18.453 Lecture 1

Lecture Plan:
- Intros
- Logistics
- ABOUT THE TOPIC
- Breakout rooms to work on examples

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INTROS:

ABOUT ME:
- COLE FRANKS
  - PLS call me COLE
- Postdoc in applied math
- study theoretical computer science

ABOUT YOU: Pls say your
- name
- Major (in english not numbers)
- year
- Draw yourself in explain.mit.edu main room

Logistics:

- lectures, recorded but attend. encourage.
- one OH W 11-12:30, another TBA.
- Gi · weekly pset. 30%
  1 quiz. 25%
  1 final 35.%
- pset in groups: non-mandatory write-ups must be done individually.

WHAT IS COMBINATORIAL OPTIMIZATION?

find $\max_{x \in X} f(x)$

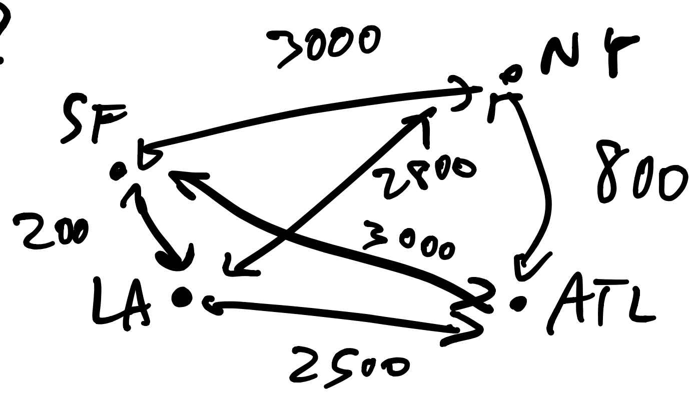
Some set

Some function

- unlike calculus, X usually <u>finite</u> e.g. $\{0,1\}^n$ or <u>discrete</u> e.g. $\mathbb{Z}$.
- However, even when X finite, too hard to check all elts: $|\{0,1\}^n| = 2^n$ ....

Famous example: Travelling Salesman problem (TSP)

— Given pairwise distances between n cities, what's shortest route to visit them all?



possible trips: $n! \gg 2^n$

Thus, we need better techniques than

simply trying all possibilities.

- Frequently this is just impossible.
  e.g. TSP is "NP hard".

- We still get lucky for many combinatorial structures!
  - Matchings
  - flows/cuts
  - TREES & MATROIDS

• SUBMODULARITY

• <u>Main tool</u>: linear programming (LP)

e.g.

$$\boxed{\begin{aligned} &\max \quad 2x + 3y \\ &\text{subject to} \quad x + y \le 2 \\ &\qquad\qquad\quad x - y \le 4 \\ &\qquad\qquad\quad x \ge 0 \\ &\qquad\qquad\quad y \ge 0. \end{aligned}}$$

• Even when we aren't lucky, LP and other tools can help approximate (e.g. TSP).

Example: Matchings

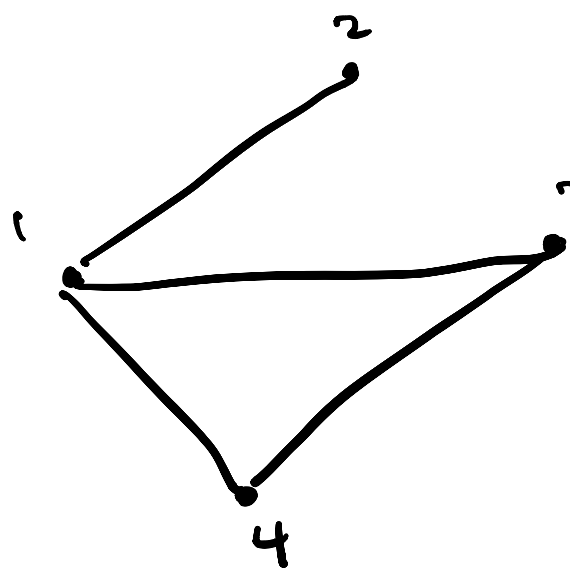
- Graph: $G = (V, E)$

$$G = (\underset{\substack{\uparrow \\ \text{vertices}}}{V}, \underset{\substack{\uparrow \\ \text{edges}}}{E})$$

