Cole Shanks
54950860

Final Exam

$\boxed{Q1}$ $\quad |re^{i(\pi/4)} - 1| = 1$

* Clearly $\boxed{r=0}$ is a solution

$$= |re^{i(\pi/4)} - e^{i(-\pi)}| = 1$$

Euler's Formula:
$re^{i\theta} = r\cos\theta + ri\sin\theta$

$$|r\cdot[\cos(\pi/4) + i\sin(\pi/4)] - \cos(-\pi) - \cancel{i\sin(-\pi)}^{\emptyset}| = 1$$

$$|r(\tfrac{1}{\sqrt{2}} + i(\tfrac{1}{\sqrt{2}})] + 1| = 1$$

$$\left|\left(\frac{r}{\sqrt{2}}+1\right) + i\left(\frac{r}{\sqrt{2}}\right)\right| = 1 \qquad \therefore \sqrt{\left(\frac{r}{\sqrt{2}}+1\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2} = 1$$

$$\rightarrow \left(\frac{r}{\sqrt{2}}+1\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2 = 1$$

$$\left(\frac{r}{\sqrt{2}}+1\right)\left(\frac{r}{\sqrt{2}}+1\right) + \frac{r^2}{2} = 1$$

$$\frac{r^2}{2} + \frac{2r}{\sqrt{2}} + 1 + \frac{r^2}{2} = 1$$

$$\rightarrow r^2 + \frac{2r}{\sqrt{2}} + 1 = 1$$

$$r^2 + \frac{2r}{\sqrt{2}} = 0$$

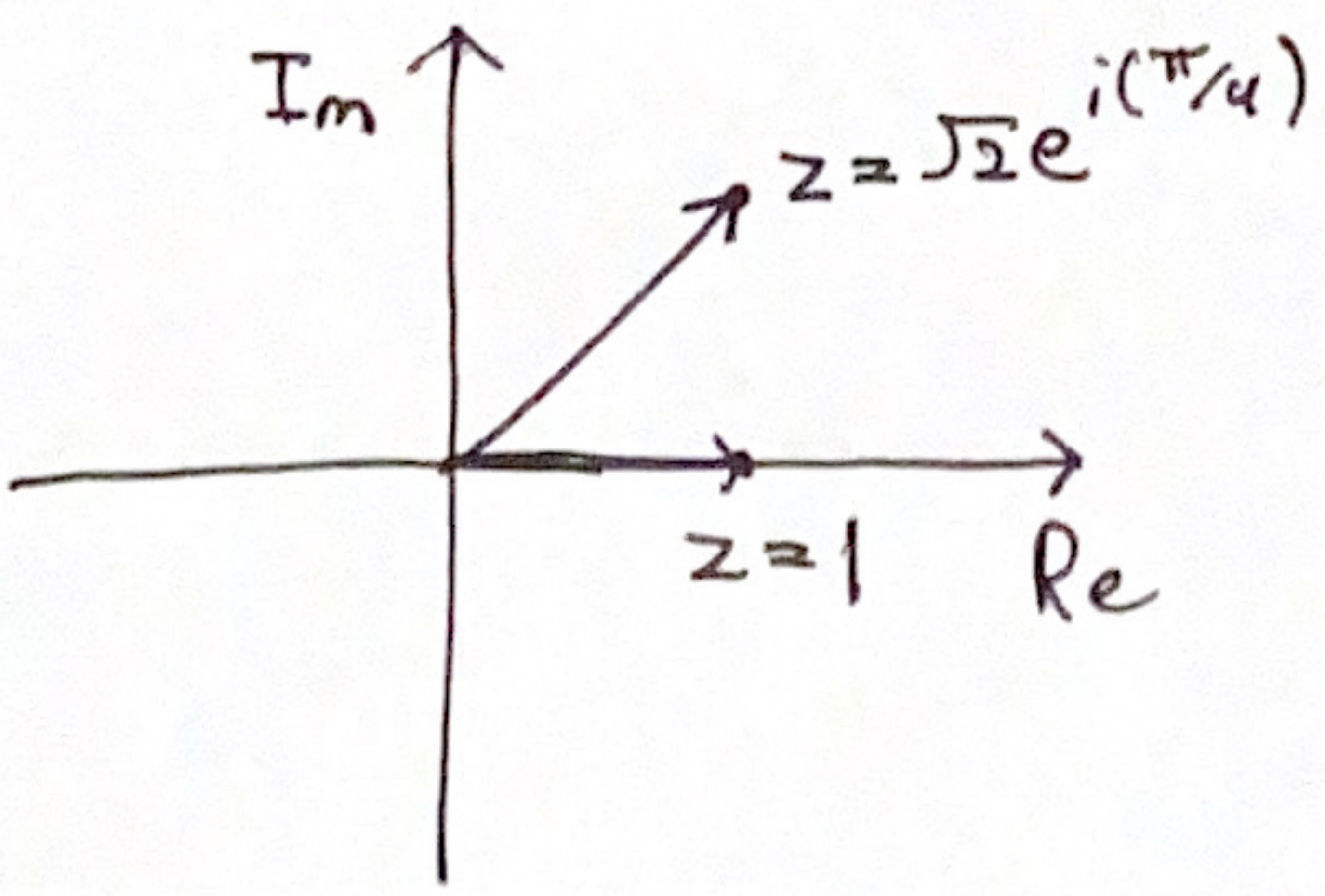
$$r(r+\sqrt{2}) = 0$$

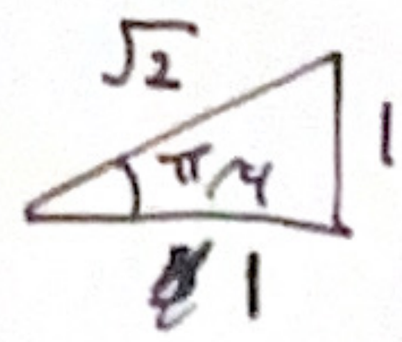
$$\cancel{\boxed{\therefore r = 0, -\sqrt{2}}}$$

* Square above and $r \geq 0$. So I can choose $r = \sqrt{2}$

$$\boxed{r = 0, \sqrt{2}}$$

-Makes sense geometrically as well





$\boxed{Q2}$

(a) $f(z) = \dfrac{z^3 \cdot e^z \sin(z)}{(1-\cos(z))^2}$; $f(0) = \dfrac{(0)(1)(0)}{(1-1)^2}$

at $z_0 = 0$, e^z is 1 so can be ignored. Expansions needed for $z^3 \cdot \sin(z)$ and $(1-\cos(z))$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots\cdots$$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} \cdots\cdots\right)$$

* Therefore in the numerator we have $z^4 \cdot \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}\right)$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots\cdots$$

$$1 - \cos(z) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} \cdots = z^2\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} \cdots\cdots\right)$$

Therefore we can re-write as

$$f(z) = \frac{z^4\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots\cdots\right)e^z}{z^4\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} \cdots\cdots\right)^2}$$

$\therefore f(z)$ has a $\boxed{\text{Removable Singularity}}$ at $z_0 = 0$

(b) $f(z) = \dfrac{z^3 e^z \sin(z)}{(1-\cos(z))^2}$; $f(2\pi) = \dfrac{(2\pi)^3 \cdot e^{2\pi} \cdot (0)}{(1-1)^2}$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\cdots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots\right)$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots \; ; \; 1-\cos(z) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} \cdots$$

$$= z^2\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} \cdots\right)$$ when squared we ~~get~~ get,

$$[1-\cos(z)]^2 = z^4\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} \cdots\right)^2$$

