Lecture 2

Plan

- Bipartite matchings
- König's theorem
  (vertex covers)
- Augmenting paths algorithm
- Hall's theorem

Bipartite matching

Recall from Tues: Graph $G = (V, E)$
vertices edges.

More terms:

If $e = (a,b) \in E$, say e incident to a,b.

or a,b endpoints of e.

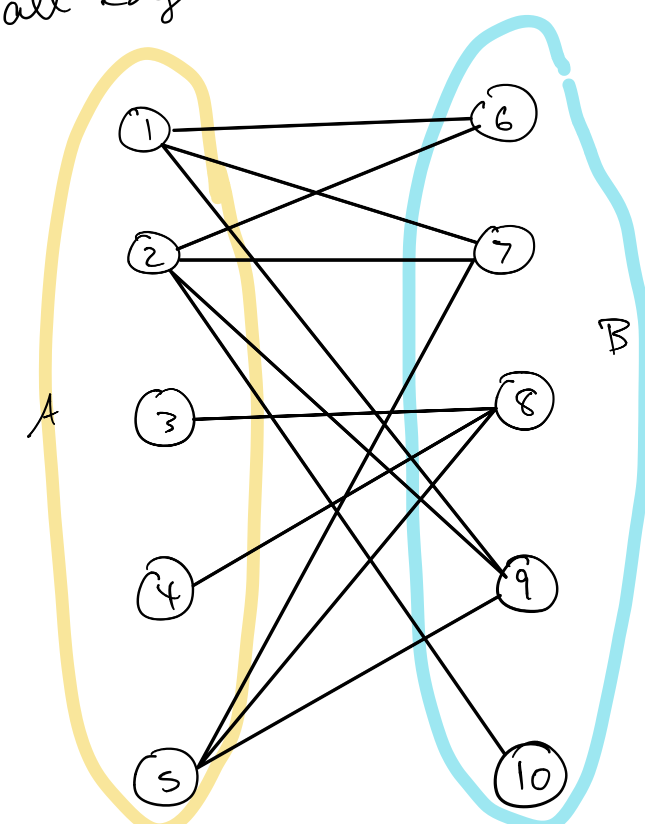
called "bipartition"

Def: Bipartite

G bipartite if V has partition A, B

s.t. all edges between A & B.

Ex:



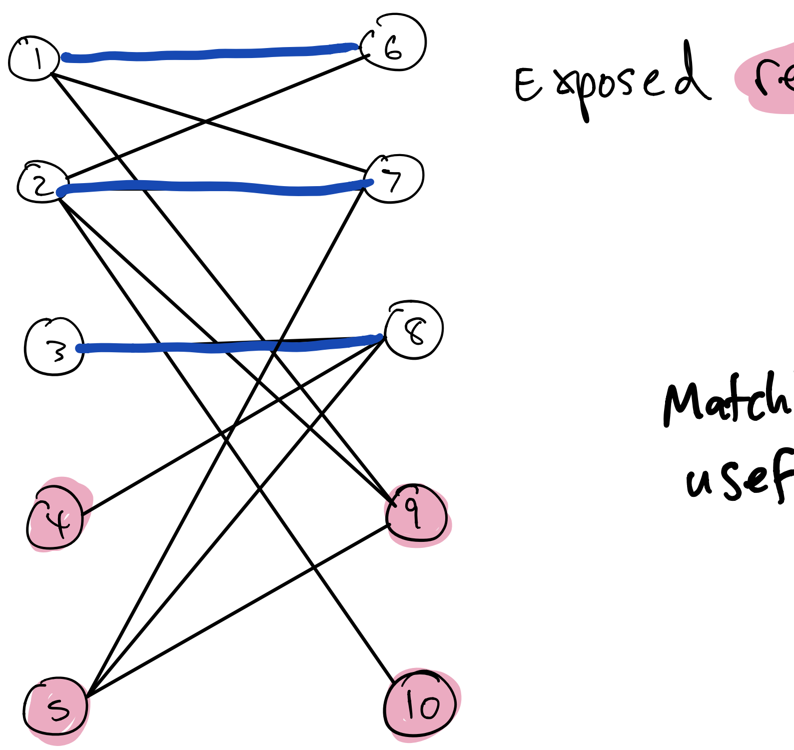
Def: Matchings

Recall <u>matching</u>: collection $M \subseteq E$ of disjoint edges.

Say v <u>exposed</u> no edge in M incident to v.

M <u>perfect</u> if no exposed vertices.

Ex.



Matchings useful!

Problem 1: Cardinality perfect matching

find matching M of maximum size.

Problem 2: Minimum weight perfect matching Given costs c_{ij} for all edges $(i,j) \in E$, find a perfect matching of minimum cost, where

$$\text{cost} = c(M) := \sum_{(i,j) \in M} c_{ij}.$$

Today we look at **Problem 1.**

König's theorem

Before building algorithms, how's one certify that a matching is optimal (largest possible)?

Use obstruction to larger matching.

