Homework 8 (50 points) Due: Friday November 15, 2024 11:59 pm
COMPSCI 733: Advanced Algorithms and Designs

Documentation: (5 points) Type your solutions using Latex

(www.overleaf.com or https://www.latex-project.org/). You may insert the images of hand-drawn figures. Submit your solutions (pdf is enough) to Canvas.

Problem 1 (15 points): Read the definition of a flow in the CLRS textbook. Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted αf , is a function from $V \times V$ to R defined by

$$(\alpha f)(u,v) = \alpha f(u,v)$$

Prove that the flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for all α in the range $0 \le \alpha \le 1$.

Proof.

$\Rightarrow$ First, we check the Capacity Constraint of: For all $u, v \in V$, $f(u,v) \le c(u,v)$.

$\Rightarrow$ Since f_1 and f_2 are flows, they satisfy the capacity constraint individually:

$$0 \le f_1(u,v) \le c(u,v) \quad \text{and} \quad 0 \le f_2(u,v) \le c(u,v) \quad \text{for all } u, v \in V.$$

$\Rightarrow$ Now, consider $f(u,v) = \alpha f_1(u,v) + (1\alpha) f_2(u,v)$.

$\Rightarrow$ Since $0 \le \alpha \le 1$, we have:

$$0 \le \alpha f_1(u,v) \le \alpha c(u,v) \quad \text{and} \quad 0 \le (1\alpha) f_2(u,v) \le (1\alpha) c(u,v).$$

$\Rightarrow$ Adding these inequalities gives:

$$0 \le f(u,v) = \alpha f_1(u,v) + (1\alpha) f_2(u,v) \le \alpha c(u,v) + (1\alpha) c(u,v) = c(u,v).$$

$\Rightarrow$ Thus, f satisfies the capacity constraint.

$\Rightarrow$ Next, we verify for f Flow Conservation where for all $u \in V$ except $\{s,t\}$,

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u).$$

$\Rightarrow$ Since f_1 and f_2 are flows, they satisfy flow conservation individually:

$$\sum_{v \in V} f_1(u,v) = \sum_{v \in V} f_1(v,u) \quad \text{and} \quad \sum_{v \in V} f_2(u,v) = \sum_{v \in V} f_2(v,u).$$

$\Rightarrow$ For $f = \alpha f_1 + (1\alpha) f_2$, we compute:

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} \left(\alpha f_1(u,v) + (1\alpha) f_2(u,v) \right).$$

⇒ By the linearity of summation, this becomes:

$$\sum_{v \in V} f(u,v) = \alpha \sum_{v \in V} f_1(u,v) + (1\alpha) \sum_{v \in V} f_2(u,v).$$

⇒ Similarly, $\sum_{v \in V} f(v,u) = \alpha \sum_{v \in V} f_1(v,u) + (1\alpha) \sum_{v \in V} f_2(v,u)$.

⇒ Since f_1 and f_2 satisfy flow conservation, we have:

$$\sum_{v \in V} f_1(u,v) = \sum_{v \in V} f_1(v,u) \quad \text{and} \quad \sum_{v \in V} f_2(u,v) = \sum_{v \in V} f_2(v,u).$$

⇒ Thus,

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u),$$

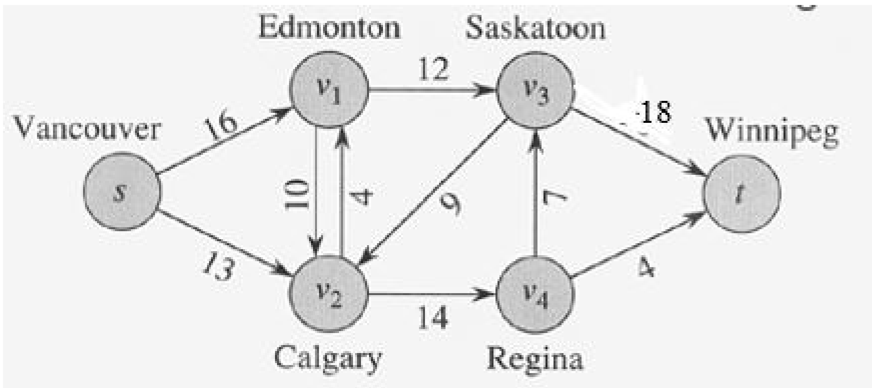
shows that f satisfies flow conservation at every vertex $u \in V except \{s,t\}$.

