Lecture 6

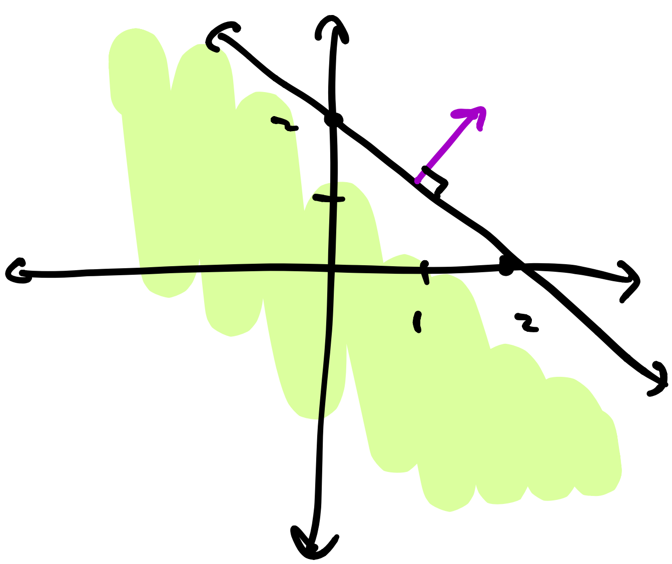
Plan:

1. Definitions
2. Solvability of systems of inequalities.
3. Linear programming duality.

Definitions

Def (Halfspace): set $\{x \in \mathbb{R}^n : a^T x \le b\}$

$a \in \mathbb{R}^n$ $\quad b \in \mathbb{R}$



$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$b = 2$.

Def (Polyhedron): Intersection of finitely many half spaces. write

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

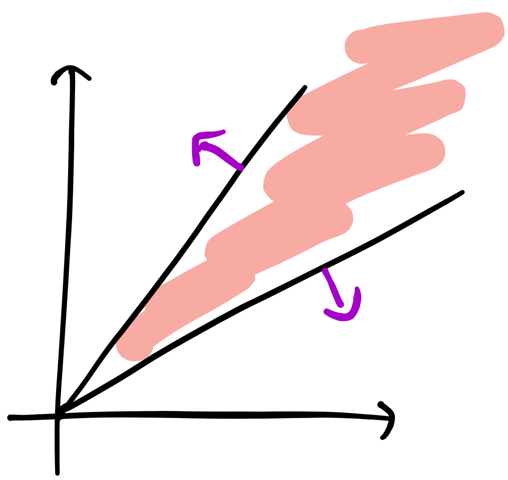
A: $m \times n$ matrix

$\leq$: coordinatewise

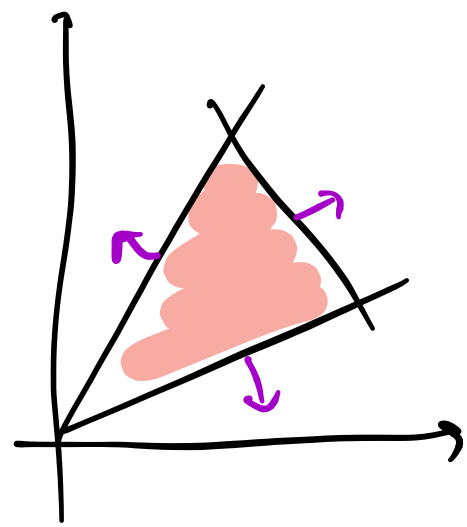
$b \in \mathbb{R}^m$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 1.5 \\ 3.5 \end{pmatrix}$$

Def (Polytope): Bounded polyhedron.



