Lecture 22

plan: 1) Finish matroid union ✓

2) Ellipsoid

Ellipsoid Algorithm

- general purpose convex opt. alg.
- poly time in many situations, e.g. linear programming. (not necessarily strongly polynomial).

For $\max\{c^T x : Ax \leq b\}$ can solve in poly$(\langle A \rangle, \langle b \rangle)$ $\sum \log|a_{ij}|, \sum \log|b_i|$ if $a_{ij}, b_i \in \mathbb{Z}$.

- Slow for LP in practice.

- Contrast w/ simplex which is fast in practice but not provably poly.

- Many consequences for complexity of combinatorial opt problems.

- There is also <u>interior point</u> methods (Karmarkar '84) that solve LP in poly time, & fast in practice.

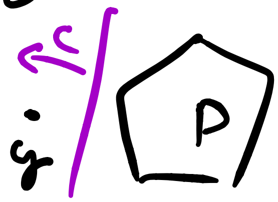
  but not as versatile for theory

Consequences

Given convex set $P \subseteq \mathbb{R}^n$, (e.g. a polyhedron), consider two problems:

- **Separation (SEP):**

  Given $y \in \mathbb{R}^n$, decide if $y \in P$, if not return **separating hyperplane** i.e. $c \in \mathbb{R}^n$ s.t. $c^T y > \max\{c^T x : x \in P\}$.

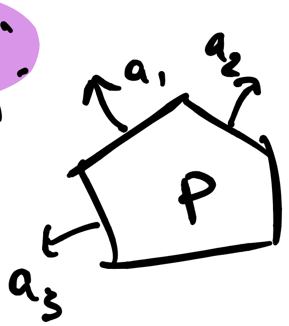


- **Optimization (OPT)**

  Given vector $c \in \mathbb{R}^n$, find
  find x maximizing $c^T x$ on P.

Examples

- **Linear programming:**



  $P = \{x : Ax \le b\}$

  how to solve **SEP**?

  $P = \{x : a_i^T x \le b_i\}$, so just check
  foreach i if $a_i^T x \le b_i$;
  if not for some i,
  output $a_i^T x \le b_i$ as separating hyperplane.

(efficient if A is part of the input)

- OPT for $P = \{x : Ax \leq b\}$ is just LP.

$$\max\ c^T x$$
$$\text{subject to } Ax \leq b$$

(SEP easy, OPT seems hard.)

- Matroid polytope:

$M = (E, \mathcal{I})$ matroid,

$P = \mathrm{conv}(\{1_S : S \in \mathcal{I}\})$

matroid polytope

we <u>know</u> a face characterization.

<u>Thm</u>

$$\boxed{P = \left\{ x \in \mathbb{R}^E : \begin{array}{l} x(S) \leq r(S) \quad \forall S \subseteq E \\ x_e \geq 0 \quad \forall e \in E \end{array} \right\}.}$$

However, exponentially many constraints!

Even if we can compute

rank function r_M, SEP not obviously efficient!

- OPT for P is just greedy algorithm for the matroid.

  OPT = max cost indep set.

(OPT easy, SEP seems hard.)

- Matroid intersection polytope:

OPT ?? SEP ??

- **Amazing Result:** → Ellipsoid method & consequences in combo. opt.

Theorem (Grötschel, Lovasz, Schrijver '81) For a family P of convex bodies,

SEP for P is poly-time solvable

⟺

OPT for P is poly time solvable.

Proof idea:

⇒ Ellipsoid algorithm: can solve OPT using calls to SEP.

$(\Leftarrow)$ Reduces to $(\Rightarrow)$ using "polar" P^* of P; we won't cover.

- Actually, if P is "nondegenerate" enough, don't need SEP, just need membership (MEM)

correction: last time I didn't say!

MEM: decide if $x \in P$.