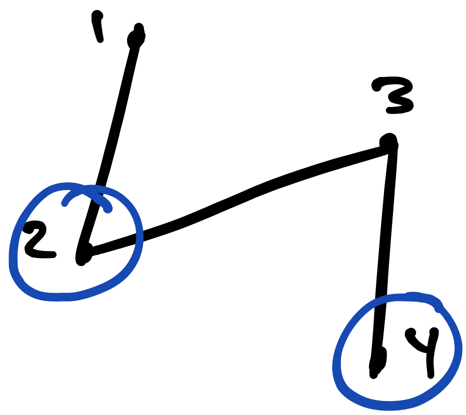
Duality again!

Def: Vertex Cover:

Set C of vertices is

a vertex cover for G if every edge $e \in E$ is incident to some $c \in C$.

Ex.



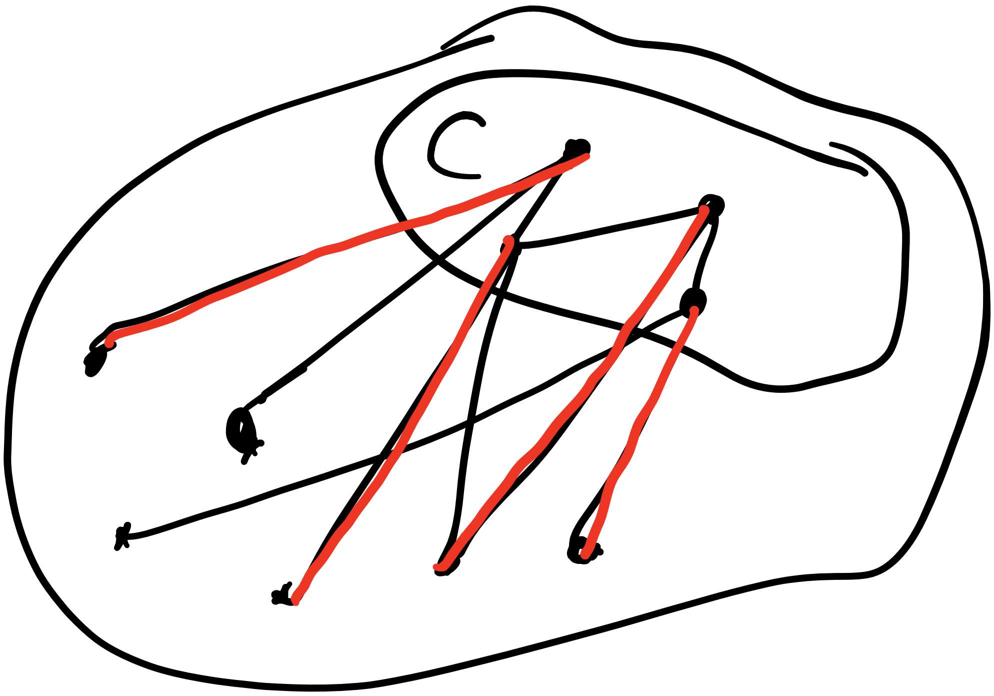
2, 4 are vertex cover,

1, 4 not.

Claim "weak duality"

$$|\text{any matching } M| \leq |\text{any vertex cover } C|$$

why? C contains at least one endpoint of every edge in M; but edges in M disjoint.



M

vertex covers are <u>only</u> obstructions for bipartite matching! strong duality.

<u>**Theorem**</u> (König 1931).

For any bipartite graph,

max |matching| = min |vertex cover|

(Sometimes called "min-max" characterization.)

we'll prove this algorithmically;

Augmenting Paths Algorithm

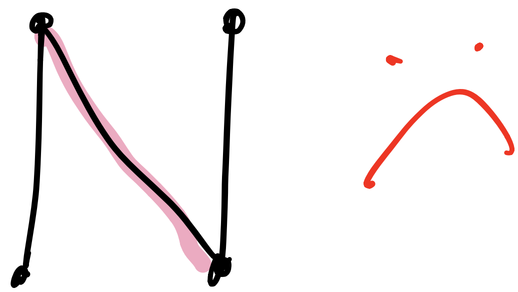
outputs matching M, cover C with

$$|M| = |C|;$$

by weak duality,
they must be max/min, respectively.

Note: greedy algorithm

won't work.



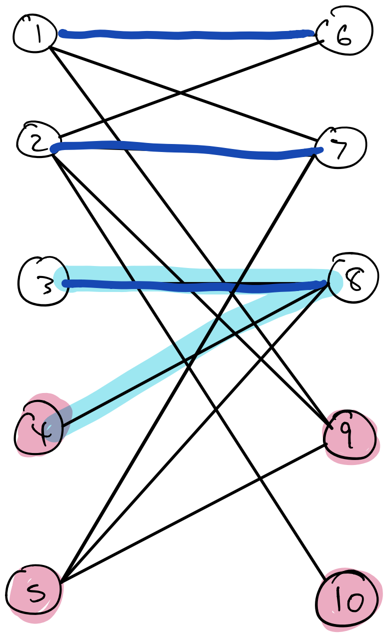
Def: Alternating path w.r.t. M

A path in G that alternates b/w edges in M and E − M.

