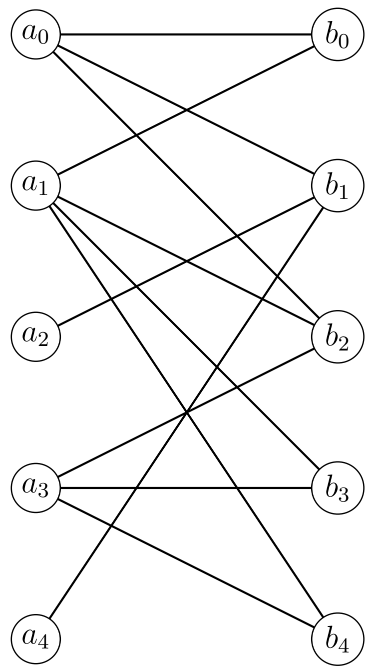
18.453 Quiz

Instructions. This is both an **in-class** quiz and a **take-home** quiz. For the *in-class* quiz, answer the following questions in the blue booklet and hand in the booklet at the end of the quiz (3:55PM on April 9th, 2019). Then take the questions home, and you should answer them again by the beginning of class on Thursday April 18th. For the take-home part, you can consult the lecture notes and other material, but you are not allowed to discuss it with your friends. Please write the solutions to the **take-home** as **neatly** as possible, and show your steps. Your grade will be the average of these two grades.

1. (a) Given a bipartite graph, state König's theorem about the size of the maximum matching in G.

   (b) Find a minimum vertex cover in the following graph, and give a short argument for its optimality.



2. An $n \times n$ matrix A is called *doubly stochastic* if all the entries of A are nonnegative, and if the entries of every row and column of A sum to 1.

   (a) Define a bipartite graph G_A on with vertex sets $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$, and with an edge connecting a_i to b_j whenever $A_{ij} > 0$. Show that if A is doubly stochastic, then G_A has a perfect matching.

   (b) Show that every $n \times n$ doubly stochastic matrix can be written as a convex combination of at most n^2 permutation matrices.

   (c) (**Extra Credit**) Can the number n^2 in part (b) be reduced? If so, how much can you reduce it?

3. Suppose $G = (V, E)$ is a 2-edge-connected graph (that is, G remains connected if you delete any single edge) with at least one perfect matching, and suppose that G has a special edge e which shows up in *every* perfect matching of G. Show that there is necessarily a nonempty set $S \subseteq V$ with the following properties:

- the number of odd components of $G \setminus S$ is exactly $|S|$,
- $G \setminus S$ has at least one even component.

4. Given a *bipartite* graph $G = (V, E)$ with bipartition $V = A \cup B$ and given an integer k, consider the set of all matchings of cardinality at most k. We know that if there was no constraint on the cardinality (for example, if $k \geq |V|/2$) then the convex hull P of all (incidence vectors of) matchings would be given by

$$P = \{x \in \mathbb{R}^{|E|} : \begin{array}{ll} \displaystyle\sum_{j \in B:(i,j)\in E} x_{ij} \leq 1 & i \in A \\ \displaystyle\sum_{i \in A:(i,j)\in E} x_{ij} \leq 1 & j \in B \\ x_{ij} \geq 0 & (i,j) \in E\} \end{array}$$

In this exercise, you will show that the convex hull P_k of all matchings of cardinality at most k is given by

$$P_k = \{x \in \mathbb{R}^{|E|} : \begin{array}{ll} \displaystyle\sum_{j \in B:(i,j)\in E} x_{ij} \leq 1 & i \in A \\ \displaystyle\sum_{i \in A:(i,j)\in E} x_{ij} \leq 1 & j \in B \\ \displaystyle\sum_{(i,j)\in E} x_{ij} \leq k & \\ x_{ij} \geq 0 & (i,j) \in E\} \end{array}$$

Here are three ways to prove it. Do **just one** of them for the in-class quiz for full credit (I would suggest doing the first one), but do **two** of them for the take-home version.

(a) Show that the underlying matrix A is totally unimodular, where $P_k = \{x : Ax \leq b, x \geq 0\}$. If you use this way, first define what a totally unimodular matrix is, specify what the matrix A look like, and explain why this implies the description of P_k.

(b) Provide a reduction between matchings of cardinality at most k in G and feasible integer flows (of any value) between two vertices s and t in an augmented graph G'. Explain why this would imply the integrality of the description of P_k.

(c) Consider any vertex x^* of P_k. First argue that x^* is in a face of dimension 1 of P (the matching polytope without restriction on the cardinality), i.e. x^* can be seen as a convex combination of incidence vectors of two *adjacent* matchings M_1 and M_2 of G. Then state (without proof) the condition for two matchings M_1 and M_2 to be adjacent on P. Finally conclude that x^* must have been the incidence vector of either M_1 or M_2.

5. (**Take-home only problem**) Consider a directed graph $G = (V, E)$ with vertices $s, t \in V$ and capacities $u(e)$ for each edge $e \in E$. Define the *flow polytope* to be the set

$$P = \{x \in \mathbb{R}^{|E|} : \sum_{e \in \delta^+(u)} x_e - \sum_{e \in \delta^-(u)} x_e = 0 \quad u \in V \setminus \{s, t\}$$
$$0 \leq x_e \leq u(e) \quad e \in E\}.$$

Suppose we start with a (non-maximal) flow x which corresponds to a *vertex* of this flow polytope P, find an augmenting path from s to t, compute the bottleneck for this path, and push that much flow along the augmenting path to make an augmented flow x'. Does the augmented flow x' necessarily correspond to a vertex of the flow polytope P?