Thm (GLS '88): → Book

Given ball of radius ε contained in P, ball of radius R containing P, (and a MEM oracle to P) can solve SEP with

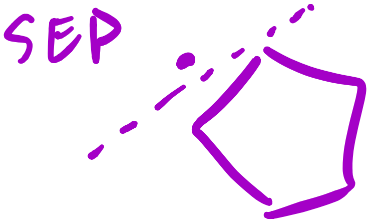
$\text{poly}(\log(\frac{1}{\varepsilon}), \log(R), n)$

if not given the ε-ball, might never hit P!

Calls to MEM.

Actually, is about approximate versions of SEP & MEM.

Proof: not covered.



OPT vs. feasibility

- First we solve simpler problem:

FEAS (feasibility) Given SEP oracle for P find some $x \in P$ or decide $P = \emptyset$.

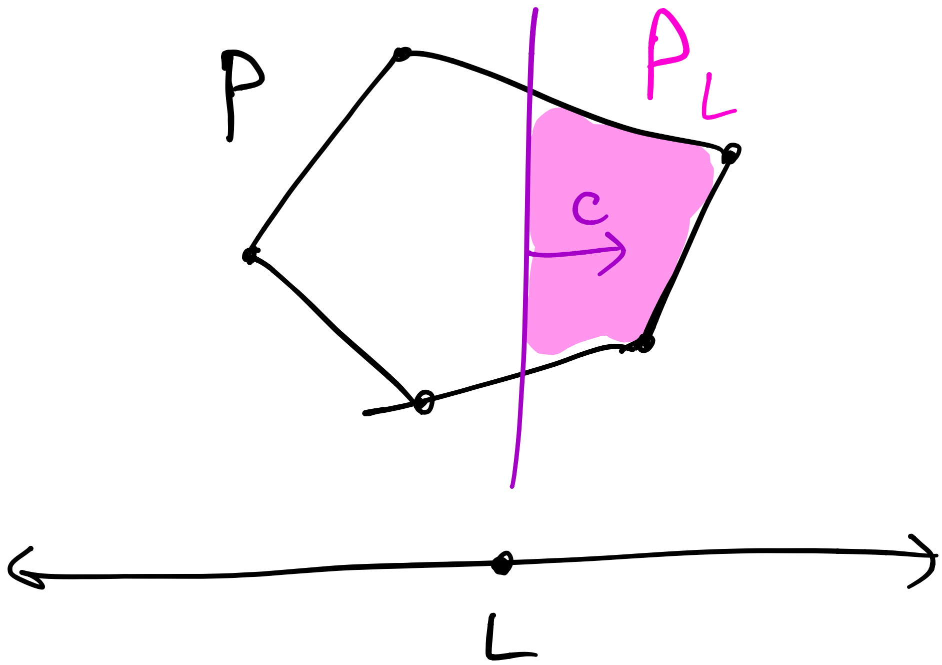
FEAS tells you if P empty or NOT.

- OPT reduces to FEAS:

binary search:

$\max\{c^T x : x \in P\} \geq L$ if

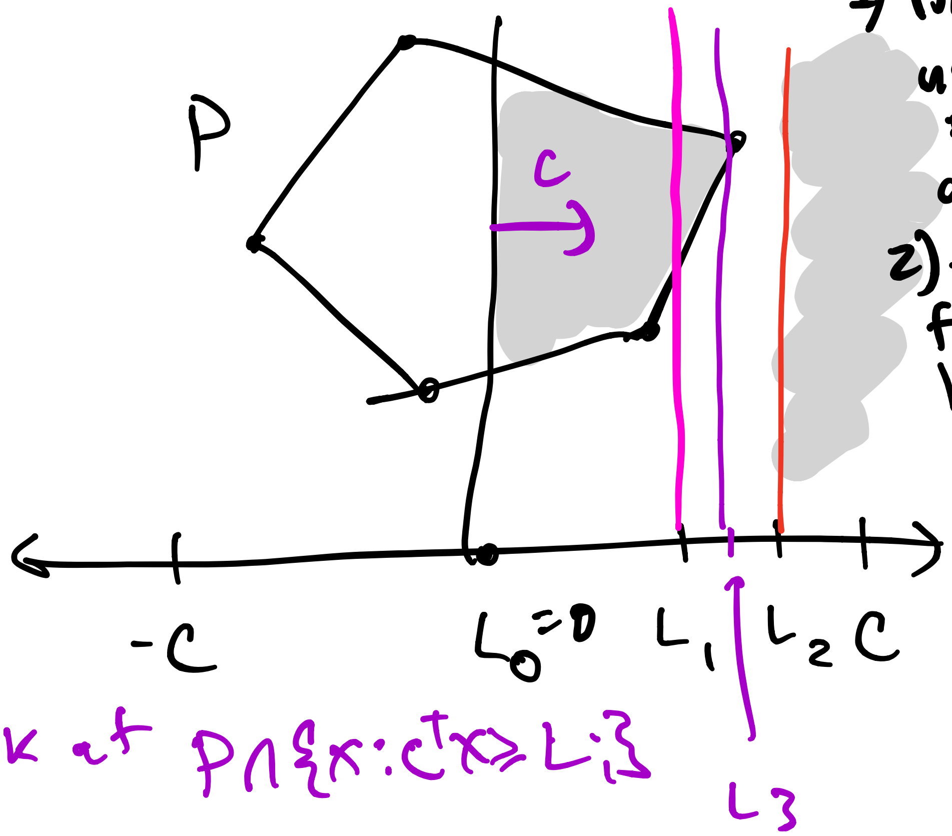
$$P_L := P \cap \{x : c^T x \geq L\} \neq \emptyset.$$



- Given a-priori bound

$$-C \leq L \leq C$$

Binary search to find max $L : P_L \neq \emptyset$.



1) find $P_{L_0} \neq \emptyset$ using FEAS alg.

2) test for L_1 between L_0, C.

look at $P \cap \{x : c^T x \geq L_i\}$

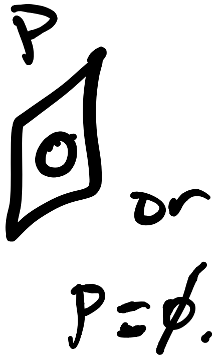
- optimizes to ε- precision in $\log\left(\frac{2C}{\varepsilon}\right)$ calls to FEAS.

- for LP, can solve exact OPT with $C, \frac{1}{\varepsilon} \leq$ exponential in bitsize of A, b.

(Details later.)

(finally!)

The Algorithm



- Solves FEAS in time $\text{poly}(\log(\frac{1}{\varepsilon}), \log R, n)$ assuming given ball $B(x_0, R)$ of radius R containing P, and either P contains ball of radius ε or $P = \emptyset$.

- <u>ε, R dependence not a big deal:</u> (& actually necessary).

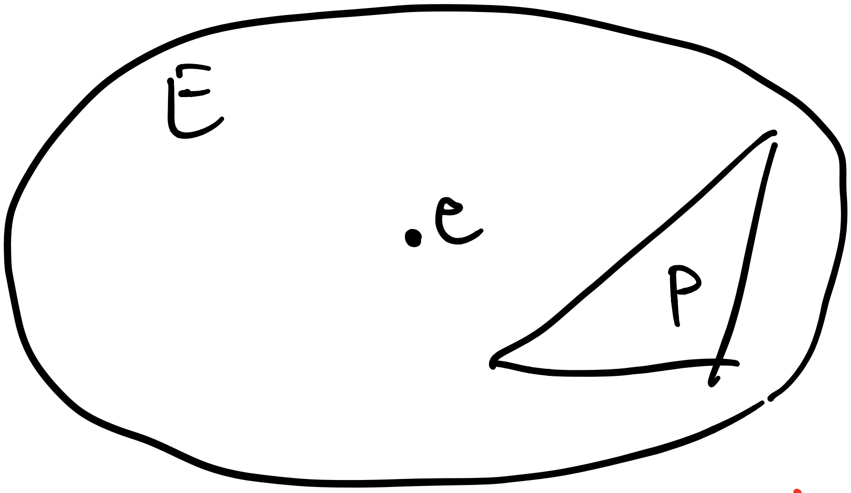
  ▷ For LP with $P = \{x : Ax \le b\}$ ε, R can be assumed exponential in bitsize of A, b.

