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Assignment 3

$\boxed{\text{Section 4.4}}$

rational function

$\boxed{Q10}$ (a) $f(z) = \frac{z}{z^2+25} = \frac{z}{(z+5i)(z-5i)}$, Domain $\boxed{f(z) \text{ is not analytic at } z = 5i, (-5i). \text{ It is analytic everywhere else}}$

These points are outside of the disk $|z| \leq 2$,
Therefore $f(z)$ is analytic everywhere within this disk.
We know from the Thereom that if $f(z)$ is analytic in a Simply connected domain, and Γ is a closed loop in this domain, then $\int_\Gamma f(z)dz = 0$. $\therefore$ $\boxed{\int_{|z|=2} f(z)\,dz = 0}$

rational function

(c) $f(z) = \frac{\cos(z)}{z^2-6z+10} \rightarrow \frac{\cos(z)}{(z-3+i)(z-3-i)}$, Domain $\boxed{f(z) \text{ is analytic everywhere except at } z = 3-i,\ z = 3+i}$

- These points lie outside of $|z| \leq 2$, $\therefore$ $\boxed{\int_{|z|=2} f(z)\,dz = 0}$

(d) $f(z) = \text{Log}(z+3) \rightarrow$ Not analytic on non-positive Real axis

$\text{Re}(f(z)) = x+3$, $x+3 < 0 \rightarrow x < -3$. Domain $\boxed{f(z) \text{ is analytic everywhere except when } z = x \text{ and } x < -3}$

- This lies outside of $|z| \leq 2$, $\therefore$ $\boxed{\int_{|z|=2} f(z)\,dz = 0}$

$\boxed{Q16}$ $f(z) = \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \cdots + \frac{A_1}{z} + g(z) \quad (k \geq 1)$

$g(z)$ is analytic inside and on $|z|=1$

$$\int_{|z|=1} f(z)\,dz = \int_{|z|=1} \left[\frac{A_k}{z^k} + \cdots + \frac{A_1}{z} + g(z)\right] dz$$

$$= \cancel{\int_{|z|=1} \frac{A_k}{z^k}\,dz}^{\,0} + \cdots \cancel{+}^{\,0} \int_{|z|=1} \frac{A_1}{z}\,dz + \cancel{\int_{|z|=1} g(z)\,dz}^{\,0 \text{ (Analytic)}}$$

$= A_k \int_{|z|=1} z^{-k}\,dz$

$= A_k \cdot 0$

$$= \int_{|z|=1} \frac{A_1}{z}\,dz = A_1 \int_{|z|=1} \frac{1}{z}\,dz = \boxed{A_1 \cdot (2\pi i)}$$

$\boxed{\text{Section 4.5}}$

$\boxed{Q4}$ $\int_C \frac{z+i}{z^3+2z^2}\,dz = \int_C \frac{z+i}{z(z^2+2z)}\,dz$

(a) C is $|z|=1$ CCW

$$= \int_{|z|=1} \frac{1}{z^2} \cdot f(z)\,dz \text{ where } f(z) = \frac{z+i}{z+2}$$

$$f'(z) = \frac{(z+2) - (z+i)}{(z+2)^2} = \frac{2-i}{(z+2)^2}, \quad f'(0) = \frac{1}{2} - \frac{1}{4}i$$

$$\int_{|z|=1} \frac{1}{z^2} f(z) = 2\pi i\left(\frac{1}{2} - \frac{1}{4}i\right) = \boxed{\pi i + \frac{1}{2}\pi}$$

(b) C is $|z+2-i|=2$ ccw

$\int_C \frac{f(z)}{z+2}\,dz$ where $f(z)$ is $\frac{z+i}{z^2}$

$f(-2) = \frac{-2+i}{(-2)^2} = \frac{-2+i}{4} = \frac{-1}{2} + \frac{1}{4}i$

$\int_C \frac{f(z)}{z+2}\,dz = 2\pi i\left(\frac{-1}{2}+\frac{1}{4}i\right) = -\pi i - \frac{1}{2}\pi = \boxed{\pi\left(\frac{-1}{2} - i\right)}$

