Lecture 8

Plan:

- Faces of Polyhedra
- State tons of facts
- Prove them

Faces of Polyhedra

Def: $a^{(1)}, \ldots, a^{(k)} \in \mathbb{R}^n$ are affinely independent if

$$\sum_{i=1}^{k} \lambda_i a^{(i)} = 0$$

and $\sum \lambda_i = 0$ imply $\lambda_1 = \ldots = \lambda_k = 0$.

(w/out $\sum \lambda_i = 0$, is just linear indp.)

linear independence $\Rightarrow$ affine independ.

Note:

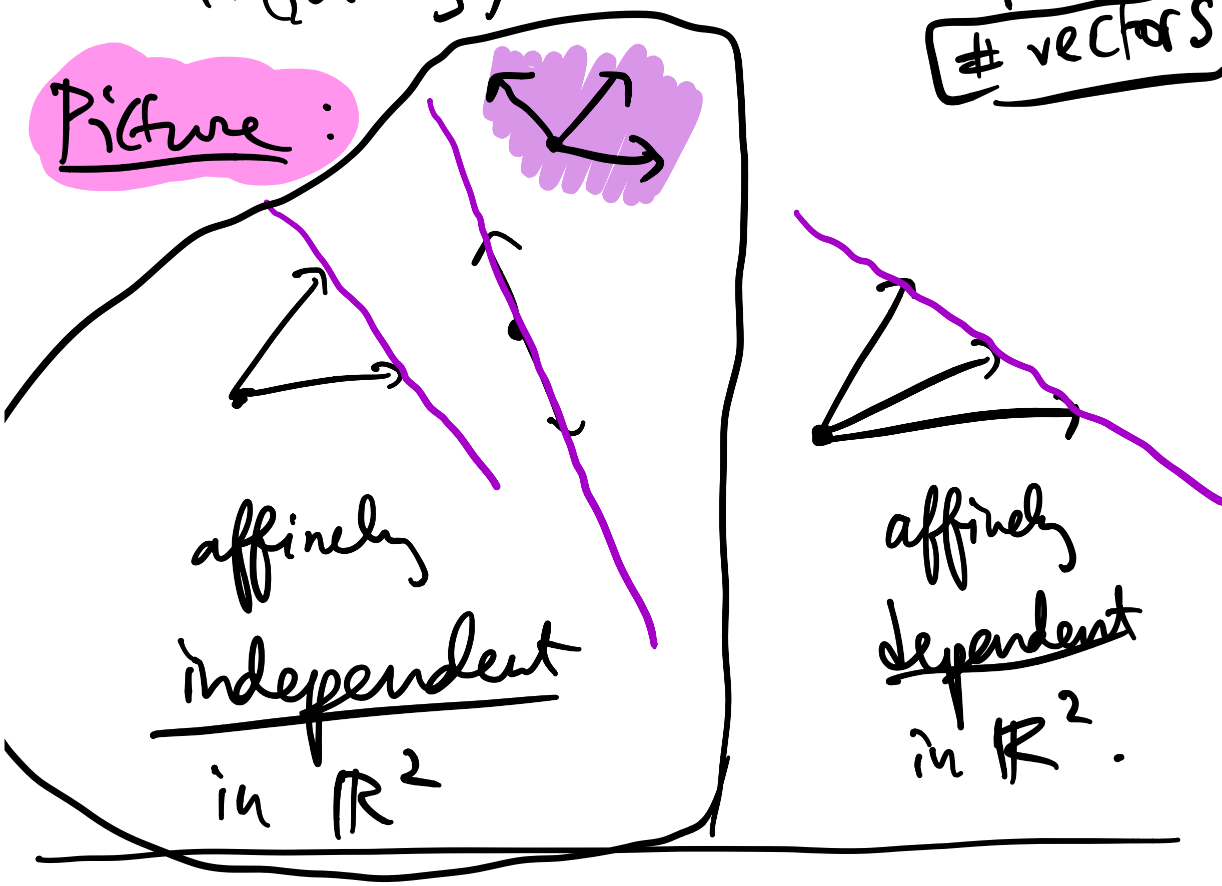
$\{a^{(i)}\}$ affinely independent iff

$\{a^{(i)}\}$ affinely independent

$$\left\{ \begin{bmatrix} a^{(i)} \\ 1 \end{bmatrix} \right\} \text{ linearly independent.}$$