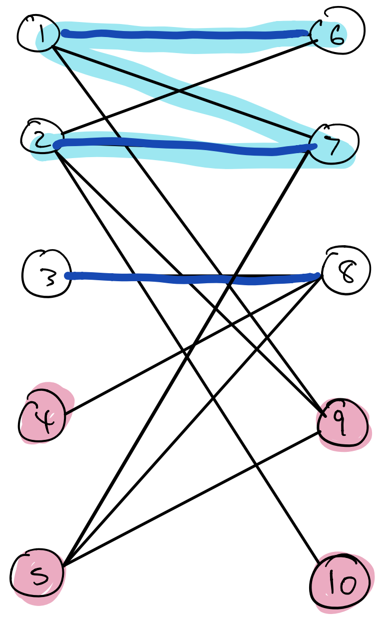
Def: Augmenting path w.r.t M

An alternating path whose first & last vertices are exposed.

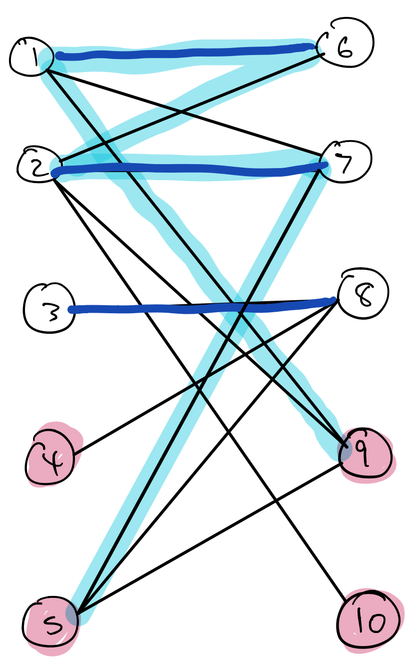
alternating



alternating



augmenting



Interesting properties

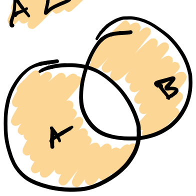
let P aug. path.

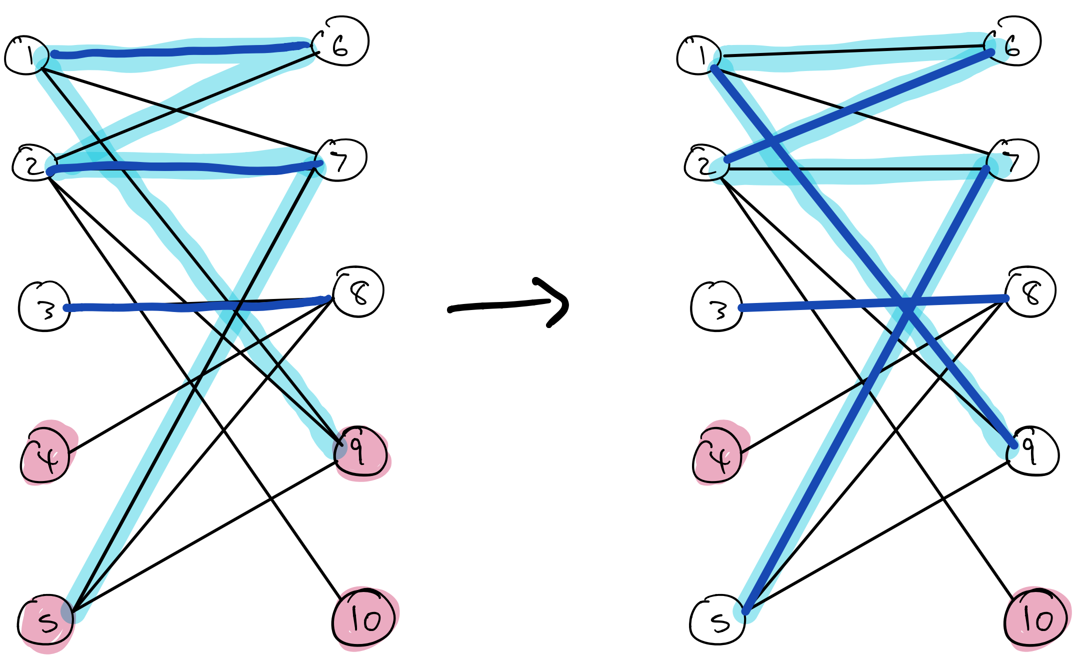
1. If P has k edges in M, has k+1 edges not in M.

2. P's endpoints are on opposite sides (parity)

3. "Switch" edges in P: replace M by symmetric difference $M' = M \Delta P$ to obtain matching M' with one more edge.

$A \Delta B$





3 4

Equiv: replace edges in $P \cap M$ by edges in $P \setminus M$.

Say we have augmented M along P.

Thm: Matching M maximum
$\Longleftrightarrow$ there are no augmenting paths w.r.t. M.

