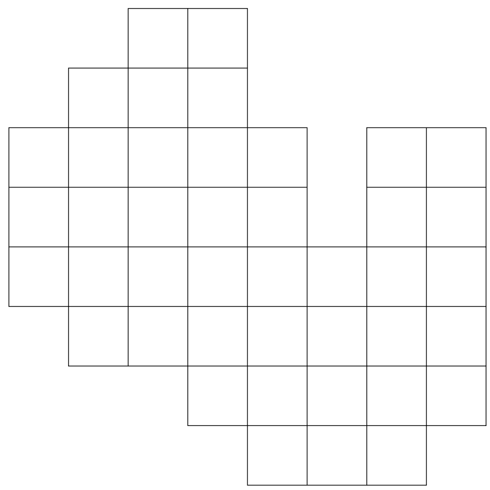
Solutions to Problem Set 1 (Do not distribute)

1-4 Consider the problem of perfectly tiling a subset of a checkerboard (i.e. a collection of unit squares, see example below) with dominoes (a domino being 2 adjacent squares).

(a) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.

(b) Can the following figure be tiled by dominoes? Give a tiling or a short proof that no tiling exists.



Solution: Consider the bipartite graph G with a vertex for each square and two squares are adjacent if they share an edge. This graph is bipartite since the squares can be colored black and white in a checkerboard pattern.

Any perfect tiling gives a perfect matching by simply selecting the edges corresponding to the dominoes selected. And vice versa.

We claim that the configuration shown in Figure 0.1 is a maximum one and so no perfect tiling exists. We will prove that the matching M corresponding to the configuration in Figure 0.1 is maximum by showing that there is no augmenting path as in the lecture. (Alternatively we could use Hall's theorem.)

Let A be the set of black squares and B the set of white squares. Orient the edges of G according to M, i.e. all the edges in M are oriented from B to A, and the edges not in M are oriented from A to B as in Figure 0.3.

Let v be the only exposed vertex of A and w be the only exposed vertex of B, and consider L to be the set of vertices reachable from v (the enclosed area in Figure 0.3). Since w is not in L we obtain that no augmenting path exists.

We can also deduce the fact that no perfect matching exists from Hall's theorem by observing that the 11 black vertices in L (the enclosed region on the right of Figure 0.3) has only 10 (white) neighbors.

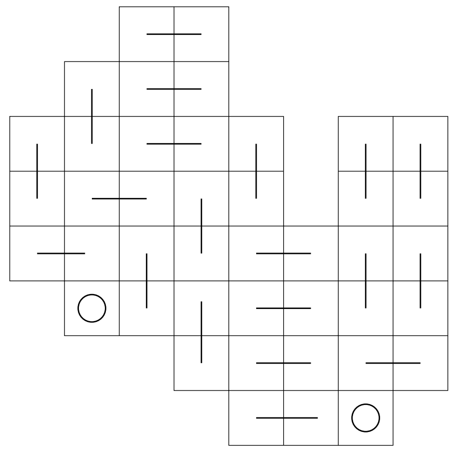


Figure 0.1: Maximum configuration of dominoes.

1-6 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B): $V = A \cup B$. Assume that, for some vertex sets $A_1 \subseteq A$ and $B_1 \subseteq B$, there exists a matching M_A covering all vertices in A_1 and a matching M_B covering all vertices in B_1. Prove that there always exists a matching covering all vertices in $A_1 \cup B_1$.

Solution: Make a bipartite graph H with edge set $M_A \cup M_B$ (and vertex set V). Then every vertex has H-degree at most 2, and every vertex in $A_1 \cup B_1$ has H-degree at least 1. So the connected components of H consist of paths and (even) cycles, and every vertex of $A_1 \cup B_1$ is contained in a nontrivial connected component of H. We will show that there is a matching M of H which covers all vertices of $A_1 \cup B_1$; since the edges of H are a subset of the edges of G, this will solve the problem.