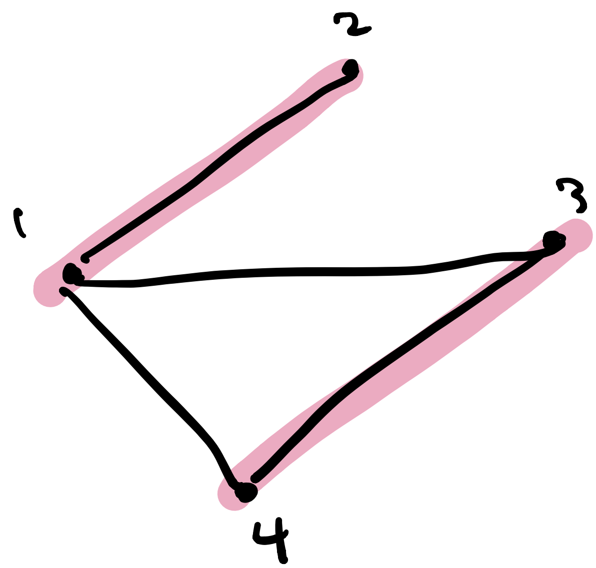
e.g. $V = \{1, 2, 3, 4\}$

$E = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{1, 4\}\}$

Depicted

$G =$ 

- <u>matching</u>: $M \subseteq E$ disjoint set of edges.

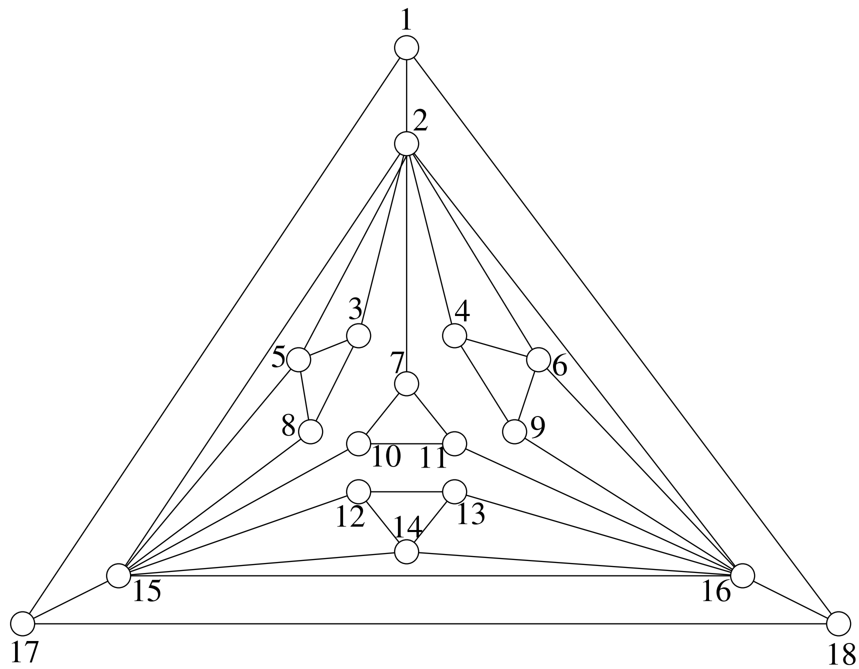


- <u>perfect matching</u>: M includes all vertices (G above has <u>perfect</u> matching).

ACTIVITY 1:

Matching illustration

A matching M in a graph $G = (V, E)$ is a set of edges with no endpoints in common. In the graph below, find a matching of maximum size.



How can you convince someone that the matching you found is indeed of maximum cardinality?

