Lectures 18 & 19

Plan:

1) briefly recap matroid polytope
2) matroid intersection activity

Matroid intersection

- Matroids very nice b/c greedy works. given $c: E \to \mathbb{R}$, $\max_{S \in \mathcal{I}} c(S) = \sum_{e \in S} c(e)$
- But greedy doesn't work for lots of problems,

  e.g. ▷ max matching,
  ▷ max stable set in graph.

$\Rightarrow$ matroids very limited!

$|2^E| = 2^{|E|}$

matroid intersection much richer.

A.K.A powerset.

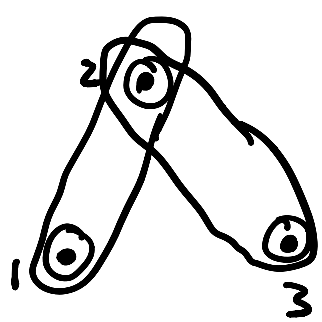
2^E = set of subsets of E e.g. $2^{\{1,2\}} = \{\{1,2\}, \{1\}, \{2\}, \emptyset\}$.

Def If $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ matroids on common ground set E, their **intersection** is just

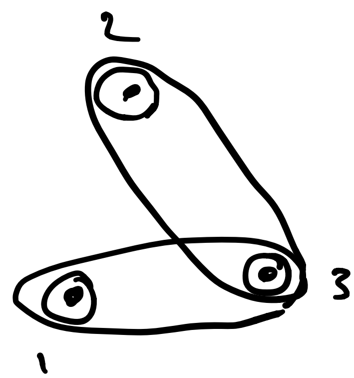
$$\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq 2^E, \quad (\mathcal{I}_1, \mathcal{I}_2 \subseteq 2^E)$$

i.e. the sets indep. in M_1 & M_2.

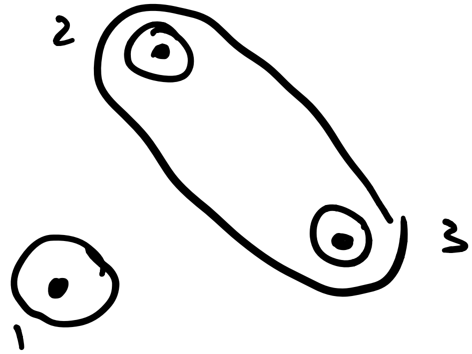
E.g. $E = \{1,2,3\}$

$\mathcal{I}_1 =$ 

$+ \emptyset$

$\mathcal{I}_2 =$ 

$+ \emptyset$

$I_1 \cap I_2 =$ 

$+ \emptyset.$

- Activity: lots of examples!
- We'll show how to find largest common independent set efficently! (Next time, probably).

Largest Common indep. set

- we give a min-max characterization
  i.e. a duality result. for L.C.I.S.
- allows us to prove our candidate
  is optimal.

- Let M_1, M_2 matroids w/ rank functions r_1, r_2.
- Let $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ be common independent set, $U \subseteq E$ any set.

Then

$$
\begin{aligned}
|S| &= |S \cap U| + |S \setminus U| \\
&= r_1(S \cap U) + r_2(S \setminus U) \\
&\leq r_1(U) + r_2(E \setminus U).
\end{aligned}
$$

b/c $S \cap U$, $S \setminus U$ indep in <u>both</u> matroids!

- max over S, min over U:

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| \leq \min_{U \subseteq E} r_1(U) + r_2(E \setminus U).$$

"strong duality": equality holds

Theorem: (Edmonds)

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| = \min_{U \subseteq E} r_1(U) + r_2(E \setminus U).$$

Remark: Enough to minimize over U closed for M_1;

- $U \leftarrow \text{span}_{M_1}(U)$ makes RHS smaller.
- $E \setminus U \leftarrow \text{span}_{M_2}(E \setminus U)$ makes RHS smaller.

But can't assume both closed.

