Lecture 24

Plan:

1) Ellipsoid for LP
2) If time, examples.

Ellipsoid for LP

- Even for feasibility of $P = \{x : Ax \leq b\}$, are issues!
- Finding starting ellipse,

- Bounding volume of P.

- To avoid numerical details, study important special case:

$$\boxed{\text{Assume } P = \qquad\qquad ,}$$

e.g.

$P =$

- can handle $\dim P < n$ by

- Given $c \in \mathbb{R}^n$, want

- What's polynomial time here?

- **Input-size:**
  - ▷ Assume $c \in \mathbb{Q}^n$

▷ assume each entry satisfies

( )

⇒

- Thus we'll take polynomial time to mean e.g.

NOT

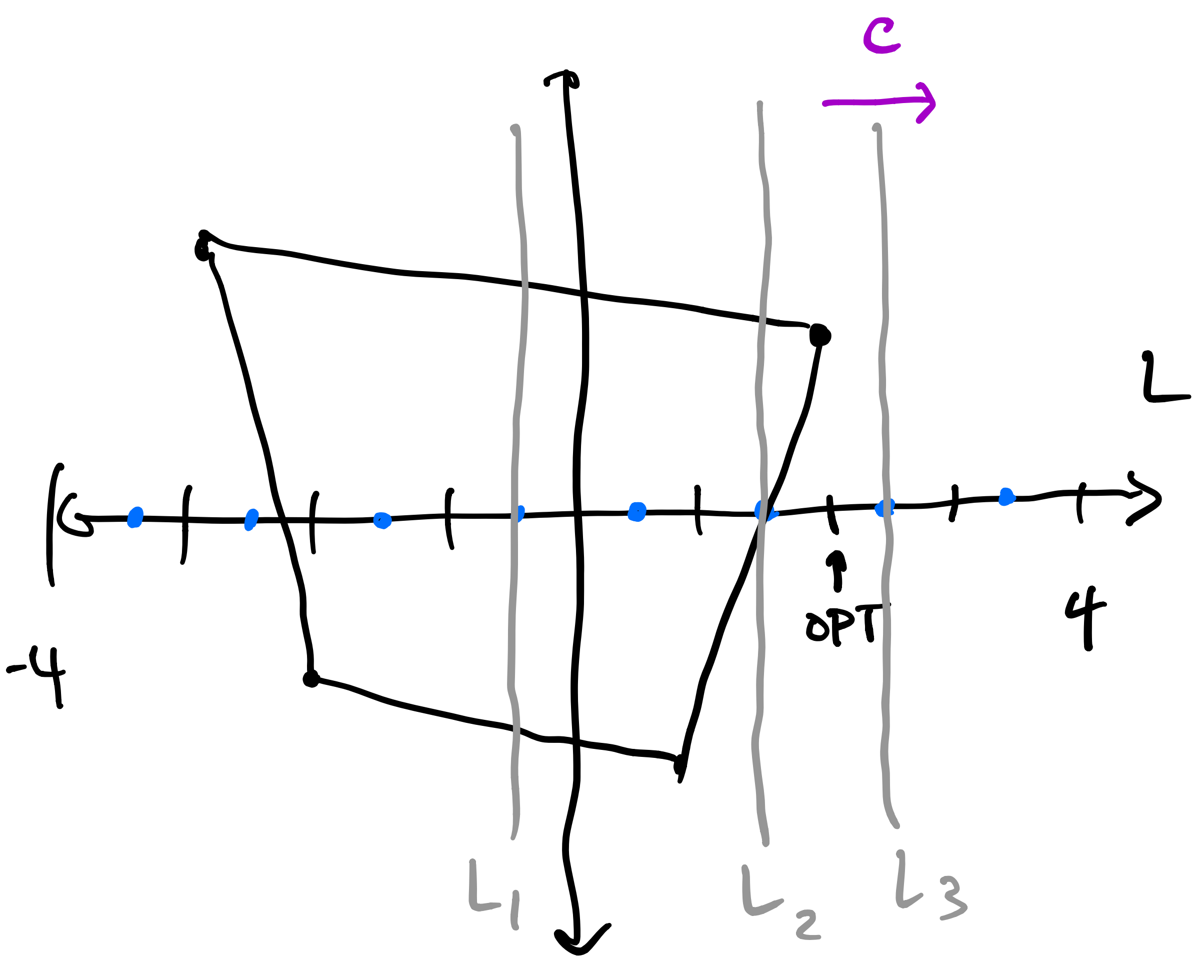
NOT

Implementing the binary Search

- How long do we need to do binary search?
  - ▷ Know
  - ▷ Know

▷ Thus

e.g. Suppose $M_n = 4$, $OPT = 2$



