Lecture 11

- Finish TU
- nonbip. matching polytope
- Introduce flows?

Total unimodularity

Recall: A T.U. $\iff$ all subdeterminants in $\{0, -1, +1\}$.

The point: when does an I.P. have same solutions as its L.P. relaxation?

e.g

$$z_{IP} := \min \quad c^T x \qquad\qquad z_{LP} := \min \quad c^T x$$

$$\text{(I.P.)} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \\ x &\in \mathbb{Z}^n \end{aligned} \qquad \text{vs} \qquad \text{(L.P.)} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

Always: $z_{IP} \geq z_{LP}$.

Main result from last lecture:

If A is TU then $z_{IP} = z_{LP}$;

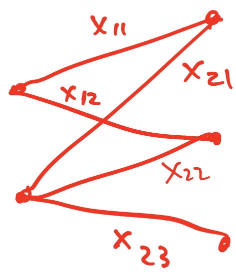
In particular, $P = \{x : Ax = b, x \geq 0\}$ integral.

Example bipartite matching.

polytope of "fractional matchings" we used for min-weight-perfect-matching:

Let (U, V) be bipartition.

$$P = \Big\{ x \in \mathbb{R}^E : \sum_{(i,j)\in E} x_{ij} = 1 \quad \forall i \in U$$



$$\sum_{(i,j)\in E} x_{ij} = 1 \quad \forall j \in V$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in E \Big\}$$

$$:= \{ x : Ax = b, \ x \geq 0 \}.$$

Integral points in P = perfect matchings in G!

Recall: Lecture on bipartite matching

$\Rightarrow$ P is integral.* i.e.

THM: (MWPM THEOREM)

$$\boxed{\text{MWPM} = \min_{\substack{M \\ \text{perfect} \\ \text{matching} \\ \text{in } G}} \sum_{(i,j) \in M} c_{ij} = \min\{c^T x : x \in P\}}$$

* technically only showed for G = complete bipartite, but also true for any bipartite G.

Another way to show it:

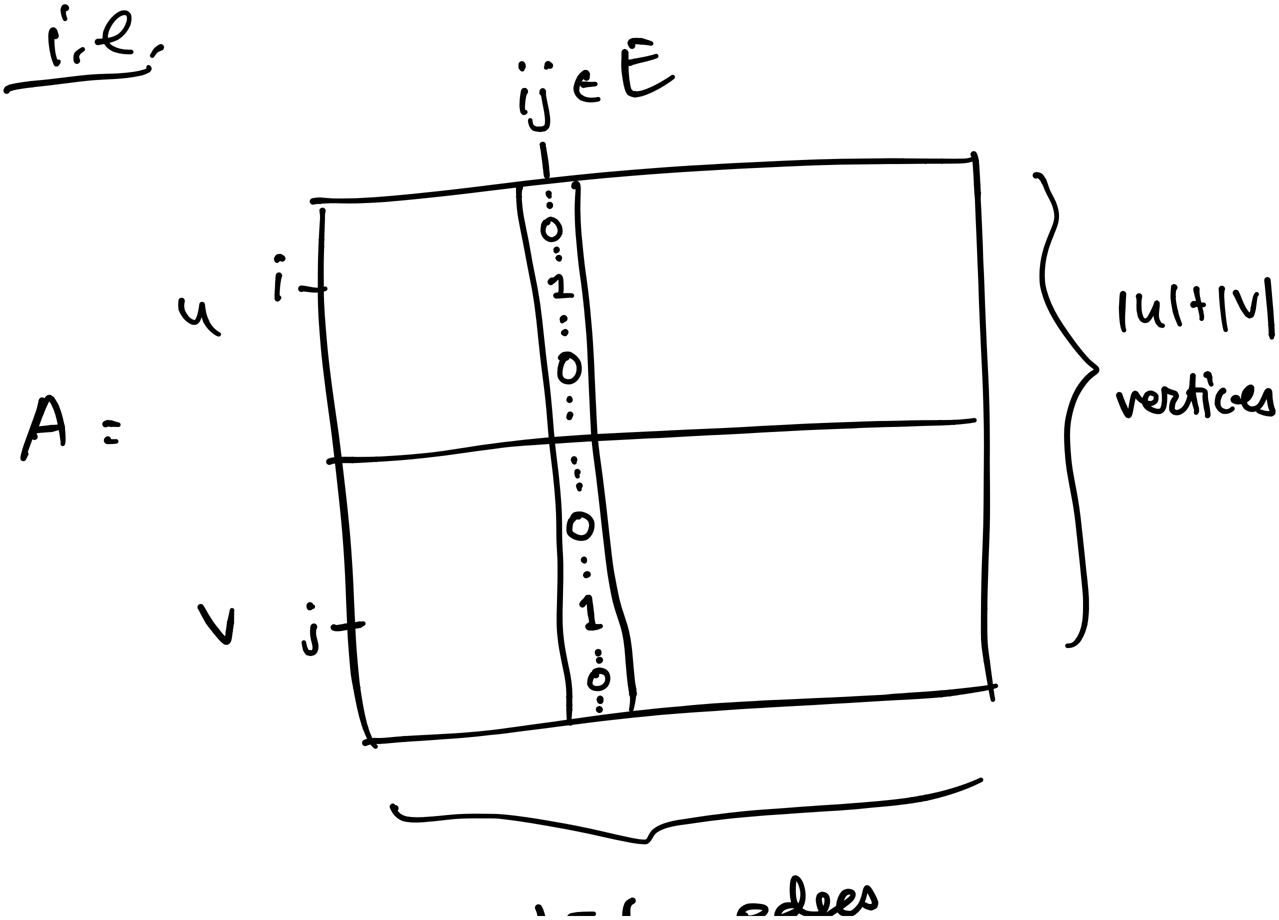
Theorem: The matrix A is totally unimodular.

