Lecture 10

Plan: 1) Discuss pset / misc. remarks; Carathéodory's theorem.

2) Total unimodularity

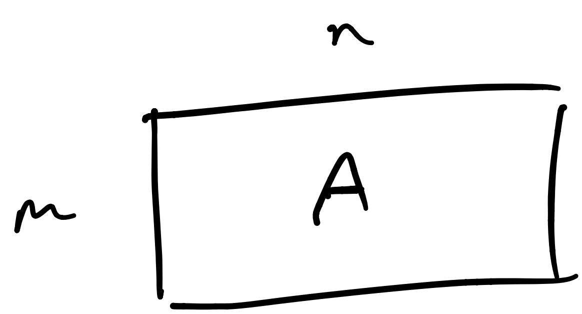
Refresher: BFS

$$P = \{x : Ax = b,\ x \geq 0\}.$$

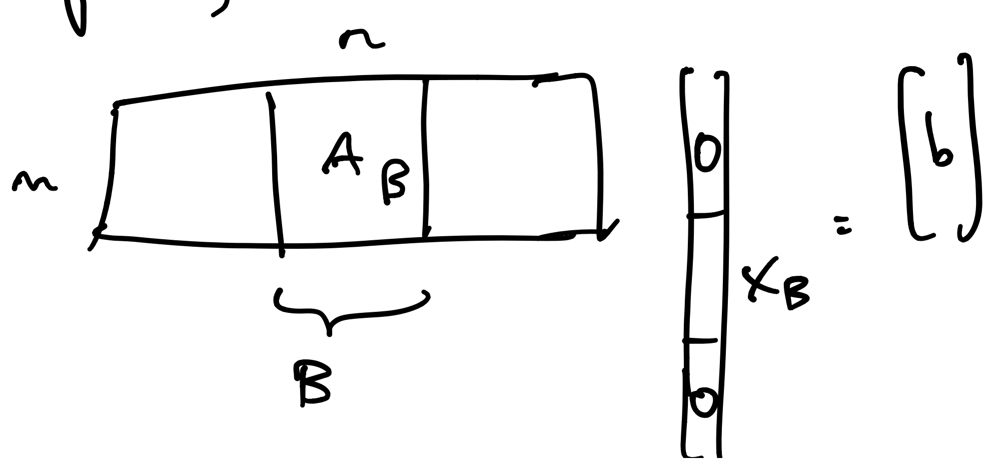
__Recall__: vertices of P are __basic feasible__ solutions, i.e.

feasible solns obtained as follows:

- Remove redundant rows from A (until rank $A = m$)



- Choose m columns B of A, (a basis for $\mathbb{R}^m$)



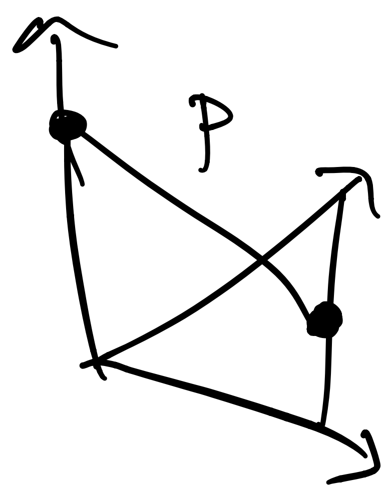
- Solve $A_B x_B = b$, set

$$x^* = \begin{bmatrix} 0 \\ \hline x_B \\ \hline 0 \end{bmatrix} \begin{matrix} \\ \}\, B \\ \\ \end{matrix}.$$

x^* is <u>bfs</u>, all vertices of P take this form.

<u>E.g.</u>: in $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



3 potential bfs:

1) $z=0, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow (x,y,z) = \boxed{\left(\frac{1}{2}, \frac{1}{2}, 0\right)}$ ✓ feasible

2) $y=0, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow (x,y,z) = \boxed{(0,0,1)}$ ✓ feas.