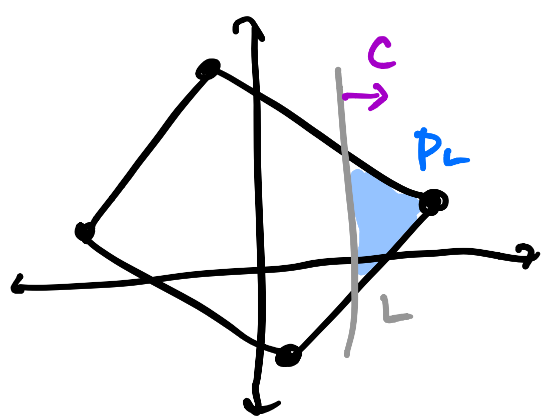
- How many steps?

$\leq$

$($

$.)$

- At each step, need to test if $L < OPT$, i.e. if



- For this, use ellipsoid.

Runtime of ellipsoid calls

- **Recall:** to test feasbility with ellipsoid, must know
  - ▷ ____________________
  - ▷ ____________________
- **To test:**
  - ▷ run ellipsoid for

steps. (   ).

$\triangleright$ if

- Thus we just need

Bounding starting ellipsoid

- Simple:

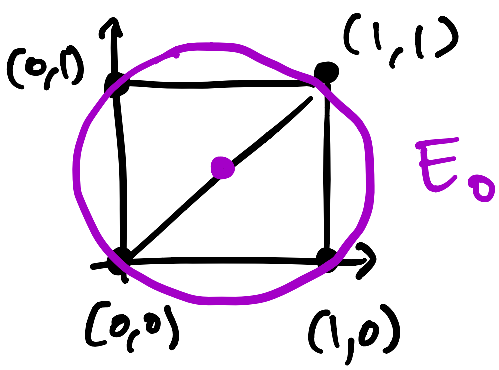
$\Rightarrow$

$\Rightarrow$

• Can use $E_0 =$

$($ $)$

e.g. for $n=2$,



• Vol $E_0 =$

$\leq$ $=$.

$\Rightarrow$ $\leq$

Bounding Vol P_L

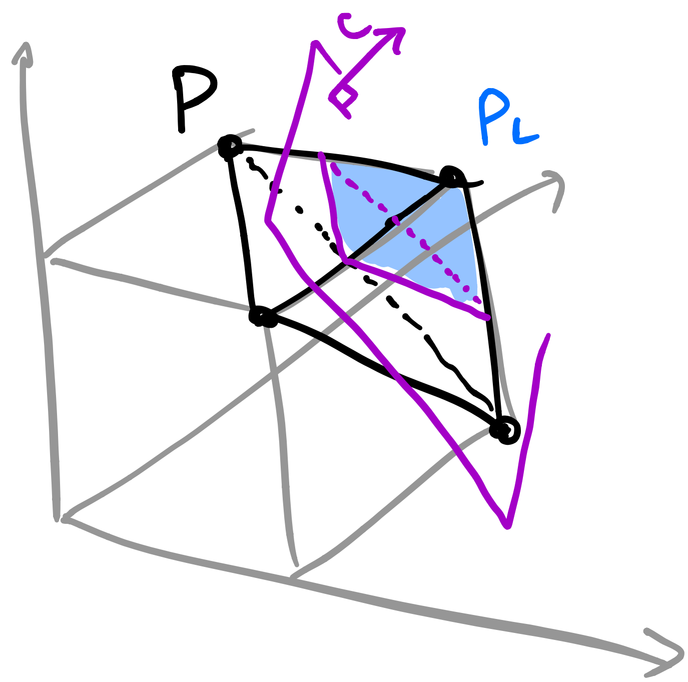
- Need to show

where

- Since $P_L \neq \emptyset$,

)

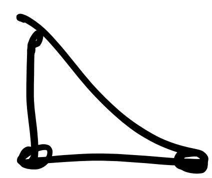
e.g. $n = 3$

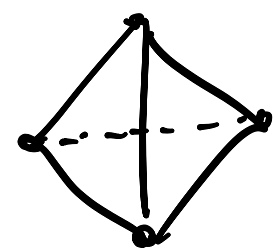


- One way (there are many):

- <u>Simplex</u> in $\mathbb{R}^n$ is

<u>e.g.</u>



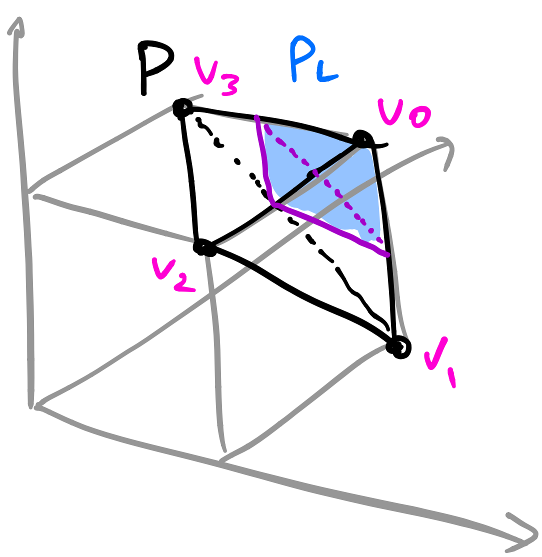


easy to compute volumes of simplices.