Cor: MWPM THEOREM.

Proof: What's A look like?

A^T is incidence matrix of G.

i.e.



$|E|$

- To show A is TU, consider square submatrix M & look at cases:

  1) if M has 0 row/col, $\det M = 0$.

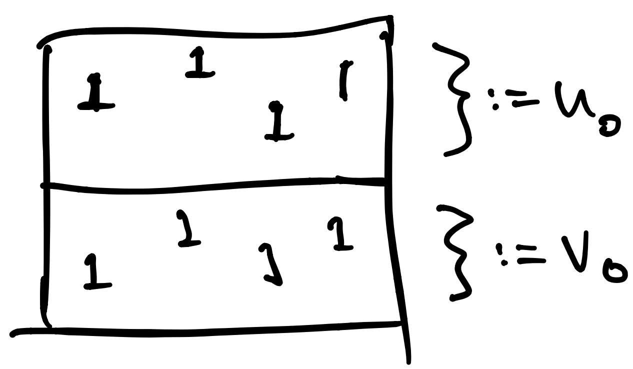
  2) if M has row/col w/ only one 1, expand down that row/col & get smaller M

smaller M.

3) M has ≥ 2 nonzero entries per row & col.

$\Rightarrow$ M has <u>exactly</u> 2 nonzero entries per column

M



$$\mathbb{1}_{u_0} = \begin{array}{c} \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \end{array} \begin{array}{l} u_0 \\ \\ v_0 \end{array} \qquad \mathbb{1}_{v_0} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ \hline 1 \\ \vdots \\ 1 \end{array}\right] \end{array} \begin{array}{l} u_0 \\ \\ v_0 \end{array}$$

T

$$\mathbb{1}_{U_0}^T M = \mathbb{1}^T$$

(add up rows of M in U_0, get $\mathbb{1}^T$).

Similarly

$$\mathbb{1}_{V_0}^T M = \mathbb{1}^T$$

$\Rightarrow$ two distinct solns. to $x^T M = \mathbb{1}^T$; rows not lin indep.

$$\Rightarrow \quad \det M = 0. \qquad \square$$

Neat side note:

$m \times n$

Def: <u>discrepancy</u> of $A \in \mathbb{R}^{m \times n}$ is

$$\min_{x \in \{\pm 1\}^n} \|Ax\|_\infty = \min_{x \in \{\pm 1\}^n} \max_{i \in m} |(Ax)_i|$$

How well A can be "balanced".

E.g. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has discrepancy 1 via $x = [1, -1, -1]$.

Fact: A is T.U. $\Leftrightarrow$ all submatrices of A have discrepancy 1.

T.U. matrices are highly "balanceable".

