18.453 lecture 4

Lecture plan:

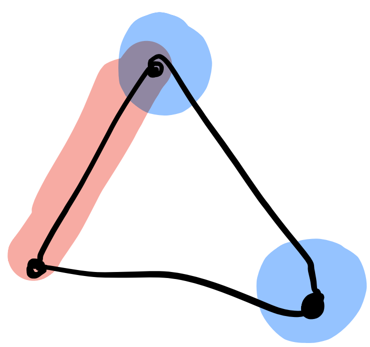
1. non-bipartite matchings.
2. Tutte-Berge
3. Algorithmic proof (Edmonds' alg)

* might not finish!

Non-bipartite Matching

- Given $G=(V,E)$;
  do <u>not</u> assume bipartite.
- want maximum matching M in G.
- König's theorem doesn't hold:

  max matching $\lneq$ min vertex cover.



- Recall from lecture 1: instead, duality w/ obstructions based on parity.

Tutte-Berge Formula

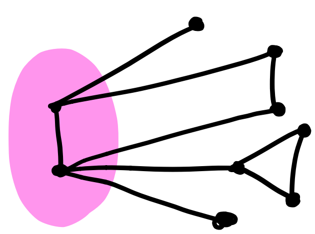
Given $U \subseteq V$,

Def

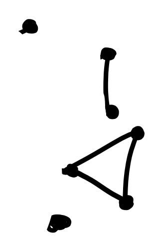
$G \setminus U :=$ G after deleting U & all adjacent edges.

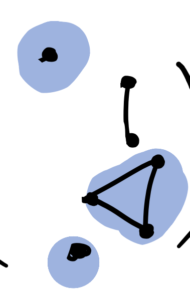
$o(G \setminus U) :=$ # odd connected components in $G \setminus U$ (# c.c.'s w/ odd # of verts).

E.g.

$G =$ 

U

$G \setminus U =$ 

$o(G \setminus U) = o\Big($  $\Big) = 3$

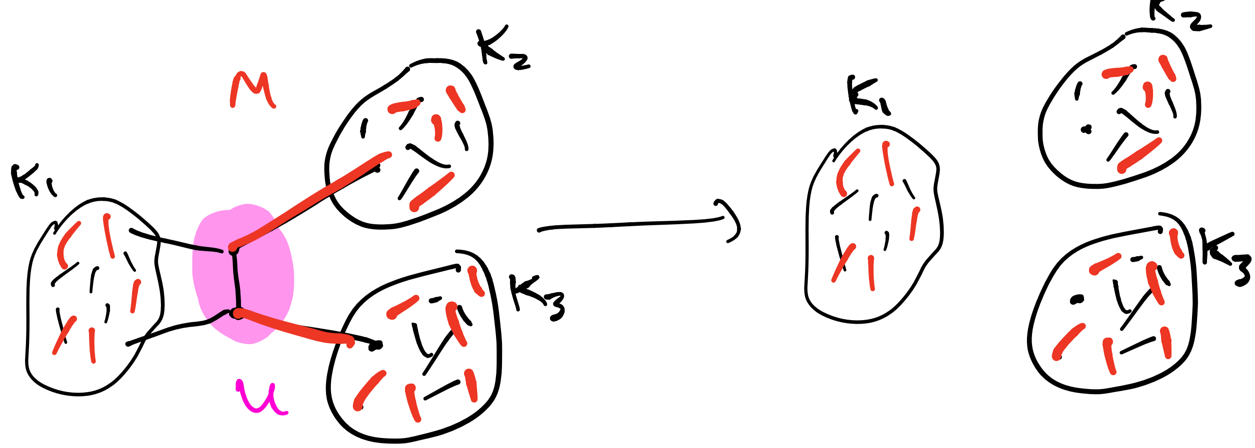
Thm (Tutte-Berge Formula):

$$\max_{\text{matching } M} \underbrace{|M|}_{\#\text{ edges}} = \min_{U \subseteq V} \underbrace{\tfrac{1}{2}}_{\text{not there in lec1}} \Big(|V| + |U| - o(G \setminus U) \Big)$$

Pf ($\leq$) i.e. "weak duality"

- Deleting U deletes $\leq |U|$ edges of M.

How many left over?



- here, # left over is at most

$$\sum_{i=1}^{3} \left\lfloor \frac{|K_i|}{2} \right\rfloor$$

- Thus, if $K_1 \dots K_k$ are connected components of $G \setminus U$,

$$* \quad |M| \leq |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor.$$

- Can rewrite:

$$\left\lfloor \frac{|K_i|}{2} \right\rfloor = \begin{cases} \frac{|K_i|}{2} & \text{if } |K_i| \text{ even} \\ \frac{|K_i|-1}{2} & \text{else.} \end{cases}$$

thus

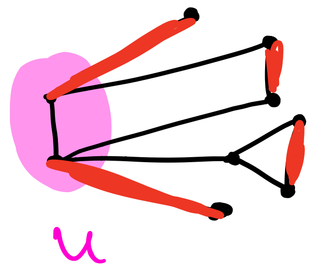
$$** \quad \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \underbrace{\sum_{i=1}^{k} \frac{|K_i|}{2}}_{} - \frac{1}{2} o(G \setminus U).$$

$$= \frac{|V| - |U| - o(G \setminus U)}{2}$$

- Plugging ** into * gives

$$|M| \le \frac{|V|}{2} + \frac{|U|}{2} - \frac{o(G \setminus U)}{2}. \quad \square$$

E.g.