- We've assumed P full-dimensional

$\Rightarrow$

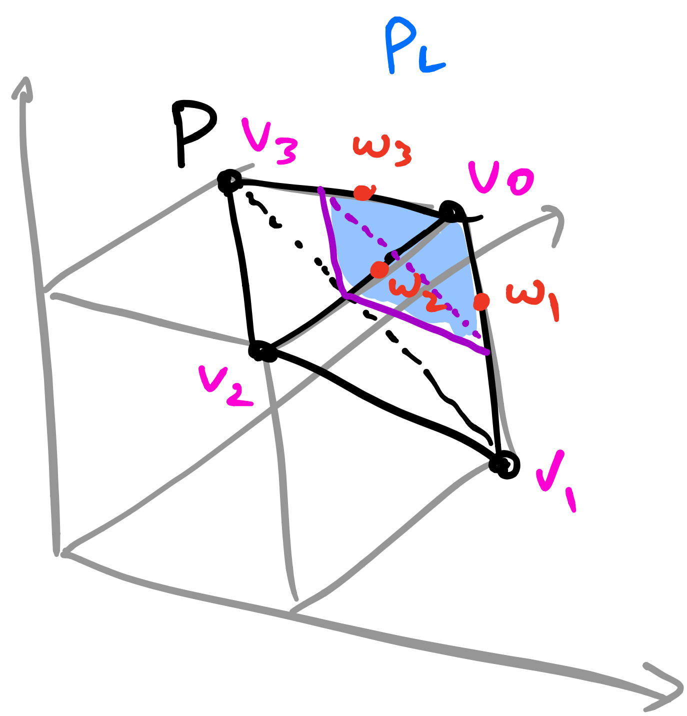
e.g. $n = 3$



- $v_1 \ldots v_n$ might not be in P, but

$$w_i = \{$$

e.g. $n=3$



- Can take $\alpha = \frac{1}{2Mn}$,

because then

$c^T w_i =$

$=$

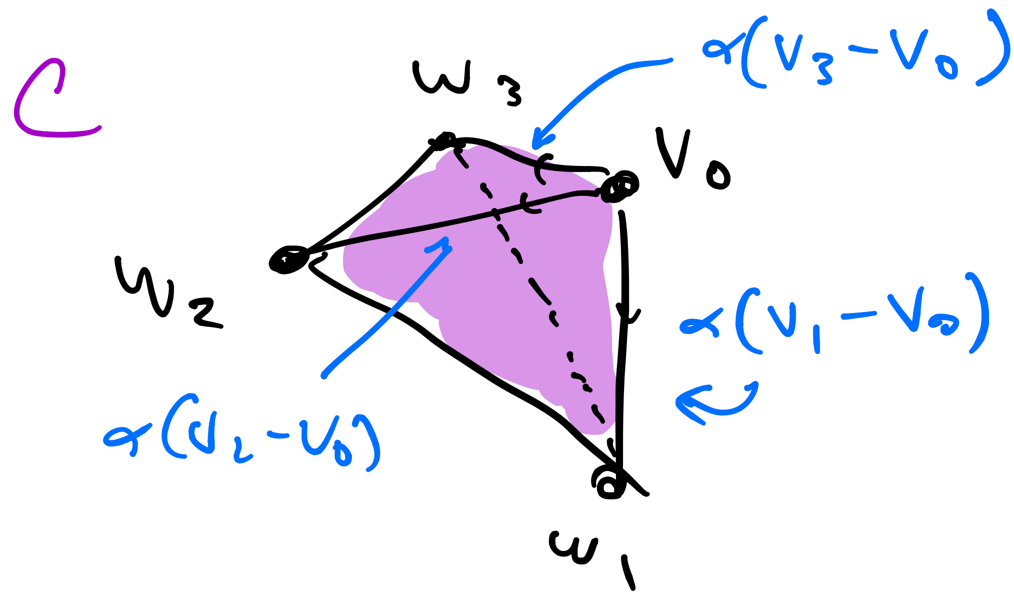
$\geq$

$\geq$

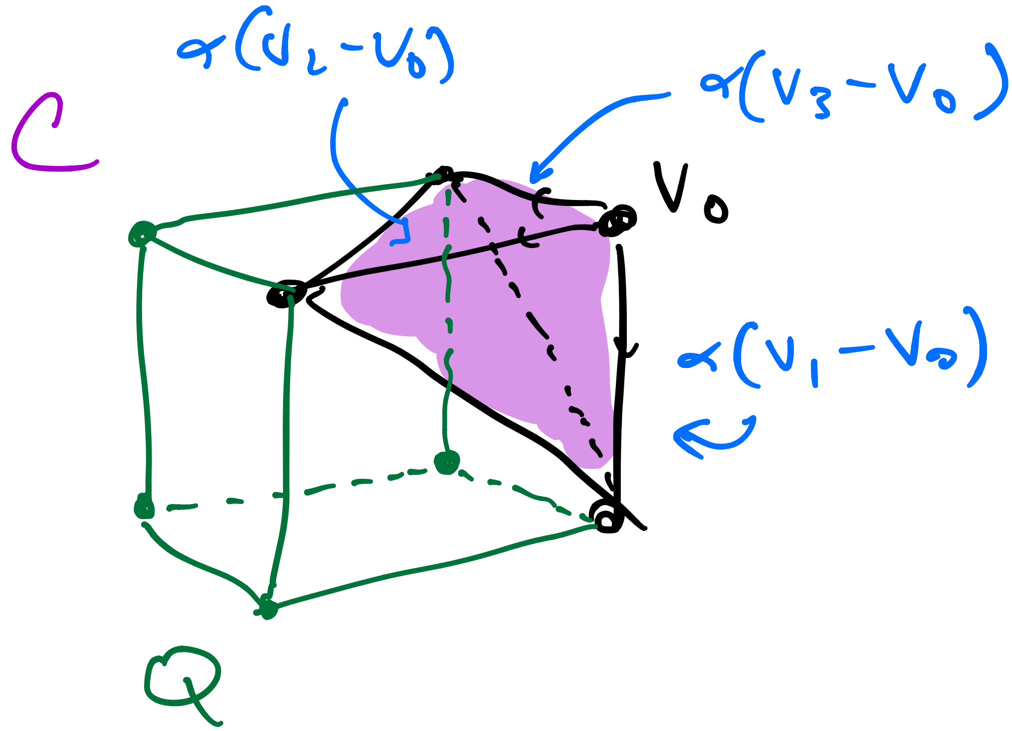
$\Rightarrow w_i \in P_L$.