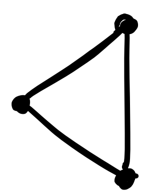
Do this w/ your breakout room in explain.mit.edu

Key Theme: Duality

loosely: $\boxed{\text{The SIMPLE obstructions are the ONLY obstructions}}$

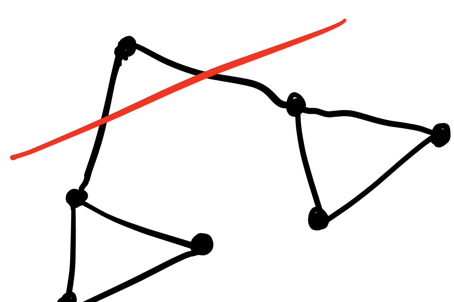
- What OBSTRUCTS matchings?

  - parity



  - parity + cuts

at least 1 vertex unmatched.



This is the only kind of obstruction!

Tutte's theorem: If $U \subseteq V$, let

$$o(U) = \{\#\text{ odd connected components if } U \text{ is removed}\}$$

Then

$$\max \left| \begin{smallmatrix}\text{perfect}\\ \text{matching in } G\end{smallmatrix} \right| = \min_{U \subseteq V} |V| + |U| - o(U).$$

Eventually we'll show how duality leads to efficient algs for matching!

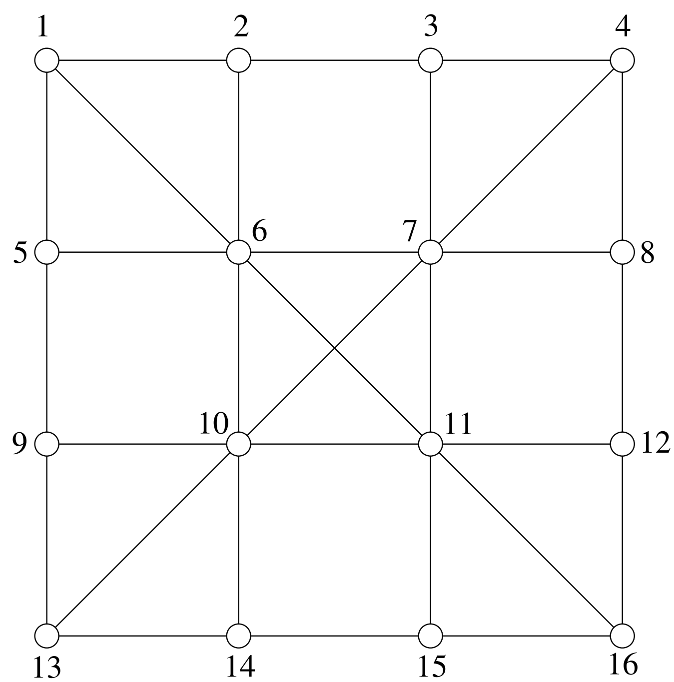
Activity No. 2:

Spanning Tree Game

A spanning tree T in a graph $G = (V, E)$ is a set of edges without any cycles that connect all vertices together. The spanning tree game is a 2-player game. Each player in turn selects an edge. Player 1 starts by deleting an edge, and then player 2 fixes an edge (which has not been deleted yet); an edge fixed cannot be deleted later on by the other player. Player 2 wins if he succeeds in constructing a spanning tree of the graph; otherwise, player 1 wins.

* Which graphs have a winning strategy for player 1? Which graphs have a winning strategy for player 2?

For this graph with 16 vertices and 30 edges, which player has a winning strategy?



- Play on this graph w/ your group.
- Find examples where P1 wins & where P2 wins.
- Try to answer *.

CASE 1: $\exists$ 2 disjoint spanning trees $A, B \in G$.

Claim: P1 wins!

