Solutions to some of the exercises

1-2 An *edge cover* of a graph $G = (V, E)$ is a subset of R of E such that every vertex of V is incident to at least one edge in R. Let G be a bipartite graph with no isolated vertex. Show that the cardinality of the minimum edge cover R^* of G is equal to $|V|$ minus the cardinality of the maximum matching M^* of G. Give an efficient algorithm for finding the minimum edge cover of G. Is this true also for non-bipartite graphs?

Let $\rho(G)$ be the size of a minimum edge cover and $\nu(G)$ the size of the maximum matching. A maximum matching covers $2\nu(G)$ vertices. Because of the connectedness, the remaining $n - 2\nu(G)$ vertices can be covered by no more than $n - 2\nu(G)$ edges. These edges and the maximum matching thus form an edge cover of size $n - \nu(G)$. On the other hand, a minimum edge cover has to be a forest (an acyclic graph). (Indeed, if it has any cycle then the removal of any edge of the cycle would still give an edge cover, of smaller cardinality.) The number of connected components of this forest is precisely $n - \rho(G)$ because every component of the forest is a tree, and a tree on k vertices has $k - 1$ edges, and one can take one edge per component to get a matching. We therefore have $\nu(G) \geq n - \rho(G)$.

The first part of the proof clearly yields an algorithm for finding a minimum edge cover given an algorithm for finding a maximum cardinality matching.

Yes, the result remains true for non-bipartite graphs. Observe the proof above carries over for non-bipartite graphs.

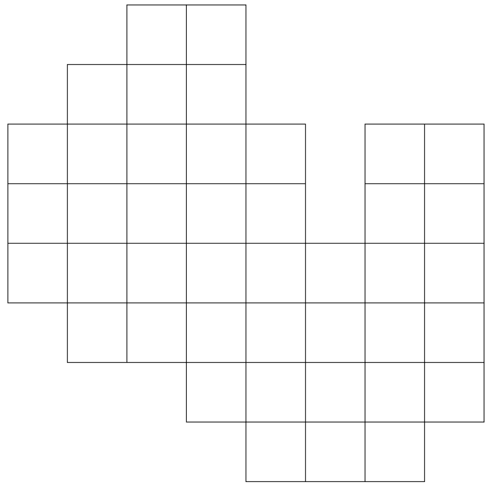
1-3 Show that in any graph $G = (V, E)$ (not necessarily bipartite), the size of *any maximal* matching M (i.e. a matching M in which one cannot add an edge while keeping it a matching) is at least half the size of a *maximum* matching M^*.

Let M^* be a maximum matching and M a maximal one. For every edge $e \in M^*$, at least one of its endpoints must be covered by edges of M. Otherwise the edge e can be added to M, which contradicts its maximality. It follows that the number of vertices covered by M is at least the number of edges in M^*, thus $2|M| \geq |M^*|$.

1-4 Consider the problem of perfectly tiling a subset of a checkerboard (i.e. a collection of unit squares, see example below) with dominoes (a domino being 2 adjacent squares).

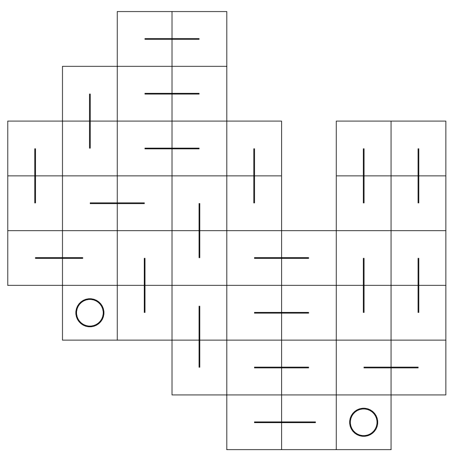
(a) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.

(b) Can the following figure be tiled by dominoes? Give a tiling or a short proof that no tiling exists.



(a) Consider the bipartite graph G with a vertex for each square and two squares are adjacent if they share an edge. This graph is bipartite since the squares can be colored black and white in a checkerboard pattern.

Any perfect tiling gives a perfect matching by simply selecting the edges corresponding to the dominoes selected. And vice versa.



(b)

Figure 0.1: Maximum configuration of dominoes.

We claim that the configuration shown in Figure 0.1 is a maximum one and so no perfect tiling exists. We will prove that the matching M corresponding to the configuration in Figure 0.1 is maximum by showing that there is no augmenting path as in the lecture. (Alternatively we could use Hall's theorem.)

Let A be the set of black squares and B the set of white squares. Orient the edges of G according to M, i.e. all the edges in M are oriented from B to A, and the edges not in M are oriented from A to B as in Figure 0.3.

Let v be the only exposed vertex of A and w be the only exposed vertex of B, and consider L to be the set of vertices reachable from v (the enclosed area in Figure 0.3). Since w is not in L we obtain that no augmenting path exists.

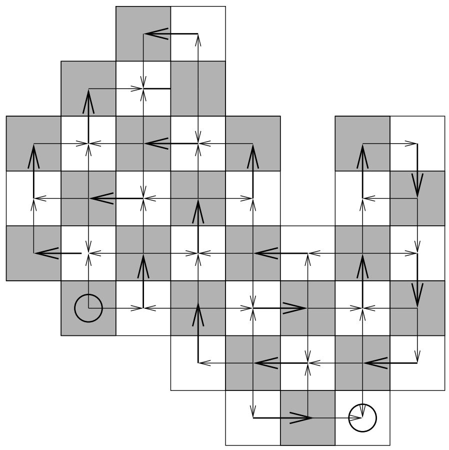


Figure 0.2: Oriented graph.

