

# Probability and Bayes Theorem

Biology 683

Lecture 3

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# Last week

1. What is the difference in a population and a sample?
2. What is the difference in a parameter and a statistic?
3. How does the central limit theorem help us?

# Today

1. Probability
2. Frequentist vs Bayesian

# Mutually exclusive events

What is the probability of drawing either an ace or a king from a deck of cards?

addition rule: if two events are mutually exclusive then the probability that either will occur is just the sum of the individual probabilities

$$\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$$
$$\Pr(\text{ace or king}) = 15\% = \frac{4}{52} + \frac{4}{52}$$

assumes standard deck with no jokers

# Independent events

What is the probability of rolling a 1 and then a 2 on a pair of dice?

multiplication rule: if two events are independent of one another then the probability of both occurring is just the product of the individual probabilities.

$$\Pr(A \text{ then } B) = \Pr(A) \times \Pr(B)$$

$$\Pr(1 \text{ then } 2) = 3\% = \frac{1}{6} \times \frac{1}{6}$$

# Non-independence

non-independent events- conditional probability: is the probability of an event given that another event has already occurred.

For instance, the probability of surviving the sinking of the Titanic was very different than the probability of surviving if you were a female

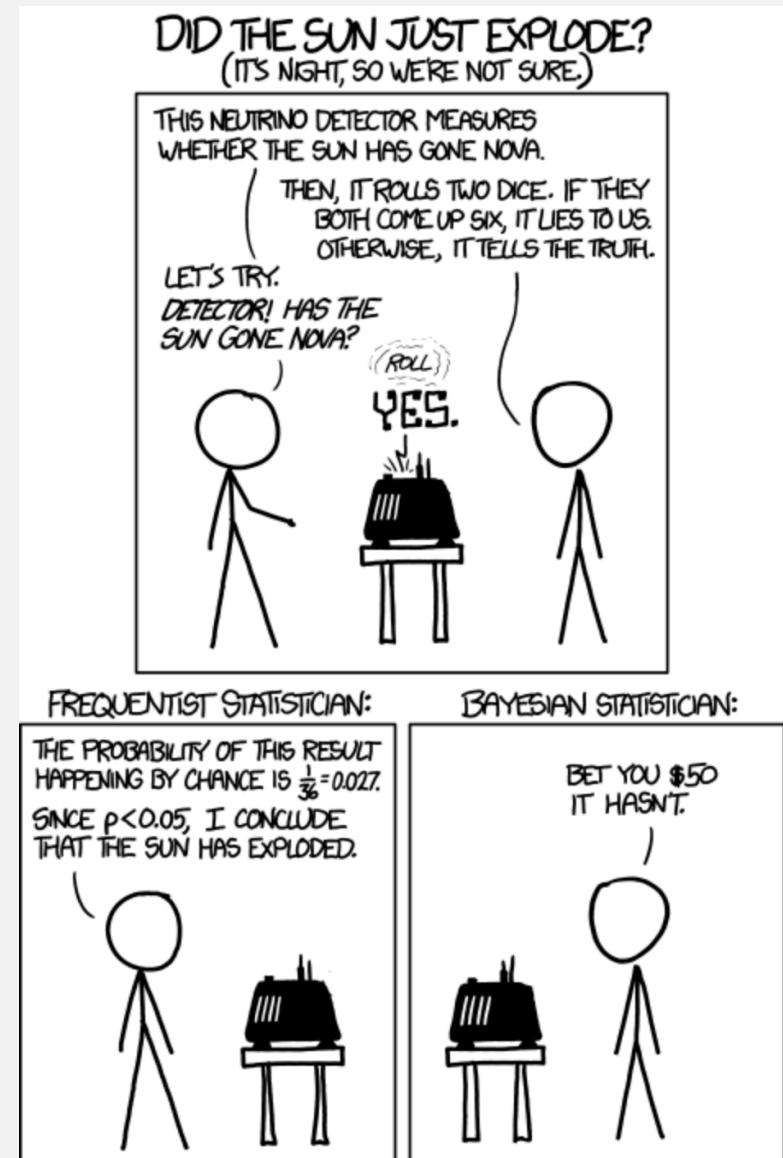
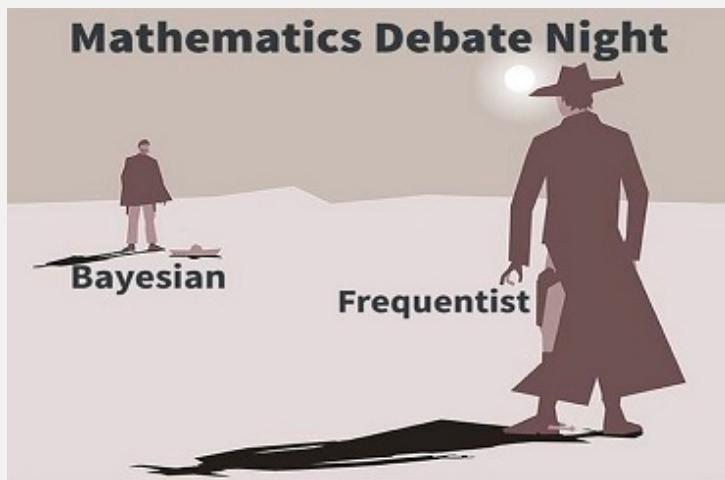
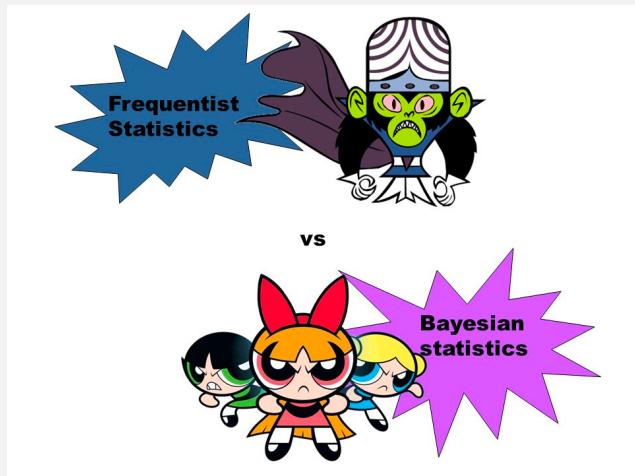
	died	lived
male	1364	367
female	126	344

