Lecture 9

Plan: Finish polyhedra

$P = \{x : Ax \le b\}$

$\begin{bmatrix} \vdots \\ - \; a_i^T \; - \\ \vdots \end{bmatrix}$

1) nonredundancy of facets
2) cones near vertices

Polyhedra Cont.

Recall: Nonredundant = Facets.

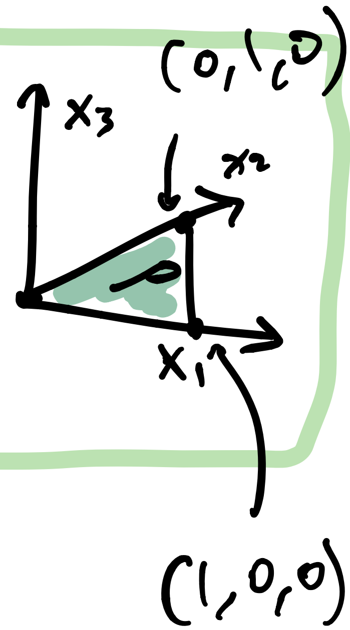
- inequality $a_i^T \le b_i$ redundant if P unchanged when it's removed.
- $I_= := \{i : a_i^T x = b_i \ \forall x \in P\}$ "equalities"
- $\{i : \exists x \in P \ a_i^T x < b_i\}$

- $I_< := \{ i : \exists x \in P \;\; a_i^T x < b_i \}$.

"real inequalities"

e.g.

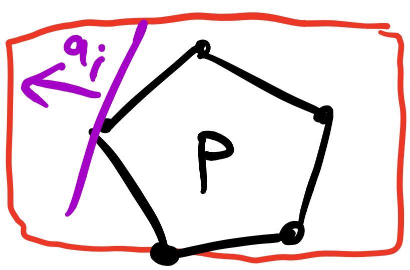
$$P = \left\{ x : \begin{array}{l} \left.\begin{array}{r} x_1 + x_2 \le 1 \\ -x_1 \le 0 \\ -x_2 \le 0 \end{array}\right\} I_< \\ \left.\begin{array}{r} x_3 \le 0 \\ -x_3 \ge 0 \end{array}\right\} I_= \end{array} \right.$$



THEN:

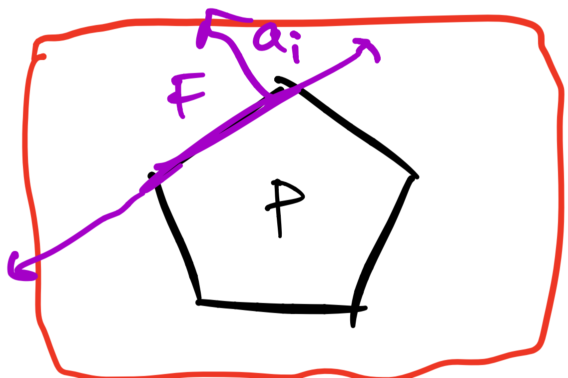
$\boxed{\text{Not facet} \Rightarrow \text{redund.}}$ face $a_i^T x = b_i$ for $i \in I_<$ not facet

$\Longrightarrow a_i^T x \le b_i$ is redundant.



($i \in I_<$ necessary, $x_3 \le 0$, $-x_3 \le 0$ don't give facets but not redundant).

Facet ⇒ non redund. F is facet of P, ⇒ $\exists i \in I_{<}$ s.t. F from $a_i^T x = b_i$.



TAKE-HOME: in minimal description of P, need

- lin. independent set of equalities ($I_=$)
- one inequality per facet ($I_<$).

Proof We only prove ⇒.

- Suppose $a_i^T x \le b_i$ not redundant * want to show corresp. face F_i facet.

- We'll do this by showing

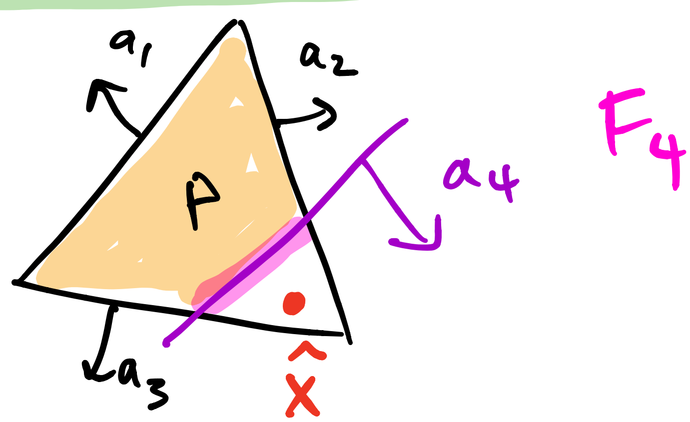
$$\boxed{\dim(F_i) = \dim(P) - 1.}$$

$\boxed{\dim(F_i) \geq \dim(P) - 1}$:

