

$$\text{Find } y'' \text{ for } y = \left(1 + \frac{2}{x}\right)^3.$$

If  $n$  is a positive or negative integer and  $f(u) = u^n$ , the power rule says that  $f'(u) = nu^{n-1}$ . If  $u$  is a differentiable function of  $x$ , then use the chain rule to extend this to the power chain rule.

$$\frac{d}{dx} f(u) = nu^{n-1} \frac{du}{dx}$$

Use the power chain rule to find  $y'$ .

$$y = \left(1 + \frac{2}{x}\right)^3$$

$$y' = 3\left(1 + \frac{2}{x}\right)^2 \frac{d}{dx}\left(1 + \frac{2}{x}\right)$$

To find  $\frac{d}{dx}\left(1 + \frac{2}{x}\right)$ , use the rules of differentiation.

$$\frac{d}{dx}\left(1 + \frac{2}{x}\right) = \left(-\frac{2}{x^2}\right)$$

$$\text{Thus, } y' = 3\left(1 + \frac{2}{x}\right)^2 \left(-\frac{2}{x^2}\right).$$

Use the derivative product rule to find  $y''$ .

$$y' = 3\left(1 + \frac{2}{x}\right)^2 \left(-\frac{2}{x^2}\right)$$

$$y'' = 3\left(1 + \frac{2}{x}\right)^2 \frac{d}{dx}\left(-\frac{2}{x^2}\right) + \left(-\frac{2}{x^2}\right) \frac{d}{dx}\left(3\left(1 + \frac{2}{x}\right)^2\right)$$

To find  $\frac{d}{dx}\left(-\frac{2}{x^2}\right)$ , use the power rule for negative integers.

$$\frac{d}{dx}\left(-\frac{2}{x^2}\right) = \left(\frac{4}{x^3}\right)$$

To find  $\frac{d}{dx}\left(3\left(1 + \frac{2}{x}\right)^2\right)$ , use the power chain rule.

$$\frac{d}{dx} 3\left(1 + \frac{2}{x}\right)^2 = 6\left(1 + \frac{2}{x}\right) \frac{d}{dx}\left(1 + \frac{2}{x}\right)$$

$$= 6\left(1 + \frac{2}{x}\right) \left(-\frac{2}{x^2}\right)$$

Simplify.

$$y'' = 3\left(1 + \frac{2}{x}\right)^2 \frac{d}{dx}\left(-\frac{2}{x^2}\right) + \left(-\frac{2}{x^2}\right) \frac{d}{dx}\left(3\left(1 + \frac{2}{x}\right)^2\right)$$

$$= 3\left(1 + \frac{2}{x}\right)^2 \left(\frac{4}{x^3}\right) + \left(-\frac{2}{x^2}\right) 6\left(1 + \frac{2}{x}\right) \left(-\frac{2}{x^2}\right)$$

$$= \frac{12}{x^3} \left(1 + \frac{2}{x}\right)^2 + \frac{24}{x^4} \left(1 + \frac{2}{x}\right)$$

$$\text{Thus, } y'' = \frac{12}{x^3} \left(1 + \frac{2}{x}\right) \left(1 + \frac{4}{x}\right).$$



## Solution

$$\frac{d}{dx} \left( \left( 5 + \frac{1}{x} \right)^3 \right) = -\frac{3(5x+1)^2}{x^4}$$

## Steps

$$\frac{d}{dx} \left( \left( 5 + \frac{1}{x} \right)^3 \right)$$

Apply the chain rule:  $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$f = u^3, \quad u = \left( 5 + \frac{1}{x} \right)$$

$$= \frac{d}{du} \left( u^3 \right) \frac{d}{dx} \left( 5 + \frac{1}{x} \right)$$

$$\frac{d}{du} \left( u^3 \right) = 3u^2$$

Show Steps

$$\frac{d}{dx} \left( 5 + \frac{1}{x} \right) = -\frac{1}{x^2}$$

Show Steps

$$= 3u^2 \left( -\frac{1}{x^2} \right)$$

$$\text{Substitute back } u = \left( 5 + \frac{1}{x} \right)$$

$$= 3 \left( 5 + \frac{1}{x} \right)^2 \left( -\frac{1}{x^2} \right)$$

$$\text{Simplify } 3 \left( 5 + \frac{1}{x} \right)^2 \left( -\frac{1}{x^2} \right): \quad -\frac{3(5x+1)^2}{x^4}$$

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$$3 \left( 5 + \frac{1}{x} \right)^2 \left( -\frac{1}{x^2} \right)$$

$$\text{Remove parentheses: } (-a) = -a$$

$$= -3 \left( 5 + \frac{1}{x} \right)^2 \frac{1}{x^2}$$

Multiply fractions:  $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$

$$= -\frac{1 \cdot 3 \left(5 + \frac{1}{x}\right)^2}{x^2}$$

Multiply the numbers:  $1 \cdot 3 = 3$

$$= -\frac{3 \left(\frac{1}{x} + 5\right)^2}{x^2}$$

$$\left(5 + \frac{1}{x}\right)^2 = \frac{(5x + 1)^2}{x^2}$$

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$$\left(5 + \frac{1}{x}\right)^2$$

Join  $5 + \frac{1}{x}$ :  $\frac{5x + 1}{x}$

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$$= \left(\frac{5x + 1}{x}\right)^2$$

Apply exponent rule:  $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

$$= \frac{(5x + 1)^2}{x^2}$$

$$= -\frac{3 \cdot \frac{(5x + 1)^2}{x^2}}{x^2}$$

Multiply  $3 \cdot \frac{(5x + 1)^2}{x^2}$ :  $\frac{3(5x + 1)^2}{x^2}$

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$$3 \cdot \frac{(5x + 1)^2}{x^2}$$