# Conditional probability

$$\Pr(\text{survival}) = 32\% = \frac{711}{2201}$$

$$\Pr(\text{survival} \mid \text{female}) = 73\% = \frac{344}{470}$$

# Frequentist and Bayesian



# Frequentist and Bayesian

	<b>Frequentist</b>	<b>Bayesian</b>
Hypothesis test	P-value	Bayes factors
Parameter estimation	Maximum likelihood with confidence interval	Posterior distribution

# Frequentist and Bayesian

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Hypothesis test	P-value	Bayes factors
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# What is a p-value?

Is the probability of finding the observed, or more extreme, results when the null hypothesis is true.

```
```{r}
cor.test(dat.f$offspring, dat.f$body)
```

```

```
Pearson's product-moment correlation

data: dat.f$offspring and dat.f$body
t = 3.3693, df = 105, p-value = 0.001055
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
0.1302191 0.4740979
sample estimates:
cor
0.3123551
```

$$t = \frac{r\sqrt{df}}{\sqrt{1 - r^2}}$$

# Frequentist and Bayesian

|                      | <b>Frequentist</b>                          | <b>Bayesian</b>        |
|----------------------|---|------------------------|
| Hypothesis test      | P-value                                     | Bayes factors          |
| Parameter estimation | Maximum likelihood with confidence interval | Posterior distribution |

# Frequentist and Bayesian

$$\text{Bayes Factor} = \frac{\Pr(D|M_1)}{\Pr(D|M_2)}$$

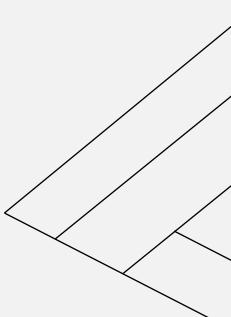
| Bayes factor | Interpretation                 |
|--------------|--------------------------------|
| >100         | Decisive evidence for $H_A$    |
| 30–100       | Very strong evidence for $H_A$ |
| 10–30        | Strong evidence for $H_A$      |
| 3–10         | Substantial evidence for $H_A$ |
| 1–3          | Anecdotal evidence for $H_A$   |
| 1            | No evidence                    |
| 1/3–1        | Anecdotal evidence for $H_0$   |
| 1/10–1/3     | Substantial evidence for $H_0$ |
| 1/30–1/10    | Strong evidence for $H_0$      |
| 1/100–1/30   | Very strong evidence for $H_0$ |
| <1/100       | Decisive evidence for $H_0$    |