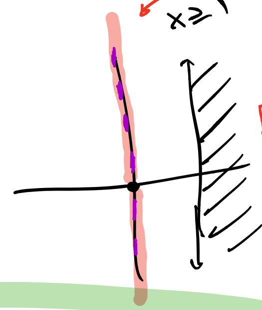
- $* \Rightarrow$ Is $\hat{x}$ (not in P) s.t.

but $\boxed{\begin{array}{l} a_i^T \hat{x} > b_i \\ a_j^T \hat{x} \leq b_j \quad \forall j \neq i \end{array}}$

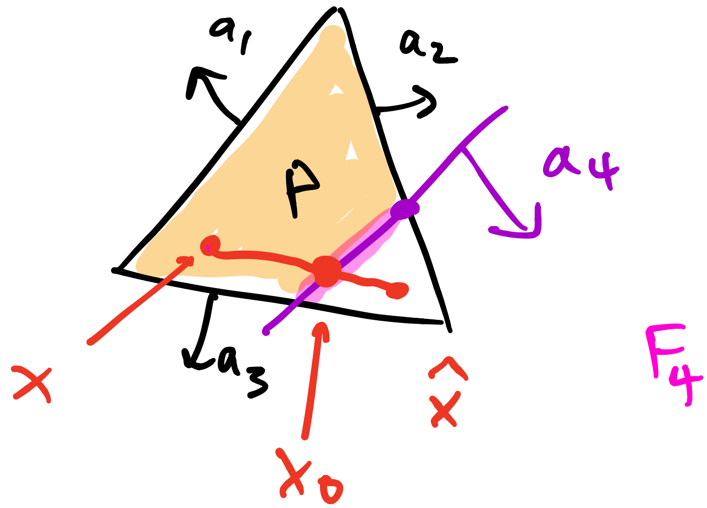
e.g. $i=4$



- $\forall x \in P$, line segment $\hat{x} \to x$ has unique $x_0 \in F_i$.



e.g. i=4

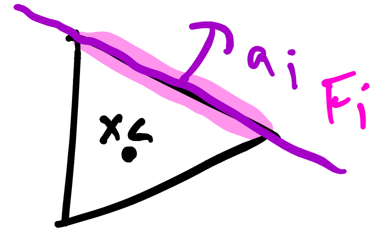


$\Rightarrow$ any point $x \in P$ contained in

$$\boxed{\mathrm{aff}(\{\hat{x}\} \cup F_i)}$$

- $\Rightarrow P \subseteq \mathrm{aff}(\{x\} \cup F_i) \Rightarrow \dim(P) \leq \dim(F_i) + 1$

$\boxed{\dim(F) \neq \dim(P):}$



- Recall $i \in I_<$.

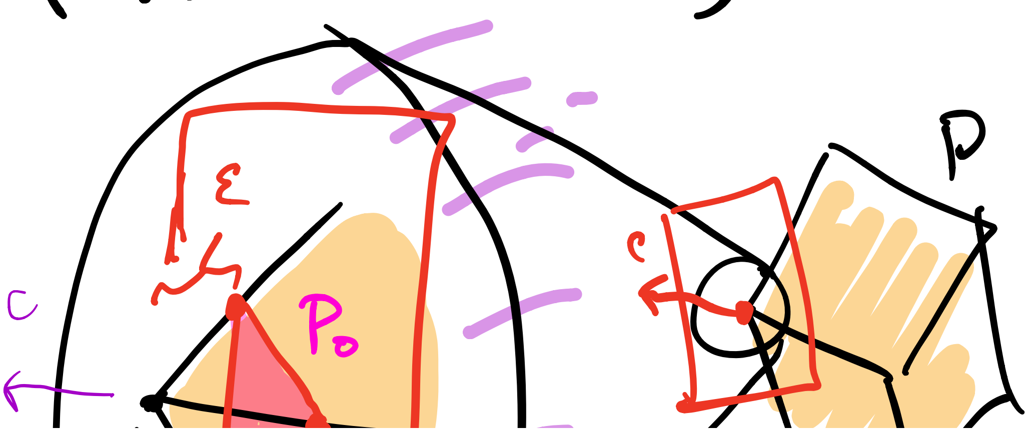
$\Rightarrow \exists\, x_< \in P$ with $a_i^T x < b_i$

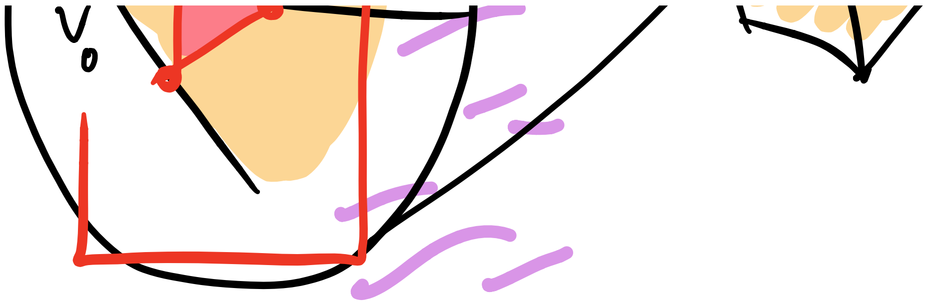
- $x_<$ can't be in aff(F_i). $\square$

b/c $F_i \subseteq \{x : a_i^T x = b_i\}$

Recall: Near vertex $=$ Cone(polytope)

(N.V.C. Theorem)





Let v_0 vertex of P from valid inequality $c^T x \le m$.