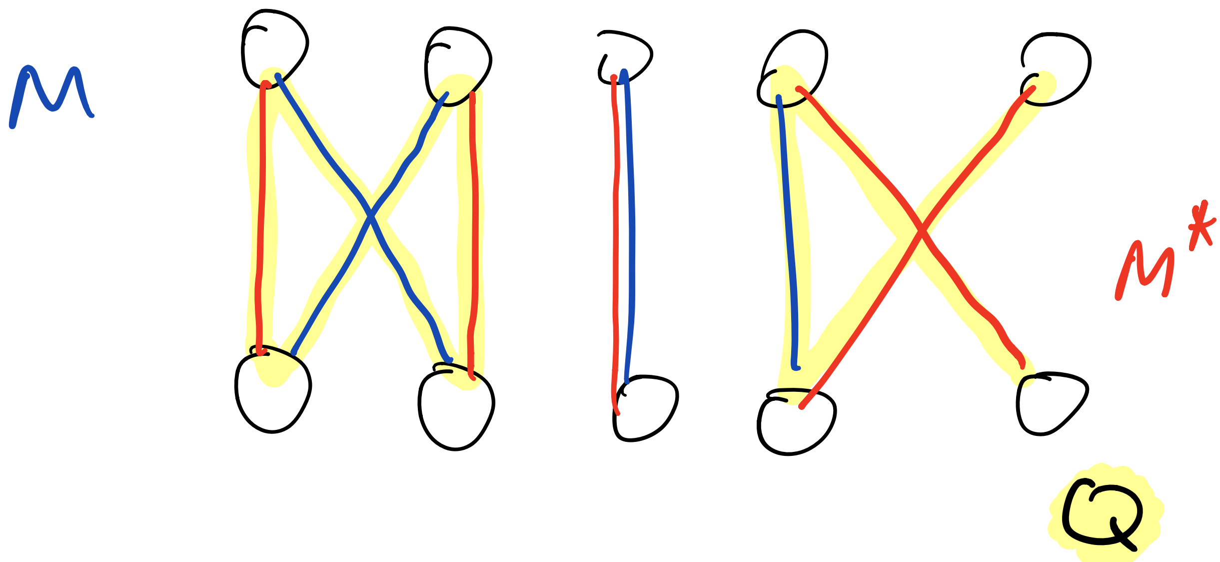
Proof: By contradiction.

$\Rightarrow$ we already showed: if augmenting path, $|M'| > |M|$; M not maximal; contradiction.

$\Leftarrow$ Assume M not maximum. Then let M^* be maximum, so $|M^*| > |M|$.

Let $Q = M \triangle M^*$

$= (M - M^*) \cup (M^* - M)$.



Then:

(1) Q has more edges from M^* than M

(because
$$\begin{array}{ccccc} |M^*| &=& |M \cap M^*| &+& |M^* - M| \\ \vee && \parallel && \vee \\ |M| &=& |M \cap M^*| &+& |M - M| \end{array}$$
)

② Every vertex $v \in G$ adjacent to ≤ 1 vertex in $M \cap Q$, ≤ 1 in $M^* \cap Q$.

(because M, M^* matchings).

③ Q partitioned into paths, cycles that alternate between M, M^*.

(② $\Rightarrow$ <u>degree</u> of $Q \le 2$

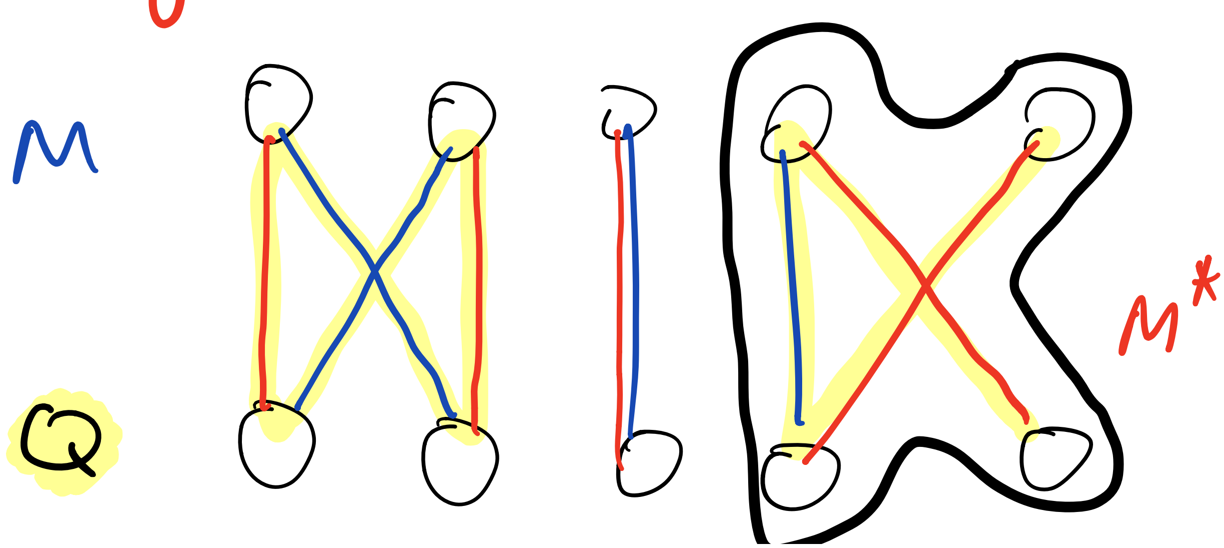
$\Rightarrow$ decomp into paths/cycles.

② $\Rightarrow$ alternating.)

4 Must be path in CQ with more edges from M^* than from M.

(b/c cycles are evenly split, and 1).