Multiply fractions:  $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$

$$= \frac{(5x + 1)^2 \cdot 3}{x^2}$$

$$= -\frac{3(5x + 1)^2}{x^2}$$

Apply the fraction rule:  $\frac{b}{a} = \frac{b}{c \cdot a}$

$$= -\frac{3(5x+1)^2}{x^2 x^2}$$

$$x^2 x^2 = x^4$$

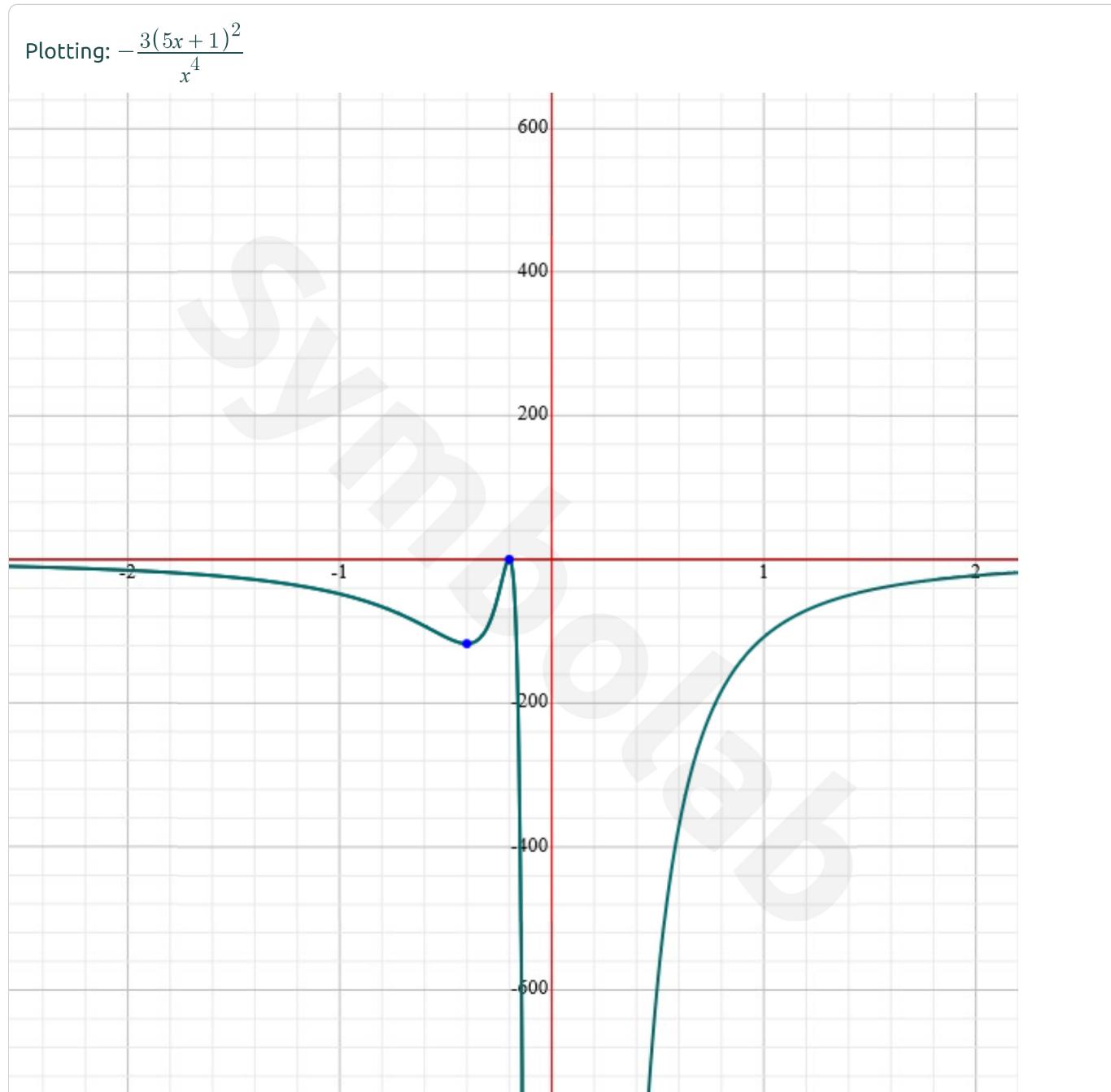
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$$= -\frac{3(5x+1)^2}{x^4}$$

$$= -\frac{3(5x+1)^2}{x^4}$$

Click to practice Chain Rule

## Graph



Find the value of  $(f \circ g)'$  at the given value.

$$f(u) = u^7 + 3, \quad u = g(x) = \sqrt{x}, \quad x = 1$$

The Chain Rule states if  $u$  is differentiable at the point  $u = g(x)$  and  $g(x)$  is differentiable at  $x$ , then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$ , and  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ . In Leibniz's notation, if  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \text{ where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

Use the Chain Rule to find  $(f \circ g)'$ .