Let ε be such that $\boxed{c^T v' \le m - \varepsilon}$ for all other vertices v'.

Then

$$P_0 = \{ x \in P : c^T x = m - \varepsilon \}$$

is a polytope & is bijection

$$\{P_0\text{'s dim } k \text{ faces}\} \overset{\text{inclusion}}{\longleftrightarrow}$$

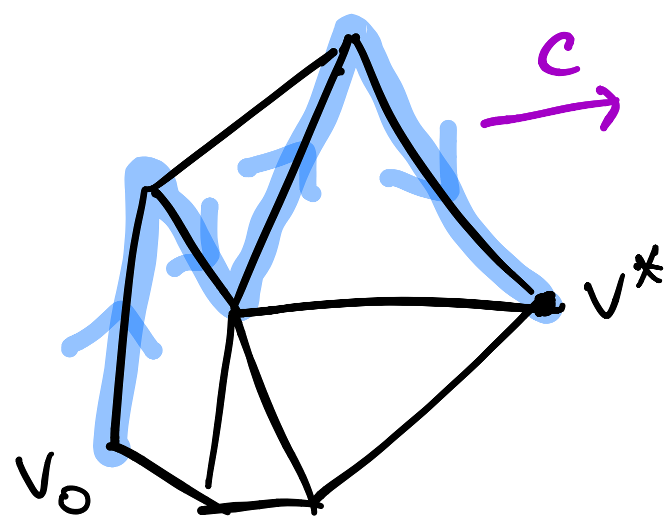
$\{P$'s dim $k+1$ faces

{P's ali...
containing v_0 }

Corollary: Graph connected

Graph of vertices & edges of polyhedron P is always connected.

In particular: if v^* max. of $c^T x$ over P, v_0 vertex, $\exists\ v_0 \to v^*$ path which doesn't decrease objective.



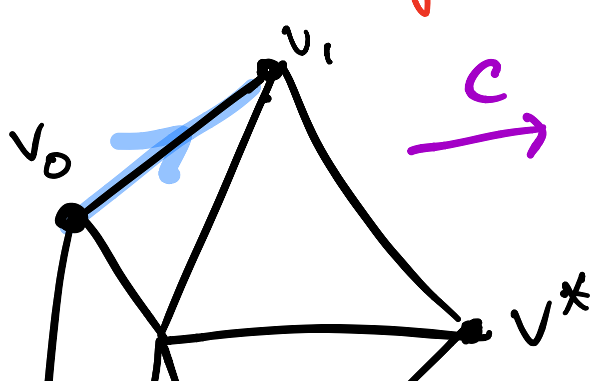
Proof of Corollary:

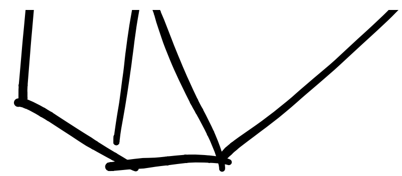
- Suppose v^* unique max of $c^T x$ over P.

- Enough to show that $\forall$ vertices $v_0 \neq v^*$, $\exists$ edge to vertex v_1 w/

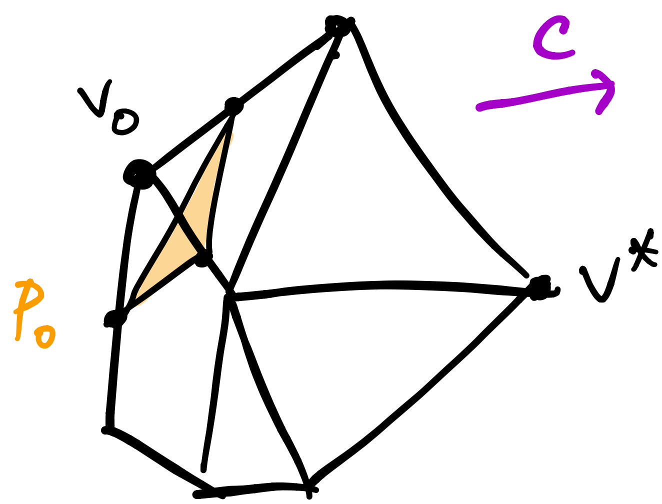
$$\boxed{c^T v_1 > c^T v_0}$$

(by finiteness of # vertices).

