| **Student:** Cole Lamers **Date:** 09/02/19 | **Instructor:** Viktoriya Shcherban **Course:** Calc 1 11:30 AM / Internet (81749&81750) Shcherban | **Assignment:** 2.3 The Precise Definition of a Limit |

For the given function f(x) and values of L, c, and $\varepsilon > 0$ find the largest open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then determine the largest value for $\delta > 0$ such that $0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon$.

$f(x) = x^2$, $L = 25$, $c = -5$, $\varepsilon = 0.3$

Solve $|f(x) - L| < \varepsilon$ to find the largest interval containing c on which the inequality holds.

$|x^2 - L| < 0.3 \rightarrow -0.3 < x^2 - 25 < 0.3$

Simplify by adding 25 to all expressions.

$-0.3 < x^2 - 25 < 0.3 \rightarrow 24.7 < x^2 < 25.3$

Since all three expressions are greater than 0, simplify by taking the square root of all expressions.

$4.9699 < x < 5.0299$

However, c is -5, not 5. Multiply $(4.9699 < x < 5.0299)$ by -1, producing $-5.0299 < -x < -4.9699$. Note: since $x > 0$, $-x < 0$.

Since interval $(-5.0299, -4.9699)$ is not symmetric with respect to -5, δ is the distance from -5 to the closer endpoint of the interval.

Thus, $\delta = 0.0299$.

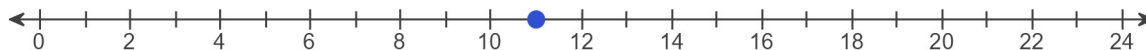|	
Student: Cole Lamers **Date:** 08/31/19	**Instructor:** Viktoriya Shcherban **Course:** Calc 1 11:30 AM / Internet (81749&81750) Shcherban

Assignment: 2.3 The Precise Definition of a Limit

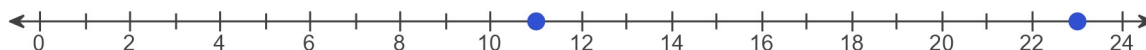Suppose that the interval (a,b) is on the x-axis with the point c inside the interval. For the given values of a, b, and c, find the value of $\delta > 0$ such that for all x, $0 < |x - c| < \delta \rightarrow a < x < b$.
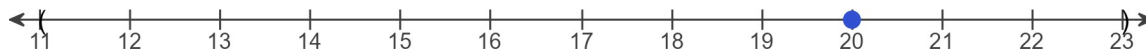
$a = 11, \quad b = 23, \quad c = 20$

Plot a on the number line.

Now plot b on the number line.

Plot c. Notice the number line is the interval (a,b).

The distance from c to b is shorter than the distance from c to a since c is closer to b on the number line.

The expression |x − c| gives the distance of any point, x, on the x-axis from the point c. To make sure x is in the interval, the distance from c to x must be less than the distance from c to either endpoint. Therefore, the distance must be less than 23 − 20 = 3.

If the distance of any point, x, on the x-axis is less than 3 from the given point c, then that point will be within the interval (a,b). Symbolically, if |x − c| < 3 → a < x < b.
Thus, $\delta = 3$.

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For the given function f(x) and values of L, c, and $\varepsilon > 0$ find the largest open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then determine the largest value for $\delta > 0$ such that $0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon$.

f(x) = 2x + 3,   L = 9,   c = 3, $\varepsilon = 0.04$

Solve $|f(x) - L| < \varepsilon$ to find the largest interval containing c on which the inequality holds.

$|(2x + 3) - 9| < 0.04 \rightarrow -0.04 < [(2x + 3) - 9] < 0.04$

Combine the constants.

$-0.04 < 2x - 6 < 0.04$

Simplify.

$5.96 < 2x < 6.04$

Complete the solution of the inequality by dividing the three expressions by 2.

$2.98 < x < 3.02$

Since the interval $2.98 < x < 3.02$ is centered on c = 3, δ is the distance from 3 to either endpoint of the interval.

Thus, the value of δ is 0.02.

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For the given function f(x) and numbers L, c, and $\varepsilon > 0$, find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$f(x) = x^2 - 7$, $L = 2$, $c = 3$, $\varepsilon = 1$

Begin by solving the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a,b) containing c on which the inequality holds for all $x \neq c$.