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{d}{du}(u^7 + 3) \cdot \frac{d}{dx}(\sqrt{x})\end{aligned}$$

To find  $\frac{d}{du}(u^7 + 3)$  and  $\frac{d}{dx}(\sqrt{x})$ , use the rules of differentiation.

$$\frac{d}{du}(u^7 + 3) = 7u^6, \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-1/2}$$

Now, substitute  $u$  in  $f'(g(x))$  and simplify.

$$\begin{aligned}&= \frac{d}{du}(u^7 + 3) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= 7u^6 \cdot \frac{1}{2}x^{-1/2} \\ &= 7(\sqrt{x})^6 \cdot \frac{1}{2}x^{-1/2} \\ &= \frac{7}{2}x^{5/2}\end{aligned}$$

$$\text{Thus, } (f \circ g)'(x) = \frac{7}{2}x^{5/2}.$$

Now, to find  $(f \circ g)'(1)$ , substitute 1 for  $x$  and simplify.

$$(f \circ g)'(1) = \frac{7}{2}$$

$$\text{Thus, } (f \circ g)'(1) = \frac{7}{2}.$$



Solution

$$\frac{d}{dx}((\sqrt{x})^7) = \frac{7x^{\frac{5}{2}}}{2}$$

## Steps

$$\frac{d}{dx}((\sqrt{x})^7)$$

Apply the chain rule:  $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$f = u^7, u = (\sqrt{x})$$

$$= \frac{d}{du}(u^7) \frac{d}{dx}(\sqrt{x})$$

$$\frac{d}{du}(u^7) = 7u^6$$

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$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

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$$= 7u^6 \frac{1}{2\sqrt{x}}$$

Substitute back  $u = (\sqrt{x})$

$$= 7(\sqrt{x})^6 \frac{1}{2\sqrt{x}}$$

$$\text{Simplify } 7(\sqrt{x})^6 \frac{1}{2\sqrt{x}}: \quad \frac{7x^{\frac{5}{2}}}{2}$$

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$$7(\sqrt{x})^6 \frac{1}{2\sqrt{x}}$$

$$(\sqrt{x})^6 = x^3$$

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$$(\sqrt{x})^6$$

Apply radical rule:  $\sqrt{a} = a^{\frac{1}{2}}$

$$= \left(x^{\frac{1}{2}}\right)^6$$

Apply exponent rule:  $(a^b)^c = a^{bc}$

$$= x^{\frac{1}{2} \cdot 6}$$

$$\frac{1}{2} \cdot 6 = 3$$

Show Steps 

$$= x^3$$

$$= 7 \cdot \frac{1}{2\sqrt{x}} x^3$$

Multiply fractions:  $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$

$$= \frac{1 \cdot 7x^3}{2\sqrt{x}}$$

Multiply the numbers:  $1 \cdot 7 = 7$

$$= \frac{7x^3}{2\sqrt{x}}$$

Apply radical rule:  $\sqrt[n]{a} = a^{\frac{1}{n}}$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$= \frac{7x^3}{2x^{\frac{1}{2}}}$$

Apply exponent rule:  $\frac{x^a}{x^b} = x^{a-b}$

$$\frac{x^3}{x^{\frac{1}{2}}} = x^{3-\frac{1}{2}}$$

$$= \frac{7x^{-\frac{1}{2}+3}}{2}$$

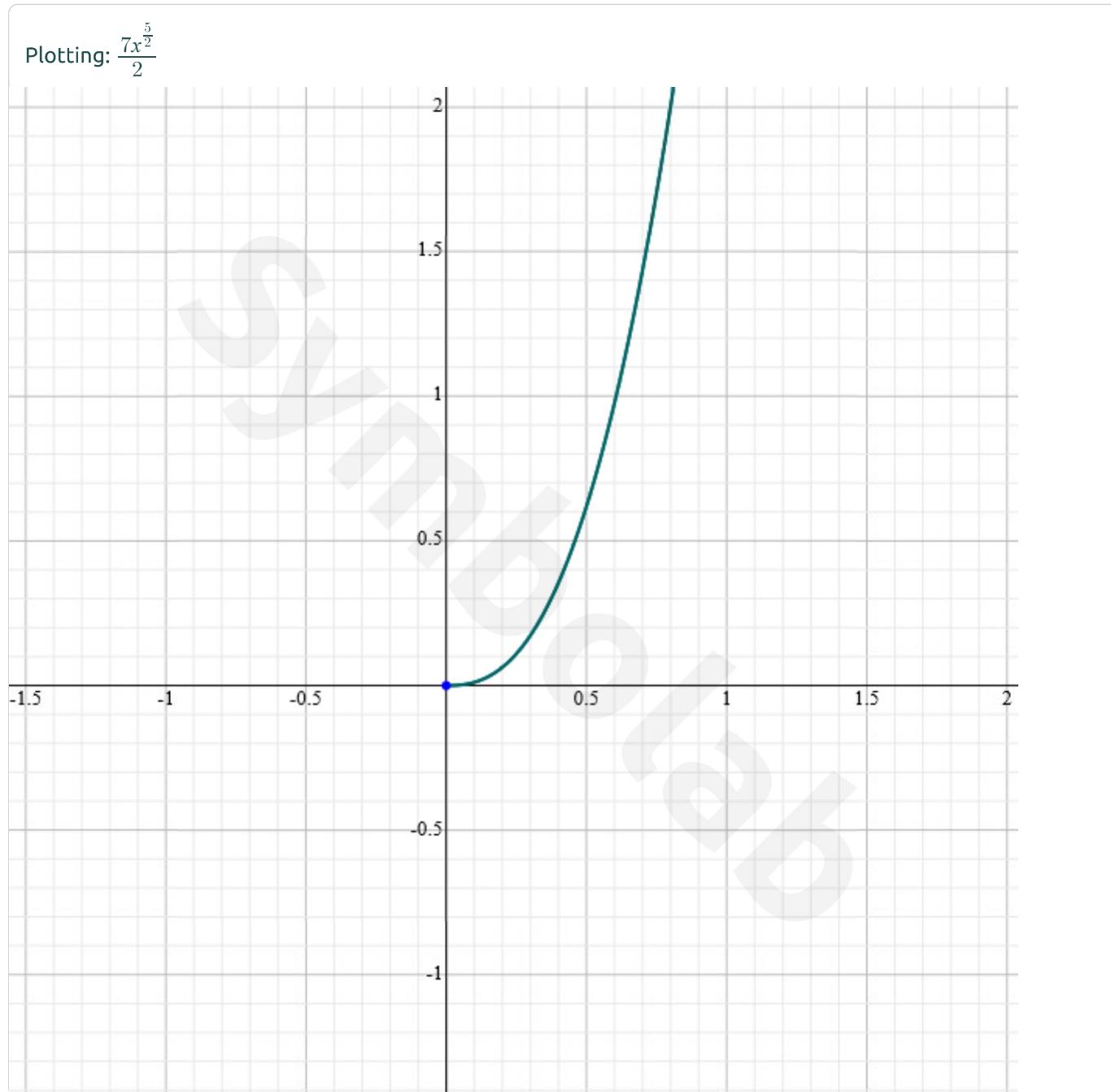
Subtract the numbers:  $3 - \frac{1}{2} = \frac{5}{2}$

