Solutions to Problem Set 2 (do not distribute)

2-2 Let $G = (V, E)$ be any graph. Given a set $S \subseteq V$, suppose that there exists a matching M covering S (i.e. S is a subset of the matched vertices in M). Prove that there exists a *maximum* matching M^* covering S as well.

Suppose for contradiction that M' has maximum size among matchings that cover S, but that M' is not a maximum matching. Then M' has an augmenting path P. Then $M'\Delta P$ is a larger matching, so to finish the contradiction we just need to show that it covers S. Vertices of S which are not on the path P will still be covered by $M'\Delta P$ since they are covered by edges of M' which are also in $M'\Delta P$. Vertices of S which are on the path P must necessarily be in the interior of the path P (since the endpoints of P are exposed by the definition of an augmenting path), and each vertex in the interior of P is contained in one edge of M and one edge of P which is not contined of M, so vertices in the interior of the path P will be covered by $M'\Delta P$ as well.

2-6 Show that any 3-regular 2-edge-connected graph $G = (V, E)$ (not necessarily bipartite) has a perfect matching. (A 2-edge-connected graph has at least 2 edges in every cutset; a cutset being the edges between S and $V \setminus S$ for some vertex set S.)

We will use the Tutte-Berge formula. Let $U \subseteq V$, and let W be any connected component of odd size of $G \setminus U$. Because G is 3-regular, there is an odd number of edges between W and U (this follows by just counting the edges incident to a vertex of W, and observing that the edges inside W will be counted twice). Moreover, those edges are a cutset. Thus, that cutset has at least 3 edges. Because this holds for every connected component of odd size, the 3-regularity of G implies that $|U| \geq o(G \setminus U)$. In other words, by the Tutte-Berge formula, $\max_M |M| = |V|/2$.

2-7 A graph $G = (V, E)$ is said to be factor-critical if, for all $v \in V$, we have that $G \setminus \{v\}$ contains a perfect matching. In parts (a) and (b) below, G is a factor-critical graph.

1. Let U be any minimizer in the Tutte-Berge formula for G. Prove that $U = \emptyset$.
2. Deduce that when Edmonds algorithm terminates the final graph (obtained from G by shrinking blossoms) must be a single vertex.
3. Given a graph $H = (V, E)$, an ear is a path $v_0 - v_1 - v_2 - \cdots - v_k$ whose endpoints (v_0 and v_k) are in V and whose internal vertices (v_i for $1 \leq i \leq k - 1$) are not in V. We allow that v_0 be equal to v_k, in which case the path would reduce to a cycle. Adding the ear to H creates a new graph on $V \cup \{v_1, \cdots, v_{k-1}\}$. The trivial case when $k = 1$ (a 'trivial' ear) simply means adding an edge to H. An ear is called odd if k is odd, and even otherwise; for example, a trivial ear is odd.

(a) Let G be a graph that can be constructed by starting from an odd cycle and repeatedly adding odd ears. Prove that G is factor-critical.

(b) Prove the converse that any factor-critical graph can be built by starting from an odd cycle and repeatedly adding odd ears.

1. Let U be a minimizer for Tutte-Berge formula. We know that $|U| = o(G \setminus U) - 1$ since the size of maximum matching is $\frac{|V|-1}{2}$. Let $v \in V(G)$ and let M_v be a near-perfect matching which exposes v. Each odd component O of $G \setminus U$ has a vertex that is unmatched inside O but might be matched with a vertex from U. Since $|U| < o(G \setminus U)$, there is an odd component O with an exposed vertex. The only exposed vertex is v, so v must lie in one of the odd connected components. This is true for all v, so $U = \emptyset$.

2. Recall the proof for the correctness of Edmonds algorithm in the note. We showed that when Edmonds algorithm terminates, then in current graph $U = \textsc{Odd}$ is the minimizer of Tutte-Berge formula. If the current graph is factor-critical, then U must be empty and this is only possible when the current graph is a single vertex. Hence, for factor-critical graph which is not a single vertex, Edmonds algorithm will always find a flower.

   We claim that for a factor-critical graph G, if B is a blossom with respect to the nearly-perfect matching M, then G/B is again factor-critical. It implies that the final graph must be a single vertex.

   Let b be the vertex of G/B for shrunken cycle B. Let v be the missing vertex of M. We may assume that $v \in B$, since we can take $M' = M \triangle P$ where P is the stem of the flower. Note that $|M \cap E(B)| = \frac{|B|-1}{2}$, i.e., every vertex in B are matched inside B except v.

