Lectures 18 & 19

Plan:

1) ~~briefly recap matroid polytope~~
2) ~~Matroid intersection~~ activity
3) largest common independent subset

Matroid intersection

- Matroids very nice b/c greedy works. given $c: E \to \mathbb{R}$, $\max_{S \in \mathcal{I}} c(S) = \sum_{e \in S} c(e)$
- But greedy doesn't work for lots of problems,
  e.g. ▷ max matching,
  ▷ max stable set in graph.

$\Rightarrow$ matroids very limited!

$|2^E| = 2^{|E|}$

A.K.A. powerset

matroid intersection much richer.

2^E = set of subsets of E e.g. $2^{\{1,2\}} = \{\{1,2\}, \{1\}, \{2\}, \emptyset\}$.

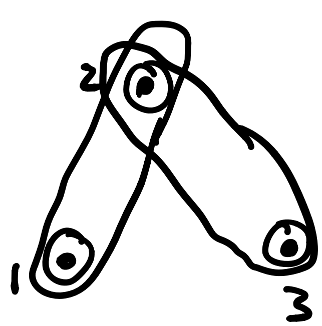
Def of $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ matroids on common ground set E, their **intersection** is just

$$\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq 2^E, \quad (\mathcal{I}_1, \mathcal{I}_2 \subseteq 2^E)$$

i.e. the sets indep. in M_1 & M_2.

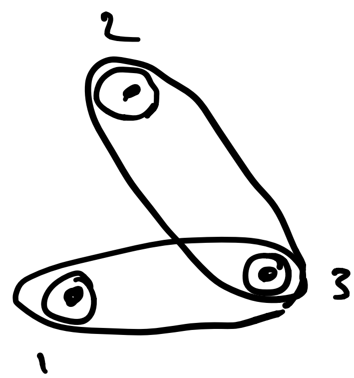
E.g. $E = \{1, 2, 3\}$

$\mathcal{I}_1 =$



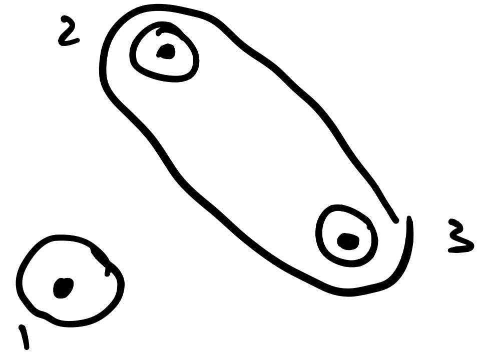
$+ \emptyset$

$\mathcal{I}_2 =$



$+ \emptyset$

$I_1 \cap I_2 =$



$+ \emptyset.$

- Activity: lots of examples!

- We'll show how to find largest common independent set efficiently! (Next time, probably).

Largest Common indep. set

- we give a min-max characterization for L.C.I.S.

i.e.

- allows us to prove:

- Let

- Let

Then

- max over S, min over $\mathcal{U}$:

$\leq$

"strong duality":

Theorem: (Edmonds)

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| =$$

Remark: Enough to minimize over

-
-

But

E.g. Special cases:

- Can show (Exercise) that orientn G w/ indegree $\leq p(v)$ possible $\Leftrightarrow$

- Can show $\exists$ colorful spaing tree $\Leftrightarrow$

Proof of Theorem

- proof is "primal-dual", i.e.

- Uses

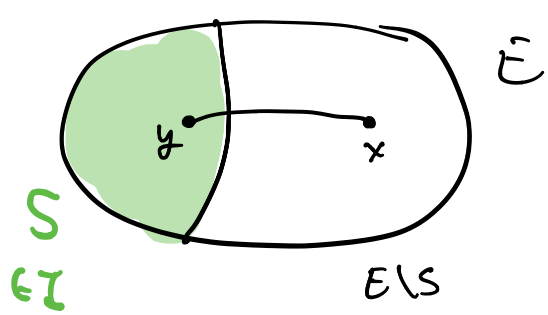
first, undirected:

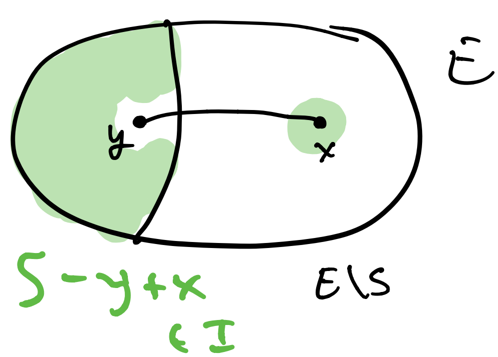
Def Given $S \in \mathcal{I}$, (undirected) exchange graph

$G_M(S) :=$ •

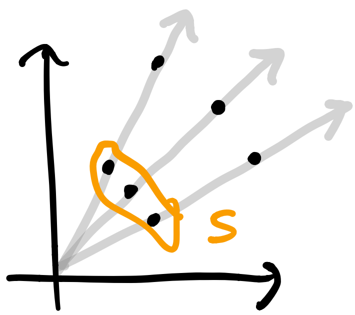
•

•





i.e. a linear matroid



equivalent:

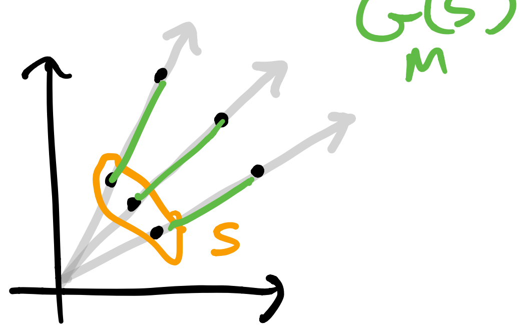
- Arises in "matroid sampling",

- For us, useful for the following reason:

Lemma: Let

Then

e.g.



proof: Exercise ( ).

and a partial converse.

Lemma: Let

suppose

Then

Proof: Let N be unique matching.

Claim: Can order

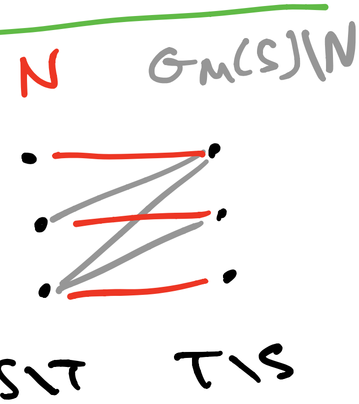
$S \setminus T$:

$T \setminus S$ =

so that

and

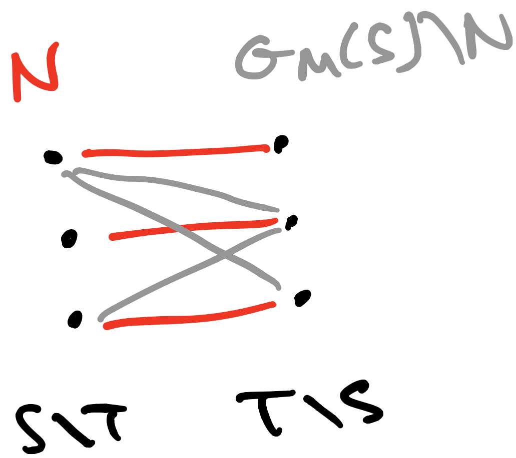
e.g.



proof of claim:

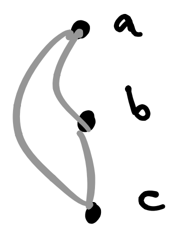
- Ignore
- Orient
- Others

e.g.



- Contract

e.g.

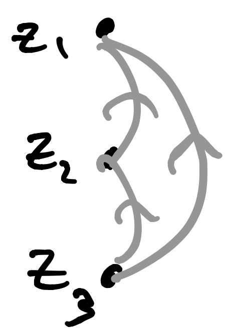


- Get acyclic directed graph

  ( ).

- Topologically order

  ( ).



