18.453 Quiz

Instructions. This is a **timed** quiz. This is meant to be done in **2** hours with access to notes and course material, but no access to collaborators. For best practice I suggest trying to complete it under these conditions. Afterwards please tell me if 2 hours felt like enough.

1. Consider a bipartite graph $G = (V, E)$ in which every vertex has degree k (a so-called k-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

   (a) by using König's theorem:

   Solution: Let the bipartition be A, B. First note that $|A| = |B|$, because the by counting the number of edges in two different ways we obtain that the number of edges is $k|A|$ and also $k|B|$. Thus a perfect matching could, in principle, exist.[1] König's theorem says that the number of edges in a maximum matching is the number of vertices in a minimum vertex cover. Thus, it is enough to show that a minimum vertex cover has size exactly $|A|$. The set A is indeed a vertex cover because G is bipartite. On the other hand, the number of edges covered by any set S is at most $k|S|$. As the number of edges is $k|A|$, we must have $|S| \geq |A|$. The size of the minimum vertex cover is $|A|$, concluding the proof.

   (b) by using the LP formulation of the min-weight perfect matching problem:

   Solution: The LP formulation of the min-weight perfect matching problem on a complete graph is as follows:

$$
(P)\qquad
\begin{array}{lll}
\text{Min} & \displaystyle\sum_{i,j} c_{ij}x_{ij} & \\
\text{subject to:} & & \\
& \displaystyle\sum_{j} x_{ij} = 1 & i \in A \\
& \displaystyle\sum_{i} x_{ij} = 1 & j \in B \\
& x_{ij} \geq 0 & i \in A, j \in B.
\end{array}
$$

   By Theorem 1.6 in the perfect matching notes, there is an integral minimizer x to the above LP, i.e. an actual perfect matching with the same value.

   We can formulate the maximum matching on our graph G (which is not necessarily a complete graph) in various ways. One way is by setting $c_{ij} = -1$ for $(i, j) \in E$ and $c_{ij} = 0$ else; for our matching we just keep the edges $M = \{(i, j) \in E : x_{ij} =$

[1] Strictly speaking, we don't need this step, but I find it comforting.

1}. Thus the size of the maximum matching is

$$
(P)\qquad
\begin{array}{lll}
\text{Max} & \displaystyle\sum_{(i,j)\in E} x_{ij} & \\
\text{subject to:} & & \\
& \displaystyle\sum_j x_{ij} = 1 & i \in A \\
& \displaystyle\sum_i x_{ij} = 1 & j \in B \\
& x_{ij} \geq 0 & i \in A, j \in B.
\end{array}
$$

As the size of the maximum matching the value of the above LP, if we can show the value of the above LP is at least $|A|$, then we will be done. The following assignment will accomplish this:

$$x_{ij} = \begin{cases} \frac{1}{k} & \text{if } (i,j) \in E \\ 0 & \text{else.} \end{cases}$$

Because G is k-regular, x is feasible. The value of this solution is $\frac{1}{k}|E| = |A|$, which completes the proof.

2. Suppose $G = (V, E)$ is a 2-edge-connected graph (that is, G remains connected if you delete any single edge) with at least one perfect matching, and suppose that G has a special edge e such that the graph obtained by removing e from G has no perfect matching. Show that there is necessarily a nonempty set $S \subseteq V$ with the following properties:

   - the number of odd components of $G \setminus S$ is exactly $|S|$,
   - $G \setminus S$ has at least one even component.

   Solution:
   Let G' be the graph obtained by removing e. Let S be a minimizer in the Tutte-Berge formula for the maximum matching in G'. Note that S is nonempty because G' is connected and has an even number of vertices (because G has a perfect matching). As G' has a matching of size $|V|/2-1$ (obtained by removing e from any perfect matching in G), the number of odd components in $G' \setminus S$ is exactly $|S| + 2$. Moreover, we must have that the number of odd components in $G \setminus S$ is at least $|S|$ because G has a perfect matching. Thus, the edge e must be between two of the odd components of $G' \setminus S$. These two odd components become an even component of $G \setminus S$, which now has exactly $|S|$ odd components.

3. Show that for any point x_0 in an unbounded polyhedron $P \subset \mathbb{R}^n$, P contains a *ray* from x_0, a set of the form $\{x_0 + \alpha y : \alpha \geq 0\}$ for some $y \in \mathbb{R}^n$. A suggested approach:

   - Show it is enough to prove this when x_0 is 0.
   - As P unbounded, there is some c such that $\max\{c^T x : x \in P\} = \infty$. Apply linear programming duality for the linear program $\max\{c^T x : x \in P\}$ to show something about the feasibility/infeasibility of the dual.
   - Apply Farkas' lemma to the infeasibility/feasibility of the dual in order to obtain the direction of the ray.

Solution: Proving the claim for the point $x_0 = 0$ in the polytope $P - x_0$ proves the original claim for x_0 in P, as we can simply translate the ray from 0 by x_0 to obtain the desired ray.

Thus we assume $x_0 = 0 \in P$. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Note that $b \geq 0$ because $0 \in P$. Let $c \in \mathbb{R}^n$ be such that $\max\{c^T x : x \in P\} = \infty$. By linear programming duality, the dual of this program $\min\{b^T y : A^T y = c, y \geq 0\}$ is infeasible. By Farkas lemma, there is a solution x to $Ax \geq 0$ with $c^T x < 0$. As $A(-x) \leq 0$, we have $A\alpha(-x) \leq 0 \leq b$ for all $\alpha > 0$. Thus P contains the ray $\{-\alpha x : \alpha > 0\}$.

An extra problem: This one shouldn't count as part of your 2 hours, but a problem like it could appear on the exam.

4. Let G be a bipartite graph with bipartition A, B and edge set E. A *fractional vertex cover* is a pair of assignments of numbers $(x_a \in \mathbb{R} : a \in A)$ and $(y_b \in \mathbb{R} : b \in B)$ to the vertices such that

$$\begin{aligned} x_a + y_b &\geq 1 \quad \forall ab \in E \\ x_a &\geq 0 \quad \forall a \in A \\ y_a &\geq 0 \quad \forall b \in B \end{aligned}$$

The *fractional vertex cover number* is

$$\tau(G) := \min\left\{\sum_{a\in A} x_a + \sum_{a \in B} y_a : (x, y) \text{ is a fractional vertex cover of } G\right\}.$$

Show that the fractional vertex cover number is the same as the vertex cover number, i.e. the size of a minimum vertex cover. **Hint:** [2]

Solution: Suppose $|A| = n_1$ and $|B| = n_2$ and $|E| = m$. Consider the matrix A and the vector b such that the set of fractional covers $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is equal to the set $\{(x, y) : A(x, y) \leq b, (x, y) \geq 0\}$. The number of columns of A is $n_1 + n_2$ and the number of rows is m. A is precisely the incidence matrix of the bipartite graph G. We recognize A^T as the matrix that appears in the linear programming formulation of the minimum weight perfect matching problem. In class we stated that A^T totally submodular. (Actually, A^T is a *submatrix* of the matrix from the minimum weight perfect matching problem, but it is still totally unimodular).

[2] Use total modularity, and that M^T is totally unimodular if and only if M^T is.