

Find  $y''$  for  $y = \left(1 + \frac{2}{x}\right)^3$ .

If  $n$  is a positive or negative integer and  $f(u) = u^n$ , the power rule says that  $f'(u) = nu^{n-1}$ . If  $u$  is a differentiable function of  $x$ , then use the chain rule to extend this to the power chain rule.

$$\frac{d}{dx}f(u) = nu^{n-1} \frac{du}{dx}$$

Use the power chain rule to find  $y'$ .

$$y = \left(1 + \frac{2}{x}\right)^3$$

$$y' = 3\left(1 + \frac{2}{x}\right)^2 \frac{d}{dx}\left(1 + \frac{2}{x}\right)$$

To find  $\frac{d}{dx}\left(1 + \frac{2}{x}\right)$ , use the rules of differentiation.

$$\frac{d}{dx}\left(1 + \frac{2}{x}\right) = \left(-\frac{2}{x^2}\right)$$

$$\text{Thus, } y' = 3\left(1 + \frac{2}{x}\right)^2 \left(-\frac{2}{x^2}\right).$$

Use the derivative product rule to find  $y''$ .

$$y' = 3\left(1 + \frac{2}{x}\right)^2 \left(-\frac{2}{x^2}\right)$$

$$y'' = 3\left(1 + \frac{2}{x}\right)^2 \frac{d}{dx}\left(-\frac{2}{x^2}\right) + \left(-\frac{2}{x^2}\right) \frac{d}{dx}\left(3\left(1 + \frac{2}{x}\right)^2\right)$$

To find  $\frac{d}{dx}\left(-\frac{2}{x^2}\right)$ , use the power rule for negative integers.

$$\frac{d}{dx}\left(-\frac{2}{x^2}\right) = \left(\frac{4}{x^3}\right)$$

To find  $\frac{d}{dx}\left(3\left(1 + \frac{2}{x}\right)^2\right)$ , use the power chain rule.

$$\begin{aligned}\frac{d}{dx}3\left(1 + \frac{2}{x}\right)^2 &= 6\left(1 + \frac{2}{x}\right) \frac{d}{dx}\left(1 + \frac{2}{x}\right) \\ &= 6\left(1 + \frac{2}{x}\right) \left(-\frac{2}{x^2}\right)\end{aligned}$$

Simplify.

$$\begin{aligned}y'' &= 3\left(1 + \frac{2}{x}\right)^2 \frac{d}{dx}\left(-\frac{2}{x^2}\right) + \left(-\frac{2}{x^2}\right) \frac{d}{dx}\left(3\left(1 + \frac{2}{x}\right)^2\right) \\ &= 3\left(1 + \frac{2}{x}\right)^2 \left(\frac{4}{x^3}\right) + \left(-\frac{2}{x^2}\right) 6\left(1 + \frac{2}{x}\right) \left(-\frac{2}{x^2}\right) \\ &= \frac{12}{x^3} \left(1 + \frac{2}{x}\right)^2 + \frac{24}{x^4} \left(1 + \frac{2}{x}\right) \\ \text{Thus, } y'' &= \frac{12}{x^3} \left(1 + \frac{2}{x}\right) \left(1 + \frac{4}{x}\right).\end{aligned}$$



Solution

$$\frac{d}{dx}\left(\left(5 + \frac{1}{x}\right)^3\right) = -\frac{3(5x+1)^2}{x^4}$$

## Steps

$$\frac{d}{dx}\left(\left(5 + \frac{1}{x}\right)^3\right)$$

Apply the chain rule:  $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$f = u^3, \quad u = \left(5 + \frac{1}{x}\right)$$

$$= \frac{d}{du}(u^3) \frac{d}{dx}\left(5 + \frac{1}{x}\right)$$

$$\frac{d}{du}(u^3) = 3u^2$$

Show Steps +

$$\frac{d}{dx}\left(5 + \frac{1}{x}\right) = -\frac{1}{x^2}$$

Show Steps +

$$= 3u^2\left(-\frac{1}{x^2}\right)$$

Substitute back  $u = \left(5 + \frac{1}{x}\right)$

$$= 3\left(5 + \frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)$$

Simplify  $3\left(5 + \frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)$ :  $-\frac{3(5x+1)^2}{x^4}$

