Homework 4

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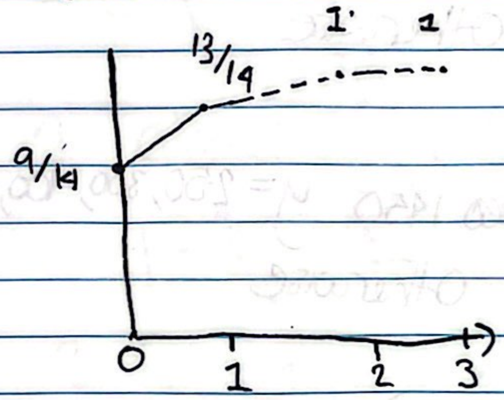
$$p_X(x) = \begin{cases} \frac{(3-x)^2}{\alpha}, & x = 0,1,2,3 \end{cases}$$

1. discrete: $\frac{3^2}{\alpha} + \frac{2^2}{\alpha} + \frac{1^2}{\alpha} + \frac{0^2}{\alpha} = 1$

$$1 = \sum_{r=0}^{3} p_X(r)$$

$$\frac{14}{\alpha} = 1, \quad \alpha = 14$$

2. CDF... sum at each



3. mean of X?

$$\mu_X = E[X] = \sum x p_X(x)$$

$$= 0 \cdot \frac{9}{14} + 1 \cdot \frac{4}{14} + 2 \cdot \frac{1}{14} + 3 \cdot \frac{0}{14}$$

$$\mu_X = 6/14 = 3/7$$

4. variance of X. $E[X^2] - [E[X]]^2 = \sigma_X^2$

so $0^2 \cdot \frac{9}{14} + 1^2 \cdot \frac{4}{14} + 2^2 \cdot \frac{1}{14} + 3^2 \cdot \frac{0}{14}$

so $\frac{4}{7} - \left(\frac{3}{7}\right)^2 = 19/49 = \sigma_X^2$

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5. $E[Y] = E[(X-2)^2]$ $\quad g(x) = (x-2)^2$

so $\sum_x p_X(x) \cdot g(x)$

$\frac{9}{14} \cdot (0-2)^2 + \frac{4}{14} \cdot (1-2)^2 + 0^2 \cdot \frac{1}{14} + 1^2 \cdot \frac{0}{14}$

$\frac{36}{14} + \frac{4}{14} = \frac{20}{7} = E[Y]$

Proble 2

1. $p_X(x) = \begin{cases} 1/4 & x = 250, 300, 400, 500 \\ 0 & \text{otherwise} \end{cases}$

by box

$p_Y(y) = \begin{cases} y/\cancel{1400}\ 1450 & y = 250, 300, 400, 500 \\ 0 & \text{otherwise} \end{cases}$

2. $E[X] = \sum x\, p_X(x)$

$\frac{250}{4} + \frac{300}{4} + \frac{400}{4} + \frac{500}{4} = 362.5$

$E[Y] = 250 \cdot \frac{250}{1450} + \frac{300^2}{1450} + \frac{400^2}{1450} + \frac{500^2}{1450}$

$= 387.9$

so $E[Y] > E[X]$ because the pmf of Y is weight with the chip count of each box.

Problem 3

$\lambda = 30/60 = 0.5/\text{minute}$ $\quad \alpha = \lambda T$

Poisson RV

$$\text{PMF} = \left\{ \frac{\alpha^k}{k!} e^{-\alpha} \quad k = 0, 1 \right.$$

1. atleast 1 distrubance over a minute

$\alpha = 0.5$

$P(X=k)$, so $P(X \geq 1) = 1 - P(0)$

$$= 1 - \frac{0.5^0 \cdot e^{-0.5}}{0!} = 1 - e^{-0.5} = 0.393$$

2. atleast 1 in 5 minutes. $\alpha = 0.5 \cdot 5 = 2.5$

$$1 - \frac{2.5^0 \cdot e^{-2.5}}{0!} = 1 - e^{-2.5} = 0.918$$

3. fewer than 5 in 15 minutes. $\alpha = 7.5$

$$\sum_{k=0}^{4} P(X=k) = P(X=0) + P(X=1) + \ldots$$

$$\frac{7.5^0 e^{-7.5}}{0!} + \frac{7.5^1 e^{-7.5}}{1} + \ldots \frac{7.5^4 e^{-7.5}}{4!}$$

$= 0.132$

4. ~~10~~ more than 10 in 15 $\alpha = 7.5$

$P(X>10) = 1 - P(X \le 10) = 1 - \sum_{k=0}^{10} \frac{7.5^k e^{-7.5}}{k!}$

$= 0.1378$

Problem 4

$f_X(x) = cx^4, \ [-2, 2]$

1. find valid c for PDF so

$1 = c\int_{-2}^{2} x^4 dx, \ c = 0.781$ or

$c\left[\frac{x^5}{5}\right]_{-2}^{2} \quad 1 = c\left(\frac{32}{5} - \frac{-32}{5}\right) = \frac{64}{5}c = 1$