(c) C is $|z-2i|=1$ ccw

$\int_C \frac{z+i}{z^2(z+2)}\,dz \rightarrow$ Zeros lie at 0 and -2. Since Neither Zero lies inside $|z-2i|=1$ and its analytic inside. $\int_C \frac{z+i}{z^2(z+2)}\,dz = \boxed{0}$

$\boxed{Q6}$ $\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz = \int_\Gamma \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz$, Γ is $|z|=3$ ccw

$$= \int_{|z|=3} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz + \int_{|z|=3} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\bigg|_{z=i} + \frac{d}{dz}\left(\frac{e^{iz}}{(z-i)^2}\right)\bigg|_{z=-i}\right]$$

$$= 2\pi i\left[\frac{e^{iz}(iz-3)}{(z+i)^3}\bigg|_{z=i} + \frac{e^{iz}(iz-1)}{(z-i)^3}\bigg|_{z=-i}\right]$$

$$= 2\pi i\left[\frac{-1}{2e}i + 0\right] = \boxed{\pi/e}$$

Q15 $F(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ analytic at each point $|z| \leq 1$ and $f(0) = 0$

$$\lim_{\Delta \to 0} \frac{F(0+\Delta z) - F(0)}{\Delta z} = 0 \to \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} - \lim_{\Delta z \to 0} \frac{f'(0)}{\Delta z}$$

$$= \frac{f(\Delta z)}{(\Delta z)^2} - \frac{f'(0)}{\Delta z}$$

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - \cancel{f(0)}^{0}}{\Delta z} - \lim_{z \to 0} \frac{f(\Delta z)}{\Delta z} \to f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{F(0+\Delta z) - F(0)}{\Delta z} = \lim_{\Delta \to 0} \frac{1}{\Delta z} \cdot f'(0) - \lim_{\Delta z \to 0} \frac{f'(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\cancel{\frac{f'(0)}{\Delta z} - \frac{f'(0)}{\Delta z}}^{0} \right) = 0$$

- $\frac{f(z)}{z}$ is differentiable and continuous for $z \neq 0$

Thrm 15:

$$G(z) = \frac{1}{2\pi i} \int_{|\gamma|=1} \frac{f(\gamma)/\gamma}{\gamma - z} d\gamma$$

$\therefore F(z)$ is analytic on $|z| \leq 1$

$\boxed{\text{Section } 4.6}$

$\boxed{Q4}$ $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $|p(z)| = M$ for $|z| = 1$

$|a_k| \leq M$ for each k

$|p(z)| = |a_0 + a_1 z + \cdots a_n z^n|$

$|p^{(k)}(0)| \leq \frac{M \cdot k!}{1!} = Mk!$

$|p^{(1)}(0)| = a_1$

$|p^{(2)}(0)| = 2!\, a_2$

$|p^{(3)}(0)| = 3!\, a_3$

$\vdots$

$|p^{(k)}(0)| = k!\, a_k \longrightarrow |a_k| \leq M$

$P(0) = a_0$

$|P(0)| = |a_0| \longrightarrow$ ~~$|P(0)|$~~ $|a_0| = |P(0)|$

$= \max_{|x| \leq 1} |P(z)|$

$= \max_{|x| \leq 1} |P(z)|$

$= M$

$\therefore \boxed{|a_0| \leq M}$ for $k = 1, 2, 3 \ldots n-1$, n

$\boxed{Q8}$ f is analytic in annulus $1 \le |z| \le 2$ and

$|f(z)| \le 3$ on $|z| = 1$ and $|f(z)| \le 12$ on $|z| = 2$

$h(z) = \dfrac{f(z)}{3z^2}$ → $f(z)$ is analytic in annulus $1 \le |z| \le 2$ $|f(z)| \le 3$

$$\cancel{\left|\frac{f(z)}{3z^2}\right|} \quad \left|\frac{f(z)}{3z^2}\right| = \frac{|f(z)|}{3|z|^2} \rightarrow |h(z)| = \frac{|f(z)|}{3|z|^2}$$

