Lecture 21

Plan: 1) finish arborescence
2) matroid union

Motivating example: recall

Spanning tree game:

Given graph G, players alternate:

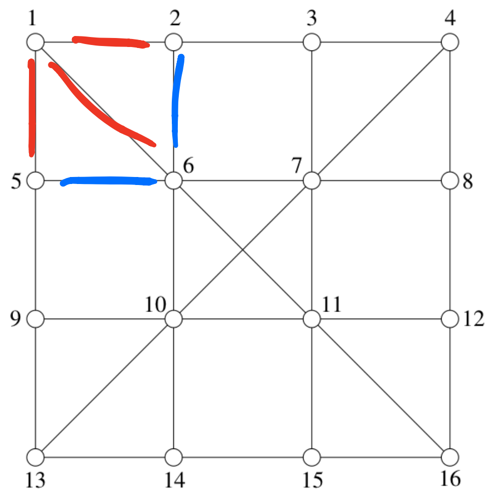
1) P1 "cuts" some edge
2) P2 "fixes" one of the remaining edges;
P1 can never cut a fixed edge.

P wins if graph becomes disconnected.

e.g. P1 win: (if P2 plays very badly).

P1

P2



Recall:

P2 wins if (A) $\exists$ 2 disjoint spanning trees in G.

P2 uses the spanning trees to maintain connectivity.

P1 wins if (B) $\exists$ partition $V_1, \ldots V_p$ of V w/

$|\text{edges w/ endpoints in different parts}| < 2(p-1)$.

$\parallel$

$\delta(V_1, \ldots V_p)$

P1 always plays edges
from $\delta(V_1, \ldots V_p)$;
at end $< p-1$ left $\rightarrow$
parts not connected.

Matroid theory shows $A \Leftrightarrow \neg B$,

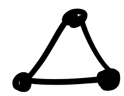
i.e. P2 wins iff $\exists$ 2 disjt. spanning trees.

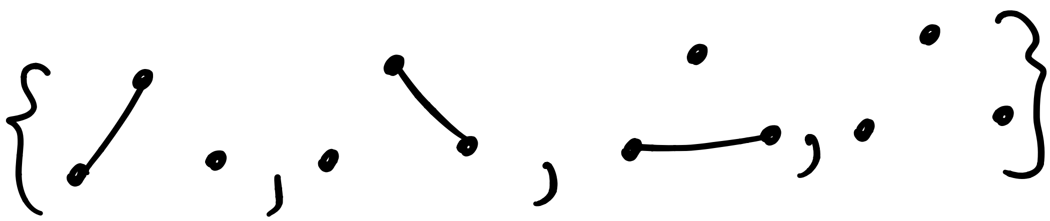
Matroid Union:

Let $M = (E, \mathcal{I})$ matroid.

Recall dual matroid $M^* = (E, \mathcal{I}^*)$

$\mathcal{I}^* = \{X \subseteq E : E \setminus X$ contains a base of $M\}$.

E.g. If $M = M_G$ for $G =$ ,

$I^* =$ 

i.e. subgraphs s.t. complement contains a spaning tree.

Theorem The dual matroid M^* is in fact a matroid with rank function

$$r_{M^*}(X) = |X| + r_M(E \setminus X) - r_M(E).$$

Proof Use

Fact: Can define a matroid using properties of rank function.

I.e. if a function $r: 2^E \to \mathbb{N}$ satisfies

R1) Monotonicity

R2) Submodularity

then $M = (E, \mathcal{I})$ where

$$\mathcal{I} = \{ S \subseteq E : r(S) = |S| \}.$$

is a matroid w/ rank function r.

Thus theorem follows by showing

A) largest element of $\mathcal{I}^*$ in X has cardinality $r_{M^*}(X)$,

B) r_{M^*} satisfies R1), R2).

left as exercise. $\square$

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e.g. <u>disjoint spanning trees:</u>

G has 2 disjoint spanning trees $\iff \max_{S \in \mathcal{I} \cap \mathcal{I}^*} |S| = |V| - 1.$

$M_G \qquad\qquad M_G^*$

and LCIS algo finds (one of) the trees!

moreover, minimax char. $\Rightarrow$

Theorem: G has two disjoint spanning trees $\Leftrightarrow$ $\forall$ partition $V_1 \ldots V_p$ of V,

$$|\delta(V_1 \ldots V_p)| \geq 2(p-1).$$

Proof Assume G connected; else trivial.