$$= \frac{7x^{\frac{5}{2}}}{2}$$

$$= \frac{7x^{\frac{5}{2}}}{2}$$

Click to practice Chain Rule

## Graph



Find the value of  $(f \circ g)'$  at the given value of  $x$ .

$$f(u) = \frac{6u}{u^2 + 2}, \quad u = g(x) = 3x^2 + 5x + 2, \quad x = 0$$

The Chain Rule states if  $u$  is differentiable at the point  $u = g(x)$  and  $g(x)$  is differentiable at  $x$ , then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$ , and  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ . In Leibniz's notation, if  $y = f(u)$  and  $u = g(x)$ , then  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ , where  $\frac{dy}{du}$  is evaluated at  $u = g(x)$ .

Use the Chain Rule to find  $(f \circ g)'$ .

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{d}{du} \left( \frac{6u}{u^2 + 2} \right) \cdot \frac{d}{dx} (3x^2 + 5x + 2)$$

To find  $\frac{d}{du} \left( \frac{6u}{u^2 + 2} \right)$  and  $\frac{d}{dx} (3x^2 + 5x + 2)$ , use the rules of differentiation.

$$\frac{d}{du} \left( \frac{6u}{u^2 + 2} \right) = \frac{12 - 6u^2}{(u^2 + 2)^2}, \quad \frac{d}{dx} (3x^2 + 5x + 2) = 6x + 5$$

Now, substitute  $u$  in  $f'(g(x))$ .

$$= \frac{d}{du} \left( \frac{6u}{u^2 + 2} \right) \cdot \frac{d}{dx} (3x^2 + 5x + 2)$$

$$= \frac{12 - 6u^2}{(u^2 + 2)^2} \cdot (6x + 5)$$

$$= \frac{12 - 6(3x^2 + 5x + 2)^2}{((3x^2 + 5x + 2)^2 + 2)^2} \cdot (6x + 5)$$

Instead of simplifying further, to find  $(f \circ g)'(0)$ , substitute 0 for  $x$  and simplify.

$$(f \circ g)'(x) = \frac{12 - 6(3x^2 + 5x + 2)^2}{((3x^2 + 5x + 2)^2 + 2)^2} \cdot (6x + 5)$$

$$(f \circ g)'(0) = \frac{12 - 6(3 \cdot 0^2 + 5 \cdot 0 + 2)^2}{((3 \cdot 0^2 + 5 \cdot 0 + 2)^2 + 2)^2} \cdot (6 \cdot 0 + 5)$$

$$= -\frac{5}{3}$$

$$\text{Thus, } (f \circ g)'(0) = -\frac{5}{3}.$$

Use implicit differentiation to find  $\frac{dy}{dx}$ .

$$x = \cot y$$

To use implicit differentiation, first differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a differentiable function of  $x$ .

Use the rules of differentiation.

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cot y)$$

To find  $\frac{d}{dx}(x)$ , use implicit differentiation and the Power Rule for Positive Integers.

$$\frac{d}{dx}(x) = 1$$

To find  $\frac{d}{dx}(\cot y)$ , use implicit differentiation and the definition of the derivative of the cotangent function.

$$\frac{d}{dx}(\cot y) = (-\csc^2 y) \frac{dy}{dx}$$

The terms with  $\frac{dy}{dx}$  are already collected on one side of the equation.

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cot y)$$

$$1 = (-\csc^2 y) \frac{dy}{dx}$$

Now, solve for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\sin^2 y$$



Solution

Implicit Derivative  $\frac{dx}{dy}$  of  $\sin(xy) = \frac{1}{5}$ :  $-\frac{x}{y}$

## Steps

$$\sin(xy) = \frac{1}{5}$$

Treat  $x$  as  $x(y)$

Differentiate both sides of the equation with respect to  $y$

$$\frac{d}{dy}(\sin(xy)) = \frac{d}{dy}\left(\frac{1}{5}\right)$$

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$$\frac{d}{dy}(\sin(xy)) = \cos(xy)\left(y\frac{d}{dy}(x) + x\right)$$

$$\frac{d}{dy}(\sin(xy))$$

Apply the chain rule:  $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$f = \sin(u), u = xy$$

$$= \frac{d}{du}(\sin(u)) \frac{d}{dy}(xy)$$

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$$\frac{d}{du}(\sin(u)) = \cos(u)$$

$$\frac{d}{dy}(xy) = y\frac{d}{dy}(x) + x$$

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$$\frac{d}{dy}(xy)$$

Apply the Product Rule:  $(f \cdot g)' = f' \cdot g + f \cdot g'$

$$f = x, g = y$$

$$= \frac{d}{dy}(x)y + \frac{d}{dy}(y)x$$

$$\frac{d}{dy}(y) = 1$$

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$$= \frac{d}{dy}(x)y + 1 \cdot x$$

Simplify

$$= y \frac{d}{dy}(x) + x$$

$$= \cos(u) \left( y \frac{d}{dy}(x) + x \right)$$

Substitute back  $u = xy$

$$= \cos(xy) \left( y \frac{d}{dy}(x) + x \right)$$

$$\frac{d}{dy}\left(\frac{1}{5}\right) = 0$$

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$$\cos(xy) \left( y \frac{d}{dy}(x) + x \right) = 0$$