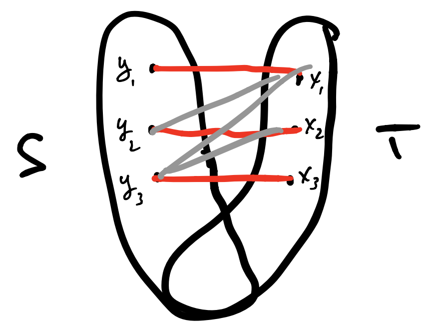
- Let

$\square$

Now, for contradiction:

- then

e.g.

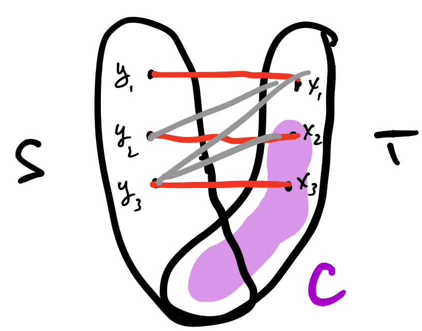


- then

( ).

- Let

e.g.



- Now we'll find that

( ).

- To show this, observe

b/c

$\Rightarrow$

$\square$

• now generalize to directed exchange graph

Def For $S \in \mathcal{I}_1 \cap \mathcal{I}_2$, (directed) exchange graph

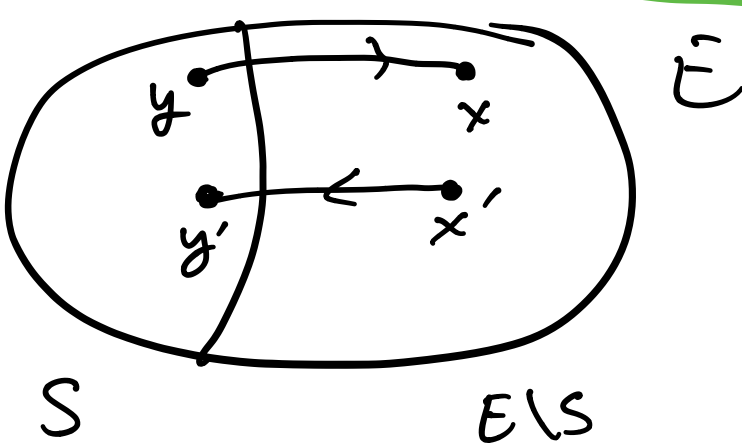
$D_{M_1, M_2}(S) :=$ •

•

•

•

Picture:



here

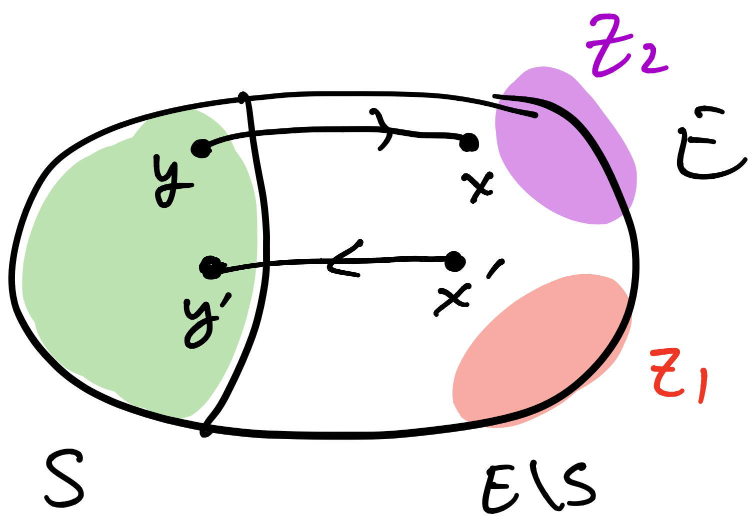
• **Note:**

- Also define:

  "sources" $Z_1 :=$

  "sinks" $Z_2 :=$

e.g.



