Lecture 15

Plan

1) Finish min T-odd cut
(see Lec14 notes]

2) Matroids.

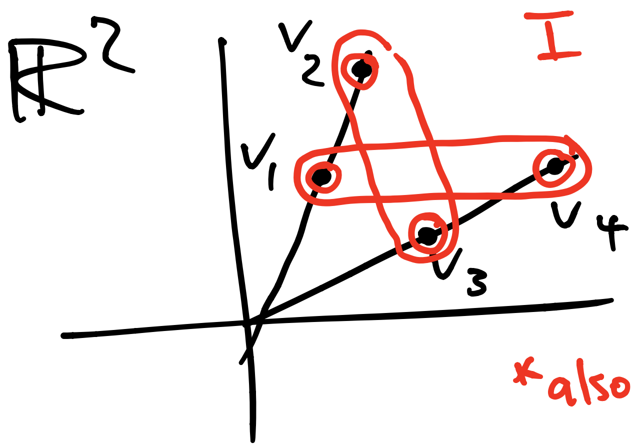
Matroids

"Tractable" set systems.

E.g. Given vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, consider set system $\mathcal{I} \subseteq 2^{[m]}$:

$$\mathcal{I} = \left\{ S \subseteq [m] : \{v_i : i \in S\} \text{ linearly independent} \right\}$$

picture:



*also $\emptyset \in \mathcal{I}$.

Properties of $\mathcal{I}$:

(P1) "Downward closed"

If $X \subseteq Y$, $Y \in \mathcal{I}$ then $X \in \mathcal{I}$.

(P2) "Exchange property"

If $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ and $|Y| > |X|$,

then there is something in Y you can add to X while keeping it independent.

Formally: $\exists\, e \in Y \setminus X$ s.t. $X \cup \{e\} \in \mathcal{I}$.

Pf of P2: $|X| = \dim \operatorname{span}\{v_i : i \in X\}$

$|Y| = \dim \operatorname{span}\{v_i : i \in Y\}$.

$\Rightarrow \exists\, j \in Y$ s.t. $v_j \notin \operatorname{span}\{v_i : i \in X\}$.

$\Rightarrow X \cup \{j\} \in \mathcal{I}$. $\square$

P1, P2 capture combinatorial structure of $\mathcal{I}$.

for matroids, we just take P1, P2 as axioms.

Def (Matroid) A matroid M is a pair $(E, \mathcal{I})$ where

- $E = E(M)$ is finite set called "ground set" of M
- $\mathcal{I} = \mathcal{I}(M) \subseteq 2^E$ "independent sets" of M

- $\mathcal{I}$ satisfies P1 and P2.

Remarks

- P2 $\Rightarrow$ all maximal independent sets have same size.
- Maximal independent set called a base of M.
- dependent := not independent.
- for $F \subseteq E$, the restriction $M|_F = \{S \in \mathcal{I} : S \subseteq F\}$ is another matroid.

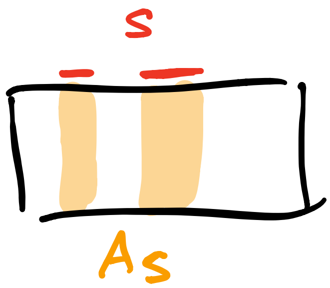
Examples

- Linear matroid: example from beginning.

A.K.A "representable"

▷ equiv def: $A \in \mathbb{R}^{n \times m}$ matrix,

$\mathcal{I} = \{$subsets S of columns s.t. submatrix A_S has rank $A_S = |S|\}$.

e.g. $A =$ 

write $M = M_A$.

▷ Actually can have $A \in \mathbb{F}^{m \times n}$ for any field $\mathbb{F}$.

▷ a <u>base</u> of M is precisely a <u>basis</u> for $\mathbb{R}^n$.

- "boring" example: <u>uniform matroid</u>: $U_{n,k} = (E, \mathcal{I})$

  where $|E| = n$ &

$$\mathcal{I} = \{\text{all subsets of size} \le k\}$$
$$= \{S \subseteq E : |S| \le k\}.$$

the **free matroid** is $U_{n,n} = 2^E$.

