Lecture 16

Plan:

1) Matroid opt.
(see lec 15 notes)

2) Matroid polytopes

More preliminaries:

Rank function

- Analogous to rank of matrices
- rank function $r_M : 2^E \to \mathbb{N}$

of matroid $M = (E, \mathcal{I})$ is

$=$

$=$

•

Examples

• **linear matroid:**

• **partition matroid:** Recall

for $E =$

$\mathcal{I} =$

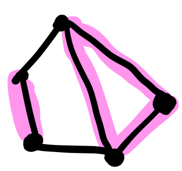
- **Graphic matroid**: M_G, $G=(V,E)$.

For

$K(V,F) :=$

e.g.

F



$r(F) =$

Properties of rank function

Let r be rank function of matroid.

(R1)

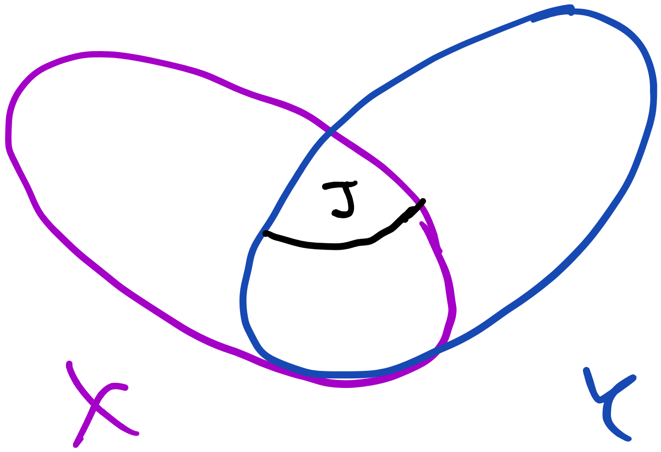
(R2) <u>monotonicity</u>:

(R3) <u>submodularity</u>:

<u>Proof of R3</u>: • Let $X, Y \subseteq E$.

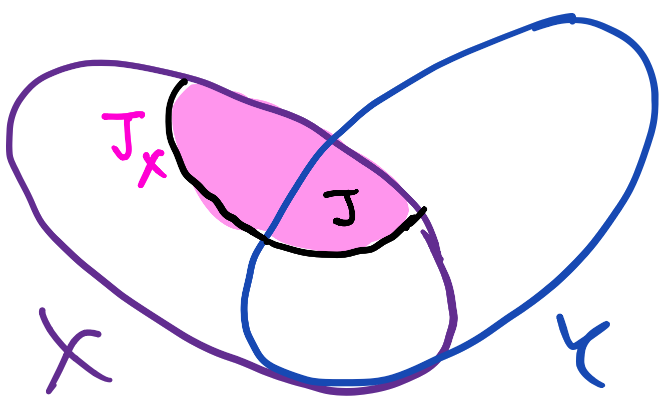
• We want to show

- Let J



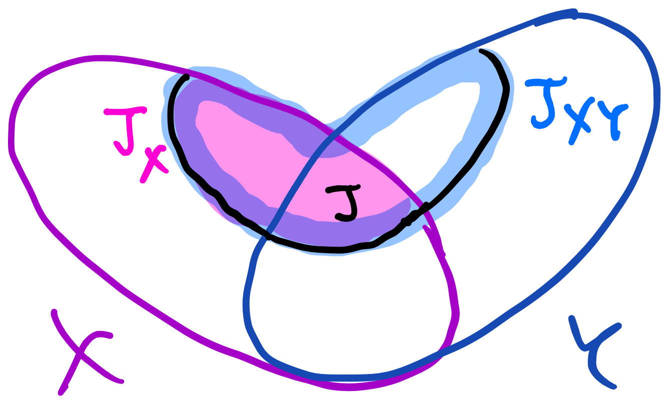
$\Rightarrow$

- Extend J to



$\Rightarrow$

• Extend J_X



$\Rightarrow$

• Note $\underset{\text{UI}}{X \cap Y} \subseteq \underset{\text{UI}}{X} \subseteq \underset{\text{UI}}{X \cup Y}$ ☆

by

i.e.

- submodularity $\iff$

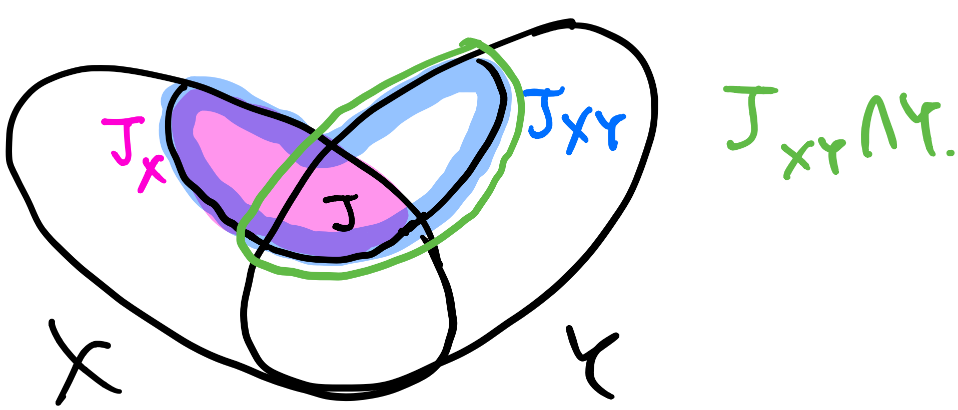
$r(X)+r(Y) \geq r(X\cap Y)+r(X\cup Y)$.