But this path is augmenting! Contradiction. □

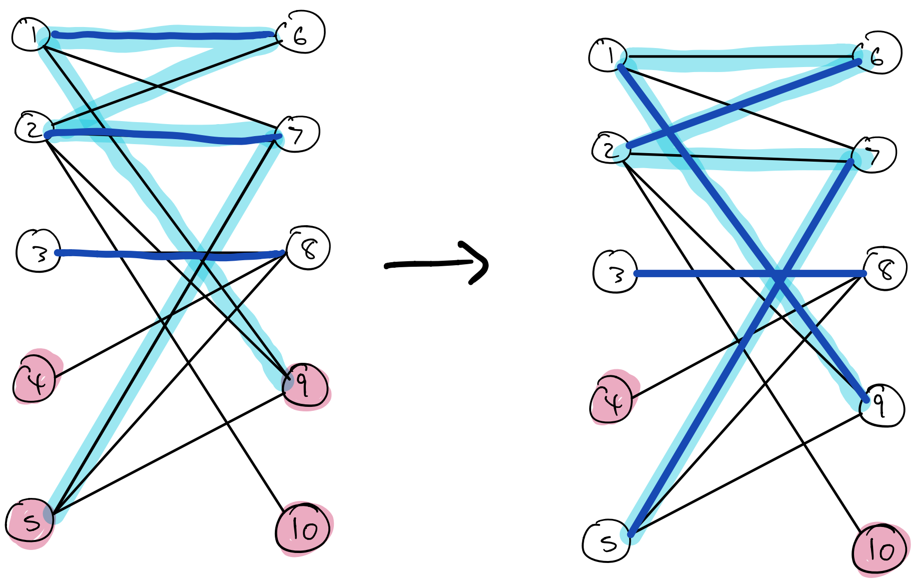


Augmenting path wrt M.

Alg: Augmenting paths.

- Begin with any matching M.
- Find augmenting path P wrt M, augment M along P.
- Stop when no more augmenting paths.

Ex:

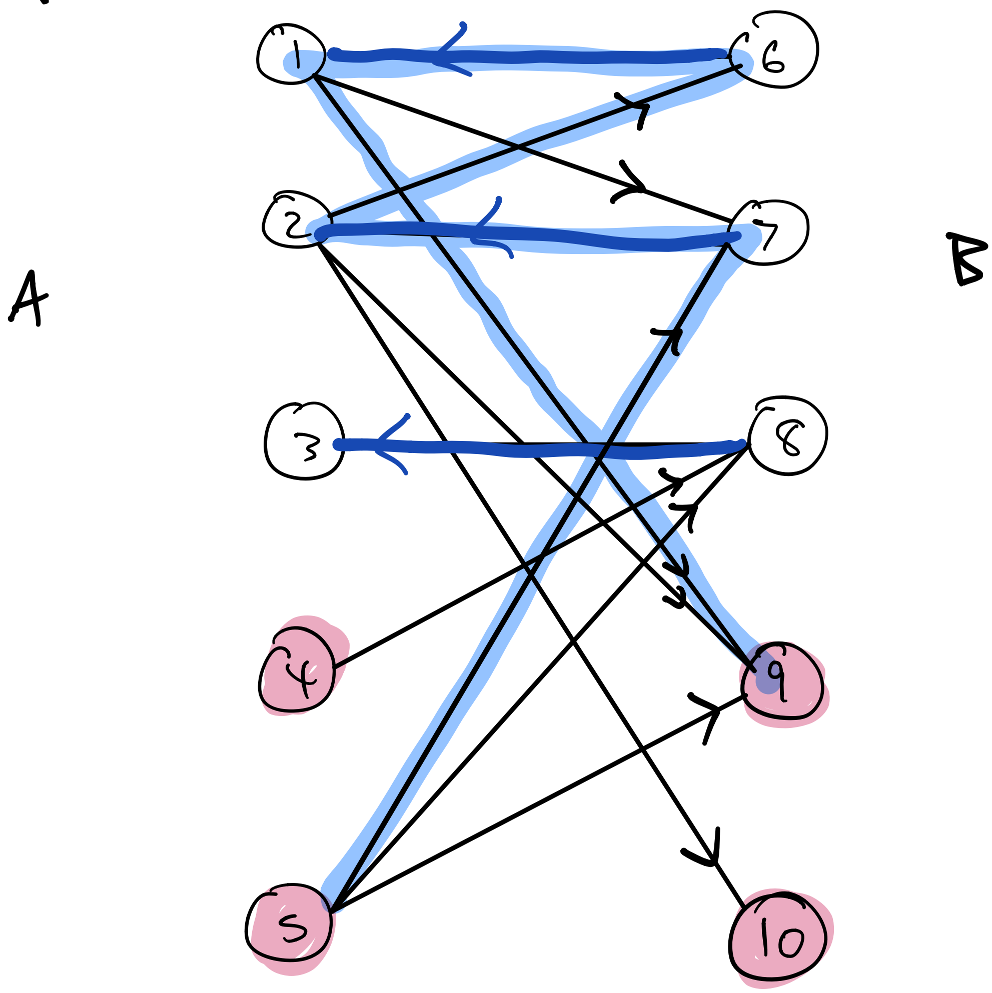


3 → 4 → Done.