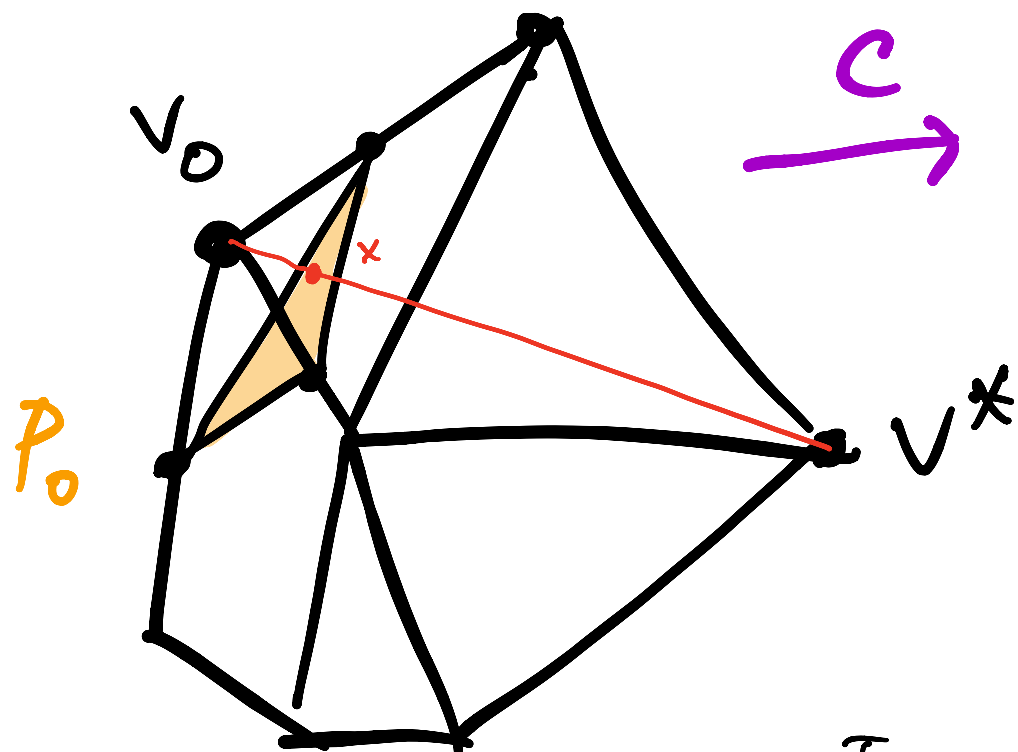




- Let P_0 be polytope from last theorem.



- Let x be intersection of P_0 and segment joining v_0, v^*.



- note that $c^T v_0 < c^T x$.

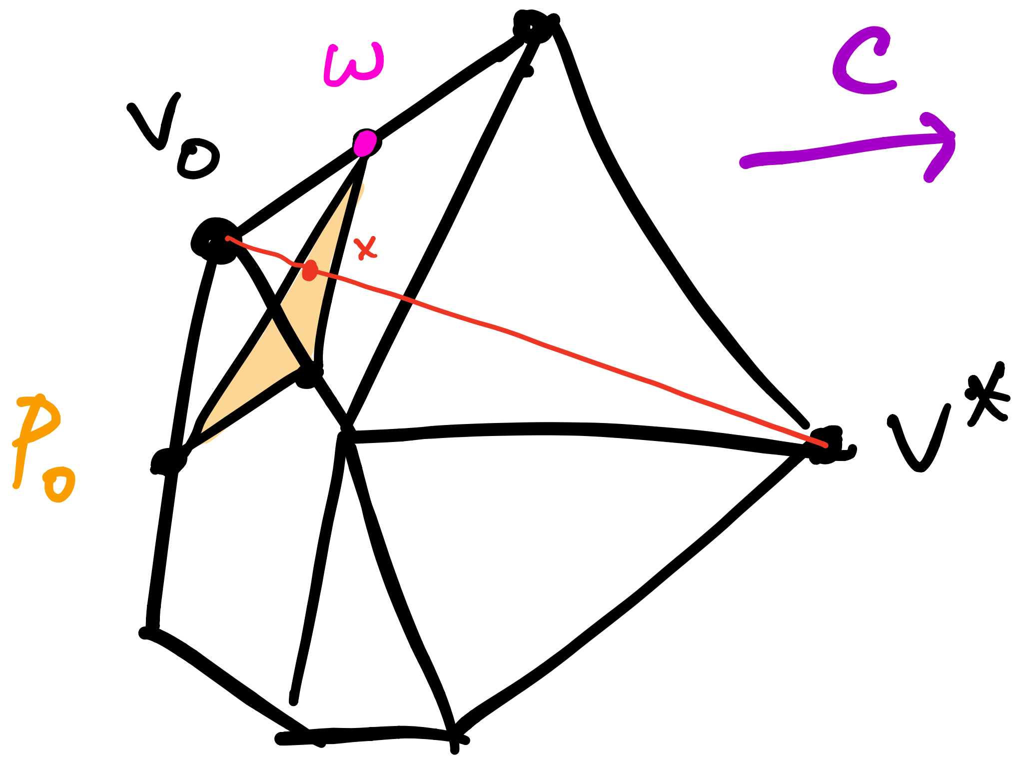
  ($c^T y$ incr. along the segment.).