$\Leftrightarrow$ $\text{aff}(\{a^{(i)}\})$ has dimension $\underset{\uparrow \ \boxed{\#\text{ vectors.}}}{K-1}$

Picture:



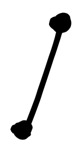
Def **Dimension** $\dim(P)$ of polyhedron P:

$-1 +$ max # affinely independent points in P.

Equivalently, dimension of affine hull $\mathrm{aff}(P)$.

Examples: $P = \emptyset$, $\dim(P) = -1$

$P =$ singleton $\quad \cdot \quad$ $\dim(P) = 0$

$P =$ line segment  $\dim(P) = 1$

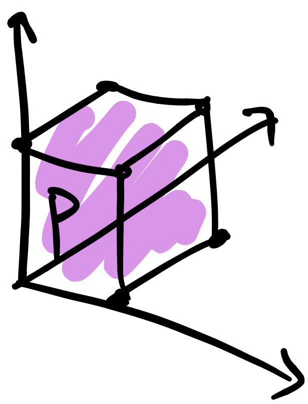
$\vdots$

$\mathrm{aff}(P) = \mathbb{R}^n$

$\dim(P) = n$;

P "full dimensional"

e.g. cube in $\mathbb{R}^3$: $\{x : 0 \le x_i \le 1\}$



$\dim P = 3$

$\dim \mathbb{R}^3 = 3$

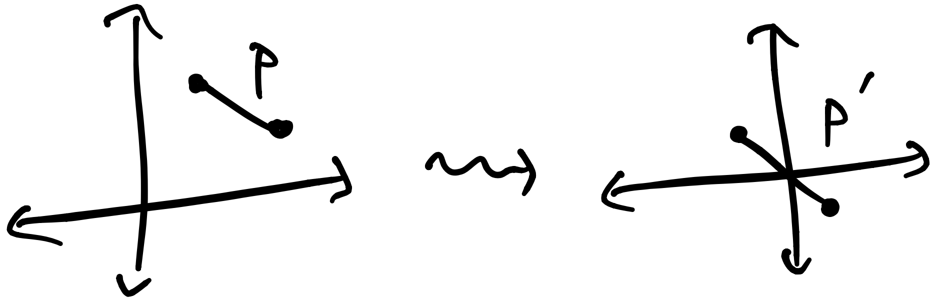
(as polyhedron).

Why affine, not linear? affine independence is translation invariant:

if I used max # lin indep points -1

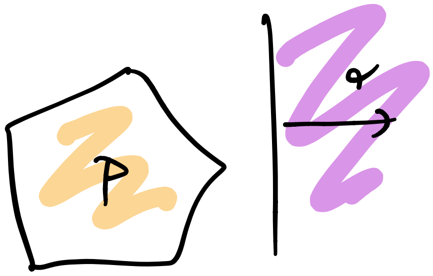
$\dim(P) = 1$

$\dim(P') = 0$



$1 = \dim(P) = \dim(P')$.

Def: $a^T x \le \beta$ is a <u>valid inequality</u> for P if $a^T x \le \beta$ for all $x \in P$.

