Lecture 15

Plan

1) Finish Min T-odd cut (see Lec14 notes]

2) Matroids.

Matroids

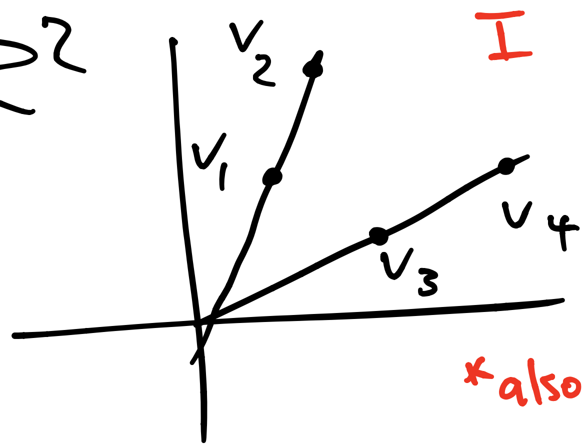
"Tractable" set systems.

E.g. Given vectors $v_1, \ldots\ldots, v_m \in \mathbb{R}^n$,

consider set system $\mathcal{I} \subseteq 2^{[m]}$:

$$\mathcal{I} = \{ \qquad\qquad\qquad\qquad \}$$

picture: $\mathbb{R}^2$



$\mathcal{I}$

*also $\emptyset \in \mathcal{I}$.

Properties of $\mathcal{I}$:

(P1) "Downward closed"

(P2) "Exchange property"

If $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ and

$|Y| > |X|$,

Formally:

Pf of P2:

$\square$

P1, P2 capture combinatorial structure of $\mathcal{I}$.

for matroids:

Def (Matroid) A matroid M is a pair $(E, \mathcal{I})$ where

- $E = E(M)$
- $\mathcal{I} = \mathcal{I}(M) \subseteq 2^E$

•

Remarks • P2 $\Rightarrow$

• Maximal independent set called a **base** of M.

• **dependent** :=

• for $F \subseteq E$, the **restriction** $M|_F$

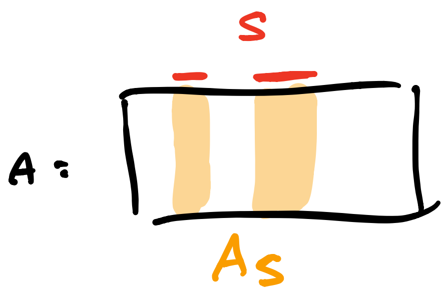
Examples

• Linear matroid:

▷ equiv def: $A \in \mathbb{R}^{n \times m}$ matrix,

$I = \{ \qquad \}$.

e.g.



▷

▷ bases of M:

- "boring" example: **uniform matroid**:

  where

  $I = \{ \qquad \}$

  $=$

the **free matroid** is

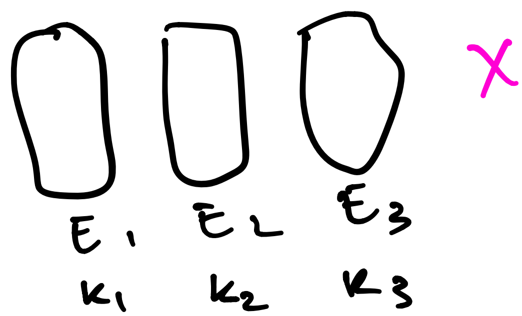
- **partition matroid**: $M = (E, I)$

where E is

$I = \{ \qquad \}$

for

e.g.



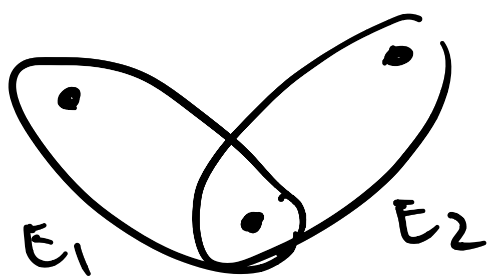
Check P2:

- Let $|X| < |Y|$, $X, Y \in I$.

Remark: if E_i not disjoint:

e.g.