Hide Steps -

$$3\left(5 + \frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)$$

Remove parentheses:  $(-a) = -a$

$$= -3\left(5 + \frac{1}{x}\right)^2\frac{1}{x^2}$$

Multiply fractions:  $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$

$$= -\frac{1 \cdot 3\left(5 + \frac{1}{x}\right)^2}{x^2}$$

Multiply the numbers:  $1 \cdot 3 = 3$

$$= -\frac{3\left(\frac{1}{x} + 5\right)^2}{x^2}$$

$$\left(5 + \frac{1}{x}\right)^2 = \frac{(5x + 1)^2}{x^2}$$

Hide Steps 

$$\left(5 + \frac{1}{x}\right)^2$$

Join  $5 + \frac{1}{x}$ :  $\frac{5x + 1}{x}$

Show Steps 

$$= \left(\frac{5x + 1}{x}\right)^2$$

Apply exponent rule:  $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

$$= \frac{(5x + 1)^2}{x^2}$$

$$= -\frac{3 \cdot \frac{(5x + 1)^2}{x^2}}{x^2}$$

Multiply  $3 \cdot \frac{(5x + 1)^2}{x^2}$ :  $\frac{3(5x + 1)^2}{x^2}$

Hide Steps 

$$3 \cdot \frac{(5x + 1)^2}{x^2}$$

Multiply fractions:  $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$

$$= \frac{(5x + 1)^2 \cdot 3}{x^2}$$

$$= -\frac{\frac{3(5x + 1)^2}{x^2}}{x^2}$$

Apply the fraction rule:  $\frac{\frac{b}{c}}{a} = \frac{b}{c \cdot a}$

$$= -\frac{3(5x+1)^2}{x^2 x^2}$$

$$x^2 x^2 = x^4$$

Show Steps 

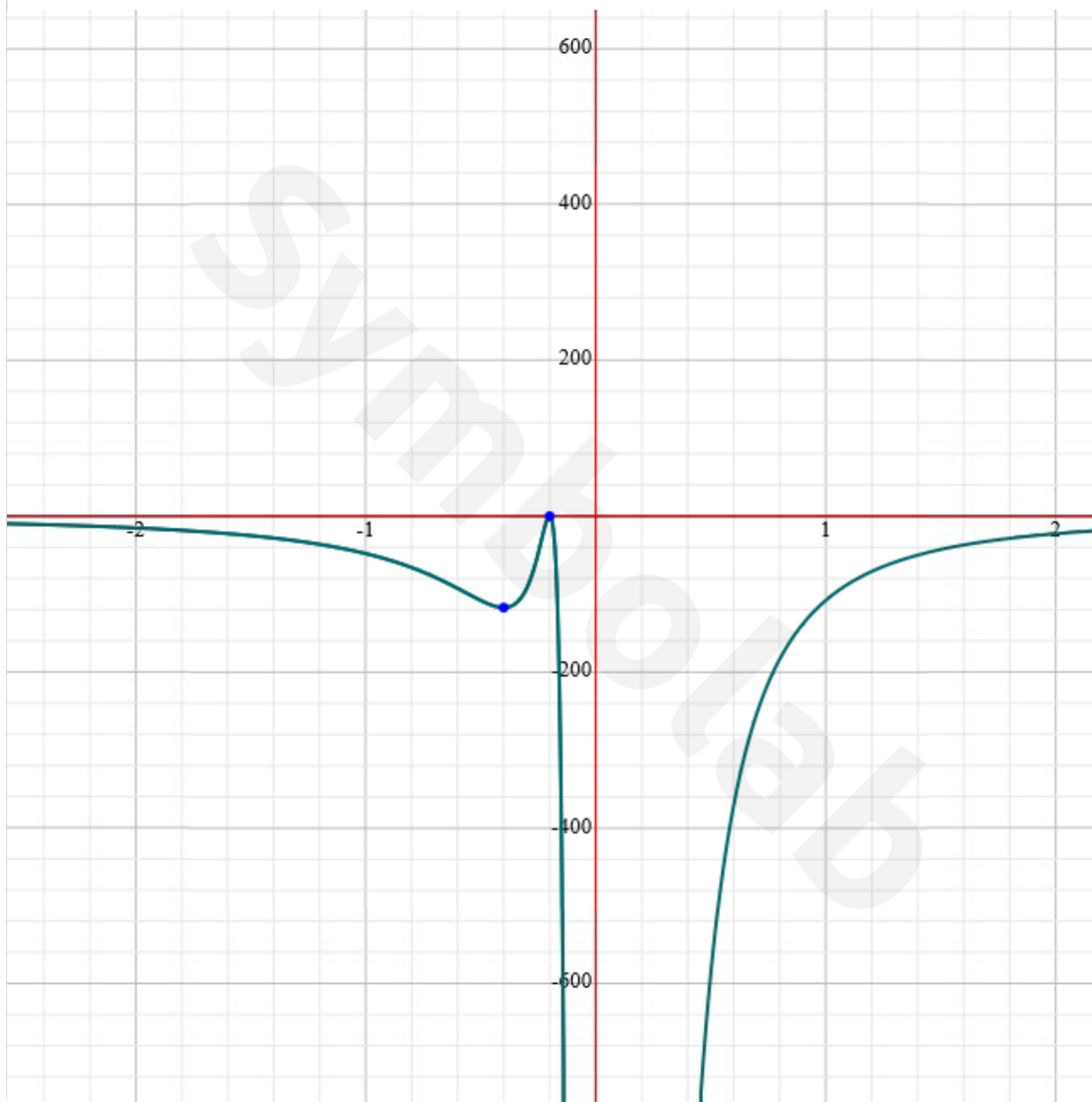
$$= -\frac{3(5x+1)^2}{x^4}$$

$$= -\frac{3(5x+1)^2}{x^4}$$

[Click to practice Chain Rule](#)

# Graph

Plotting:  $-\frac{3(5x+1)^2}{x^4}$



Find the value of  $(f \circ g)'$  at the given value.

$$f(u) = u^7 + 3, \quad u = g(x) = \sqrt{x}, \quad x = 1$$

The Chain Rule states if  $u$  is differentiable at the point  $u = g(x)$  and  $g(x)$  is differentiable at  $x$ , then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$ , and  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ . In Leibniz's notation, if  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \quad \text{where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

Use the Chain Rule to find  $(f \circ g)'$ .

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{d}{du}(u^7 + 3) \cdot \frac{d}{dx}(\sqrt{x})\end{aligned}$$

To find  $\frac{d}{du}(u^7 + 3)$  and  $\frac{d}{dx}(\sqrt{x})$ , use the rules of differentiation.

$$\frac{d}{du}(u^7 + 3) = 7u^6, \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-1/2}$$

Now, substitute  $u$  in  $f'(g(x))$  and simplify.

$$\begin{aligned}&= \frac{d}{du}(u^7 + 3) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= 7u^6 \cdot \frac{1}{2}x^{-1/2} \\ &= 7(\sqrt{x})^6 \cdot \frac{1}{2}x^{-1/2} \\ &= \frac{7}{2}x^{5/2}\end{aligned}$$

$$\text{Thus, } (f \circ g)'(x) = \frac{7}{2}x^{5/2}.$$

Now, to find  $(f \circ g)'(1)$ , substitute 1 for  $x$  and simplify.