$G =$ 

$|M| = 4$, $\quad \frac{1}{2}(|V| + |U| - o(G\setminus U))$
$= \frac{1}{2}(9 + 2 - 3) = 4.$

Proof of $\ge$?

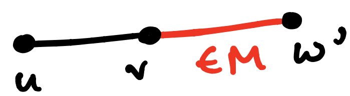
- Beautiful algorithm due to Edmonds.
- <u>challenge</u>: though still true that

  M max $\Longleftrightarrow$ no augmenting path w.r.t M.

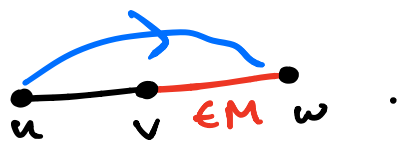
  <u>finding</u> the paths is hard.

- Why? Natural approach repeats vertices. (Fails)

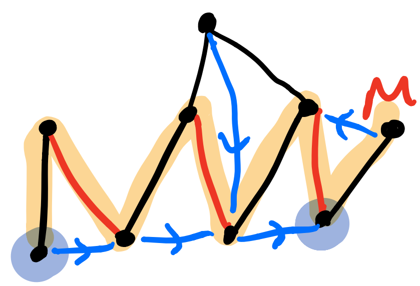
<u>Natural approach</u>: whenever you see



add directed edge uw:



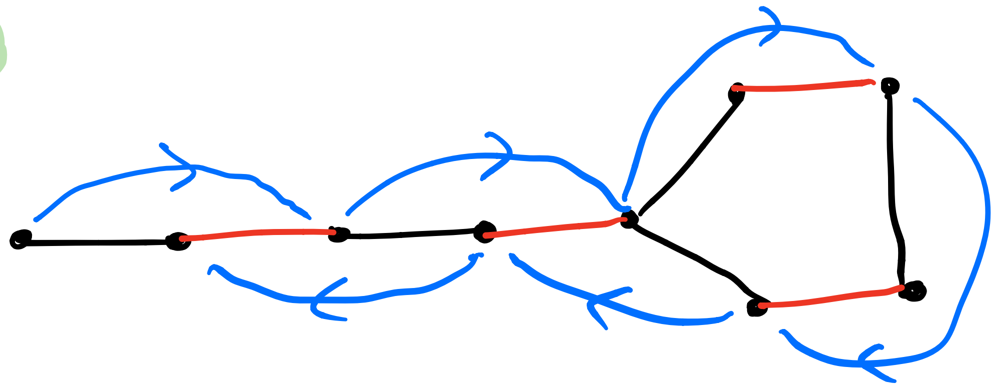
<u>E.g.</u>



Then, <u>start</u> at exposed vertex & look for vertex adjacent to an exposed vertex in blue digraph.

<u>Problem</u>: can lead to repeated vertices.

E.g.



when we first repeat, have found a

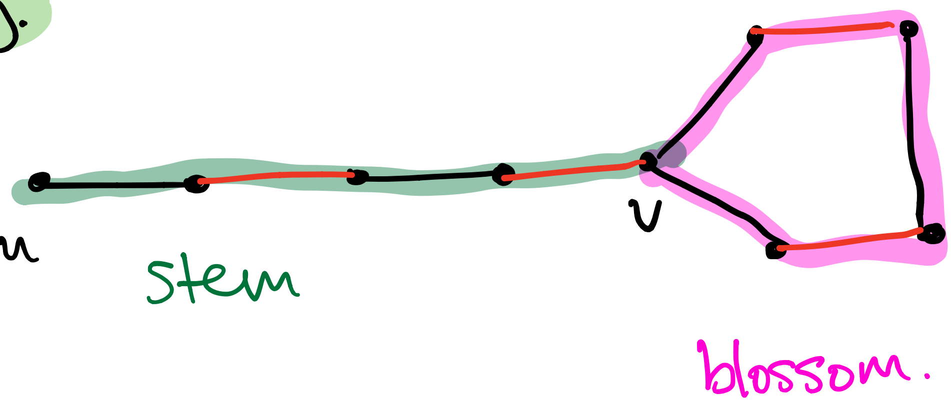
flower (with respect to M):

Stem: even-length alternating path from exposed vertex u to vertex v

+

Blossom: odd-length cycle intersects stem in v alternating *except* for edges incident to v.

E.g.



Algorithm idea:

At each step, have matching M.

- find aug. path or flower w.r.t M or show neither exists.
- If neither exists, Matching is maximum. (b/c no aug. path)
- if aug. path, augment & repeat.