Remove the absolute value sign and rewrite the inequality as a compound inequality.

$$\left| \left(x^2 - 7\right) - 2 \right| < 1$$
$$-1 < \left(x^2 - 7\right) - 2 < 1$$

Simplify the center expression and isolate the x-term.

$$-1 < \left(x^2 - 7\right) - 2 < 1$$
$$-1 < x^2 - 9 < 1$$
$$8 < x^2 < 10$$

Isolate x by taking the square root of all three parts.

$$8 < x^2 < 10$$
$$2\sqrt{2} < x < \sqrt{10}$$

Therefore, for the open interval $\left(2\sqrt{2}, \sqrt{10}\right)$, which contains $x = 3$, the inequality $\left| \left(x^2 - 7\right) - 2 \right| < 1$ holds.

Now choose a positive value for δ that places the open interval $(c - \delta, c + \delta)$ centered on c inside the interval $\left(2\sqrt{2}, \sqrt{10}\right)$.

Use the smallest distance between $c = 3$ and the endpoints of the interval $\left(2\sqrt{2}, \sqrt{10}\right)$.

The distance to the first endpoint is $3 - 2\sqrt{2} \approx 0.1716$.
The distance to the second endpoint is $\sqrt{10} - 3 \approx 0.1623$.

The largest possible value for δ is $\sqrt{10} - 3$.

Therefore, for all x satisfying $0 < |x - 3| < \sqrt{10} - 3$, the inequality $\left| \left(x^2 - 7\right) - 2 \right| < 1$ holds.

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For the given function f(x), the point c, and a positive number ε, find L = $\lim_{x \to c}$ f(x). Then find a number $\delta > 0$ such that for all x,

$0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon$.

    f(x) = 8 − 3x, c = 4, ε = 0.03

Notice that f(x) is a linear function. Since a linear function is a simple polynomial function and it is defined for all x, the limit L = $\lim_{x \to c}$ is the value of f(x) at c.

Evaluate the function at c = 4.

L = 8 − 3(4) = − 4

To find δ, begin by solving the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a,b) containing c on which the inequality holds for all x ≠ c.

Remove the absolute value sign and rewrite the inequality as a compound inequality.

$$|(8 - 3x) - (-4)| < 0.03$$
$$-0.03 < (8 - 3x) - (-4) < 0.03$$

Simplify the center expression and isolate the x-term.

$$-0.03 < (8 - 3x) - (-4) < 0.03$$
$$-0.03 < -3x + 12 < 0.03$$
$$-12.03 < -3x < -11.97$$

Then isolate x in the center. Notice that the direction of the inequalities has been changed.

$$-12.03 < -3x < -11.97$$
$$4.01 > x > 3.99$$

Therefore, for x in the interval (3.99,4.01), the inequality $|(8 - 3x) - (-4)| < 0.03$ holds. Now choose a positive value for δ that places the open interval $(c - \delta, c + \delta)$ centered on c inside the interval (3.99,4.01).

Use the smallest distance between c = 4 and the endpoints of the interval (3.99,4.01).

The distance to the first endpoint is 4 − 3.99 = 0.01.
The distance to the second endpoint is 4.01 − 4 = 0.01.

The largest possible value for δ is 0.01.

Therefore, for all x satisfying $0 < |x - 4| < 0.01$, the inequality $|(8 - 3x) - (-4)| < 0.03$ holds.

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Give an ε-δ proof of the limit fact.

$$\lim_{x \to 0} (5x + 1) = 1$$

We begin by giving the precise meaning of limit.

To say that $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x) - L| < \varepsilon$, provided that $0 < |x - c| < \delta$; that is,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

To establish the proof, we first perform a preliminary analysis to find the appropriate choice of δ.

Let ε be any positive number.

We must produce a $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |(5x + 1) - 1| < \varepsilon$.

By simplifying the inequality on the right, we will determine the value of δ needed for the inequality on the left.

$$|(5x + 1) - 1| < \varepsilon \Leftrightarrow |5x| < \varepsilon$$
$$\Leftrightarrow |5|\,|x| < \varepsilon$$
$$\Leftrightarrow 5|x| < \varepsilon$$
$$\Leftrightarrow |x| < \frac{\varepsilon}{5}, \text{ or equivalently, } |x - 0| < \frac{\varepsilon}{5}$$

Now we see that we should choose $\delta = \dfrac{\varepsilon}{5}$. We can now construct the formal proof of the statement $\lim_{x \to 0} (5x + 1) = 1$.

Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{5}$. Then $0 < |x - 0| < \delta$ implies the following chain of equalities and an inequality.

$|(5x + 1) - 1|$	$=$	$|5x|$	Simplify inside absolute value bars.
	$=$	$|5|\,|x|$	Rewrite as product of absolute values.
	$=$	$5|x|$	Simplify.
	$=$	$5|x - 0|$	Rewrite expression inside absolute value bars.
	$<$	5δ	Apply the condition $0 < |x - 0| < \delta$.
	$=$	ε	Substitute $\delta = \dfrac{\varepsilon}{5}$ and multiply.

The result of the above chain of equalities and an inequality is that $|(5x + 1) - 1| < \varepsilon$.

Therefore, we have proven that $\lim_{x \to 0} (5x + 1) = 1$.

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Give an ε-δ proof of the limit fact.

$$\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right) = 4$$

We begin by giving the precise meaning of limit.

To say that $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x) - L| < \varepsilon$, provided that $0 < |x - c| < \delta$; that is,

$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$

To establish the proof, we first perform a preliminary analysis to find the appropriate choice of δ.

Let ε be any positive number. Insert the information from the problem into the precise definition of limit.

We must produce a $\delta > 0$ such that $0 < |x - 2| < \delta \Rightarrow \left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon.$

By simplifying the inequality on the right, we will determine the value of δ needed for the inequality on the left. Simplify the expression inside the absolute value bars.

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon \Leftrightarrow \left| \frac{(x + 2)(x - 2)}{x - 2} - 4 \right| < \varepsilon \qquad \text{Factor the difference of two squares.}$$

$$\Leftrightarrow |x + 2 - 4| < \varepsilon \qquad \text{Simplify the rational expression.}$$

$$\Leftrightarrow |x - 2| < \varepsilon \qquad \text{Combine like terms.}$$

We next choose the value δ. Based on the analysis above, should choose $\delta = \varepsilon$. With $\delta = \varepsilon$, we have $0 < |x - 2| < \delta$ because of the result from the previous step.

We can now construct the formal proof of the statement $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right) = 4$.

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then $0 < |x - 2| < \delta$ implies the following chain of equalities and an inequality.

Begin with the left side of the inequality we want to prove.

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| = |(x + 2) - 4| \qquad \text{Factor the difference of two squares and divide by the common factor.}$$

$$= |x - 2| \qquad \text{Combine like terms.}$$

$$< \delta \qquad \text{Now apply the condition } 0 < |x - 2| < \delta.$$

$$= \varepsilon \qquad \text{Finally, substitute, using } \delta = \varepsilon.$$

The result of the above chain of equalities and an inequality is that $\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon.$

Therefore, we have proven that $\lim\limits_{x \to 2} \left(\dfrac{x^2 - 4}{x - 2} \right) = 4$.

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For the given function f(x) and the given values of c and $\varepsilon > 0$, find L = $\lim\limits_{x \to c}$ f(x).

Then determine the largest value for $\delta > 0$ such that $0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon$.

$$f(x) = \frac{x^2 - 81}{x - 9}, \quad c = 9, \quad \varepsilon = 0.07$$

Begin by determining the value of the limit L.

$$L = \lim_{x \to 9} \frac{x^2 - 81}{x - 9} = \lim_{x \to 9} \frac{(x + 9)(x - 9)}{x - 9} = \lim_{x \to 9} (x + 9) = 18$$

Now that L is known, solve $|f(x) - L| < \varepsilon$ to find the largest interval containing c on which the inequality holds and use this information to determine δ.

$$\left| \frac{x^2 - 81}{x - 9} - 18 \right| < 0.07 \rightarrow 17.93 < (x + 9) < 18.07$$

So, the interval on the x-axis is 8.93 < x < 9.07, x ≠ 9 because f(x) is not defined at x = 9.

Now, $|x - c| < \delta$ is $|x - 9| < \delta$, which implies $-\delta < (x - 9) < \delta$, or $9 - \delta < x < 9 + \delta$, x ≠ 9.

Comparing the intervals or $9 - \delta < x < 9 + \delta$ and 8.93 < x < 9.07, $9 - \delta = 8.93$ and $9 + \delta = 9.07$. Thus, $\delta = 0.07$.