$$(f \circ g)'(1) = \frac{7}{2}$$

$$\text{Thus, } (f \circ g)'(1) = \frac{7}{2}.$$



## Solution

$$\frac{d}{dx}((\sqrt{x})^7) = \frac{7x^{\frac{5}{2}}}{2}$$

## Steps

$$\frac{d}{dx}((\sqrt{x})^7)$$

Apply the chain rule:  $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$f = u^7, u = (\sqrt{x})$$

$$= \frac{d}{du}(u^7) \frac{d}{dx}(\sqrt{x})$$

$$\frac{d}{du}(u^7) = 7u^6$$

Show Steps +

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Show Steps +

$$= 7u^6 \frac{1}{2\sqrt{x}}$$

Substitute back  $u = (\sqrt{x})$

$$= 7(\sqrt{x})^6 \frac{1}{2\sqrt{x}}$$

$$\text{Simplify } 7(\sqrt{x})^6 \frac{1}{2\sqrt{x}}: \frac{7x^{\frac{5}{2}}}{2}$$

Hide Steps -

$$7(\sqrt{x})^6 \frac{1}{2\sqrt{x}}$$

$$(\sqrt{x})^6 = x^3$$

Hide Steps -

$$(\sqrt{x})^6$$

Apply radical rule:  $\sqrt{a} = a^{\frac{1}{2}}$

$$= \left(x^{\frac{1}{2}}\right)^6$$

Apply exponent rule:  $(a^b)^c = a^{bc}$

$$= x^{\frac{1}{2} \cdot 6}$$

$$\frac{1}{2} \cdot 6 = 3$$

Show Steps 

$$= x^3$$

$$= 7 \cdot \frac{1}{2\sqrt{x}} x^3$$

Multiply fractions:  $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$

$$= \frac{1 \cdot 7x^3}{2\sqrt{x}}$$

Multiply the numbers:  $1 \cdot 7 = 7$

$$= \frac{7x^3}{2\sqrt{x}}$$

Apply radical rule:  $\sqrt[n]{a} = a^{\frac{1}{n}}$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$= \frac{7x^3}{2x^{\frac{1}{2}}}$$

Apply exponent rule:  $\frac{x^a}{x^b} = x^{a-b}$

$$\frac{x^3}{x^{\frac{1}{2}}} = x^{3-\frac{1}{2}}$$

$$= \frac{7x^{-\frac{1}{2}+3}}{2}$$

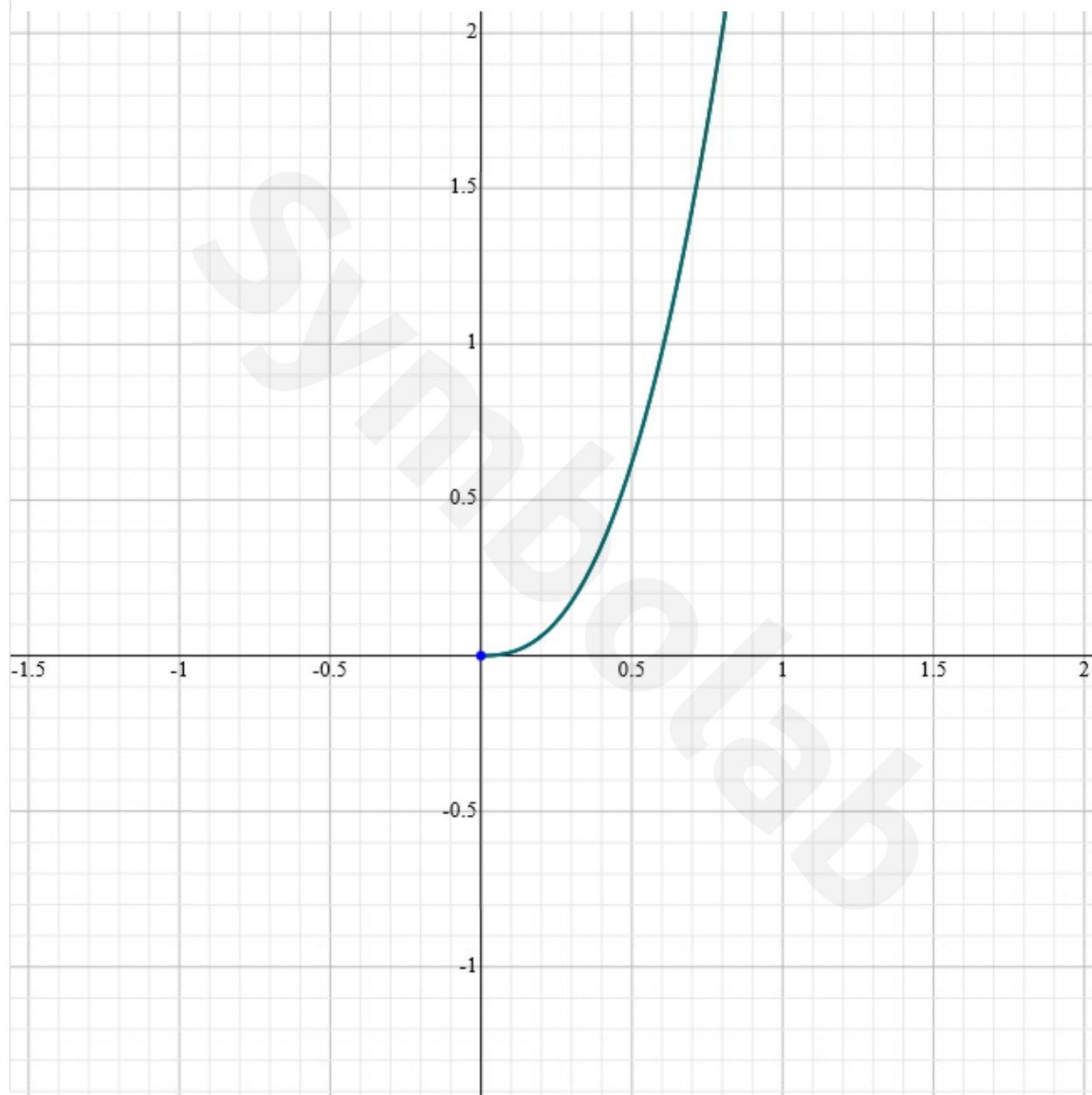
Subtract the numbers:  $3 - \frac{1}{2} = \frac{5}{2}$

$$= \frac{7x^{\frac{5}{2}}}{2}$$

$$= \frac{7x^{\frac{5}{2}}}{2}$$

[Click to practice Chain Rule](#)

