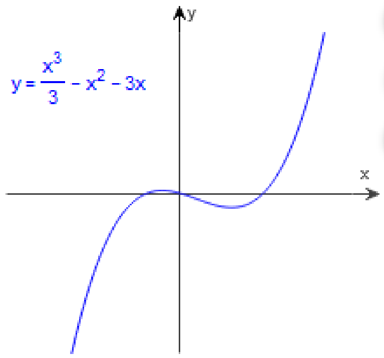
Identify the inflection points and local maxima and minima of the function graphed below. Identify the intervals on which it is concave up and concave down.




A point where the graph of the function has a tangent line and where the concavity changes is a point of inflection.

Solve $y'' = 0$ for x to find the point of inflection.

To find the second derivative, first find the first derivative.

$$y = \frac{x^3}{3} - x^2 - 3x$$
$$y' = x^2 - 2x - 3$$

Now find the second derivative.

$$y'' = 2x - 2$$

To determine where an inflection point occurs, solve $y'' = 0$.

$$2x - 2 = 0$$
$$x = 1$$

Evaluate y at $x = 1$.

$$y = \frac{(1)^3}{3} - (1)^2 - 3(1) = -\frac{11}{3}$$

Thus, the curve $y = \frac{x^3}{3} - x^2 - 3x$ has a point of inflection at $\left(1, -\frac{11}{3}\right)$.

To find any local extrema, solve $y' = 0$.

$$x^2 - 2x - 3 = 0$$
$$x = -1, 3$$

Use the Second Derivative Test for Local Extrema to find the local extrema.

The Second Derivative Test for Local Extrema assumes $f''$ is continuous on an open interval that contains $x = c$.
1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function may have a local maximum, a local minimum, or neither.

Evaluate $y''$ at $x = -1$.

$2(-1) - 2 = -4$

Evaluate $y''$ at $x = 3$.

$2(3) - 2 = 4$

Since $y'' < 0$ at $x = -1$, y has a local maximum at $x = -1$.

Since $y'' > 0$ at $x = 3$, y has a local minimum at $x = 3$.

Evalute y at $x = -1$.

$$y = \frac{(-1)^3}{3} - (-1)^2 - 3(-1) = \frac{5}{3}$$

Evalute y at $x = 3$.

$$y = \frac{(3)^3}{3} - (3)^2 - 3(3) = -9$$

Thus, y has a local maximum of $\frac{5}{3}$ at $x = -1$ and a local minimum of $-9$ at $x = 3$.

To determine where the graph is concave up and concave down, use The Second Derivative Test for Concavity.

The Second Derivative Test for Concavity first assumes $y = f(x)$ is twice-differentiable on an interval I.
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Use the point of inflection to subdivide the domain into intervals in which $y''$ is either positive or negative.

The intervals are $(-\infty,1),(1,\infty)$.

For the interval $(-\infty,1)$, evaluate $y''$ at $x = 0$.

$2(0) - 2 = -2$

For the interval $(1,\infty)$, evaluate $y''$ at $x = 2$.

$2(2) - 2 = 2$

Since $y'' < 0$ at $x = 0$, the graph of y over the interval $(-\infty,1)$ is concave down.

Since $y'' > 0$ at $x = 2$, the graph of y over the interval $(1,\infty)$ is concave up.

Thus, the function is concave down over the interval $(-\infty,1)$ and concave up over the interval $(1,\infty)$.

Find the coordinates of any local extreme points and inflection points. Use these to graph the function $y = x^2 - 6x + 8$.

Here are the strategies for graphing $y = f(x)$.

1. Identify the domain of f and any symmetries the curve may have.
2. Find $y'$ and $y''$.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in steps 3-5, and sketch the curve.

The domain of y is $(-\infty, \infty)$, and there are no symmetries about either axis or the origin.

Find $y'$ and $y''$.

$$y = x^2 - 6x + 8$$
$$y' = 2x - 6$$
$$y'' = 2$$

To find the critical point of y, solve $y' = 0$.

$$2x - 6 = 0$$
$$x = 3$$

Use the Second Derivative Test for Local Extrema to determine whether a local extremum occurs at the critical point.