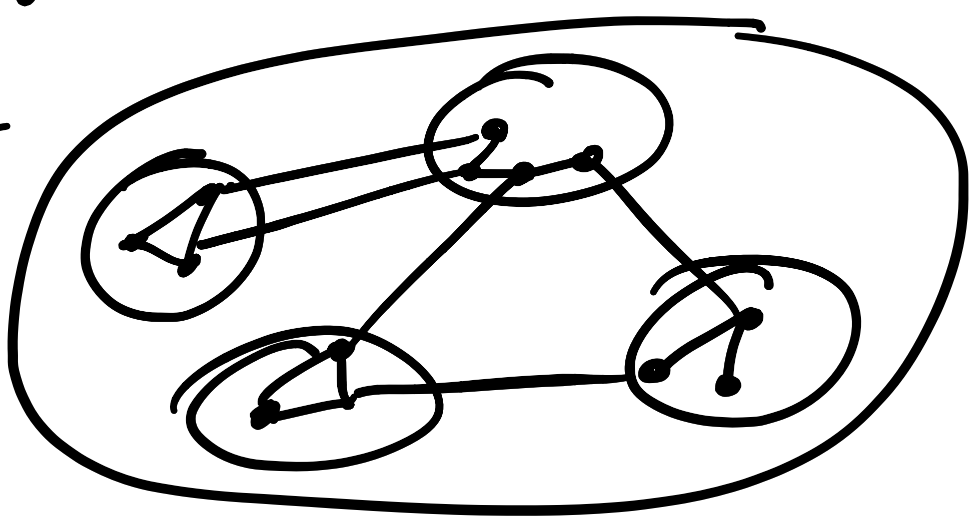
- when P1 cuts from A, P2 adds edge from B to A so A is still spanning tree. "Exchange property"
- A, B will be disjoint except fixed edges.
- in the end, $A = B$ is spanning tree remaining.

CASE 2: no 2 disjoint spanning trees.

claim: P2 wins!

Duality: simple obstruction for 2 disjt. spanning trees.

G



4 parts, only

$2\cdot(4-1)-1 = 5$

edges between

them. But a

spanning tree would

have $\geq p-1$ edges

between p parts –

& 2 spanning trees

would have $2(p-1)$

edges!

Thus, partition into p parts w/ $< 2(p-1)$ edges between the parts is an obstruction to 2 disjoint spanning trees.

Thm (Lehman): This is the <u>ONLY</u> obstruction - duality!

Back to Case 2:

Show P2 wins:

- no 2 disjt spanning trees, $\Rightarrow \exists$ partition into p parts w/ $< 2(p-1)$ edges between parts.
- P2 can delete $\geq \frac{1}{2}$ of these - not enough left to connect up the parts. $\square$

Algorithmically? How to find the trees/partition?

Spanning trees example of matroid;
(set system w/ "exchange property";
generalizes set of bases of vector space)

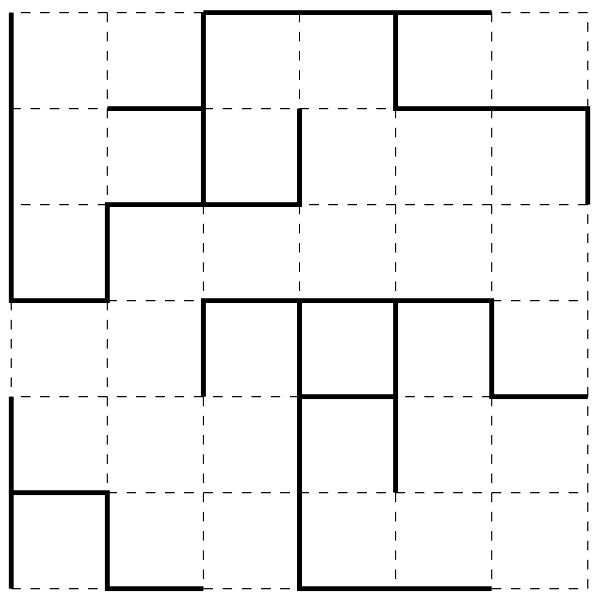
2 disjt. spanning trees example
of matroid intersection
which we will solve later in the course

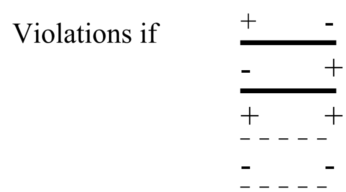
Activity 3:

Massachusetts Institute of Technology
18.453: Combinatorial Optimization
Instructor: Cole Franks **Notes:** Michel Goemans and Zeb Brady
February 15, 2021

Spin Glass

Consider the 7×7 grid drawn below, where each edge has been made thick (solid) or thin (dashed). Given an assigment of signs (+ or -) to the vertices of this grid, a thick edge is *violated* if the endpoints have two different signs. A thin edge is violated if the endpoints have identical signs. The goal is to find an assignment of signs which minimize the total number of violations in the grid.