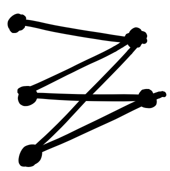
(Non-bipartite) Matching polytope

We saw (lecture 1, prev. example) that if $G=((U,V),E)$ bipartite, then the convex hull of p.m.'s is

$$P=\Big\{x\in\mathbb{R}^E : \sum_{(i,j)\in E} x_{ij}=1 \quad \forall i\in U$$

"degree constraints".

$$\sum_{(i,j)\in E} x_{ij}=1 \quad \forall j\in V$$

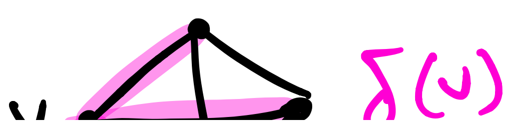


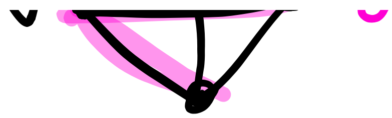
$$x_{ij}\geq 0 \quad \forall (i,j)\in E\Big\}$$

But for nonbipartite?

- Degree constraints enough?

Def $\delta(v)=\{e : v\in e\}$

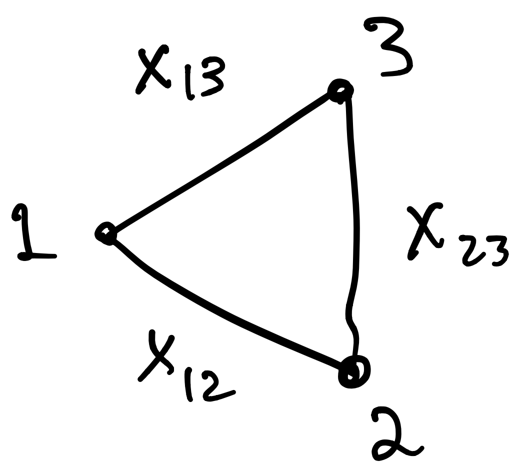




Could we have conv(matchings) =

$$P = \left\{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \le 1 \quad \forall v \in V \right.$$
$$\left. x \geq 0 \quad \forall e \in E \right\} ?$$

No: E.g.



- $x_{12} = x_{13} = x_{23} = \frac{1}{2}$ feasible;
- Sum is $\frac{3}{2}$ but every matching has sum ≤ 1.

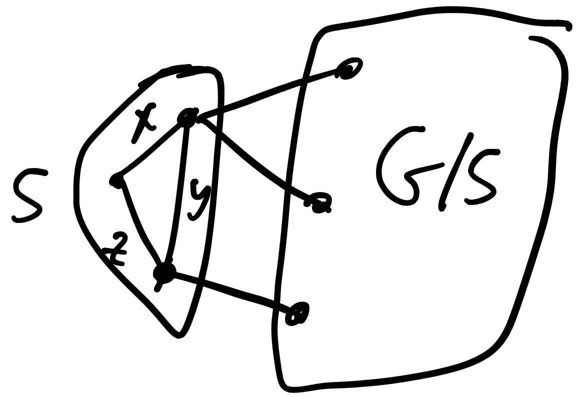
Need another constraint:

"ODD SET CONSTRAINT"

If $|S|$ odd, then

$$\sum_{e \in E(S)} x_e \le \frac{|S|-1}{2}$$

e.g.



$x+y+z \le 2$.

ODD set constraints hold for $x \in \mathrm{conv}(\text{matchings!})$.

THM (Edmonds) Let

$X = \{ 1_M : M \text{ matching in } G \}$.

Then $\mathrm{conv}(X) = P$ where

$$P = \Big\{ x : \sum_{e \in \delta(v)} x_e \le 1 \quad \forall v \in V.$$

degree constraints

$$\sum_{e \in E(S)} x_e \le \frac{|S|-1}{2} \quad \forall S \subseteq V,\ |S| \text{ odd}$$

odd set constraints.

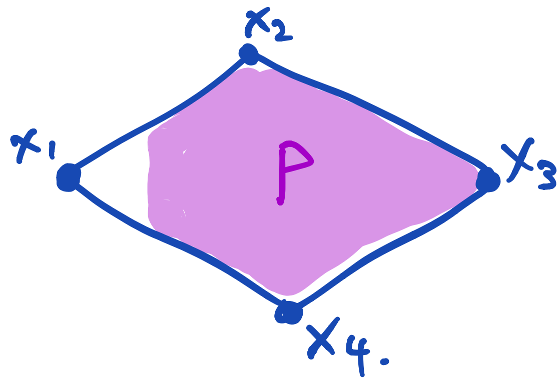
$$x_e \ge 0 \quad \forall e \in E. \Big\}$$

Proof: Idea: show they have the same facets.