# Frequentist and Bayesian

|                      | <b>Frequentist</b>                          | <b>Bayesian</b>        |
|----------------------|---|------------------------|
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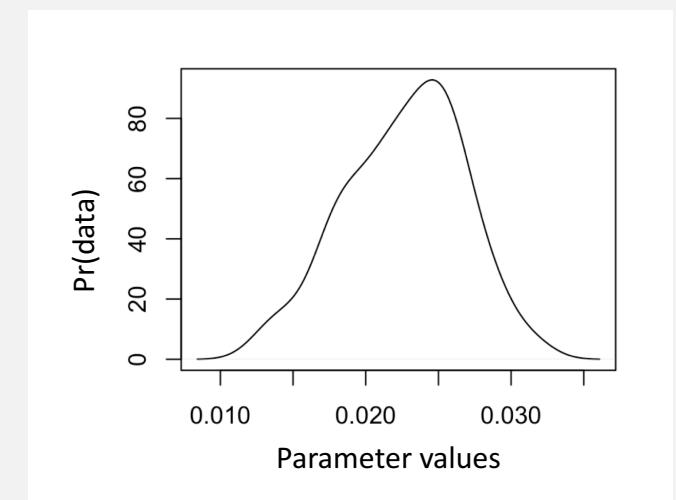
# Frequentist parameter estimate

Everything is fixed except parameter of interest. Shape of likelihood determines estimate of the statistic



TAAATATAGAGCTATGTTT  
TAAGTATAGAGCTATGTTT  
TAAGTATAGAGCTATGTTT  
TAAGTATAGAGCTATGTTT  
TAAATATAGAGCTATGTTT

|   | A        | G        | C        | T        |
|---|----------|----------|----------|----------|
| A |          | $\beta$  | $\alpha$ | $\alpha$ |
| G | $\beta$  |          | $\alpha$ | $\alpha$ |
| C | $\alpha$ | $\alpha$ |          | $\beta$  |
| T | $\alpha$ | $\alpha$ | $\beta$  |          |

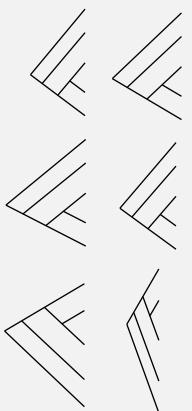


# Frequentist and Bayesian

|                      | <b>Frequentist</b>                          | <b>Bayesian</b>        |
|----------------------|---|------------------------|
| Hypothesis test      | P-value                                     | Bayes factors          |
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# Bayesian parameter estimate

We create a marginal estimate of the statistic of interest



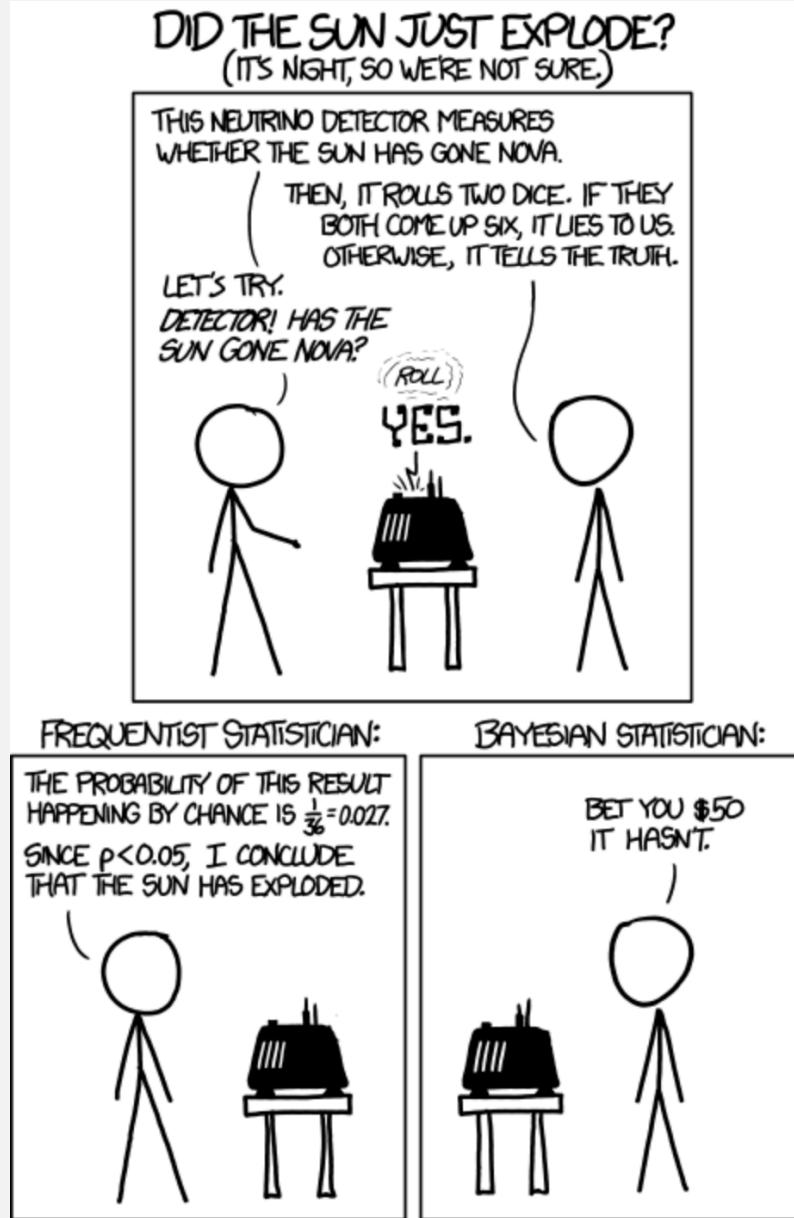
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TAAGTATAAGAGCTATGTTT  
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TAAGTATAAGAGCTATGTTT  
TAAGTATAAGAGCTATGTTT  
TAAATATAAGAGCTATGTTT

