}"'] \end{tikzcd}$$

for E.   for unit sphere

e.g.



- Now

$$\frac{\text{vol } E'}{\text{vol } E} = \frac{\text{vol } T^{-1} E'_0}{\text{vol } E}$$

$$= \frac{\text{vol } E'_0}{\text{vol } E(0, I)} \leq e^{-\frac{1}{2(n+1)}}.$$

Completes proof of volume part. $\square$.

How to compute E'?

- We pretty much saw how in the proof of volume lemma.

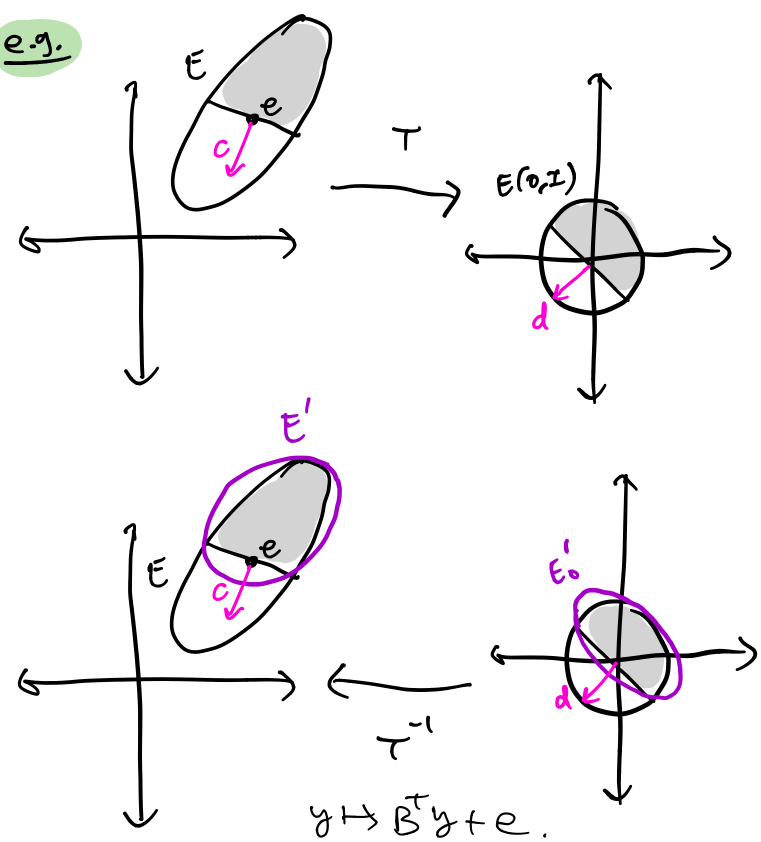
- Let's carefully compute where things go under T, T^{-1}.

If $E = E(e, A)$, recall

$$T : x \longmapsto y := (B^{-1})^T (x - e).$$

has $TE = E(0, I)$, where $A = B^T B$.

e.g.



- First find d. Under T,

$$\{x : c^T x \le c^T e\} \xrightarrow[T]{} \{y : c^T(e + B^T y) \le c^T e\}$$

$$= \{y : c^T B^T y \le 0\} = \{y : d^T y \le 0\},$$

for $d = \dfrac{Bc}{\sqrt{c^T B^T B c}} = \dfrac{Bc}{\sqrt{c^T A c}}.$

- Recall that

$$E_0' = E\left(\frac{-d}{n+1}, \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} d d^T\right)\right)$$

- Let $b = B^T d = \dfrac{Ac}{\sqrt{c^T A c}}$;

Applying T^{-1} to E_0' yields

$$E' = E\left(e - \frac{1}{n+1} b, \frac{n^2}{n^2-1} B^T\left(I - \frac{2}{n+1} dd^T\right)B\right).$$

$$= E\left(e - \frac{1}{n+1} b, \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1} bb^T\right)\right).$$

$(T^{-1} y = B^T y + e.)$

Ellipsoid (Concretely):

- Initialize $E = E(e, A) = E_0 \supseteq P$.
- while $e \notin P$:
  - ▷ Let $c^T x \leq c^T e$ valid inequality for P (returned by SEP oracle).
  - ▷ Let $b = \frac{Ac}{\sqrt{c^T A c}}$.

▷ Set $e \leftarrow e - \frac{1}{n+1} b$.

▷ Set $A \leftarrow \frac{n^2}{n^2-1}\left(A_k - \frac{2}{n+1} b b^T\right)$.

Analysis summary:

After k iterations,

$$\mathrm{vol}\, E \le e^{-\frac{k}{2(n+1)}} \mathrm{vol}\, E_0 .$$

$\Rightarrow$

terminates in $\le$

$$2(n+1) \log \frac{\mathrm{vol}\, E_0}{\mathrm{vol}\, P}$$

steps.

Linear programming: