18.453 final. This exam is closed book. Be neat! **In any problem, you can refer to results we have covered in class, but you need to state them precisely.** Don't worry; the exam is probably pretty challenging but that way, the learning process continues... Have a great summer!

1. (a) Define when a matrix A is totally unimodular.

   (b) State precisely the main property of a system of linear inequalities whose underlying matrix is totally unimodular.

   (c) Let A be a $0-1$ matrix. We say that A has the consecutive-one property if for all rows of A, the value 1 appears consecutively (and the remaining entries are 0). Show that any consecutive-ones matrix is totally unimodular.

2. (a) Give a complete description in terms of linear inequalities of the matching polytope for an arbitrary graph $G = (V, E)$. Argue that your stated inequalities are valid for the matching polytope, but you do *not* need to prove that they form a complete description of it.

3. (a) Consider a directed graph $G = (V, E)$ with nonnegative (upper) capacities $u : E \to \mathbb{R}$ (and no lower capacities). For any two vertices $s, t \in V$, define $\lambda_{st} \in \mathbb{R}$ to be the maximum flow value from s to t. Given any 3 vertices $s, t, u \in V$, show that $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$.

   (b) If the graph is undirected, the previous result still holds: $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$ for all s, t, u. Furthermore, $\lambda_{st} = \lambda_{ts}$. Now, consider the complete graph K_V on the vertex set V with weight λ_{uv} on edge (u, v) for all u, v. Let T be a *maximum weight* spanning tree on K_V with respect to these weights λ_{uv}. Argue that for every $(s, t) \notin T$, we have
$$\lambda_{s,t} = \min_{(u,v)\in P_{st}} \lambda_{uv}$$
   where P_{st} denotes the (edges of K_V) of the unique path in T between s and t. (This implies the somewhat surprising result that, over all pairs (s, t), λ_{st} can take at most $|V| - 1$ values (those along the edges of T).)

4. For a matching M in a graph $G = (V, E)$, let $V(M)$ denote the vertices matched in M.

   (a) Suppose that we are given a set $S \subseteq V$ and a matching M covering, i.e. such that $S \subseteq V(M)$. Given $v \notin S$, how would you decide (efficiently) whether there exists a matching M' such that $S \cup \{v\} \subseteq V(M')$. You can use building blocks we have seen in lectures (state them, but no need to reprove them), but you should justify any additional statements.

   (b) Consider a (not necessarily bipartite) graph $G = (V, E)$ and a profit function $p : V \to \mathbb{R}_+$. (The profit function is defined on the vertices of G.) Our goal is to find a matching M which maximizes $\sum_{v\in V(M)} p(v)$. How would you solve this problem? Justify your solution.

5. Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) (so $V = A \cup B$). Suppose we are also given a matroid $M = (A, \mathcal{I})$ defined on A (one of the sides of

the bipartition). We would like to restrict our attention to *independent matchings* M which are those matchings M such that $\{a \in A : \exists b \in B \text{ with } (a, b) \in M\} \in \mathcal{I}$. The maximum independent matching problem is the problem of finding an independent matching of maximum cardinality. How can this problem be solved efficiently? Justify your answer. (You can use as building blocks things we have seen in lectures (state them precisely though); anything else needs to be justified.)