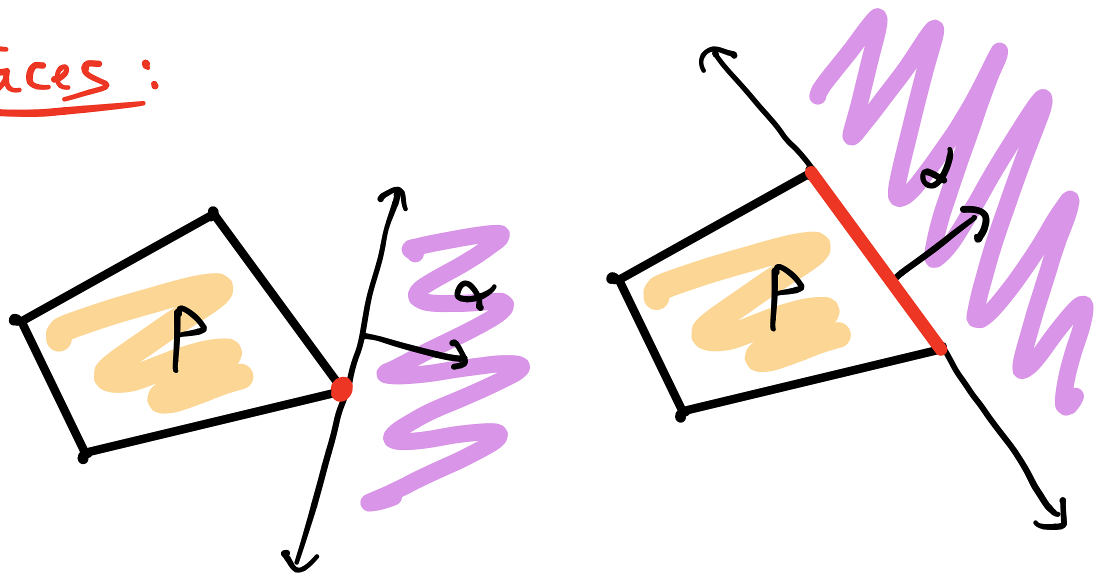


or

Def A <u>face</u> of a polyhedron P is $\{x \in P : \qquad \}$ for

$\alpha^T x \le \beta$ valid.

Faces:



Properties:

- Faces are polyhedra
- Empty face & entire P are called trivial faces
- else F nontrivial

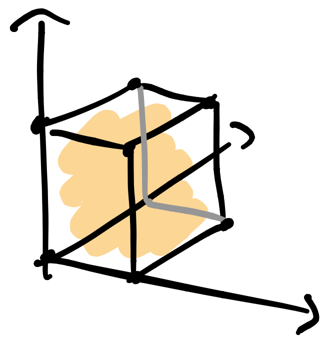
$\le \dim(F) \le$

- $F : \dim(F) = \dim(P) - 1$ called facets.
- $F : \dim(F) = 0$ called vertices

- $F : \dim(F) = 0$ called **vertices**

<u>Ex</u>: list the 28 faces of the cube

$$P = \{x \in \mathbb{R}^3 : \qquad \}$$

$P =$ 

<u>Fact</u>: ∞ many valid ineqs, but # faces finite!

EVERYTHING ABOUT POLYHEDRA

Let $A = \begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_m & - \end{bmatrix} \in \mathbb{R}^{m \times n}$.

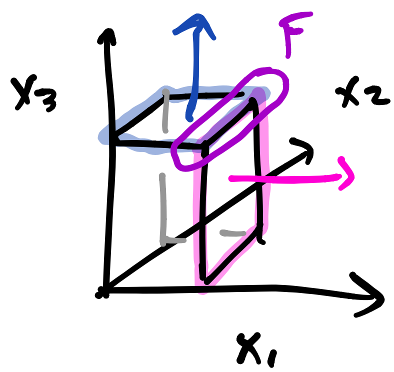
- **Face Characterization:**

Any nonempty face of $P = \{x : Ax \leq b\}$ is

$$\left\{ x : \qquad \right\}$$

for some set $I \subseteq \{1, \ldots, m\}$.

