MATH 300, Summer 2021.
Solutions to Quiz 3, 7:30pm sitting

Problem 1. Evaluate

$$\int_\Gamma \frac{e^{\pi z}}{(z^2+1)^2}\, dz\,,$$

where Γ is the positively oriented circle $|z-2i| = 2$. Express your answer in the form $a+bi$, where a and b are real numbers.

Solution: Write z^2+1 as $(z+i)(z-i)$. Note that one of the roots of this polynomial, $z = i$, is inside Γ, where as the other root, $z = -i$, is outside Γ. Keeping this in mind, we write $\dfrac{e^{\pi z}}{(z^2+1)^2}$ as $\dfrac{f(z)}{(z-i)^2}$, where $f(z) = e^{\pi z}(z+i)^{-2}$ is analytic on and inside Γ. Hence, by Cauchy's integral formula,

$$\int_\Gamma \frac{e^{\pi z}}{(z^2+1)^2}\, dz = \int_\Gamma \frac{f(z)}{(z-i)^2} = 2\pi i f'(i)\,. \tag{1}$$

By the product rule,

$$f'(z) = e^{\pi z}(-2)(z+i)^{-3} + \pi e^{\pi z}(z+i)^{-2} = e^{\pi z}(z+i)^3(-2+\pi(z+i))$$

and thus $f'(i) = e^{\pi i}(2i)^{-3}(-2+2\pi i)$. Since $e^{\pi i} = -1$ (Euler's identity) and $i^3 = -i$, this simplifies to

$$f'(i) = \frac{1}{8i}(-2+2\pi i) = \frac{1}{4i}(-1+\pi i).$$

The formula (1), now yields

$$\int_\Gamma \frac{e^{\pi z}}{(z^2+1)^2}\, dz = 2\pi i f'(i) = \frac{\pi}{2}(-1+\pi i) = -\frac{\pi}{2} + \frac{\pi^2}{2} i\,.$$

Problem 2. Let $f(z)$ be an entire function, i.e., a complex-valued function analytic in the entire complex plane. Suppose $f(z) = 0$ for every z on the unit circle, $|z| = 1$. Are the following assertions true or false? If true, give a proof. If false, give a counterexample.

(a) $f(z_1) = 0$ for every z_1 inside the unit disk, i.e., for every z_1 such that $|z_1| < 1$?

(b) $f(z_2) = 0$ for every z_2 outside the unit disk, i.e., for every z_2 such that $|z_2| > 1$?

Solution: (a) True. Proof: By the Maximum Modulus Principle,

$$|f(z_1)| \leq \text{maximal value of } |f(z)| \text{ on the unit circle}$$

for any z_1 in the unit disk. But $f(z) = 0$ for any z on the unit circle. We conclude that $|f(z_1)| \leq 0$; hence, $f(z_1) = 0$ for every z_1 in the unit disk.

(b) True. Proof: Consider the Taylor series $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ at 0. By part (a), $f^{(n)}(0) = 0$ for every $n \geqslant 0$. Here $f^{(0)} = f$ and $f^{(n)}$ denotes the nth derivative of f; in particular,

$f^{(0)} = f$. Hence, $a_n = \dfrac{f^{(n)}(0)}{n!} = 0$ for every $n \geqslant 0$, We conclude that $f(z) = 0$ in the entire complex plane.

Problem 3. Consider the series $\displaystyle\sum_{j=1}^{\infty} \frac{z^j}{1+z^{2j}}$.

(a) Does this series converge for every z in the unit disk, i.e., for every z such that $|z| < 1$?

(b) Does this series converge for every z outside the unit disk, i.e., for every z such that $|z| > 1$?

Solution: (a) Yes, the series converges by the ratio test. Proof: Let $a_j = \dfrac{z^j}{1+z^{2j}}$. Then $a_{j+1} = \dfrac{z^{j+1}}{1+z^{2j+2}}$ and

$$a_{j+1}/a_j = \frac{z^{j+1}(1+z^{2j})}{z^j(1+z^{2j+2})} = z\frac{1+z^{2j}}{1+z^{2j+2}}.$$

As $\lim_{j\to\infty} z^{2j} = \lim_{j\to\infty} z^{2j+2} = 0$, we see that $\displaystyle\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}| = |z| < 1$, and the series converges by the ratio test (Theorem 2 on p. 237 in the text).

(b) Once again, the series converges by the ratio test. Proof: The calculation we carried out in part (a) shows that

$$a_{j+1}/a_j = \frac{z+z^{2j+1}}{1+z^{2j+2}} = \frac{z^{-1-2j}+z^{-1}}{z^{-2-2j}+1}$$

where the last equality is obtained by dividing top and bottom by z^{2j+2}. Since $|z| > 1$, we see that $\lim_{j\to\infty} z^{-k} = 0$ for any positive integer k. In particular,

$$\lim_{j\to\infty} z^{-1-2j} = \lim_{j\to\infty} z^{-1} = \lim_{j\to\infty} z^{-2-2j} = 0$$

and hence,

$$\lim_{j\to\infty} |a_{j+1}/a_j| = \lim_{j\to\infty} |\frac{z^{-1-2j}+z^{-1}}{z^{-2-2j}+1}| = \lim_{j\to\infty} |\frac{0+0}{0+1}| = 0.$$

The series converges by the ratio test.

Problem 4. (a) Expand $f(z) = \dfrac{z+2}{(z-1)(z-4)}$ into a Laurent series in the annulus $1 < |z| < 4$.

(b) What is the coefficient of z^{-20} in this series? What is the coefficient of z^3?

Solution: First we find the partial fraction decomposition of $\dfrac{z+2}{(z-1)(z-4)}$. Write
$$\frac{z+2}{(z-1)(z-4)} = \frac{a}{z-1} + \frac{b}{z-4},$$
multiply both sides by $(z-1)(z-4)$,
$$z+2 = a(z-4) + b(z-1).$$
Substituting $z=1$ into this formula, we obtain $-3a = 3$ or $a=-1$. Substituting $z=4$, we obtain $6 = 3b$ or $b = 2$. Thus
$$\frac{z+2}{(z-1)(z-4)} = -\frac{1}{z-1} + \frac{2}{z-4}$$
We now rewrite
$$\frac{1}{z-1} = \frac{1}{z}\,\frac{1}{1-1/z} = \frac{1}{z}\sum_{j=0}^{\infty}\big(\frac{1}{z}\big)^j.$$
Note that the above series converges whenever $|z|>1$. Similarly,
$$\frac{2}{z-4} = \big(-\frac{1}{2}\big)\,\frac{1}{1-z/4} = -\frac{1}{2}\sum_{j=0}^{\infty}\Big(\frac{1}{4^j}\Big)z^j,$$
where the series converges when $|z|<4$.

(a) In the annulus $1<|z|<4$ both series converge, and we obtain the following Laurent expansion
$$f(z) = -\sum_{k=1}^{\infty} z^{-k} - \frac{1}{2}\sum_{j=0}^{\infty}\Big(\frac{z}{4}\Big)^j.$$

(b) The coefficient of z^{-20} is -1. The coefficient of z^3 is $-\dfrac{1}{2}\Big(\dfrac{1}{4}\Big)^3 = -\dfrac{1}{2^7} = -\dfrac{1}{128}$.