E.g. Special cases:

- Can show (Exercise) that $\exists$ orientation G w/ indegree $\leq p(v)$ possible $\Leftrightarrow \forall S \subseteq V, \ |E(S)| \leq \sum_{v \in V} p(v)$.

  (as in Pset 4).

- Can show $\exists$ colorful spanning tree $\leftrightarrow$ deleting c colors produces $\leq c+1$ c.c.'s.

Proof of Theorem

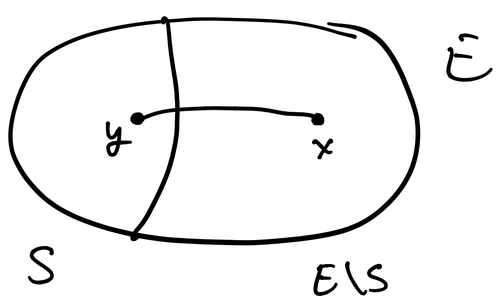
- proof is "primal-dual", i.e. each step increase $|S|$ or output U s.t. $r_1(U) + r_2(E \setminus U) = |S|$
- Uses "directed exchange graph".

we'll build up to definition.

first, undirected from one matroid $M = (E, \mathcal{I})$.

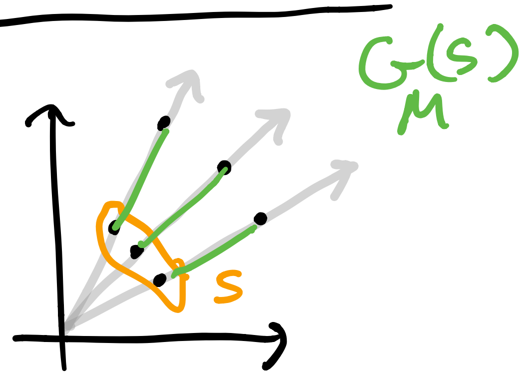
Def Given $S \in \mathcal{I}$, (undirected) exchange graph

$G_M(S) :=$
- bipartite graph,
- parts S, $E \setminus S$
- $(\underset{S}{\underset{\uparrow}{y}}, \underset{E\setminus S}{\underset{\uparrow}{x}})$ edge if $S - y + x$



i.e. (y,x) edge if adding y, deleting x results in indep. set.

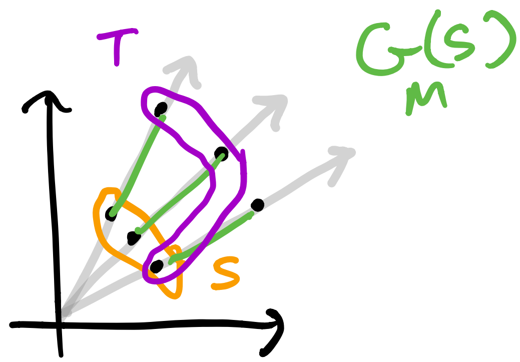
i.g. a <u>linear matroid</u>



- Arises in "<u>matroid sampling</u>", sampling uniformly random base
- For us, useful for the following reason:

<u>**Lemma**</u>: Let $S, T \in \mathcal{I}$, $|S| = |T|$. Then $G_M(S)$ has a perfect matching between $S \setminus T$ and $T \setminus S$.

e.g.



<u>**proof**</u>: Exercise (repeatedly apply exchange axiom).

and a partial converse.

Lemma: Let $S, T \in \mathcal{I}$ s.t. $|S| = |T|$, suppose $G_M(S)$ has a unique perfect matching between $S \setminus T$ & $T \setminus S$.

$\boxed{\text{Then } T \in \mathcal{I}.}$

Proof: Let N be unique matching.