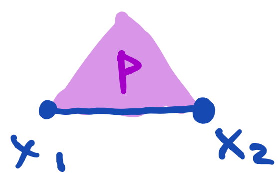
$\boxed{\subseteq}$ $\text{conv}(X) \subseteq P$ ("showed" before)

$\boxed{\supseteq}$ To show $P \subseteq \text{conv}(X)$,
show every facet of $\text{conv}(X)$ comes from inequality of P.
($\Rightarrow$ P has <u>more</u> constraints.
$\Rightarrow$ containment).

*



* Caveat: need conv(X) full-dimensional for this proof strategy to work.



Showing [2]:

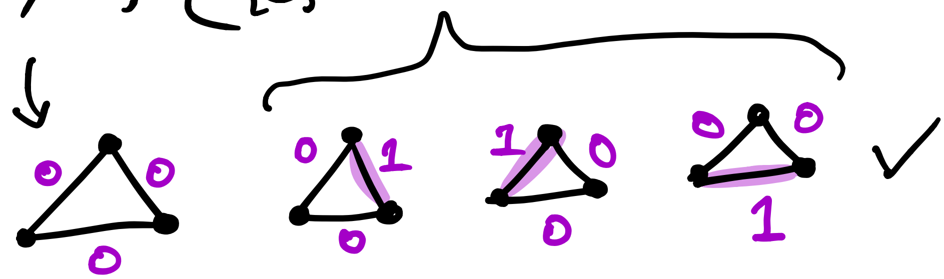
- Step 1: Show $\boxed{\dim \operatorname{conv}(X) = |E|.}$

Recall:

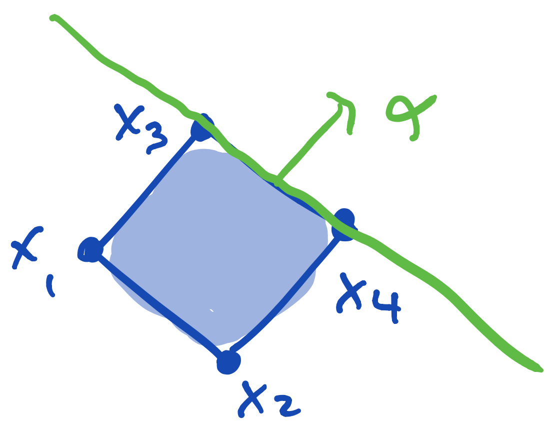
$$\dim \operatorname{conv}(X) = (\#\text{ affinely indep points}) - 1\ ;$$

$|E|+1$ affinely independent points:

$1_\emptyset$, $\{1_{\{e\}} : e \in E\}$



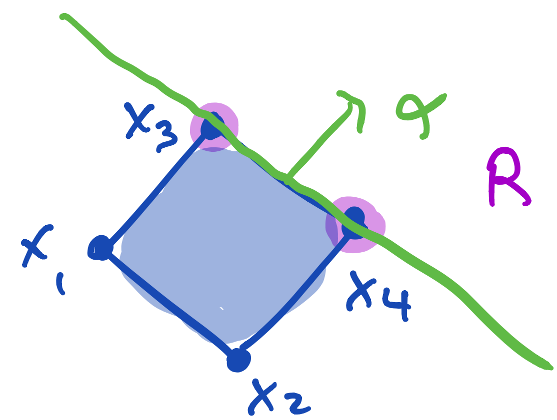
- **Step 2:** Now consider face F of $\mathrm{conv}(X)$ from inequality $\alpha^T x \le \beta$.



Need to show F contained in face from inequality of P.

Note: $F = \mathrm{conv}(R)$ where

$R = \{x \in X : \alpha' x = \beta\}$.



If R empty, done. Assume not.

- **Case (a):** α has negative entry α_e.

  $\Rightarrow x_e = 0$ (else $\alpha^T x$ can be increased).

  $\Rightarrow F \subseteq$ face from ineq. $x_e \geq 0$. ✓

assume $\alpha \geq 0$ for remaining cases.

- **Case (b):** Some vertex v covered by every $x \in R$, i.e.

$$\boxed{\sum x_e = 1 \quad \forall x \in R.}$$

$e \in \delta(v)$

$\Rightarrow F \subseteq$ face from degree constraint

$$\sum_{e \in \delta(v)} x_e \leq 1.$$

For remaining case:

Assume $\forall v$, is $x_v \in R$ not covering v.

