Lecture 24

Plan:

1) Ellipsoid for LP
2) If time, examples.

Ellipsoid for LP

- Even for feasibility of $P = \{x : Ax \leq b\}$, are issues!
- Finding starting ellipse,

- Bounding volume of P.

- To avoid numerical details, study important special case:

$$\boxed{\text{Assume } P = \operatorname{conv}(X) \text{ where } X \subseteq \{0,1\}^n, \ \& \ \dim P = n.}$$

e.g.

$$P = \operatorname{conv}\{\mathbb{1}_M : M \text{ matching in } G\} \subseteq \mathbb{R}^E.$$

- can handle $\dim P < n$ by eliminating vars if given $\operatorname{aff}(P)$; tricky if not!

- Given $c \in \mathbb{R}^n$, want to compute

$$OPT = \max\{c^T x : x \in P\}$$

in polynomial time given separation oracle for P.

- What's polynomial time here?

- <u>Input-size</u>:
  - ▷ Assume $c \in \cancel{\mathbb{Q}^n}$ $\cancel{\mathbb{Z}^n}$!

$\mathbb{Q}^n$ because we must store in a machine,

$\mathbb{Z}^n$ by clearing denominator.

▷ assume each entry satisfies

$$|C_i| \leq M;$$

need $\lceil \log_{10} M \rceil$ digits to write down any entry,

(i.e. $\approx \log_2 M$ bits)

$\Rightarrow$ total input size $\leq n \lceil \log_2 M \rceil$.

- Thus we'll take polynomial time to mean poly$(n, \log M)$ steps / calls to separation oracle.

e.g. $10 n^3 (\log M)^2$,

NOT $2^n \log M$,

NOT $n M^2$.

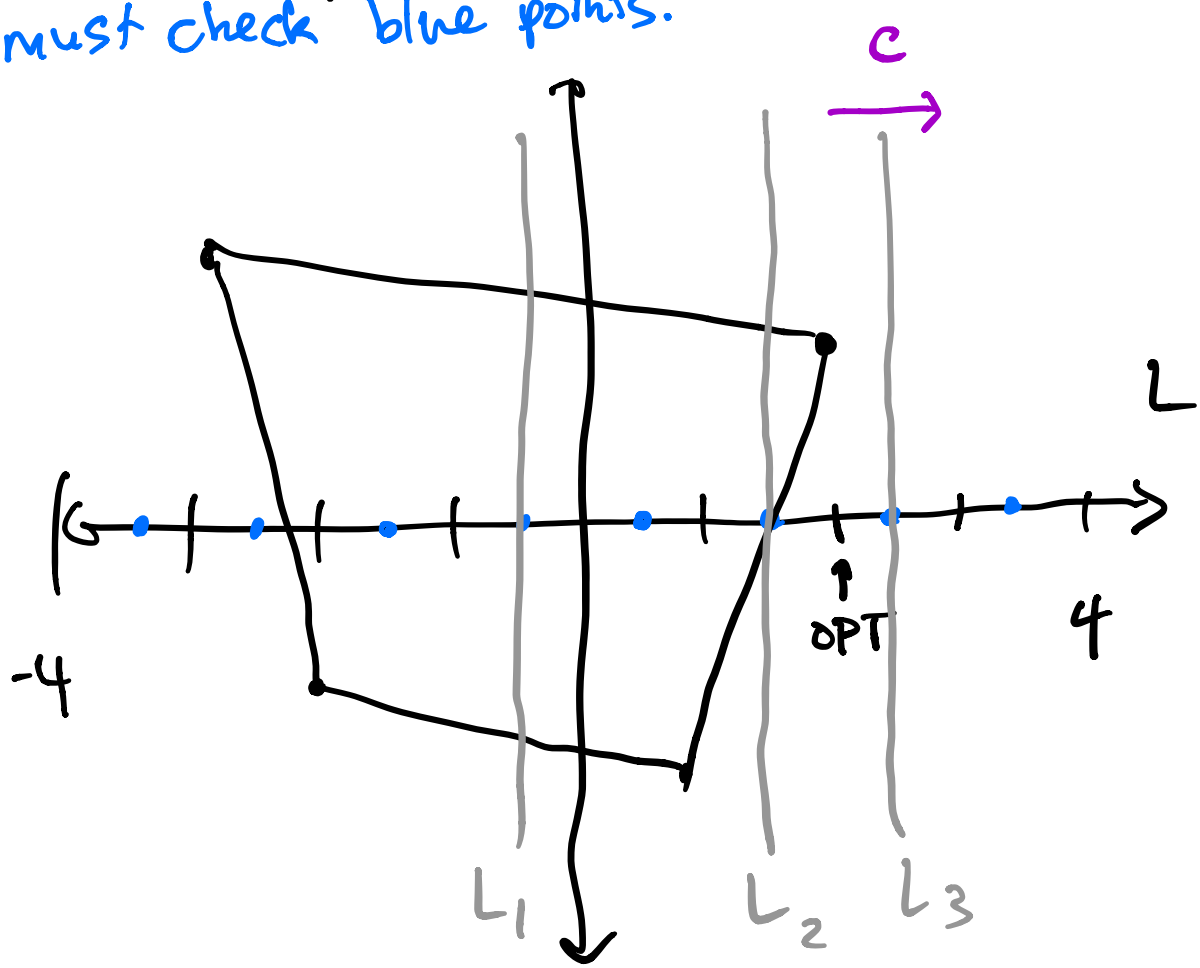
Implementing the binary search.