For convenience, write  $\frac{d}{dy}(x)$  as  $x'$

$$\cos(xy) \left( yx' + x \right) = 0$$

$$\text{Isolate } x': \quad x' = -\frac{x}{y}$$

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$$x' = -\frac{x}{y}$$

Write  $x'$  as  $\frac{d}{dy}(x)$

$$\frac{d}{dy}(x) = -\frac{x}{y}$$

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Use implicit differentiation to find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  using the following equation.

$$5x^2 + 3y^2 = 16$$

To find  $\frac{dy}{dx}$  using implicit differentiation, first differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a function of  $x$  and using the chain rule.

$$\begin{aligned} 5x^2 + 3y^2 &= 16 \\ \frac{d}{dx}(5x^2) + \frac{d}{dx}(3y^2) &= \frac{d}{dx}(16) \end{aligned}$$

Use the constant multiple rule and the power rule to find  $\frac{d}{dx}(5x^2)$ .

$$10x + \frac{d}{dx}(3y^2) = \frac{d}{dx}(16)$$

Again use the constant multiple rule and the power rule to find  $\frac{d}{dx}(3y^2)$ . Treat  $y$  as a differentiable function of  $x$ .

$$10x + 6y \frac{dy}{dx} = \frac{d}{dx}(16)$$

Note that the right side of the original equation is the constant function, 16, and the derivative of a constant is 0.

$$10x + 6y \frac{dy}{dx} = 0$$

Now solve the equation  $10x + 6y \frac{dy}{dx} = 0$  for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = -\frac{5x}{3y}$$

Therefore,  $\frac{dy}{dx} = -\frac{5x}{3y}$ .

To find  $\frac{d^2y}{dx^2}$ , differentiate the expression for  $\frac{dy}{dx}$  using the quotient rule.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{5x}{3y}\right) \\ &= -\frac{d}{dx}\left(\frac{5x}{3y}\right) \quad \text{Use the constant multiple rule.} \\ &= -\frac{15y - 5x\left(3\frac{dy}{dx}\right)}{9y^2} \quad \text{Use the quotient rule.} \end{aligned}$$

Substitute  $-\frac{5x}{3y}$  for  $\frac{dy}{dx}$ .

$$\begin{aligned} -\frac{15y - 5x\left(3\frac{dy}{dx}\right)}{9y^2} &= -\frac{15y - 15x\left(-\frac{5x}{3y}\right)}{9y^2} \\ &= -\frac{15y \cdot y + \frac{25x^2}{y} \cdot y}{9y^2 \cdot y} \quad \text{Multiply the numerator and denominator by } y. \\ &= -\frac{15y^2 + 25x^2}{9y^3} \quad \text{Simplify.} \end{aligned}$$

$$\text{Therefore, } \frac{d^2y}{dx^2} = -\frac{15y^2 + 25x^2}{9y^3}.$$



## Solution

Implicit Derivative  $\frac{d^2y}{dx^2}$  of  $x^2 + 2y^2 = 1$ :  $-\frac{2y^2 + x^2}{4y^3}$

## Steps

$$x^2 + 2y^2 = 1$$

Treat  $y$  as  $y(x)$

Implicit Derivative  $\frac{dy}{dx}$  of  $x^2 + 2y^2 = 1$ :  $-\frac{x}{2y}$

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$$x^2 + 2y^2 = 1$$

Differentiate both sides of the equation with respect to  $x$

$$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2 + 2y^2) = 2x + 4y\frac{d}{dx}(y)$$

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$$\frac{d}{dx}(1) = 0$$

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$$2x + 4y\frac{d}{dx}(y) = 0$$

For convenience, write  $\frac{d}{dx}(y)$  as  $y'$

$$2x + 4yy' = 0$$

Isolate  $y'$ :  $y' = -\frac{x}{2y}$

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$$y' = -\frac{x}{2y}$$

Write  $y'$  as  $\frac{d}{dx}(y)$

$$\frac{d}{dx}(y) = -\frac{x}{2y}$$

$$\frac{d}{dx}(y) = -\frac{x}{2y}$$

Implicit Derivative  $\frac{d^2y}{dx^2}$  of  $\frac{d}{dx}(y) = -\frac{x}{2y}$ :  $-\frac{y - x \frac{d}{dx}(y)}{2y^2}$

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Differentiate both sides of the equation with respect to  $x$

$$\frac{d^2}{dx^2}(y) = \frac{d}{dx}\left(-\frac{x}{2y}\right)$$

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$$\frac{d}{dx}\left(-\frac{x}{2y}\right) = -\frac{y - x \frac{d}{dx}(y)}{2y^2}$$

$$\frac{d^2}{dx^2}(y) = -\frac{y - x \frac{d}{dx}(y)}{2y^2}$$

$$\frac{d^2}{dx^2}(y) = -\frac{y - x \frac{d}{dx}(y)}{2y^2}$$

Subsstitue  $\frac{d}{dx}(y) = -\frac{x}{2y}$

$$\frac{d^2}{dx^2}(y) = -\frac{y - x\left(-\frac{x}{2y}\right)}{2y^2}$$

Simplify  $-\frac{y - x\left(-\frac{x}{2y}\right)}{2y^2} = -\frac{2y^2 + x^2}{4y^3}$

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$$-\frac{y - x\left(-\frac{x}{2y}\right)}{2y^2}$$

Apply rule  $-(-a) = a$

$$= -\frac{y + x \frac{x}{2y}}{2y^2}$$

$$x \frac{d^2y}{dx^2} = \frac{x^2}{2y}$$

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$$= -\frac{y + \frac{x^2}{2y}}{2y^2}$$

$$\text{Join } y + \frac{x^2}{2y}: \quad \frac{2y^2 + x^2}{2y}$$

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$$= -\frac{\frac{2y^2 + x^2}{2y}}{2y^2}$$

$$\text{Simplify } \frac{\frac{2y^2 + x^2}{2y}}{2y^2}: \quad \frac{2y^2 + x^2}{4y^2 y}$$

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$$= -\frac{2y^2 + x^2}{4y^2 y}$$

$$4yy^2 = 4y^3$$

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$$= -\frac{2y^2 + x^2}{4y^3}$$

$$\frac{d^2}{dx^2}(y) = -\frac{2y^2 + x^2}{4y^3}$$

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If  $x^3 + y^3 = 56$ , find the value of  $\frac{d^2y}{dx^2}$  at the point  $(-2, 4)$ .