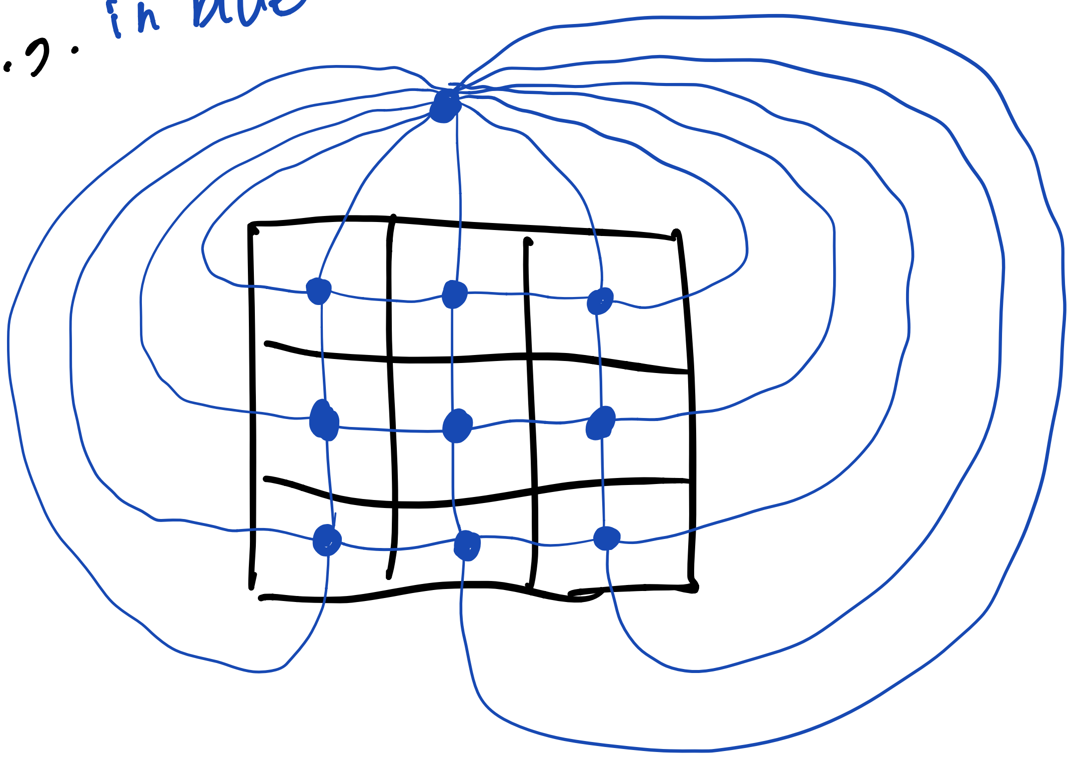
As you'll probably realize, although finding a "good" assignment of signs might not be that difficult, providing a proof that your solution is optimal is in fact much more challenging (and a short proof exists for any instance).

In your groups, try to find the best assignment you can, and try to see what LOWER bounds you can prove.

Note: pink edges form graph. G
degree of frustrated placquettes
in G is ≥ 1.