Terminates after $\leq |M| \leq \frac{|V|}{2}$ augmentations.

BUT HOW TO FIND THE PATHS?

Reduces to finding path in a directed graph.

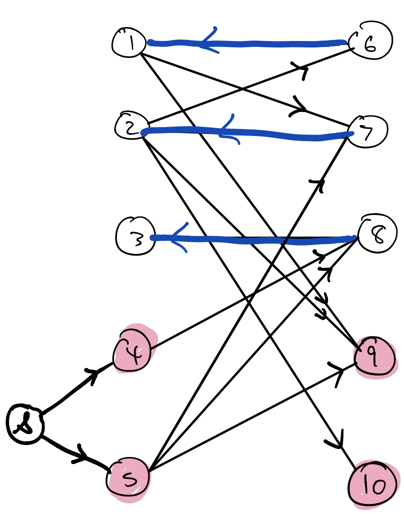


Direct e $A \to B$ if $e \in M$, $B \to A$ else.

Augmenting path is precisely a directed path from exposed vertex in A to exposed vertex in B.

suggests to use depth-first search. (DFS)

Subroutine for Aug paths:



* direct graph as above.
* attach vertex s to exposed vertices in A
* do DFS until hit exposed vertex in B.
* Trace back path.

Takes $O(|E|)$ time to find augmenting path in G.

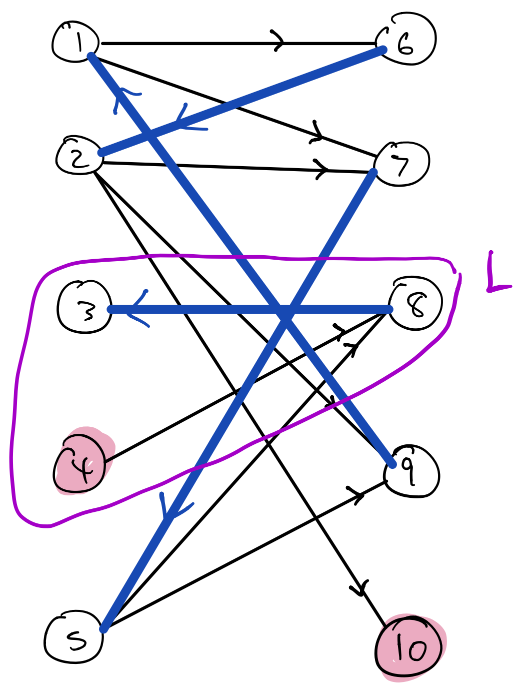
Thus, complexity is $O(|V||E|)$.

possible to get $O(\sqrt{n}\, m)$; Hopcroft-Karp.

Vertex Covers

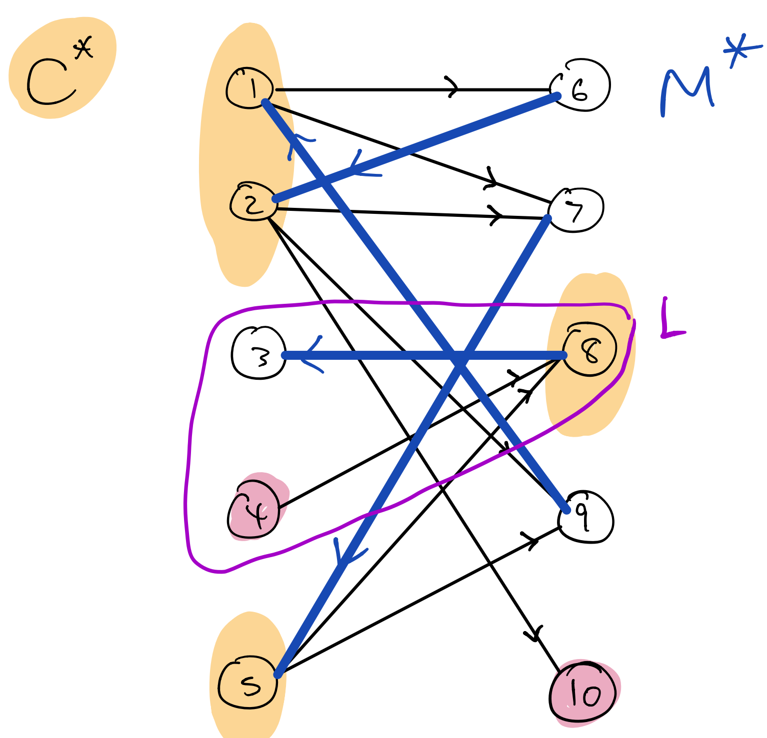
If no augmenting path wrt M, aug. path subroutine outputs a vertex cover. How?

Let L be set of vertices reachable by directed path from exposed vert in A.



Claim: When the algorithm terminates, $C^* = (A - L) \cup (B \cap L)$ is a vertex cover & $|C^*| = |M^*|$

matching returned by alg.

