# Lecture 17

## Matroid polytope!

1) finish algo proof (see lec16)
2) TU proof
3) Facets

Next time: Matroid intersect.

- Why do we care about matroid polytope?
  example of polymatroid, polytope crucial for submodular minimization.

## T.U. Proof

Recall matroid polytope

$$\boxed{P_M := \operatorname{conv}(X)}.$$

Want to show $P_M = P$ where

$$\boxed{P = \{x \in \mathbb{R}^E : \quad \text{(rank}_S\text{)} \quad x(S) \le r(S) \quad \forall S \subseteq E \quad \text{(nonnegativity)} \quad x_e \geq 0 \quad \forall e \in E\}.}$$

Here $x(S) = \sum_{e \in S} x_e = \mathbb{1}_S \cdot x$

**T.U. proof** Say $|E| = m$.

- Note that $X = \{\text{integral points in } P\}$.

Why?
- $x \in \mathbb{Z}^E \cap P \Rightarrow (x_i) \in \{0,1\}$, (rank $\{i\}$)
- $\mathbb{1}_S \in P \Rightarrow |S| \le r(S) \Rightarrow |S| = r(S) \Rightarrow S \in \mathcal{I}$. (rank$_S$).

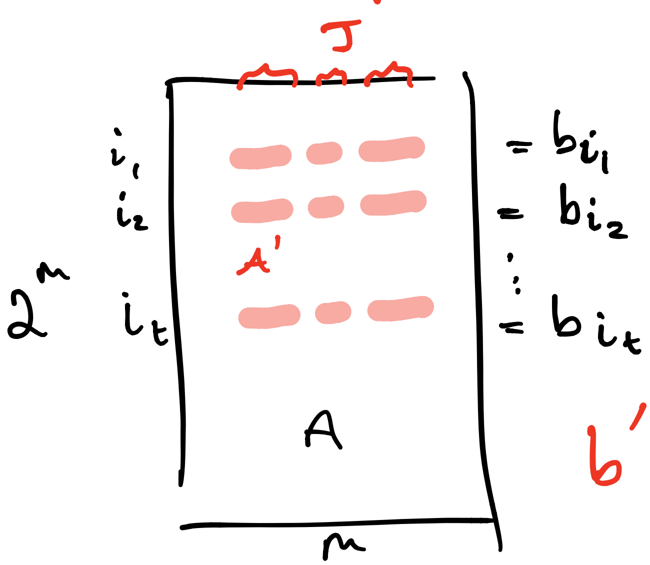
$\Rightarrow$ enough to show $P$ integral.

- If $P=\{x : Ax \le b,\ x \ge 0\}$,

is $A$ TU? <u>No!</u>

- Recall that vertices come from $n$ <u>tight</u> constraints.

(some from rows of $A$, some from $x \ge 0$).




▷ i.e. vertices come from setting $x_{E \setminus J} = 0$, solving $A' x_J = b'$ for remaining entries.

- Instead of showing $A$ T.U., show vertices come from T.U. submatrix $A'$.

$\Rightarrow$ vertices integral because

$x_J = (A')^{-1} b'$ for some integral $b'$,
$x_{E \setminus J} = 0$.

- In fact, submatrix $A'$ will be even more special:

rows of $A'$ $\Longleftrightarrow$ subsets of $E$,

we can make these subsets form a chain:

$$S_1 \subseteq \cdots\cdots \subseteq S_t.$$

$$\Rightarrow \quad A' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & 0 & & & 1 \end{bmatrix}$$

which is clearly TU.

(or one sees $\{x : A'x\} \subseteq \mathbb{Z}^m$ directly).

- This will be helpful for matroid intersection also.

- Actually, we'll prove something stronger:

**Claim** Let $F$ be a face of $P$.
Then $\exists$ chain $C$,

subset $J \subseteq E$ s.t.

$$F = \{ x \in \mathbb{R}^E : x(S) \leq r(S) \ \forall S \subseteq E,$$

(tight chain) $$x(S) = r(S) \ \forall S \in C,$$

$$x_e \geq 0 \ \forall e \in J,$$

(set $x_{E \setminus J} \to 0$) $$x_e = 0 \ \forall e \in E \setminus J. \}.$$

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Proof uses crucial lemma from submodularity of rank.

**Lemma**: $\forall x \in P$, the tight constraints

$$T := \{ S : x(S) = r(S) \}$$

are closed under $\cap$ and $\cup$.

I.e. if $R, S \in \mathcal{T}$,

$S \cup R \in \mathcal{T}$ & $S \cap R \in \mathcal{T}$.

