18.453 Practice Final

Instructions. This is practice for a **timed** final. This is meant to be done in **3** hours with access to notes and course material, but no access to collaborators. For best practice I suggest trying to complete it under these conditions. Afterwards please tell me if 3 hours felt like enough.

1. Answer true or false. For items **not** marked with *, if true, provide a concise reason (no rigor necessary) and if false, exhibit a counterexample.

   (a) Every matching that is not maximum in a graph G has an augmenting path.*

   (b) If A, b are integral, then the linear program $\max\{c^T x : Ax \leq b\}$ has an integral maximizer.

   (c) The set of matchings in a bipartite graph forms a matroid.

   (d) Given a bipartite graph, the set of subgraphs of degree at most two is the intersection of two matroids.

   (e) Given a separation oracle for a polyhedron $P \subset [0,1]^n$, it is always possible to test feasibility of P with polynomially many calls to the separation oracle.

2. For $k \leq n$ an integer, define a *k-bounded permutation* on $\{1, ..., n\}$ to be a permutation σ such that $|\sigma(i) - i| \leq k$ for all $i \in \{1, ..., n\}$.

   Suppose we are given an integer $k \leq n$ and costs $c(i)$ for $i \in \{1, ..., n\}$, and our goal is to find a k-bounded permutation σ on $\{1, ..., n\}$ minimizing $\sum_{i=1}^{n} c(i)\sigma(i)$. Give a polynomial-time algorithm for this problem (there is no need to give the most efficient algorithm, but the algorithm should be polynomial in n and k). (You can refer to any algorithm we have seen in class.)

3. (a) Consider a directed graph $G = (V, E)$ with nonnegative (upper) capacities $u : E \to \mathbb{R}$ (and no lower capacities). For any two vertices $s, t \in V$, define $\lambda_{st} \in \mathbb{R}$ to be the maximum flow value from s to t. Given any 3 vertices $s, t, u \in V$, show that $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$.

   (b) If the graph is undirected, the previous result still holds: $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$ for all s, t, u. Furthermore, $\lambda_{st} = \lambda_{ts}$. Now, consider the complete graph K_V on the vertex set V with weight λ_{uv} on edge (u, v) for all u, v. Let T be a *maximum weight* spanning tree on K_V with respect to these weights λ_{uv}. Argue that for every $(s, t) \notin T$, we have

$$\lambda_{s,t} = \min_{(u,v) \in P_{st}} \lambda_{uv}$$

   where P_{st} denotes the (edges of K_V) of the unique path in T between s and t. (This implies the somewhat surprising result that, over all pairs (s, t), λ_{st} can take at most $|V| - 1$ values (those along the edges of T).)

4. Consider a bipartite graph $G = (A, B, E)$ with parts A, B and edges $E \subseteq A \times B$. Suppose we have a matroid $M_A = (A, \mathcal{I}_A)$ on A with rank function r_A. Define a family of sets $\mathcal{I}_B$ to be the collection of sets $T \subseteq B$ such that there exists a matching M of G with vertex set $V(M) = S \cup T$, such that $S \subseteq A$ and $S \in \mathcal{I}_A$.

   Prove that $M_B = (B, \mathcal{I}_B)$ is a matroid. (For **half credit**, you can do this in the special case where every vertex of A has degree 1, so that G is the graph of a function from A to B.)

5. Let $x \in [0,1]^n$ be an unknown vector, and we suppose have access to a separation oracle for the set $S = [x_1, x_1 + 0.1] \times \cdots \times [x_n, x_n + 0.1] \subset \mathbb{R}^n$. Can we find a point in S in time polynomial in n, and if so, how? (You can refer to any algorithm we have seen in class).