Lecture 7

Plan:

0. Discuss pset

1. Finish LP duality (prev notes).

2. Faces of Polyhedra

Faces of Polyhedra

Def: $a^{(1)}, \ldots, a^{(k)} \in \mathbb{R}^n$ are

affinely independent if

$$\sum_{i=1}^{k} \lambda_i a^{(i)} = 0$$

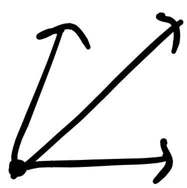
and $\sum_i \lambda_i = 0$ imply $\lambda_1 = \ldots = \lambda_k = 0$.

(w/out $\sum \lambda_i = 0$, is just linear independence).

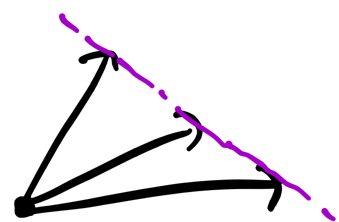
$\{a^{(i)}\}$ affinely independent if

$$\left\{ \begin{bmatrix} a^{(i)} \\ 1 \end{bmatrix} \right\} \text{ \underline{linearly} independent.}$$

Picture :



affinely
<u>independent</u>
in $\mathbb{R}^2$



affinely
<u>dependent</u>
in $\mathbb{R}^2$.

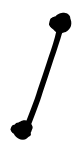
Def <u>Dimension</u> $\dim(P)$ of polyhedron P :

$-1 +$ max # affinely independent points in P.

Equivalently, dimension of affine hull $\mathrm{aff}(P)$.

Examples: $P = \emptyset$, $\dim(P) = -1$.

P = singleton $\cdot$ $\dim(P) = 0$

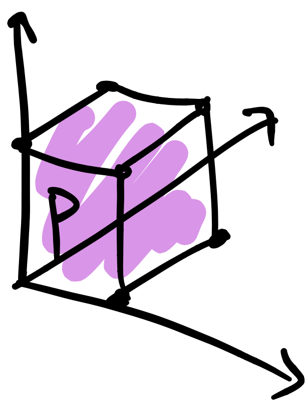
P = line segment  $\dim(P) = 1$

$\vdots$

$\mathrm{aff}(P) = \mathbb{R}^n$

$\dim(P) = n$;
P "full dimensional"

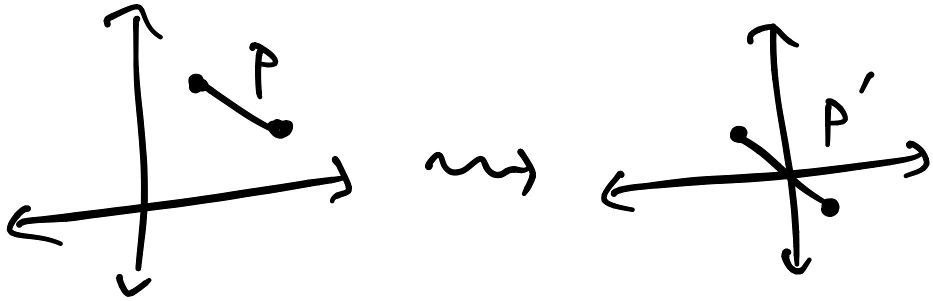
e.g. cube in $\mathbb{R}^3$:



$\dim P = 3$

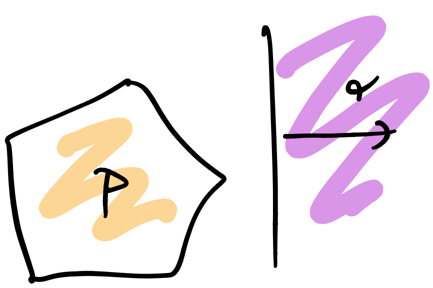
$\dim \mathbb{R}^3 = 3$
(as polyhedron).

Why affine, not linear? affine independence is translation invariant:



$$\dim P = \dim P'.$$

Def: $\alpha^T x \leq \beta$ is a <u>valid inequality</u> for P if $\alpha^T x \leq \beta$ for all $x \in P$.