$$|h(z)| \le \frac{3}{3 \cdot 1^2} = 1$$

$$|h(z)| \le 1$$

$|z| = 2$

$$|h(z)| = \frac{|f(z)|}{3|z|^2}$$

$$|h(z)| \le \frac{12}{3 \cdot 2^2} = 1$$

$$|h(z)| \le 1$$

$$\boxed{\therefore |f(z)| \le 3|z|^2 \text{ for } 1 \le |z| \le 2}$$

$\boxed{Q14}$ - f is analytic in bounderd D and continuous up to the boundary

- f(z) is analytic in D.
- 1/f(z) is analytic in D

|1/f(z)| takes on its max value on the boundary, beceuse |f(z)| gets ~~the~~ minimum value on bandary

→ By Minimum module principal

f(z) cannot be 0 ~~because then the function will blow up~~

Ex: if $f(z) = z^4$, ~~then~~ on $|z| < 1$, then min val $|f(z)| = 0$ is at $z = 0$, not at the boundary $|z| = 1$

$\boxed{\text{Section 5.1}}$

$\boxed{\text{Q8}}$ (a)

$\boxed{\text{Q14}}$

(a) $\sum_{j=1}^{\infty} \frac{1}{j(j+i)}$, $c_j = \frac{1}{j(j+1)} \rightarrow |c_j| = \left|\frac{1}{j(j+1)}\right| = \left|\frac{1}{j^2+j \cdot i}\right|$

$c_j \leq \frac{1}{j^2} = M$

$|c_j| \leq M_j \rightarrow \sum_{j=1}^{\infty} M_j = \sum_{j=1}^{\infty} \frac{1}{j^2}$ Converges $(P = 2 > 1)$

Comparison Test: $\sum_{j=1}^{\infty} c_j = \sum_{j=1}^{\infty} \frac{1}{j(j+1)}$ $\boxed{\text{Converges}}$

(b) $\sum_{k=1}^{\infty} \frac{\sin(k^2)}{k^{3/2}}$, $C_k = \frac{\sin(k^2)}{k^{3/2}} \rightarrow |C_k| = \left|\frac{\sin(k^2)}{k^{3/2}}\right|$

$C_k \leq \frac{}{k^{3/2}}$

$|C_k| \leq M_k \rightarrow \sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ Converges $(P = \frac{3}{2} > 1)$

Comparison Test: $\sum_{k=1}^{\infty} \frac{\sin(k^2)}{k^{3/2}}$ $\boxed{\text{Converges}}$

(c) $\sum_{k=1}^{\infty} \frac{k^2 i^k}{k^4+1}$

$C_k = \frac{k^2 i^k}{k^4+1} \rightarrow |C_k| = \left|\frac{k^2 i^k}{k^4+1}\right| = \frac{k^2}{k^4+1} = \frac{1}{k^4} \cdot \frac{k^2}{1+k^{-4}}$

$C_k < \frac{1}{k^2}$

$|C_k| < M_k \rightarrow \sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $(P=2>1)$

Comparison Test: $\sum_{k=1}^{\infty} \frac{k^2 i^k}{k^4+1}$ $\boxed{\text{Converges}}$

(d) $\sum_{k=2}^{\infty} (-1)^k \frac{5k+8}{k^3-1}$

$C_k = (-1)^k \frac{5k+8}{k^3-1} \rightarrow |C_k| = \left|(-1)^k \left(\frac{5k+8}{k^3-1}\right)\right| = \frac{5k+8}{k^3-1}$

$C_k < \left(k\left(5+\frac{8}{k}\right) / k^3(1-1/k^3)\right)$

$< 1/k^2$

$|C_k| < M_k \rightarrow \sum_{k=2}^{\infty} M_k = \sum_{k=2}^{\infty} \frac{1}{k^2}$ Converges $(P=2>1)$

Comparison Test: $\sum_{k=2}^{\infty} (-1)^k \left(\frac{5k+8}{k^3-1}\right)$ $\boxed{\text{Converges}}$