- Case (C):
  - Let E_+ be edges where $\alpha > 0$, i.e.

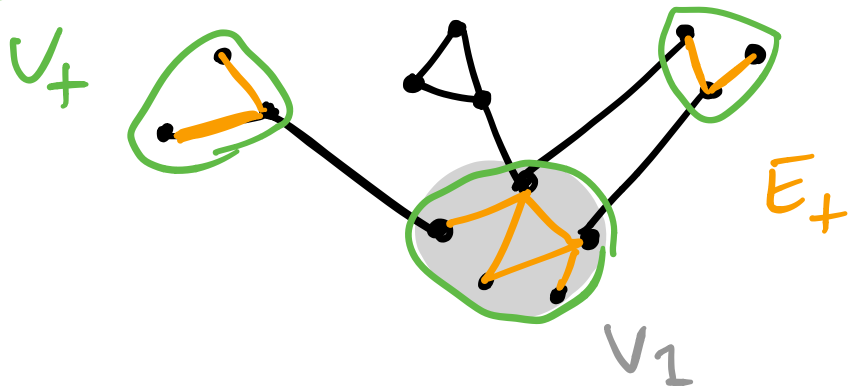
$$E_+ = \{e \in E : \alpha_e > 0\};$$

Case 1 $\Rightarrow \alpha_e = 0$ for $e \in E \setminus E_+$.

  - Let V_+ = vertex set of E_+,

    (V_1, E_1) any connected component of (V_+, E_+).

e.g.

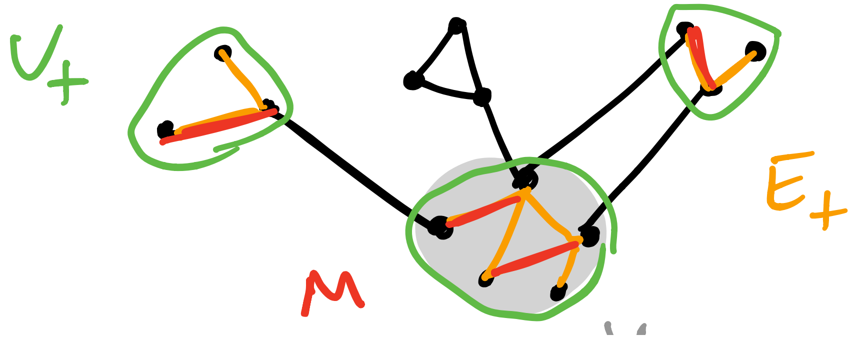


Claim: F contained in face from odd set constraint w/ $S = V_1$.

i.e. $$\sum_{e \in E(V_1)} x_e = \frac{|S|-1}{2} \quad \forall x \in R$$

equiv: all $M \in R$ have $\frac{|S|-1}{2}$ edges in V_1.

e.g.



V1

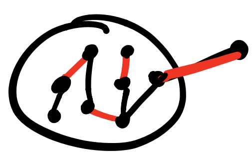
Idea of proof of claim:

Show

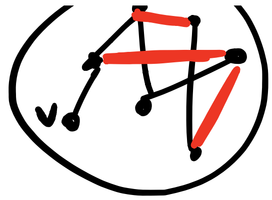
* $\boxed{\text{no matching } M \in R \text{ excludes two vertices } u, v \in V_1.}$

Why is * enough?

* $\Rightarrow$ (i): any matching $M \in R$ missing ≥ 1 vertex of V_1 can't have edges departing V_1 (else removal $\rightsquigarrow M'$ contradicting *)

V_1  doesn't happen

(i) $\Rightarrow$ (ii): $|V_1|$ odd (because $\exists\, M \in R$ missing some $v \in V_1$; is near-perfect in E_+

V_1 

(i),(ii) $\Rightarrow$ $\boxed{\text{Every extremal } M \text{ has } \frac{|V_1|-1}{2} \text{ edges of } E_+;}$

