Large deviation principle for quantum marginal and moment map tomography

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Motivation

We study concentration properties of certain families of probability distributions derived from the decomposition of tensor product representations. The following specific instances have been considered in the literature in different contexts.

1. **Multiplicities:** Given a representation π of a compact group K, what is the asymptotic distribution of the multiplicities of the irreducible representations in $\pi^{\otimes m}$? For the group SU(2) this was studied in [CLR88] and used in the calculation of free energies in quantum statistical mechanics, and subsequently generalized to compact semisimple Lie groups in [Duf90].
2. **Sums of IID random variables:** Let $X_1, X_2, \ldots$ be identically distributed, bounded, integer-valued random variables. What is the probability that the sample mean $\frac{1}{m}(X_1 + \cdots + X_m)$ falls in some fixed interval not containing the expected value of X_1? The decay rate is given by Cramér's large deviation theorem.
3. **Spectrum estimation:** The empirical Young diagram measurement (EYD) [ARS88, KW01] is based on the decomposition of $\mathcal{H}^{\otimes m}$ into irreducible representations of $U(\mathcal{H})$, associating the outcome $\frac{1}{m}\lambda$ with the integer partition λ. Applying the measurement to the state $\rho^{\otimes m}$ results in a distribution that has a sharp peak at the ordered spectrum of ρ.
4. **Tomography:** Keyl proposed a refinement of the EYD measurement for the purpose of tomograpy, and proved a large deviation principle in that setting [Key06].
5. **Universal entanglement concentration:** In the Matsumoto–Hayashi protocol, the experimenters Alice and Bob in distant labs share many copies of a pure entangled state $|\psi\rangle_{AB}$, and both perform local EYD measurements to distill maximally entangled states [MH07]. The analysis of the performance of the protocol relies on the asymptotic distribution of the measurement outcomes.
6. **Local spectrum estimation:** More generally, we can ask how the outcomes a local EYD measurement are concentrated around the marginal spectra. These outcomes are correlated in a nontrivial way, and the study of their asymptotic behavior lead to new information quantities [Vra20].

Result for bipartite states [FW20, BCV20]

We generalize Keyl and Werner's measurements to representations of arbitrary compact groups K. For concreteness, we first present for $\mathcal{H} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ and $K = \mathrm{U}(n_1) \times \mathrm{U}(n_2)$.

- **State:** Bipartite state $\rho \in S(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$.
- **Measurement:** Apply Keyl's tomography measurement *locally* to $\rho^{\otimes m}$.
- **Targets:** Estimate marginals ρ_1, ρ_2.

Then the density at (σ_1, σ_2) **decays exponentially** away from ρ_1, ρ_2 as $\exp(-m\, I_\rho(\sigma_1, \sigma_2))$ where I_ρ is an explicit optimization problem:[a]

$$\begin{aligned} I_\rho(\mathrm{diag}(x), \mathrm{diag}(y)) = \sup_{P,Q \succ 0} \; & - \log \mathrm{Tr}(P \otimes Q)\rho \\ & + \sum_{i=1}^{n} (x_i - x_{i+1}) \log \mathrm{pm}_i\, P \\ & + \sum_{i=1}^{n} (y_i - y_{i+1}) \log \mathrm{pm}_i\, Q, \end{aligned}$$

where pm_i denotes the i^{th} leading principal minor and x, y are nonincreasing positive sequences of reals with $x_{n+1} = y_{n+1} := 0$.

Rate is larger than maxima of rates for each EYD measurement!

[a] The formula for non-diagonal inputs follows by equivariance of the distribution under the action of $U(n_1) \times U(n_2)$.

General result [FW20, BCV20]

- **State:** State $\rho \in S(\mathcal{H})$, $\mathcal{H}$ unitary representation of compact, connected Lie group K.
- **Measurement:** Constructed analogously from irreps of K in $\mathcal{H}^{\otimes m}$, applied to $\rho^{\otimes m}$. 1
- **Targets:** *Moment map* $J(\rho) \in i\mathfrak{k}^*$, analogous to marginals. Captures the expectation values of any observable in the representation of the Lie algebra $\mathfrak{k}$. 1

Then

(i) the outcomes converge to $J(\rho)$ in probability.

(ii) the outcomes satisfy a **large deviation principle** 1 with rate function I_ρ that can be expressed as an optimization problem over the Lie algebra of K.

In particular, I_ρ is computable to arbitrary precision.

Example: U(1) and sum of IID random variables

Let $\pi : U(1) \to \mathcal{H}$ be a finite dimensional representation of U(1).

- **State:** $\rho \rightsquigarrow$ bounded random variable X on $\mathbb{Z}$.
- **Moment map:** $J(\rho) = \mathbb{E}[X]$.
- **Keyl measurement:** applied to $\rho^{\otimes m}$ obtain $X_m := \frac{1}{m}(\underbrace{X + \cdots + X}_{m \text{ i.i.d copies}})$.