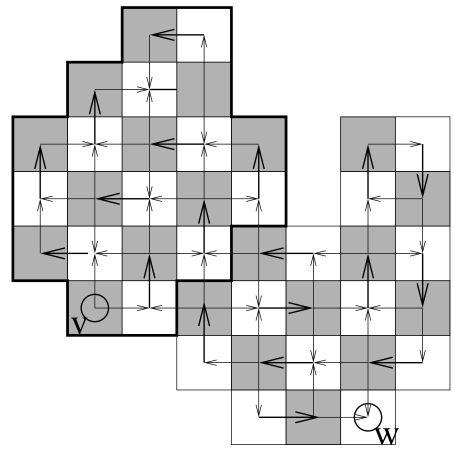


Figure 0.3: Set of reachable vertices from v.

We can also deduce the fact that no perfect matching exists from Hall's theorem by observing that the 11 black vertices in L (the enclosed region on the right of Figure 0.3) has only 10 (white) neighbors.

1-5 Consider a $m \times n$ checkerboard where m is even, and cells are alternatively colored black and white. Show that if we remove arbitrarily one black cell and one white cell, the resulting $mn - 2$ cells can be covered by dominoes.

The $m \times n$ checkerboard when m is even has a Hamiltonian cycle. After removing two cells of different colors, we get two even paths which can each be covered by a matching.

1-6 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B): $V = A \cup B$. Assume that, for some vertex sets $A_1 \subseteq A$ and $B_1 \subseteq B$, there exists a matching M_A covering all vertices in A_1 and a matching M_B covering all vertices in B_1. Prove that there always exists a matching covering all vertices in $A_1 \cup B_1$.

Let $G = (V, E) = (A \cup B, E)$, subsets $A_1 \subset A, B_1 \subset B$ and matchings M_A, M_B that cover A_1 and B_1, respectively. We construct a matching M that covers $A_1 \cup B_1$.

Clearly, the edge set $M = M_A \cup M_B$ covers $A_1 \cup B_1$, but it is not necessarily a matching. We show how to delete edges from M to make it into a matching. We know $M_A \triangle M_B$ is a union of disjoint cycles and alternating paths. The vertices with some incident edge from both $M_A \setminus M_B$ and from $M_B \setminus M_A$ are the only ones where M *fails to be a matching*. We show how to delete some edges from $M_A \triangle M_B$, so M is still a matching and no vertices are uncovered. We do so in each component of $M_A \triangle M_B$.

- **Cycle:** Since G is bipartite, the cycle has even length. Therefore, we can delete every other edge and the desired properties hold.
- **Path of odd number of edges:** We can delete every other edge starting from the edge that is adjacent to the last edge of the path. The desired properties hold. Note that this is possible only because the path has odd number of edges.
- **Path of even number of edges:** In this case, we can delete every other edge but one endpoint will be covered and the other uncovered. We need to prove that both endpoints cannot be in $A_1 \cup B_1$. Thus, we delete every other edge so the endpoint that is not in $A_1 \cup B_1$ is uncovered.

  We do so by contradiction: assume both endpoints are in $A_1 \cup B_1$. As the path has an even number of edges, and G is bipartite, then both endpoints must belong to the same bipartition set (A or B). W.l.o.g say they both belong to A, and thus also belong to A_1. Note that each vertex in A_1 has exactly one incident edge from M_A; thus the path we are analyzing (that is a connected component of $M_A \triangle M_B$) must contain these two edges. However, this path is of even length, and is alternating, so the end-edges cannot be from the same matching M_A ($\Longrightarrow\Longleftarrow$). This shows that our initial assumption is wrong, i.e., it must happen that both endpoints do not belong to $A_1 \cup B_1$, as desired.

1-7 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) $(V = A \cup B)$. Let $\mathcal{I} = \{X \subseteq A :$ there exists a matching M of G such that all vertices of X are matched$\}$.

Show that

(a) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

(b) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$ then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

(Later in the class, we will discuss matroids, and properties (i) and (ii) form the definition of independent sets of a matroid.)

(a) Let $Y \subset X \in \mathcal{I}$. Since X is an independent set, there exists a matching M_X that covers X. This matching also covers Y. Hence Y is an independent set.

(b) Let $X, Y \in \mathcal{I}$ with $|X| < |Y|$. It follows that there exist matchings M_X and M_Y such that M_X covers X and M_Y covers Y. Consider the graph $G' = (V, M_X \Delta M_Y)$. The set of edges of G' is the union of paths and cycles.

If M_X covers some element y in $Y \setminus X$. Then $X + y$ is an independent set.

Otherwise, all the vertices in $Y \setminus X$ are of degree 1 in G'. Since $|Y| > |X|$, we have $|Y \setminus X| > |X \setminus Y|$. Therefore, by the previous observation, there are more degree 1 vertices in $Y \setminus X$ than in $X \setminus Y$. It follows that there exists a path P in the decomposition of G' starting in a vertex $y \in Y \setminus X$ and not ending in X. We conclude that $M_X \Delta P$ is a matching of G that covers $X \cup \{y\}$. Thus, $X + y$ is an independent set.