An ordering result:

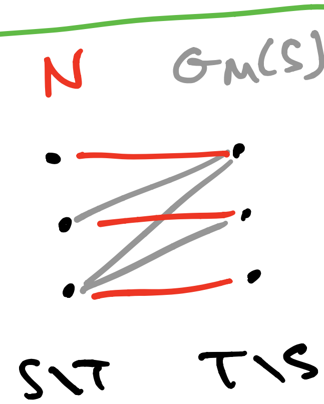
Claim: Can order

$S \setminus T = \{y_1, \dots, y_k\}$

$T \setminus S = \{x_1, \dots, x_k\}$

so that $N = \{(y_1, x_1), \dots, (y_k, x_k)\}$ and $(y_i, x_j) \notin G_M(S)$ for $i < j$.

e.g.

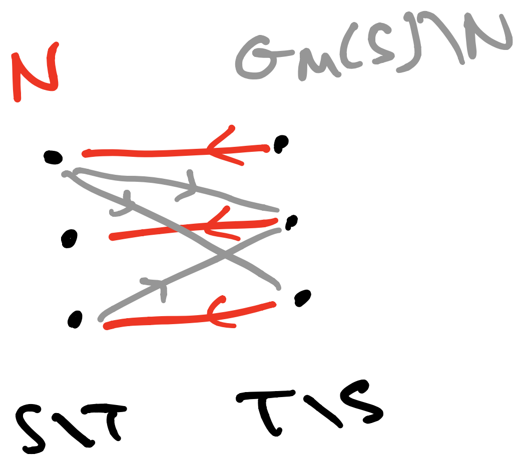


Claim 1 really just about graphs.

proof of claim:

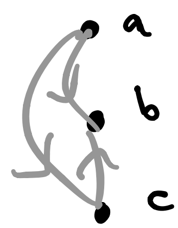
- Ignore edges not between $S\setminus T$, $T\setminus S$.
- Orient N from $T \to S$,
- Others $S \to T$.

e.g.



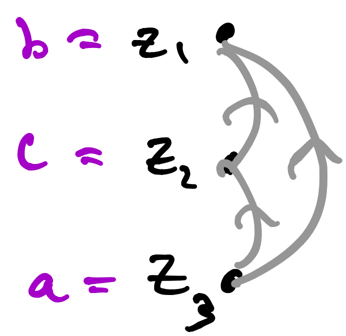
- Contract vertices of N

e.g.



- Get <u>acyclic</u> directed graph (else original graph has alternating cycle, contradicts N's uniqueness).

- Topologically order contracted graph so all edges point backwards. (possible b/c acyclic).

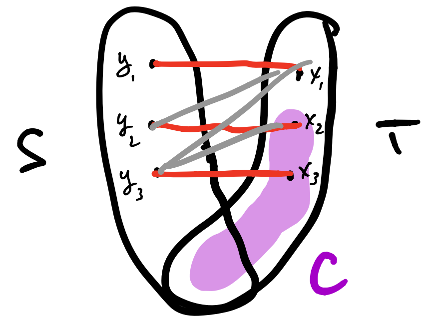


- Let $x_i \in T \setminus S$, $y_i \in S \setminus T$ be vertices contracted to get z_i. $\square$

Now, for contradiction: suppose $T \notin \mathcal{I}$.

- then T contains circuit C.

e.g.

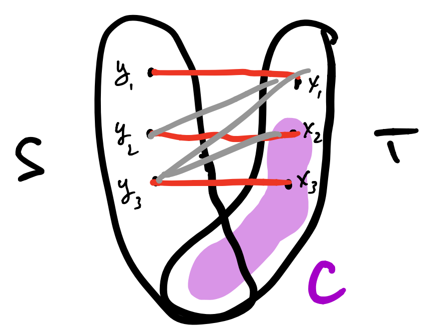


- C intersects $T \setminus S$.

(else $C \subseteq S$, contradicts $S \in \mathcal{I}$).

- Let x_i be first element in C.

e.g.



here $i=2$.

- Now we'll find that x_i in $\text{span}(S - y_i)$, contradicting $(y_i, x_i) \in G_M(S)$! (by def. of $G_M(S)$).