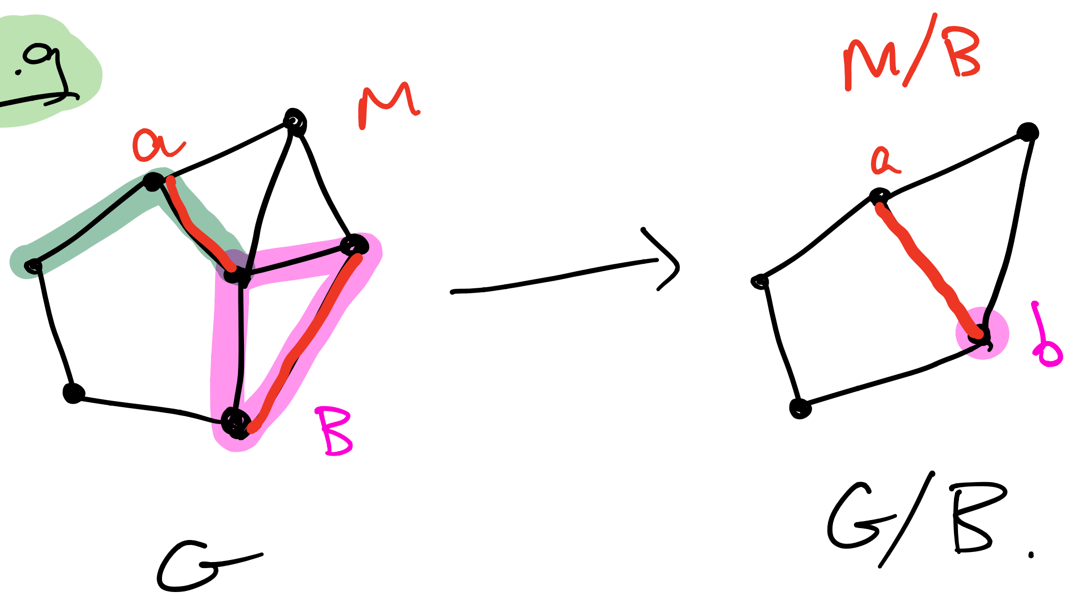
- if <u>flower</u>, let B be blossom.

Create graph G/B (not $G \setminus B$)
called <u>contraction</u> where

(1.) B shrunk to single vert. b

(2.) edges (u,v) $u \notin B$, $v \in B$
replaced by $(u,b) \in G/B$.

<u>E.g</u>



Note: is matching M/B in G/B

and $|M| - |M/B| = \frac{|B|-1}{2}$

(i.e. # edges of M in B).

Crucial Theorem: Let B be a blossom w.r.t. M. Then

$$\boxed{M \text{ max matching in } G} \iff \boxed{M/B \text{ max matching in } G/B.}$$

Proof will be **algorithmic**:

If bigger matching in G/B than

M/B, can use it to find bigger matching in G than M.

Theorem → Algorithm: recursion!

Assuming we can find either aug path or blossom, can recurse to increase size of M/B in G/B.

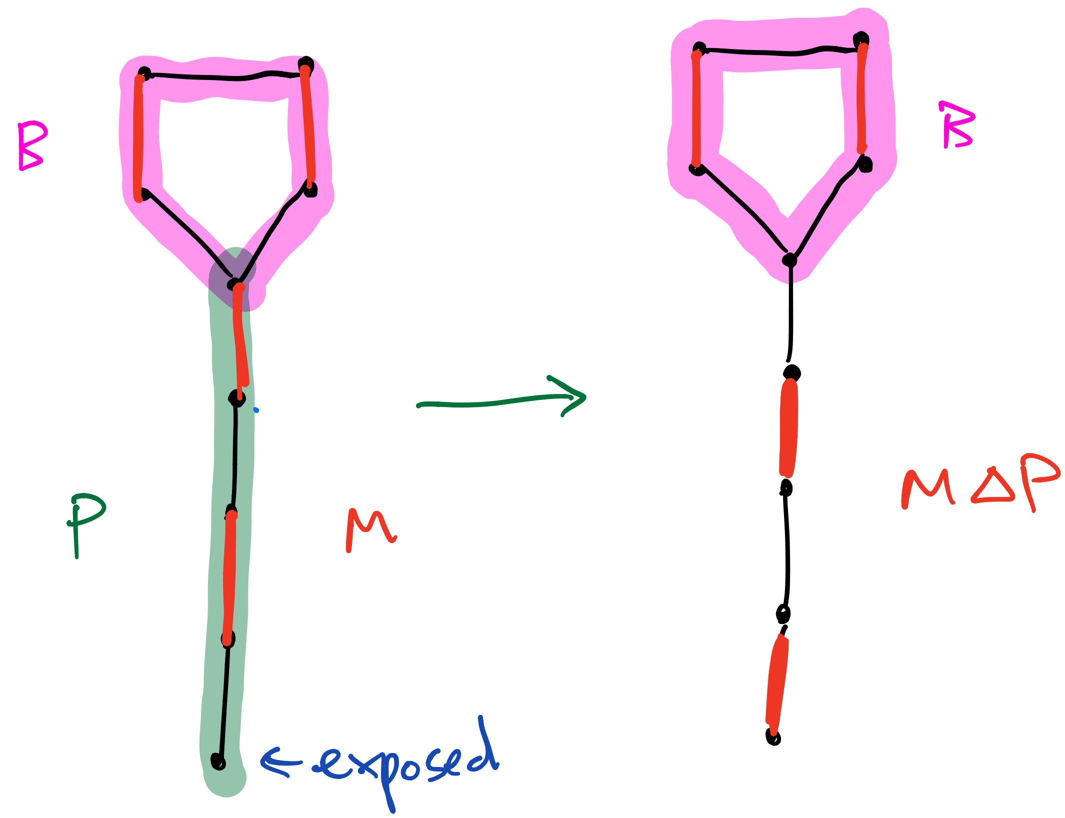
- if not possible: M maximum.
- else: use new matching in G/B to increase M

Proof of Crucial Theorem:

0. W.L.O.G. assumption:

B has empty stem?!