CAAATATAAGAGCTATGTTT  
CAAATATAAGAGCTATGTTT  
TAAATATAAGAGCTATGTTT  
TAAGTATAAGAGCTATGTTT  
TAACTATAAGACCTATGTTT

|   | A        | G        | C        | T        |
|---|----------|----------|----------|----------|
| A |          | $\beta$  | $\alpha$ | $\alpha$ |
| G | $\beta$  |          | $\alpha$ | $\alpha$ |
| C | $\alpha$ | $\alpha$ |          | $\beta$  |
| T | $\alpha$ | $\alpha$ | $\beta$  |          |

|        |             |         |            |            |
|--------|-------------|---------|------------|------------|
| Tree4  | alignment8  | model6  | 1.0391601  | 2.25225446 |
| Tree8  | alignment4  | model6  | 1.04903411 | 2.11859404 |
| Tree10 | alignment7  | model5  | 1.03117276 | 2.88413248 |
| Tree10 | alignment3  | model4  | 1.02935202 | 2.26183042 |
| Tree9  | alignment8  | model3  | 1.04700046 | 2.00114012 |
| Tree2  | alignment10 | model2  | 1.01892741 | 2.60891431 |
| Tree2  | alignment6  | model3  | 1.01524482 | 2.11437827 |
| Tree8  | alignment2  | model1  | 1.02230859 | 2.30319668 |
| Tree5  | alignment7  | model1  | 1.02593538 | 2.13290411 |
| Tree8  | alignment10 | model5  | 1.00762542 | 2.13242719 |
| Tree0  | alignment4  | model5  | 1.01884236 | 2.42583388 |
| Tree9  | alignment8  | model4  | 1.00252288 | 2.71467137 |
| Tree4  | alignment0  | model8  | 1.02480351 | 2.5452598  |
| Tree8  | alignment8  | model10 | 1.03066934 | 2.38768005 |
| Tree1  | alignment3  | model5  | 1.046732   | 2.8957133  |
| Tree9  | alignment5  | model6  | 1.0226392  | 2.79771149 |
| Tree10 | alignment0  | model8  | 1.03210411 | 2.08787975 |
| Tree2  | alignment6  | model8  | 1.00233853 | 2.15930312 |
| Tree4  | alignment6  | model3  | 1.0266862  | 2.21811627 |
| Tree2  | alignment0  | model6  | 1.02813474 | 2.01792657 |
| Tree1  | alignment1  | model6  | 1.00101131 | 2.0549765  |
| Tree7  | alignment7  | model2  | 1.00596282 | 2.20547933 |
| Tree5  | alignment3  | model9  | 1.00241339 | 2.62082855 |
| Tree5  | alignment9  | model9  | 1.03602053 | 2.25983463 |
| Tree0  | alignment5  | model4  | 1.00629136 | 2.86463569 |
| Tree8  | alignment5  | model7  | 1.02207157 | 2.79890432 |
| Tree8  | alignment3  | model5  | 1.01522322 | 2.66694394 |
| Tree9  | alignment8  | model9  | 1.02316269 | 2.75648806 |
| Tree4  | alignment5  | model10 | 1.0229115  | 2.28025282 |
|        |             |         | 1.02547847 |            |

# Frequentist and Bayesian



# Frequentist approach

You have a null hypothesis and calculate the probability of observing your data under that hypothesis.

Null: The sun has not exploded

$$\Pr(\text{det yes} \mid \text{sun is ok}) = 0.027 = \frac{1}{6} \times \frac{1}{6}$$

this is less than the typical  $\alpha$  level of 0.05 so we reject the null that the sun is ok.