- To show this, observe

  $\forall x \in C - x_i$, $x \in \text{span}(S - y_i)$.

  b/c $(y_i, x) \notin G_M(S)$ by ordering.

$$\Rightarrow \text{span}(S - y_i) \supseteq \text{span}(C - x_i) \ni x_i$$

by properties of span

b/c C circuit.

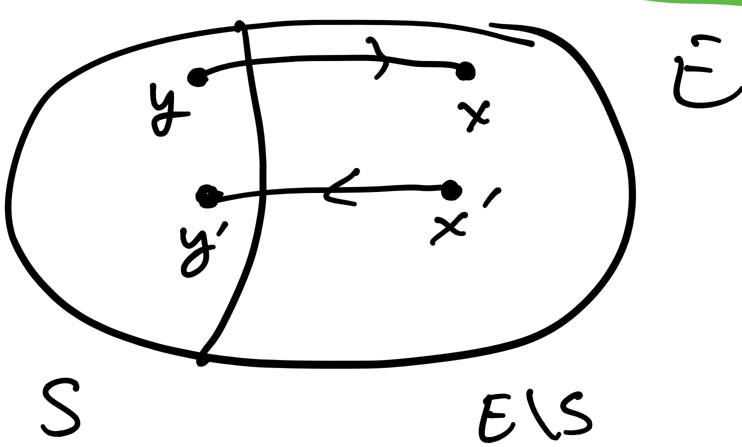
$\square$

now generalize to <u>directed exchange graph</u> from <u>two</u> matroids $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$.

Def For $S \in \mathcal{I}_1 \cap \mathcal{I}_2$, (directed) <u>exchange graph</u>

$D_{M_1, M_2}(S) :=$
- directed bipartite graph,
- parts S, $E \setminus S$
- (y, x) edge if $S - y + x \in \mathcal{I}_1$
- (x, y) edge if $S - y + x \in \mathcal{I}_2$.

Picture:



here

$S - y + x \in \mathcal{I}_1$, $S - y' + x' \in \mathcal{I}_2$

- <u>Note</u>: $G_{M_1}(S)$, $G_{M_2}(S)$ are "<u>subgraphs</u>".

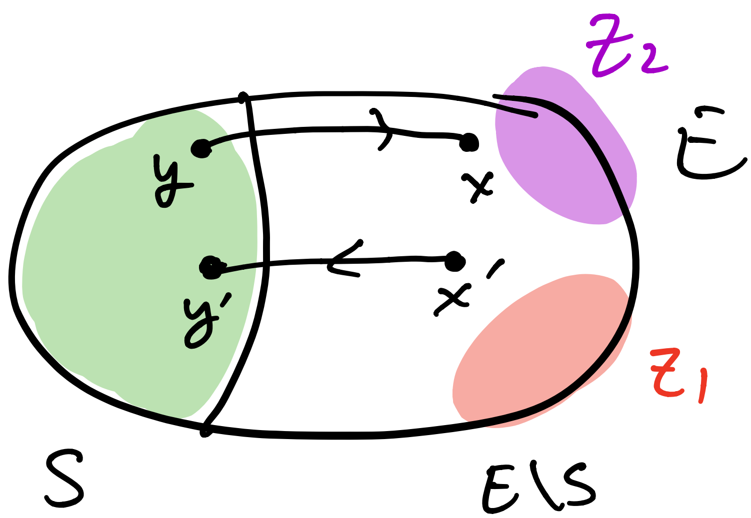
rightwards edges. leftwards.

- **Also define:**

"sources" $Z_1 := \{ x \notin S : S + x \in I_1 \}$

"sinks" $Z_2 := \{ x \notin S : S + x \in I_2 \}$.

e.g.