Algorithm

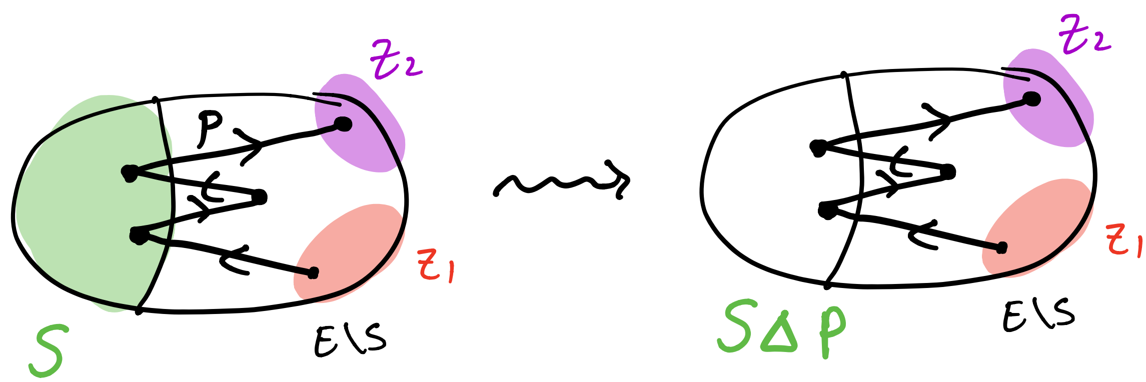
Initialize .

▷ Repeat:

  ▷ if :

▷ $P :=$.

▷ Replace ( ).



▷ else:

▷ return

$U = \{ \quad \}.$

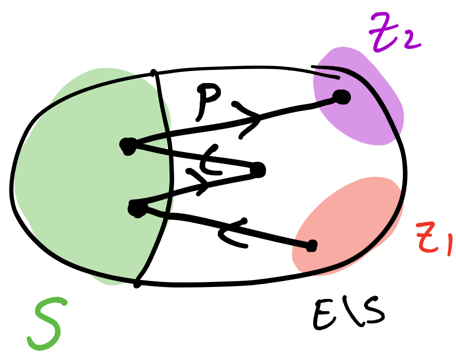
Correctness:

- **Claim 1:**
- **Claim 2:**

Proof of Claim 1:

- Recall P shortest path;

e.g.



- Enough to show:

- We first show

- To do this, define new matroid $M_1' = (E', I')$

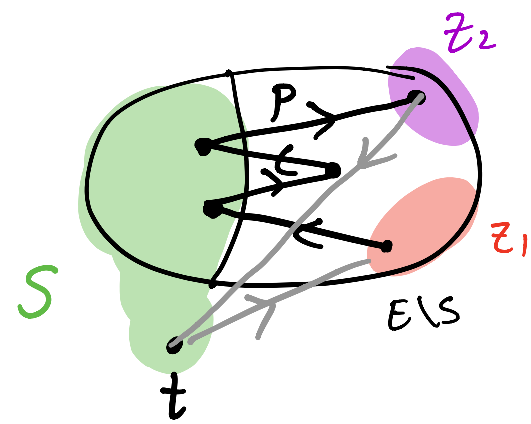
$E' =$ and $I' =$

i.e.

- Define M_2' analogously

consider

~~e.g.~~

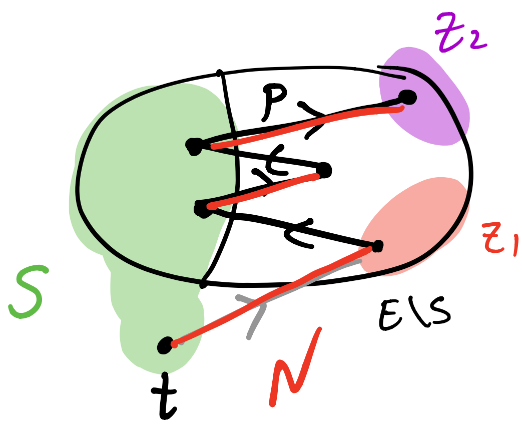


- Note

- View

- Observe

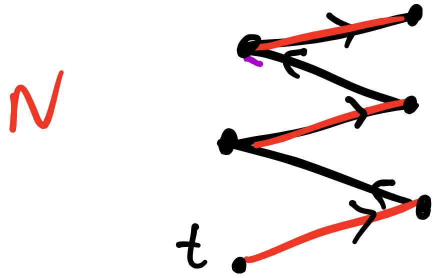
e.g.



