18.453 final. This exam is closed book. Be neat! **In any problem, you can refer to results we have covered in class, but you need to state them precisely.** If you can't solve one part of a problem, but need the result for a later part of the problem, you can assume that the earlier part of the problem has been proved for the sake of the later part. Have a great summer!

1. For $k \leq n$ an integer, define a *k-bounded permutation* on $\{1, ..., n\}$ to be a permutation σ such that $|\sigma(i) - i| \leq k$ for all $i \in \{1, ..., n\}$.

   Suppose we are given an integer $k \leq n$ and costs $c(i)$ for $i \in \{1, ..., n\}$, and our goal is to find a k-bounded permutation σ on $\{1, ..., n\}$ minimizing $\sum_{i=1}^{n} c(i)\sigma(i)$. Give a polynomial-time algorithm for this problem (there is no need to give the most efficient algorithm, but the algorithm should be polynomial in n and k). (You can refer to any algorithm we have seen in class.)

2. Let $M = (E, \mathcal{I})$ be a matroid with rank function r and suppose we have a cost function $c : E \to \mathbb{R}_{\geq 0}$ (for simplicity we are assuming that all the costs are nonnegative). We are interested in finding a base B of maximum total cost, i.e. maximizing $\sum_{e \in B} c(e)$.

   Consider the following greedy algorithm, different from the one covered in lecture, where instead of starting from an empty set and repeatedly adding elements of highest cost, we start from a full set and repeatedly remove elements of lowest cost.

   - ▷ Sort the elements (from smallest to largest) such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m)$ where $m = |E|$
   - ▷ $S = E$
   - ▷ For $j = 1$ to m
     - ▷ if $r(S \setminus \{e_j\}) = r(E)$ then $S \leftarrow S \setminus \{e_j\}$
   - ▷ Output S

   Prove that this algorithm returns a maximum cost basis in the matroid.

3. (a) Consider a directed graph $G = (V, E)$ with nonnegative (upper) capacities $u : E \to \mathbb{R}$ (and no lower capacities). For any two vertices $s, t \in V$, define $\lambda_{st} \in \mathbb{R}$ to be the maximum flow value from s to t. Given any 3 vertices $s, t, u \in V$, show that $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$.

   (b) If the graph is undirected, the previous result still holds: $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$ for all s, t, u. Furthermore, $\lambda_{st} = \lambda_{ts}$. Now, consider the complete graph K_V on the vertex set V with weight λ_{uv} on edge (u, v) for all u, v. Let T be a *maximum weight* spanning tree on K_V with respect to these weights λ_{uv}. Argue that for every $(s, t) \notin T$, we have
   $$\lambda_{st} = \min_{(u,v) \in P_{st}} \lambda_{uv}$$
   where P_{st} denotes the edges (of K_V) of the unique path in T between s and t. (This implies the somewhat surprising result that, over all pairs (s, t), λ_{st} can take at most $|V| - 1$ values (those along the edges of T).)

4. Consider a bipartite graph $G = (A, B, E)$ with parts A, B and edges $E \subseteq A \times B$. Suppose we have a matroid $M_A = (A, \mathcal{I}_A)$ on A with rank function r_A. Define a family of sets $\mathcal{I}_B$ to be the collection of sets $T \subseteq B$ such that there exists a matching M of G with vertex set $V(M) = S \cup T$, such that $S \subseteq A$ and $S \in \mathcal{I}_A$.

   (a) Prove that $M_B = (B, \mathcal{I}_B)$ is a matroid. (For **partial credit**, you can do this in the special case where every vertex of A has degree 1, so that G is the graph of a function from A to B.)

   (b) Prove the following generalization of König's Theorem, which gives a formula for the rank of $M_B = (B, \mathcal{I}_B)$:

$$\max_{T \in \mathcal{I}_B} |T| = \min_{C \text{ a vertex cover of } G} r_A(C \cap A) + |C \cap B|.$$

5. (**Extra Credit**)

   Given a matrix $A \in \mathbb{R}^{m \times n}$ with entries in $\{-1, 0, 1\}$, we can associate a directed bipartite graph D_A with parts $\{r_1, ..., r_m\}$ and $\{c_1, ..., c_n\}$, which has an edge directed from r_i to c_j when $A_{ij} = +1$, has an edge directed from c_j to r_i when $A_{ij} = -1$, and has no edges between r_i, c_j when $A_{ij} = 0$.

   Define a *circuit* of D_A to be a connected subgraph $C \subseteq D_A$ such that every vertex of C has degree 2. We say that a circuit C of D_A is *odd* if we can flip the directions of an odd number of edges of C to make it into a directed cycle, and otherwise we say that C is *even*.

   (a) Suppose that D_A has an *odd* circuit C. Show that it is possible to replace some of the entries of A by 0s to get a matrix A' which is *not* totally unimodular. (Hint: consider the case where D_A consists of just a single circuit.)

   (b) Suppose that every circuit of D_A is *even*. Show that it is possible to negate some of the rows of A to get a matrix A'' with the property that for *every* $k \leq m$, the sum of the first k rows of A'' is a row vector with all entries in $\{-1, 0, 1\}$. (Hint: pair up the nonzero entries of each column of A into groups of two, and try to arrange for the corresponding pairs of entries of A'' to have opposite signs.)

   (c) Prove Commoner's sufficient condition for total unimodularity: if every circuit of D_A is *even*, then A is totally unimodular.

   For part (b), you can use the following fact without proof: if you have a system of equations in variables $x_i \in \{-1, +1\}$, each of the form $x_i = x_j$ or of the form $x_i = -x_j$, then the system has a solution if and only if there are no "bad cycles" of the form $x_i = \pm x_j = \pm x_k = \cdots = -x_i$, with each equality coming from one of the equations of your system (possibly with both sides negated).