1-8 Consider the following 2-person game on a (not necessarily bipartite) graph $G = (V, E)$. Players 1 and 2 alternate and each selects a (yet unchosen) edge e of the graph so that e together with the previously selected edges form a simple path. The first player unable to select such an edge loses. Show that if G has a *perfect* matching then player 1 has a winning strategy.

Assume that there exists a perfect matching M^* in G. Then consider the following strategy for Player 1:

- In the first move, select any edge $e_o \in M^*$.
- If Player 2 selects an edge $e = (v, w)$, where w is a new endpoint of the path, then select the edge of M^* that covers w.

It is easy to see that after every turn of Player 2, the path P constructed so far is an even alternating path for M^*, and thus it has only one vertex w that has not been covered yet by edges of $M^* \cap P$. Furthermore, this vertex must be an endpoint of the last edge added by Player 2. Since M^* is a perfect matching, Player 1 can always add the edge of M^* that covers w to the path. The only problem for this move could be that the new edge form a cycle, but this is not possible since all the vertices in $V(P) \setminus \{w\}$ were already covered by edges of the matching.

Since Player 1 can always select an edge, he cannot lose, thus this is a winning strategy.

1-9 Deduce Hall's theorem from König's theorem.

It is easy to see that the condition is necessary. To see that it is sufficient, consider a (minimum) vertex cover C of size equal to the maximum matching, by means of König's theorem and assume the condition given by Hall's theorem. For a contradiction, suppose that $|C| < |A|$. We have that $N(A \setminus C) \subseteq C \cap B$ (by definition of a vertex cover). Thus

$$|N(A \setminus C)| \leq |C \cap B| = |C| - |C \cap A| < |A| - |C \cap A| = |A \setminus C|.$$

This is a contradiction.

1-10 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B). For $X \subseteq A$, define $\mathrm{def}(X) = |X| - |N(X)|$ where $N(X) = \{b \in B : \exists a \in X \text{ with } (a, b) \in E\}$. Let

$$\mathrm{def}_{max} = \max_{X \subseteq A} \mathrm{def}(X).$$

Since $\mathrm{def}(\emptyset) = 0$, we have $\mathrm{def}_{max} \geq 0$.

(a) Generalize Hall's theorem by showing that the maximum size of a matching in a bipartite graph G equals $|A| - \mathrm{def}_{max}$.

(b) For any 2 subsets $X, Y \subseteq A$, show that

$$\mathrm{def}(X \cup Y) + \mathrm{def}(X \cap Y) \geq \mathrm{def}(X) + \mathrm{def}(Y).$$

(a) Clearly, the size of a maximum matching cannot be more than $|A| - \text{def}_{max}$ (since any matching has at most $|A| - |X|$ edges incident to $A - X$ and at most $|N(X)|$ edges incident to X).

Conversely, consider the minimum vertex cover C and let $X = A \setminus C$. Observe that $N(X) \subseteq C \cap B$, and thus

$$\text{def}(X) = |X| - |N(X)| \geq |A \setminus C| - |C \cap B| = |A| - |C \cap A| - |C \cap B| = |A| - |C|.$$

Therefore $\text{def}_{max} \geq |A| - |C|$ and the result follows from König's theorem.

(b) This is a simple counting argument. First of all,

$$|X \cup Y| + |X \cap Y| = |X| + |Y|.$$

Furthermore,

$$|N(X \cup Y)| + |N(X \cap Y)| \leq |N(X)| + |N(Y)|,$$

since every vertex b in B contributes at least as much to the right-hand-side than to the left-hand-side. Indeed, if $b \in N(X \cup Y) \setminus N(X \cap Y)$, it should be either in $N(X)$ or in $N(Y)$, while if $b \in N(X \cap Y)$, it should be in both $N(X)$ and in $N(Y)$.

1-11 Let $S = \{1, 2, \cdots, n\}$. Let A_k be the set of all subsets of S of cardinality k (thus $|A_k| = \binom{n}{k}$). Let $k < \frac{n}{2}$. Consider the graph G_k with bipartition A_k and A_{k+1}, and with $E = \{(a, b) | a \in A_k, b \in A_{k+1} \text{ and } a \subset b\}$.

(a) Prove that the maximum matching in G_k has size A_k (remember $k < n/2$).

(b) Prove *Sperner's lemma.* The maximum number of subsets of S such that no subset is contained into another is $\binom{n}{\lfloor n/2 \rfloor}$.

(a) Let X be a subset of A_k. Note that any vertex in A_k has degree $n - k$ in G_k. So, the number of edges between X and $N(X)$ is $(n - k)|X|$. On the other hand, the number of edges adjacent to $N(X)$ is $(k + 1)|N(X)|$ since any vertex in A_{k+1} has degree $k + 1$. Thus, $(n - k)|X| \leq (k + 1)|N(X)|$. Since $k < \frac{n}{2}$, we have

$$|X| \leq \frac{k + 1}{n - k}|N(X)| \leq |N(X)|.$$

By Hall's Theorem, there is a matching in G_k covering A_k.