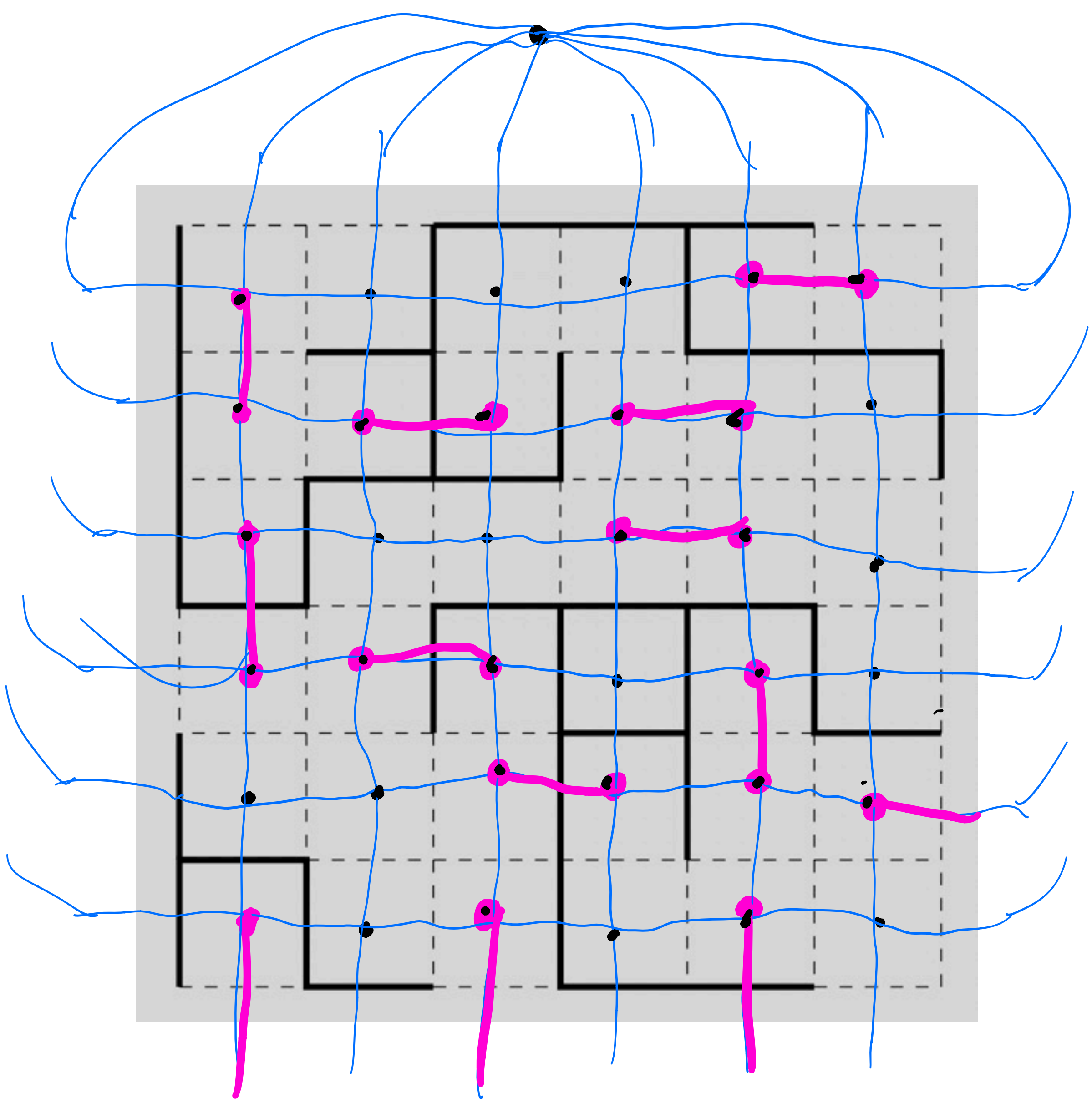
Make Dual graph:
each square is vertex, edges b/w
neighboring squares, one vertex
for outside.

e.g. in blue



TURNS OUT: min # violations is just min # edges of graph

- w/ odd degree on frustrated plaquettes,
- even degree on unfrustrated.