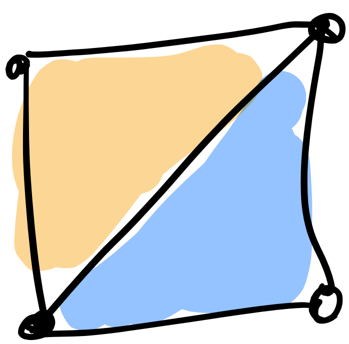
3) $x=0, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow (x,y,z) = \boxed{(0,0,1)}$ ✓ feas.

<u>Another example:</u>

Caratheodory's Theorem

Every point in $\mathrm{conv}(P) \subseteq \mathbb{R}^n$ can be written as a convex combination of $\leq n+1$ vertices of P.



Proof Let $x \in P$,

- consider ways to write x as convex combo of vertices $v_1, \ldots, v_t$ of P.
- Assume affine hull of vertices is $\mathbb{R}^n$, else can translate/rotate P so contained in $\mathbb{R}^{n'}$, $n' < n$.

$$Q = \{\lambda : \sum \lambda_i v_i = x$$
$$\sum \lambda_i = 1$$
$$\lambda_i \geq 0.\}$$

$$= \{\lambda : \begin{array}{c} n+1 \left\{ \begin{bmatrix} | & & | \\ v_1 & \cdots & v_t \\ | & & | \\ \hline 1 & 1 & 1 \; 1 \end{bmatrix} \right. \end{array} \lambda = x, \; \lambda \geq 0\}.$$

$$\underbrace{}_{A}$$

$\mathrm{aff}(v_1, \dots, v_t) = \mathbb{R}^n \iff \mathrm{rank}\, A = n+1$

(no redundant rows).

Can use λ^* where λ^* is vertex;

So can take $\lambda^* = \underline{BFS}$!

BFS λ^* have only $n+1$

nonzero coordinates.

Total unimodularity

- Consider discrete set $X \subseteq \mathbb{R}^n$.

E.g.
- $X \in \mathbb{R}^{n \times n}$ incidence vectors of matchings
- $X \in \mathbb{R}^{n \times n}$ incidence vectors of independent (a.k.a. stable sets) in graph.

- To optimize linear functions over X, enough to do so over $\mathrm{conv}(\mathcal{X})$.

- For this, want simple polyhedral description

$$\boxed{\mathrm{conv}(\mathcal{X}) = \underbrace{\{x : Ax \le b\}}_{P}.}$$

$\boxed{\text{Given } \underline{\text{proposed } A, b}, \text{ how to prove } \underline{\mathrm{conv}(X) = P?}}$

- Easy to show $\mathrm{conv}(\mathcal{X}) \subseteq P$; just check $Ax \le b$ for $x \in \mathcal{X}$.

- How about $P \subseteq \text{conv}(X)$?

<u>Harder</u>!

- One way is <u>Algorithmically</u>:

Enough to show $\forall c \in \mathbb{R}^n$

$$\boxed{\max_{x \in X} c^T x = \boxed{\max_{x \in P} c^T x}}$$ ← primal

By weak duality, enough to exhibit <u>dual</u> feasible y and $x \in X$ with

$$c^T x = b^T y.$$

e.g. what we did for min-weight perfect matching (MWPM).

- Today, another way: show extreme points of P integral.

E.g. if

$$X = \{x \in \mathbb{Z}^n : x \in P\}.$$

i.e.

$$X = \{\text{feasible solution of I.P.}\}.$$

In this case:

$$\boxed{\text{all extreme pts of } P \in \mathbb{Z}^n} \Leftrightarrow \boxed{P = \text{conv}(X)}$$

if this happens, say P

integral.

Method of showing this!

Total Unimodularity

This is true when matrix A

is very <u>special</u>.

<u>Def</u>: matrix A is <u>totally unimodular</u> (TU) if every square submatrix has determinant $-1, 0,$ or $+1$.