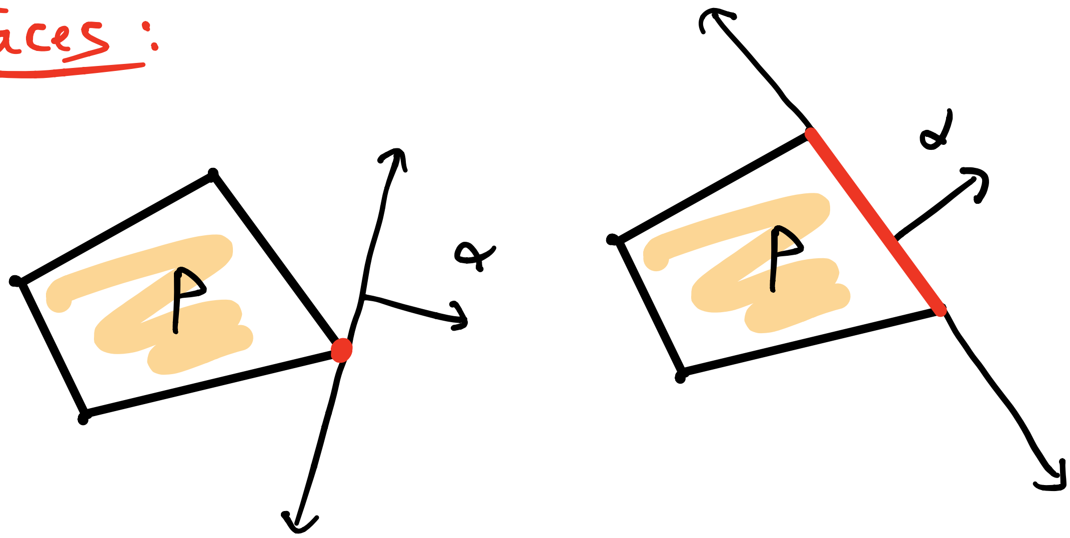


or

Def A <u>face</u> of a polyhedron P is $\{x \in P : \alpha^T x = \beta\}$ for

$\alpha^T x \leq \beta$ valid.

Faces:



Properties:

- Faces are polyhedra
- Empty face & entire P are called trivial faces

dim F = -1

dim F = dim P

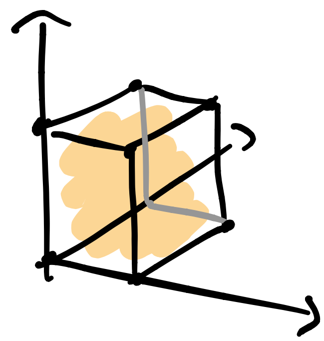
- else F nontrivial

$$0 \leq \dim(F) \leq \dim(P) - 1$$

- $F : \dim(F) = \dim(P) - 1$ called facets.

Ex: list the 28 faces of the cube

$$P = \{x \in \mathbb{R}^3 : 0 \le x_i \le 1\}$$

P = 

Fact: ∞ many valid ineqs, but # faces finite!

Theorem: Let $A \in \mathbb{R}^{m \times n}$.

$$A = \begin{bmatrix} & \vdots & \\ - & a_i^T & - \\ & \vdots & \end{bmatrix}$$

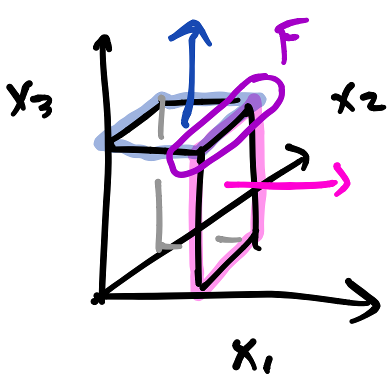
Any nonempty face of $P = \{x : Ax \le b\}$ is

$$\left\{ x : \begin{array}{ll} a_i^T x = b_i & \forall i \in I \\ a_i^T x \le b_i & \forall i \notin I \end{array} \right\}$$

for some set $I \subseteq \{1, \ldots, m\}$.

$\Rightarrow$ # nonempty faces of P is $\le 2^m$.

E.g. cube



$$F = \left\{ x : \begin{array}{c} x_3 = 1 \\ x_1 = 1 \\ 0 \le x_2 \le 1 \end{array} \right\}$$

Proof Consider valid inequality

$\alpha^T x \leq b$ giving nonempty face F.

- F = optimum solutions to bounded LP

$$\boxed{(P) \quad \begin{array}{ll} \max & \alpha^T x \\ \text{subject to} & Ax \leq b \end{array}}$$

- Let y^* optimal solution to dual.

- **Complementary slackness:**

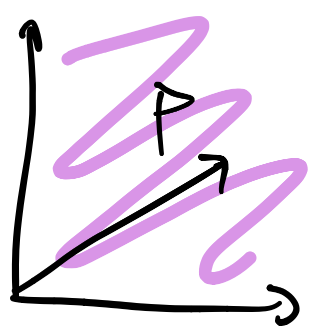
optimal solns F are

$$\boxed{\{x : a_i^T x = b_i \text{ for } i : y_i^* > 0\}.}$$

Thus we can take $I = \{i : y_i^* > 0\}$. $\square$

Ex: positive orthant $\{x \in \mathbb{R}^n : x_i \geq 0\} = P$ has $2^n + 1$ faces (theorem is tight!)

- How many of dim k?



For polytopes can bound # faces in terms of # vertices.

"upper bound theorem"

For extreme points (dimension 0 faces) can just use equalities.