First, $y''$ evaluated at $x = 3$ is 2.

Now, y evaluated at $x = 3$ is $-1$.

Since $y'' > 0$ at $x = 3$, y has a local minimum at $(3, -1)$.

Use the First Derivative Test for Monotonic Functions to determine on which intervals the function is increasing or decreasing.

The critical point subdivides the domain of y into intervals $(-\infty,3)$ and $(3,\infty)$ on which $y'$ is either positive or negative. Determine the sign of $y'$ by evaluating y at a convenient point in each subinterval..

For the subinterval $(-\infty,3)$, evaluate $y'$ at $x=0$.

$y'=2(0)-6=-6$

For the subinterval $(3,\infty)$, evaluate $y'$ at $x=4$.

$y'=2(4)-6=2$

Since $y'<0$ at $x=0$, y decreases on $(-\infty,3)$.

Since $y'>0$ at $x=4$, y increases on $(3,\infty)$.

To find the points of inflection of y, solve $y''=0$.

Since $2\neq 0$, y has no points of inflection.

Use the Second Derivative Test for Concavity to determine the concavity of the curve.

Since there are no points of inflection, there are no changes in concavity.

Since $y''>0$ on $(-\infty,\infty)$, the graph of y over $(-\infty,\infty)$ is concave up.

There are no asymptotes.

To find the y-intercept, evaluate y at $x=0$.

$y=(0)^2-6(0)+8=8$

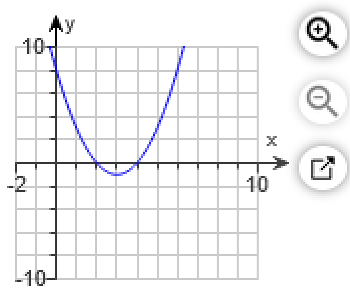
To find the x-intercept(s), solve $y=0$.

$$\begin{aligned} x^2-6x+8&=0\\ x&=2,4 \end{aligned}$$

The y-intercept is $(0,8)$.

The x-intercepts are $(2,0)$ and $(4,0)$.

The graph of $y=x^2-6x+8$ is shown below.



Find the coordinates of any local extreme points and inflection points. Use these to graph the function $y = x^3 - 3x + 3$.

Here are the strategies for graphing $y = f(x)$.

1. Identify the domain of f and any symmetries the curve may have.
2. Find $y'$ and $y''$.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in steps 3-5, and sketch the curve.

The domain of y is $(-\infty, \infty)$, and there are no symmetries about either axis or the origin.

Find $y'$.

$$y = x^3 - 3x + 3$$
$$y' = 3x^2 - 3$$

Find $y''$.

$$y'' = 6x$$

To find the critical point(s) of y, solve $y' = 0$.

$$3x^2 - 3 = 0$$
$$x = \pm 1$$

Use the Second Derivative Test for Local Extrema to determine whether any local extrema occur at the critical points.

First, evaluate $y''$ at the critical points.

$$y'' = 6(-1) = -6$$

$$y'' = 6(1) = 6$$

Now, evalutate y at the critical points.

$y = (-1)^3 - 3(-1) + 3 = 5$

$y = (1)^3 - 3(1) + 3 = 1$

Since $y'' < 0$ at $x = -1$, y has a local maximum at $(-1,5)$.

Since $y'' > 0$ at $x = 1$, y has a local minimum at $(1,1)$.

Use the First Derivative Test for Monotonic Functions to determine on which intervals the function is increasing or decreasing.

The critical points subdivide the domain of y into intervals $(-\infty, -1)$, $(-1,1)$, and $(1,\infty)$ on which y ' is either positive or negative. Determine the sign of $y'$ by evaluating y at a convenient point in each subinterval..

For the subinterval $(-\infty, -1)$, evaluate $y'$ at $x = -3$.

$y' = 3(-3)^2 - 3 = 24$

For the subinterval $(-1,1)$, evaluate $y'$ at $x = 0$.

