Solutions to Problem Set 5

5-5 Given a family $A_1, A_2, \cdots, A_n$ of sets (they are not necessarily disjoint), a *transversal* is a set T such that $T = \{a_1, a_2, \cdots, a_n\}$, the a_i's are distinct, and $a_i \in A_i$ for all i. A partial transversal is a transversal for $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ for some subfamily of the A_i's.

Show that the family of all partial transversals forms a matroid (on the ground set $E = \cup A_i$). (Hint: Think of bipartite matchings.)

(I1) It is easy to see that the first axiom ($I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$) is satisfied, since subset of a partial transversal is again a partial transversal.

(I2) We would like to prove the following:

If X and Y are partial transversals with $|X| < |Y|$, then there exists $y \in Y$ such that $X \cup \{y\}$ is a partial transversal.

Let us construct a bipartite graph $G = (V, E)$ as following. Let $A = \cup A_i$ and $B = \{A_i : i \in [n]\}$. Here $V = A \cup B$ and (A, B) forms a bipartition of G. A pair of vertices $a \in A$ and $A_i \in B$ forms an edge if $a \in A_i$. Note that $T \subseteq A$ is a partial transversal if and only if there exists a matching M in G which matches every vertex in T.

Now let X and Y be two partial transversals with $|X| < |Y|$ and let M and N be matchings in G which covers X and Y respectively. We may assume that $|M| = |X|$ and $|N| = |Y|$ by omitting excessive edges. Then there exists an augmenting path P in $M \cup N$ since $|M| < |N|$. Note that $M' = M \triangle P$ is a matching in G with $|M'| > |M|$, and covers $X \cup \{y\}$ where $y \in Y \setminus X$ is an end-point of P. Hence, $X \cup \{y\}$ is a partial transversal.

5-7 A family $\mathcal{F}$ of sets is said to be *laminar* if, for any two sets $A, B \in \mathcal{F}$, we have that either (i) $A \subseteq B$, or (ii) $B \subseteq A$ or (iii) $A \cap B = \emptyset$. Suppose that we have a laminar family $\mathcal{F}$ of subsets of E and an integer $k(A)$ for every set $A \in \mathcal{F}$. Show that $(E, \mathcal{I})$ defines a matroid (a *laminar* matroid) where:

$$\mathcal{I} = \{X \subseteq E : |X \cap A| \leq k(A) \text{ for all } A \in \mathcal{F}\}.$$

It is easy to see that $(E, \mathcal{I})$ satisfies the first axiom (I_1) that if $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$. For (I_2), consider $X, Y \in \mathcal{I}$ and $|Y| > |X|$, in order to show the second axiom (I_2), we need to show that there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$. Let us call a set $S \in \mathcal{F}$ maximal in $T \subseteq E$, $T \neq S$, if $S \subset T$ and S is not contained in any other element of $\mathcal{F}$ that is properly contained in T. Suppose that $A_1, \ldots, A_n$ are the maximal sets in E. Set $A^* = E \backslash (A_1 \cup \cdots A_n)$. Since $|Y| > |X|$, we must have $|Y \cap A^*| > |X \cap A^*|$, or $|Y \cap A_i| > |X \cap A_i|$ for some i. In case $|Y \cap A^*| > |X \cap A^*|$, there is an element $e \in (Y \cap A^*) \in (X \cap A^*)$, and $X \cup \{e\} \in \mathcal{I}$. So we only need to study the case that $|Y \cap A_i| > |X \cap A_i|$ for some i. Without lost of generality we may assume $|Y \cap A_1| > |X \cap A_1|$.