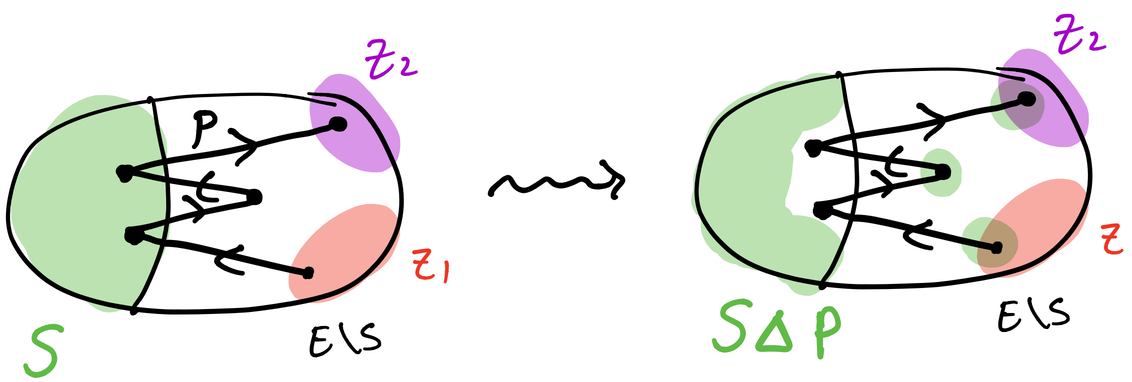
Algorithm Initialize $S = \emptyset$.

▷ Repeat:

▷ if $\exists$ directed path from sources Z_1 to sinks Z_2:

▷ $P :=$ a shortest such path.

▷ Replace $S \leftarrow S \Delta \bar{P}$,
($\bar{P} :=$ vertices on path).



▷ else: (i.e. no path)

▷ return

$U = \{z \in E :$ sinks Z_2 reachable from $z\}$.

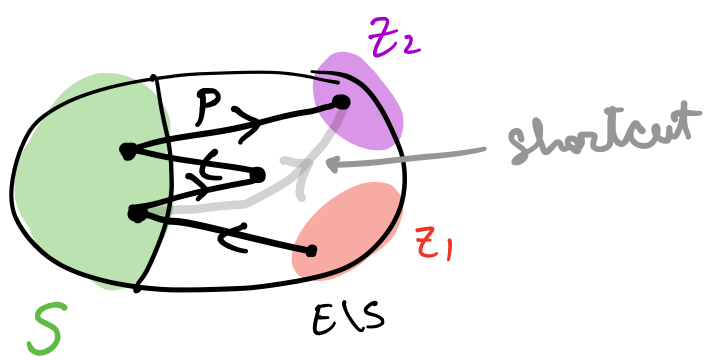
Correctness:

- Claim 1: S remains in $\mathcal{I}_1 \cap \mathcal{I}_2$
- Claim 2: $|S| = r_1(U) + r_2(E \setminus U)$.

Proof of Claim 1:

- Recall P shortest path; in particular no "shortcuts":

e.g.



- Enough to show:
  no shortcuts $\Rightarrow S \Delta P \in \mathcal{I}_1 \cap \mathcal{I}_2$.

- We first show $S \Delta P \in \mathcal{I}_1$.

- To do this, define new matroid $M_1' = (E', \mathcal{I}')$ from M_1 by adding new element t to E & defining

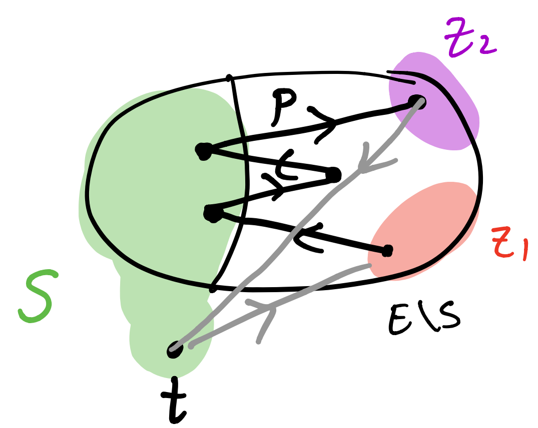
$E' = E + t$ and $\mathcal{I}' = \{J : J - t \in \mathcal{I}_1\}$.

i.e. just making t "independent from everything else"... get $R, R+t$ s.t. $R \in \mathcal{I}$.