# Bayesian approach



$$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$$

$$\Pr(nova|det\ yes) = \frac{\Pr(det\ yes|nova) \times \Pr(nova)}{\Pr(det\ yes)}$$

$$\Pr(nova|det\ yes) << 0.00000000037 \quad \frac{1 \times 0.0000000001}{0.027}$$

# Bayesian approach

Imagine a genetic mutation present in 1 person per 100,000. If you take a test that correctly returns a positive result 99.9% of the time when someone is infected but has a false positive rate of 0.5%.

How likely are you to have the disease?

Should you be concerned?

Bayes' theorem provides a natural way to think about this.

**Lets do this!**

# Bayesian approach

$$\Pr(\text{inf}|\text{pos. test}) = \frac{\Pr(\text{pos. test}|\text{inf}) \times \Pr(\text{inf})}{\Pr(\text{pos. test})}$$

$$\Pr(\text{inf}|\text{pos. test}) = 0.001 = \frac{0.999 \times 0.00001}{0.00501}$$

Which means that you have only a 0.1% chance of having the disease even if you test positive.

What if your doctor noticed a symptom that made them give you this test what changes?

# Priors

$$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$$

$\Pr(A)$  this is the prior. It is powerful because it allows you to incorporate previous knowledge into your analysis, but it can lead to very bad inference if you are careless.

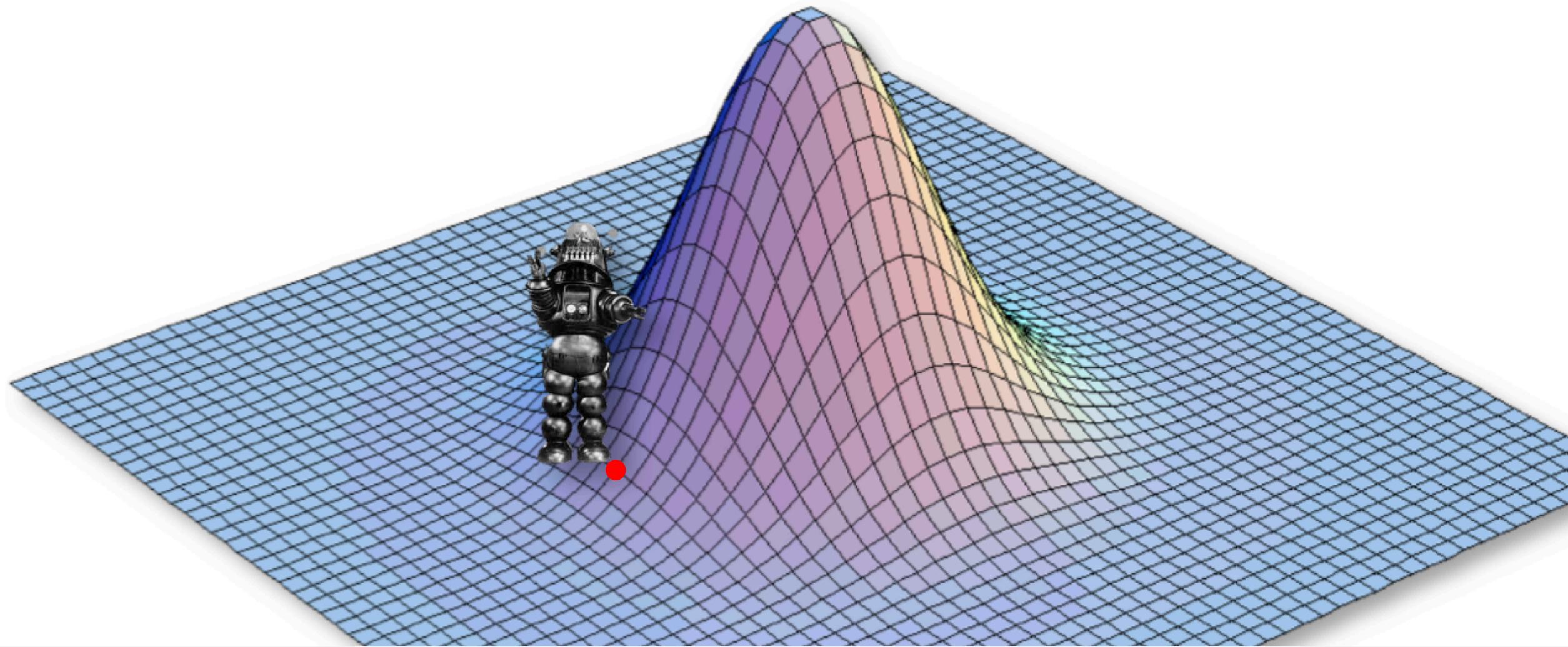
# Bayesian methods in practice (MCMC)

In practice our models are quite complex with many parameters we would like to estimate. Each of these has its own prior distribution. The way that we explore the space of solutions is using an MCMC.