Let $B_1, \ldots, B_m$ be the maximal sets in A_1 and let $B^* = A_1 \backslash (B_1 \cup \ldots B_m)$. Since $|Y \cap A_1| > |X \cap A_1|$, we have $|Y \cap B^*| > |X \cap B^*|$ or $|Y \cap B_i| > |X \cap B_i|$ for some i. Again if $|Y \cap B^*| > |X \cap B^*|$, there is an element $e \in (Y \cap A^*) \in (X \cap A^*)$, and $X \cup \{e\} \in \mathcal{I}$. Otherwise we can repeat this process for B_i satisfying $|Y \cap B_i| > |X \cap B_i|$. Since the ground set E is finite, we can find the required e in a finite number of steps, and we are done.

5-8 We are given n jobs that each take one unit of processing time. All jobs are available at time 0, and job j has a profit of c_j and a deadline d_j. The profit for job j will only be earned if the job completes by time d_j. The problem is to find an ordering of the jobs that maximizes the total profit. First, prove that if a subset of the jobs can be completed on time, then they can also be completed on time if they are scheduled in the order of their deadlines. Now, let $E(M) = \{1, 2, \cdots, n\}$ and let $\mathcal{I}(M) = \{J \subseteq E(M) : J \text{ can be completed on time}\}$. Prove that M is a matroid and describe how to find an optimal ordering for the jobs.

First solution. Let $(j_1, j_2, ..., j_k)$ be a sequence of jobs ordered in increasing order on their deadlines, i.e., $d_{j_1} \leq d_{j_2} \leq \ldots \leq d_{j_k}$. If they could not be completed in time, there must exist some i for which $d_{j_i} < i$ (because j_1 will finish at time 1, j_2 will finish at time 2, etc.) However, this would imply that $d_{j_1}, d_{j_2}, ...d_{j_i} < i$. In other words, there are i jobs with deadline less than i; therefore at least i jobs need to be completed by the time $i-1$. This implies that the sequence of jobs is infeasible. Thus the contrapositive of what we just proved is that if a sequence of jobs can be completed in some order, then they can be completed in order of their deadlines.

Now we prove M is a matroid by checking the two axioms.

I1 If $Y \in \mathcal{I}$ and $X \subset Y$, then $X \in I$.

This is obvious: if a set of jobs can be completed in time, then a subset of the jobs can also be completed in time.

I2 If $X \in I, Y \in I$ and $|Y| > |X|$ then $\exists e \in Y \setminus X : X \cup \{e\} \in I$.

Suppose both sets of jobs are ordered by deadline. Let $y = |Y|, x = |X|$ and e be one of the jobs with latest deadline in $Y \setminus X$. Suppose e is in position $y - k$ of Y. Let $K = \{j_{i_1}, j_{i_2}, ..., j_{i_k}\}$ be the set of jobs ordered by deadline $(d_{i_1} \leq d_{i_2} \leq ... \leq d_{i_k})$ that appear after e in Y.

Since e is in position $y - k$ in Y, we have $d_e \geq y - k$. Also $d_{i_t} \geq y - k + t$ because j_{i_t} is in position $y - k + t$. In order to prove $X + e \in \mathcal{I}$, we need to prove there is no q for which the job in position q in X has deadline q for $q \geq d_e$ (This is the only way for

$X + e$ to be infeasible.) For the sake of contradiction, assume such q exists and job is x_q. Suppose there are n elements from K to the right of x_q.

If $n = k$, then x_q has at least k elements to its right. This means that it is in position at most $x - k$. However, $x - k < y - k \leq q$. This is a contradiction since x_q is in position q in X.

If $n < k$, then x_q is to the right of $j_{i_{k-n}}$, which has deadline $d_{i_{k-n}} \geq y-k+(k-n) = y-n$. Therefore the deadline of x_q, which is q, satisfies $q \geq y - n > x - n$. However, since there are at least n elements to the right of the q element of X, we have $q \leq x - n$. Again a contradiction.

The contradiction proves that $X + e \in \mathcal{I}$, as desired.

Finally, to find an optimal scheduling, consider the value of each job j_i being its reward c_i. The greedy algorithm in matroid M then finds the optimal configuration.