<u>why w.l.o.g?</u> If P nonempty, look at $M \Delta P$.



- $M \Delta P$ has empty stem & blossom B.

- Proving theorem for $M \Delta P$ also proves for $\underline{M}$:

M maximum in G

P alternating $|M| = |M \Delta P|$

$\Longleftrightarrow$ $M \Delta P$ maximum in G

Theorem for empty stem

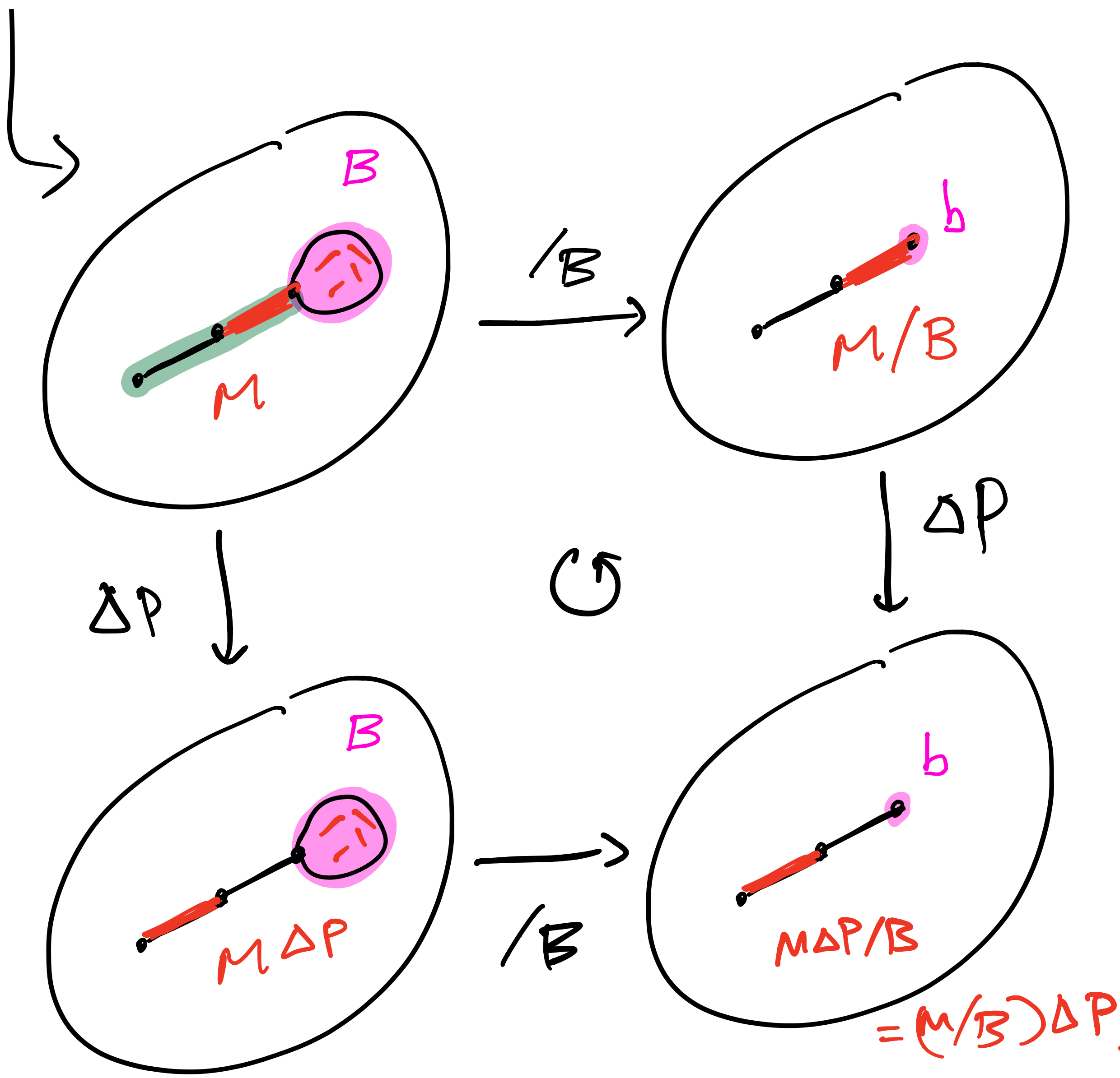
$\Longleftrightarrow$ $M \Delta P / B$ max in G/B

$M \Delta P / B = (M/B) \Delta P$

$\Longleftrightarrow$ $(M/B) \Delta P$ max in G/B

P still alternating.