- How long do we need to do binary search?
  - ▷ Know $OPT \in \mathbb{Z}$;
  - ▷ Know $|OPT| \leq nM$

- Thus can <u>exactly</u> solve using binary search on only $L = k + \frac{1}{2}$, $k \in \mathbb{Z}$, $|L| \le M_n$.

e.g. Suppose $M_n = 4$, $OPT = 2$

must check blue points.



Know $L_2 \le OPT \le L_3$, $OPT \in \mathbb{Z} \Rightarrow OPT = 2$.

- How many steps?

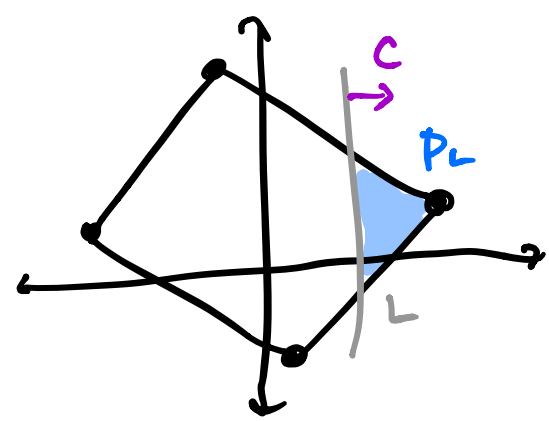
$$\leq \log_2 |2Mn|!$$

(at each step, # points remaining to check is cut in half.)

- At each step, need to test if $L < OPT$, i.e. if

$$P_L : \{x \in P : c^T x \geq L\}$$

is nonempty.



- For this, use ellipsoid.

Runtime of ellipsoid calls

- **Recall:** to test feasibility with ellipsoid, must know
  starting ellipsoid $E_0 \supseteq P_L$
  volume lower bound $\mathrm{vol}\, P_L \geq \delta$
  for all $P_L \neq \emptyset$.

- **To test:**

  ▷ run ellipsoid for

  $$2(n+1) \ln \frac{\mathrm{vol}\, E_0}{\delta}$$

  steps. (or until find $x \in P_L$).

▷ if haven't found $x \in P_L$, output that $P_L = \emptyset$.

- Thus we just need a bound on $\mathrm{vol}\, P_L$ and $\mathrm{vol}\, E_0$.