polyhedron

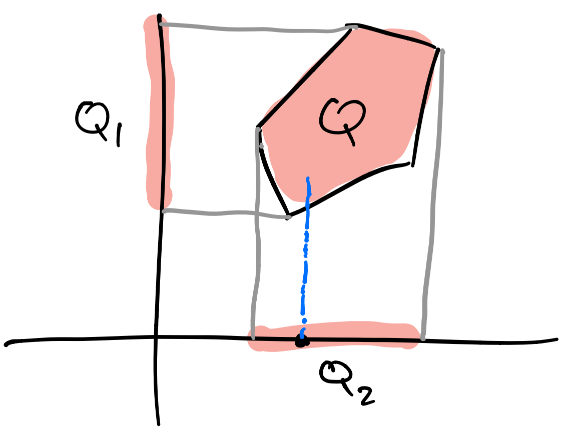


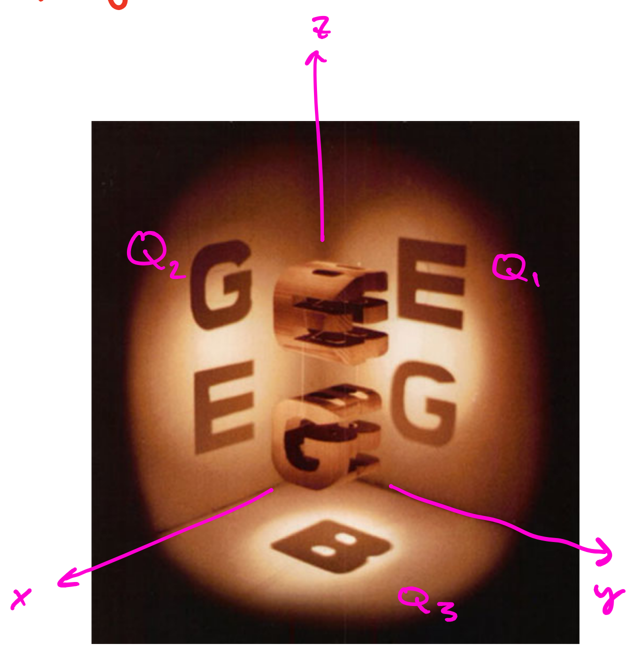
polytope.

Def: If $Q \subseteq \mathbb{R}^n$ is a set, (coordinate) <u>projection</u> $Q_k \subseteq \mathbb{R}^{n-1}$ is

$$Q_k := \{(x_1, \dots x_{k-1}, x_{k+1}, \dots x_n) : x \in Q \text{ for some } x_k \in \mathbb{R}\}.$$

- (is some way to choose x_k to "complete")
- Special case of projection to subspace.





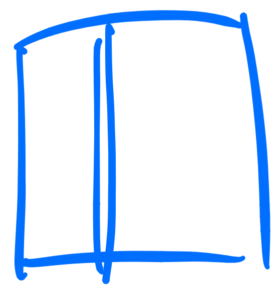
Claim: P polyhedron $\Rightarrow P_k$ polyhedron.

Proof: Give P_k's inequalities.

Method:

Fourier-Motzkin elimination

Let $P = \{x : Ax \leq b\}$



k

- $S_+ = \{i : a_{ik} > 0\}$
- $S_- = \{i : a_{ik} < 0\}$
- $S_0 = \{i : a_{ik} = 0\}$.

$S_0, S_-, S_+ \subseteq [m]$

E.g. $n=3, m=4, k=2$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ -4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S_+ = \{1,2\}$

$S_- = \{3\}$

$S_0 = \{4\}$.

- Any x in P_k satisfies

  (*) $\quad a_i^T x \le b_i$

  for all $i \in S_0$.

  (b/c these ineqs. don't involve x_k).

- Can take linear combination of S_+, S_- ineqs to eliminate coefficient of x_k:

if $i \in S_+$, $l \in S_-$,

$$(**)\quad a_{ik}\Big(\sum_j a_{\ell j} x_j\Big) - a_{\ell k}\Big(\sum_j a_{ij} x_j\Big) \le a_{ik} b_\ell - a_{\ell k} b_i.$$

for all $x \in P_k$.

$\boxed{\textbf{Claim: } (*) \text{ for } i \in S_0,\ (**) \text{ for } i \in S_+,\ \ell \in S_- \text{ describe } P_k.}$

Still Need to show:

for any $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ satisfying $(*)$ and $(**)$ there is x_k s.t. $x \in P$. **Why?**

(i) for $i \in S_+$, $a_i^T x \le b_i$

is upper bound on x_k.

$$a_{ik} x_k + \sum_{j \ne k} a_{ij} x_j \le b_i$$

$$\Longleftrightarrow \quad x_k \le \boxed{\frac{b_i - \sum_{j \ne k} a_{ij} x_j}{a_{ik}}} \quad (U)$$

(ii) for $l \in S_-$, is lower bd.

$$a_{lk} x_k + \sum_{j \ne k} a_{lj} x_j \le b_l$$

$$\Longleftrightarrow \quad x_k \ge \boxed{\frac{b_l - \sum_{j \ne k} a_{lj} x_j}{a_{lk}}} \quad (L)$$