$\Longleftrightarrow$ (M/B) max in G/B.



Finally, start proof of <u>crucial thm</u>:

$\boxed{M \text{ max. in } G}$

$\iff \boxed{M/B \text{ max in } G/B}$

Contrapos: Suppose M/B not max, show M not max.

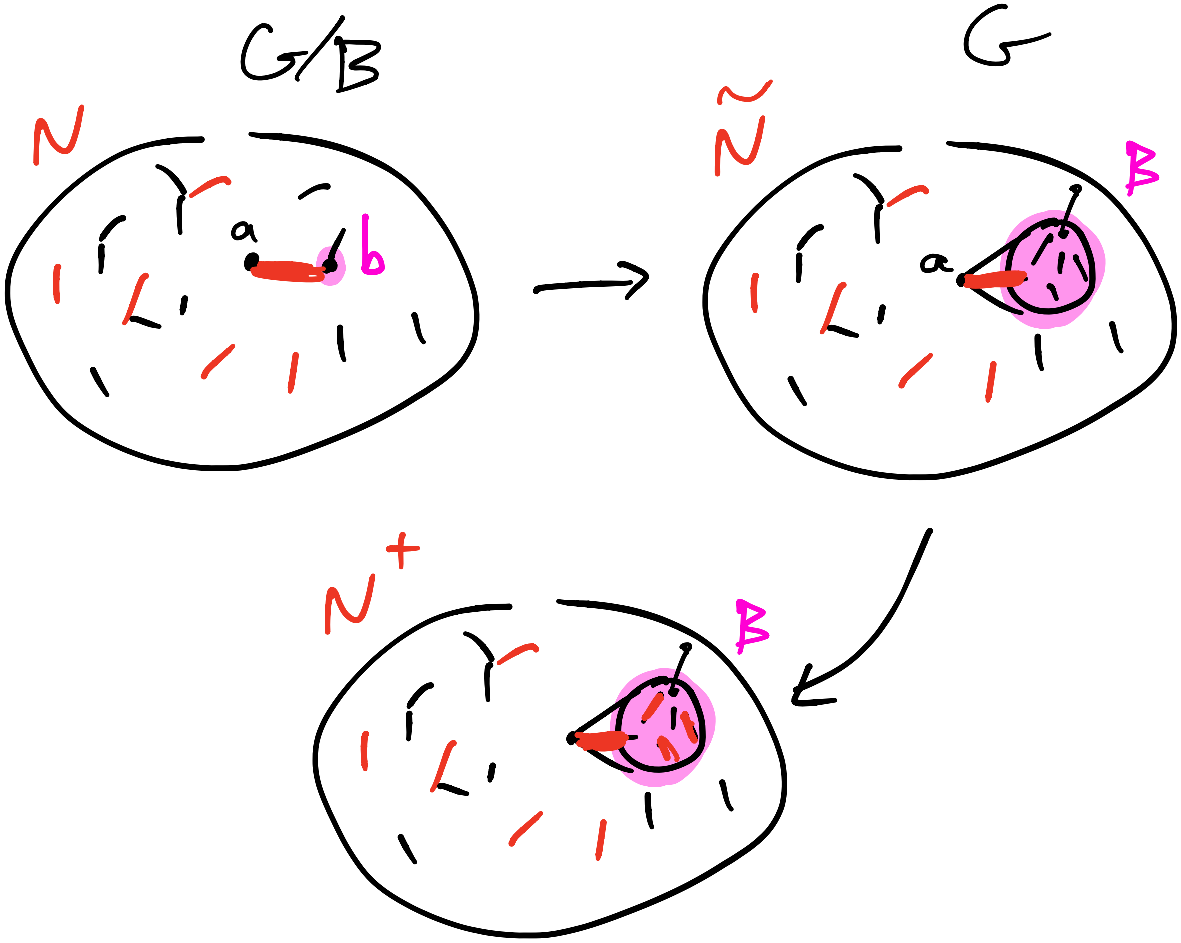
1. ($\Rightarrow$):

Suppose N is matching in G/B larger than M/B.

- pull back N to matching $\tilde{N}$ in G: $\tilde{N}/B = N$
  $\tilde{N}$ incident to ≤ 1 vertex of B.

- Expand to matching $N^{\uparrow}$ in G: add $\frac{1}{2}(|B|-1)$

edges in B.



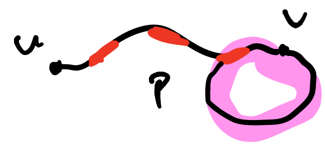
$|N^+|$ exceeds $|M|$ by same amt. $|N|$ exceeds $|M/B|$.

2. ($\Leftarrow$)

Contrapos: if M not max, then M/B is not max.