Bounding starting ellipsoid

- <u>Simple</u>: $P \subseteq [0,1]^n$,

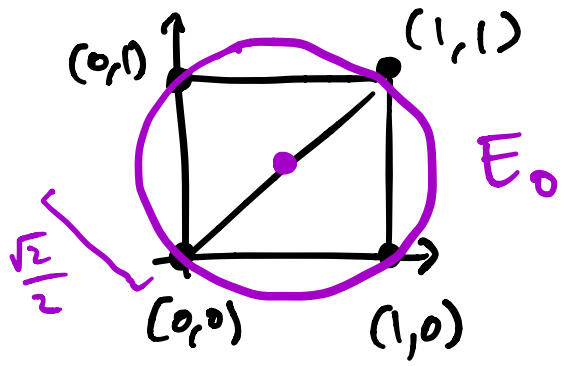
  $\Rightarrow P_L \subseteq [0,1]^n$;

  $\Rightarrow$ any $E_0 \supseteq [0,1]^n$ is ok.

- Can use E_0 = ball centered at $\frac{1}{2}$, radius $\frac{1}{2}\sqrt{n}$.

(Eo goes through all points of $\frac{1}{2}\sqrt{n}$.)

e.g. for $n=2$,



- $$\text{Vol } E_0 = \left(\frac{\sqrt{n}}{2}\right)^n \cdot \text{vol}(\text{unit ball})$$

$$\leq \left(\frac{\sqrt{n}}{2}\right)^n 2^n = n^{\frac{n}{2}}.$$

$$\Rightarrow \ln \text{vol } E_0 \leq \frac{n}{2} \ln n.$$

Bounding Vol P_L

- Need to show

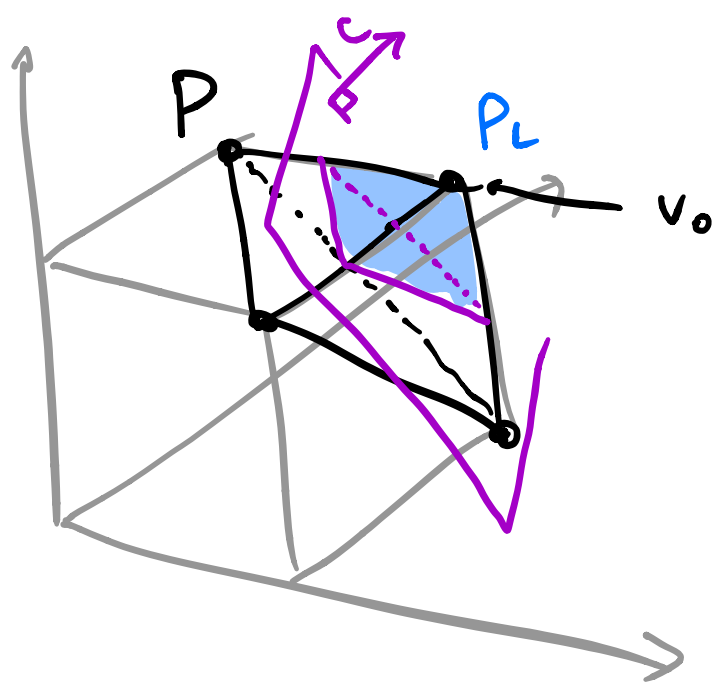
  $P_L \neq \emptyset \Rightarrow \text{Vol } P_L > \delta$.

  where $\log \frac{1}{\delta} = \text{poly}(n, M)$.

- Since $P_L \neq \emptyset$, contains some optimal vertex $v_0 \in \{0,1\}^n$ of P.

  $(c^T v_0 = OPT)$.

e.g. $n = 3$

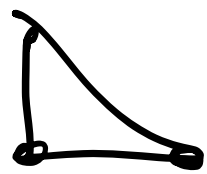


- One way (there are many) is to find a <u>simplex</u> in the "corner" of P_L.

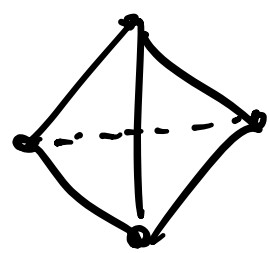
- <u>Simplex</u> in $\mathbb{R}^n$ is convex hull of $n+1$ affinely independent points.

<u>e.g.</u>

triangle in $\mathbb{R}^2$,



tetrahedron in $\mathbb{R}^3$



easy to compute volumes of simplices.