(iii) (**) says

every such upper bound (U) on x_k

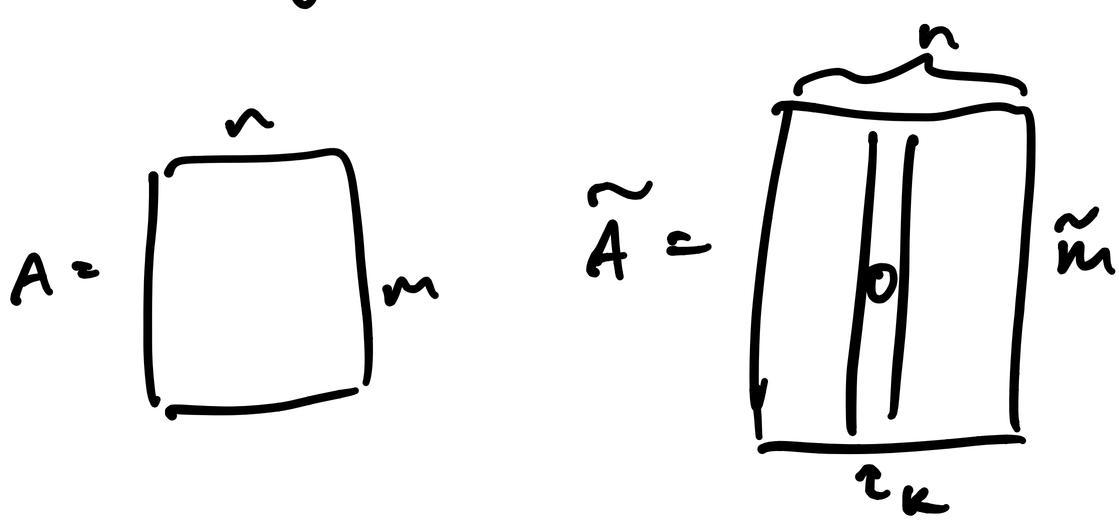
is bigger than lower bound (L) on x_k.

$\Rightarrow$ is some choice of x_k satisfying all. $\square$

Summary of FM elim:

$$\boxed{\begin{array}{l}\text{system } Ax \leq b \\ \rightsquigarrow \text{new system } \tilde{A}x \leq \tilde{b}\end{array}}$$

① New inequalities don't involve x_k



② Inequalities of $\tilde{A}x \leq \tilde{b}$

are nonnegative linear combinations of those in $Ax \leq b$.

③ $Ax \leq b \Rightarrow \tilde{A}x \leq \tilde{b}$.

④ $\tilde{A}x \leq \tilde{b} \Rightarrow \exists y$ s.t.

$A(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots x_n) \leq b$

(3),(4) $\Rightarrow$ $\boxed{Ax \leq b \text{ solvable} \iff \tilde{A}x \leq \tilde{b} \text{ solvable.}}$

MORE DEFS

Def: for $a^{(1)}, \ldots, a^{(k)} \in \mathbb{R}^n$,

<u>linear</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \qquad \lambda_i \in \mathbb{R}$$

<u>affine</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \qquad \sum_{i=1}^{k} \lambda_i = 1$$

<u>conical</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \qquad \lambda_i \geq 0$$

<u>convex</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \quad \lambda_i \geq 0 \,, \; \Sigma \lambda_i = 1$$

(affine + conical)

<u>linear hull</u>: lin(S) = all linear combinations of elements of S.

↳ span(S).

<u>affine hull</u>: aff(S) = " affine

<u>conical</u> " cone(S) = " conical

<u>convex</u> " conv(S) = " convex.

Def (Equiv. def. of polytope):

A polytope is the convex hull of finitely many points.

Why equivalent? $S = \{a^{(1)}, \dots, a^{(k)}\}$

$P = \operatorname{conv}(S) \Rightarrow$ P bounded polyhedron:

because P is projection of polyhedron in $\mathbb{R}^{n+k}$:

$$\tilde{P} = \left\{ (x, \lambda) : \begin{array}{l} x - \sum_k \lambda_k a^{(k)} = 0 \\ \sum_k \lambda_k = 1 \\ \lambda_k \geq 0. \end{array} \right\}$$

project out last k coords. of $\tilde{P}$ to get P.

<u>we already saw</u>: proj. of polyhedron is polyhedron! (Fourier-Motzkin).

<u>P bounded</u>: convex combos have

$$\left\| \sum \lambda_k a^{(k)} \right\| \leq \sum \lambda_k \left\| a^{(k)} \right\|$$

$$\leq \max_k \left\| a^{(k)} \right\|.$$

($\Leftarrow$)? <u>Later in notes</u>.