   Let $u \in V(G/B)$. We want to show that there is a nearly-perfect matching of G/B missing u. If $u = b$, then M/B is a nearly-perfect matching of G/B missing u. If $u \neq b$, then there is a nearly-perfect matching M_u of G missing u. So we can find an even M_v-alternating path connecting u and v by taking $M_u \triangle M_v$. Let s be the first vertex in B from u. Let P be the subpath connecting u and s. This subpath is still even, since every vertex in $B \setminus \{v\}$ is matched inside B. Hence, $(M/B) \triangle P$ is a nearly-perfect matching of G/B missing u.

3. (a) Use induction on the number of ears. If G is just an odd cycle, then G is factor-critical. Assume that $G = (V, E)$ is factor-critical, and $G' = (V', E')$ be a graph obtained by adding an odd ear $v_0 - v_1 - \ldots - v_k$ to G.

   Let $v \in V'$. If $v \in V$, then there is a matching M of G which covers $V \setminus \{v\}$. So $M \cup \{v_1v_2, v_3v_4, \ldots, v_{k-2}v_{k-1}\}$ covers $V' \setminus \{v\}$. Otherwise, $v = v_i$ for some i with $1 \leq i \leq k-1$. If i is even, then let M be a matching of G which covers $V \setminus \{v_0\}$ and let $M' = M \cup \{v_0v_1, \ldots, v_{i-2}v_{i-1}, v_{i+1}v_{i+2}, \ldots, v_{k-2}v_{k-1}\}$. If i is odd, then let M be a matching of G which covers $V \setminus \{v_k\}$ and let $M' = M \cup \{v_1v_2, \ldots, v_{i-2}v_{i-1}, v_{i+1}v_{i+2}, \ldots, v_{k-1}v_k\}$. In either cases, M' covers $V' \setminus \{v\}$. So G' is factor-critical.

(b) Suppose that G is factor-critical. For any v, fix a near-perfect matching M_v misses v. Note that if uv is an edge, then $M_u \triangle M_v$ contains an even alternating path from u to v. Together with uv we obtain an odd cycle. This establishes the existence of an initial odd cycle.
Fix a vertex v and M_v. We proceed by induction. Let H be the subgraph of G defined by the odd ear decomposition we found so far. We will add an odd ear to H until $H = G$, while maintaining that $v \in H$ and that no edge in M_v crosses $V(H)$ (connects $V(H)$ and $V(G) \setminus V(H)$).
If $V(H) = V(G)$, then we can add remaining edges to H since each edge is a trivial odd ear. Otherwise, there is an edge ab such that $a \in H$, $b \notin H$ and $ab \notin M_v$ since G is connected. Consider $M_b \triangle M_v$. It contains an even alternating path from b to v. Let xy be the first edge on the path from b such that $x \notin H$ but $y \in H$. By induction hypothesis, xy is not in M_v. Hence, the subpath P from b to y must be of even length. So $P \cup \{ab\}$ is an odd ear connecting a and y.

P4 Consider $S = \{(1,0,1),(0,1,1),(1,1,2),(0,2,2)\} \subseteq \mathbb{R}^3$. Describe $\text{lin}(S)$, $\text{aff}(S)$, $\text{cone}(S)$ and $\text{conv}(S)$ (as a polyhedron, in terms of the linear equalities/inequalities).

(1) $\text{lin}(S) = \{(x,y,z) \in \mathbb{R}^3 : z = x + y\}$.
(2) $\text{aff}(S) = \{(x,y,z) \in \mathbb{R}^3 : z = x + y\}$.
(3) $\text{cone}(S) = \{(x,y,z) \in \mathbb{R}^3 : z = x + y, x \geq 0, y \geq 0\}$.
(4) $\text{conv}(S) = \{(x,y,z) \in \mathbb{R}^3 : z = x + y, 0 \leq x \leq 1, 1 \leq z \leq 2\}$.

P5 Suppose you are given a description of a polyhedron P as the solution set to a system of linear inequalities/equalities. Describe a procedure for finding a description of the conic hull, $\text{cone}(P)$, as the solution set of a system of linear inequalities and equalities. (Hint: Introduce a new variable and use Fourier-Motzkin elimination to get rid of it.)