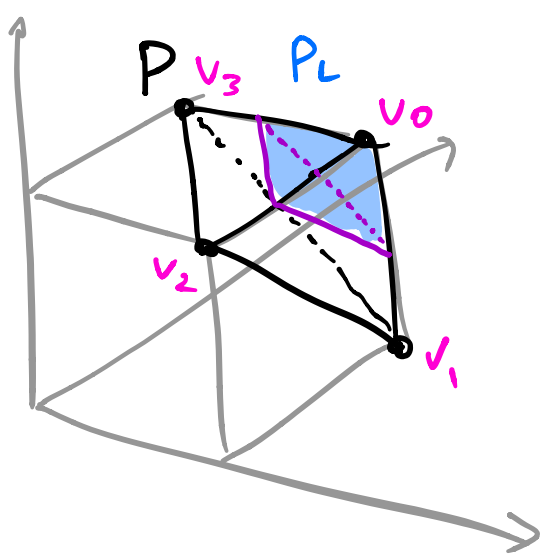
- We've assumed P full-dimensional

$\Rightarrow \exists\, v_1, \dots, v_n \in \underbrace{\{0,1\}^n}_{X} \cap P$

s.t. $\operatorname{conv}\{v_0, \dots, v_n\}$ full-dimensional.

simplex.

e.g. $n = 3$

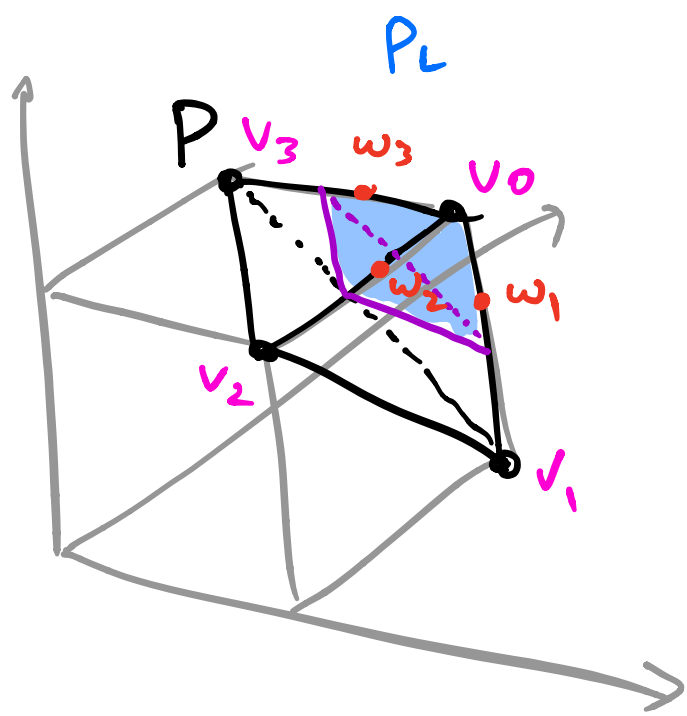


- $v_1 \ldots v_n$ might not be in P, but we can "truncate" simplex.

$$w_i = \begin{cases} v_i & \text{if } c^T v_i \geq L \\ v_0 + \alpha(v_i - v_0) & \text{else} \end{cases}$$

<u>is</u>, for small enough $\alpha > 0$.

<u>e.g.</u> $n = 3$



- Can take $\alpha = \frac{1}{2Mn}$,

because then

$$c^T w_i = c^T v_0 + \alpha c^T (v_i - v_0)$$

$$= OPT + \alpha c^T (v_i - v_0)$$

$$\geq OPT - \alpha M n$$

$|c_i| \leq M$, $v_i - v_0 \in [-1, 1]^n$.