- **partition matroid**: $M = (E, \mathcal{I})$ where E is disjt union $E_1 \dot\cup \ldots \dot\cup E_\ell$,

$$\mathcal{I} = \{X \subseteq E : |X \cap E_i| \le k_i\}$$

for some fixed $k_1, \ldots, k_\ell$.

e.g.

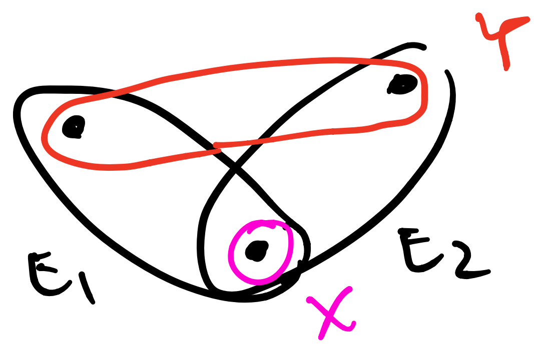
Check P2: Let $|X| < |Y|$, $X, Y \in \mathcal{I}$. Must be some i s.t. $|X \cap E_i| < |Y \cap E_i|$;

thus $k_i > |X \cap E_i|$; may add any $y \in Y \cap E_i \setminus X$.

Remark: if E not disjoint, not matroid.

e.g.

$k_1 = 1$

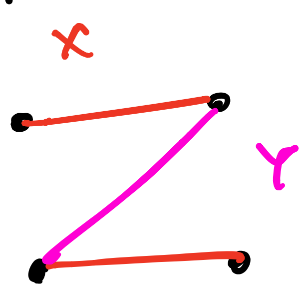


$k_2 = 1$

fails P2

• **Nonexample:** set of matchings in a graph.

e.g.

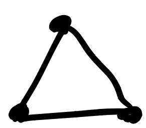


fails P2

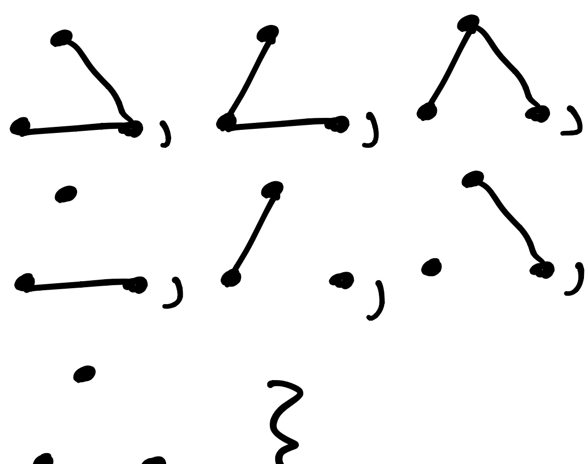
- **graphic matroids**:

  Given graph $G = (V, E)$,
  let $M(G) = (E, I)$ where

$$I = \{\text{forests in } G\}$$
$$= \{\text{acyclic subgraphs of } G.\}$$

e.g. $G =$ 

$I = \{$

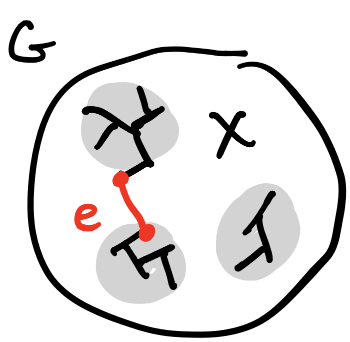


$\}$

Checkin P2:

- F forest $\Rightarrow |V| - |F| = \#$ connected components.
- X, Y forests, $|X| < |Y| \Rightarrow$
  Y has 1 fewer c.c.

$\Rightarrow$ edge e of Y connects two CC's of X.



$\Rightarrow X \cup \{e\}$ larger forest.

▷ <u>bases</u> are the spanning trees.

