Solutions to Problem Set 3

Problem 1 We will prove strong linear programming duality in a more geometric manner. Consider the linear program of maximizing $c^T x$ over the polyhedron $P = \{x : Ax \leq b\}$ where A has rows $(a_i^T : i = 1, ..., m)$. Assume that the linear program is bounded. We want to show that there is some $y \geq 0$ such that $A^T y = c$ satisfying $b^T y = \max\{c^T x : x \in P\}$.

(a) Let v_0 be a vertex maximizing $c^T x$. Let I be the set $\{i : a_i^T v_0 = b_i\}$ of constraints that are tight for v_0. Using that 0 is a maximizer, show that there is no vector x such that $a_i^T x \leq 0$ for all $i \in I$ and $c^T x > 0$.

(b) Conclude that there is some vector $y \geq 0$ such that $y_j = 0$ for $j \notin I$ and and $A^T y = c$. **Hint:** [1]

(c) Show that this y satisfies $b^T y = c^T v_0$, completing the proof.

Solution:

(a) Suppose there is x such that $c^T x > 0$ and $a_i^T x \leq 0$ for all $i \in I$. Then for $\varepsilon > 0$ sufficiently small, one has $a_i^T (v_0 + \varepsilon x) \leq b_i$, but $c^T (v_0 + \varepsilon x) > c^T v_0$, which contradicts the choice of v_0.

(b) Let us write A_I for the submatrix of A whose rows are $a_i^T, i \in I$. Recall the Farkas lemma: out of two systems

$$\begin{cases} A_I x \leq 0 \\ c^T x > 0 \end{cases} \quad \text{and} \quad \begin{cases} A_I^T y_I = c \\ y_I \geq 0 \end{cases}$$

exactly one has a solution. The former doesn't hence there is a vector $y_I = (y_i)_{i \in I} \geq 0$ such that $A_I^T y_I = c$. Extend it to a vector $y = (y_j)_{1 \leq j \leq m}$ by setting $y_j = 0$ for $j \notin I$.

(c) We have

$$b^T y = \sum_{i \in I} b_i y_i = v_0^T A_I^T y_I = v_0^T c = c^T v_0.$$

3-10 Show that two vertices u and v of a polytope P are adjacent if and only there is a unique way to express their midpoint $(\frac{1}{2}(u+v))$ as a convex combination of vertices of P.

Solution: First suppose u, v are adjacent, and assume for contradiction that there exist vertices $w_1, ..., w_n$ (at least one of which is not u or v) and weights $\lambda_1, ..., \lambda_n > 0$, $\sum \lambda_i = 1$, such that

$$\frac{u+v}{2} = \lambda_1 w_1 + \cdots + \lambda_n w_n.$$

[1] Apply Farkas' lemma.

Since u, v are adjacent, there is a cost vector c such that the line segment connecting u, v is exactly the set of points x of P which maximize $c^T x$. But then

$$c^T u = \frac{c^T u + c^T v}{2} = \lambda_1 c^T w_1 + \cdots + \lambda_n c^T w_n < \lambda_1 c^T u + \cdots + \lambda_n c^T u = c^T u,$$

a contradiction.

Now suppose that u, v are *not* adjacent, and let F be the minimal face of P containing u and v (F is defined by the set of all inequalities of P that have equality at both u and v). Since F is a polytope, F is the convex hull of its vertices, so F must have at least one vertex w which is not u or v. Let L be the intersection of the line connecting w to $\frac{u+v}{2}$ with F (note $w \neq \frac{u+v}{2}$ since w is a vertex). Since L is a polytope defined by some system of equations describing a line together with the inequalities describing the facets of F, the vertices of L come from setting some inequalities corresponding to facets of F to equalities. Suppose p is the second vertex of L (the first is w), and suppose the corresponding facet of F comes from the inequality $a^T x \leq b$, with equality $a^T p = b$ at p. By the minimality of F, at least one of $a^T u, a^T v$ is strictly less than b, so $p \neq \frac{u+v}{2}$. Thus $\frac{u+v}{2}$ can be written as a convex combination of w and p with a nonzero weight on w. Since p can be written as a convex combination of vertices of P, we see that $\frac{u+v}{2}$ can be written as a convex combination of vertices of P with a nonzero weight on w.

3-12 A *stable set* S (sometimes, it is called also an independent set) in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. If we let P denote the convex hull of all (incidence vectors of) stable sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for P.

(a) Give a graph G for which P is *not* equal to

$$\begin{aligned} \{x \in \mathbb{R}^{|V|} : \quad & x_i + x_j \leq 1 && \text{for all } (i,j) \in E \\ & x_i \geq 0 && \text{for all } i \in V\} \end{aligned}$$

(b) Show that if the graph G is bipartite and has no isolated vertices then P equals

$$\begin{aligned} \{x \in \mathbb{R}^{|V|} : \quad & x_i + x_j \leq 1 && \text{for all } (i,j) \in E \\ & x_i \geq 0 && \text{for all } i \in V\}. \end{aligned}$$

Solution:

(a) Take G to be the triangle, with vertex set $V = \{1, 2, 3\}$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. The vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ satisfies the given inequalities, but the sum of its coordinates is $\frac{3}{2}$, which is larger than the sum of the coordinates of any vertex of P, since every stable subset of the triangle has size at most 1.