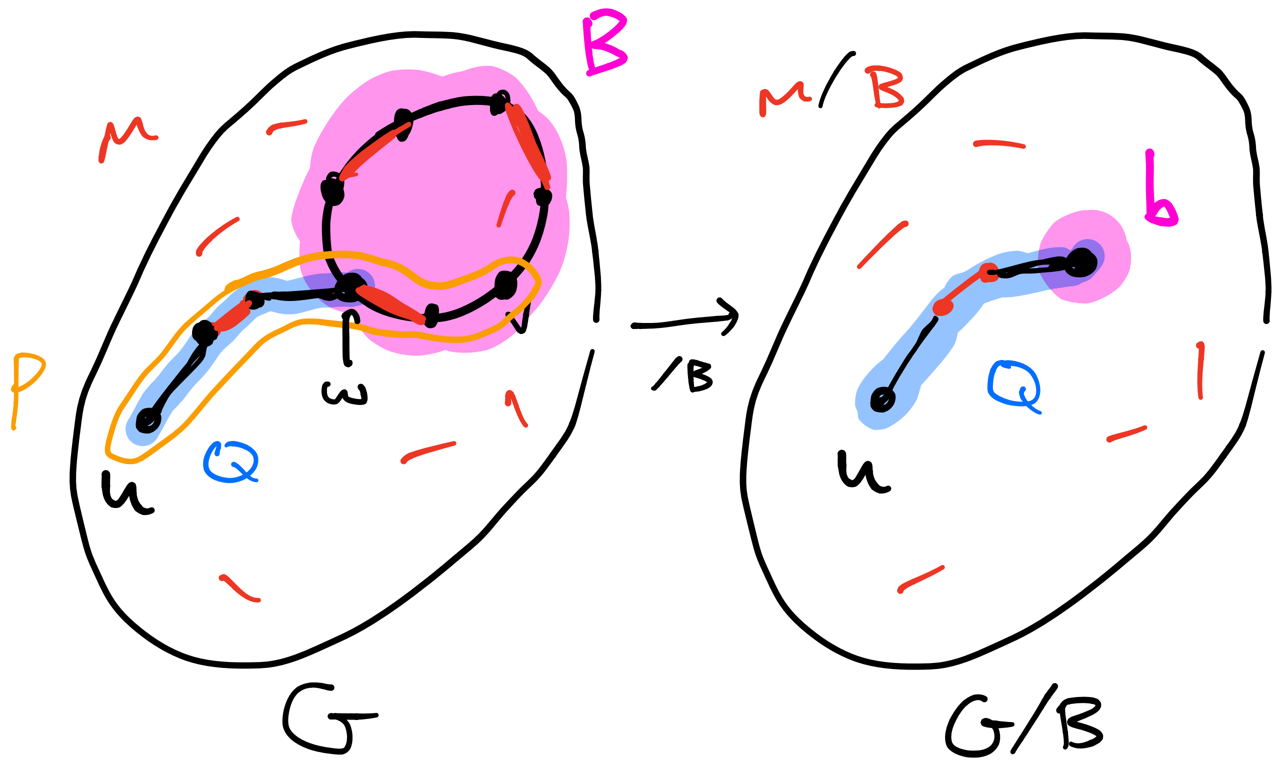
Suppose M not max in G.

- then $\exists$ aug path P between exposed verts $u, v \in G$.



- wlog $u \notin B$, B has only 1 exposed vertex. (stem is empty).
- $$w := \begin{cases} \text{first vertex of } P \text{ in } B \text{ (starting from } u\text{)} \\ v \quad \text{if } P, B \text{ share no vertices.} \end{cases}$$
- $Q :=$ part of P between u, w.
- Q augmenting path in M/B

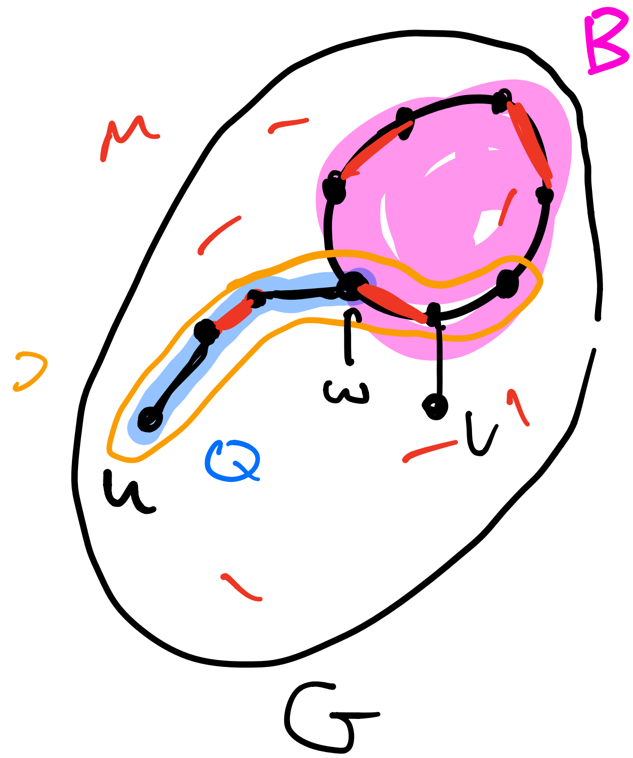
b exposed in M/B bc stem is empty.



Also: if P, B vertex disjoint, then P aug. M/B.