- We only show $\Leftarrow$; leave $\Rightarrow$ as exercise.

Plan: use Minimax theorem for

$$M = M_G, \quad M^* = (E, \mathcal{I}^*).$$

- Let $n = |V|$.

- G has 2 edge disjt. spanning trees $\Leftrightarrow$

$$\max_{S \in \mathcal{I} \cap \mathcal{I}^*} |S| = n-1$$

- $r_M(F) = n - \underset{\substack{\uparrow \\ \#\text{cc's in } (V,F).}}{K(F)}$

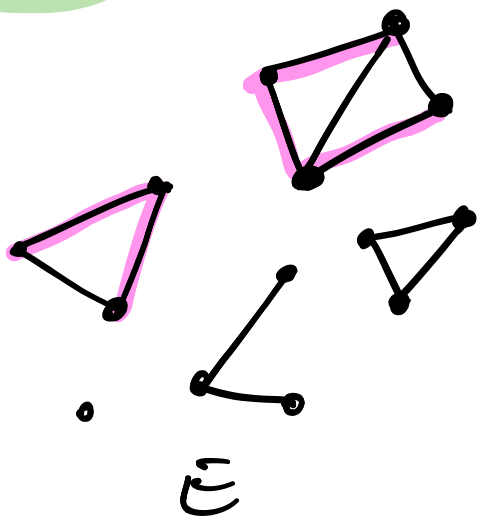
min-max:

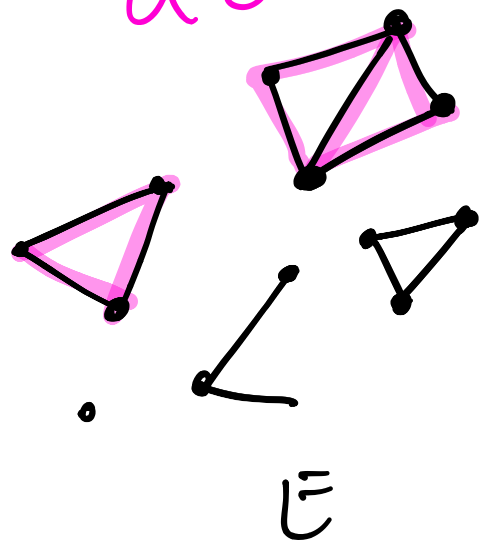
- **Matroid Intersection theorem:**

$$\max_{S \in \mathcal{I} \cap \mathcal{I}^*} |S| = \min_{U \subseteq E} r_M(U) + r_{M^*}(E \setminus U)$$

- Recall: we may restrict M to U closed in M, i.e. $U = \mathrm{span}(U)$, that is U is union of subgraphs induced by it's c.c.'s.

e.g.





$$\Rightarrow \star = \min_{U \subseteq E \ \underline{\text{closed}} \text{ in } M} r_M(U) + r_{M^*}(E \setminus U)$$

$$= \min_{U \text{ closed in } M} \Big((n - k(U)) + \big(|E \setminus U| + k(E) - k(U) \big) \Big).$$

def of I^*

$$= \min_{U \text{ closed}} \left[n + 1 + |E \setminus U| - 2k(U) \right]$$

$k(E) = 1$, i.e. G conn.

$$= \min_{\substack{V_1, \dots, V_p \\ \text{CC's of } U}} \left[n + 1 + |\delta(V_1, \dots, V_p)| - 2p \right]$$

U closed in M.

- $|\delta(V_1 \dots V_p)| \geq 2p - 2 \ \forall$ partitions

$\Rightarrow$ above is $\geq n - 1$,

$\Rightarrow \exists$ 2 disjt. spanning trees $\square$.