- b/c P_0 polytope,

  $$\boxed{P_0 = \mathrm{conv}(\text{vertices of } P_0).}$$

  $\Rightarrow \exists$ vertex w of P_0 with

  $$c^T v_0 < c^T x \leq c^T w$$



WHY? <u>Simple but powerful principle:</u>

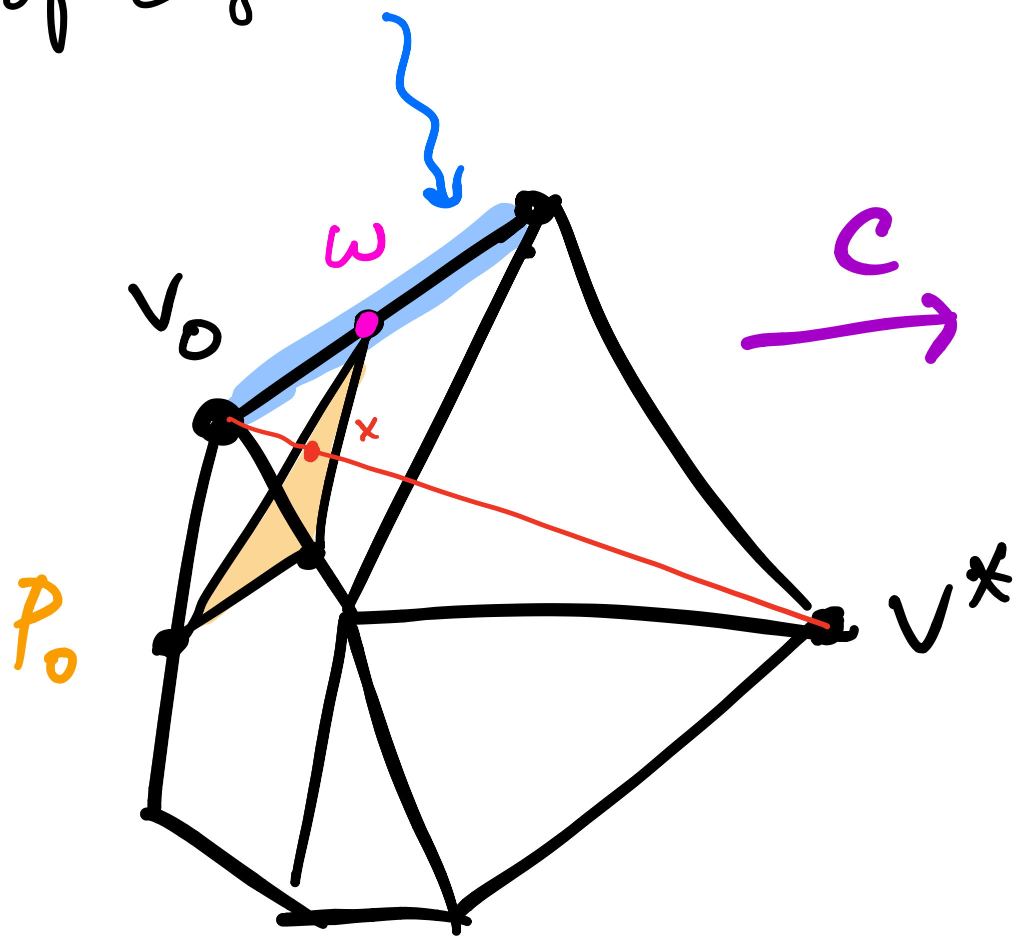
$$x = \sum_{\substack{\omega \\ \text{vertices} \\ \text{of } P_0}} \lambda_\omega \omega, \quad \sum \lambda_\omega = 1$$

$$\Rightarrow c^T x = \sum_{\substack{\omega \\ \text{vertices of } P_0}} \lambda_\omega c^T \omega \qquad \text{"weighted average"}$$

$\Rightarrow$ <u>some</u> ω with $c^T\omega \geq c^T x$.

- ω is intersection

- By bijection,
of edge e with P_0.



- e must be bounded
(because $c^T y$ ~~is~~ increasing along $e \Rightarrow e$ ends in some vertex v_1).