$$\geq \left(L + \frac{1}{2}\right) - \frac{1}{2} \geq L$$

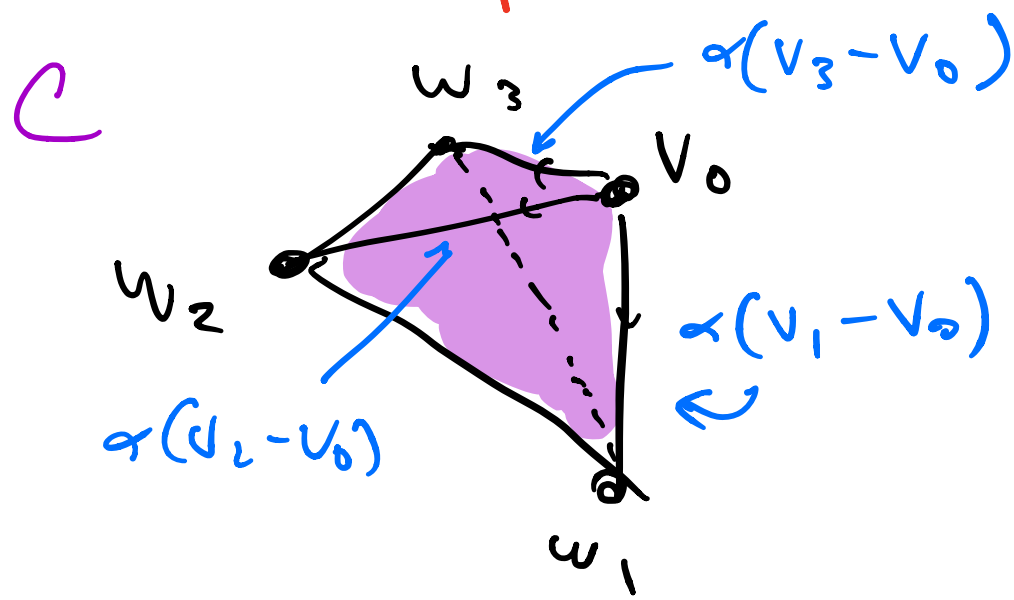
$OPT \geq L + \frac{1}{2}$,

$\Rightarrow w_i \in P_L$.

- Now $\text{vol}\, P_L \geq \text{vol}\, C$,

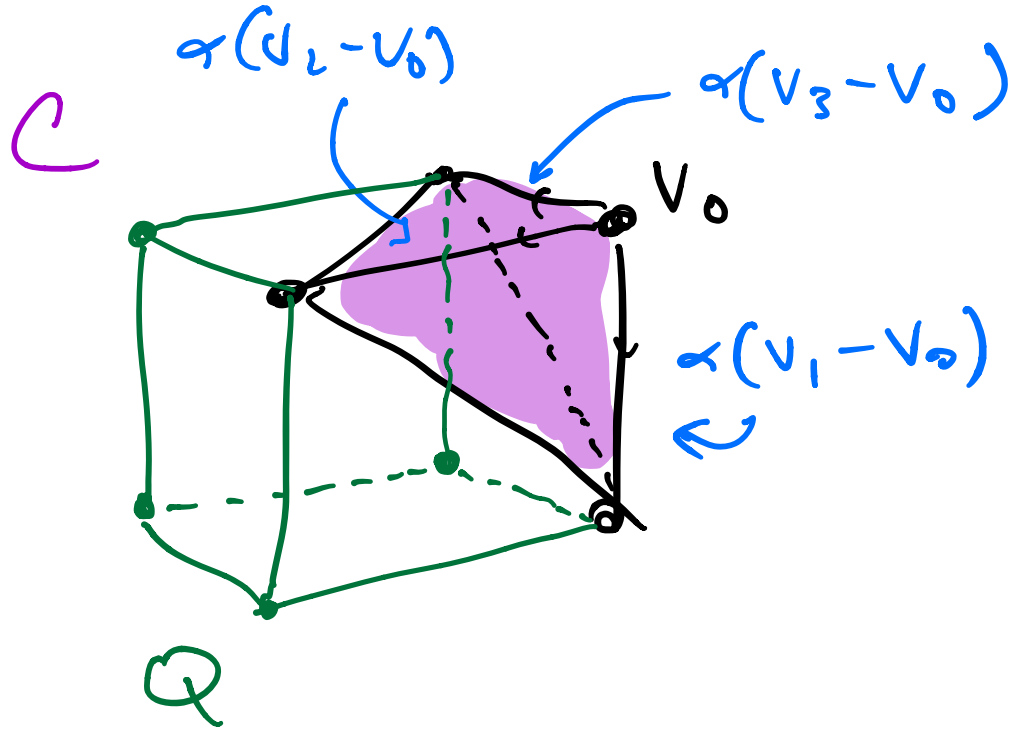
$C := \mathrm{conv}\{v_0, \ldots, v_n\}$.