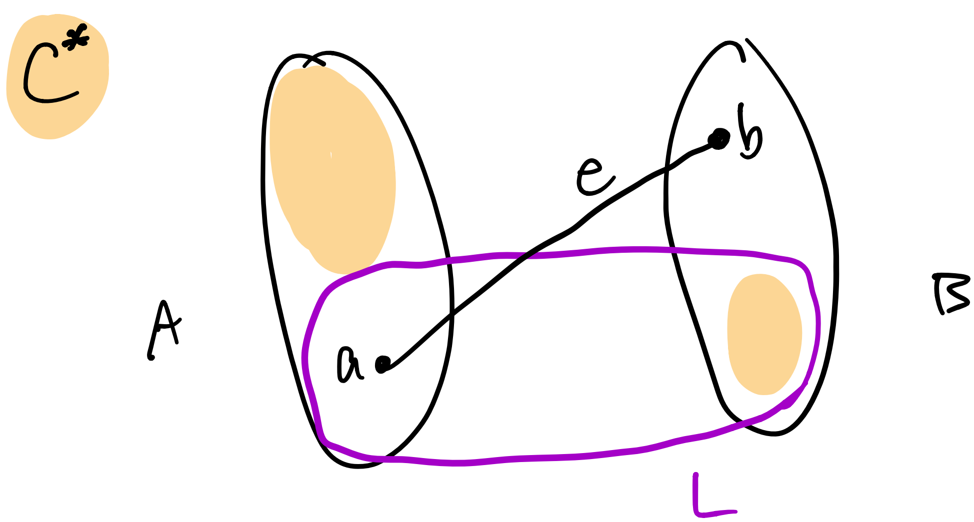


Corollary: König's Theorem

$$\max|\text{matching}| = \min|\text{vertex cover}|$$

Proof of Claim: First show C^* is cover. Assume not.

- Then exists $e=(a,b)\in E$ with $a\in A\cap L$, $b\in(B-L)$



- $e \notin M$.

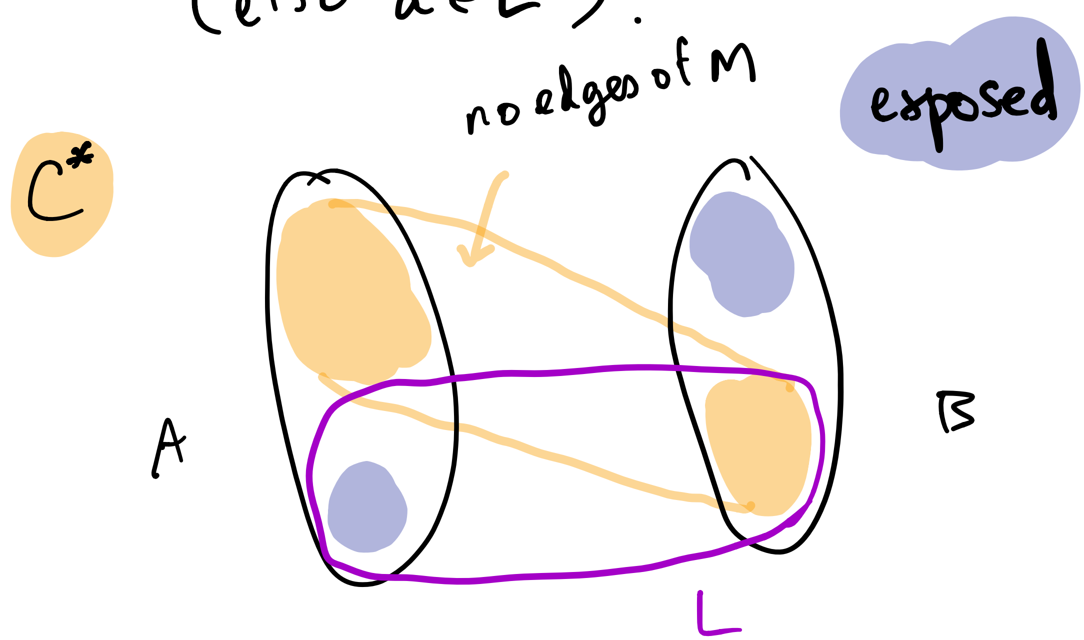
(because if $e \in M$,
then e is only $B \to A$ edge
to a because M matching.
thus, a not reachable if
$b \notin L$; contradicts $a \in L$.

Thus, b reachable;
contradicts $b \notin L$.

Now to prove $|M^*| = |C^*|$.

Enough to show $|C^*| \leq |M^*|$;
(b/c weak duality says $|M^*| \leq |C^*|$.)
To prove, observe:

- No vertex in $A-L$ exposed (by def of L.)
- No vertex in $B \cap L$ exposed (else $\exists$ augmenting path).
- No edge of M b/w $a \in A-L$, $b \in B \cap L$ (else $a \in L$).

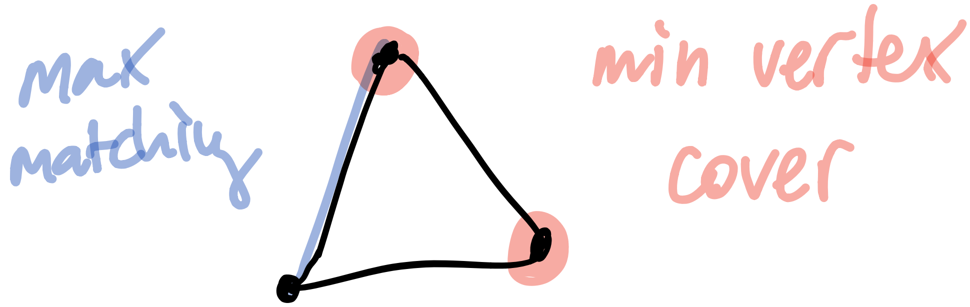


Conclude: every vertex in C^* matched, and no edge of M fully within C^*. hence $|C^*| \leq |M^*|$. $\square$

What about general graphs?