- Now $\mathrm{vol}\, P_L \geq \mathrm{vol}\, C$,

$C' =$



• simplex C =



- vol C =

Exercise;

- vol Q =

- $\mathrm{vol}\, Q' \geq 1$,

$\mathrm{vol}\, Q' =$

$\geq$

- So together:

$\text{vol } P_L \geq \qquad =$

$\geq \qquad =$

Thus we may take

$\delta =$

Overall Runtime

- **# steps of ellipsoid**

$$\le 2(n+1) \ln \frac{\text{vol } E_0}{\delta}$$

$\le$

$=$

$=$

- **# steps of binary search**

$\le \log_2(2nM)$

$=$

- **Overall:**

$$\boxed{O(\qquad\qquad)}$$

$=$

To summarize...

Theorem: Given a separation oracle for $P = \mathrm{conv}(S)$, $S \subseteq \{0,1\}^n$,

$($

)

- **Side Remark:** is not strongly polynomial;

- **Éva Tardos '86**: can Solve LP's $\max\{c^T x : Ax \le b\}$ in time

- Thus if $A \in \{-1,0,1\}^{m \times n}$,

Example: nonbip. matching

- Given G, cost $c: E \to \mathbb{R}$, find
- Equivalently, optimize $c^T x$ over

$P =$

- Recall: Matching polytope given by

$$P = \Big\{ x \in \mathbb{R}^E : \qquad \forall v \in V. \qquad \forall S \subseteq V,\ |S| \text{ odd} \qquad \forall e \in E. \Big\}$$

- P is full-dimensional
- However, separation oracle nontrivial! ( )
- Can implement using

Matching polytope SEP oracle:

- Check degree constraints;

- Next check odd set constraints.

How?

▷ For x satisfying degree constraints, need to decide if

$\forall S \subseteq V, |S|$ odd,

▷ Assume WLOG $|V|$ even

( ).

▷ For $v \in V$, define

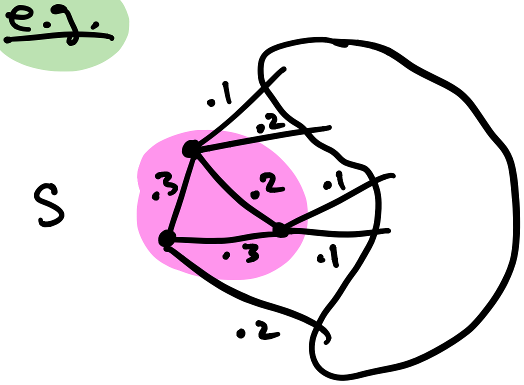
$S_v =$

▷ **observe:** Given $S \subseteq V$,

$=$

Pf

e.g.



Thus

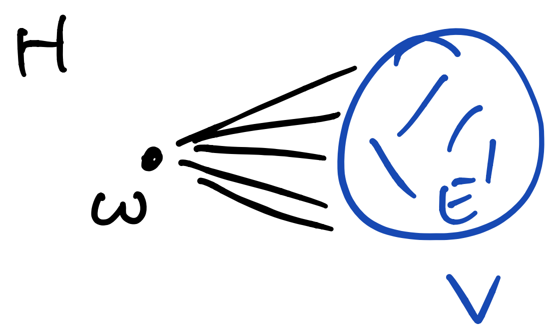
$\star \iff$

▷ Define new graph H with

vertex set =

edge set =

picture:



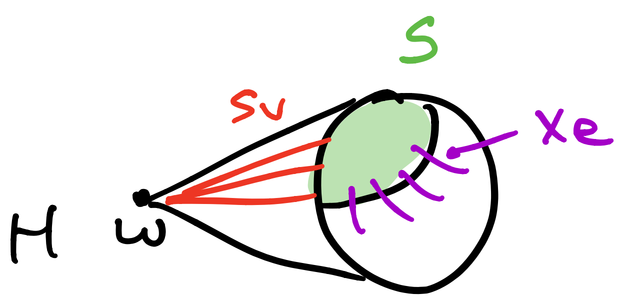
▷ Define edge weights

$$u_e = \begin{cases} \\ \\ \end{cases}$$

▷ For a cut S in H, may assume

▷ cut $S \subseteq V$ in H has value

$$\sum_{e \in \delta_H(S)} u_e =$$



▷ Thus

$\Leftrightarrow$

▷ we have seen how to compute min T-odd cut ;

▷ if

▷ if not,

□