"simplex".



- simplex C = corner of parallelepiped Q with sides

$\alpha(v_1 - v_0), \ldots \alpha(v_n - v_0)$



- $\operatorname{vol} C = \frac{1}{n!} \operatorname{vol} Q$

Exercise; WLOG $Q = [0,1]^n$,

$C = \{x \in Q : \sum x_i \leq 1\}$.

- $\operatorname{vol} Q = \alpha^n \cdot \operatorname{vol} Q'$,

Q' parallelipiped w/ sides $v_1 - v_0, \ldots, v_n - v_0$.

- $\operatorname{vol} Q' \geq 1$, because sides lin indep. in $\mathbb{R}^n$.

$$\operatorname{vol} Q = \left| \det \begin{bmatrix} | & & | \\ v_1 - v_0 & \cdots & v_n - v_0 \\ | & & | \end{bmatrix} \right|$$

≥ 1.

- So together:

$$\text{vol } P_L \geq \frac{1}{n!}\alpha^n = \frac{1}{n!}\left(\frac{1}{2nM}\right)^n.$$

$$\geq \frac{1}{n^n}\frac{1}{(2nM)^n} = \frac{1}{(2nM)^n}.$$

Thus we may take

$$\delta = \frac{1}{(2nM)^n}.$$

Overall Runtime

- **# steps of ellipsoid**

$$\leq 2(n+1)\ln\frac{\mathrm{vol}\,E_0}{\delta}$$

$$\leq 2(n+1)\left(\ln\left(n^{\frac{n}{2}}\right)+\ln\left((2nM)^n\right)\right)$$

$$= 2(n+1)\left(\frac{3n}{2}\ln(n)+n\ln(2M)\right).$$

$$= O\left(n^2(\ln n+\ln M)\right).$$

- **# steps of binary search**

$$\leq \log_2(2nM)$$

$$= O(\ln(n)+\ln(M)).$$

- **Overall:**

$$\boxed{O\left(n^2 (\ln n + \ln M)^2\right)}$$

$= \text{poly}(n, \ln M)$ SEP calls.

To summarize...

Theorem: Given a separation oracle for $P = \text{conv}(S)$, $S \subseteq \{0,1\}^n$, s.t. $\dim P = n$, can maximize $c^T x$ over P (& hence S) in polynomial time.

($O(n^2 (\ln n + \ln M)^2)$ calls

to SEP oracle.)

- **Side Remark:** is not <u>strongly</u> polynomial; # iterations depends on c (albeit polynomially).

- **Éva Tardos '86**: can solve LP's $\max\{c^T x : Ax \le b\}$ in time

$$\text{poly}(\text{input size of } A)$$

notably independent of c, b!

- Thus if $A \in \{-1,0,1\}^{m \times n}$, e.g. can solve LP's in strongly polynomial time!

Example: nonbip. matching

- Given G, cost $c: E \to \mathbb{R}$, find max cost matching M.
- Equivalently, optimize $c^T x$ over matching polytope,

$$P = \mathrm{conv}(\{1_M : M \text{ matching in } G\}).$$

- Recall: Matching polytope given by

$$P = \Big\{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \le 1 \quad \forall v \in V.$$

degree constr.

$$\sum_{e \in E(S)} x_e \le \frac{|S|-1}{2} \quad \forall S \subseteq V, \ |S| \text{ odd}$$

odd set constraints.

$$x_e \ge 0 \quad \forall e \in E. \Big\}$$

- P is full-dimensional (exercise).
- However, separation oracle nontrivial! (Exponentially many constraints!)
- Can implement using min-T-odd cut. (Padberg-Rao).