Finally: Augmenting M/B along Q

⇒ M/B not maximum □



$\hat{M}/B = M^*$

$\uparrow$ B is a blossom for M

maybe not for $\hat{M}$.

Subtlety: thm <u>doesn't</u> say

maximum matching M^* in G/B

$\rightsquigarrow$ max matching $\hat{M}$ in G
by adding $\frac{|B|-1}{2}$ edges
from B to M^*!

Ex. find example of above:
i.e. blossom B of M so that M^* max
in G/B but adding the $\frac{|B|-1}{2}$ edges
to M^* in G doesn't result in max matching $\hat{M}$
in G

~~explain why no contradiction.~~

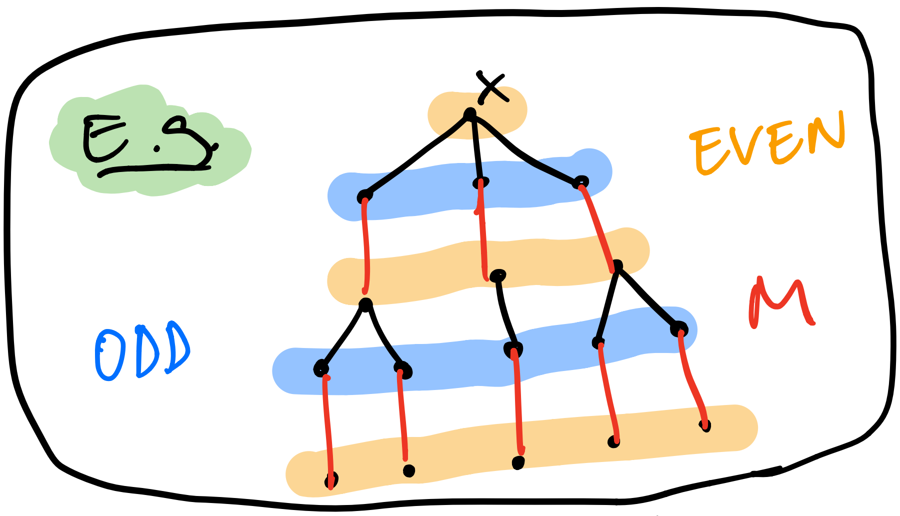
Edmonds' Algorithm

- Label exposed vertices EVEN;

Keep others unlabelled initially.
( )

- Maintain alternating forest:
graph in which each connected component is alternating tree (AT)
i.e. tree s.t. paths to root are

(i)
(ii)



- Process EVEN vertices one at a time.

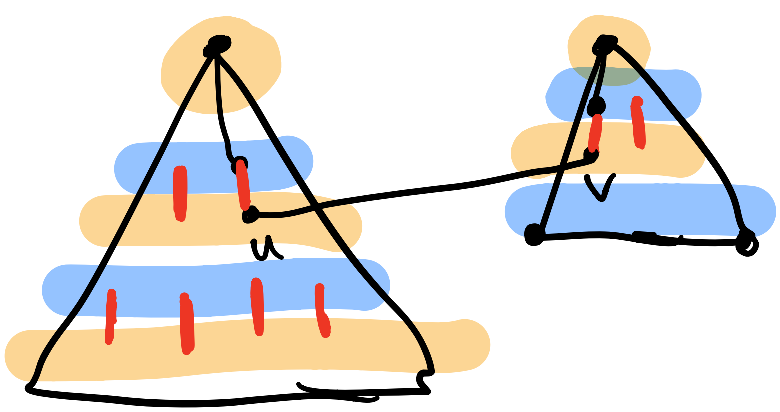
(a) If edge (u,v) with v unlabelled, label v **ODD**. $\boxed{v \text{ not exposed.}}$

(b/c); label v's mate w **EVEN**.

Add

(b) if ∃ edge (u,v) s.t.
v EVEN and v
belongs to different AT
than u,

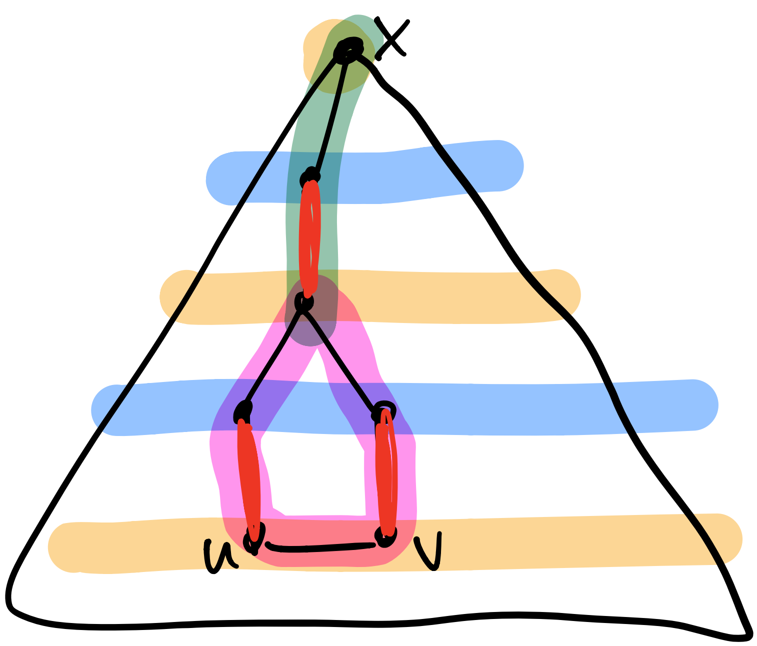
Then



have found aug path; increase
M, start over with new M.