(b) The easy direction is checking that each indicator vector x of a stable set S satisfies the given inequalities, which follows immediately from the definition of a stable set. For the other direction - showing that each vector satisfying our system of inequalities is contained in the convex hull P - we give two different proofs.

Vertex Proof. Let $A \in \mathbb{R}^{E\times V}$ be the matrix given by

$$A_{ev} = \begin{cases} 1 & v \in e, \\ 0 & v \notin e, \end{cases}$$

and let $b \in \mathbb{R}^E$ be the vector of all 1s, so our system of inequalities can be written in the form

$$\{x \in \mathbb{R}^V : \quad \begin{aligned} Ax &\le b \\ x &\ge 0\}. \end{aligned}$$

Note that A^T is the matrix coming from the bipartite matching polytope, which we have already shown is totally unimodular. Since the transpose of a T.U. matrix is T.U., every vertex of the polyhedron defined by the system $\{Ax \le b, x \ge 0\}$ is integral, and since this polyhedron is bounded (each x_v is bounded below by 0 and above by 1 as long as v is incident to at least one edge) it is the convex hull of its vertices. Let x be a vertex of the polyhedron, we will show it is the indicator vector of a stable set. Since x is integral and each coordinate of x is bounded between 0 and 1, x is certainly the indicator vector of *some* set S - explicitly, $S = \{v \in V \mid x_v = 1\}$. If there was any edge e between two vertices v, w of S, we would have $x_v + x_w = 2$, contradicting the inequality $x_v + x_w \le 1$ corresponding to the edge e, so in fact S must be a stable set.

Facet Proof. First we check that we are not missing any equalities, by showing that $\dim(P) = |V|$. To see this, note that every set S with $|S| \le 1$ is stable, so P contains the $|V|+1$ affinely independent points $(0, 0, ...0)^T$, $(1, 0, ..., 0)^T$, $(0, 1, ..., 0)^T$, ..., $(0, 0, ..., 1)^T$.

Now suppose that F is a facet of P, defined by maximizing some cost $c^T x$ over vetices of P. We will show that the set $\{x \in P \mid c^T x \text{ is maximal}\}$ is contained in some facet of the polyhedron defined by the given system of inequalities. There are two cases.

First case: for some $v \in V$, we have $c_v < 0$. In this case, every x corresponding to a stable set S which maximizes $c^T x$ must have $x_v = 0$, since otherwise the set $S \setminus \{v\}$ is also stable, and if x' is the corresponding vector, then $c^T x' = c^T x - c_v > c^T x$. Thus the face of P corresponding to the cost vector c must be contained in the facet corresponding to the inequality $x_v \ge 0$.

Second case: for some $v \in V$ we have $c_v > 0$. Suppose for contradiction that for each edge $e = \{v, w\}$ containing v, there is some stable set S_w which doesn't contain v or w, but such that if x_w is the corresponding indicator vector, then

$c^T x_w$ maximizes $c^T x$ over x in P. Let W be any subset of the set of neighbors of v, we will show by induction on $|W|$ that there is a stable set S_W which doesn't contain v or any vertex from W, but such that the corresponding indicator vector x_W maximizes $c^T x_W$. Taking W to be the set $N(v)$ of all neighbors of v, we will get a stable set $S_{N(v)}$ not containing v or any neighbor of v and maximizing our cost function, but then adding v to this stable set gives us a stable set with a strictly larger cost, giving us our contradiction.

For the inductive step, suppose $W = X \cup Y$, and that we have already constructed stable sets S_X, S_Y maximizing our cost function, not containing v, and s.t. $S_X \cap X = S_Y \cap Y = \emptyset$. Let H be the induced subgraph of G with vertex set $S_X \cup S_Y$, and let C be the set of vertices of H which are connected to some element of X in H. Let $A, B \subseteq V$ be the two parts of G, and suppose $v \in A$. Then by induction on the length of the shortest path (in H) connecting a vertex c in C to X, we see that $c \in B \iff c \in S_Y$ and $c \in A \iff c \in S_X$. In particular, no vertex of C is in Y. Additionally, we see that both $S_X \Delta C$ and $S_Y \Delta C$ are stable sets, and the sum of their costs is equal to the sum of the costs of S_X and S_Y, so they both maximize our cost function as well. Thus we can take $S_W = S_Y \Delta C$, which has no elements of X (since neither $S_Y \cap X = C \cap X$ by the definition of C) and no elements of Y (since $S_Y \cap Y = \emptyset$ and $C \cap Y = \emptyset$). This completes the inductive step, which as we saw above gives us the required contradiction.