To use implicit differentiation, first differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a differentiable function of  $x$ .

Use implicit differentiation and the rules of differentiation to find  $\frac{dy}{dx}$ .

$$x^3 + y^3 = 56$$
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(56)$$

To find  $\frac{d}{dx}(x^3)$ , use the Power Rule for Positive Integers.

$$\frac{d}{dx}(x^3) = 3x^2$$

To find  $\frac{d}{dx}(y^3)$ , use implicit differentiation and the Power Rule for Positive Integers.

$$\frac{d}{dx}(y^3) = 3y^2 \frac{dy}{dx}$$

To find  $\frac{d}{dx}(56)$ , use the Derivative of a Constant Function Rule.

$$\frac{d}{dx}(56) = 0$$

Simplify.

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(56)$$
$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

Now, collect the terms with  $\frac{dy}{dx}$  on one side of the equation.

$$3y^2 \frac{dy}{dx} = -3x^2$$

Solve for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = \frac{-x^2}{y^2}$$

Use implicit differentiation and the Derivative Quotient Rule to find  $\frac{d^2y}{dx^2}$ .

$$\frac{d^2y}{dx^2} = \frac{(y^2)(-2x) - (-x^2) \left( 2y \frac{dy}{dx} \right)}{(y^2)^2}$$

Simplify.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2xy^2 + 2x^2y \frac{dy}{dx}}{y^4} \\ &= \frac{-2xy + 2x^2 \frac{dy}{dx}}{y^3}\end{aligned}$$

Substitute  $\frac{dy}{dx}$  to express  $\frac{d^2y}{dx^2}$  in terms of x and y.

$$\frac{d^2y}{dx^2} = \frac{-2xy + 2x^2 \left( -\frac{x^2}{y^2} \right)}{y^3}$$

Simplify.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2xy + 2x^2 \left( -\frac{x^2}{y^2} \right)}{y^3} \cdot \frac{y^2}{y^2} \\ &= \frac{-2xy^3 - 2x^4}{y^5}\end{aligned}$$

Factor the numerator and simplify using the original equation  $x^3 + y^3 = 56$ .

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2xy^3 - 2x^4}{y^5} \\ &= \frac{-2x(y^3 + x^3)}{y^5} \\ &= \frac{-112x}{y^5}\end{aligned}$$

Finally, evaluate  $\frac{d^2y}{dx^2}$  at  $(x,y) = (-2,4)$ .

$$\frac{d^2y}{dx^2} \Big|_{(-2,4)} = \frac{-112(-2)}{(4)^5} = \frac{7}{32}$$

Thus, the value of  $\frac{d^2y}{dx^2}$  at the point  $(-2,4)$  is  $\frac{7}{32}$ .

The original 24 m edge length  $x$  of a cube decreases at the rate of 6 m/min.

- a. When  $x = 3$  m, at what rate does the cube's surface area change?
- b. When  $x = 3$  m, at what rate does the cube's volume change?

- a. When  $x = 3$  m, at what rate does the cube's surface area change?

Begin by identifying the variables. Let  $t$  represent time,  $x$  represent the edge length of the cube at time  $t$ ,  $S$  represent the surface area of the cube at time  $t$ , and  $V$  represent the volume of the cube at time  $t$ .

Since a function relating the surface area of a cube to the length of one of its edges is not given, set up a function relating a cube's surface area,  $S$ , to its edge length,  $x$ .

$$S = 6x^2$$

Assume that  $S$  and  $x$  are differentiable functions of  $t$ . Differentiate the function  $S = 6x^2$  with respect to time. Remember that  $x$  is a function of time. Therefore, applying the chain rule,  $\frac{dS}{dt} = \frac{dS}{dx} \frac{dx}{dt}$ .

$$\frac{dS}{dt} = 12x \frac{dx}{dt}$$

The derivative  $\frac{dx}{dt}$  and the specific length of the edge,  $x = 3$  m, are needed to determine the rate at which the cube's surface area is changing. Determine  $\frac{dx}{dt}$ .

$$\frac{dx}{dt} = -6 \text{ m/min}$$

Use the values obtained in the previous step to find  $\frac{dS}{dt}$ .

$$\begin{aligned}\frac{dS}{dt} &= 12x \frac{dx}{dt} \\ &= 12(3)(-6) \\ &= -216 \text{ m}^2/\text{min}\end{aligned}$$

Thus the surface area of the cube is changing at a rate of  $-216 \text{ m}^2/\text{min}$  when the edge is 3 m and the length of the edge is decreasing at a rate of 6 m/min.

- b. When  $x = 3$  m, at what rate does the cube's volume change?

Since a function relating the volume of a cube to the length of one of its edges is not given, begin by setting up a function relating a cube's volume,  $V$ , to its edge length,  $x$ .

$$V = x^3$$

Differentiate the function  $V = x^3$  with respect to time. Remember that  $x$  is a function of time and apply the chain rule to obtain the following equation.