<u>e.g.</u> $\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, <u>not</u> $\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

Important because:

Theorem: (TU theorem)

Suppose A totally unimodular. Then $\forall$ integral b,

$$P = \{x : Ax \leq b, x \geq 0\}$$

is integral.

Pre-proof remarks:

- Same proof shows thm also holds for

$$P = \{x : Ax \geq b, x \geq 0\}$$

or

$$P = \{x : Ax = b, x \geq 0\}.$$

- <u>Converse</u>: if $P = \{x : Ax \le b, x \ge 0\}$ is integral for <u>all</u> integral b then A is TU.

(but converse <u>not</u> true for $\{x : Ax = b, x \ge 0\}$).

<u>Proof</u>: First, reduce to equality by adding slack:

let

$$Q = \{(x,s) : Ax + Is = b, x \ge 0, s \ge 0\}.$$

<u>Ex</u>: Q integral $\Leftrightarrow$ P integral.

<u>Fact</u>: $\tilde{A} = [A \mid I]$ TU

$\Leftrightarrow$ A TU.

Because if submatrix has cols of identity, either they are 0 or we can expand down them. e.g.

$$\det \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus: Forget about $\tilde{A}$, assume

$$P = \{Ax = b, x \geq 0\}.$$

where A is TU.

Recall: vertices are BFS!

Claim: BFS integral.

why? $A_B x_B = b$

$$\Rightarrow \quad x_B = A_B^{-1} b$$

But

$$\boxed{A_B^{-1} = \frac{1}{\det(A_B)} A_B^{adj}}$$

where A_B^{adj} is

adjugate matrix -

entries are subdeterminants.

in particular <u>integral</u>!

b, A_B^{adj} integral, $|\det A_B| = 1$

$\Rightarrow \quad x = A_B^{-1} b$ integral. $\square$

<u>Example</u> <u>bipartite matching.</u>

polytope of "fractional matchings" we used for min-weight-perfect-matching:

Let (U, V) be bipartition.

$$P = \left\{ x : \sum_j x_{ij} = 1 \quad \forall i \in U \right.$$
$$\sum_i x_{ij} = 1 \quad \forall j \in V$$
$$\left. x_{ij} \geq 0 \quad \forall i \in U, j \in V. \right\}$$

$$= \{ x : Ax = b,\ x \geq 0 \}.$$

Theorem: The matrix A is totally unimodular.

is totally

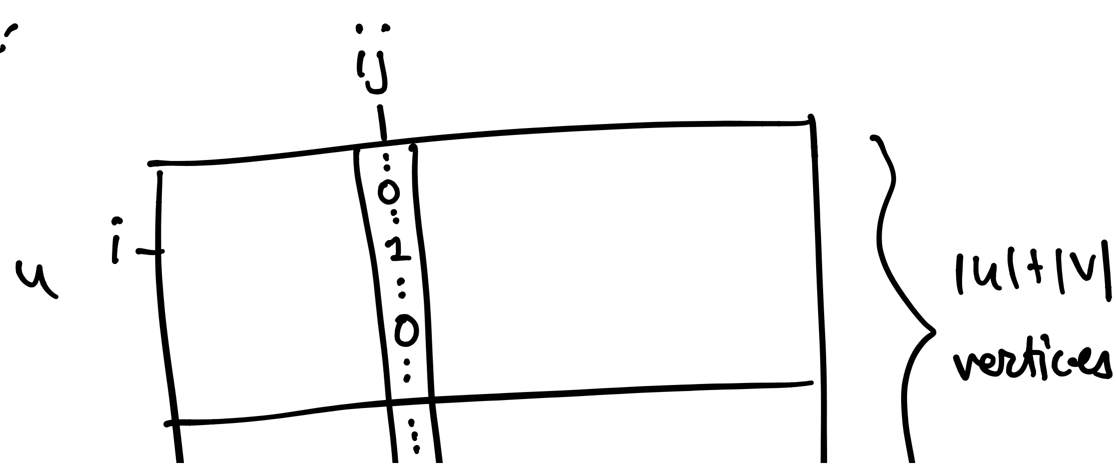
(for non-complete graphs, the matrix would be submatrix of A; still TU).