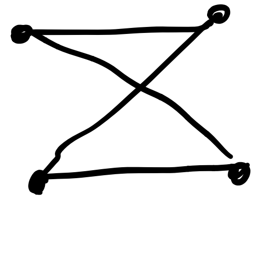
$k_1 = 1$



$k_2 = 1$

- **Another Nonexample:**

e.g.

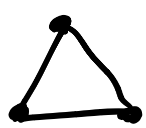


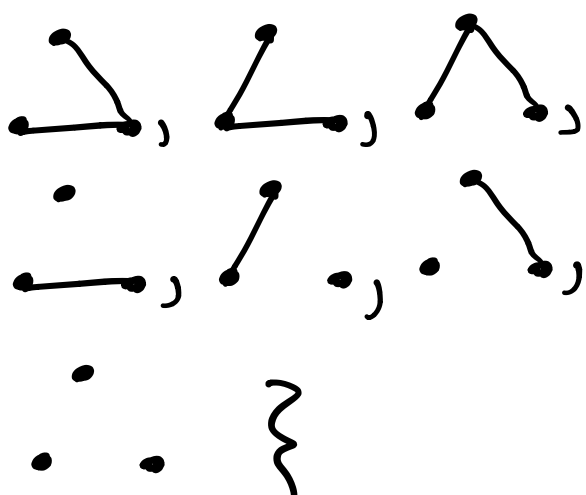
- **graphic matroids:**

Given graph $G=(V,E)$,
let $M(G) = (E, I)$ where

$I = \{ \qquad \qquad \qquad \}$

$=$

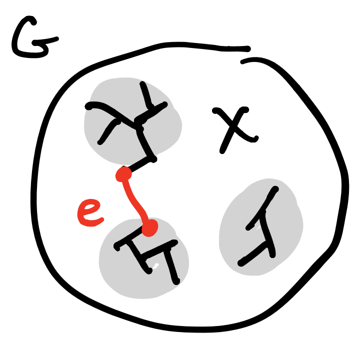
e.g. $G =$ 

$I = \{$  $\}$

Checkin P2:

- F forest $\Rightarrow$
- X, Y forests, $|X|<|Y| \Rightarrow$

$\Rightarrow$



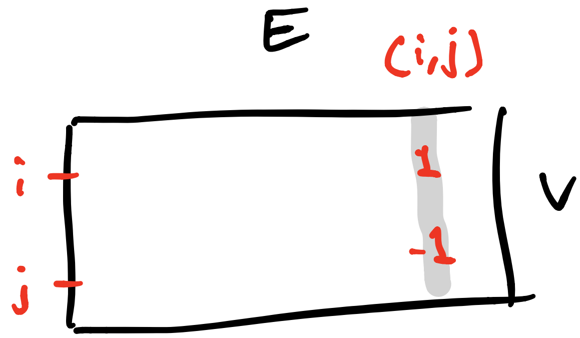
$\Rightarrow$

▷ <u>bases</u>:

▷ <u>graphic $\Rightarrow$ linear</u>:

Pf: $M = M_A$ where

$A =$



Ex. Check:

$\Longleftrightarrow$

$\square$

▷ Graphic $\Rightarrow$ regular:

Say M **regular** if

Note:

Fact: matroid M regular $\Longleftrightarrow$

Circuits

- **circuit** : =

e.g. ▷ in graphic matroid:

▷ in partition matroid, circuits are just subsets $C \subseteq E_i$ with $|C| = k+1$.

e.g.

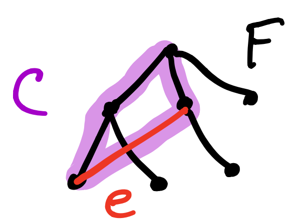
Note:

There's exactly one way to do the reverse:

Theorem (unique circuit property)

- Let $M = (E, \mathcal{I})$ matroid.
-
- Then:

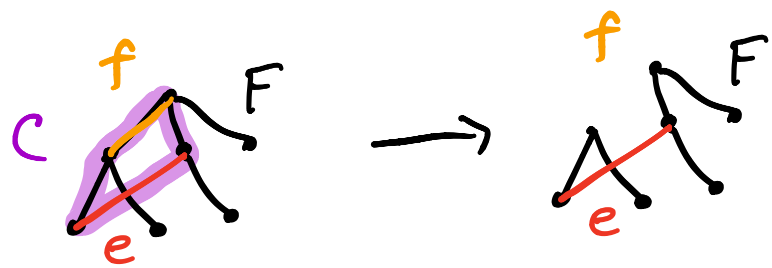
e.g. If F is a forest, $F + e$ isn't:



Remark: uniqueness shows

how to make more independent sets:

e.g.