- Define M_2' analogously (same t);

consider $D_{M_1, M_2}(S+t)$.

e.g.



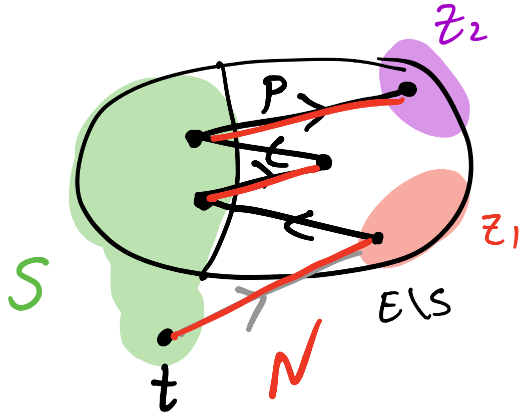
- Note $D_{M_1', M_2'}(S+t)$ is just $D_{M_1, M_2}(S)$ plus edges $t \to Z_1$, and $t \leftarrow Z_2$.

- View $G_{M_1'}(S+t)$ as subgraph of $D_{M_1,M_2}(S+t)$

- $G_{M_1'}(S+t)$ contains perfect matching N between

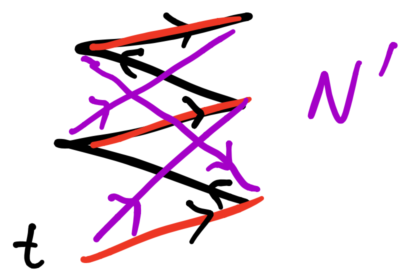
$$S \cap P + t \quad \& \quad P \setminus S.$$

e.g.



(by edge $t \to$ first vertex of P + edges of P starting in S.).

- And N unique such matching by no shortcut property.



(edges in $G_{M'}(S+t)$ all point left; any other matching would yield shortcut.)

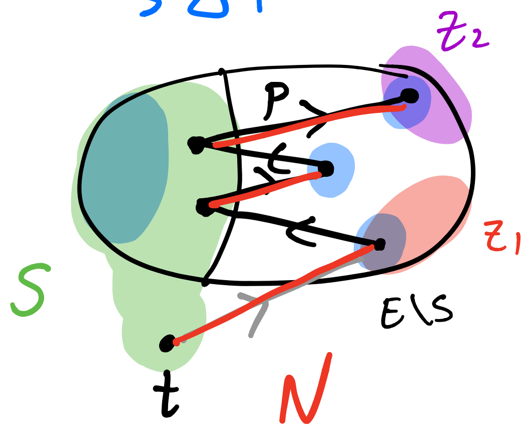
- Unique perfect matching lemma

$\Rightarrow S \Delta P \in \mathcal{I}_1'$

$\Rightarrow S \Delta P \in \mathcal{I}_1$. (by def of $\mathcal{I}'$).

$S \Delta P$

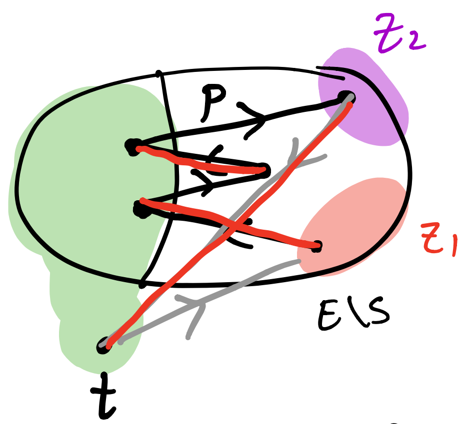
e.g.