( ).

- And

.



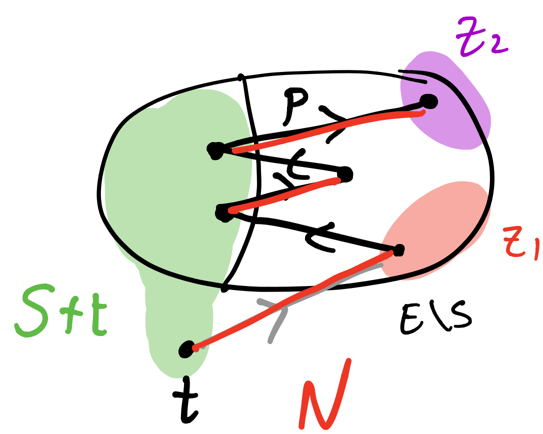
( )

- Unique perfect matching lemma

$\Rightarrow$

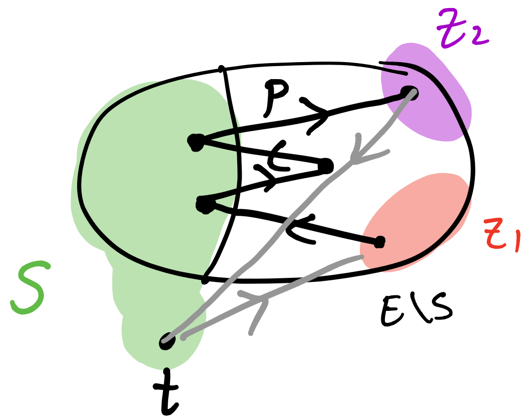
$\Rightarrow$ ( ).

e.g.



- To show

e.g.



•

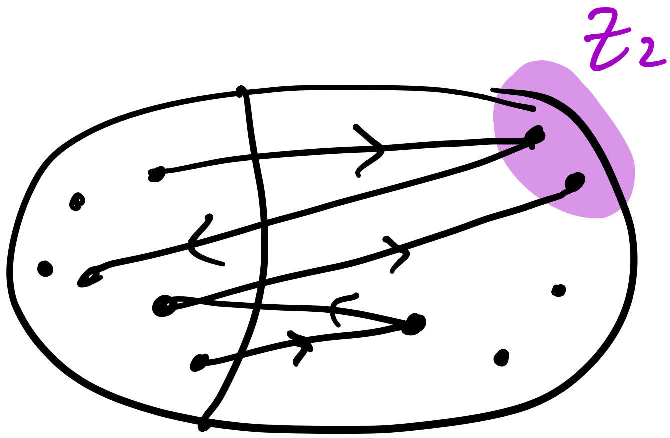
$\square$.

Proof of Claim 2:

• Want to show

where $u =$

e.g.



- First note and

- Enough to show

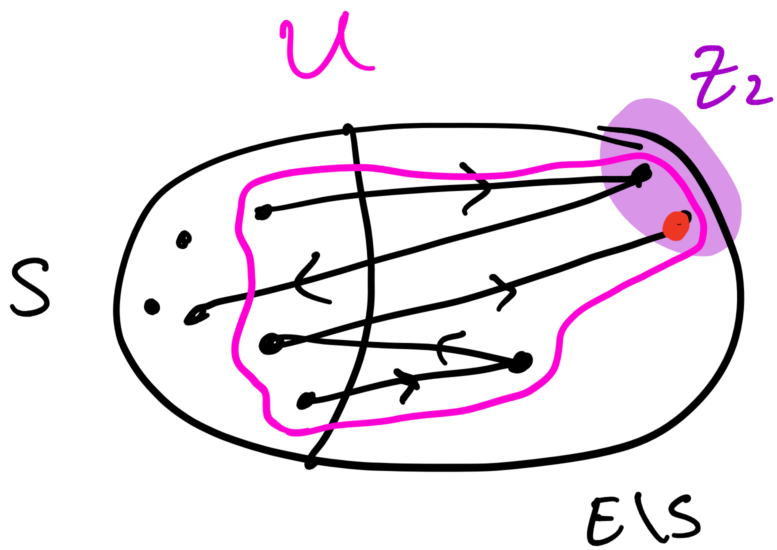
( )