using some tricks (we'll see these tricks for a special case.)

Algorithm idea:

- Set $E = E_0$ ellipsoid guaranteed to contain P.

  (e.g. E_0 = outer ball $B(0,R)$).



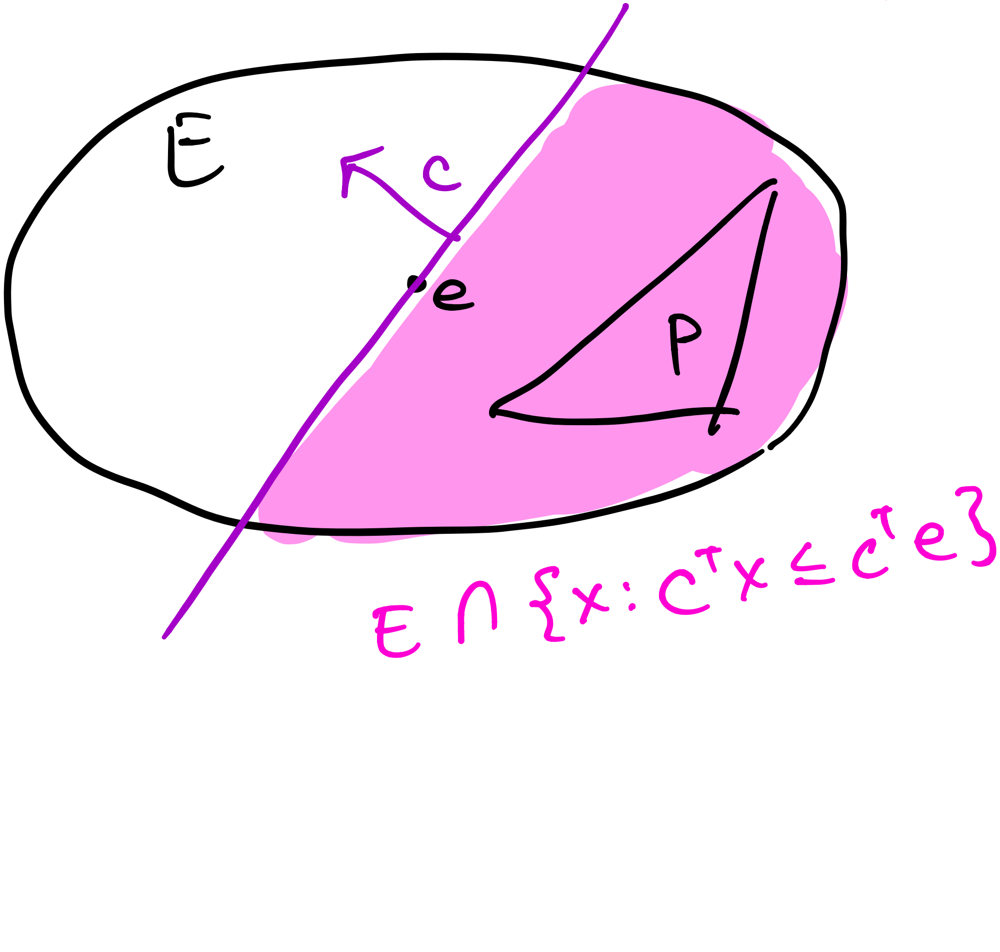
- While $e \notin P$: (if so, just return e & done).