Still have weak duality,

but not strong duality:

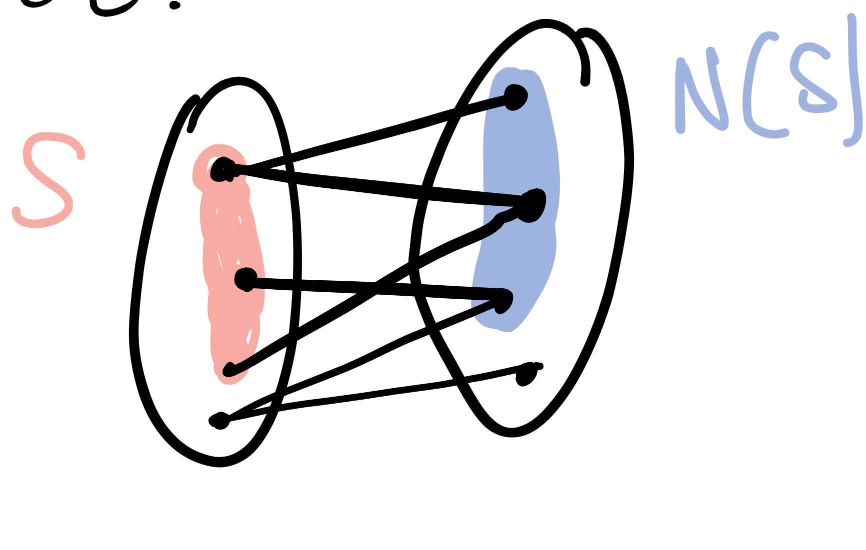


Need Tutte's theorem from lec 1; we'll discuss later.

Hall's theorem

Hall's theorem is another "duality" characterization of the existence of a perfect matching.

Def: neighborhood $N(S)$ of a set S is b s.t. $\exists a \in S$ w/ $(a, b) \in E$.

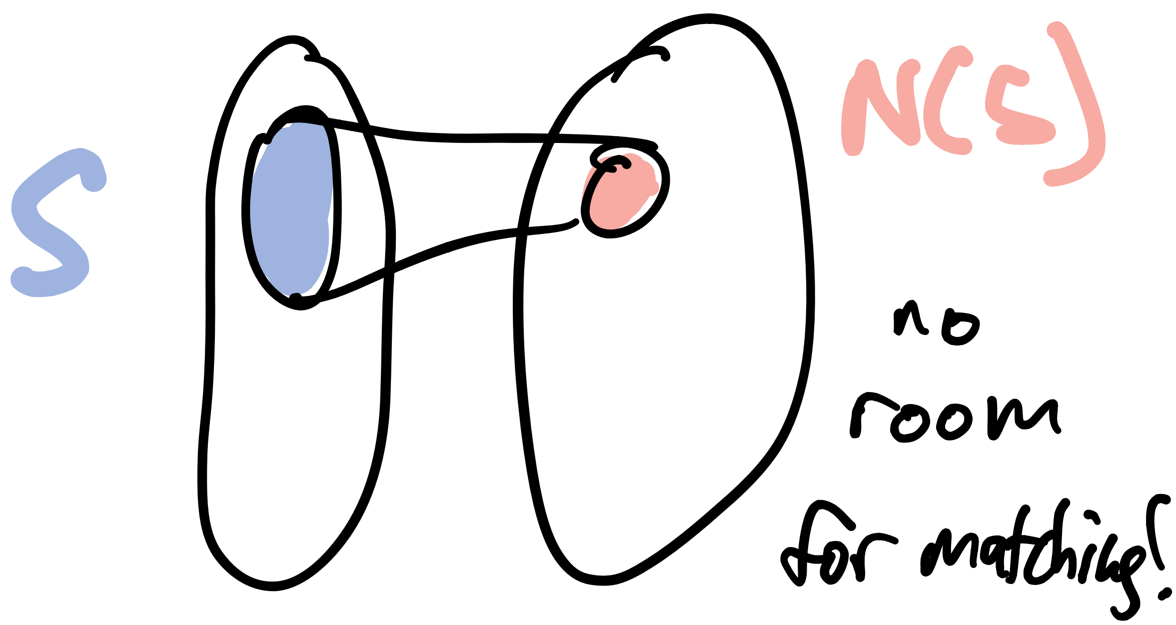


THM (Hall) Bipartite G with bipartition A, B has perfect matching

$$\iff$$

$$\forall S \subseteq A, \quad |N(S)| \geq |S|.$$

Clearly, $|N(S)| \geq |S|$ necessary;



i.e. $|N(S)|<|S|$ obstructs p.m.'s; weak duality. Hall says strong duality here also.

Hall's follows from König's theorem.

See exercises in source.