## Graph

Plotting:  $\frac{7x^{\frac{5}{2}}}{2}$ 

Find the value of  $(f \circ g)'$  at the given value of  $x$ .

$$f(u) = \frac{6u}{u^2 + 2}, \quad u = g(x) = 3x^2 + 5x + 2, \quad x = 0$$

The Chain Rule states if  $u$  is differentiable at the point  $u = g(x)$  and  $g(x)$  is differentiable at  $x$ , then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$ , and  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ . In Leibniz's notation, if  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \quad \text{where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

Use the Chain Rule to find  $(f \circ g)'$ .

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{d}{du} \left( \frac{6u}{u^2 + 2} \right) \cdot \frac{d}{dx} (3x^2 + 5x + 2)\end{aligned}$$

To find  $\frac{d}{du} \left( \frac{6u}{u^2 + 2} \right)$  and  $\frac{d}{dx} (3x^2 + 5x + 2)$ , use the rules of differentiation.

$$\frac{d}{du} \left( \frac{6u}{u^2 + 2} \right) = \frac{12 - 6u^2}{(u^2 + 2)^2}, \quad \frac{d}{dx} (3x^2 + 5x + 2) = 6x + 5$$

Now, substitute  $u$  in  $f'(g(x))$ .

$$\begin{aligned}&= \frac{d}{du} \left( \frac{6u}{u^2 + 2} \right) \cdot \frac{d}{dx} (3x^2 + 5x + 2) \\ &= \frac{12 - 6u^2}{(u^2 + 2)^2} \cdot (6x + 5) \\ &= \frac{12 - 6(3x^2 + 5x + 2)^2}{((3x^2 + 5x + 2)^2 + 2)^2} \cdot (6x + 5)\end{aligned}$$

Instead of simplifying further, to find  $(f \circ g)'(0)$ , substitute 0 for  $x$  and simplify.

$$\begin{aligned}(f \circ g)'(x) &= \frac{12 - 6(3x^2 + 5x + 2)^2}{((3x^2 + 5x + 2)^2 + 2)^2} \cdot (6x + 5) \\ (f \circ g)'(0) &= \frac{12 - 6(3 \cdot 0^2 + 5 \cdot 0 + 2)^2}{((3 \cdot 0^2 + 5 \cdot 0 + 2)^2 + 2)^2} \cdot (6 \cdot 0 + 5) \\ &= -\frac{5}{3}\end{aligned}$$

$$\text{Thus, } (f \circ g)'(0) = -\frac{5}{3}.$$

Use implicit differentiation to find  $\frac{dy}{dx}$ .

$$x = \cot y$$

To use implicit differentiation, first differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a differentiable function of  $x$ .

Use the rules of differentiation.

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cot y)$$

To find  $\frac{d}{dx}(x)$ , use implicit differentiation and the Power Rule for Positive Integers.

$$\frac{d}{dx}(x) = 1$$

To find  $\frac{d}{dx}(\cot y)$ , use implicit differentiation and the definition of the derivative of the cotangent function.

$$\frac{d}{dx}(\cot y) = (-\csc^2 y) \frac{dy}{dx}$$

The terms with  $\frac{dy}{dx}$  are already collected on one side of the equation.

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cot y)$$

$$1 = (-\csc^2 y) \frac{dy}{dx}$$

Now, solve for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\sin^2 y$$



## Solution

Implicit Derivative  $\frac{dx}{dy}$  of  $\sin(xy) = \frac{1}{5}$ :  $-\frac{x}{y}$

### Steps

$$\sin(xy) = \frac{1}{5}$$

Treat  $x$  as  $x(y)$

Differentiate both sides of the equation with respect to  $y$

$$\frac{d}{dy}(\sin(xy)) = \frac{d}{dy}\left(\frac{1}{5}\right)$$

$$\frac{d}{dy}(\sin(xy)) = \cos(xy) \left( y \frac{d}{dy}(x) + x \right)$$

Hide Steps

$$\frac{d}{dy}(\sin(xy))$$

Apply the chain rule:  $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$$f = \sin(u), \quad u = xy$$

$$= \frac{d}{du}(\sin(u)) \frac{d}{dy}(xy)$$

$$\frac{d}{du}(\sin(u)) = \cos(u)$$

Show Steps

$$\frac{d}{dy}(xy) = y \frac{d}{dy}(x) + x$$

Hide Steps

$$\frac{d}{dy}(xy)$$

Apply the Product Rule:  $(f \cdot g)' = f' \cdot g + f \cdot g'$

$$f = x, \quad g = y$$

$$= \frac{d}{dy}(x)y + \frac{d}{dy}(y)x$$

$$\frac{d}{dy}(y) = 1$$

Show Steps 

$$= \frac{d}{dy}(x)y + 1 \cdot x$$

Simplify

$$= y \frac{d}{dy}(x) + x$$

$$= \cos(u) \left( y \frac{d}{dy}(x) + x \right)$$

Substitute back  $u = xy$

$$= \cos(xy) \left( y \frac{d}{dy}(x) + x \right)$$

$$\frac{d}{dy}\left(\frac{1}{5}\right) = 0$$

Show Steps 

$$\cos(xy) \left( y \frac{d}{dy}(x) + x \right) = 0$$

For convenience, write  $\frac{d}{dy}(x)$  as  $x'$

$$\cos(xy) (yx' + x) = 0$$

$$\text{Isolate } x' : x' = -\frac{x}{y}$$

Show Steps 

$$x' = -\frac{x}{y}$$

Write  $x'$  as  $\frac{d}{dy}(x)$

$$\frac{d}{dy}(x) = -\frac{x}{y}$$

[Click to practice Implicit Derivatives](#)