▷ <u>graphic $\Rightarrow$ linear</u>:

Pf: Let A be the vertex-edge incidence matrix of G, i.e.

$$A = \begin{array}{cc} & \begin{array}{cc} E & \quad (i,j) \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} \quad & 1 \\ \quad & -1 \end{bmatrix} \end{array} V$$

Ex. Check subset of cols lin. indep. $\iff$ subgraph contains no cycle. $\square$

▷ Graphic $\Rightarrow$ regular:

Say M regular if M linear over every field $\mathbb{F}$.

$-1 \leftarrow$ additive inverse of 1 in $\mathbb{F}$.

Note: matrix A is T.U.

Fact: matroid M regular $\iff$ $M = M_A$ over $\mathbb{R}$ for T.U. matrix A.

Circuits

- **circuit** := minimal dependent set.

  inclusion

e.g.
- in graphic matroid, the circuits are the cycles.
- in partition matroid, circuits are just subsets $C \subseteq E_i$ with $|C| = k+1$.

e.g.

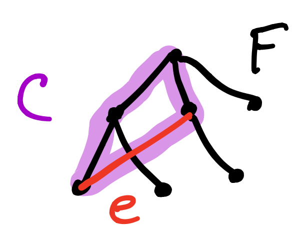
Note: removing any element of a circuit yields an indep. set.

There's exactly one way to do the reverse:

Theorem (unique circuit property)

- Let $M = (E, \mathcal{I})$ be a matroid.
- Let $S \in \mathcal{I}$, $e \in E$ s.t. $\underbrace{S+e}_{S \cup \{e\}.} \notin \mathcal{I}$.
- Then: there exists a unique circuit $C \subseteq S+e$.

e.g. If F is a forest, $F+e$ isn't, then $F+e$ contains unique cycle.



Remark: the uniqueness shows

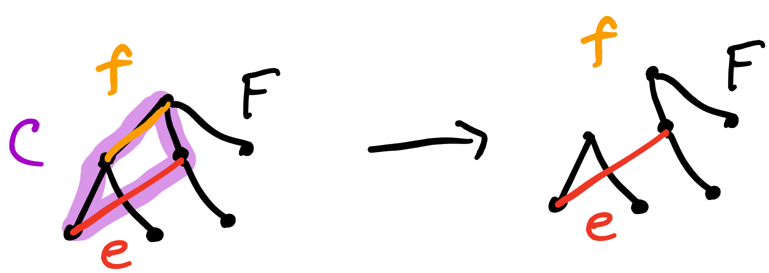
how to make more independent sets. Let

$$C \subseteq S+e \text{ circuit}, \; f \in C \setminus e.$$

then $S+e-f \in \mathcal{I}$.

Pf: Else $S+e-f$ contains circuit $C' \notin C$.

e.g.

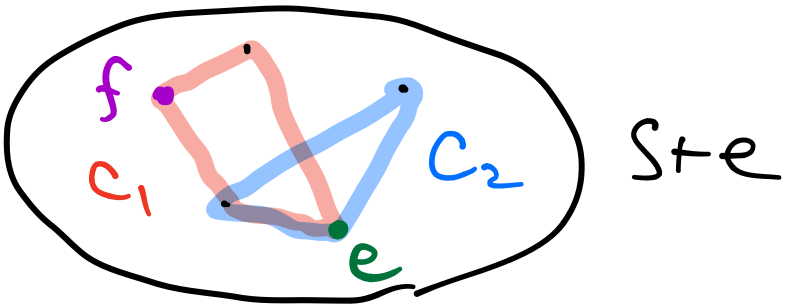


Proof of UCP:

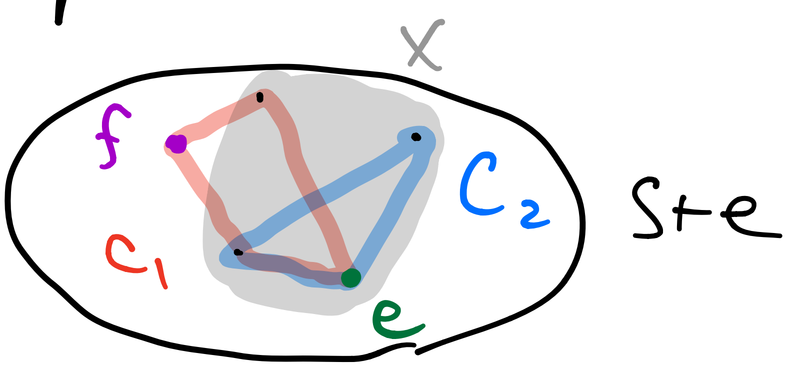
- suppose $S+e$ contains distinct circuits $C_1 \neq C_2$.
- Minimality $\Rightarrow C_1 \subsetneq C_2$; take $f \in C_1 \setminus C_2$

Note: $C_2 \subseteq S+e-f$.