- To show $S \Delta P \in \mathcal{I}_2$, instead

find matching in $G_{M_2'}(S+t)$
using edge from <u>last</u> vertex in P to t.

e.g.



- finishes proof of Claim 1. $\square$.

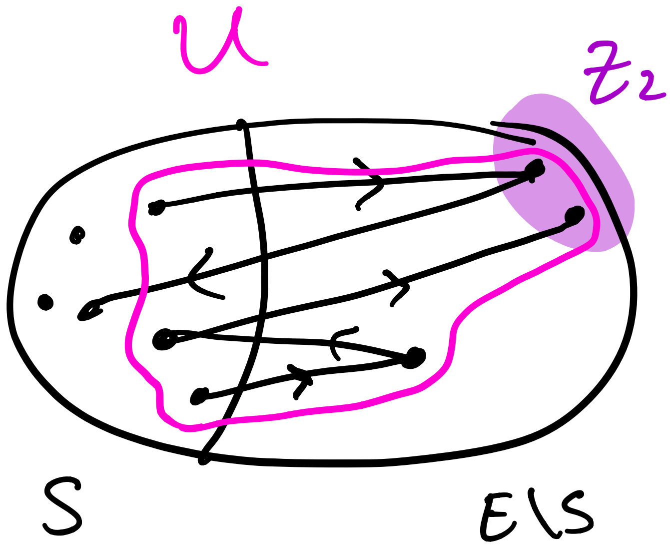
Proof of Claim 2:

- Want to show at termination,

$$|S| = r_1(U) + r_2(E \setminus U)$$

where U = everything that can reach Z_2.

e.g.



- First note $Z_2 \subseteq U$, and $Z_1 \cap U = \emptyset$.

  clear

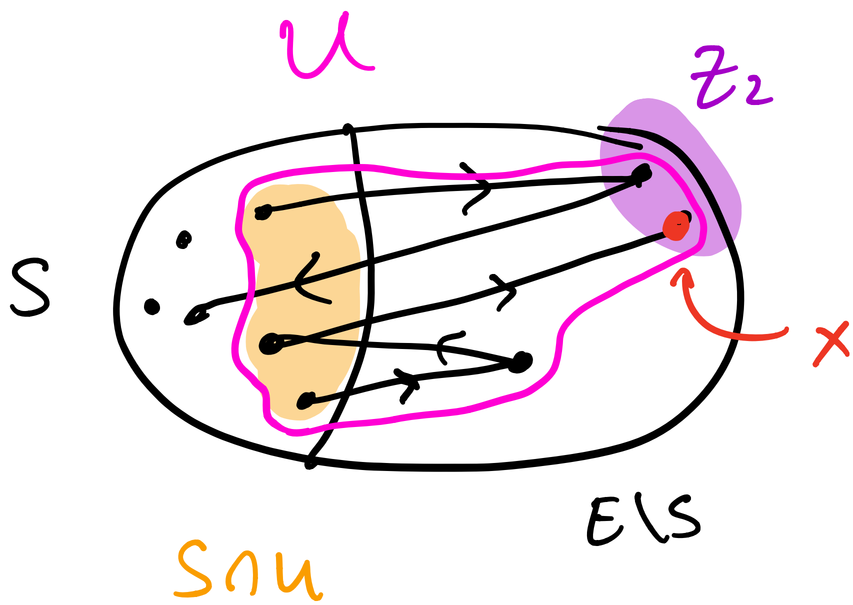
  else alg. not done.

- Enough to show $r_1(U) = |S \cap U|$, $r_2(E \setminus U) = |S \setminus U|$.

  (then $|S| = |S \cap U| + |S \setminus U|$
  $= r_1(U) + r_2(E \setminus U)$.)

- Suppose $|S \cap U| \neq r_1(U)$.