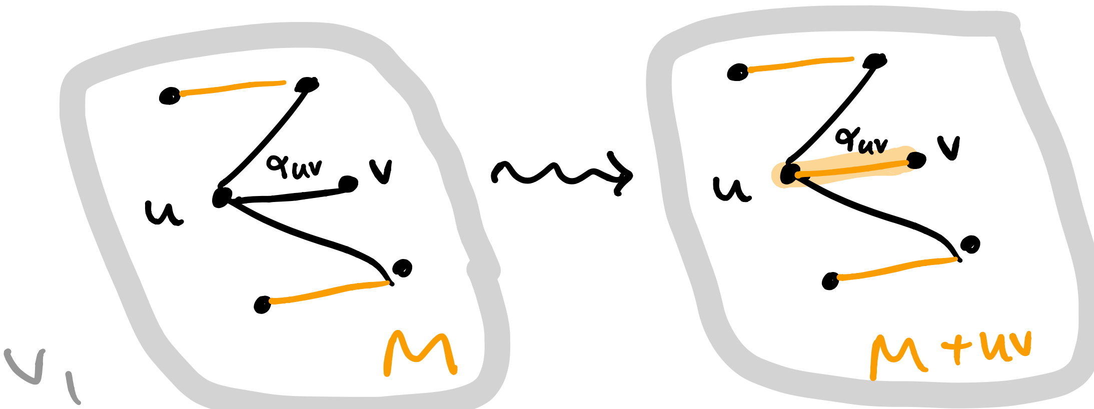
else can remove edges of M not in E_+ to miss ≥ 2 vertices of V_1. ✓.

<u>Proof of $*$</u>:

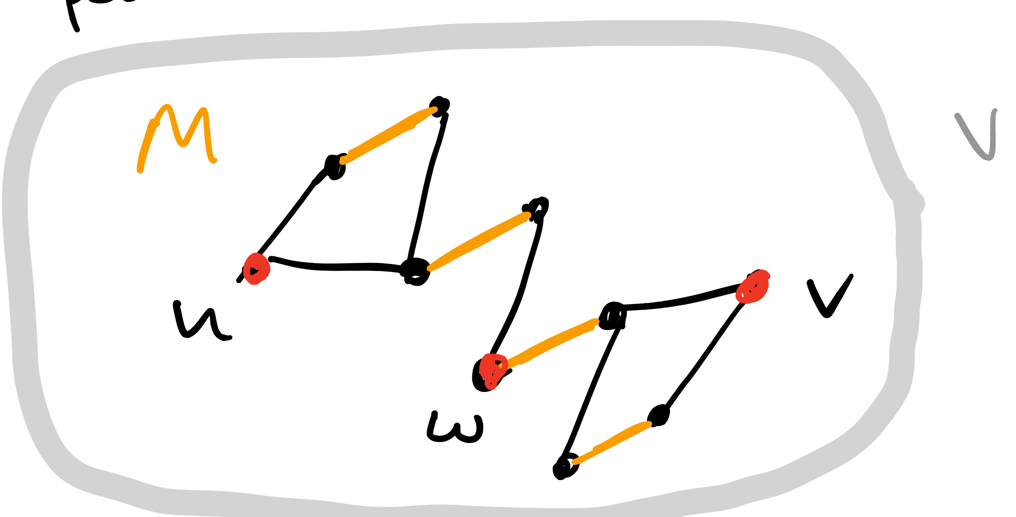
- <u>Suppose not</u>: Among elts of R excluding some two verts $u, v \in V_1$ let $M \in R$ be matching where u, v closest in (V_+, E_+).

(we'll create M_1 missing two closer vertices for a contradiction.)

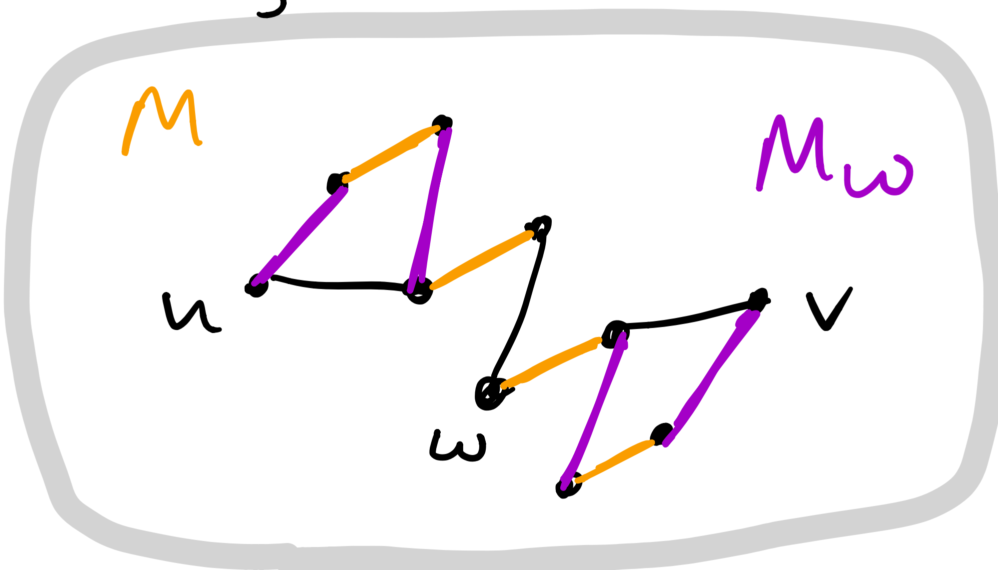
• If dist = 1, then $(u,v) \in E$, and $M + uv$ would violate $\alpha^T x \le \beta$ b/c $\alpha_{uv} > 0$.