(b) For a collection $\mathcal{C}$ of subsets of S, we call it a *chain* if for any $x, y \in \mathcal{C}$ either $x \subset y$ or $y \subset x$. In other words, chain is a sequence of subsets $a_1 \subset a_2 \subset \ldots \subset a_k$. On the other hand, we call a collection $\mathcal{F}$ of subsets of S an *antichain*, if no subset is contained in another. Note that any chain and antichain can share at most one element.

We claim that the collection of all subsets of S can be partitioned into $\binom{n}{\lceil n/2 \rceil}$ chains. It implies that the size of antichain is at most $\binom{n}{\lceil n/2 \rceil}$, since antichain can have at most one element from each chain.

Recall part (a). We know that G_k has a matching covering A_k if $k < \lceil \frac{n}{2} \rceil$. Similarly, if $k \geq \lceil \frac{n}{2} \rceil$ then G_k has a matching covering A_{k+1}. Let M be the union of those matchings in G_k for $k = 0, 1, \ldots, n-1$. Note that M consists of disjoint paths, and for each path there are indices k and ℓ such that the path is of the form $a_k a_{k+1} \ldots a_\ell$ where $a_j \in A_j$ for $j = k, \ldots, \ell$ and $a_j a_{j+1} \in M$. Moreover, each path contains exactly one element from $A_{\lceil \frac{n}{2} \rceil}$. Since each path is a chain, we have $\binom{n}{\lceil n/2 \rceil}$ disjoint chains covering all subsets of S.

1-15 Consider a bipartite graph $G = (V, E)$ in which every vertex has degree k (a so-called k-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

(a) by using König's theorem,

(b) by using the linear programming formulation we have derived in this section.

Let A, B be the bipartition of V.

(a) Because of k-regularity, we have $|A| = |B|$. Let $n = |A|$. By König's theorem, let C be a minimum vertex cover of size equal to the maximum matching. Then, $N(A \setminus C) \subseteq B \cap C$, and because of k-regularity, $|A \setminus C| \leq |B \cap C|$. Similarly, $|B \setminus C| \leq |A \cap C|$. Adding the inequalities we get $|V \setminus C| \leq |C|$, which implies that $|C| \geq n$.

(b) Any integer solution of the LP formulation

$$
\begin{array}{lll}
\text{Min} & \sum_{i,j} c_{ij} x_{ij} & \\
\text{subject to:} & & \\
& \sum_{j} x_{ij} = 1 & i \in A \\
& \sum_{i} x_{ij} = 1 & j \in B \\
& x_{ij} \geq 0 & i \in A, j \in B
\end{array}
$$

is a perfect matching. Also, all the extreme points (if any) of the LP are integral (see lecture notes on bipartite matching). Thus, it is enough to prove that the LP is feasible (so it will have at least one extreme point), and this is indeed the case as $x_{ij} = 1/k$ for all edges (i, j) is a feasible solution.

(One needs to add some assumption for the result to be true for non-bipartite graphs; indeed a cycle on 3 vertices does not have a perfect matching.)

1-17

Given a graph $G = (V, E)$, its edge coloring number is the smallest number of colors needed to color the edges in E so that any two edges having a common endpoint have a different color.

(a) Show that the edge coloring number of a *bipartite* graph G is always equal to its maximum degree Δ (i.e. the maximum over all vertices v of the number of edges incident to v). (Use the previous problem.)

(b) Give an example of a non-bipartite graph for which the edge coloring number is (strictly) greater than Δ.

(a) Clearly the edge coloring number is at least Δ since the Δ edges incident to a vertex of maximum degree have to be colored by different colors. To show the reverse inequality, first we will transform the graph G into a Δ-regular graph. For this purpose, first add vertices if needed so that both sides of the bipartition have the same number of vertices. Then add edges to the graph (in any way) so that every vertex has now degree Δ. In the resulting graph H, we know by the previous exercise that there exists a perfect matching. Deleting this perfect matching, we still have a regular graph, now a $\Delta - 1$-regular graph. We can therefore again extract a perfect matching, delete it and proceed. In this process, we have partitioned H into Δ perfect matchings, and thus the edges of H can be colored with Δ colors. Since G is a subgraph of H, restricting this coloring to the edges of G gives a valid coloring with (at most, and thus exactly) Δ colors.

(b) Consider a cycle on 3 vertices.

2-2 Let $G = (V, E)$, $S \subseteq V$ and a matching M that covers S. We construct a maximum matching M^* that covers S as follows. Start with matching $M_0 = M$. In a general step, we have that if M_i is not maximum, then there exists an augmenting path P_i. If we let $M_{i+1} = M_i \triangle P_i$, observe that the set of covered vertices by M_{i+1} includes the set of covered vertices by M_i. We iterate this procedure until M_n is a maximum matching, in which case we are done.

Observation: We do not present an efficient algorithm to find a desired maximum matching M^* from M. The statement of the problem does not ask for one.