E.g. cube



$F = \{ x : \qquad \}$

- **Facet Maximality:** The facets are the maximal nontrivial faces of a nonempty polyhedron P.

For vertices:

③ Vertex Characterization:

Suppose x^* extreme point of $P = \{x : Ax \leq b\}$

Then $\exists I$ s.t. x^* is the unique soln to

moreover, any such unique solution $x^* \in P$ is extreme.

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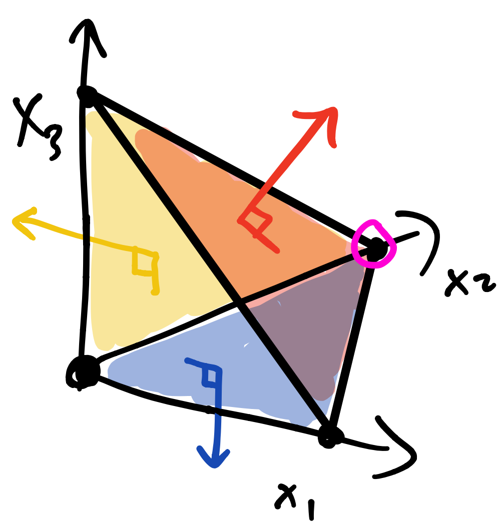
e.g. simplex $(0,1,0)$ is intersection of 3 constraints



- **Vertex minimality:** For rank$(A) = n$, minimal nontrivial faces of polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \le b\}$$

are the vertices.

- **Polytopes = convex hulls**

If a polyhedron P is bounded then $P = \mathrm{conv}(\{\text{extreme points of } P\})$.

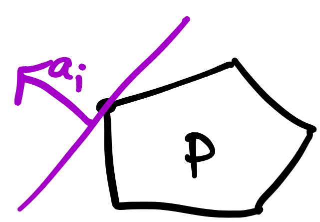
(special case of Krein-Milman theorem.)

- **Facets Characterize**

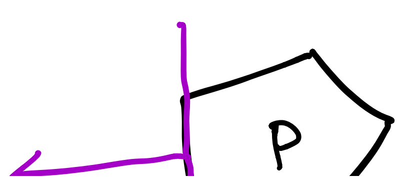
- inequality $a_i^T \leq b_i$ redundant if P unchanged when it's removed.
- $I_= := \{i : a_i^T x = b_i \ \forall x \in P\}$
- $I_< := \{i : \exists x \in P \ a_i^T x < b_i\}$.

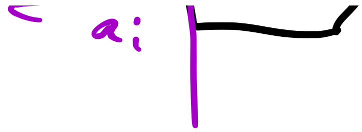
THEN:

(Sufficiency:) If face $a_i^T x \leq b_i$ for $i \in I_<$ is not facet, then $a_i^T x \leq b_i$ is redundant.