- Suppose

-

$\Rightarrow$

$\Rightarrow$

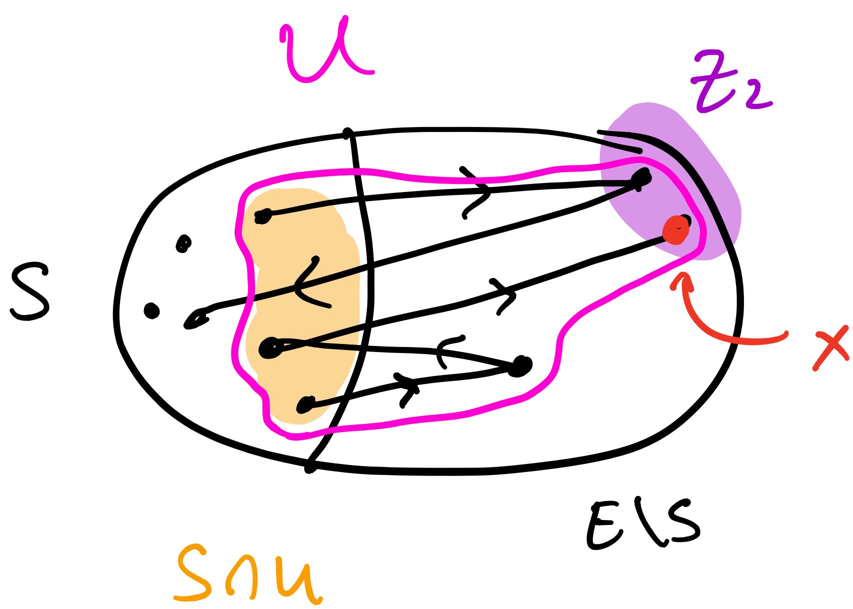


• $S \in \mathcal{I}_1 \Rightarrow$

(

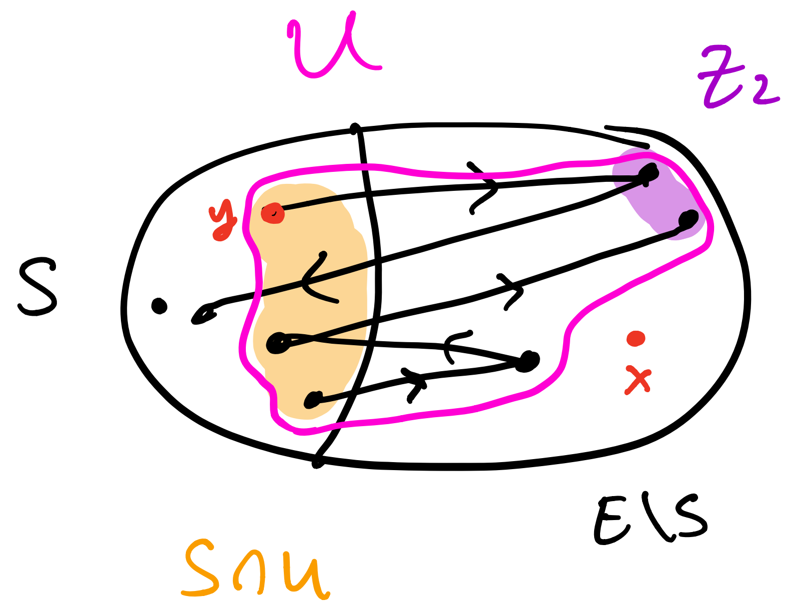
:)

• But then



- Case $r_2(E\setminus U) \neq |S\setminus U|$

similar;



$\square$.