Q20

$$\left|\sum_{j=0}^{n} z^j - \frac{1}{1-z}\right| = \left|\frac{(1-z)\sum_{j=0}^{n} z^j - 1}{1-z}\right| = \left|\frac{z^n}{1-z}\right|$$

$$\left|\frac{z^n}{1-z}\right| < \varepsilon \longrightarrow \text{No } n \text{ for such that } |z|<1 \quad \left|\frac{z^n}{1-z}\right| < \varepsilon$$

$\therefore \sum_{j=0}^{\infty} z^j$ $\boxed{\text{doesn't converge uniformly}}$ to $\frac{1}{1-z}$

Section 5.2

Q14 Let $f(z)$ be analytic in $|z-z_0|<R$

and $f^{(k)}(z_0)=0$ for $k=0,1,2,3,\ldots n$

Taylor Series Expansion:

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k,\quad f^{(k)}(z_0)=0 \text{ for every } k=0,1,2,\ldots n$$

$\therefore \boxed{f(z)=0 \text{ for all } D}$

Q18

$\boxed{\text{Section 5.3}}$

$\boxed{Q2}$ $\sum_{j=0}^{\infty} a_j (z-z_0)^j$, $\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = L$

$C_j = a_j(z-z_0)^j$, $C_{j+1} = a_{j+1}(z-z_0)^{j+1}$

$$\lim_{j\to\infty}\left|\frac{C_{j+1}}{C_j}\right| = \lim_{j\to\infty}\left|\frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j}\right| = \lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right||z-z_0|$$
$$= L|z-z_0|$$

Ratio Test: $\sum_{j=0}^{\infty} c_j$, as $j\to\infty$ $\left|\frac{c_{j+1}}{c_j}\right|$ approaches L

$L<1$ Converge
$L>1$ Diverge

$L|z-z_0|<1 \to |z-z_0|<\frac{1}{L}$ - Converge

$L|z-z_0|>1 \to |z-z_0|>\frac{1}{L}$ - Diverge $\quad \therefore \boxed{R = 1/L}$

$\boxed{Q4}$ $\sum_{j=0}^{\infty} a_j z^j$ that converges ~~to~~ at $z=2+3i$ and diverges at $z=3-i$

$|2+3i| = \sqrt{13}$ → If it ~~diverges~~ converges at $z=2+3i$, $R\le\sqrt{13}$, $|z-z_0|\le\sqrt{13}$

$|3-i| = \sqrt{10}$ → If it diverges at $z=3-i$, Radius of convergence <10

- Radius of convergence $<\sqrt{10}$
- $R\le\sqrt{13}$

$\therefore$ $\boxed{\text{Does not exist}}$

$\boxed{Q6}$

$$f(z) = \begin{cases} \dfrac{\sin(z)}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

(a) <u>Maclaurin expansion:</u>

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots$$

$$\frac{\sin z}{z} = \frac{z}{z} - \frac{z^3}{z \cdot 3!} + \frac{z^5}{z \cdot 5!} + \cdots\cdots$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\cdots$$

$$\therefore \boxed{f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\cdots}$$

(b) $f(z) = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \dfrac{z^6}{7!} + \cdots\cdots$

$f(0) = 1 \mp \cancel{0} + \cancel{0} \mp \cancel{0} + \cdots\cdots$ Differentiable for $z \neq 0$

at $z_0 = 0$

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1 - \frac{(\Delta z)^2}{3!} + \frac{(\Delta z)^4}{5!} - \cdots\cdots - 1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(-\frac{(\Delta z)}{3!} + \frac{(\Delta z)^3}{5!} - \cdots\cdots \right) = -\frac{0}{3!} + \frac{(0)^3}{5!} - \cdots\cdots = 0$$

$\therefore$ $\boxed{f(z) \text{ is analytic at the origin}}$

(c)

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

<u>Taylor series Expansion :</u>

$$f(z) = 1 + \frac{f^{1}(0)}{1!} z^1 + \frac{f^{2}(0)}{2!} z^2 + \frac{f^{3}(0)}{3!} z^3 + \frac{f^{4}(0)}{4!} z^4 + \cdots$$

$$\boxed{\begin{aligned} f^{(3)}(0) &= 0 \\ f^{(4)}(0) &= 1/5 \end{aligned}}$$