$y' = 3(0)^2 - 3 = -3$

For the subinterval $(1,\infty)$, evaluate $y'$ at $x = 3$.

$y' = 3(3)^2 - 3 = 24$

Since $y' > 0$ at $x = -1$, y increases on $(-\infty, -1)$.

Since $y' < 0$ at $x = 0$, y decreases on $(-1,1)$.

Since $y' > 0$ at $x = 1$, y increases on $(1,\infty)$.

To find the points of inflection of y, first solve $y'' = 0$.

$$6x = 0$$
$$x = 0$$

Use the Second Derivative Test for Concavity to determine the concavity of the curve.

For the subinterval $(-\infty,0)$, recall that $y''$evaluated at $x=-1$ is $-6$ and that $y''$ evaluated at $x=1$ is 6.

Since $y''<0$ at $x=-1$, y is concave down on $(-\infty,0)$.

Since $y''>0$ at $x=1$, y is concave up on $(0,\infty)$.

Thus, y is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

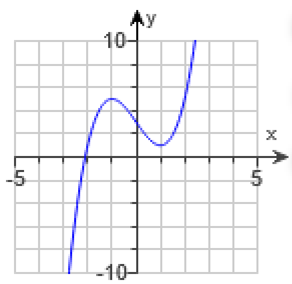
Evaluate y at $x=0$.

$y=(0)^3-3(0)+3=3$

Thus, the function has an inflection point at (0,3).

There are no asymptotes.

The graph of $y=x^3-3x+3$ is shown below.



Sketch the graph of the given function by determining the appropriate information and points from the first and second derivatives. Use a graphing calculator to check the graph.

$y = x^3 - 2x^2 - 15x + 10$

The first step in sketching a graph of y is to find the first and second derivatives, as well as to identify the domain of y. Determine the first derivative.

$$\begin{aligned} y &= x^3 - 2x^2 - 15x + 10 \\ y' &= 3x^2 - 4x - 15 \end{aligned}$$

Next determine the second derivative.

$$\begin{aligned} y' &= 3x^2 - 4x - 15 \\ y'' &= 6x - 4 \end{aligned}$$

The domain of y is all real numbers.

The critical values of x occur where $y' = 0$ and where $y'$ does not exist. Since $y' = 3x^2 - 4x - 15$ is a polynomial, it is defined for all real numbers.

Solve $y' = 0$.

$$\begin{aligned} 3x^2 - 4x - 15 &= 0 \\ (3x+5)(x-3) &= 0 \\ x &= -\frac{5}{3} \quad \text{or} \quad x = 3 \end{aligned}$$

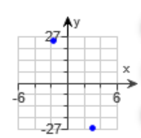
Find the function values for $x = -\frac{5}{3}$ and $x = 3$. Substitute these x-values into y and simplify. Calculate the y-value corresponding to $x = -\frac{5}{3}$.

$$\begin{aligned} y &= x^3 - 2x^2 - 15x + 10 \\ &= \left(-\frac{5}{3}\right)^3 - 2\left(-\frac{5}{3}\right)^2 - 15\left(-\frac{5}{3}\right) + 10 && \text{Substitute } x = -\frac{5}{3}. \\ &= \frac{670}{27} && \text{Simplify.} \end{aligned}$$

Calculate the y-value corresponding to $x = 3$.

$$\begin{aligned} y &= x^3 - 2x^2 - 15x + 10 \\ &= (3)^3 - 2(3)^2 - 15(3) + 10 && \text{Substitute } x = 3. \\ &= -26 && \text{Simplify.} \end{aligned}$$

The points $\left(-\frac{5}{3}, \frac{670}{27}\right)$ and $(3, -26)$ are possible relative extrema. Plot these two points on the graph of y.




Next, determine what type of extrema these points are. To do this, evaluate $y''$ for each critical value of x.

If the value of $y''$ at the critical value is less than 0, then the value of y at the critical value is a relative maximum. If the value is greater than 0, then the value of y at the critical value is a relative minimum.