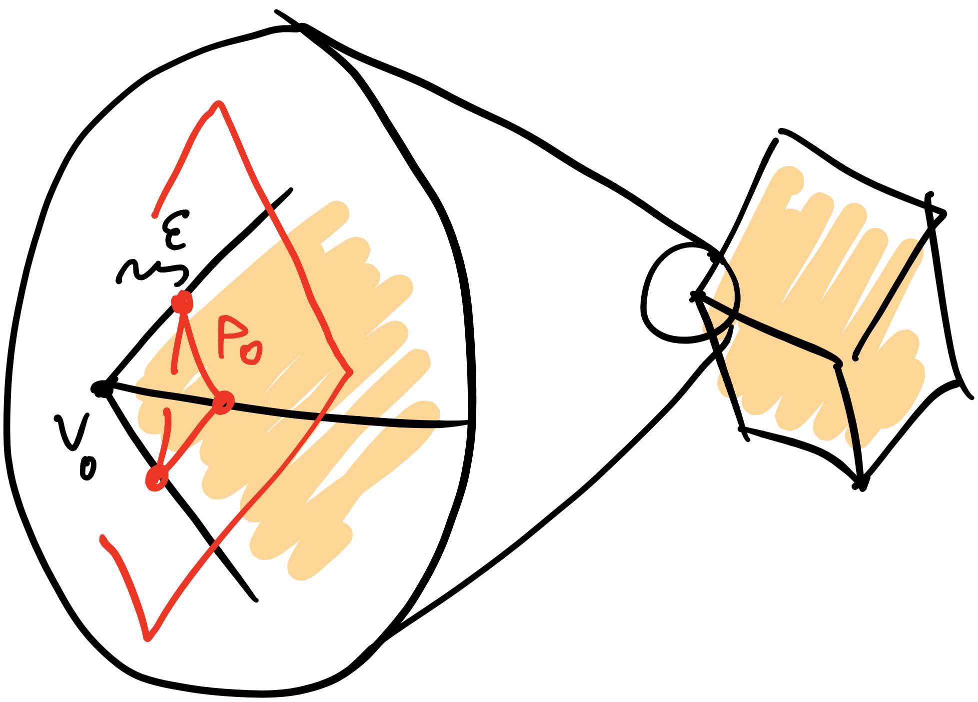


(Necessity:) If F is facet of P, $\exists$ $i \in I_<$ such that F is induced by $a_i^T \leq b_i$.





Near vertices = cones over polytopes



Let v_0 vertex of P from valid inequality $c^T x \le m$.

Let ε be such that $c^T v' < m - \varepsilon$

for all other vertices v'.

Then

$$P_0 = \{x \in P : c^T x = m - \varepsilon\}$$

is a polytope & is bijection

$\{P_0$'s dim k faces$\}$

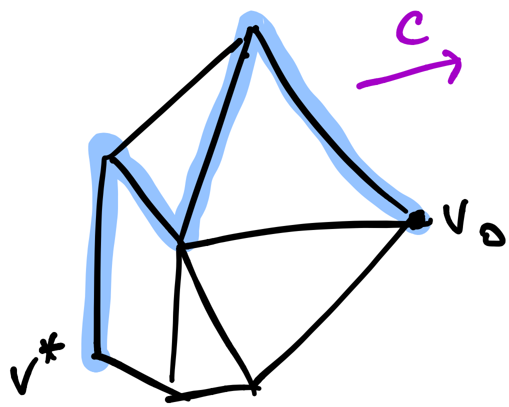
$$\longleftrightarrow$$

$\{P$'s dim k faces containing $v_0\}$.

P's "graph" connected: Graph of vertices & edges of polyhedron P

is always connected.

In particular: if v^* minimum of $c^T x$ over P, $\exists$ path from any v_0 to v with decreasing objective along path.



PROOFS

Recall face characterization:

Let $A \in \mathbb{R}^{m \times n}$, $A = \begin{bmatrix} & \vdots & \\ - & a_i^T & - \\ & \vdots & \end{bmatrix}$

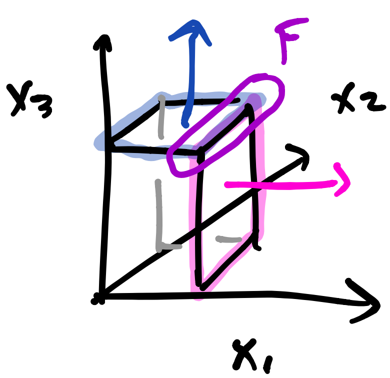
Any nonempty face of $P = \{x : Ax \leq b\}$ is

$$\left\{ x : \qquad \right\}$$

for some set $I \subseteq \{1, \ldots, m\}$.

$\Rightarrow$

E.g. cube



$$F = \left\{ x : \qquad \right\}$$

Proof Consider valid inequality $a^T x \le b$ giving nonempty face F.

- F = optimum solutions to bounded LP

(P) max

subject to

- Let y^* optimal solution to dual.