Solvability of Systems of Inequalities

Linear algebra: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$;

$$Ax = b \quad \text{has no solution}$$

$$\iff A^T y = 0,\ b^T y \neq 0 \text{ for some } y \in \mathbb{R}^n.$$

Why? $\mathrm{Col}(A) = \{\text{possible } b\text{'s}\} = \mathrm{Null}(A^T)^\perp$.

A form of duality: y's obstruct x's.

For inequalities:

Theorem (Theorem of the Alternatives) (TOTA).

$\boxed{\begin{array}{l} Ax \le b \text{ has no solution } \Longleftrightarrow \\ \exists\, y \in \mathbb{R}^m \text{ s.t. } y \ge 0,\ A^T y = 0,\ b^T y < 0, \end{array}}$

Proof: ($\Leftarrow$) simplest: Suppose $Ax \le b$;

then

$$\boxed{0 > b^T y = y^T b \ge y^T A x = 0^T x = 0,}$$

contradiction.

($\Rightarrow$) Fourier-Motzkin Elim.

- Eliminate all variables - get infeasible $\tilde{A}x \leq \tilde{b}$.
- $\tilde{A} = 0$ ($\tilde{m} \times n$ zero matrix)
- $0^{\tilde{m} \times n} x \leq \tilde{b}$ infeasible $\Rightarrow$ some $\tilde{b}_i < 0$.
- But then $0 \cdot x \leq b_i$ is a nonnegative linear combo.

$$\sum y_i (a_i^T x \leq b_i) = 0 \cdot x \leq b_i.$$

i.e. $\boxed{A^T y = 0, y \geq 0, b \cdot y < 0}$ $\square$

Variant (mixed $=/\le$)

$$\underbrace{\begin{matrix} a_1^T x \le b_1 \\ a_2^T x \le b_2 \\ a_3^T x = b_3 \end{matrix}}_{Ax \,\triangle\, b} \text{ has no soln.} \iff \exists\, y \text{ s.t. } A^T y = 0,\ b^T y < 0. \quad \underbrace{\begin{matrix} y_1 \ge 0 \\ y_2 \ge 0 \\ y_3 \end{matrix}}_{y \,\square\, 0.}$$

$\triangle \rightsquigarrow \square$: $= \rightsquigarrow$ unconstrained.

Ex: prove variant.

Another variant.

Farkas Lemma: $Ax = b$, $x \ge 0$ has no solution

$x \geq 0$

$\Leftrightarrow \exists y$ s.t. $A^T y \geq 0$, $b^T y < 0$.

Ex. prove Farkas from TOTA.

LP Duality

Linear "program":
maximizing linear function over polyhedron.

$$\boxed{\begin{array}{ll} & \max: \quad \underbrace{c^T x}_{\text{objective}} \\ (P) & \\ & \text{subject to}: \quad \underbrace{Ax \leq b}_{\text{constraints}}. \end{array}}$$

- x **feasible** if satisfies constraints.
- If no x is feasible, say (P) **infeasible**, has value $-\infty$.
- If value $+\infty$, say (P) **unbounded**. Else (P) **bounded**.
- value finite $\iff$ (P) neither infeas. nor unbounded.
- Many equivalent forms also LP's, e.g. with constraints $\geq, =, \leq$.

  e.g. min-weight perfect matching was
  $$\min\{c^T x : Ax = b, x \geq 0\}.$$

The **Dual** of (P):

$$\text{(D)} \qquad \begin{aligned} \text{min: } & b^T y \\ \text{subject to: } & A^T y = c \\ & y \geq 0. \end{aligned}$$

- (D) said to be **dual**, (P) **primal**.
- Terminology for (D) analogous except **unbounded** if value $= -\infty$, **infeas** if value $= +\infty$.

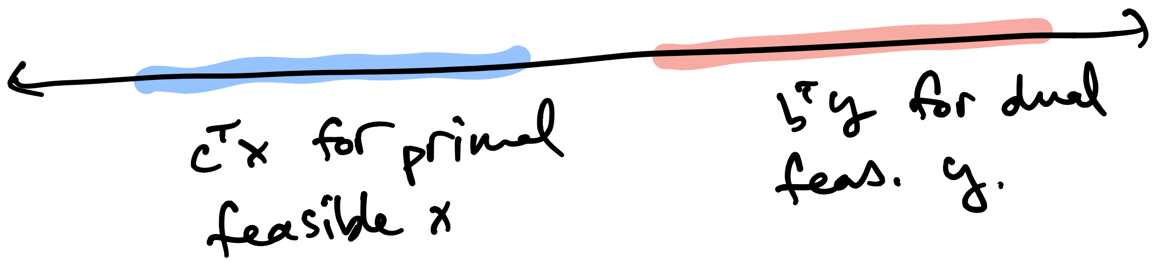
Note: primal / dual vars different: if $A \in \mathbb{R}^{m \times n}$, then

n primal vars, m dual vars.