2-3 Let U be a minimizer set and M be a maximum matching. Note that each edge in M is either an edge of some $G[K_i]$ or it is adjacent to some vertex in U. Also note that the number of edges in M adjacent to U is at most $|U|$ and the number of edges in M

from $G[K_i]$ is at most $\left\lfloor \frac{|K_i|}{2} \right\rfloor$. It follows that:

$$\begin{aligned}|M| &= |[M \cap \{e \in E : e \text{ adjacent to some } v \in U\}]| + \sum_{i=1}^{k} |M \cap E(G[K_k])| \\ &\leq |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \frac{1}{2}(|V| + |U| - o(G \setminus U)) = |M|,\end{aligned}$$

where the last equality holds because M is maximum and U is minimizer to the Tutte-Berge formula.

The previous formula implies that all the inequalities are equalities (if some of them is strict, we would have a contradiction). In particular we have that:

$$|[M \cap \{e \in E : e \text{ adjacent to some } v \in U\}]| = |U|,$$
$$\text{and for every } i,\ |M \cap E(G[K_k])| = \left\lfloor \frac{|K_i|}{2} \right\rfloor.$$

(a) So M has exactly $|U|$ edges adjacent to some vertex in U, and for every i, M contains exacly $\left\lfloor \frac{|K_i|}{2} \right\rfloor$ edges from $G[K_i]$. In particular, $G[K_i]$ is perfectly matched for even components K_i and near-perfect matched for odd components.

(b) From the previous analysis, every vertex u in U is matched to some vertex in $G \backslash U$. Since all the vertices of the even components are already matched, u must be matched to some vertex in an odd component K_i of $G \setminus U$.

(c) Finally, since all the vertices in U are matched to some vertex outside U, and all the vertices in each even component are perfectly matched, we obtain that the only unmatched vertices must be in odd components of $G \setminus U$.

1-20 For the assignment problem, the greedy algorithm (which repeatedly finds the minimum cost edge disjoint from all the previously selected edges) can lead to a solution whose cost divided by the optimum cost can be arbitrarily large (even for graphs with 2 vertices on each side of the bipartition).

Suppose now that the cost comes from a metric, even just a line metric. More precisely, suppose that the bipartition is $A \cup B$ with $|A| = |B| = n$ and the ith vertex of A (resp. the jth vertex of B) is associated with $a_i \in \mathbb{R}$ (resp. $b_j \in B$). Suppose that the cost between these vertices is given by $c_{ij} = |a_i - b_j|$.

Consider the greedy algorithm: select the closest pair of vertices, one from A and from B, match them together, delete them, and repeat until all vertices are matched. For these line metric instances, is the cost of the greedy solution always upper bounded by a constant (independent of n) times the optimum cost of the assignment? If so, prove it; if not, give a family of examples (parametrized by n) such that the corresponding ratio becomes arbitrarily large.

The greedy algorithm can provide solutions which are arbitrarily far away from the optimum. Reingold and Tarjan (SIAM J. on Computing, Vol. 10, 1981) show instances on a line for which the ratio between the greedy algorithm and the optimum cost matching is a factor more than $n^{0.58}$.

3-1 We have:

$$Ax = b, x \geq 0 \text{ has no solution}$$
$$\text{iff}$$
$$Ax \leq b, -Ax \leq -b, -Ix \leq 0 \text{ has no solution}$$
$$\text{iff}$$
$$Cx \leq d \text{ has no solution where } C = \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} \text{ and } d = \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$
$$\text{iff}$$
$$\exists (p,q,r) \geq 0 : A^T p - A^T q - Ir = 0, b^T p - b^T q < 0$$
$$\text{iff}$$
$$\exists (p,q) \geq 0 : A^T(p-q) \geq 0, b^T(p-q) < 0$$
$$\text{iff}$$
$$\exists y \geq 0 : A^T y \geq 0, b^T y < 0.$$

3-2 Write the primal (P) in standard form

$$\max c^T x$$
$$\text{s.t. } Ax \leq b$$

and its dual (D) as

$$\min b^T y$$
$$\text{s.t. } A^T y = c$$
$$y \geq 0$$

We can rewrite (D) by noting that $\min b^T y = \max(-b^T)y$ and that $A^T y = c$ is equivalent to $A^T y \leq c$ and $-A^T y \leq -c$.

$$\max(-b^T)y$$

$$\text{s.t.} \quad \begin{bmatrix} A^T \\ -A^T \\ -I \end{bmatrix} y \leq \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}$$

Then the dual of (D) is (DD) below

$$\min \begin{bmatrix} c^T & -c^T & 0 \end{bmatrix} y'$$

$$\text{s.t.} \quad \begin{bmatrix} A & -A & -I \end{bmatrix} y' = -b$$

$$y' \geq 0$$

If in (P), $x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$, then $y \in \mathbb{R}^m \implies y' \in \mathbb{R}^{2n+m}$. Rewrite y' as

$$y' = \begin{bmatrix} x_- \\ x_+ \\ z \end{bmatrix},$$

where each x_+, x_- has dimension n and z has dimension m; also let $x = x_+ - x_-$. Thus, we can rewrite (DD) as

$$\min c^T x_+ - c^T x_-$$

$$\text{s.t.} \quad Ax_+ - Ax_- - z = -b$$

$$x_+ \geq 0$$

$$x_- \geq 0$$

$$z \geq 0,$$

or equivalently,

$$\max c^T x$$

$$\text{s.t.} \quad Ax + z = b$$

$$z \geq 0$$

Of course, this is equivalent to (P) since the last two constraints above can be replaced by the single inequality $Ax \leq b$.

3-9 First consider the situation in which M_1 and M_2 are such that $M_1 \Delta M_2$ have more than one connected component. Consider one of these connected components, say $S \subseteq V$, and partition M_1 and M_2 into $M_1 = M_{1s} \cup M_{1t}$ and $M_2 = M_{2s} \cup M_{2t}$ where M_{1s} and M_{2s} correspond to the edges within S. By definition $M_{1s} \cup M_{2s} \neq \emptyset$. Now define two other matchings by $M_3 = M_{1s} \cup M_{2t}$ and $M_4 = M_{2s} \cup M_{1t}$. Observe that

$$\chi(M_1) + \chi(M_2) = \chi(M_3) + \chi(M_4)$$

which implies that any face that contains M_1 and M_2 will also contain M_3 and M_4, and thus cannot be an edge.

Conversely, suppose that $M_1 \Delta M_2$ has only one connected component, and say that this component has k_1 edges from M_1 and k_2 edges from M_2. We must have that $|k_1 - k_2| \leq 1$. Now consider the following cost function:

$$c_e = \begin{cases} 1 & e \in M_1 \cap M_2 \\ -1 & e \notin (M_1 \cup M_2) \\ k_2 & e \in M_1 \setminus M_2 \\ k_1 & e \in M_2 \setminus M_1. \end{cases}$$

Notice that $c(M_1) = c(M_2) = b$ where $b := |M_1 \cap M_2| + 2k_1k_2$ and for any other matching M we have that $c(M) < b$. Thus the valid inequality $c^T x \leq b$ induces a face with only teh incidence vectors of M_1 and M_2 has vertices. Thus the line segment between M_1 and M_2 defines an edge.

3-17 Let $I_i = [t_i, u_i) \subseteq \mathbb{R}$ be the time interval of activity i.

(a) We could write the integer program:

$$\begin{aligned} \text{Max} \quad & \sum_i p_i x_i \\ \text{subject to:} \quad & \\ & x_i + x_j \leq 1 && \forall i, j : I_i \cap I_j \neq \emptyset \\ & x_i \in \{0, 1\} && \forall i. \end{aligned}$$

And this would be correct for this subquestion. However, if we relax the integrality constraints some of the extreme points may be fractional (consider for example 3 identical intervals).

A better formulation as an integer program is:

$$\begin{aligned} \text{Max} \quad & \sum_i p_i x_i \\ \text{subject to:} \quad & \\ & \sum_{j : t_i \in I_j} x_j \leq 1 && \forall i \\ & x_i \in \{0, 1\} && \forall i. \end{aligned}$$

This is a valid formulation of the problem as any feasible (integer) solution cannot have two intervals, say I_k and I_l, that overlap; indeed the constraint for $i = k$ or for $i = l$ would be violated.

(b) Sort the rows of A according to the left endpoint t_i that defines them. This shows that A has the "consecutive ones property" along columns, that is, every column is simply a (possible empty) group of zeros followed by a (possible empty) group of ones followed by a (possibly empty) group of zeros. Indeed, for the column that corresponds to x_j, we have a one in row i if $t_i \in I_j$, and this implies that the ones will be consecutive.

Now, we will show that any matrix with the consecutive ones property is totally unimodular. Consider any square submatrix T of A; it also has the consecutive ones property. For every row of T except the last, subtract the next row. This operation doesn't change the determinant, but leaves at most one 1 and one -1 in each column, while making the others 0. If there is a column with exactly one nonzero entry, then by expanding the determinant along that entry, we can consider a smaller sized matrix T. Otherwise, every column has exactly one 1 and one -1. If we pre-multiply such a matrix by the all ones vector, we get the 0 vector. That is, the matrix is singular and the determinant is 0. This shows that in all cases the determinant of T is 0, 1 or -1.

4-2 Let G be any graph on the vertex set V. Let us define $E = V$ and $\mathcal{I} = \{S \subseteq E : S \text{ is covered by some matching } M\}$. To prove $(E, \mathcal{I})$ is a matroid, we need to verify two axioms.

(I_1) If $Y \in \mathcal{I}$ and $X \subseteq Y$ then $X \in \mathcal{I}$.

If M is a matching covering Y, since $X \subseteq Y$, then M also covers X. By definition, $X \in \mathcal{I}$.

(I_2) If $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ and $|Y| > |X|$ then there exists $v \in Y \setminus X : X \cup \{v\} \in \mathcal{I}$

Let M_X, M_Y be matchings covering X and Y, respectively. If there exists $v \in Y \setminus X$ such that v is a vertex covered by M_X, then clearly $X \cup \{v\} \in I$. Then let us assume such vertex v does not exist. This assumption allows us to say that a vertex covered by both M_X and M_Y is either in both X and Y, in X but not in Y, or in neither X nor Y. In other words, a vertex covered by M_X and M_Y cannot be in $Y \backslash X$. We wish to find a matching M'_X covering $X \cup \{v\}$, for some $v \in Y \backslash X$.

Consider the edges in $M_X \Delta M_Y$. We know they form cycles or simple paths. Since $|Y| > |X|$, then some cycle or some path must have more vertices from Y than from X. In cycles, all vertices are covered by both M_X and M_Y. As determined above, this implies that they cannot be in $Y \backslash X$. Similarly, we see that the inner vertices of a path cannot be in $Y \backslash X$. Thus, some path P has one of its endpoints v belonging to $Y \backslash X$. It is thus clear that $M'_X = M_X \triangle P$ is a matching covering $X \cup \{v\}$, for $v \in Y \backslash X$, as desired.

4-7 It is easy to see that $(E, \mathcal{I})$ satisfies the first axiom (I_1) that if $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$. For (I_2), consider $X, Y \in \mathcal{I}$ and $|Y| > |X|$, in order to show the second axiom (I_2), we need to show that there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$. Let us call a set $S \in \mathcal{F}$ maximal in $T \subseteq E$, $T \neq S$, if $S \subset T$ and S is not contained in any other element of $\mathcal{F}$ that is properly contained in T. Suppose that $A_1, \ldots, A_n$ are the maximal sets in E. Set $A^* = E \backslash (A_1 \cup \cdots A_n)$. Since $|Y| > |X|$, we must have

$|Y \cap A^*| > |X \cap A^*|$, or $|Y \cap A_i| > |X \cap A_i|$ for some i. In case $|Y \cap A^*| > |X \cap A^*|$, there is an element $e \in (Y \cap A^*) \in (X \cap A^*)$, and $X \cup \{e\} \in \mathcal{I}$. So we only need to study the case that $|Y \cap A_i| > |X \cap A_i|$ for some i. Without lost of generality we may assume $|Y \cap A_1| > |X \cap A_1|$.

Let $B_1, \ldots, B_m$ be the maximal sets in A_1 and let $B^* = A_1 \backslash (B_1 \cup \ldots B_m)$. Since $|Y \cap A_1| > |X \cap A_1|$, we have $|Y \cap B^*| > |X \cap B^*|$ or $|Y \cap B_i| > |X \cap B_i|$ for some i. Again if $|Y \cap B^*| > |X \cap B^*|$, there is an element $e \in (Y \cap A^*) \in (X \cap A^*)$, and $X \cup \{e\} \in \mathcal{I}$. Otherwise we can repeat this process for B_i satisfying $|Y \cap B_i| > |X \cap B_i|$. Since the ground set E is finite, we can find the required e in finite number of steps, and we are done.

4-8 Let $(j_1, j_2, ..., j_k)$ be a sequence of jobs ordered in increasing order on their deadlines, i.e., $d_{j_1} \leq d_{j_2} \leq \ldots \leq d_{j_k}$. If they could not be completed in time, there must exist some i for which $d_{j_i} < i$ (because j_1 will finish at time 1, j_2 will finish at time 2, etc.) However, this would imply that $d_{j_1}, d_{j_2}, ...d_{j_i} < i$. In other words, there are i jobs with deadline less than i; therefore at least i jobs need to be completed by the time $i - 1$. This implies that the sequence of jobs is infeasible. Thus the contrapositive of what we just proved is that if a sequence of jobs can be completed in some order, then they can be completed in order of their deadlines.

Now we prove M is a matroid by checking the two axioms.

I1 If $Y \in \mathcal{I}$ and $X \subset Y$, then $X \in I$.

This is obvious: if a set of jobs can be completed in time, then a subset of the jobs can also be completed in time.

I2 If $X \in I, Y \in I$ and $|Y| > |X|$ then $\exists e \in Y \setminus X : X \cup \{e\} \in I$.

Suppose both sets of jobs are ordered by deadline. Let $y = |Y|, x = |X|$ and e be one of the jobs with latest deadline in $Y \setminus X$. Suppose e is in position $y - k$ of Y. Let $K = \{j_{i_1}, j_{i_2}, ..., j_{i_k}\}$ be the set of jobs ordered by deadline $(d_{i_1} \leq d_{i_2} \leq ... \leq d_{i_k})$ that appear after e in Y.

Since e is in position $y - k$ in Y, we have $d_e \geq y - k$. Also $d_{i_t} \geq y - k + t$ because j_{i_t} is in position $y - k + t$. In order to prove $X + e \in \mathcal{I}$, we need to prove there is no q for which the job in position q in X has deadline q for $q \geq d_e$ (This is the only way for $X + e$ to be infeasible.) For the sake of contradiction, assume such q exists and job is x_q. Suppose there are n elements from K to the right of x_q.

If $n = k$, then x_q has at least k elements to its right. This means that it is in position at most $x - k$. However, $x - k < y - k \leq q$. This is a contradiciton since x_q is in position q in X.

If $n < k$, then x_q is to the right of $j_{i_{k-n}}$, which has deadline $d_{i_{k-n}} \geq y-k+(k-n) = y-n$. Therefore the deadline of x_q, which is q, satisfies $q \geq y - n > x - n$. However, since there are at least n elements to the right of the q element of X, we have $q \leq x - n$. Again a contradiction.

The contradiction proves that $X + e \in \mathcal{I}$, as desired.

Finally, to find an optimal scheduling, consider the value of each job j_i being its reward c_i. The greedy algorithm in matroid M then finds the optimal configuration.

PS4-1/13 Problem 1 from Problem Set 4 from 2013.

The set of sources is $So = \{f, i\}$, the set of sinks is $Si = \{e, h\}$. In the exchange graph, one needs to find a path of minimum distance from So to Si, for example, $P = f \to a \to h$. From this path, the larger independent set obtained is $S \triangle P = \{b, c, d, f, h\}$.

5-1 Our goal is to prove Konig's theorem using the matroid intersection theorem. Recall that Konig's theorem states that if $G = (A \cup B, E)$ is a bipartite graph, then the size of its maximum matching equals the size of its minimum vertex cover.

Consider $M_A = (E, \mathcal{I}_A)$ to be the matroid on the set of edges with $\mathcal{I}_A = \{I \subset E : |\delta(v) \cap I| \leq 1 \text{ for all } v \in A\}$. Define $M_B = (E, \mathcal{I}_B)$ similarly with B replacing A. The correspondence between an independent set in both matroids, i.e., an element $I \in \mathcal{I}_A \cap \mathcal{I}_B$, and a matching of M. Thus $\max_{S \in I_1 \cap I_2} |S|$ is the size of the maximum matching in G. We use the matroid intersection theorem with the matroids M_A, M_B, thus obtaining

$$\max_{S \in \mathcal{I}_A \cap \mathcal{I}_B} |S| = \min_{U \subset E} [r_A(U) + r_B(E \setminus U)]$$

We only need to show that the right hand side of the above expression equals the size of the minimum vertex cover in G. Observe that both M_A, M_B are partition matroids, for which we know the rank function. In this case, for any $U \subset E$, $r_A(U)$ is the number of vertices of A adjacent to some edge of U. Let $A(U) \subset A$ be this set of vertices adjacent to some $e \in U$. Likewise, $r_B(E \setminus U)$ is the number of vertices of B NOT adjacent to any edge of U and let $B(U) \subset B$ to this set of vertices. It is clear that $A(U) \cup B(U)$ is a vertex cover of G of size $r_A(U) + r_B(E \setminus U)$. This shows that the right hand side of the equation is at least the size of the minimum vertex cover. Thus, we have proven that the size of the maximum matching is at least the size of the minimum vertex cover. The other inequality is obvious (and indeed holds for any non-necessarily bipartite graph), so Konig's theorem follows.

PS5-1-/13 Problem 1 from Problem Set 5 from 2013.

Notice that one can begin by checking whether $w_i > w_n + g'$, where $g' = \sum_{i=1}^{n-1} g_{ni}$, for some $i < n$. Obviously if this was the case, team n could not win. Therefore, let us assume $w_i \leq w_n + \sum_{i=1}^{n-1} g_{ni}$ for all $i < n$. Moreover, if there was an outcome where team n won, then it could only get better if it won all of its games, so let us assume that all g_{ni} games to be played between teams n and i are won by team n. If we let x_{ij} to be the number of games between i and j won by i, then team n has a chance of winning iff there are positive integers x_{ij} with $x_{ij} + x_{ji} = g_{ij} = g_{ji}$, $i \neq j$, and $w_i + \sum_{j=1}^{n-1} x_{ij} \leq w_n + g'$ for all $i < n$.

Consider the graph G on $V = \{s,t\} \cup \{v(i,j)\}_{1\leq i<j\leq n-1} \cup \{w(i)\}_{1\leq i\leq n-1}$ with edges classified in these categories (all lower capacities are $l(e)=0$).

- All edges e from s to $v(i,j)$ with $u(e) = g_{ij}$.
- All edges e_1, e_2 from $v(i,j)$ to $w(i)$ and to $w(j)$ with $u(e_1) = u(e_2) = \infty$.
- All edges e from $w(i)$ to t with $u(e) = w_n + g' - w_i$.

We claim that team n can win iff the maximum flow from s to t is $\sum_{i<j} g_{ij}$. Indeed a maximum flow with that value exists iff there exists an integer flow $x : E \to \mathbb{Z}$ with that flow value exists (since all capacities are integers). If we let $y_{ij} = x(v(i,j), w(i))$, such integer flow satisfies $y_{ij} \geq 0$, $g_{ij} = y_{ij} + y_{ji}$ and $\sum_{j=1}^{n-1} x_{ij} \leq w_n + g' - w_i$. As asserted above, this is equivalent to team n having some chance.

PS5-3/13 Problem 3 from Problem Set 5 from 2013.

Let G be an undirected graph with minimum degree $\delta(G) \geq k$. We can consider G as a digraph by replacing an edge $\{u,v\}$ by two arcs $(u,v), (v,u)$ with all $l(e) = 0$ and $u(e) = 1$. We know that we can find k edge-disjoint paths between vertices $s \neq t$ from an integral flow from s to t of value at least k. Thus it will be enough to find two vertices s, t and a flow with value at least k between them.

Consider the following algorithm. Choose any vertex $v_1 \in V$ and let $S = \{v_1\}$. Iteratively, find $v_i = \arg\max_{v \in V \setminus S} c(S, \{v\})$ and let $S = S \cup \{v_i\}$. We get an ordering of the vertices $v_1, v_2, \ldots, v_n$. As shown in the lecture notes, $\{v_n\}$ induces a minimum (v_{n-1}, v_n)-cut; the value of such minimum cut equals to the outdegree of v_n, which is at least k by assumption. From duality, the maximum flow value between v_{n-1} and v_n is at least k, as desired.