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

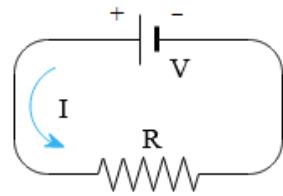
Recall that  $\frac{dx}{dt} = -6 \text{ m/min}$ . Determine how fast the volume is changing when  $x = 3$  m.

Substitute these values into the function for  $\frac{dV}{dt}$  to find how fast the volume is changing when  $x = 3$  m.

$$\begin{aligned}\frac{dV}{dt} &= 3x^2 \frac{dx}{dt} \\ &= 3(3)^2(-6) \\ &= -162 \text{ m}^3/\text{min}\end{aligned}$$

Thus, the volume of the cube is changing at a rate of  $-162 \text{ m}^3/\text{min}$  when the edge is 3 m and the length of the edge is decreasing at a rate of 6 m/min.

The voltage  $V$  (volts), current  $I$  (amperes), and resistance  $R$  (ohms) of an electric circuit like the one shown here are related by the equation  $V=IR$ . Suppose that  $V$  is increasing at the rate of 5 volt/sec while  $I$  is decreasing at the rate of  $\frac{1}{6}$  amp/sec. Let  $t$  denote time in seconds. Answer the following questions.



- a. What is the value of  $\frac{dV}{dt}$ ?

$\frac{dV}{dt}$  is the rate of change in the voltage,  $V$ , as given in the problem statement.

$$\frac{dV}{dt} = 5 \text{ volt/sec}$$

- b. What is the value of  $\frac{dI}{dt}$ ?

$\frac{dI}{dt}$  is the rate of change in the current,  $I$ . Since the current is decreasing, the rate is negative.

$$\frac{dI}{dt} = -\frac{1}{6} \text{ amp/sec}$$

- c. What equation relates  $\frac{dR}{dt}$  to  $\frac{dV}{dt}$  and  $\frac{dI}{dt}$ ?

The equation  $V=IR$  relates the three variables. Therefore, differentiate this equation with respect to  $t$  using the chain rule, where  $V$ ,  $I$ , and  $R$  are all functions of  $t$ .

Use the product rule on the right side.

$$\frac{dV}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt}$$

Then solve  $\frac{dV}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt}$  for  $\frac{dR}{dt}$ .

$$I \frac{dR}{dt} = \frac{dV}{dt} - R \frac{dI}{dt}$$

$$\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - R \frac{dI}{dt} \right)$$

Since  $R$  was not given in the problem statement, use  $V=IR$  to find an expression for  $R$  in terms of  $V$  and  $I$ .

$$R = \frac{V}{I}$$

Substitute this expression for  $R$  in  $\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - R \frac{dI}{dt} \right)$ .

$$\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right)$$

- d. Find the rate at which  $R$  is changing when  $V=36$  volts and  $I=3$  amp. Is  $R$  increasing, or decreasing?

Substitute the known values into the rate equation.

$$\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right)$$

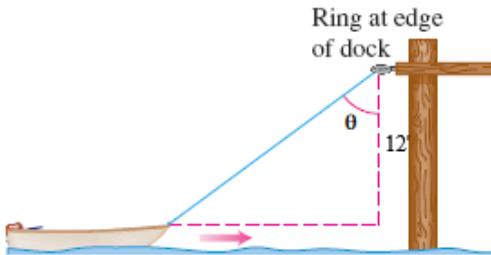
$$\frac{dR}{dt} = \frac{1}{3} \left( 5 - \frac{36}{3} \left( -\frac{1}{6} \right) \right)$$

Simplify the right side.

$$\frac{dR}{dt} = \frac{1}{3} \left( 5 - \frac{36}{3} \left( -\frac{1}{6} \right) \right) = \frac{7}{3} \text{ ohm/sec}$$

Since  $\frac{dR}{dt}$  is positive, the resistance is increasing.

A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 12 feet above the bow. The rope is hauled in at the rate of 1 ft/sec. Complete parts a. and b.

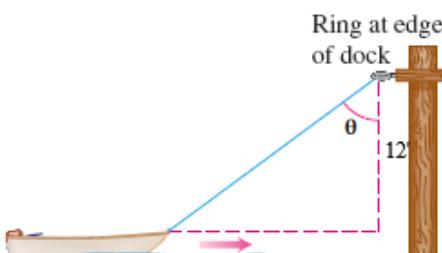


a. How fast is the boat approaching the dock when 15 ft of rope are out?

Name the variables and constants and label them in your diagram. Use t for time. Assume that all variables are differentiable functions of t.

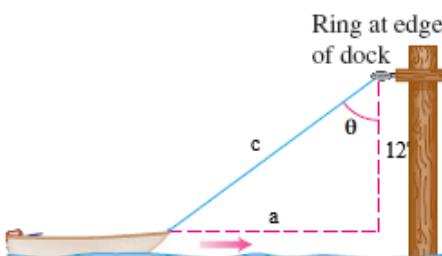
Note that the rope, the dock, and the water form a right triangle.

The distance from the top of the dock to the water is constant at 12 feet.



The horizontal distance from the boat to the dock is a variable. Name it a.

The length of rope (the distance from the dinghy to the top of the ring on the dock) is also a variable. Name it c.



Write down the numerical information in terms of the symbols already defined. Translate "The rope is hauled in at the rate of 1 ft/sec."

$$\frac{dc}{dt} = -1 \text{ ft/sec}$$

Next, translate, "How fast is the boat approaching the dock" using the symbols already defined. This is the quantity that part a. asks for.

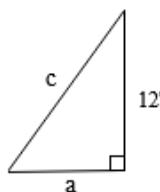
"How fast is the boat approaching the dock" asks for  $-\frac{da}{dt}$ .