Weak duality: For feasible solns x, y to (P), (D),

$$c^T x \leq b^T y$$

Picture:



Proof: $c^T x = y^T A x \leq b^T y.$

$\uparrow$ $Ay = c$

$\uparrow$ $Ax \leq b$, $y \geq 0$

The dual was defined this way precisely so this would happen.

Corollary:

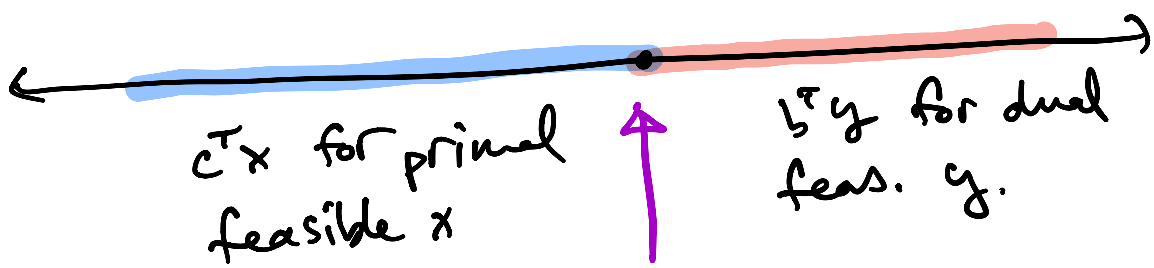
(P) unbounded $\Rightarrow$ (D) infeasible

(D) unbounded $\Rightarrow$ (P) infeasible

Theorem (Strong Duality)

Suppose (P), (D) feasible.

Then optimal values are the same.



Equiv: $\exists$ primal feas x, dual feas y

s.t. $c^T x = b^T y$.

Many proofs:

- e.g. Brauer fixed-point $\Rightarrow$ von-Neumann minimax $\Rightarrow$ strong duality
- Fourier-Motzkin Elim.

TODAY: Proof using TOTA.

IDEA: write down bigger system encoding

(i) x primal feasible

(ii) y dual feasible

(iii) $c^T x \geq b^T y$.

Use TOTA to show feasible.

Proof: Suppose x^* feasible for (D), y^* feasible for (P).

For contradiction, assume following is <u>infeasible</u>:

<u>System</u>:

$$
\begin{aligned}
Ax \quad\quad\quad\quad &\leq b && \text{(i)} \\
A^T y &= c && \\
-Iy &\leq 0 && \text{(ii)} \\
-c^T x + b^T y &\leq 0 && \text{(iii)}
\end{aligned}
$$

Matrix:

$$
\underbrace{\begin{bmatrix} A & 0 \\ 0 & A^T \\ 0 & I \\ -c^T & b^T \end{bmatrix}}_{\tilde{A}}
\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\tilde{x}}
\begin{matrix} \le \\ = \\ \le \\ \le \end{matrix}
\underbrace{\begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}}_{\tilde{b}}
$$

TOTA: $Ay \,\Delta\, b$ infeasible $\iff$

$\exists\ \tilde{y}$ s.t. $\boxed{\tilde{A}^T\tilde{y} = 0,\ \tilde{y} \,\square\, 0,\ \tilde{b}^T y < 0.}$

$$\overbrace{\left[\begin{array}{c|c|c|c} A^T & 0 & 0 & -c \\ \hline 0 & A & I & b \end{array}\right]}^{\tilde{A}^T} \overbrace{\begin{bmatrix} s \\ \hline t \\ \hline u \\ \hline v \end{bmatrix}}^{\tilde{y}} = 0 ,$$

$$\begin{array}{l} \begin{bmatrix} s \\ \hline t \\ \hline u \\ \hline v \end{bmatrix} \begin{array}{l} \geq 0 \\ \\ \geq 0 \\ \geq 0 \end{array} \end{array} \qquad \left[\begin{array}{c|c|c|c} b & c & 0 & 0 \end{array}\right] \begin{bmatrix} s \\ \hline t \\ \hline u \\ \hline v \end{bmatrix} < 0.$$

Writing these out:

$$A^T s - v c = 0$$

$$A t - u - v b = 0$$

$$b^T s + c^T t < 0.$$

$$s \geq 0$$
$$u \geq 0$$
$$v \geq 0.$$

Case 1: $v = 0$.