Any conic combination of points in P can be written as a nonnegative overall scale factor times a convex combination of points in P. Since P is closed under convex combinations, a point x is in the conic hull of P if and only if there is a scale factor $\lambda \geq 0$ such that $x \in \lambda P$, where $\lambda P = \{\lambda z \mid z \in P\}$. If $\lambda > 0$, this is equivalent to $\lambda^{-1} x \in P$. If P is described in the form $P = \{x \mid Ax \leq b\}$, then $\lambda^{-1} x \in P$ if and only if we have

$$A(\lambda^{-1} x) \leq b,$$

and from $\lambda > 0$, this occurs if and only if

$$Ax \leq \lambda b.$$

If P is nonempty, then we see that $x \in \text{cone}(P)$ if and only if either $x = 0$ or there exists $\lambda > 0$ such that $Ax \le \lambda b$ (if P is a polytope, this can be simplified to $x \in \text{cone}(P) \iff \exists \lambda \ge 0\ Ax \le \lambda b$, but the example $P = \{(a, b) \in \mathbb{R}^2 \mid a \ge 0, b = 1\}$ shows that if P is a polyhedron, the strict inequality can be necessary). Since the system

$$\begin{aligned} \lambda &> 0 \\ Ax &\le \lambda b \end{aligned}$$

is linear in the coordinates of x and in λ, we can eliminate λ from this system using Fourier-Motzkin elimination to get a system of linear inequalities in x. Explicitly, if the ith row of the matrix A is a_i^T and the ith coordinate of b is b_i, then we divide the set of indices I into three sets:

$$\begin{aligned} I_= &= \{i \mid b_i = 0\}, \\ I_< &= \{i \mid b_i < 0\}, \\ I_> &= \{i \mid b_i > 0\}. \end{aligned}$$

After elimination (remembering to also use the inequality $\lambda > 0$), the inequalities describing the nonzero points in $\text{cone}(P)$ are

$$\begin{aligned} a_i^T x &\le 0 && \forall i \in I_=, \\ b_j a_i^T x &\le b_i a_j^T x && \forall i \in I_<, j \in I_>, \\ a_i^T x &< 0 && \forall i \in I_<. \end{aligned}$$

Note that all of these inequalities are "scale-free".

2-5 Given a graph $G = (V, E)$, an *inessential* vertex is a vertex v such that there exists a *maximum* matching of G not covering v. Let B be the set of all inessential vertices in G (e.g., if G has a perfect matching then $B = \emptyset$). Let C denote the set of vertices not in B but adjacent to at least one vertex in B (thus, if $B = \emptyset$ then $C = \emptyset$). Let $D = V \setminus (B \cup C)$. The triple $\{B, C, D\}$ is called the Edmonds-Gallai partition of G. Show that $U = C$ is a minimizer in the Tutte-Berge formula. (In particular, this means that in the Tutte-Berge formula we can assume that U is such that the union of the odd connected components of $G \setminus U$ is precisely the set of inessential vertices.)

Let M be a maximum matching of G, and let G_0 be the result of iteratively shrinking blossoms until no blossoms remain. Let X be the set of exposed vertices in G, let Even be the set of vertices which can be reached from an exposed vertex by an alternating path of even length, let Odd be the set of vertices which can be reached from an exposed vertex by an alternating path but which are not in Even, and let Free be the remaining vertices. We will show that $B = \text{Even}, C = \text{Odd}$, and $D = \text{Free}$, and that the partition remains the same if we compute it in G_0 instead of in G; we will see that every blossom is contained in Even, and that blossom contractions respect the partition. In class, we showed that the set of odd vertices of G_0 could be taken as the minimizer U in the Tutte-Berge formula, so this will complete the proof.

To see that $B = \text{Even}$, note that for any vertex $v \in \text{Even}$ we can flip the even alternating path from X to v to get a maximum matching that doesn't cover v, and conversely if there is a maximum matching M' not covering v, then in $M \Delta M'$ (which is a union of alternating paths and alternating cycles) v must be an endpoint of an even length alternating path to an exposed vertex. To see that $C = \text{Odd}$, we just have to prove that every neighbor u of a vertex v in Even has an alternating path to X: if u is on the even alternating path from X to v then this is obvious, and otherwise we can make an alternating path from X to u by extending the even alternating path from X to v by the edge from v to u (which is necessarily not in the matching, since the last edge of the even alternating path to v is in M).

Finally, we just need to show that the partition into Even, Odd, and Free doesn't change when we contract a blossom B. Note first that every vertex in a blossom B is necessarily in Even (by the definition of a blossom), and that the contracted blossom will also be in Even. If there is an alternating path in G_0 passing through the shrunk blossom, then we can build a corresponding alternating path in G of the same parity, since the alternating path in G_0 will either end at the blossom vertex or pass through the edge of the matching connecting the stem to the blossom. If there is an alternating path P from X to v in G which intersects the blossom, then we can consider the portion of P from the last time P intersects the corresponding flower until it reaches v, and replace the prefix by part of the flower to make a path P' which is still alternating and has the same parity as P.