Next, the phrase "when 15 ft of rope are out," means the moment when  $c = 15$ .

Now, write an equation that relates the variables a and c.

Use the Pythagorean theorem to write an equation.

$$a^2 + 12^2 = c^2$$



Next, differentiate the equation with respect to t.

$$\frac{d}{dt}(a^2) + \frac{d}{dt}(12^2) = \frac{d}{dt}(c^2)$$

To find  $\frac{d}{dt}(a^2)$ , use implicit differentiation and the power rule.

$$\frac{d}{dt}(a^2) = 2a \frac{da}{dt}$$

To find  $\frac{d}{dt}(12^2)$ , use the constant function rule.

$$\frac{d}{dt}(12^2) = 0$$

To find  $\frac{d}{dt}(c^2)$ , use implicit differentiation and the power rule.

$$\frac{d}{dt}(c^2) = 2c \frac{dc}{dt}$$

The results of the last three steps yield the following equation.

$$2a \frac{da}{dt} + 0 = 2c \frac{dc}{dt}$$

Since the quantity that we are looking for is  $\frac{da}{dt}$ , solve the equation for  $\frac{da}{dt}$ .

$$\frac{da}{dt} = \frac{c}{a} \frac{dc}{dt}$$

Next determine the values to substitute for  $a$ ,  $c$ , and  $\frac{dc}{dt}$ . Recall that the problem gives us the values  $\frac{dc}{dt} = -1$  and  $c = 15$ .

Find  $a$  by letting  $c = 15$  in the original equation,  $a^2 + 12^2 = c^2$ , and solving.

$$a^2 + 12^2 = 15^2 \quad \text{Substitute.}$$

$$a^2 = 15^2 - 12^2 \quad \text{Subtract } 12^2 \text{ from both sides}$$

$$\begin{aligned} a &= \pm \sqrt{81} && \text{Simplify and take the square root of both sides.} \\ &= \pm 9 \end{aligned}$$

Now substitute the values for  $a$ ,  $c$ , and  $\frac{dc}{dt}$  to find  $\frac{da}{dt}$ , the rate at which the dinghy is approaching the dock.

$$\frac{da}{dt} = \frac{c}{a} \frac{dc}{dt}$$

$$= \frac{15}{9}(-1) \quad \text{Substitute.}$$

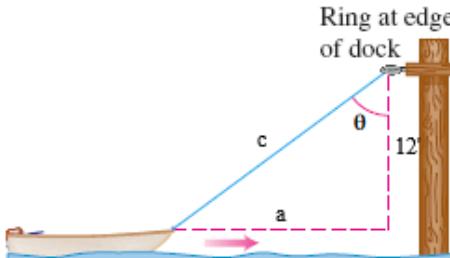
$$= -\frac{5}{3} \quad \text{Simplify.}$$

Thus, the distance between the dinghy and the dock is changing at a rate of  $-\frac{5}{3}$  ft/sec when 15 feet of rope are out.

b. At what rate is the angle  $\theta$  changing at this instant? (see the figure)

b. At what rate is the angle  $\theta$  changing at this instant? (see the figure)

Notice that the angle  $\theta$  is already labeled in the diagram.



Translate "At what rate is the angle changing at this instant?" This is the quantity that part b. asks for.

"At what rate is the angle changing at this instant" asks for  $\frac{d\theta}{dt}$ .

Write an equation relating  $\theta$ ,  $c$ , and the constant leg of the right triangle.

$$\cos \theta = \frac{12}{c} \quad \text{Use cosine to relate the constant leg of the triangle and } c.$$

Next, differentiate the equation with respect to  $t$ .

$$\frac{d}{dt} \cos \theta = \frac{d}{dt} \left( \frac{12}{c} \right)$$

To find  $\frac{d}{dt}(\cos \theta)$ , use implicit differentiation.

$$\frac{d}{dt}(\cos \theta) = -\sin \theta \frac{d\theta}{dt}$$

To find  $\frac{d}{dt} \left( \frac{12}{c} \right)$ , use implicit differentiation and the power rule.

$$\frac{d}{dt} \left( \frac{12}{c} \right) = -\frac{12}{c^2} \cdot \frac{dc}{dt}$$

The results from the last three steps yield the following equation.

$$-\sin \theta \frac{d\theta}{dt} = -\frac{12}{c^2} \cdot \frac{dc}{dt}$$

Solve the equation for  $\frac{d\theta}{dt}$ .

$$\frac{d\theta}{dt} = \frac{12}{c^2 \sin \theta} \cdot \frac{dc}{dt}$$

Next determine the values for  $a$ ,  $c$ ,  $\sin \theta$ , and  $\frac{dc}{dt}$ . Recall that the problem gives the values  $c = 15$ ,  $\frac{dc}{dt} = -1 \frac{\text{ft}}{\text{sec}}$ . Also note that in part a.  $a$  was found to be 9.

Find  $\sin \theta$  by letting  $c = 15$ .

$$\sin \theta = \frac{9}{15}$$

Now substitute the values for  $a$ ,  $c$ ,  $\sin \theta$ , and  $\frac{dc}{dt}$  to find the rate at which the angle is changing at this instant.

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{12}{c^2 \sin \theta} \cdot \frac{dc}{dt} \\ &= \frac{12}{(15)^2 \frac{9}{15}} \cdot (-1) \quad \text{Substitute.}\end{aligned}$$

Now, simplify the expression.

$$\begin{aligned}\frac{12}{(15)^2 \frac{9}{15}} \cdot (-1) &= \frac{(-1)(12)}{(15)(9)} \quad \text{Simplify.} \\ &= -\frac{4}{45} \text{ rad/sec} \quad \text{Multiply and simplify.}\end{aligned}$$

Therefore, the angle is changing at a rate of  $-\frac{4}{45}$  rad/sec.