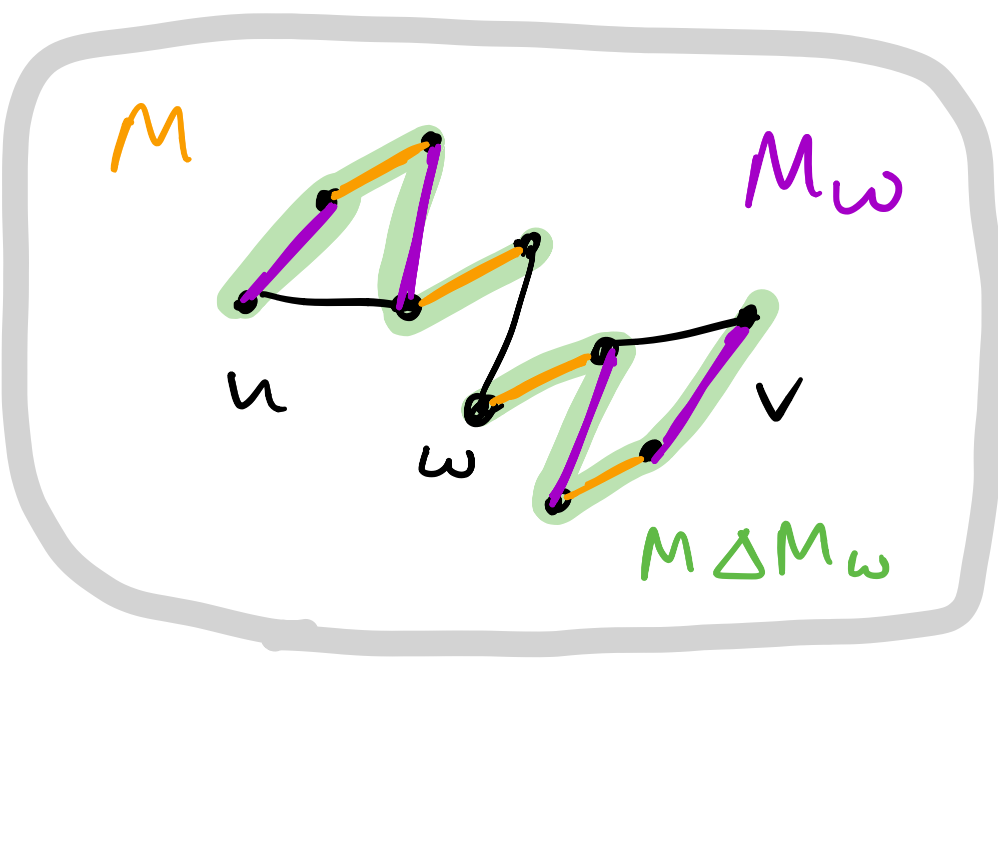
• Thus, distance ≥ 2. Let $w \notin \{u,v\}$ on shortest u-v path.