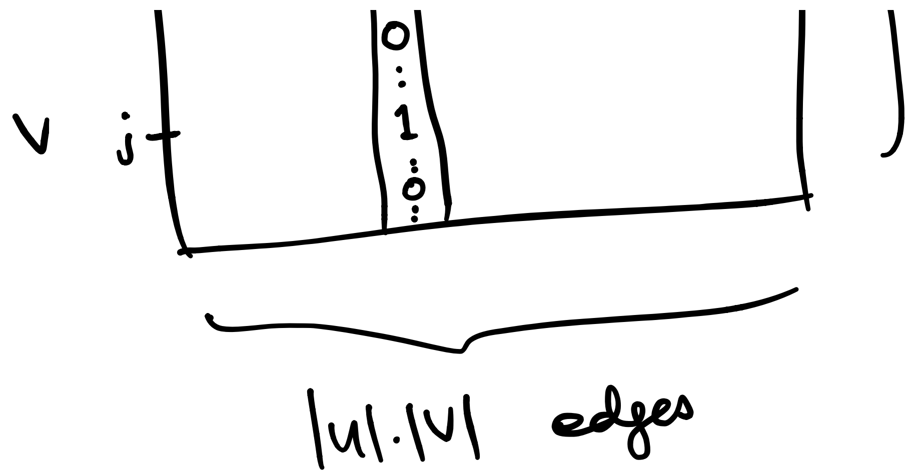
Cor: $\text{MWPM} = \min\{c^T x : x \in P\}$.

Proof: What's A look like?

A^T is incidence matrix of complete bipartite graph.

i.e.

$A =$ 



- To show A is TU, consider square submatrix M & look at cases:

  1) if M has 0 row/col,

  $$\det M = 0.$$

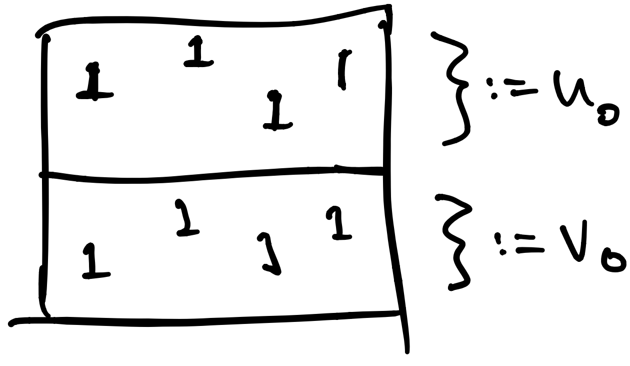
  2) If M has row/col w/ only one 1,

expand down that row/col & get smaller M.

3) M has ≥ 2 nonzero entries per row & col.

$\Rightarrow$ M has <u>exactly</u> 2 nonzero entries per column

M



$$\mathbb{1}_{u_0} = \begin{array}{c} \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] \end{array} \begin{array}{l} u_0 \\ \\ v_0 \end{array} \qquad \mathbb{1}_{v_0} = \begin{array}{c} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right] \end{array} \begin{array}{l} u_0 \\ \\ v_0 \end{array}$$

$$\mathbb{1}_{u_0}^T M = \mathbb{1}^T$$

(add up rows of M in U_0, get $\mathbb{1}^T$).

similarly

$$\mathbb{1}_{v_0}^T M = \mathbb{1}^T$$

$\Rightarrow$ two distinct solns. to $x^T M = \mathbb{1}^T$;

rows <u>not</u> lin indep.

$\Rightarrow \quad \det M = 0. \qquad \square$