Our formula for the rate function specializes to

$$I_\rho(x) = \sup_{t \in \mathbb{R}} tx - \ln \sum_{n \in \mathbb{Z}} \Pr[X = n] e^{tn},$$

the Legendre–Fenchel transform of the logarithmic moment generating function of X_i.

Example: Multiplicities

Let $\pi : K \to U(\mathcal{H})$ be any representation and $\rho = I$. Then $\mathrm{Tr}\, \Pi_\lambda \rho^{\otimes m}$ is equal the multiplicity of the representation $\mathcal{H}_\lambda$ in $\mathcal{H}^{\otimes m}$ times $\dim \mathcal{H}_\lambda$. Since the multiplicites are exponentially large, while $\dim \mathcal{H}_\lambda$ is bounded by $\mathrm{poly}(m)$, the asymptotic behavior of the multiplicities and the expectation values $\mathrm{Tr}\, \Pi_\lambda \rho^{\otimes m}$ is the same.

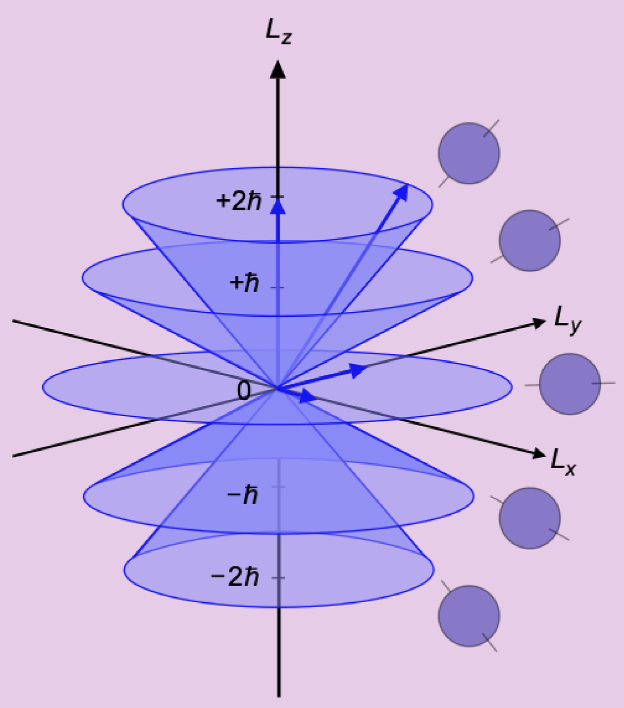
Our expression simplifies to

$$I_\rho(u x_0 u^{-1}) = \sup_{a \in \mathfrak{a}} \langle x_0, a \rangle - \ln \mathrm{Tr}\, \pi(\exp a).$$

where $u \in K$ and the real vector space $\mathfrak{a} \subset i\mathfrak{k}$ can be identified with the Lie algebra of a maximal commutative subgroup of K.

Example: SU(2) and angular momentum

- **State:** $\rho :=$ state of particle of some finite spin.
- **Moment map:** $J(\rho)$ is 2×2 Hermitian matrix, $\langle L_x \rangle \sigma_x + \langle L_y \rangle \sigma_y + \langle L_z \rangle \sigma_z$. encoding expectation values of angular momentum operators L_x, L_y, L_z.
- **EYD measurement:** measure total angular momentum L of the m copies $\rho^{\otimes m}$.



Our formula for the exponential rate of decay of $\Pr[L = 0]$ specializes to

$$I_\rho(0) = \sup_{P \succ 0} \frac{1}{m} \log \det P - \log \mathrm{Tr}\, \pi(P)\rho.$$

Example: U(d) and state tomography

Let $K = U(d)$, π the standard representation on $\mathbb{C}^d$ and ρ a state. The supremum can be found by differentiation, which gives

$$I_\rho(\mathrm{diag}(x)) = \sum_{i=1}^{d} \left[x_i \ln x_i - x_i \left(\ln \frac{\mathrm{pm}_i(\rho)}{\mathrm{pm}_{i-1}(\rho)} \right) \right]$$

when x is positive and nonincreasing diagonal, and ∞ otherwise.

Techniques

First prove for $U(1)$: Recall state $\rho \rightsquigarrow$ bounded integer random variable X.

- **Reduce** to the case $x = 0$, $\mathbb{E}[X = 0]$ so that $I_\rho(x) = 0$.
- **Local central limit theorem:** For k in some arithmetic progression,
$$\Pr[X_m = 0] = \Theta\left(\frac{1}{\sqrt{m}}\right).$$
- **Proof of LCLT:** folklore Fourier analysis.

Proof for arbitrary K: generalize Fourier analytic proof of LCLT to Fourier analysis on compact groups.

Supplement + references: 1, Full papers: 2