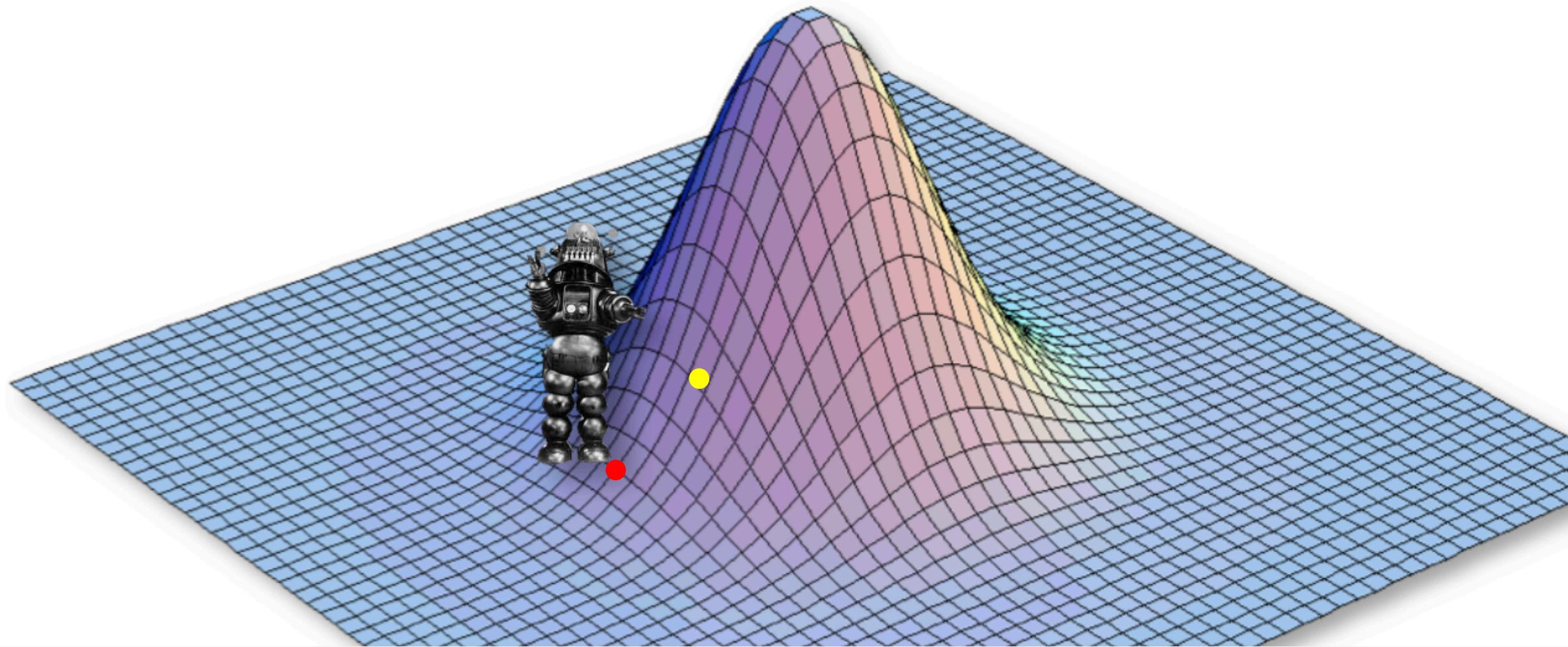
- 1) Pick an arbitrary starting point
- 2) Calculate the current probability  $p_t$
- 3) Generate a change to one of our parameters
- 4) Calculate the new probability  $p_{t+1}$
- 5) Accept new parameter value with probability  $\frac{p_{t+1}}{p_t}$
- 6) Return to step 3 (repeat for a really long time)