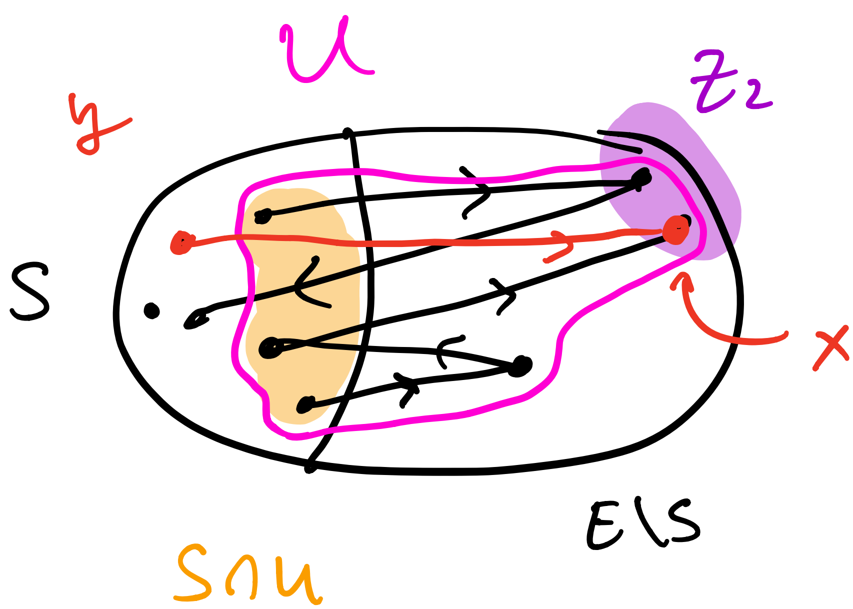
- $S \cap U \subseteq U$, $S \cap U$ indep
  $\Rightarrow |S \cap U| < r_1(U)$.
  $\Rightarrow \exists\, x \in U \setminus S$ s.t. $(S \cap U) + x \in \mathcal{I}_1$.



- $S \in \mathcal{I}_2 \Rightarrow$ can add elts of $S \setminus U$ to $(S \cap U) + x$ until we obtain $S + x - y \in \mathcal{I}_1$ for $y \in S \setminus U$.

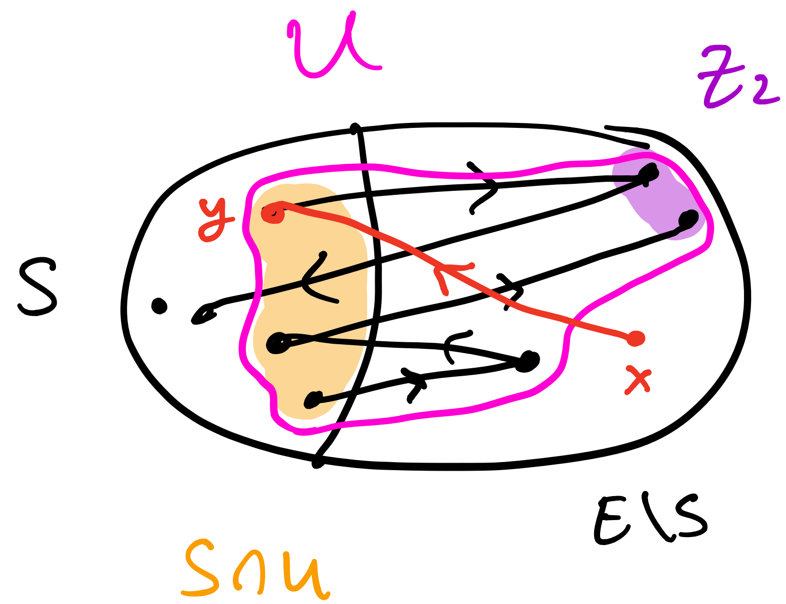
(repeatedly use exchange axiom.)

- But then (y, x) in $D_{M_1, M_2}(S)$, so $y \in U$, contradicts $y \in S \setminus U$.



• Case $r_2(E\setminus U) \neq |S\setminus U|$

similar; contradiction will look like



left as exercise. $\square$.