(c) If is edge labeled (u,v) with v labeled EVEN & V in U's AT,

then:



(c, cont.) Shrink to $G \backslash B$,
recursively find max.
matching in $G \backslash B$.

Start over w/ new M.

Correctness: Suppose none of a, b, c apply anymore for the EVEN vertices.

Claim: Current matching M_K

is max in current $G_k = (V_k, E_k)$

Proof of Claim: Consider $U = \text{ODD}$ and consider the upper bound from Tutte-Berge for G_k,

$|M_k| \leq$

- No edges b/w EVEN vertices, ( ).

  (else

  & no edges b/w EVEN & unlabelled, ( ).

  (else

- Thus, EVEN are singleton components in $G_k \setminus U$, so $o(G_k \setminus \underline{ODD})$ $*$

- Further, all <u>unlabelled</u> vertices matched, so

$$** \begin{cases} |M_k| = |ODD| + \frac{1}{2}(\qquad\qquad) \\ \quad = \frac{1}{2}(\qquad\qquad) \end{cases}$$

- Plug $*$ into $**$:

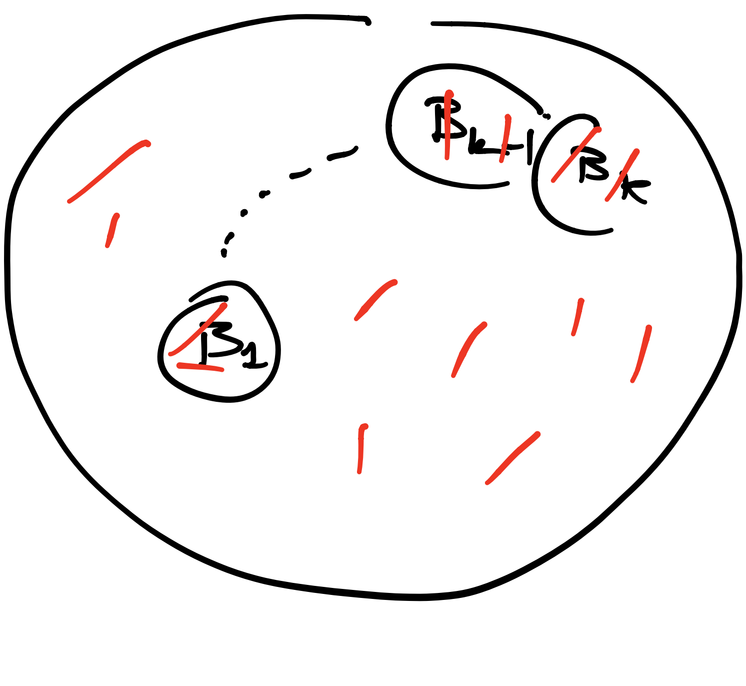
$$|M_k| = \frac{1}{2}(\qquad\qquad)$$

$$= \frac{1}{2}(\qquad\qquad).$$

<u>Tutte Berge (upper bound)</u> $\Rightarrow$

M' max in G'. Apply crucial theorem repeatedly for $B_k\, B_{k-1} \ldots B_1$

Shows:



Running time.

- Algorithm performs $\boxed{\leq \quad}$ augmentations of matching "outer loop"
- between two augmentations, "inner loop" shrinks blossom $\boxed{\leq \quad}$ times (

).

- Time to construct AT is $\boxed{\quad}$, $m := |E|$.

So overall, $\boxed{}$.

Proof of Tutte-Berge ($\geq$)

We showed: TB holds for graph G_K for which alg. terminates.

- Recall

  G_i:

  M_i:

  G_0:

- TB holds for G_K, i.e.

$|M_k| = \frac{1}{2}\Big(\qquad\qquad\qquad \Big)$

where $U =$ ODD,

b/c $G_k \setminus$ ODD $=$ EVEN;

- Unshrink B_i one by one.

induct backwards.

$$\boxed{\text{In step } G_i \longrightarrow G_{i-1}:}$$

(i) $|V_{i-1}| =$

and

$|M_{i-1}| =$

(ii) Unshrinking B_i adds even # (   ) vertices to some C.C. of $G_i \setminus U$, so # odd/even components stays same.

i.e. $\boxed{\quad}$

(iii) Using this, when $i \leftarrow i-1$

the RHS & LHS of

$$|M_i| = \frac{1}{2}\left(|V_i| + |U| - o(G_i \setminus U)\right)$$

increase by .

By induction,

$$\boxed{|M_0| = \frac{1}{2}\left(\qquad \right).}$$

$\square$

Corollary of Tutte-Berge:

G has p.m. iff

$\forall U, \ o(G \setminus U) \leq |U|$.

this is called

Tutte's matching thm.