Evaluate $y'' = 6x - 4$ at $x = -\frac{5}{3}$ and at $x = 3$.

$$\begin{aligned} y'' &= 6\left(-\frac{5}{3}\right) - 4 \\ &= -14 \end{aligned} \qquad \begin{aligned} y'' &= 6(3) - 4 \\ &= 14 \end{aligned}$$

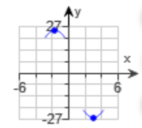
Since $y'' < 0$ at $x = -\frac{5}{3}$, the point $\left(-\frac{5}{3}, \frac{670}{27}\right)$ is a relative maximum.

Since $y'' > 0$ at $x = 3$, the point $(3, -26)$ is a relative minimum.

If the value of $y''$ at the critical value $x_0$ is less than 0, then y has a maximum at $x_0$, and y is increasing to the left of $x_0$ and decreasing to the right of $x_0$. If the value is greater than 0, then y has a minimum at $x_0$, and y is decreasing to the left of $x_0$ and increasing to the right of $x_0$.

Since $y'' = -14$ at $x = -\frac{5}{3}$ and $y'' = 14$ at $x = 3$, y is increasing on the intervals $x < -\frac{5}{3}$ and $x > 3$, and is decreasing on the interval $-\frac{5}{3} < x < 3$.

Add curves to the graph of y to indicate that $(3, -26)$ is a relative minimum and $\left(-\frac{5}{3}, \frac{670}{27}\right)$ is a relative maximum. These curves will also indicate where the graph is increasing and decreasing.




Now find the points of inflection. Determine the candidates for points of inflection by finding where $y'' = 0$ or where $y''$ does not exists. Find the function values at these points.

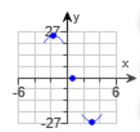
Note that $y''$ exists for all values of x since the function y is a polynomial. Solve $y'' = 0$ for x.

$$\begin{aligned} 6x - 4 &= 0 \\ x &= \frac{2}{3} \end{aligned}$$

Evaluate y at $x = \frac{2}{3}$.

$$\begin{aligned} y &= x^3 - 2x^2 - 15x + 10 \\ &= \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 - 15\left(\frac{2}{3}\right) + 10 && \text{Substitute } x = \frac{2}{3}. \\ &= -\frac{16}{27} && \text{Simplify.} \end{aligned}$$

Thus, the point $\left(\frac{2}{3}, -\frac{16}{27}\right)$ is a possible point of inflection. Add this point to the graph of y.

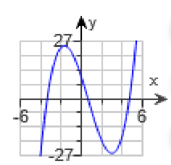



For $\left(\frac{2}{3}, -\frac{16}{27}\right)$ to be a point of inflection, the function must change concavity at that point. Substitute the critical values found above into $y''$ to determine where the graph is concave up $(y'' > 0)$ and where it is concave down $(y'' < 0)$.

Because $y'' = -14$ at $x = -\frac{5}{3}$ and $y'' = 14$ for $x = 3$, the function is concave down on $x < \frac{2}{3}$ and concave up on $x > \frac{2}{3}$.

Because y is concave down on $x < \frac{2}{3}$ and concave up on $x > \frac{2}{3}$, y changes concavity at $x = \frac{2}{3}$. Therefore, there is a point of inflection at $x = \frac{2}{3}$.

Finally, sketch the graph using all of the information found above and plotting extra points as needed. The graph is shown on the right.



Find and graph the coordinates of any local extreme points and inflection points of the function $y = x^4 - 6x^2$.

Here are the strategies for graphing $y = f(x)$.
1. Identify the domain of f and any symmetries the curve may have.
2. Find $y'$ and $y''$.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in steps 3-5, and sketch the curve.

The domain of y is $(-\infty, \infty)$. The graph is symmetric about the y-axis.

Find $y'$ and $y''$.

$y = x^4 - 6x^2$
$y' = 4x^3 - 12x$
$y'' = 12x^2 - 12$

To find the critical point(s) of y, solve $y' = 0$.