$$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$$

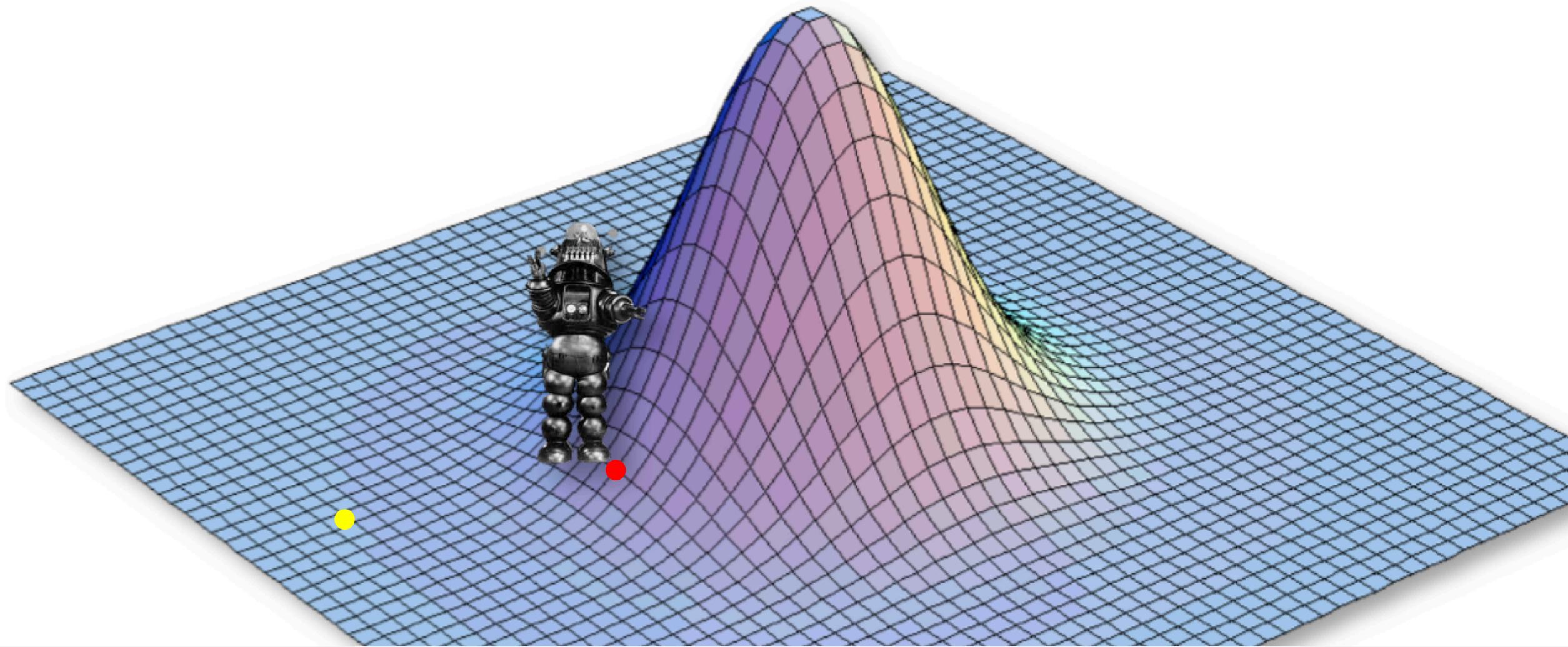
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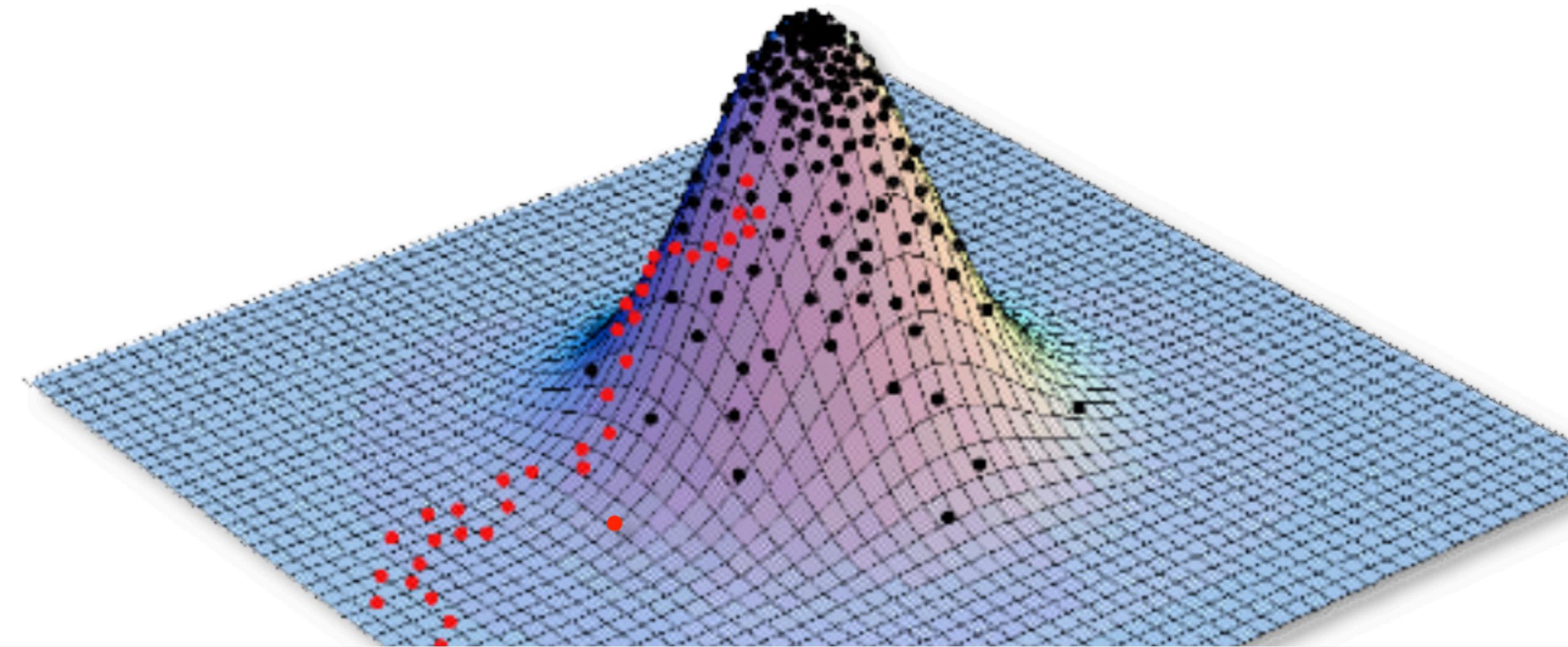
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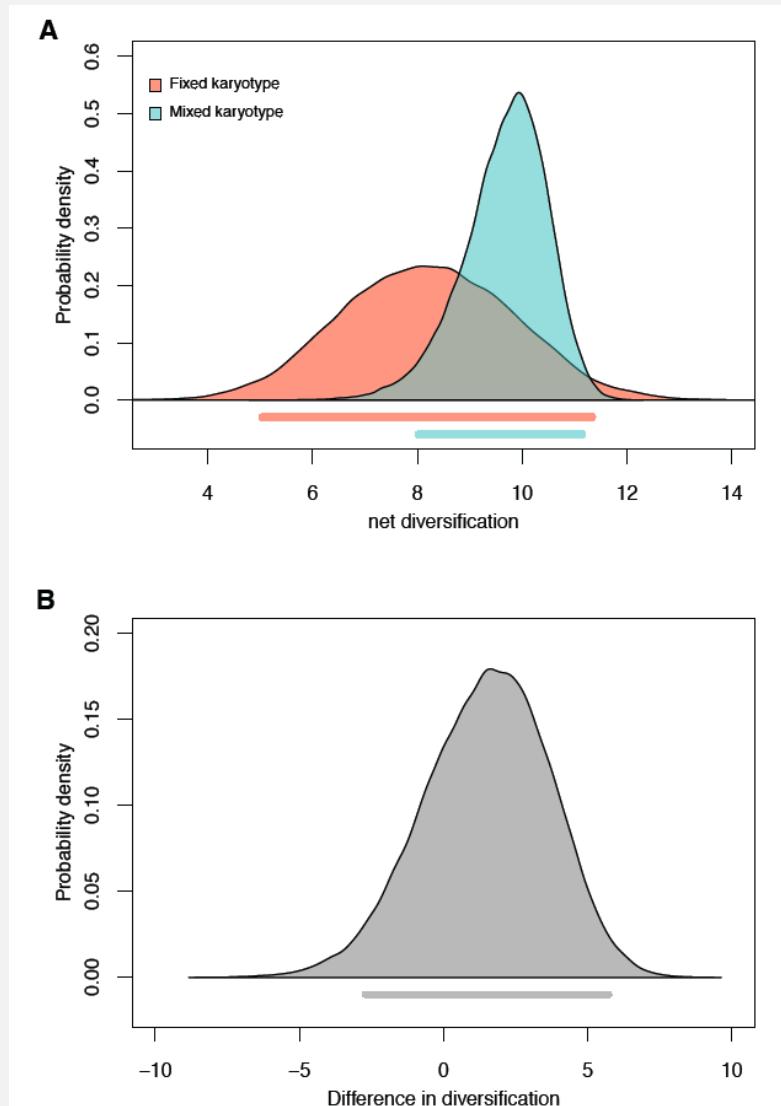
# Bayesian methods in practice (MCMC)



# Bayesian methods in practice (MCMC)



# The power of Bayesian approaches



With a Bayesian approach we can take into account all types of uncertainty and be more conservative.

- Phylogenetic
- Model selection uncertainty
- Parameter value uncertainty
- Uncertainty in measurements

# For Thursday

Read chapter WS 5 and chapter 2 of McElreath

Bring laptop to class!

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# Setting matters