Theorem Let x^* extreme point for $P = \{x : Ax \le b\}$.

$$A = \begin{bmatrix} & \vdots & \\ - & a_i^T & - \\ & \vdots & \end{bmatrix}$$

Then $\exists I$ s.t. x^* is the unique soln to

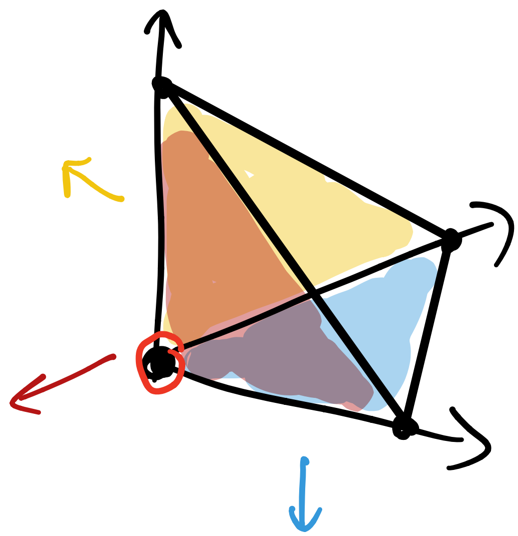
$$\boxed{a_i^T x = b \quad \forall i \in I.}$$

moreover, any such unique solution x^* is extreme.

is

e.g. simplex

0 is intersection of constraints $x_1 = 0$
$x_2 = 0$
$x_3 = 0$.



Proof: Given extreme point x^*,

- define $I = \{i : a_i^T x = b_i\}$.
- Note for $i \notin I$, $a_i^T x < b_i$
- By "faces theorem", x^* uniquely defined by

(*) $$a_i^T x = b_i \quad i \in I$$

(**) $$a_i^T x \leq b_i \quad i \notin I.$$

- Suppose $\exists$ other soln. $\hat{x}$ to (*).
- Because $a_i^T x < b_i$ for $i \notin I$,

$$(1-\varepsilon) x^* + \varepsilon X$$

still satisfies (*), (* *).

- Contradicts F having only one point $\square$.

Basic Feasible Solutions:

For $$P = \{ Ax = b \,,\ x \geq 0 \}$$

can describe feasible points very explicitly.

(Can convert all polyhedra to this form).

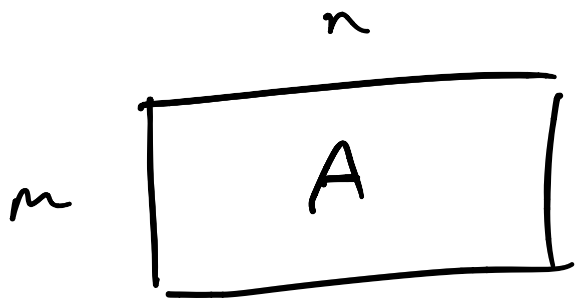
Corollary: Every extreme point of P can be obtained by setting

$$x_j = 0 \text{ for } j \in J$$

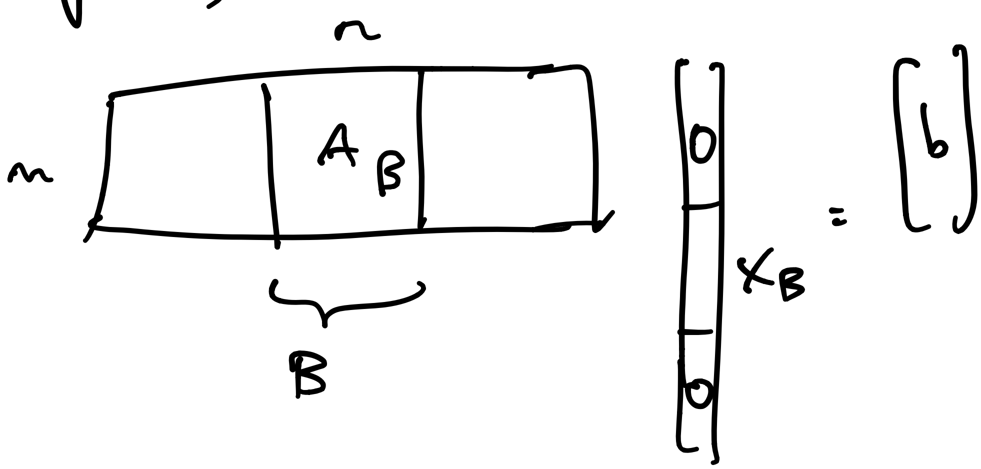
and finding unique solution to $Ax = b$ for remaining variables.

Can say more: Extreme points of $P = \{x : Ax = b, x \geq 0\}$ are the basic feasible solutions (bfs), feasible solns obtained as follows:

- Remove redundant rows from A (assume A has full row rank).



- Choose m columns B of A, (a basis for $\mathbb{R}^m$)



- Solve $A_B x_B = b$,
  set

$$x^*_i = \begin{cases} (x_B)_i & i \in B \\ 0 & \text{else.} \end{cases}$$

$$\boxed{\{\text{bfs}\} = \{\text{extreme pts}\}}.$$

Corollary: The facets are the maximal nontrivial faces of a nonempty polyhedron P.

Pf: Exercise.

Corollary: