18.453 Practice Final

Instructions. This is practice for a **timed** final. This is meant to be done in **3** hours with access to notes and course material, but no access to collaborators. For best practice I suggest trying to complete it under these conditions. Afterwards please tell me if 3 hours felt like enough.

1. Answer true or false. For items **not** marked with *, if true, provide a concise reason (no rigor necessary) and if false, exhibit a counterexample.

   (a) Every matching that is not maximum in a graph G has an augmenting path.*

   (b) If A, b are integral, then the linear program $\max\{c^T x : Ax \leq b\}$ has an integral maximizer.

   (c) The set of matchings in a bipartite graph forms a matroid.

   (d) Given a bipartite graph, the set of subgraphs of degree at most two is the intersection of two matroids.

   (e) Given a separation oracle for a polyhedron $P \subset [0,1]^n$, it is always possible to test feasibility of P with polynomially many calls to the separation oracle.

   Answers.

   (a) Yes.

   (b) No. For the following program the only maximizer is fractional:

   $$\max\left\{ x_1 \;\middle|\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

   (c) No. For the complete graph with parts $\{a_1, a_2\}, \{b_1, b_2\}$ the exchange axiom is violated for the independent sets $X = \{\{a_1, b_1\}\}$, $Y = \{\{a_1, b_2\}, \{a_2, b_1\}\}$.

   (d) Yes. For a graph $G = (V, E)$ with bipartition $V = A \sqcup B$, the set of subgraphs of degree at most two is the intersection of two partition matroids $(E, \mathcal{I}_A)$ and $(E, \mathcal{I}_B)$ defined as follows. Let $E = \bigsqcup_{a \in A} E_a$ be the partition of edges according to their left endpoint; then $\mathcal{I}_A = \{E' \subset E \mid |E' \cap E_a| \leq 2 \ \forall a \in A\}$. Similarly one defines $\mathcal{I}_B$.

   (e) No. It is impossible to tell a single point from the empty set, because the separation oracle might output the same.

2. For $k \leq n$ an integer, define a *k-bounded permutation* on $\{1, ..., n\}$ to be a permutation σ such that $|\sigma(i) - i| \leq k$ for all $i \in \{1, ..., n\}$.

   Suppose we are given an integer $k \leq n$ and costs $c(i)$ for $i \in \{1, ..., n\}$, and our goal is to find a k-bounded permutation σ on $\{1, ..., n\}$ minimizing $\sum_{i=1}^{n} c(i)\sigma(i)$. Give a polynomial-time algorithm for this problem (there is no need to give the most efficient algorithm, but the algorithm should be polynomial in n and k). (You can refer to any algorithm we have seen in class.)

   Solution: This can be cast as an instance of the minimum cost perfect matching problem. A permutation σ is none other than a matching in the complete bipartite graph with bipartition $A \cup B$ equal to two disjoint copies of $[n]$, where there is an edge between i and $\sigma(i)$. The restriction that the permutation is k-bounded requires that the permutaiton correspond to a matching in the subgraph $G = (A \cup B, E)$ where $E = \{(i, j) : i \in A, j \in B, |i - j| \leq k\}$. If we set the cost of the edge (i, j) to be $c_{i,j} = c(i)j$, then the minimum cost perfect matching in G yields the minimum cost k-bounded permutation. This can be solved in (strongly) polynomial time using the Hungarian algorithm.

3. (a) Consider a directed graph $G = (V, E)$ with nonnegative (upper) capacities $u : E \to \mathbb{R}$ (and no lower capacities). For any two vertices $s, t \in V$, define $\lambda_{st} \in \mathbb{R}$ to be the maximum flow value from s to t. Given any 3 vertices $s, t, u \in V$, show that $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$.

   (b) If the graph is undirected, the previous result still holds: $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$ for all s, t, u. Furthermore, $\lambda_{st} = \lambda_{ts}$. Now, consider the complete graph K_V on the vertex set V with weight λ_{uv} on edge $\{u, v\}$ for all u, v. Let T be a *maximum weight* spanning tree on K_V with respect to these weights λ_{uv}. Argue that for every $\{s, t\} \notin T$, we have

$$\lambda_{st} = \min_{\{u,v\} \in P_{st}} \lambda_{uv}$$

   where P_{st} denotes (the edges of K_V of) the unique path in T between s and t. (This implies the somewhat surprising result that, over all pairs (s, t), λ_{st} can take at most $|V| - 1$ values (those along the edges of T).)

Solution.

(a) Consider a minimal cut separating s and u, that is, a partition $V = S \sqcup U$ such that $s \in S$, $u \in U$, $\lambda_{su} = \sum_{e \in \delta^+(S)} u(e)$. If $t \in S$, then the same cut separates t and u, and so $\lambda_{su} \geq \lambda_{tu}$. Otherwise $t \in U$, the cut separates s and t, and $\lambda_{su} \geq \lambda_{st}$. In either case, $\lambda_{su} \geq \min(\lambda_{st}, \lambda_{tu})$.

(b) Using the previous part, one shows by induction that $\lambda_{st} \geq \min_{\{u,v\} \in P_{st}} \lambda_{uv}$. Let $\{u, v\}$ be a specific edge delivering the minimum in the right-hand side. We'd like to argue that $\lambda_{st} \leq \lambda_{uv}$. Indeed, if not, then $\lambda_{st} > \lambda_{uv}$, and one can replace the edge $\{u, v\}$ in the tree T by the edge $\{s, t\}$. The resulting subgraph is still a spanning tree, whereas its weight gets increased, which contradicts the choice of T. Therefore, $\lambda_{st} = \lambda_{uv}$.

4. Consider a bipartite graph $G = (A, B, E)$ with parts A, B and edges $E \subseteq A \times B$. Suppose we have a matroid $M_A = (A, \mathcal{I}_A)$ on A with rank function r_A. Define a family of sets $\mathcal{I}_B$ to be the collection of sets $T \subseteq B$ such that there exists a matching M of G with vertex set $V(M) = S \cup T$, such that $S \subseteq A$ and $S \in \mathcal{I}_A$.

   Prove that $M_B = (B, \mathcal{I}_B)$ is a matroid. (For **half credit**, you can do this in the special case where every vertex of A has degree 1, so that G is the graph of a function from A to B.)

   Solution: We need to show that $\mathcal{I}_B$ is downward-closed and satisfies the exchange property. Downward closure follows because if T is matched to independent set $S \in \mathcal{I}_A$, then $T' \subset T$ is matched to a subset of $S' \subset S$ which is also independent.

   For the exchange property, we must show that given $T', T \in \mathcal{I}_B$ with $|T'| > |T|$, there is some element $t \in T' \setminus T$ such that $T + t \in \mathcal{I}_B$. That is, $t \in T' \setminus T$ such that $T + t$ can be matched to an independent set in $\mathcal{I}_A$. Let M be a matching between T, S for $S \in \mathcal{I}_A$, and let M' be a matching between T', S' for $S' \in \mathcal{I}_A$. We'll use alternating paths in $M \cup M'$ to gradually modify M until we get the matching we want.

   As $|S'| > |S|$, the basis exchange property for I_A says there is an element a in $S' \setminus S$ such that $S + a$ is independent. The element a is covered by M' but not M, so has degree 1 in $M' \cup M$. As $M' \cup M$ has maximum degree two, the connected component containing a is a path P. Moreover, it must be alternating in M. If this path ends in B, then we may augment M along P to cover an element t of $T' \setminus T$, and we are done. If instead the path ends at some element $a' \in A$, then $a \in S \setminus S'$. The symmetric difference of $M \Delta P$ matches T to the independent set $S + a - a'$. Replace $S \leftarrow S + a - a'$ and M by $M \Delta P$. The size of the intersection $S \cap S'$ has increased by one. Continue this process until we are done (we have covered an element of $T' \setminus T$) or $S \subset S'$. Once $S \subset S'$, we can take t to be any element of $T' \setminus T$.
   Note: In retrospect this problem was too hard. It's still good practice, but don't let it scare you.

5. Let $x \in [0,1]^n$ be an unknown vector, and we suppose have access to a separation oracle for the set $S = [x_1, x_1 + 0.1] \times \cdots \times [x_n, x_n + 0.1] \subset \mathbb{R}^n$. Can we find a point in S in time polynomial in n, and if so, how? (You can refer to any algorithm we have seen in class).

Solution.

We exploit the ellipsoid method. The starting ellipsoid E_0 is the ball of radius $\sqrt{n}/2$ centered at $(1/2, \ldots, 1/2)$. Recall Claim 7.2 from the notes: the volumes of successive ellipsoids E_k in the method decay exponentially,

$$\mathrm{Vol}(E_k) \leq \mathrm{Vol}(E_0) \exp\left(-\frac{k}{2(n+1)}\right).$$

If by k^{th} step a point in S is not found, then $S \subset E_k$, and

$$0.1^n = \mathrm{Vol}(S) < \mathrm{Vol}(E_k) \leq \mathrm{Vol}(E_0) \exp\left(-\frac{k}{2(n+1)}\right).$$

It follows that $k < 2n(n+1)\log 10$. Hence, after $2n(n+1)\log 10$ iterations of the ellipsoid method a point in S will be found.