Use implicit differentiation to find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  using the following equation.

$$5x^2 + 3y^2 = 16$$

To find  $\frac{dy}{dx}$  using implicit differentiation, first differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a function of  $x$  and using the chain rule.

$$\begin{aligned} 5x^2 + 3y^2 &= 16 \\ \frac{d}{dx}(5x^2) + \frac{d}{dx}(3y^2) &= \frac{d}{dx}(16) \end{aligned}$$

Use the constant multiple rule and the power rule to find  $\frac{d}{dx}(5x^2)$ .

$$10x + \frac{d}{dx}(3y^2) = \frac{d}{dx}(16)$$

Again use the constant multiple rule and the power rule to find  $\frac{d}{dx}(3y^2)$ . Treat  $y$  as a differentiable function of  $x$ .

$$10x + 6y \frac{dy}{dx} = \frac{d}{dx}(16)$$

Note that the right side of the original equation is the constant function, 16, and the derivative of a constant is 0.

$$10x + 6y \frac{dy}{dx} = 0$$

Now solve the equation  $10x + 6y \frac{dy}{dx} = 0$  for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = -\frac{5x}{3y}$$

Therefore,  $\frac{dy}{dx} = -\frac{5x}{3y}$ .

To find  $\frac{d^2y}{dx^2}$ , differentiate the expression for  $\frac{dy}{dx}$  using the quotient rule.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left( -\frac{5x}{3y} \right) \\ &= -\frac{d}{dx} \left( \frac{5x}{3y} \right) && \text{Use the constant multiple rule.} \\ &= -\frac{15y - 5x \left( 3 \frac{dy}{dx} \right)}{9y^2} && \text{Use the quotient rule.} \end{aligned}$$

Substitute  $-\frac{5x}{3y}$  for  $\frac{dy}{dx}$ .

$$\begin{aligned} -\frac{15y - 5x \left( 3 \frac{dy}{dx} \right)}{9y^2} &= -\frac{15y - 15x \left( -\frac{5x}{3y} \right)}{9y^2} \\ &= -\frac{15y \cdot y + \frac{25x^2}{y} \cdot y}{9y^2 \cdot y} && \text{Multiply the numerator and denominator by } y. \\ &= -\frac{15y^2 + 25x^2}{9y^3} && \text{Simplify.} \end{aligned}$$

$$\text{Therefore, } \frac{d^2y}{dx^2} = -\frac{15y^2 + 25x^2}{9y^3}.$$



## Solution

Implicit Derivative  $\frac{d^2y}{dx^2}$  of  $x^2 + 2y^2 = 1$ :  $-\frac{2y^2 + x^2}{4y^3}$

### Steps

$$x^2 + 2y^2 = 1$$

Treat  $y$  as  $y(x)$

Implicit Derivative  $\frac{dy}{dx}$  of  $x^2 + 2y^2 = 1$ :  $-\frac{x}{2y}$

Hide Steps

$$x^2 + 2y^2 = 1$$

Differentiate both sides of the equation with respect to  $x$

$$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2 + 2y^2) = 2x + 4y \frac{d}{dx}(y)$$

Show Steps

$$\frac{d}{dx}(1) = 0$$

Show Steps

$$2x + 4y \frac{d}{dx}(y) = 0$$

For convenience, write  $\frac{d}{dx}(y)$  as  $y'$

$$2x + 4yy' = 0$$

Isolate  $y'$ :  $y' = -\frac{x}{2y}$

Show Steps

$$y' = -\frac{x}{2y}$$

Write  $y'$  as  $\frac{d}{dx}(y)$

$$\frac{d}{dx}(y) = -\frac{x}{2y}$$

$$\frac{d}{dx}(y) = -\frac{x}{2y}$$

Hide Steps 

Implicit Derivative  $\frac{d^2y}{dx^2}$  of  $\frac{d}{dx}(y) = -\frac{x}{2y}$ :  $-\frac{y - x\frac{d}{dx}(y)}{2y^2}$

Differentiate both sides of the equation with respect to  $x$

$$\frac{d^2}{dx^2}(y) = \frac{d}{dx}\left(-\frac{x}{2y}\right)$$

Show Steps 

$$\frac{d}{dx}\left(-\frac{x}{2y}\right) = -\frac{y - x\frac{d}{dx}(y)}{2y^2}$$

$$\frac{d^2}{dx^2}(y) = -\frac{y - x\frac{d}{dx}(y)}{2y^2}$$

$$\frac{d^2}{dx^2}(y) = -\frac{y - x\frac{d}{dx}(y)}{2y^2}$$

Substitute  $\frac{d}{dx}(y) = -\frac{x}{2y}$

$$\frac{d^2}{dx^2}(y) = -\frac{y - x\left(-\frac{x}{2y}\right)}{2y^2}$$

Hide Steps 

Simplify  $-\frac{y - x\left(-\frac{x}{2y}\right)}{2y^2}$ :  $-\frac{2y^2 + x^2}{4y^3}$

$$-\frac{y - x\left(-\frac{x}{2y}\right)}{2y^2}$$

Apply rule  $-(-a) = a$

$$= -\frac{y + x\frac{x}{2y}}{2y^2}$$

$$x \frac{x}{2y} = \frac{x^2}{2y}$$

Show Steps 

$$= -\frac{y + \frac{x^2}{2y}}{2y^2}$$

$$\text{Join } y + \frac{x^2}{2y}: \frac{2y^2 + x^2}{2y}$$

Show Steps 

$$= -\frac{\frac{2y^2 + x^2}{2y}}{2y^2}$$

$$\text{Simplify } \frac{\frac{2y^2 + x^2}{2y}}{2y^2}: \frac{2y^2 + x^2}{4y^2y}$$

Show Steps 

$$= -\frac{2y^2 + x^2}{4y^2y}$$

$$4yy^2 = 4y^3$$

Show Steps 

$$= -\frac{2y^2 + x^2}{4y^3}$$

$$\frac{d^2}{dx^2}(y) = -\frac{2y^2 + x^2}{4y^3}$$

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If  $x^3 + y^3 = 56$ , find the value of  $\frac{d^2y}{dx^2}$  at the point  $(-2, 4)$ .