- Thus ends at some vertex v_1,

$$\boxed{c^T v_1 > c^T v_0} \quad \square.$$

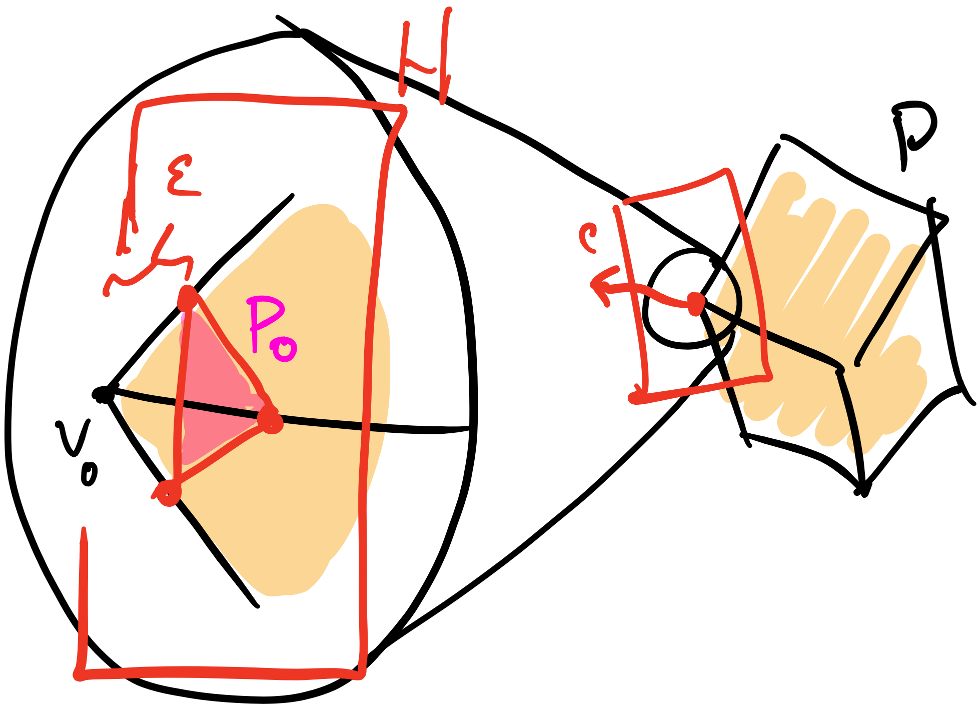
Proof of N.V.C:

Recall: if vertex v_0 given by $c^T x = M$,

then

$$P_0 = P \cap \{x : c^T x = M - \varepsilon\}$$

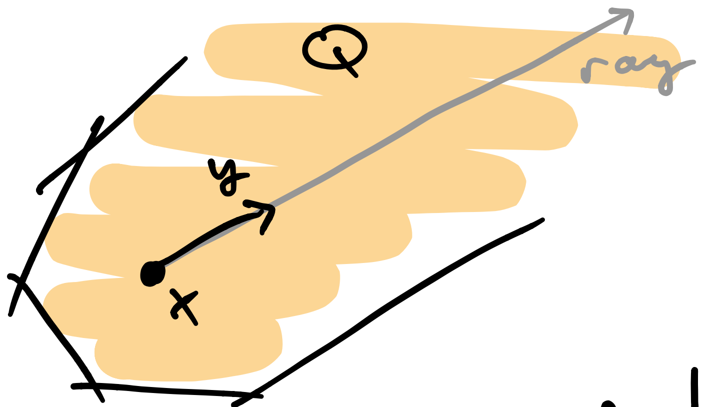
for small ε. H



① P_0 bounded.

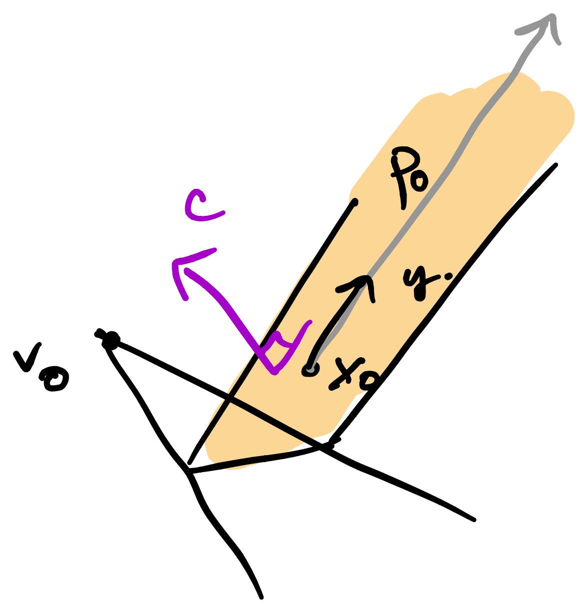
• **Exercise:** If Q unbounded polyhedron, $x \in Q$, then Q contains ray from x:

$\{x + \alpha y : \alpha \geq 0\}$.