$\Rightarrow A^T s = 0.$

$\Rightarrow y^* + \alpha s$ dual feasible for all $\alpha \in \mathbb{R}$.

Similarly,

$$At = 0$$

$\Rightarrow x - \alpha t$ primal feas.

By weak duality,

$$c^T(x^* - \alpha t) \le b^T(y^* + \alpha s).$$

$$\Longleftrightarrow$$

$$c^T x^* - b^T y^* \le \underbrace{\alpha\,(b^T s + c^T t)}_{\xrightarrow[\alpha \to +\infty]{} -\infty}.$$

<u>Contradiction.</u>

<u>Case 2</u> $\quad v > 0.$

<u>Recall</u>: $A^T s - vc = 0$

$$At - u - vb = 0$$

$$b^T s + c^T t < 0 \,. \quad *$$

$$s \geq 0$$

$$u \geq 0$$

$$v \geq 0 \,.$$

<u>Divide</u> through by v,

<u>Rename</u> $\frac{s}{v} \leftarrow s$, $\frac{t}{v} \leftarrow t$, $\frac{u}{v} \leftarrow u$

$$A^T s - c = 0$$

$$At - u = -b$$

$$b^T s + c^T t < 0$$

$$s \geq 0$$

$$u \geq 0$$

$\Rightarrow$ s dual feasible,

$-t$ primal feasible.

$\underset{\substack{\uparrow \\ \text{weak duality}}}{\Longrightarrow}$ $c^T(-t) \le b^T s$; contradicts $*$. $\square$

Ex Dual of dual is primal.

Ex. Strong duality holds when either (P) or (D) feas.

i.e. if (P) feas but dual infeas, then (P) unbounded. (both values $+\infty$).

Ex: find example where both infeasible.

What do optimal solutions

$$x^*, y^*$$

look like? look at

$$c^T x = y^T A x \leq b y.$$

Theorem (Complementary Slackness)

Suppose x primal feas., y dual feas.

x optimum in (P)

y optimum in (D)

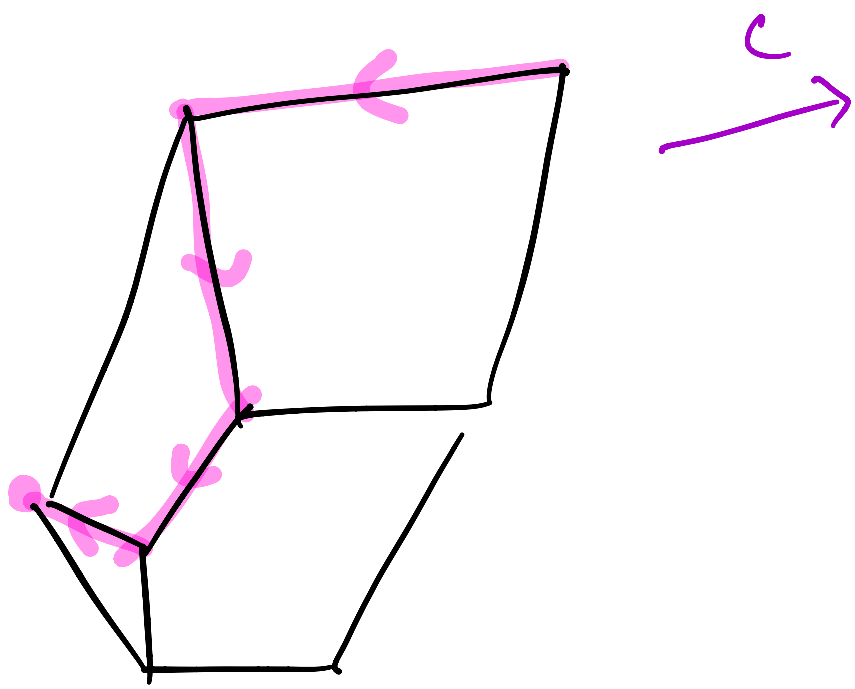
$\Longleftrightarrow$

$\forall i$ either $y_i = 0$

or $(Ax)_i = 0$.

(or both).

Simplex method:



Doesn't provably run in polynomial time.

There are poly-time algorithms:

- interior point
- ellipsoid.