Pf:

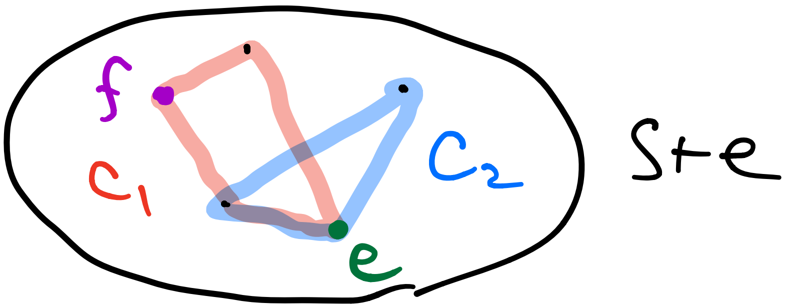
Proof of UCP:

- suppose $S+e$ contains distinct circuits $C_1 \neq C_2$.

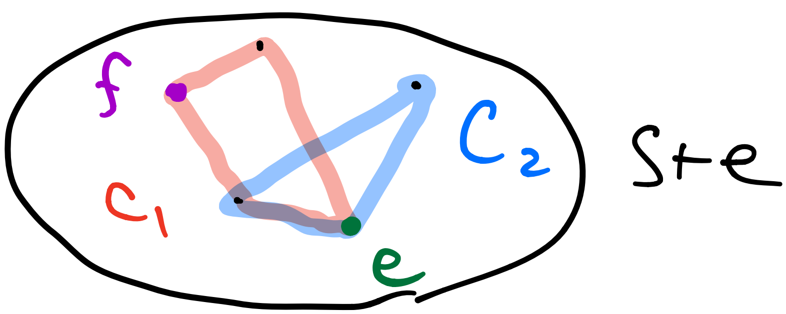
- Minimality $\Rightarrow$

Note:

▷



• $C_1 - f$ independent ⇒



• Both S, X maximal independent within $S+e$ ⇒

- $e \in X \Rightarrow$

- $\Rightarrow$ $\square$

Matroid optimization

- Given cost function $c : E \to \mathbb{R}$,

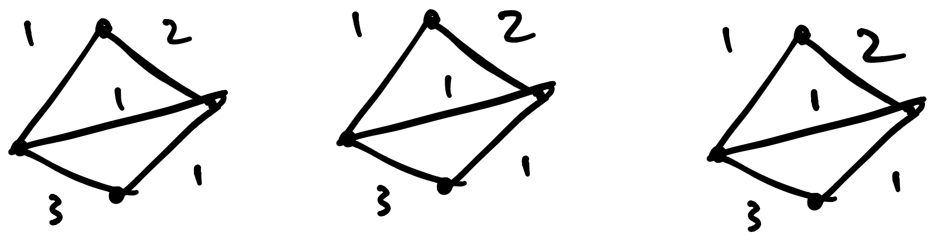
-

- if some $c(e) < 0$:

- if $c \geq 0$:

e.g. for graphic matroids:

Recall: M.S.T. has simple greedy algorithm:



- **Fact:**

- Actually, for all k:

Algorithm Let $|E|=m$.

▷ Sort E by cost:

▷

- For $j=1$ to m:
  - if ______________________ then:
    -
    -
- Output $S_1, \ldots, S_k$.

Thm: For any matroid $M=(E,\mathcal{I})$, above alg. finds indep. set S_k such that

Proof: Suppose not.

- Let $\delta_k =$

- Suppose $T_k =$

- Let $p :=$

- Let $A =$

  $B =$

- $|A| > |B| \Rightarrow$

- But

$\square$

Next time: