Lecture 16

Plan:

1) Matroid opt.
(see lec 15 notes)

2) Matroid polytopes

More preliminaries:

Rank function

- Analogous to rank of matrices
- rank function $r_M : 2^E \to \mathbb{N}$

of matroid $M=(E,I)$ is

$$\boxed{r_M(X) := \max\{|Y| : Y \subseteq X,\ Y \in I\}.}$$

= the size of largest independent set contained in X.

= size of any indep set that is max'l in X

- sometimes just r.

Examples

- **linear matroid**: $r(X) = \text{rank}(A_X)$.
- **partition matroid**: Recall for $E = E_1 \dot\cup \cdots \dot\cup E_\ell$,

$$I = \{X \subseteq E : |X \cap E_i| \leq k_i \ \forall i = 1 \ldots \ell\}.$$

$$\boxed{r(X) = \sum_{i=1}^{\ell} \min\{|E_i \cap X|, k_i\}.}$$

- **Graphic matroid**: M_G, $G=(V,E)$.

For $F \subseteq E$,

$$r(F) = n - K(V,F)$$

$K(V,F) :=$ # connected components of graph w/ vertices V, edges F.

e.g.

$$r(F) = 5 - 2 = 3$$

Properties of rank function

Let r be rank function of matroid.

(R1) $0 \leq r(X) \leq |X|$

(R2) <u>monotonicity</u>:

$$X \subseteq Y \Rightarrow r(X) \leq r(Y).$$

(R3) <u>submodularity</u>:

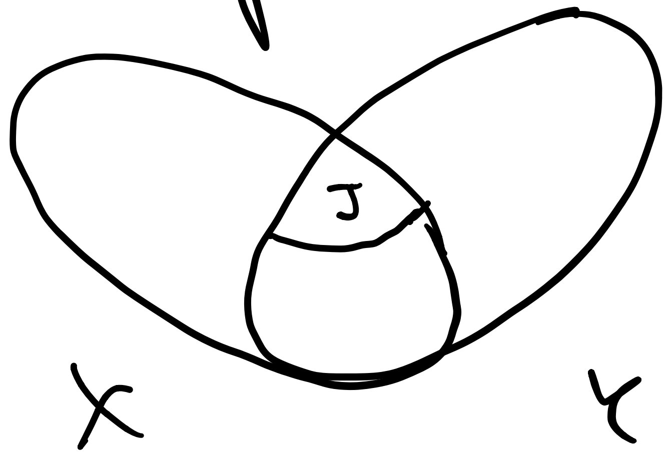
$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cup Y)$$

<u>Proof of R3</u>: • Let $X, Y \subseteq E$.

• We want to show

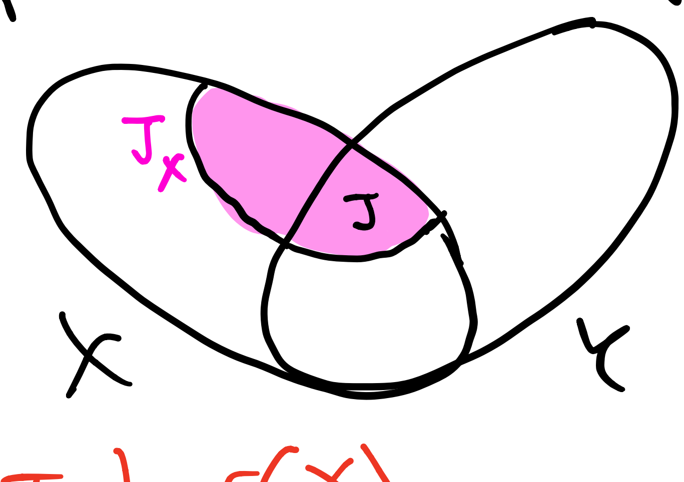
$$r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y).$$

- Let J max'l indep. subset of $X \cap Y$.



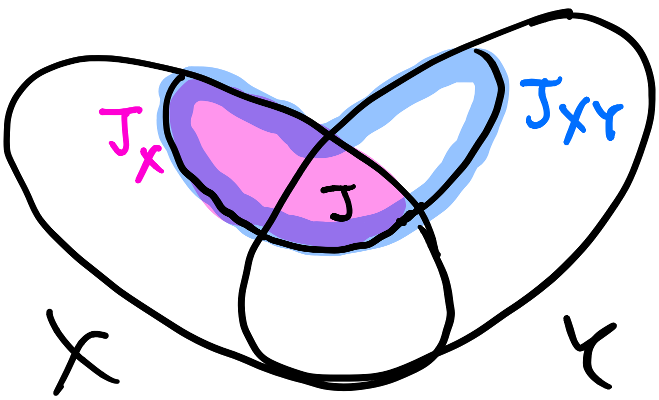
$\Rightarrow |J| = r(X \cap Y)$ by ☆

- Extend J to max'l indep subset J_X of X.



$\Rightarrow |J_X| = r(X)$.

• Extend J_X to max'l indep. J_{XY} of $X \cup Y$.



$\Rightarrow |J_{X \cup Y}| = r(X \cup Y)$.

• Note

$$\begin{array}{ccccc} X \cap Y & \subseteq & X & \subseteq & X \cup Y \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ J & \subseteq & J_X & \subseteq & J_{XY} \\ \| & & \| & & \\ J_{XY} \cap (X \cap Y) & & J_X \cap X & & \end{array}$$

by maximality of J in $X \cap Y$, J_X in X.

i.e. get chain of J's by intersecting J_{XY} with $X \cap Y$, X.

- <u>submodularity</u> $\iff$

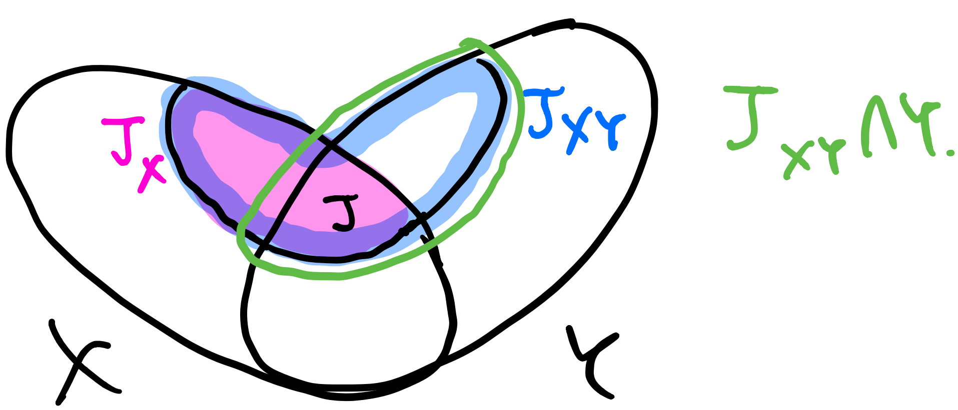
$$r(X)+r(Y) \geq r(X\cap Y)+r(X\cup Y).$$

$$\iff |J_X| + r(Y) \geq |J| + |J_{XY}|.$$

$$\iff r(Y) \geq |J| + |J_{XY}| - |J_X|.$$

- To prove, exhibit indep set in Y: use $J_{XY} \cap Y$.

$$\Rightarrow r(Y) \geq |J_{XY} \cap Y|.$$



- **Claim:**

$$|J_{XY} \cap Y| = |J| + |J_{XY}| - |J_X|.$$

Pf of Claim: $|J_{XY} \cap Y|$

$$= |(J_{XY} \cap Y) \setminus X| + |(J_{XY} \cap X) \cap Y|$$

$$= |(J_{XY} \setminus X) \cap Y| + |J_{XY} \cap (X \cap Y)|.$$

$$\Big\downarrow J_{XY} \subseteq X \cup Y \qquad\qquad \Big\downarrow$$

$$= |J_{XY} \setminus X| + |J|.$$

$$= |J_{xy} \setminus J_x| + |J|$$

$$= |J_{xy}| - |J_x| + |J|. \qquad \square$$

Span: analogous to span in linear algebra.

- Given matroid $M = (E, I)$, span of $S \subseteq E$ is

$$\text{span}(S) := \{e \in E : r(S+e) = r(S)\}.$$

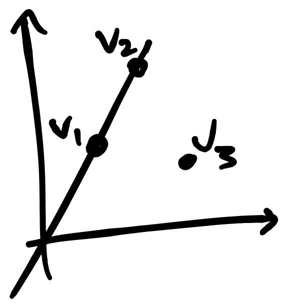
i.e. the elements that do not increase the rank of S when added.

e.g. For a linear matroid from $v_1, \ldots, v_m \subseteq \mathbb{R}^n$,

$$\text{span}(S) = \{ j : j \in \text{span}\{v_i : i \in S\} \}.$$

linear algebra def.

$\mathbb{R}^2$



$\operatorname{span}(\{1\}) = \{1,2\}.$

- **Claim:** $r(S) = r(\operatorname{span}(S))$.

(rank is preserved by adding all elts that don't increase rank)

Pf:

- Take $J \subseteq S$ max'l indep.
- Suppose $r(\operatorname{span}(S)) > |J|$;

(exchange axiom) $\Rightarrow \exists\, e \in \operatorname{span}(S) \setminus J$ s.t. $J + e \in \mathcal{I}$.

$\Rightarrow r(S+e) \geq r(J+e) = |J| + 1 > r(S)$;

contradicts $e \in \operatorname{span}(S)$. $\square$

- Say S is **closed** if $S = \operatorname{span}(S)$;
  closed S also called **flats**.

Matroid polytope

- Let $M = (E, \mathcal{I})$ matroid.
- Let $X = \{ 1_S : S \in \mathcal{I} \}$.
  $= \{$indicator vectors of independent sets$\}$.

- $P_M := \operatorname{conv}(X)$ is the **matroid polytope**.

What are the inequalities of P_M?

Theorem: For r rank function of M, let

$$P = \{ x \in \mathbb{R}^{|E|} :$$

(rank) $\quad x(S) \leq r(S) \quad \forall S \subseteq E$

(nonnegativity) $\quad x_e \geq 0 \quad \forall e \in E \}.$

Here $x(S) := \sum_{e \in S} x_e$.

$$\boxed{\text{Then } P_M = P.}$$

Notes:

- Check $X \subseteq P$ by checking constraints:

For $1_{S'} \in X$, $S \subseteq E$,

$$1_{S'}(S) = |S' \cap S| \leq r(S)$$

because $S' \cap S \subseteq S$ independent.

- Thus $P_M := \operatorname{conv}(X) \subseteq P$.
- Harder to show $P \subseteq \operatorname{conv}(X) = P_M$.
  ▷ use the "3 techniques".

Algorithmic proof:

- based on greedy algorithm.
- From $\operatorname{conv}(X) \subseteq P$, have

$$\max\{c^T x : x \in X\} \leq \max\{c^T x : x \in P\}.$$

- Enough to show equality; would follow if we found <u>dual feas</u> y, $x \in X$ s.t.

$$c^T x = b^T y.$$

(because then

$c^T x \leq \max\{c^T x : x \in P\} \leq b^T y$

has equalities).

- What's the dual?
- For convenience, use "symmetric" duality:

$$\text{(primal)} \quad \begin{aligned} \max\ & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned} \quad = \quad \begin{aligned} \min\ & b^T y \\ & A^T y \geq 0 \\ & y \geq 0. \end{aligned} \quad \text{(dual)}$$

- <u>Our primal:</u>

$$\max\ c^T x$$

$$x(S) \leq r(S) \quad \forall S \subseteq E$$

$$x_e \geq 0 \quad \forall e \in E$$

$$A = \begin{array}{cc} & e \\ S & \begin{bmatrix} & & \\ & 1_S(e) & \\ & & \end{bmatrix} \end{array} 2^E$$

(rows of A are indicator vectors of all subsets).

- **Dual:** $\min \sum_{S \subseteq E} r(S) y_S$

$$\sum_{S : e \in S} y_S \geq c(e) \quad \forall e \in E$$

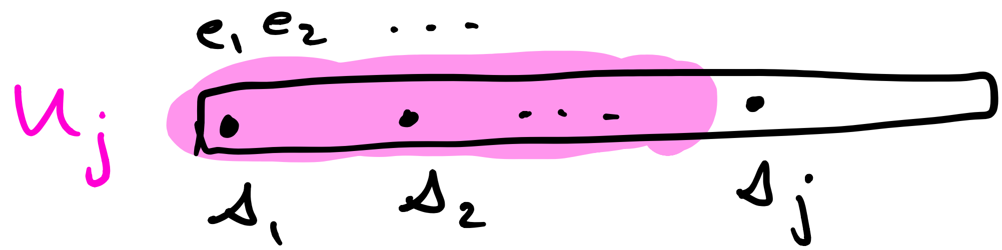
$$y_S \geq 0 \quad \forall S \subseteq S.$$

- Thus we need $S' \in \mathcal{I}$, y_S s.t.

$$c^T 1_S = \sum_{S \subseteq E} r(S) y(S).$$

- Consider cost c_i
- max cost indep set = last set S_k returned by greedy up to last nonnegative cost $c(e_q)$.
- Need y with same dual value as S_k;
- For $j \leq k$, let $S_j = \{s_1, \dots, s_j\}$.
- U_j := <u>all</u> elements up to & excluding s_j

$= \{e_1, \dots, e_\ell\}$ for $e_{\ell+1} = s_j$



- Note that $r(U_j) = j$.

▷ Indep of $S_j \Rightarrow r(U_j) \geq j$;

▷ If $r(U_j) > j$ then $\exists\, e \in U_j \setminus S_j$ s.t. $S_j + e \in \mathcal{I}$; but greedy would add e before j. ※.

- For $j = 1 \dots k$, set

$$y_{U_j} := c(s_j) - c(s_{j+1})$$

where $c(s_{k+1}) := 0$.

- Set all other $y_S := 0$.

- **Claim 1:** y dual feasible.

Pf: ▷ $y \geq 0$ because c sorted.

▷ $\forall e \in E$, let $t = \min\{j : e \in U_j\}$.

Then

$$\sum_{S \ni e} y_S = \sum_{j=t}^{k} y_{U_j} = c(S_t) \geq c(e)$$

telescope sorted

- **Claim 2:** $\sum_{S \subseteq E} r(S) y_S = c(S_k)$.

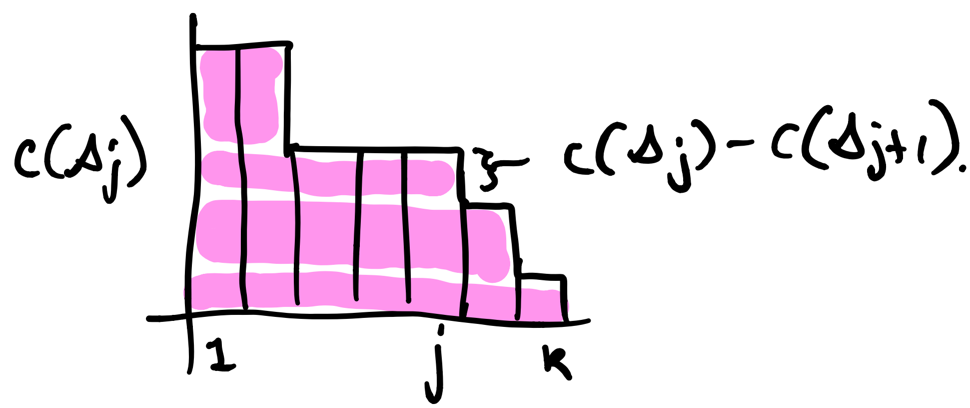
Pf: $\sum_{S \subseteq E} r(S) y_S = \sum_{j=1}^{k} r(U_j) y_{U_j}$

$$= \sum_{j=1}^{k} j \cdot \left(c(\Delta_j) - c(\Delta_{j+1}) \right)$$

$$= \sum_{j=1}^{k} \left[j - (j-1) \right] c(\Delta_j)$$

$$= \sum_{j=1}^{k} c(\Delta_j) = c(S_k). \quad \square$$

Intuition: $c(S_k)$ is area



$$c(S_k) = \sum_{j=1}^{k} c(\Delta_j) = \sum_{j=1}^{k} j \cdot \left(c(\Delta_j) - c(\Delta_{j+1}) \right).$$

Vertex proof