It's enough to show that every nontrivial connected component P of H has a matching M_P which covers the vertices of $A_1 \cup B_1$ which are contained in P. If P is an even cycle or an odd path (with an even number of vertices), then P has a perfect matching, and we take M_P to be a perfect matching of P. If P is an even path (with an odd number of vertices), then every matching of P leaves some vertex of P uncovered, so we must show that in this case at least one of the endpoints of P is not contained in $A_1 \cup B_1$. Note that since P has length greater than 1, no edge of P can be in $M_A \cap M_B$, and the edges of P alternate between edges of M_A and M_B. Since P has even length, the endpoints of P are either both from A or both from B, while exactly one of the first and last edges of P is from M_A and the other is from M_B. Thus, P either has an endpoint in A which is not covered by M_A or an endpoint in B which is not covered by M_B, and we can choose M_P to be the matching of P which covers all of P other than this endpoint.

1-7 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B) $(V = A \cup B)$.

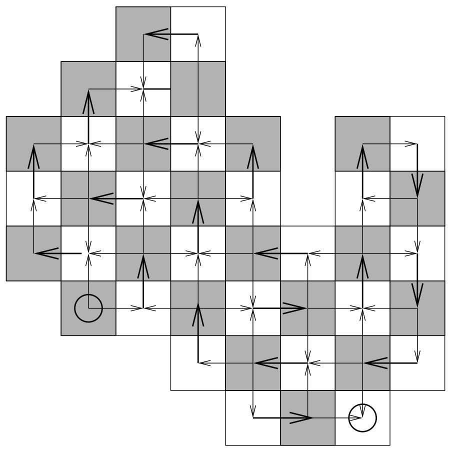


Figure 0.2: Oriented graph.

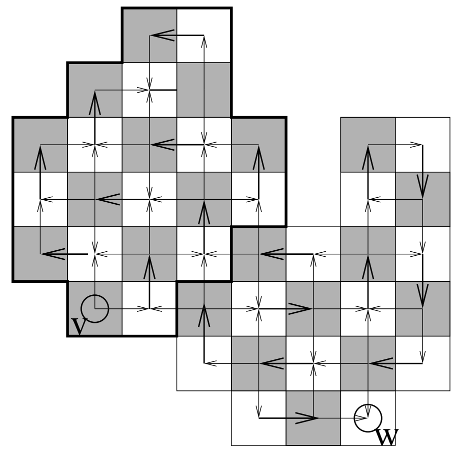


Figure 0.3: Set of reachable vertices from v.

Let $\mathcal{I} = \{X \subseteq A :$ there exists a matching M of G such that all vertices of X are matched$\}$.

Show that

(a) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

(b) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$ then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

(Later in the class, we will discuss matroids; properties (i) and (ii) form the definition of independent sets of a matroid.)

Solution:

(a) Let $Y \subset X \in \mathcal{I}$. Since X is an independent set, there exists a matching M_X that covers X. This matching also covers Y. Hence Y is an independent set.

(b) Let $X, Y \in \mathcal{I}$ with $|X| < |Y|$. It follows that there exist matchings M_X and M_Y such that M_X covers X and M_Y covers Y. Consider the graph $G' = (V, M_X \Delta M_Y)$. The set of edges of G' is the union of paths and cycles.

If M_X covers some element y in $Y \setminus X$. Then $X + y$ is an independent set.

Otherwise, all the vertices in $Y \setminus X$ are of degree 1 in G'. Since $|Y| > |X|$, we have $|Y \setminus X| > |X \setminus Y|$. Therefore, by the previous observation, there are more degree 1 vertices in $Y \setminus X$ than in $X \setminus Y$. It follows that there exists a path P in the decomposition of G' starting in a vertex $y \in Y \setminus X$ and not ending in X. We conclude that $M_X \Delta P$ is a matching of G that covers $X \cup \{y\}$. Thus, $X + y$ is an independent set.

1-10 Consider a bipartite graph $G = (V, E)$ with bipartition (A, B). For $X \subseteq A$, define $\mathrm{def}(X) = |X| - |N(X)|$ where $N(X) = \{b \in B : \exists a \in X$ with

$(a, b) \in E\}$. Let

$$\mathrm{def}_{max} = \max_{X \subseteq A} \mathrm{def}(X).$$

Since $\mathrm{def}(\emptyset) = 0$, we have $\mathrm{def}_{max} \geq 0$.

(a) Generalize Hall's theorem by showing that the maximum size of a matching in a bipartite graph G equals $|A| - \mathrm{def}_{max}$.

(b) For any 2 subsets $X, Y \subseteq A$, show that

$$\mathrm{def}(X \cup Y) + \mathrm{def}(X \cap Y) \geq \mathrm{def}(X) + \mathrm{def}(Y).$$

(c) (Optional:) Consider a 0-1 matrix. How are the following things related? 1. The maximal number of rooks (as in chess) that can be placed on a 1 in the matrix without attacking one another. 2. The minimal number of vertical or horizontal lines that contain all the 1's in the matrix. 3. The $a \times b$ all-zero submatrix with $a + b$ largest.

(a) Clearly, the size of a maximum matching cannot be more than $|A| - \mathrm{def}_{max}$ (since any matching has at most $|A| - |X|$ edges incident to $A - X$ and at most $|N(X)|$ edges incident to X).

Conversely, consider the minimum vertex cover C and let $X = A \setminus C$. Observe that $N(X) \subseteq C \cap B$, and thus

$$\mathrm{def}(X) = |X| - |N(X)| \geq |A \setminus C| - |C \cap B| = |A| - |C \cap A| - |C \cap B| = |A| - |C|.$$

Therefore $\mathrm{def}_{max} \geq |A| - |C|$ and the result follows from König's theorem.

(b) This is a simple counting argument. First of all,

$$|X \cup Y| + |X \cap Y| = |X| + |Y|.$$

Furthermore,

$$|N(X \cup Y)| + |N(X \cap Y)| \leq |N(X)| + |N(Y)|,$$

since every vertex b in B contributes at least as much to the right-hand-side than to the left-hand-side. Indeed, if $b \in N(X \cup Y) \setminus N(X \cap Y)$, it should be either in $N(X)$ or in $N(Y)$, while if $b \in N(X \cap Y)$, it should be in both $N(X)$ and in $N(Y)$.

(c) Associate a bipartite graph (with bipartition (A, B)) to a given 0-1 matrix (of size, say, $n \times m$) as follows: its vertices are the rows (A) and columns (B), and an edge between a row and a column is drawn if the corresponding matrix entry is 1. Under this correspondence, we have the following dictionary.

- Placing rooks on 1s (without attacking one another) $\longleftrightarrow$ a matching in the graph.

- Vertical or horizontal lines that contain all the 1's in the matrix $\longleftrightarrow$ a vertex cover in the graph.
- An $a \times b$ all-zero submatrix $\longleftrightarrow$ sets $A' \subset A, |A'| = a$, $B' \subset B, |B'| = b$ such that $N(A') \cap B' = \emptyset$.

With that in mind, we can relate those three numbers as follows. Let x_1 be the maximal number of rooks placed on 1s, x_2 be the minimal number of lines that contain all the 1's, and x_3 be the largest $a+b$ such that there is an $a \times b$ all-zero submatrix. For x_3, it is convenient to assume that degenerate $0 \times m$ matrices are allowed, so $x_3 \geq m$. Then by Kőnig's theorem, $x_1 = x_2$; and by extended Hall's theorem, $x_3 = n + m - x_1$.

1-18 We have shown that there always exists a solution x to the linear program (P) with all components integral. Reprove this result in the following way.

Take a (possibly non-integral) optimum solution x^*. If there are many optimum solutions, take one with as few non-integral values x^*_{ij} as possible. Show that, if x^* is not integral, there exists a cycle C with all edges $e = (i, j) \in C$ having a non-integral value x^*_{ij}. Now show how to derive another optimum solution with fewer non-integral values, leading to a contradiction.

(Optional:) Conclude that the doubly stochastic matrices (the set of $n \times n$ nonnegative matrices with unit row and column sums) are the convex hull of the permutation matrices (the matrices with exactly n ones, no two in the same row or column).

Let us first recall the linear program (P):

$$\text{Min} \quad \sum_{i,j} c_{ij} x_{ij}$$

subject to:

$$\begin{aligned} \sum_j x_{ij} &= 1 && i \in A \\ \sum_i x_{ij} &= 1 && j \in B \\ x_{ij} &\geq 0 && i \in A, j \in B \end{aligned}$$

Let x^* be an optimum solution of (P) with the fewest non-integral values x^*_{ij}. We may assume that there exists at least one ij with non-integral x^*_{ij} (otherwise, we are done). Let $G = (V, E)$ be the bipartite graph with bipartition (A, B) $(V = A \cup B)$ and the edge set

$$E = \{(i, j) \in A \times B : x^*_{ij} \text{ is not integral}\}.$$

We claim that for every $i \in V$ the degree of i with respect to G is not equal to one. Suppose not. Then there is $i \in A$ and $j' \in B$ (or change the role of A and B) such that $x^*_{ij'}$ is non-integral, and x^*_{ij} is integral for all $j \in B \setminus \{j'\}$. Since $\sum_{j\in B} x^*_{ij} = 1$, we have

$$x^*_{ij'} = 1 - \sum_{j \in B\setminus\{j'\}} x^*_{ij}.$$

This is a contradiction because the left-hand side is non-integral but the right-hand side is integral.

Now, note that every connected component of G is either an isolated vertex or a graph in which all vertices have degree at least 2. Moreover, there exists a connected component of the latter type because G has at least one edge. This implies that there exists a cycle C in G.

Let us construct an optimum solution x' with fewer non-integral values than x^*. Let $e_1, e_2, \ldots, e_\ell$ be edges of C in cyclic order. Since G is bipartite, the length ℓ of C must be an even number. Let

$$x'_{ij} = \begin{cases} x^*_{ij} - t & \text{if } ij = e_k \text{ for some odd } k \\ x^*_{ij} + t & \text{if } ij = e_k \text{ for some even } k \\ x^*_{ij} & \text{otherwise.} \end{cases}$$

Note that x' is a feasible solution as long as every entry of x' is positive, i.e.

$$-\min_{\text{odd } k} x^*_{e_k} \le t \le \min_{\text{even } k} x^*_{e_k}$$

Moreover, the objective function has the value

$$\sum_{ij} c_{ij} x'_{ij} = \sum_{ij} c_{ij} x^*_{ij} + t \sum_{k=1}^{\ell} (-1)^k c_{e_k}.$$

Since x^* is an optimum solution, we must have

$$\sum_{k=1}^{t} (-1)^k c_{e_k} = 0$$

because otherwise we can set t to be a value such that the objective value of x' is smaller than that of x^*. This implies that x'_{ij} is another optimum solution as long as it is feasible.

Set t to be $\min_{\text{even } k} x^*_{e_k}$. Let k' be an index such that $t = x^*_{e_{k'}}$. Then, $x'_{e_{k'}} = 0$ so x' has fewer non-integral values than x^* which contradicts the minimality of x^*.

For the optional part, we notice that the feasible solutions of program (P) constitute exactly the set of doubly stochastic matrices. We need to conclude that it is the same as the convex hull of the permutation matrices. This follows from the following standard facts from convexity.

- Every bounded polyhedron can be equivalently described as the convex hull of its *extreme points*, or *vertices*, that is, points that are in the set, and that do not belong to the relative interior of any straight line interval in the set. This is properly explained later in the course.
- Every extreme point can be obtained as an optimal solution in (P) for some linear functional $\sum c_{ij}x_{ij}$. This can be seen by taking coefficients c_{ij} such that the face of P corresponding to this direction contains that extreme point.

1-11 Let $S = \{1, 2, \cdots, n\}$. Let A_k be the set of all subsets of S of cardinality k (thus $|A_k| = \binom{n}{k}$). Let $k < \frac{n}{2}$. Consider the graph G_k with bipartition A_k and A_{k+1}, and with $E = \{(a, b) | a \in A_k, b \in A_{k+1} \text{ and } a \subset b\}$.

(a) Prove that the maximum matching in G_k has size A_k (remember $k < n/2$).

(b) Prove *Sperner's lemma*. The maximum number of subsets of S such that no subset is contained into another is $\binom{n}{\lfloor n/2 \rfloor}$.

(a) Let X be a subset of A_k. Note that any vertex in A_k has degree $n - k$ in G_k. So, the number of edges between X and $N(X)$ is $(n-k)|X|$. On the other hand, the number of edges adjacent to $N(X)$ is $(k+1)|N(X)|$ since any vertex in A_{k+1} has degree $k+1$. Thus, $(n-k)|X| \le (k+1)|N(X)|$. Since $k < \frac{n}{2}$, we have

$$|X| \le \frac{k+1}{n-k}|N(X)| \le |N(X)|.$$

By Hall's Theorem, there is a matching in G_k covering A_k.

(b) For a collection $\mathcal{C}$ of subsets of S, we call it a *chain* if for any $x, y \in \mathcal{C}$ either $x \subset y$ or $y \subset x$. In other words, chain is a sequence of subsets $a_1 \subset a_2 \subset \ldots \subset a_k$. On the other hand, we call a collection $\mathcal{F}$ of subsets of S an *antichain*, if no subset is contained in another. Note that any chain and antichain can share at most one element.