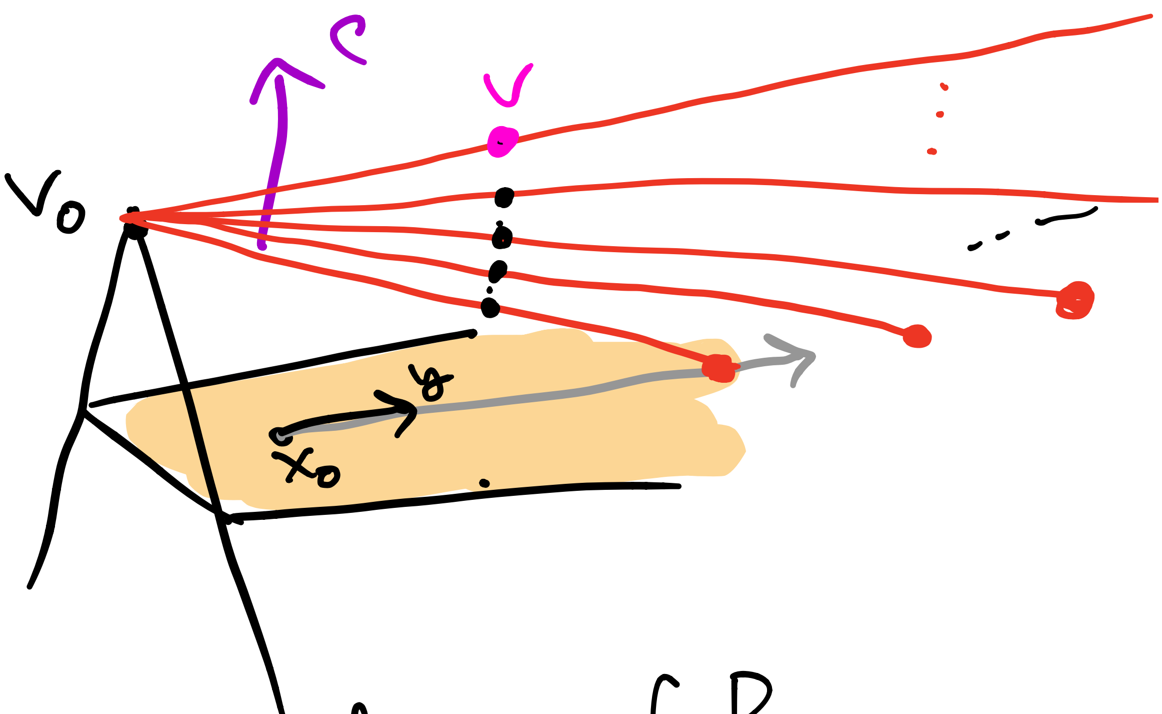


- Suppose P_0 unbounded, let $x_0 \in P_0$.

$\Rightarrow P_0$ contains ray $\{x_0 + \alpha y : \alpha \geq 0\}$



- as $P_0 \subseteq H = x_0 + C^\perp$, $\boxed{y \in C^\perp}$
- use ray to construct another minimizer v contradicting uniqueness:



- By closedness of P

$$\{v_0 + \alpha y : \alpha > 0\} \subseteq P$$

but $c'x$ constant along it.

Contradicts uniqueness of v_0.

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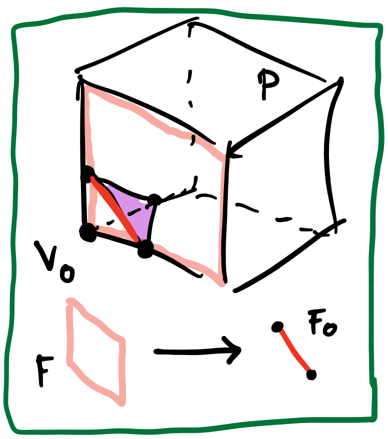
2. The bijection:



$$\boxed{\text{face } F \ni v_0 \text{ of } P} \longrightarrow \boxed{F_0 := F \cap H \cong F \cap P_0}$$

a! **onto:** If F_0 face of P_0

$\Rightarrow \exists F$ face of P, $F \in v_0$ s.t.

$$\boxed{F_0 = F \cap H.}$$

- Let F_0 nonempty face of P_0.

$$F_0 = \begin{cases} a_i^T x = b_i & i \in I \\ c^T x = m - \epsilon & \\ a_j^T x \le b_j & j \notin I \end{cases}$$

Let

$$F = \begin{cases} a_i^T x = b_i & i \in I \\ a_j^T x \le b_j & j \notin I. \end{cases}$$

(remove middle equality)

- F is a face of P by face char;
just need $v_0 \in F$.

ε.

- Recall that v_0 was only vertex v with
  $$\boxed{c^T v \geq m - \epsilon}$$

- But $c^T x$ bounded above on F
  $\Rightarrow$ reaches some max $\geq m - \epsilon$ at vertex V of F.
- V_0 only such vertex $\Rightarrow V_0 \in F$.

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⑥ Dimensions

(implies $\boxed{\text{one to one}}$ as corollary).