Second solution. Here is a shorter way to prove that M is a matroid. In fact, M consists of partial transversals (see Exercise 5.5) for the following family of sets:

$$X_D = \{1 \leq j \leq n \mid d_j \geq D\}, \quad D = 1, 2, \ldots.$$

Indeed, a partial transversal for a subfamily $X_{D_1}, X_{D_2}, \ldots$ with $1 \leq D_1 < D_2 < \ldots$ is a collection of jobs $j_{D_1} \in X_{D_1}, j_{D_2} \in X_{D_2}$, and it is easy to see that this collection of jobs can be done in time. For the other direction, if certain jobs $j_1, j_2, ..., j_k$ can be done in time (the indexing is in increasing order on deadlines) then $j_1 \in X_1, j_2 \in X_2, \ldots, j_k \in X_k$, and this is a partial transversal.

Third solution. Yet another way to show M is matroid is to use Exercise 5.7. To each set

$$A_D = \{1 \leq j \leq n \mid d_j \leq D\}, \quad D = 1, 2, \ldots,$$

assign an integer

$$k(A_D) = D.$$

Notice that the family $\mathcal{F} = \{A_1, A_2, \ldots\}$ is laminar (in fact, nested), and the matroid in question is the corresponding laminar matroid (at most D jobs can be done by the time D).

P4 Show the derivation of Theorem 6.3 from Theorem 6.1, from the notes on matroid intersection.

Given a graph $G = (V, E)$ and an edge coloring described by a partition $E = E_1 \cup \cdots \cup E_k$, we can associate two matroids with underlying set E: the graphical matroid attached to G, which we will call M_1, and the partition matroid associated to the partition (allowing at most one element per part in an independent set), which we will call M_2. A colorful spanning tree is then exactly a set $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|S| = |V| - 1$. So there exists a colorful spanning tree if and only if we have

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| \geq |V| - 1.$$

By the Matroid Intersection Theorem (Theorem 6.1), this happens if and only if we have

$$\min_{U \subseteq E} [r_1(U) + r_2(E \setminus U)] \geq |V| - 1.$$

From the discussion immediately after Theorem 6.1, we see that we just need to check that

$$r_1(U) + r_2(E \setminus U) \geq |V| - 1$$

for every set U such that $E \setminus U$ is closed for M_2.

A set is closed for M_2 if and only if it can be written as a union of color classes, so we can suppose that $E \setminus U = E_{i_1} \cup \cdots \cup E_{i_c}$ for some number c. Then $r_2(E \setminus U) = c$, and U is the set of edges that remain after we delete the edges of the c colors $i_1, ..., i_c$. Then

$$r_1(U) + r_2(E \setminus U) = (|V| - \kappa(V, U)) + c,$$

so there exists a colorful spanning tree if and only if we have

$$|V| - \kappa(V, U) + c \geq |V| - 1$$

for every subset $U \subseteq E$ obtained by deleting c color classes (for all c). Rearranging, we see this inequality is equivalent to

$$\kappa(V, U) \leq c + 1$$

for each such U.

5-12 (**Extra Credit**) Let $M = (E, \mathcal{I})$ be a matroid and let P be the corresponding matroid polytope, i.e. the convex hull of characteristic vectors of independent sets. Show that two independent sets I_1 and I_2 are adjacent on P if and only if either (i) $I_1 \subseteq I_2$ and $|I_1| + 1 = |I_2|$, or (ii) $I_2 \subseteq I_1$ and $|I_2| + 1 = |I_1|$, or (iii) $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$ and $I_1 \cup I_2 \notin \mathcal{I}$.

First, let us prove that the conditions are sufficient.

Consider two independent set I_1 and I_2 such that (i) holds. Let f be the only element in $I_2 \setminus I_1$, and consider the weight function $c : E \to \mathbb{R}$ given by:

$$c(e) = \begin{cases} 1, & \text{if } e \in I_1, \\ 0, & \text{if } e = f, \\ -1, & \text{if } e \notin I_2. \end{cases}$$

For this cost, the only maximum weight independent sets are exactly I_1 and I_2. Therefore I_1 and I_2 are adjacent. The case where (ii) holds is analogous.