$\iff$

$\iff$

- To prove,

$\Rightarrow$



- **Claim:**

$$\boxed{|J_{XY} \cap Y| =}$$

Pf of Claim: $|J_{XY} \cap Y|$

$= \qquad\qquad +$

$= \qquad\qquad +$

$= \qquad\qquad +$

$=$ $+$

$=$ $+$. $\square$

Span:

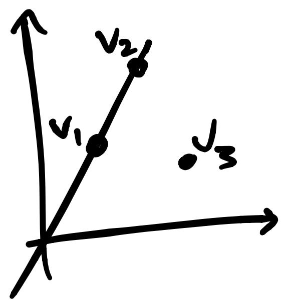
- Given $M=(E,\mathcal{I})$, **span** of $S\subseteq E$ is

$$\boxed{\text{span}(S) := \qquad\qquad\qquad\qquad}$$

i.e.

e.g. linear matroid, $v_1 \dots v_m \subseteq \mathbb{F}^n$:

$\text{span}(S) = \{ \qquad\qquad\qquad \}$



$\mathbb{R}^2$

$\text{span}(\{\,\})=$

- **Claim:** $r(S)=$

( )

Pf:
- Take J
- Suppose

( ) $\Rightarrow$

$\Rightarrow$

contradicts $\square$

- Say S is closed if ;

  A.K.A

Matroid polytope

- Let $M = (E, \mathcal{I})$ matroid.
- Let $X = \{ \}$.

  $= \{ \}$.

- the matroid polytope is

$$\boxed{P_M :=}$$

?

- some constraints :

Theorem: For r rank function of M, let

$P = \{$

$\}.$

Here

$\boxed{\text{Then}}$

Notes:

- We saw

  b/c

- Harder to show

Algorithmic proof:

-
- $\mathrm{conv}(X) \subseteq P \Rightarrow$

- Enough to show

(because

).

- What's the dual?
-

=

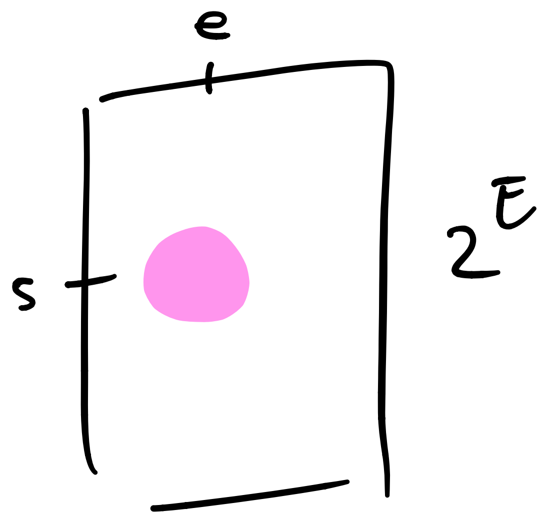
(primal)

(dual)

- **Our primal:**

$\max c^T x$

$A =$



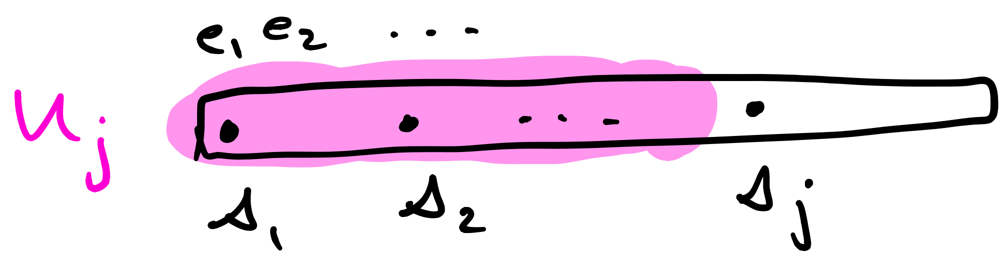
( ).

- **Dual:** min

- Thus we need

- Consider cost c.
- max cost indep set =
- Need
- For $j \leq k$,
- $u_j :=$

$=$



- Note

▷

▷

- For $j = 1 \dots k$, set

$$y_{u_j} :=$$

where

- Set

- **Claim 1:** y dual feasible.

Pf: ▷

▷

- **Claim 2:** $\sum_{S \subseteq E} r(S) y_S = c(S_k)$.

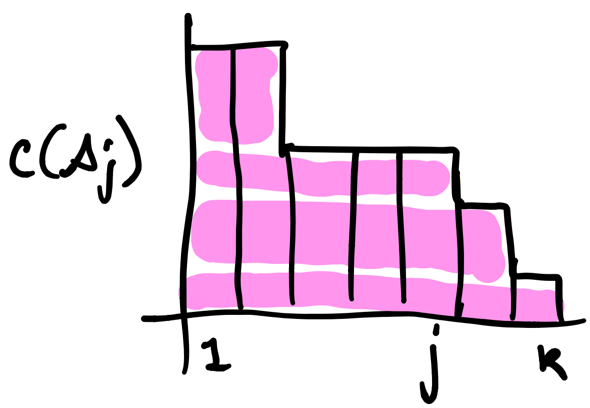
Pf: $\sum_{S \subseteq E} r(S) y_S =$

$=$

$=$

$=$ $\square$

Intuition: $c(S_k)$ is <u>area</u>



Vertex proof