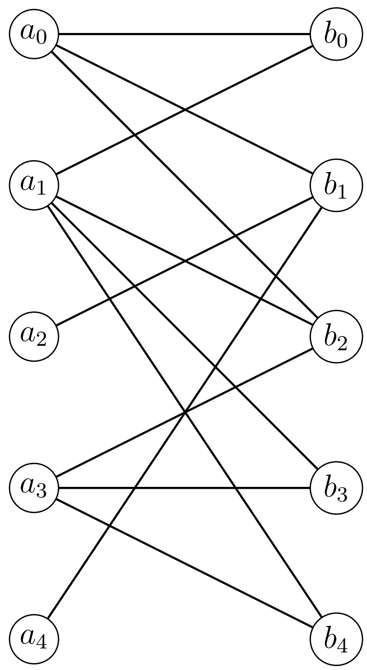
18.453 Quiz, Spring 2021

Instructions. This is a **timed** quiz. Turn this quiz by submitting a pdf or image to gradescope at most **2** hours after opening it. You may use notes and course material, but collaboration is not allowed. You may quote anything stated in the course material or lecture notes without justification, including exercises (except exercises that appear in the quiz itself). Questions are worth 4 points unless otherwise mentioned.

Failure to adhere to these rules may result in a grade of zero for the assignment, and referral to the Committee on Discipline.

1. Find a minimum vertex cover in the following graph, and give a short argument for its optimality.



Blank page

2. Let U be any minimizer in the Tutte-Berge formula. Let $K_1, \cdots, K_k$ be the connected components of $G \setminus U$.

   Show that, for *any* maximum matching M, we must have that M contains exactly $\lfloor \frac{|K_i|}{2} \rfloor$ edges from $G[K_i]$ (the subgraph of G induced by the vertices in K_i).[1]

[1] i.e. $G[K_i]$ is perfectly matched for the even components K_i and near-perfectly matched for the odd components.

Blank page

3. (a) Do there exist three points in $\mathbb{R}^2$ that are affinely independent? If so, draw an example.

   (b) Do there exist two points in $\mathbb{R}^2$ that are affinely independent but linearly *dependent*? If so, draw an example.

   (c) Draw the linear hull $\operatorname{lin}(S)$, affine hull $\operatorname{aff}(S)$, conic hull $\operatorname{cone}(S)$, and convex hull $\operatorname{conv}(S)$ of the set of points

$$S = \{(0,1), (2,1), (1,3), (1,2)\} \subset \mathbb{R}^2,$$

   and give a description of each in the form $\{x : Ax \leq b\}$.[2] No justification is needed.

[2]If the set you are trying to describe is equal to all of $\mathbb{R}^2$, i.e. has no inequalities, just write that it is equal to $\mathbb{R}^2$.

Blank page

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

(a) State the dual of the following LP: [3]

$$\begin{aligned} &\text{Min} \quad (1,2,1,3) \cdot x \\ &\text{subject to:} \\ &\qquad Ax = (2,3,1)^T \\ &\qquad x \geq 0. \end{aligned}$$

(b) Suppose b is in $\mathbb{Z}^3$. Show that either the polyhedron $P = \{x : Ax \leq b, x \geq 0\}$ is empty or all the vertices of P are integral.

[3] To avoid confusion, I'll let you know that you do **not** need the answer from part (a) for part (b).

Blank page

5. **Extra credit: (2 pts)** The incidence matrix of a directed graph $G = (V, E)$ is a $|E| \times |V|$ matrix with a row for each edge (u, v) with a 1 in the u entry and a -1 in the v entry and zeroes elsewhere. For example, the indicence matrix of the directed 3-cycle $(1, 2), (2, 3), (3, 1)$ is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Show that the indidence matrix of a directed graph is totally unimodular.

Blank page