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**Proof of claim from lemma:**

- From polyhedra, we know

$$F = \{x \in \mathbb{R}^E : x(S) \le r(S) \ \forall S \subseteq E, \\ x(S) = r(S) \ \forall S \in \mathcal{T}, \\ x_e \ge 0 \ \forall e \in J, \\ x_e = 0 \ \forall e \in E \setminus J. \}.$$

i.e. face comes from setting constraints to equality.

- Enough to show we

can replace $T$ by a chain $C$;

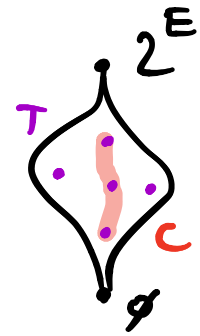
- "can replace" means they yield equivalent equalities, i.e.

$\mathrm{span}(T) :=$

$\mathrm{span}(1_S : S \in T) = \mathrm{span}(1_S : S \in C)$.

$=: \mathrm{span}(C)$.

- To show, let $C$ be a <u>maximal</u> subchain of $T$.




i.e. $C \subseteq T$, $C$ chain,

$\forall S \in T \ \exists R \in C$ s.t. $S \notin R$, $R \not\subseteq S$.

- We claim $\mathrm{span}(C) = \mathrm{span}(T)$.

- Suppose $\mathrm{span}(C) \subsetneq \mathrm{span}(T)$.

$\Rightarrow \exists\, S \in T \setminus C$ s.t. $1_S \notin \mathrm{span}(C)$, but $S$ can't be added to $C$ & still have $C$ be chain. (b/c $C$ maximal).

$\Rightarrow$ The set of "chain violations"

$$V(S) = \{R \in C : R \not\subseteq S \;\&\; S \not\subseteq R\}$$

is nonempty.

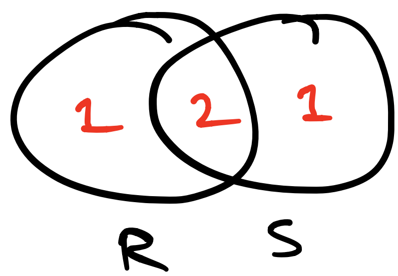
- Among all such $S$, take one with $|V(S)|$ as small as possible. (we know $|V(S)| > 0$ b/c $C$ max'l).

- Let $R \in V(S)$.

• **Lemma** $\Rightarrow$ $S \cup R,\ S \cap R \in T$.

$\Rightarrow$ $1_R,\ 1_S,\ 1_{S\cup R},\ 1_{S\cap R}$ linearly dependent:

$$1_R + 1_S = 1_{S\cup R} + 1_{S\cap R}.$$




• Since $1_R \in \mathrm{Span}(C)$, $1_S \notin \mathrm{span}(C)$, we must <u>not</u> have

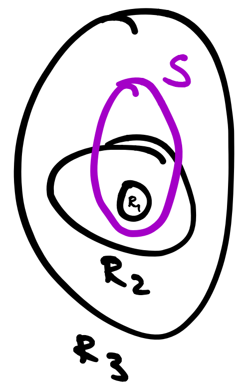
$$1_{R\cup S},\ 1_{R\cap S} \in \mathrm{Span}(C).$$

else $1_S = 1_{S\cup R} + 1_{S\cap R} - 1_R \in \mathrm{span}(C)$.

- Let $1_B \notin \mathrm{span}(C)$, where $B = R \cup S$ or $R \cap S$.

- But $|V(B)| < |V(S)|$, because $V(B) \subseteq V(S)$ (Exercise).

  and $R \in V(S) \setminus V(B)$. (by definition of $B$).




- $\Rightarrow$ contradiction. $\square$

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**Corollary**: Let $x$ vertex of $P$. $\exists$ chain $C$, subset $J$ s.t. $x$ solves

$$x(S) = r(S) \quad \forall\, S \in C,$$
$$x_e = 0, \quad \forall\, e \notin J.$$

uniquely.

$\Rightarrow$ integrality of $x$. (double check yourself).

**Proof of lemma** want to show

$$T := \{S : x(S) = r(S)\}$$

closed under $\cap$ and $\cup$.

- Proof is submodularity + algebra.

- $r(S) + r(T) = x(S) + x(T)$ (1)

$= x(S \cap T) + x(S \cup T)$ (2)

$\leq r(S \cap T) + r(S \cup T)$ (3)

$\leq r(S) + r(T)$ (4)

- (1) because $S, R \in T$
- (2) b/c $1_S + 1_T = 1_{S \cap T} + 1_{S \cup T}$.

  AKA $x(\cdot)$ **modular.**

- (3) because $x \in P \Rightarrow$

  $x(S \cap T) \leq r(S \cap T)$,

  $x(S \cup T) \leq r(S \cup T)$.

- (4) is submodularity.
- ☆ must be all equalities!

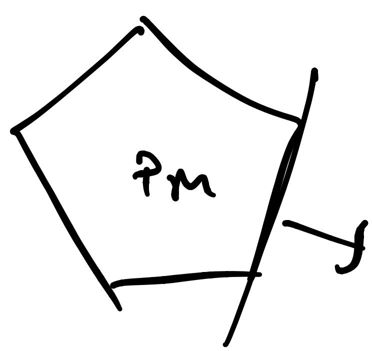
$\Rightarrow x(S \cap T) = r(S \cap T)$,

$x(S \cup T) = r(S \cup T)$. $\square$.

we'll skip facet proof; see pdf.

# Facets of $P_M$

- which of the $2^{|E|}$ inequalities
$$x(S) \le r(S)$$
define facets of $P_M$?




- For simplicity, assume
$$r(\{e\}) = 1.$$
(e s.t. $r(\{e\}) = 0$ is boring anyway).

$\Rightarrow \dim P_M = |E|$,

$x_e = 0 \rightsquigarrow$ facet:

The points $\{1_f : f \neq e\}$ are $|E|-1$ affinely indep pts on $x_e = 0$.

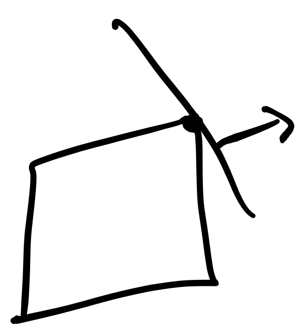
i.e. all <u>nonnegativity constraints</u> are facets.

- <u>Rank constraints</u>? $x(S) \leq r(S)$.

  ▸ if $S$ not <u>closed</u>, i.e. $S \subsetneq \mathrm{span}(S)$, then not facet b c redundant: follows from

$$x(\mathrm{span}(S)) \leq r(S) = r(\mathrm{span}(S))$$

and $x \geq 0$.



- If $S$ <u>seperable</u>, i.e. $U$, $\emptyset \subsetneq U \subsetneq S$, s.t.

$$S = r(U) + r(S \setminus U),$$

then also <u>not facet</u>:

$$x(S) \le r(S) \Leftarrow x(U) \le r(U),$$
$$x(S \setminus U) \le r(S \setminus U).$$

- <u>Fact</u>: $S \rightsquigarrow$ facet $\Leftrightarrow$ $S$ <u>closed</u> & <u>inseparable</u>.

<u>Proof omitted</u>

- <u>E.g.</u> <u>Graphic matroid</u> $M(G)$;

▷ **Exercise** $F \subseteq E$ inseparable $\iff$

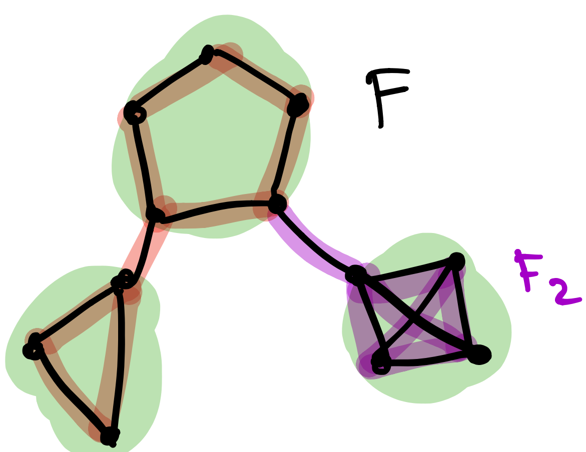
$(V, F)$ is either

▷ single edge OR

▷ is 2-connected.

E.g.

$|V| = 12$



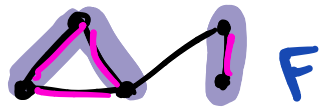

$F = F_1 \dot\cup F_2$

separable;

$r(F) = 11$ $r_{F_2} = 4$.

$r(F_1) = 7$

▷ Span$(F)$ = all edges w/ both endpts in same C.C.




▷ Thus $F$ closed & inseparable $\Leftrightarrow$ $F$ single edge or $F = E(S)$, $E(S)$ 2-connected.

$\Rightarrow$ "Forest polytope" is minimally described by

$$P = \{x \in \mathbb{R}^E : x(E(S)) \leq |S|-1 \quad \forall S \subseteq V, E(S) \text{ 2-connected or } |S|=2,$$
$$0 \leq x_e \leq 1 \quad e \in E\}.$$

"spanning tree" polytope:

$$P = \Big\{ x \in \mathbb{R}^E : \begin{array}{l} x(E) = |V| - 1 \\ x(E(S)) \quad |S| - 1 \\ \forall\, S \subsetneq V,\ E(S) \\ \text{2-conn or } |S| = 2, \\ 0 \le x_e \le 1 \quad e \in E \end{array} \Big\}.$$