By the above argument, there must be some edge $e = \{v, w\}$ containing v such that every stable set maximizing our cost function contains at least one of the vertices v, w. Thus, the face of P corresponding to the cost vector c is contained in the facet corresponding to the inequality $x_v + x_w \leq 1$.

3-13 Let $e_k \in \mathbb{R}^n$ ($k = 0, \dots, n-1$) be a vector with the first k entries being 1, and the following $n-k$ entries being -1. Let $S = \{e_0, e_1, \dots, e_{n-1}, -e_0, -e_1, \dots, -e_{n-1}\}$, i.e. S consists of all vectors consisting of $+1$ followed by -1 or vice versa. In this problem set, you will study conv(S).

(a) Consider any vector $a \in \{-1, 0, 1\}^n$ such that (i) $\sum_{i=1}^n a_i = 1$ and (ii) for all $j = 1, \dots, n-1$, we have $0 \leq \sum_{i=1}^j a_i \leq 1$. (For example, for $n = 5$, the vector $(1, 0, -1, 1, 0)$ satisfies these conditions.) Show that $\sum_{i=1}^n a_i x_i \leq 1$ and $\sum_{i=1}^n a_i x_i \geq -1$ are valid inequalities for conv(S).

(b) How many such inequalities are there?

(c) Show that any such inequality defines a facet of conv(S).
(This can be done in several ways. Here is one approach, but you are welcome to use any other one as well. First show that either e_k or $-e_k$ satisfies this inequality at equality, for any k. Then show that the resulting set of vectors on the hyperplane are affinely independent (or uniquely identifies it).)

(d) Show that the above inequalities define the entire convex hull of S.

Solution:

(a) Fix $a \in \{-1, 0, 1\}^n$ satisfying $\sum_{i=1}^n a_i = 1$ and $0 \leq \sum_{i=1}^j a_i \leq 1$ for each $j = 1, \dots, n-1$. It is enough to show that

$$-1 \leq \sum_{i=1}^n a_i (e_k)_i \leq 1$$

for each $k = 0, \dots, n-1$ (it is symmetric for $-e_k$'s).
Note that $(e_k)_i = 1$ if $i \leq k$ and $(e_k)_i = -1$ if $i > k$. We have

$$\sum_{i=1}^n a_i (e_k)_i = \sum_{i=1}^k a_i - \sum_{i=k+1}^n a_i = 2 \sum_{i=1}^k a_i - 1.$$

Since $\sum_{i=1}^k a_i$ is 0 or 1, it is between -1 and 1.

(b) Fix $a \in \{-1, 0, 1\}^n$ as in the previous part. Let $b_j = \sum_{i=1}^j a_i$ for $j = 1, \dots, n$. Then, $b_j \in \{0, 1\}$ for any $j = 1, \dots, n-1$ and $b_n = 1$ by definition of a. On the other hand, if we are given $b \in \{0, 1\}^n$ with $b_n = 1$, we can find the corresponding $a \in \{-1, 0, 1\}^n$ by letting $a_1 = b_1$ and $a_i = b_i - b_{i-1}$ for $i = 2, \dots, n$. This is a bijection between a's and b's. Hence, there are 2^{n-1} such a's and 2^n inequalities.

(c) First note that $a^T e_k$ is either -1 or 1, since $a^T e_k = 2 \sum_{i=1}^k a_i - 1$. Let b as defined in (b). Then, $a^T e_k = 1$ if and only if $b_k = 1$ (we say $b_0 = 0$). Thus,

$$\begin{aligned} \{x \in S \mid a^T x = 1\} &= \{e_k \mid b_k = 1\} \cup \{-e_k \mid b_k = 0\} \\ \{x \in S \mid a^T x = -1\} &= \{e_k \mid b_k = 0\} \cup \{-e_k \mid b_k = 1\}. \end{aligned}$$

So each inequality defines distinct hyperplanes, because they contain different set of extreme points. Moreover, if we choose exactly one vector from each $\{e_k, -e_k\}$, then they are affinely independent. For, note that it is enough to show that $\{e_0, \dots, e_{n-1}\}$ are linearly independent, and they are indeed linearly independent since $\{\frac{1}{2}(e_1 - e_0), \frac{1}{2}(e_2 - e_1), \dots, \frac{1}{2}(e_{n-1} - e_{n-2}), -\frac{1}{2}(e_{n-1} + e_0)\}$ is the standard basis of $\mathbb{R}^n$.

(d) Note that 0 is in the interior of $\text{conv}(S)$. Hence, no facet can contain $\{e_k, -e_k\}$ for any $k = 0, \dots, n-1$ (otherwise it will contain 0). Since $\text{conv}(S)$ is full-dimensional, any facet should contain at least n extreme points, i.e., it contains exactly one from each $\{e_k, -e_k\}$. So there are at most 2^n facets of $\text{conv}(S)$. On the other hand, we showed in (c) that each of 2^n inequalities defines distinct facet. Hence they define $\text{conv}(S)$.