- Complementary slackness:

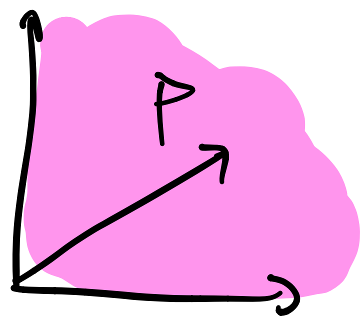
optimal solns F are

$$\{x : \quad \}.$$

Thus we can take $I = \{i : y_i^* > 0\}$. $\square$

Ex: positive orthant $\{x \in \mathbb{R}^n : x_i \geq 0\} = P$
has $2^n + 1$ faces

- How many of dim k?



For polytopes can also bound # faces in terms of # vertices.
"upper bound theorem"
Dehn-Sommerville equation

Facet minimality:

Pf: Exercise to prove from face theorem.

Recall vertex characterization:

Let x^* extreme point for

$$A = \begin{bmatrix} & \vdots & \\ — & a_i^T & — \\ & \vdots & \end{bmatrix}$$

$$P = \{x : Ax \le b\}.$$

Then $\exists\, I$ s.t. x^* is the unique soln to

$$\boxed{\quad}.$$

moreover, any such unique solution x^* is extreme.

Proof: Given extreme point x^*,

- define $I = \{ i : \qquad \}$.

- Note for $i \notin I$, $\boxed{\qquad\qquad}$.

- By "faces theorem", x^* uniquely defined by

(*) $\qquad\qquad\qquad\qquad$ $i \in I$

(**) $\qquad\qquad\qquad\qquad$ $i \notin I$.

- Suppose $\exists$ other soln. $\hat{x}$ to (*).

- Because ______ for $i \notin I$,

still satisfies (*), (* *) for

- Contradicts F having only one point. $\square$.

Basic Feasible Solutions:

For $P = \{ \quad , \quad \}$

can describe extreme points very explicitly.

( ).

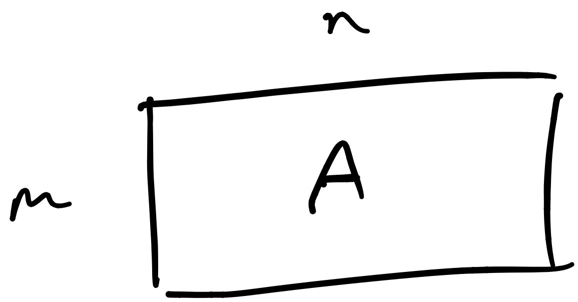
Corollary of Vertex Thm: Extreme pts. of $P = \{ \quad \}$ come from setting

$\boxed{\quad}$

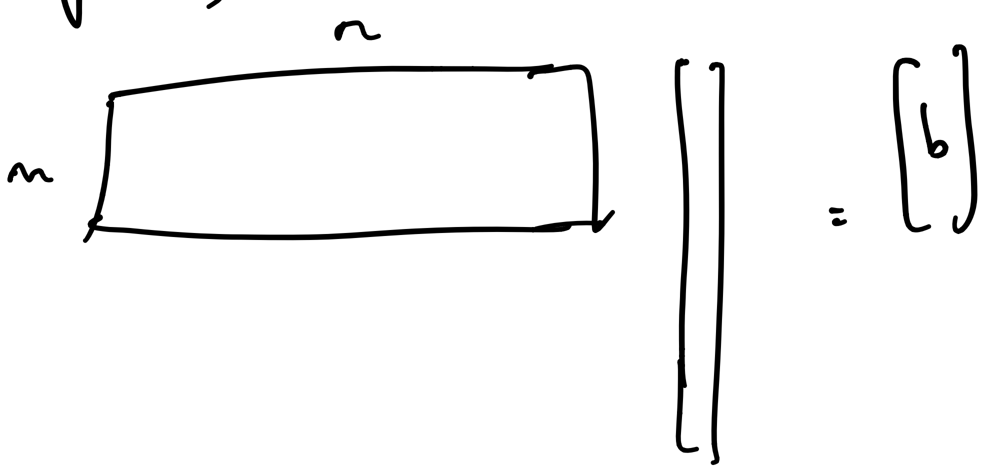
and finding unique solution to $=$ for remaining variables.