⇒ Since $f = \alpha f_1 + (1\alpha) f_2$ satisfies both the capacity constraint and flow conservation, f is a valid flow in G.

∴ The set of flows in G is convex.

□

Problem 2: (15 points) Show the execution of the Ford-Fulkerson method on the following flow network with the source, "s", and the sink, "t".Each edge is labeled with its capacity. You can follow the example given on the slides and may include pictures of hand-drawn figures.



- We start by initializing all flow values $f(u, v) = 0$ for each edge (u, v) in the graph. Then by find the Augmenting Paths.

⇒ First Augmenting Path: $s \to v_1 \to v_3 \to t$

- Residual capacities: $s \to v_1$: 16, $v_1 \to v_3$: 12, $v_3 \to t$: 18
- Minimum residual capacity along this path: $\min(16, 12, 18) = 12$
- For the Flow update, we send a flow of 12 units along this path.
- Update the flows and residual capacities:
- $s \to v_1$: $f = 12$, residual capacity becomes $16 - 12 = 4$
- $v_1 \to v_3$: $f = 12$, residual capacity becomes $12 - 12 = 0$
- $v_3 \to t$: $f = 12$, residual capacity becomes $18 - 12 = 6$

⇒ Second Augmenting Path: $s \to v_2 \to v_4 \to t$

- Residual capacities: $s \to v_2$: 13, $v_2 \to v_4$: 14, $v_4 \to t$: 4
- Minimum residual capacity along this path: $\min(13, 14, 4) = 4$
- Flow update : We send a flow of 4 units along this path.
- Update the flows and residual capacities:
- $s \to v_2$: $f = 4$, residual capacity becomes $13 - 4 = 9$
- $v_2 \to v_4$: $f = 4$, residual capacity becomes $14 - 4 = 10$

- $v_4 \to t$: $f = 4$, residual capacity becomes $4 - 4 = 0$

⇒ Third Augmenting Path: $s \to v_1 \to v_2 \to v_4 \to t$

- Residual capacities: $s \to v_1$: 4, $v_1 \to v_2$: 10, $v_2 \to v_4$: 10
- $v_4 \to t$: 0 (no capacity available)
- We cannot use this path because $v_4 \to t$ has no residual capacity left.

⇒ Fourth Augmenting Path: $s \to v_2 \to v_1 \to v_3 \to t$

- Residual capacities: $s \to v_2$: 9, $v_2 \to v_1$: 4, $v_1 \to v_3$: 12, $v_3 \to t$: 6
- Minimum residual capacity along this path: $\min(9, 4, 12, 6) = 4$

⇒ We send a flow of 4 units along this path for the Flow Update.

- Update the flows and residual capacities:
- $s \to v_2$: $f = 8$, residual capacity becomes $9 - 4 = 5$
- $v_2 \to v_1$: $f = 4$, residual capacity becomes $4 - 4 = 0$
- $v_1 \to v_3$: $f = 12$, residual capacity becomes $12 - 4 = 8$
- $v_3 \to t$: $f = 16$, residual capacity becomes $6 - 4 = 2$
- Now we look for another augmenting path, but no further augmenting paths exist with positive residual capacity. Thus, from here, we stop the algorithm.

⇒ Here we look at the Total flow from s to t is the sum of the flows along all augmenting paths:

- First path $s \to v_1 \to v_3 \to t$: Flow = 12
- Second path $s \to v_2 \to v_4 \to t$: Flow = 4
- Third path $s \to v_1 \to v_2 \to v_4 \to t$: not valid because $v_4 \to t$ has no residual capacity left after the second path.
- Fourth path $s \to v_2 \to v_1 \to v_3 \to t$: Flow = 4
- Thus, the total maximum flow is $12 + 4 + 4 = 20$.

⇒ The Residual Network capacities for the remaining edges are as follows:

- $s \to v_1$: 4
- $s \to v_2$: 5
- $v_1 \to v_2$: 10
- $v_2 \to v_1$: 4

- $v_2 \to v_4$: 10
- $v_1 \to v_3$: 8
- $v_3 \to v_2$: 9
- $v_4 \to v_3$: 7
- $v_4 \to t$: 0
- $v_3 \to t$: 2

$\therefore$ The maximum flow from source s to sink t in the network is 20 units. This is the result of sending flow along the augmenting paths we identified.

Problem 3: (15 points)

Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a minimum $s - t$ cut with respect to these capacities $\{c_e : e \in E\}$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum $s-t$ cut with respect to these new capacities $\{1 + c_e : e \in E\}$.

Proof.

Let $G = (V, E)$ be a flow network with source s, sink t, and positive integer capacities c_e for each edge $e \in E$. Suppose (A, B) is a minimum s-t cut in G with respect to these capacities $\{c_e : e \in E\}$. We will show that adding 1 to each capacity does not necessarily preserve (A, B) as a minimum s-t cut.

Counterexample:

- Consider a flow network G with vertices $V = \{s, u, t\}$ and edges:
  - $e_1 = (s, u)$ with capacity $c_{e_1} = 1$,
  - $e_2 = (u, t)$ with capacity $c_{e_2} = 1$,
  - $e_3 = (s, t)$ with capacity $c_{e_3} = 2$.
- The structure of the network is as follows:

$$s \xrightarrow{1} u \xrightarrow{1} t, \quad s \xrightarrow{2} t$$

- We evaluate the capacity of each possible s-t cut:
  - **Cut** $(\{s\}, \{u, t\})$**:** This cut separates s from u and t.

$$\text{Capacity} = c_{e_1} = 1$$

  - **Cut** $(\{s, u\}, \{t\})$**:** This cut separates s and u from t.

$$\text{Capacity} = c_{e_2} = 1$$

  - **Cut** $(\{s\}, \{t\})$**:** This cut separates s directly from t.

$$\text{Capacity} = c_{e_3} = 2$$

  The minimum s-t cut in this network is $(\{s\}, \{u, t\})$ or $(\{s, u\}, \{t\})$, both with a capacity of 1.

- Now suppose we add 1 to each edge capacity. The new capacities are:
  - For $e_1 = (s, u)$, the new capacity is $c'_{e_1} = 1 + 1 = 2$,

- For $e_2 = (u, t)$, the new capacity is $c'_{e_2} = 1 + 1 = 2$,
- For $e_3 = (s, t)$, the new capacity is $c'_{e_3} = 2 + 1 = 3$.

- With the updated capacities, we reevaluate the cuts:
  - Cut $(\{s\}, \{u, t\})$: Capacity $= c'_{e_1} = 2$
  - Cut $(\{s, u\}, \{t\})$: Capacity $= c'_{e_2} = 2$
  - Cut $(\{s\}, \{t\})$: Capacity $= c'_{e_3} = 3$

- The new minimum s-t cut is now $(\{s\}, \{u, t\})$ or $(\{s, u\}, \{t\})$, both with a capacity of 2. The minimum cut has changed in value, even though the original cut sets remain the same.

- By adding 1 to each edge's capacity, the minimum cut value changed from 1 to 2.

$\therefore$ The cut (A, B) from the original network is not necessarily a minimum s-t cut with the modified capacities. This proves that the statement is false.

$\square$