We'll show $S+e-f \in \mathcal{I}$; contradicts $C_2 \subseteq S+e-f$.



- $C_1 - f$ independent $\Rightarrow$ $C_1 - f$ extends to maximal independent $X \subseteq S+e$.



- Both S, X maximal independent within $S+e$ $\Rightarrow |X| = |S|$.

- As $e \in X$ (b/c $e \in C_i - f \subseteq X$), we have $X = S + e - f$
- $\Rightarrow S + e - f$ indep, contradiction. $\square$

Matroid optimization

- Given cost function $c : E \to \mathbb{R}$, we want independent set S of maximum cost

$$c(S) = \sum_{e \in S} c(e).$$

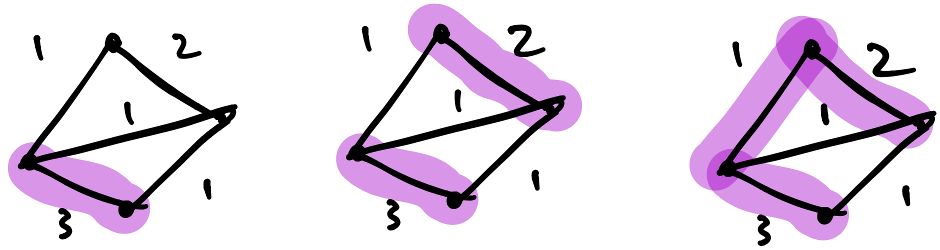
- Tractability of this problem is the main reason matroids are useful.

- if some $c(e) < 0$, can restrict to $M|_{E-e}$. Thus assume $c \geq 0$.

- if $c \geq 0$, need only consider <u>bases</u>.

<u>e.g.</u> for graphic matroids, on connected graphs, this is the maximum spanning tree (M.S.T.) problem.

<u>Recall</u>: M.S.T. has simple <u>greedy</u> algorithm: keep adding largest element that doesn't introduce cycle.



Kruskal's algorithm.

- **Fact**: the greedy algorithm works for any matroid.
- Actually, for all k can find best indep set of size k.

Algorithm

Let $|E| = m$.

▷ Sort E so $c(e_1) \geq \ldots \geq c(e_m)$.

▷ $S_0 := \phi$, $k = 0$.

- For $j = 1$ to m:
  - if $S_k + e_j \in \mathcal{I}$, then
    - $S_{k+1} := S_k + e_j$
    - $k \leftarrow k+1$.
- Output $S_1 \ldots S_k$.

Thm: For any matroid $M = (E, \mathcal{I})$, the above alg. finds an indep. set S_k s.t.

$$c(S_k) = \max_{\substack{|S| = k, \\ S \in \mathcal{I}}} c(S).$$

Proof: Suppose not.

- Let $S_k = \{s_1, \dots, s_k\}$ with

$$c(s_1) \geq c(s_2) \geq \dots \geq c(s_k).$$

- Suppose $T_k = \{t_1, \dots, t_k\}$,

$$c(t_1) \geq c(t_2) \geq \dots \geq c(t_k).$$

has <u>greater</u> cost.

- Let p first index where $c(t_p) > c(s_p)$.

- Let $A = \{t_1, \dots, t_p\}$

$$B = S_{p-1} = \{s_1, \dots, s_{p-1}\}.$$

- Because $|A| > |B|$, $\exists\ t_i \notin B$ s.t. $B + t_i \in I$. (by P1).
- But $c(t_i) \geq c(t_p) > c(s_p)$, so t_i should have been added to S_{p-1} instead of s_p; contradiction. $\square$.

Next time: rank function.