$c = 5/64$

2. Find CDF

$F_X(x) = \int_{-2}^{x} \frac{5}{64} t^4 dt = \frac{5}{64} \cdot \left[\frac{t^5}{5}\right]_{-2}^{x}$

$F_X(x) = \frac{1}{64} \cdot (t^5 + 32)$

3. Find the mean of X. Symmetric around 0 thus $E[X] = 0$

4. Find Var(X)?

$E[X^2] - E[X]$

$E[X^2] = \int_{-2}^{2} x^2 \cdot f_x \, dx = \frac{5}{64}\int_{-2}^{2} x^6 dx$

$= 20/7 - 0, \; Var(x) = 20/7$

5. $E[(X-1)^2]$

$= \frac{5}{64}\int_{-2}^{2}(x^2-2x+1)\cdot x^4 \, dx = \frac{27}{7}$

Problem 5

$X \sim Exp(\lambda)$ so $f_x(x) = \begin{cases} 0, & x<0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$

1. $E[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx \rightarrow$ Integration By Parts

$\left[-xe^{-\lambda x}\right]\Big|_0^\infty + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]\Big|_0^\infty = \frac{1}{\lambda}$

$E[X] = \frac{1}{\lambda}$

2. $Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx \; \ldots \; IPB \quad u = x^2$

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$$E[X^2] = \left[-x^2 e^{-\lambda x}\right]\Big|_0^\infty + 2\int_0^\infty x e^{-\lambda x}\,dx$$

$$0 + 2\int_0^\infty x\cancel{\lambda} e^{-\lambda x}\,dx \quad E[X^2] = \frac{2}{\lambda^2}$$

SO

$$E[X^2] = \frac{2}{\lambda^2},\ E[X] = \frac{1}{\lambda}$$

$$\frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

2. Memoryless property...

$$P(X > t + \Delta t \mid X > t) = P(X > \Delta t)$$

- exponential distro has memoryless as previous events do not affect a greater likelihood of occurrence
- It is mention, but not explored in the textbook

Problem 6

1. $\mu = 82$, $\sigma^2 = 64$, $\sigma = 8$

$Z = \frac{85-82}{8} = 0.375$

$P(Z \geq 0.375) = 1 - P(Z < 0.375$

$= 0.3538$

$P(B) = 0.5794$

$P(C) = 0.067$

2. $P(X \geq 85 \mid X \geq 80) = \frac{P(X \geq 85)}{P(X \geq 80)}$

$Z_{80} = \frac{80-82}{8} = -\frac{1}{4} \rightarrow P(Z \geq -1/4)$

so $P(X \geq 85 \mid X \geq 80) = 0.591$

3. $P(X \geq 85) = 0.25$

$P(70 \leq X \leq 85) = 0.52$

$\mu = 77.84$, $\sigma = 10.61$

Problem 7

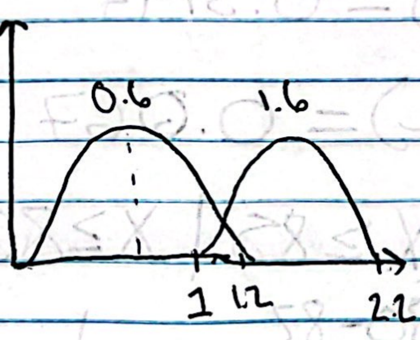
$X = S_i + N$, $S_0 = 0$, $S_1 = 1$, $N \sim \text{Uniform}[0, b]$

1. $\mu = \dfrac{(0+b)}{2}$, $\sigma^2 = \dfrac{(b-0)^2}{12}$

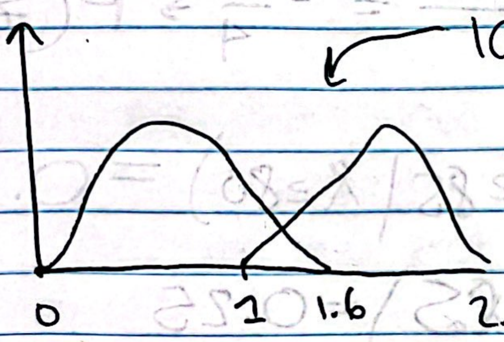
2. $T_0, S_0 = 0$, so $X = N$ so $[0, b]$

   $T_1, S_1 = 1$, $X = 1 + N$ $[1, 1+b]$

3. $b = 1.2$



4.



5. ~~At~~ From $[0, 1.1]$ choose $\hat{S}_0$

   $[1.1, 2.2]$ choose $\hat{S}_1$

6. $\hat{S}_0 = [0, 1.3)$, $\hat{S}_1 = [1.3, 2.6]$

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7. $P(T_0) = 0.4, P(T_2) = 0.6 \quad P(\text{error}) =$

$P(T_2) \cdot P(\text{error} \mid T_0) + P(T_1) P(\text{err} \mid T_1)$

$P(E_{0.2}) = 0.655 \quad P(E_{1.6}) = .147$