Now, assume that I_1 and I_2 satisfy (iii). For this case let f be the only element in $I_2 \setminus I_1$ and g be the only element in $I_1 \setminus I_2$. Consider the weight function $c : E \to \mathbb{R}$ given by:

$$c(e) = \begin{cases} 2, & \text{if } e \in I_1 \cap I_2, \\ 1, & \text{if } e = f, \text{ or } e = g \\ -1, & \text{if } e \notin I_1 \cup I_2. \end{cases}$$

For this cost, the only maximum weight independent sets are exactly I_1 and I_2, and so they are adjacent in the matroid polytope.

Now let us prove that the conditions are necessary.

Assume that I_1 and I_2 are a pair of adjacent independent sets and let $c : E \to \mathbb{R}$ be a cost function that is maximized only by I_1 and I_2. In particular note that $c(e) \geq 0$ for every element in $I_1 \cup I_2$. Assume w.l.o.g. that $|I_1| \leq |I_2|$.

Case 1:$|I_2| > |I_1|$. By the exchange axiom $(I3)$, there exists an element $f \in I_2 \setminus I_1$ such that $I_1 + f$ is an independent set and, by a previous observation, it has weight greater or equal than the weight of I_1. Since I_1 is optimum it follows that so is $I_1 + f$. Since I_2 and I_1 are the only optima, it follows that $I_2 = I_1 + f$. Therefore, (i) holds.

Case 2: $|I_2| = |I_1|$. Let f be the element in $I_1 \Delta I_2 = I_1 \setminus I_2 \cup I_2 \setminus I_1$ with minimum cost. Assume w.l.o.g. that $f \in I_1$. Clearly, $I_1 - f$ is an independent set and $|I_1 - f| < |I_2|$. It follows that there exists an element $g \in I_2 \setminus I_1$ such that $I_1 - f + g$ is an independent set. By choice of f, $c(I_1 - f + g) = c(I_1) - c(f) + c(g) \geq c(I_1)$. But then $I_1 - f + g$ is also a maximum weight independent set. Since I_2 and I_1 were the only optima, it follows that $I_2 = I_1 - f + g$, which implies that $|I_2 \setminus I_1| = |I_2 \setminus I_2| = 1$.

To conclude that (iii) holds, we only need to show that $I_1 \cup I_2 \notin \mathcal{I}$. But this is easy to see since, in other case, using that $c(e) \geq 0$ for every $e \in I_1 \cup I_2$, we would have that $c(I_1 \cup I_2) \geq c(I_1)$. This implies that $I_1 \cup I_2$ is another optimum (different from I_1 and I_2), which contradicts the adjacency condition of I_1 and I_2.

P6 (**Extra Credit**) Use Theorem 6.8 from the notes on matroid intersection to show that if $G = (V, E)$ is a graph with $|E| \geq 2|V| - 2$, such that for every nontrivial subset $S \subset V$ the number of edges of G with both endpoints in S is at most $2|S| - 2$, then G has two edge-disjoint spanning trees.

Let $V = V_1 \cup \cdots \cup V_p$ be a partition of the vertex set V into p parts, we must show that

$$\delta(V_1, ..., V_p) \geq 2(p - 1).$$

Note that an edge is counted in $\delta(V_1, ..., V_p)$ if and only if it is not completely contained in some part V_i, so we have

$$\delta(V_1, ..., V_p) = |E| - \sum_{i=1}^{p} |E(V_i)|,$$

where $E(V_i)$ is the set of edges completely contained in the part V_i. By our assumptions, we have

$$|E| - \sum_{i=1}^{p} |E(V_i)| \geq 2|V| - 2 - \sum_{i=1}^{p} (2|V_i| - 2) = 2p - 2,$$

so the condition of Theorem 6.8 is satisfied, and we are done.