We claim that the collection of all subsets of S can be partitioned into $\binom{n}{\lceil n/2 \rceil}$ chains. This implies that the size of antichain is at most $\binom{n}{\lceil n/2 \rceil}$, since an antichain can have at most one element from each chain.

Recall part (a). We know that G_k has a matching covering A_k if $k < \lceil \frac{n}{2} \rceil$. Similarly, if $k \ge \lceil \frac{n}{2} \rceil$ then G_k has a matching covering A_{k+1}. Let M be the union of those matchings in G_k for $k = 0, 1, \ldots, n-1$. Note that M consists of disjoint paths, and for each path there are indices k and ℓ such that the path is of the form $a_k a_{k+1} \ldots a_\ell$ where $a_j \in A_j$ for $j = k, \ldots, \ell$ and $a_j a_{j+1} \in M$. Moreover, each path contains exactly one element from $A_{\lceil \frac{n}{2} \rceil}$. Since each path is a chain, we have $\binom{n}{\lceil n/2 \rceil}$ disjoint chains covering all subsets of S.

1-20 For the assignment problem, the greedy algorithm (which repeatedly finds the minimum cost edge disjoint from all the previously selected edges) can lead to a solution whose cost divided by the optimum cost can be arbitrarily large (even for graphs with 2 vertices on each side of the bipartition).

Suppose now that the cost comes from a metric, even just a line metric. More precisely, suppose that the bipartition is $A \cup B$ with $|A| = |B| = n$ and the ith vertex of A (resp. the jth vertex of B) is associated with $a_i \in \mathbb{R}$ (resp. $b_j \in B$). Suppose that the cost between these vertices is given by $c_{ij} = |a_i - b_j|$.

Consider the greedy algorithm: select the closest pair of vertices, one from A and from B, match them together, delete them, and repeat until all vertices are matched. For these line metric instances, is the cost of the greedy solution always upper bounded by a constant (independent of n) times the optimum cost of the assignment? If so, prove it; if not, give a family of examples (parametrized by n) such that the corresponding ratio becomes arbitrarily large.

We will give a family of examples (based on the Cantor set) where the ratio between the cost of the greedy solution and the cost of the optimum assignment becomes arbitrarily large. Our kth example will have $n = 2^k$ vertices in A and in B, and cost ratio slightly smaller than $2 \cdot (\frac{3}{2})^k - 1$.

Fix some small $\epsilon > 0$. We define sets $A_k, B_k \subseteq \mathbb{R}$ inductively, as follows. For $k = 0$, we set $A_0 = \{0\}$ and $B_0 = \{1 + \epsilon\}$. For $k \geq 1$, we let

$$A_k = A_{k-1} \cup ((\max(B_{k-1}) + 3^{k-1}) + A_{k-1})$$

and

$$B_k = B_{k-1} \cup ((\max(B_{k-1}) + 3^{k-1}) + B_{k-1}) = (1 + \epsilon) + A_k.$$

For example, we have

$$A_1 = \{0, 2 + \epsilon\}, \qquad B_1 = \{1 + \epsilon, 3 + 2\epsilon\}$$

and

$$A_2 = \{0, 2 + \epsilon, 6 + 2\epsilon, 8 + 3\epsilon\}, \qquad B_2 = \{1 + \epsilon, 3 + 2\epsilon, 7 + 3\epsilon, 9 + 4\epsilon\}.$$

Let $a_0, ..., a_{2^k-1}$ be the elements of A_k, in sorted order, and define $b_0, ..., b_{2_k-1}$ similarly. Inducting on k, it's easy to see that the greedy algorithm assigns each a_i to b_{i-1} for $0 < i < 2^k$, and assigns a_0 to b_{2^k-1}, with total cost $3^k - 2^k + (3^k + 2^k\epsilon)$.

On the other hand, the optimal assignment is at least as good as what we get by assigning each a_i to b_i, which has total cost $2^k \cdot (1 + \epsilon)$. This gives a cost ratio of

$$\frac{2 \cdot 3^k - 2^k + 2^k\epsilon}{2^k + 2^k\epsilon},$$

and this goes to infinity as k gets large.