To use implicit differentiation, first differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a differentiable function of  $x$ .

Use implicit differentiation and the rules of differentiation to find  $\frac{dy}{dx}$ .

$$\begin{aligned}x^3 + y^3 &= 56 \\ \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(56)\end{aligned}$$

To find  $\frac{d}{dx}(x^3)$ , use the Power Rule for Positive Integers.

$$\frac{d}{dx}(x^3) = 3x^2$$

To find  $\frac{d}{dx}(y^3)$ , use implicit differentiation and the Power Rule for Positive Integers.

$$\frac{d}{dx}(y^3) = 3y^2 \frac{dy}{dx}$$

To find  $\frac{d}{dx}(56)$ , use the Derivative of a Constant Function Rule.

$$\frac{d}{dx}(56) = 0$$

Simplify.

$$\begin{aligned}\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(56) \\ 3x^2 + 3y^2 \frac{dy}{dx} &= 0\end{aligned}$$

Now, collect the terms with  $\frac{dy}{dx}$  on one side of the equation.

$$3y^2 \frac{dy}{dx} = -3x^2$$

Solve for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = \frac{-x^2}{y^2}$$

Use implicit differentiation and the Derivative Quotient Rule to find  $\frac{d^2y}{dx^2}$ .

$$\frac{d^2y}{dx^2} = \frac{(y^2)(-2x) - (-x^2)\left(2y \frac{dy}{dx}\right)}{(y^2)^2}$$

Simplify.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2xy^2 + 2x^2 y \frac{dy}{dx}}{y^4} \\ &= \frac{-2xy + 2x^2 \frac{dy}{dx}}{y^3}\end{aligned}$$

Substitute  $\frac{dy}{dx}$  to express  $\frac{d^2y}{dx^2}$  in terms of x and y.

$$\frac{d^2y}{dx^2} = \frac{-2xy + 2x^2 \left( -\frac{x^2}{y^2} \right)}{y^3}$$

Simplify.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2xy + 2x^2 \left( -\frac{x^2}{y^2} \right)}{y^3} \cdot \frac{y^2}{y^2} \\ &= \frac{-2xy^3 - 2x^4}{y^5}\end{aligned}$$

Factor the numerator and simplify using the original equation  $x^3 + y^3 = 56$ .

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2xy^3 - 2x^4}{y^5} \\ &= \frac{-2x(y^3 + x^3)}{y^5} \\ &= \frac{-112x}{y^5}\end{aligned}$$

Finally, evaluate  $\frac{d^2y}{dx^2}$  at  $(x,y) = (-2,4)$ .

$$\left. \frac{d^2y}{dx^2} \right|_{(-2,4)} = \frac{-112(-2)}{(4)^5} = \frac{7}{32}$$

Thus, the value of  $\frac{d^2y}{dx^2}$  at the point  $(-2,4)$  is  $\frac{7}{32}$ .

The original 24 m edge length  $x$  of a cube decreases at the rate of 6 m/min.

- When  $x=3$  m, at what rate does the cube's surface area change?
- When  $x=3$  m, at what rate does the cube's volume change?

- When  $x=3$  m, at what rate does the cube's surface area change?

Begin by identifying the variables. Let  $t$  represent time,  $x$  represent the edge length of the cube at time  $t$ ,  $S$  represent the surface area of the cube at time  $t$ , and  $V$  represent the volume of the cube at time  $t$ .

Since a function relating the surface area of a cube to the length of one of its edges is not given, set up a function relating a cube's surface area,  $S$ , to its edge length,  $x$ .

$$S = 6x^2$$

Assume that  $S$  and  $x$  are differentiable functions of  $t$ . Differentiate the function  $S = 6x^2$  with respect to time. Remember that  $x$  is a function of time. Therefore, applying the chain rule,  $\frac{dS}{dt} = \frac{dS}{dx} \frac{dx}{dt}$ .

$$\frac{dS}{dt} = 12x \frac{dx}{dt}$$

The derivative  $\frac{dx}{dt}$  and the specific length of the edge,  $x=3$  m, are needed to determine the rate at which the cube's surface area is changing. Determine  $\frac{dx}{dt}$ .

$$\frac{dx}{dt} = -6 \text{ m/min}$$

Use the values obtained in the previous step to find  $\frac{dS}{dt}$ .

$$\begin{aligned}\frac{dS}{dt} &= 12x \frac{dx}{dt} \\ &= 12(3)(-6) \\ &= -216 \text{ m}^2/\text{min}\end{aligned}$$

Thus the surface area of the cube is changing at a rate of  $-216 \text{ m}^2/\text{min}$  when the edge is 3 m and the length of the edge is decreasing at a rate of 6 m/min.

- When  $x=3$  m, at what rate does the cube's volume change?

Since a function relating the volume of a cube to the length of one of its edges is not given, begin by setting up a function relating a cube's volume,  $V$ , to its edge length,  $x$ .

$$V = x^3$$

Differentiate the function  $V = x^3$  with respect to time. Remember that  $x$  is a function of time and apply the chain rule to obtain the following equation.