▷ get separating hyperplane

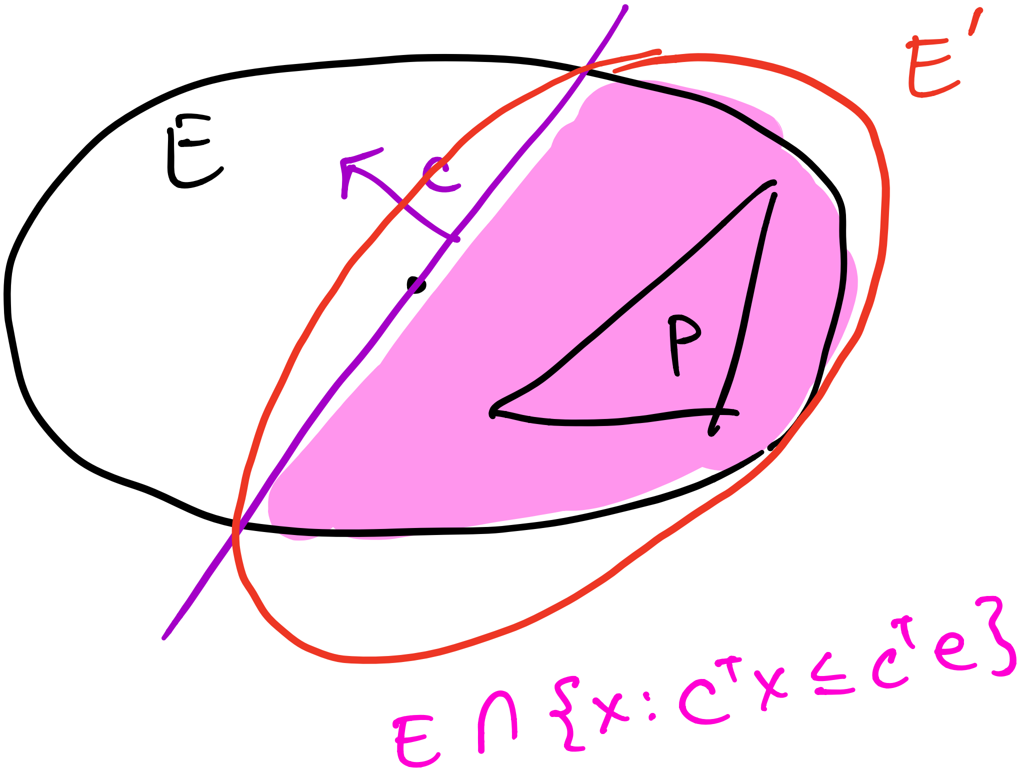
$c^T x \le d$ (valid for P but not for e).

From SEP oracle.

(actually, assume $d = c^T e$. by translating the hyperplane).



▷ Let E' "smaller ellipse" containing $E \cap \{x : c^T x \le c^T e\}$.



(Can take E' to be minimum volume ellipsoid containing $E \cap \{x : c^T x \le c^T e\}$.)

can find E' efficiently!

▷ Set $E \leftarrow E'$.

Runtime:

- **Volume Lemma:**

$$\mathrm{vol}(E') \le e^{-\frac{1}{2(n+1)}} \mathrm{vol}(E).$$

- As E always contains P, alg. must terminate in

$$\le 2(n+1) \log\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right).$$

iterations.

(if P contains ball of radius ε, cont in ball of radius R, $\le 2(n+1) \log\left(\left(\frac{R}{\varepsilon}\right)^n\right)$

$\leq 2(n+1)n\log\left(\frac{R}{\varepsilon}\right).$

If after $2(n+1)n\log\left(R/\varepsilon\right)$ iterations the algorithm hasn't terminated, output $P=\emptyset$.

Issues:

- How to compute E'
- Proof of volume lemma
- Bounding R, ε.