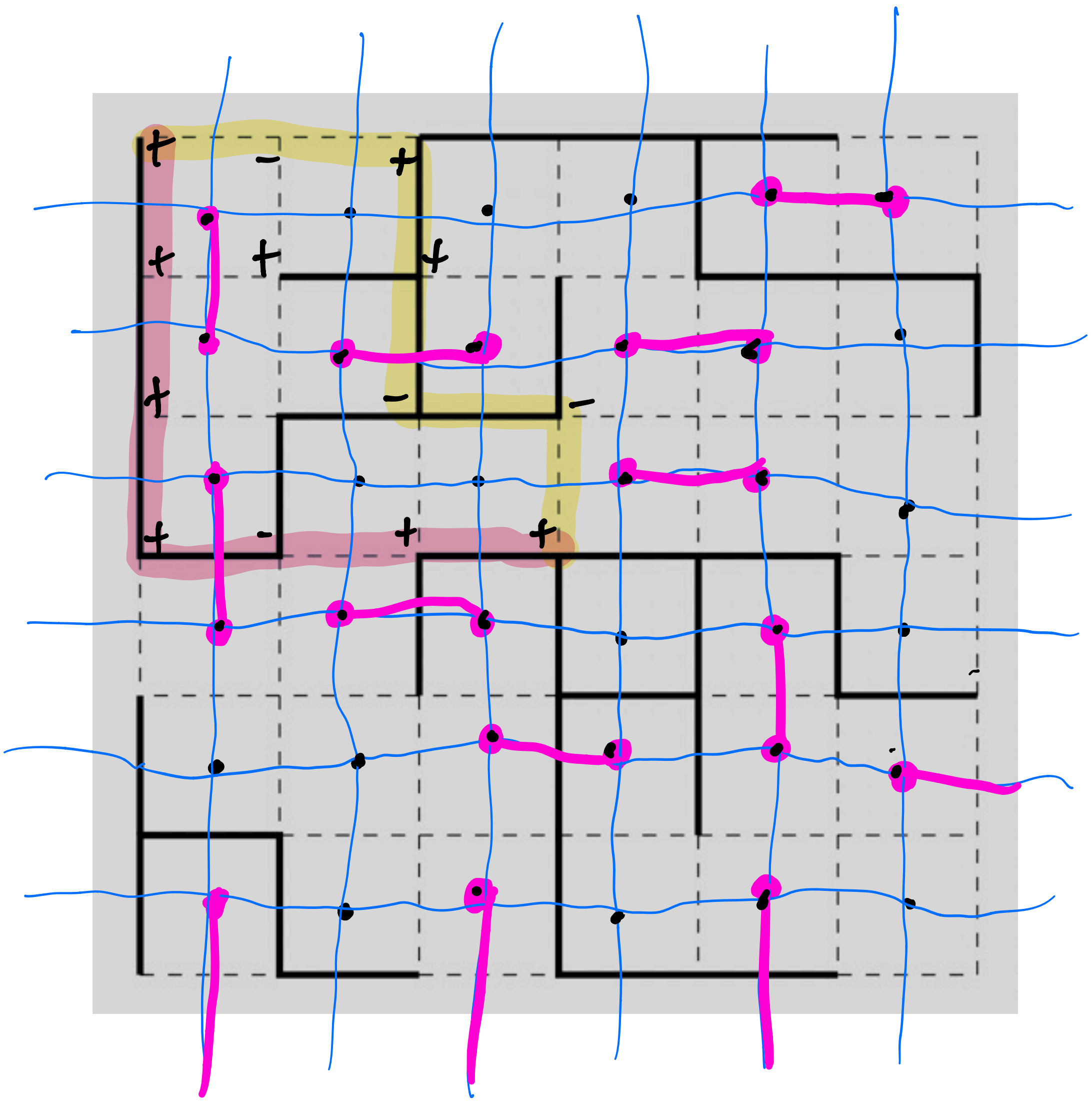
Can assume it's just a union of edge disjoint chains;

Cost is min cost matching in **weighted** graph G with

$V =$ Frusted placquettes

$w(u,v) =$ distance between u, v in <u>dual</u> graph

How to get signs?



how do we know consistent?

b/c of parity conditions!