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

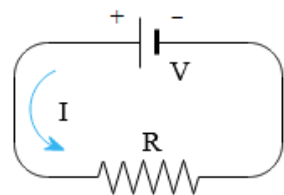
Recall that  $\frac{dx}{dt} = -6 \text{ m/min}$ . Determine how fast the volume is changing when  $x=3$  m.

Substitute these values into the function for  $\frac{dV}{dt}$  to find how fast the volume is changing when  $x=3$  m.

$$\begin{aligned}\frac{dV}{dt} &= 3x^2 \frac{dx}{dt} \\ &= 3(3)^2(-6) \\ &= -162 \text{ m}^3/\text{min}\end{aligned}$$

Thus, the volume of the cube is changing at a rate of  $-162 \text{ m}^3/\text{min}$  when the edge is 3 m and the length of the edge is decreasing at a rate of 6 m/min.

The voltage  $V$  (volts), current  $I$  (amperes), and resistance  $R$  (ohms) of an electric circuit like the one shown here are related by the equation  $V = IR$ . Suppose that  $V$  is increasing at the rate of 5 volt/sec while  $I$  is decreasing at the rate of  $\frac{1}{6}$  amp/sec. Let  $t$  denote time in seconds. Answer the following questions.



- a. What is the value of  $\frac{dV}{dt}$ ?

$\frac{dV}{dt}$  is the rate of change in the voltage,  $V$ , as given in the problem statement.

$$\frac{dV}{dt} = 5 \text{ volt/sec}$$

- b. What is the value of  $\frac{dI}{dt}$ ?

$\frac{dI}{dt}$  is the rate of change in the current,  $I$ . Since the current is decreasing, the rate is negative.

$$\frac{dI}{dt} = -\frac{1}{6} \text{ amp/sec}$$

- c. What equation relates  $\frac{dR}{dt}$  to  $\frac{dV}{dt}$  and  $\frac{dI}{dt}$ ?

The equation  $V = IR$  relates the three variables. Therefore, differentiate this equation with respect to  $t$  using the chain rule, where  $V$ ,  $I$ , and  $R$  are all functions of  $t$ .

Use the product rule on the right side.

$$\frac{dV}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt}$$

Then solve  $\frac{dV}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt}$  for  $\frac{dR}{dt}$ .

$$I \frac{dR}{dt} = \frac{dV}{dt} - R \frac{dI}{dt}$$

$$\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - R \frac{dI}{dt} \right)$$

Since  $R$  was not given in the problem statement, use  $V = IR$  to find an expression for  $R$  in terms of  $V$  and  $I$ .

$$R = \frac{V}{I}$$

Substitute this expression for  $R$  in  $\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - R \frac{dI}{dt} \right)$ .

$$\frac{dR}{dt} = \frac{1}{I} \left( \frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right)$$

- d. Find the rate at which  $R$  is changing when  $V = 36$  volts and  $I = 3$  amp. Is  $R$  increasing, or decreasing?

Substitute the known values into the rate equation.

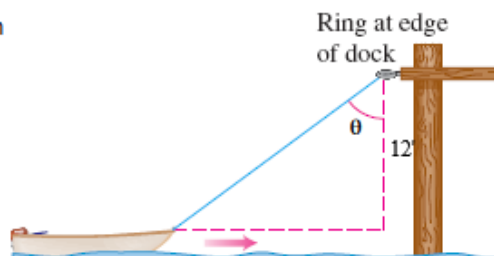
$$\begin{aligned} \frac{dR}{dt} &= \frac{1}{I} \left( \frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right) \\ \frac{dR}{dt} &= \frac{1}{3} \left( 5 - \frac{36}{3} \left( -\frac{1}{6} \right) \right) \end{aligned}$$

Simplify the right side.

$$\frac{dR}{dt} = \frac{1}{3} \left( 5 - \frac{36}{3} \left( -\frac{1}{6} \right) \right) = \frac{7}{3} \text{ ohm/sec}$$

Since  $\frac{dR}{dt}$  is positive, the resistance is increasing.

A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 12 feet above the bow. The rope is hauled in at the rate of 1 ft/sec. Complete parts **a.** and **b.**

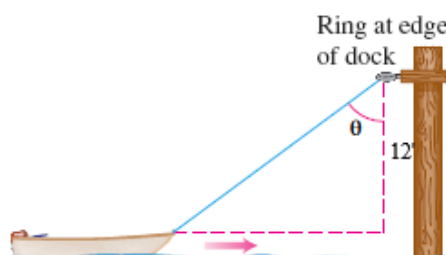


**a.** How fast is the boat approaching the dock when 15 ft of rope are out?

Name the variables and constants and label them in your diagram. Use  $t$  for time. Assume that all variables are differentiable functions of  $t$ .

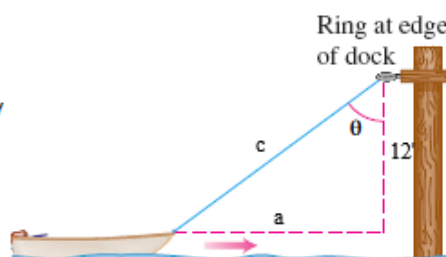
Note that the rope, the dock, and the water form a right triangle.

The distance from the top of the dock to the water is constant at 12 feet.



The horizontal distance from the boat to the dock is a variable. Name it  $a$ .

The length of rope (the distance from the dinghy to the top of the ring on the dock) is also a variable. Name it  $c$ .



Write down the numerical information in terms of the symbols already defined. Translate "The rope is hauled in at the rate of 1 ft/sec."

$$\frac{dc}{dt} = -1 \text{ ft/sec}$$

Next, translate, "How fast is the boat approaching the dock" using the symbols already defined. This is the quantity that part **a.** asks for.

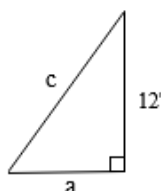
"How fast is the boat approaching the dock" asks for  $-\frac{da}{dt}$ .