Matching polytope SEP oracle:

- Check degree constraints; if violated return a violated ineq.
- Next check odd set constraints.

How?

▷ For x satisfying degree constraints, need to decide if

$$\sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2} \quad \star$$

$\forall S \subseteq V$, $|S|$ odd, & if not, produce violated S.

▷ Assume WLOG $N \mid$ even (else add isolated vertex).

▷ For $v \in V$, define

$$s_v = 1 - \sum_{e \in \delta(v)} x_e .$$

▷ **Observe:** Given $S \subseteq V$,

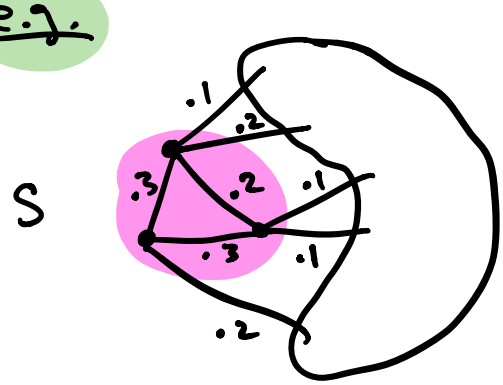
$$\sum_{v \in S} s_v + \sum_{e \in \delta(S)} x_e = |S| - 2 \sum_{e \in E(S)} x_e$$

Pf

$$|S| - \sum_{v \in S} \sum_{\delta(v)} x_e + \sum_{\delta(S)} x_e = |S| - 2\sum_{E(S)} x_e - \cancel{\sum_{\delta(S)} x_e + \sum_{\delta(S)} x_e}$$

($e \in E(S)$ double-counted).

e.g.



Thus

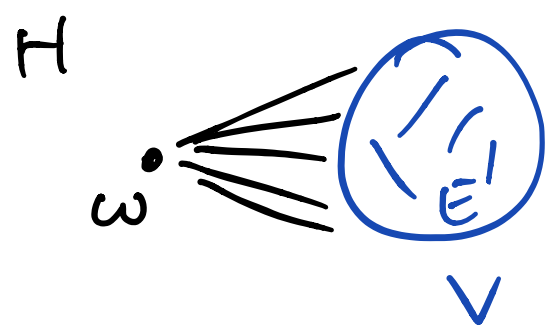
$$\star \iff \sum_{v \in S} s_v + \sum_{e \in \delta(S)} x_e \geq 1.$$

▷ Define new graph H with

vertex set $= V +$ new vertex w,

edge set $= E +$ all edges b/w w, V

picture:



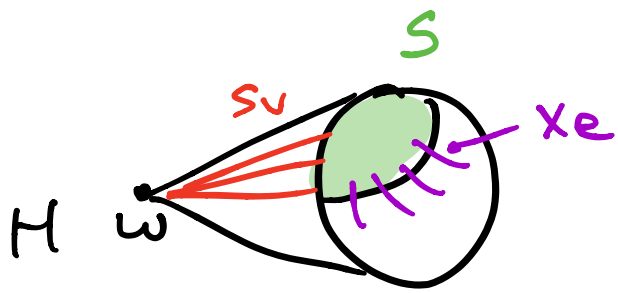
▷ Define edge weights

$$u_e = \begin{cases} x_e & \text{if } e \in E, \\ s_v & \text{if } e = (v, w). \end{cases}$$

▷ For a cut S in H, may assume $w \notin S$ by taking complements.

▷ cut $S \subseteq V$ has value

$$\sum_{e \in \delta_H(S)} u_e = \sum_{v \in S} s_v + \sum_{e \in \delta(S)} x_e .$$



▷ Thus $\sum_{v \in S} s_v + \sum_{e \in \delta(S)} x_e \geq 1$ for all odd $S \subseteq V$. $\Leftrightarrow$ min V-odd cut in H has value ≥ 1.

- we have seen how to compute min T-odd cut; do so for $T = V$.
  - if $\exists$ V-odd cut S with value < 1, S is violated inequality
  - if not, $x \in P$. $\square$