Can say more: Extreme points of $P = \{x : Ax = b, x \geq 0\}$ are the basic feasible solutions (bfs), feasible solns obtained as follows:

- Remove redundant rows from A ( )



• Choose m columns B of A, ( )



• Solve $A_B x_B = 0$, set

$$x^*_i = \begin{cases} & i \in B \\ & \text{else} \end{cases}$$

$$\boxed{\{\text{bfs}\} = \{\text{extreme pts}\}}.$$

Recall vertex minimality

If $\operatorname{rank} A = n$, vertices are minimal nontrivial faces.

Remark: Rank condition necessary; otherwise P has no vertices! (Exercise.)

Proof: Let F min'l face of P.

- Face characterization $\Rightarrow \exists I$

$$F = F_I = \left\{ x : \begin{array}{l} a_i^T x = b_i \quad \forall i \in I \\ a_j^T x \le b_j \quad \forall j \notin I \end{array} \right\}$$

and adding any elt to I makes F_I empty.

- Consider two cases:

(a) Inequalities not needed:

$$F = \{x : a_i^T x = b_i \text{ for } i \in I\}.$$

☆ Claim: $\forall j \in I$,

$$a_j \in \operatorname{lin}(a_i : i \in I).$$

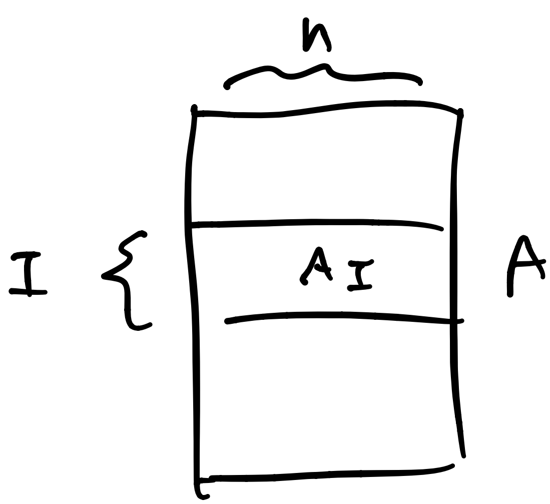
(else $a_i^T x$ for $i \in I$ doesn't determine $a_j^T x$, so

$a_j^T x = b_j + 1$ has solution

in F, contradicting $a_j^T x = b_j$)

- **Equivalently**: submatrix A_I w/ rows in I satisfies

$$\text{row}(A_I) = \text{row}(A);$$



hence $\text{rank}(A_I) = \text{rank}(A) = n$.

- **Thus**: $a_i^T x = b_i$ for $i \in I$ has **unique** soln, so F is single point, i.e. a vertex.

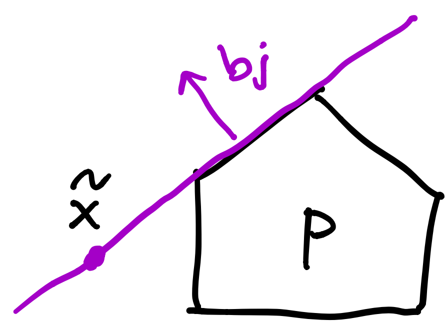
(b) Some inequality needed:

- $\exists\, j \notin I$ s.t. $\exists\, \tilde{x}$ w/

$$a_i^T \tilde{x} = b_i \quad i \in I$$

$$a_j^T \tilde{x} > b_j$$

(b/c $\tilde{x} \notin P$).