$4x^3 - 12x = 0$
$x = -\sqrt{3}, 0, \sqrt{3}$

Use the Second Derivative Test for Local Extrema to determine whether any local extrema occur at the critical points.

First, evaluate $y''$ at the critical points.

$y'' = 12\left(-\sqrt{3}\right)^2 - 12 = 24$

$y'' = 12(0) - 12 = -12$

$y'' = 12\left(\sqrt{3}\right)^2 - 12 = 24$

Now, evaluate y at the critical points.

$y = \left(-\sqrt{3}\right)^4 - 6\left(-\sqrt{3}\right)^2 = -9$

$y = (0)^4 - 6(0)^2 = 0$

$y = \left(\sqrt{3}\right)^4 - 6\left(\sqrt{3}\right)^2 = -9$

Since $y'' > 0$ at $x = -\sqrt{3}$, y has a local minimum at $\left(-\sqrt{3}, -9\right)$.

Since $y'' < 0$ at $x = 0$, y has a local maximum at $(0,0)$.

Since $y'' > 0$ at $x = \sqrt{3}$, y has a local minimum at $\left(\sqrt{3}, -9\right)$.

Use the First Derivative Test for Monotonic Functions to determine on which intervals the function is increasing or decreasing.

The critical points subdivide the domain of y into intervals $\left(-\infty, -\sqrt{3}\right)$, $\left(-\sqrt{3}, 0\right)$, $\left(0, \sqrt{3}\right)$, and $\left(\sqrt{3}, \infty\right)$ on which $y'$ is either positive or negative. Determine the sign of $y'$ by evaluating y at a convenient point in each subinterval.

For the subinterval $\left(-\infty, -\sqrt{3}\right)$, evaluate $y'$ at $x = -2$.

$y' = 4(-2)^3 - 12(-2) = -8$

For the subinterval $\left(-\sqrt{3}, 0\right)$, evaluate $y'$ at $x = -1$.

$y' = 4(-1)^3 - 12(-1) = 8$

For the subinterval $\left(0, \sqrt{3}\right)$, evaluate $y'$ at $x = 1$.

$y' = 4(1)^3 - 12(1) = -8$

For the subinterval $\left(\sqrt{3}, \infty\right)$, evaluate $y'$ at $x = 2$.

$y' = 4(2)^3 - 12(2) = 8$

Since $y' < 0$ at $x = -2$, y decreases on $\left(-\infty, -\sqrt{3}\right)$.

Since $y' > 0$ at $x = -1$, y increases on $\left(-\sqrt{3}, 0\right)$.

Since $y' < 0$ at $x = 1$, y decreases on $\left(0, \sqrt{3}\right)$.

Since $y' > 0$ at $x = 2$, y increases on $\left(\sqrt{3}, \infty\right)$.

To find the points of inflection of y, first solve $y'' = 0$.

$$\begin{aligned} 12x^2 - 12 &= 0 \\ x &= \pm 1 \end{aligned}$$

Use the Second Derivative Test for Concavity to determine the concavity of the curve.

Recall that $y''$ evaluated at $x = -\sqrt{3}$ is 24, $y''$ evaluated at $x = 0$ is $-12$, and $y''$ evaluated at $x = \sqrt{3}$ is 24.

Since $y'' > 0$ at $x = -\sqrt{3}$, y is concave up on $(-\infty, -1)$.

Since $y'' < 0$ at $x = 0$, y is concave down on $(-1, 1)$.

Since $y'' > 0$ at $x = \sqrt{3}$, y is concave up on $(1, \infty)$.

Thus, y is concave up on $(-\infty, -1)$, $(1, \infty)$ and concave down on $(-1, 1)$.

Evaluate y at $x = -1$ and $x = 1$.

$y = (-1)^4 - 6(-1)^2 = -5$

$y = (1)^4 - 6(1)^2 = -5$

Thus, the function has inflection points at $(-1, -5)$ and $(1, -5)$.

There are no asymptotes.

The graph of $y = x^4 - 6x^2$ is shown below.