Next, the phrase "when 15 ft of rope are out," means the moment when  $c = 15$ .

Now, write an equation that relates the variables  $a$  and  $c$ .

Use the Pythagorean theorem to write an equation.

$$a^2 + 12^2 = c^2$$



Next, differentiate the equation with respect to  $t$ .

$$\frac{d}{dt}(a^2) + \frac{d}{dt}(12^2) = \frac{d}{dt}(c^2)$$

To find  $\frac{d}{dt}(a^2)$ , use implicit differentiation and the power rule.

$$\frac{d}{dt}(a^2) = 2a \frac{da}{dt}$$

To find  $\frac{d}{dt}(12^2)$ , use the constant function rule.

$$\frac{d}{dt}(12^2) = 0$$

To find  $\frac{d}{dt}(c^2)$ , use implicit differentiation and the power rule.

$$\frac{d}{dt}(c^2) = 2c \frac{dc}{dt}$$

The results of the last three steps yield the following equation.

$$2a \frac{da}{dt} + 0 = 2c \frac{dc}{dt}$$

Since the quantity that we are looking for is  $\frac{da}{dt}$ , solve the equation for  $\frac{da}{dt}$ .

$$\frac{da}{dt} = \frac{c}{a} \frac{dc}{dt}$$

Next determine the values to substitute for  $a$ ,  $c$ , and  $\frac{dc}{dt}$ . Recall that the problem gives us the values  $\frac{dc}{dt} = -1$  and  $c = 15$ .

Find  $a$  by letting  $c = 15$  in the original equation,  $a^2 + 12^2 = c^2$ , and solving.

$$\begin{aligned} a^2 + 12^2 &= 15^2 && \text{Substitute.} \\ a^2 &= 15^2 - 12^2 && \text{Subtract } 12^2 \text{ from both sides} \\ a &= \pm \sqrt{81} && \text{Simplify and take the square root of both sides.} \\ &= \pm 9 \end{aligned}$$

Now substitute the values for  $a$ ,  $c$ , and  $\frac{dc}{dt}$  to find  $\frac{da}{dt}$ , the rate at which the dinghy is approaching the dock.

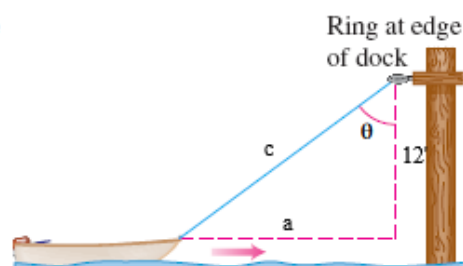
$$\begin{aligned} \frac{da}{dt} &= \frac{c}{a} \frac{dc}{dt} \\ &= \frac{15}{9}(-1) && \text{Substitute.} \\ &= -\frac{5}{3} && \text{Simplify.} \end{aligned}$$

Thus, the distance between the dinghy and the dock is changing at a rate of  $-\frac{5}{3}$  ft/sec when 15 feet of rope are out.

**b.** At what rate is the angle  $\theta$  changing at this instant? (see the figure)

b. At what rate is the angle  $\theta$  changing at this instant? (see the figure)

Notice that the angle  $\theta$  is already labeled in the diagram.



Translate "At what rate is the angle changing at this instant?" This is the quantity that part **b.** asks for.

"At what rate is the angle changing at this instant" asks for  $\frac{d\theta}{dt}$ .

Write an equation relating  $\theta$ ,  $c$ , and the constant leg of the right triangle.

$$\cos \theta = \frac{12}{c} \quad \text{Use cosine to relate the constant leg of the triangle and } c.$$

Next, differentiate the equation with respect to  $t$ .

$$\frac{d}{dt} \cos \theta = \frac{d}{dt} \left( \frac{12}{c} \right)$$

To find  $\frac{d}{dt}(\cos \theta)$ , use implicit differentiation.

$$\frac{d}{dt}(\cos \theta) = -\sin \theta \frac{d\theta}{dt}$$

To find  $\frac{d}{dt} \left( \frac{12}{c} \right)$ , use implicit differentiation and the power rule.

$$\frac{d}{dt} \left( \frac{12}{c} \right) = -\frac{12}{c^2} \cdot \frac{dc}{dt}$$

The results from the last three steps yield the following equation.

$$-\sin \theta \frac{d\theta}{dt} = -\frac{12}{c^2} \cdot \frac{dc}{dt}$$

Solve the equation for  $\frac{d\theta}{dt}$ .

$$\frac{d\theta}{dt} = \frac{12}{c^2 \sin \theta} \cdot \frac{dc}{dt}$$

Next determine the values for  $a$ ,  $c$ ,  $\sin \theta$ , and  $\frac{dc}{dt}$ . Recall that the problem gives the values  $c = 15$ ,  $\frac{dc}{dt} = -1 \frac{\text{ft}}{\text{sec}}$ . Also note that in part **a.**  $a$  was found to be 9.

Find  $\sin \theta$  by letting  $c = 15$ .

$$\sin \theta = \frac{9}{15}$$

Now substitute the values for  $a$ ,  $c$ ,  $\sin \theta$ , and  $\frac{dc}{dt}$  to find the rate at which the angle is changing at this instant.

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{12}{c^2 \sin \theta} \cdot \frac{dc}{dt} \\ &= \frac{12}{(15)^2 \frac{9}{15}} \cdot (-1) \quad \text{Substitute.}\end{aligned}$$

Now, simplify the expression.

$$\begin{aligned}\frac{12}{(15)^2 \frac{9}{15}} \cdot (-1) &= \frac{(-1)(12)}{(15)(9)} \quad \text{Simplify.} \\ &= -\frac{4}{45} \text{ rad/sec} \quad \text{Multiply and simplify.}\end{aligned}$$

Therefore, the angle is changing at a rate of  $-\frac{4}{45}$  rad/sec.