- want to show
  $$\boxed{\dim F_0 = \dim F - 1}.$$

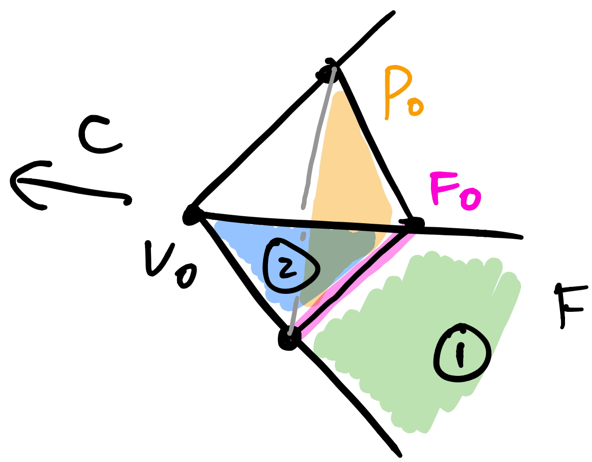
- Enough to show

$\subseteq$

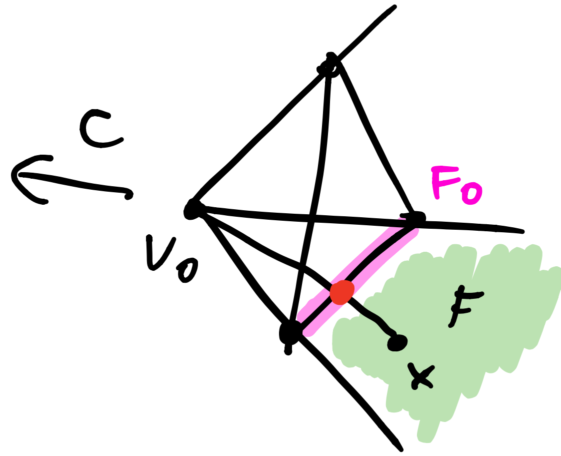
$\Rightarrow$ $\geqslant$

$($ $).$

Cases: 1 $\boxed{}$, 2 $\boxed{}$



1 If $c^T x \leq m - \epsilon$, segment $x \to v_0$ clearly hits F_0, thus $x \in$ $\boxed{}$ ✓



② Else, x is in polyhedron

$F' = F \cap \{ \qquad \}$.

- F' is bounded $(\qquad)$.

$\Rightarrow F' = \text{conv}(\qquad)$

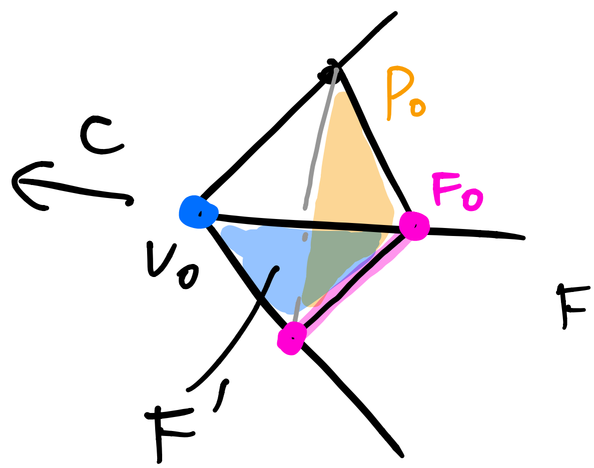
- vertices of F' are all either

a)

b)

(

).



• $\Rightarrow F' \subseteq \mathrm{conv}($ $).$

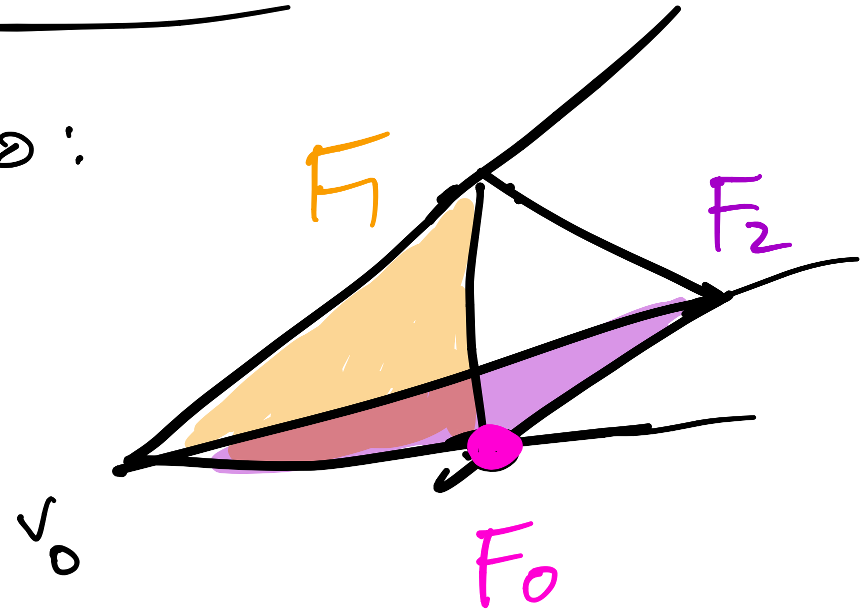
$\subseteq$

$\boxed{\text{one-to-one:}}$ can two

distinct faces

F_1, F_2 contain v_0.

No:



Say $\dim F_1 \subsetneq F_2$.

$\boxed{}$ is another facet of dimension $< \dim F_2$.

$F_0 \subsetneq$

$\Rightarrow$

contradicts $\square$