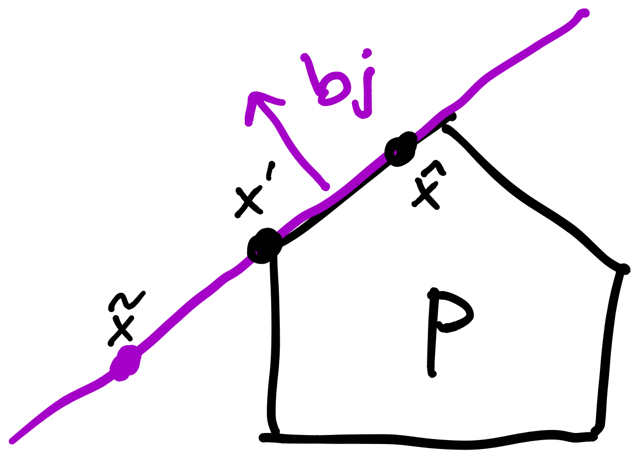
- F nontrivial $\Rightarrow \exists\, \hat{x} \in F$.

$\hat{x}$ satisfies $a_i^T \hat{x} = b_i$

$$a_j^T \hat{x} \le b_j.$$

- Consider convex combination

$$x' = \lambda \tilde{x} + (1-\lambda)\hat{x}.$$



- x' satisfies <u>one more equality</u>

(else can move further to x') $\square$

Finally we can show equiv b/w bounded polyhedra & convex hulls. (polytopes).

Recall: $P = \{Ax \leq b\}$ bounded

then $P = \text{conv}(\underbrace{\text{Xtreme pts. of } P}_{\overset{!!}{=} X})$.

Proof: Use TOTA.

- $X \subseteq P \Rightarrow \text{conv}(X) \subseteq P$.
- Assume for contradiction that $\text{conv}(X) \subsetneq P$.
- Let $\tilde{x} \in P \setminus \text{conv}(X)$.
- Then

$$\sum_{v \in X} \lambda_v v = \tilde{x}$$

$$\sum_{v \in X} \lambda_v = 1$$

$v \in X$

$\lambda_v \geq 0 \quad \forall v \in X$

has no solution.

- TOTA $\Rightarrow$

$$A : \begin{bmatrix} \text{v's} \\ 11 \cdots\cdots 11 \\ -I \end{bmatrix} \begin{bmatrix} | \\ \lambda \\ | \end{bmatrix} \underbrace{\begin{matrix} = \\ = \\ \leq \end{matrix}}_{\Delta} \underbrace{\begin{bmatrix} \tilde{x} \\ 1 \\ 0 \end{bmatrix}}_{b}$$

has no soln $\Longleftrightarrow$

$$A^T y = 0, \quad b^T y < 0, \quad y \sqsupseteq 0$$

has soln. i.e.

$A^T \qquad y$

$, \qquad ,$

$$\begin{bmatrix} \vdots & \vert & \\ -\,v^T\,- & 1 & -I \\ \vdots & \vert & \end{bmatrix}\begin{bmatrix} c \\ t \\ s \end{bmatrix} = 0 \qquad (*)$$

for $s \geq 0$, and $\underbrace{y^T b < 0}_{(**)}$.

<u>OR</u>

$(*)$ $t + c \cdot v \geq 0 \quad \forall v \in X$

$(**)$ $t + c \cdot \tilde{x} < 0$.

- P bounded $\Rightarrow$

  $\min\{c^T x : x \in P\} = z^* > -\infty$.

- Face induced by $c^T x \geq z^*$ nonempty, but contains no vertex.

(because *, ** ⇒ objective less on $\tilde{x}$ than any vertex.)

- Contradicts vertex minimality!

(it applies b/c rank $A = n$, else $\exists\, y \neq 0$ w $Ay = 0$, contradicting boundedness of P.

$\square$.