(General) matroid union

- Let $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ matroids.

Def. The <u>matroid union</u>

$M_1 \cup M_2 = (E, \mathcal{I})$ where

$$\mathcal{I} = \{X \cup Y : X \in \mathcal{I}_1, Y \in \mathcal{I}_2\}.$$

<u>Careful</u>: $\mathcal{I} \neq \mathcal{I}_1 \cup \mathcal{I}_2$.

<u>Theorem</u>: $M_1 \cup M_2$ is a matroid, has rank function

$$r_{M_1 \cup M_2}(S) = \min_{U \subseteq S} \{ |S \setminus U| + r_{M_1}(U) + r_{M_2}(U) \}$$

Proof

Part 1: Clt's a matroid.

- Let $X, Y \in \mathcal{I}$, $|X| < |Y|$,

  and $X = X_1 \dot\cup X_2$, $Y = Y_1 \dot\cup Y_2$

  disjt

  for $X_i, Y_i \in \mathcal{I}_i$,

- Need to show $\exists e \in Y \setminus X$

  s.t. $X + e \in \mathcal{I}$.

- Assume among such choices of X_i, Y_i, ours maximize

  $$|X_1 \cap Y_1| + |X_2 \cap Y_2|.$$

- Since $|Y| > |X|$, assume $|Y_1| > |X_1|$ switching $1 \leftrightarrow 2$ if necessary.

$\Rightarrow \exists\ e \in Y_1 \setminus X_1$ s.t. $X_1 + e \in \mathcal{I}_1$

- $e \notin X_2$, or else $X_1 \leftarrow X_1 + e$, $X_2 \leftarrow X_2 - e$ increases $|X_1 \cap Y_1| + |X_2 \cap Y_2|$.

($e \notin Y_2$ by disjointness of Y_i).

$\Rightarrow X + e \in \mathcal{I}_1$. End part 1 $\triangle$

Part 2: Rank function.

$$r_{M_1 \cup M_2}(S) = \min_{U \subseteq S} \left\{ |S \setminus U| + r_{M_1}(U) + r_{M_2}(U) \right\}$$

- $\leq$ clear, because for $S \in \mathcal{I}$,

$$|S| = |S \setminus U| + \underbrace{|S \cap U|}_{\leq r_{M_1}(U) + r_{M_2}(U)}$$

$S \cap U = X_1 \dot\cup X_2$, $X_i \in \mathcal{I}_i$.

$$\leq |S \setminus U| + r_{M_1}(U) + r_{M_2}(U)$$

- For $\geq$, use <u>matroid intersection</u>.

- First prove for $S = E$; proof for other S follows by considering restrictions $M_1|_S$, $M_2|_S$.

- Let X base of $M_1 \cup M_2$.
  $\Rightarrow X = X_1 \dot\cup X_2$.

- May assume $r_{M_2}(X_2) = r_{M_2}(E)$;
  (by adding to X_2/removing from X_1).
  $\Rightarrow |X| = |X_1| + r_{M_2}(E)$.

- Then $X_1 \in \mathcal{I}_1$, and $X_1 \in \mathcal{I}_2^*$ (i.e. X_1 is indep. in the dual M_2^*)

  because $E \setminus X_1$ cont. base X_2 of M_2.

- I.e. $X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*$.
- matroid intersection theorem for M_1, M_2^*:

$$
\begin{aligned}
r_{M_1 \cup M_2}(E) &= |X| \\
&\geq \max_{X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*} \left(|X_1| + r_{M_2}(E) \right) \\
\text{MIT} \quad &= \min_{U \subseteq E} r_{M_1}(U) + r_{M_2}(E \setminus U) + r_{M_2}(E) \\
&= \min_{U \subseteq E} r_{M_1}(U) + |E \setminus U| + r_{M_2}(U) \\
&\qquad \cancel{- r_{M_2}(E) + r_{M_2}(E)} \\
&= \min_{U \subseteq E} |E \setminus U| + r_{M_1}(U) + r_{M_2}(U).
\end{aligned}
$$

end part 2 Δ.