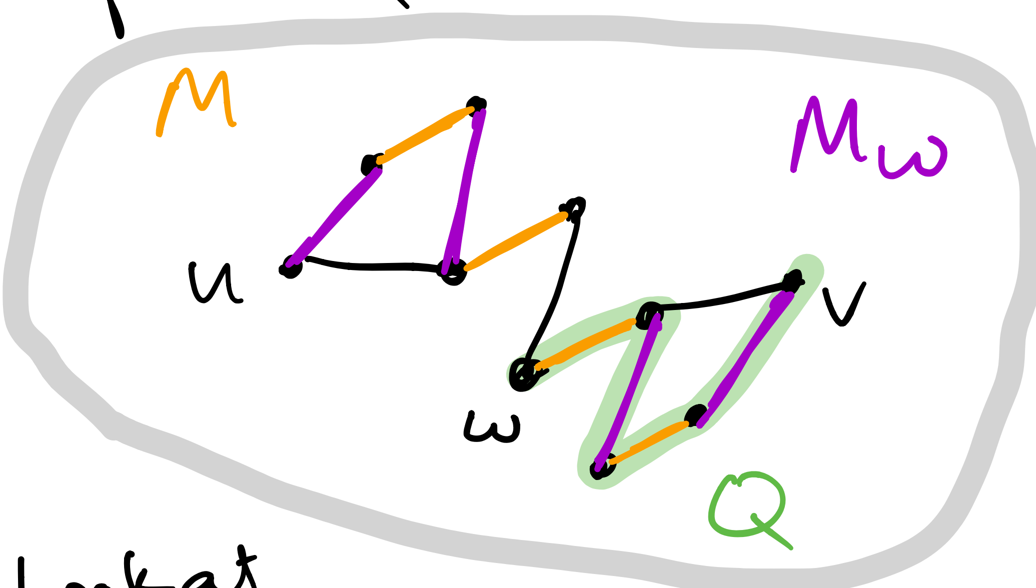
- Case (b) $\Rightarrow \exists\ M_w \in K$ missing w.



- look at $M_w \triangle M$.
  is symmetric diff of matchings.

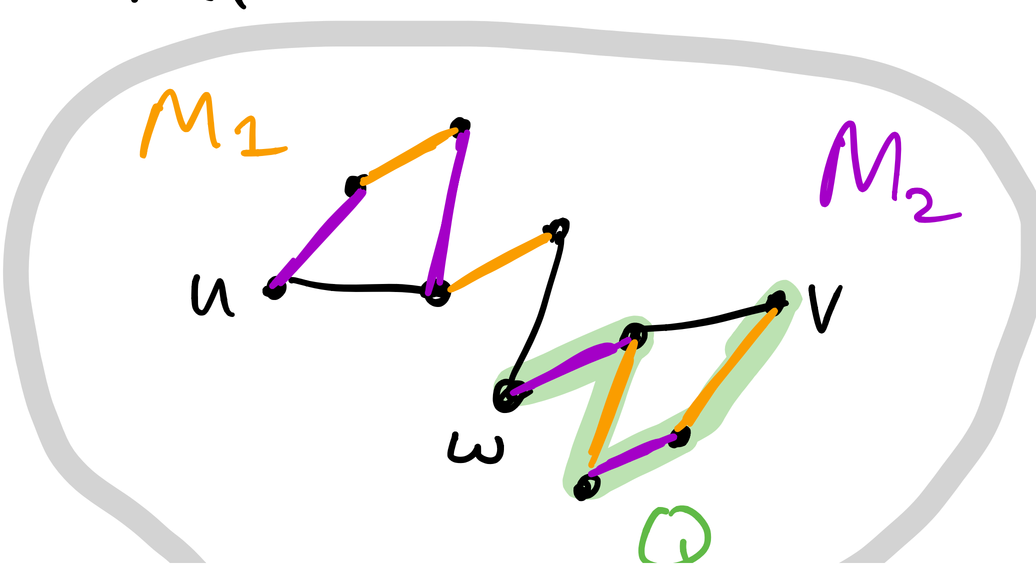


- $M_\omega \Delta M$ is union of paths & cycles, contains some path Q ending at ω.



- Look at

$$M_1 = M \Delta Q, \quad M_2 = M \Delta Q.$$



- Both matchings $\Rightarrow \sum_{e\in M_i} \alpha_e \le \beta$.

But

$$\boxed{\sum_{e\in M_1}\alpha_e + \sum_{e\in M_2}\alpha_e = \sum_{e\in M}\alpha_e + \sum_{e\in M_w}\alpha_e = 2\beta}$$

$\Rightarrow$ Both $M_1, M_2 \in R$!

- But M_1 doesn't cover w; & doesn't cover one of u or v. (whichever wasn't in Q).

$\Rightarrow$ M_1 doesn't cover two vertices closer than u, v; contradiction. $\square$

Flows