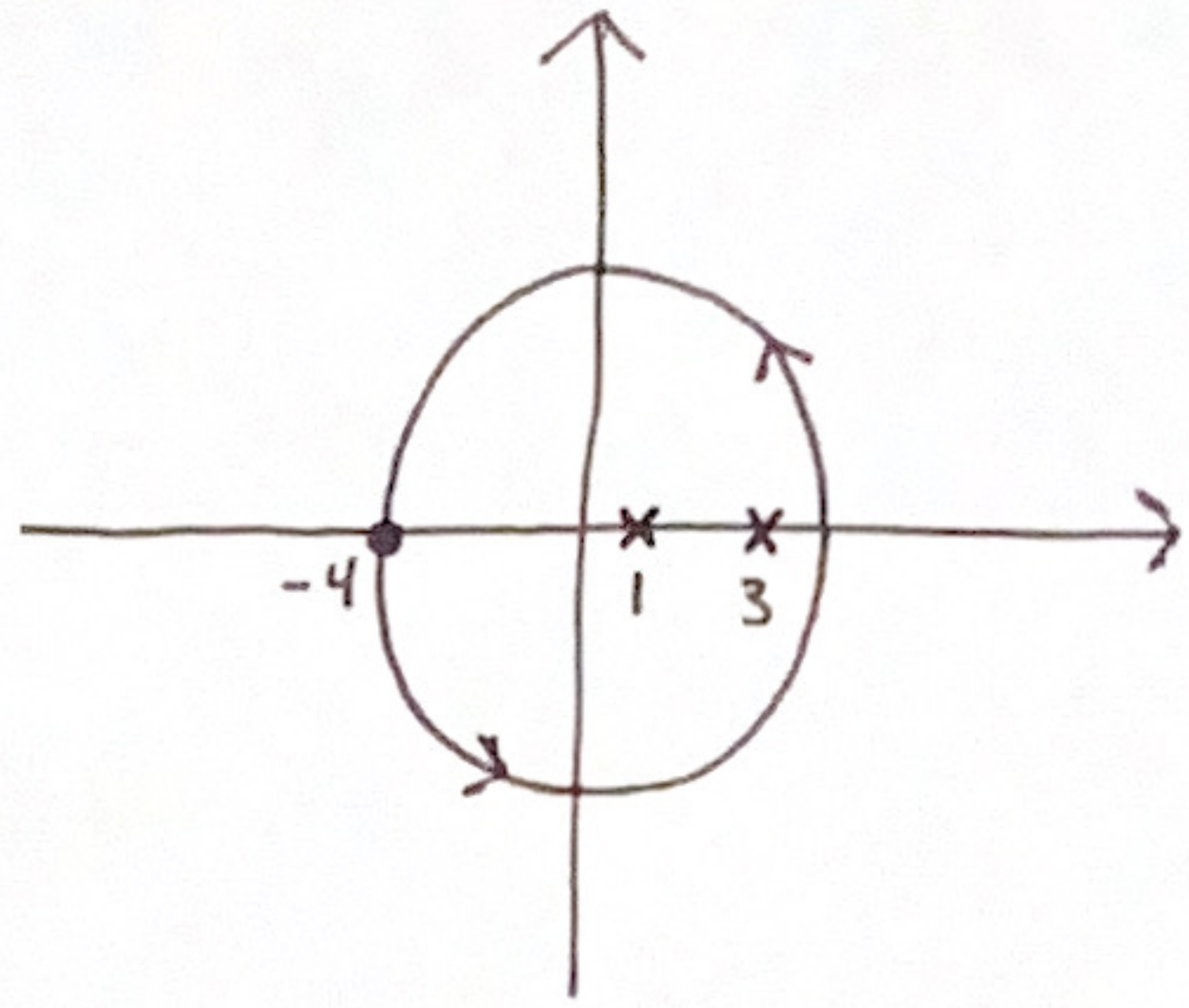
In the numerator we can factor out z

In the denominator we can factor out z^4

$\therefore$ $f(z)$ has a $\boxed{\text{Pole of order 3}}$ at $z_0 = 2\pi$

Q3

$$\int_\gamma \frac{\sin(\pi z)}{(z-1)(z-3)}\,dz$$



Partial Fractions:

$$\frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$A(z-3)+B(z-1) = 1$

let $z=3$; $B(2)=1 \to \boxed{B=1/2}$

let $z=1$; $-2A = 1 \to \boxed{A=-1/2}$

So $\frac{1}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{1/2}{z-3}$

$$\int_\gamma \frac{\sin(\pi z)}{(z-1)(z-3)}\,dz = -\frac{1}{2}\int_\gamma \frac{\sin(\pi z)}{z-1}\,dz + \frac{1}{2}\int_\gamma \frac{\sin(\pi z)}{z-3}\,dz$$

Cauchy Integral Formula: $\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

$$= \left(\frac{-1}{2}\right)(2\pi i)[\sin(\pi)] + \frac{1}{2}(2\pi i)\sin(3\pi)$$

$$= \left(\frac{-1}{2}\right)(2\pi i)(0) + \frac{1}{2}(2\pi i)(0)$$

$$= 0 + 0$$

$$= \boxed{0}$$

$\boxed{Q4}$ $|f(z) - \sin(z)| > 1; \quad z \in \mathbb{C}$

- $f(z)$ is entire
- $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\ldots$

Cauchy Integral Formula:

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{(w-z)^2}\, dw$$

$$|f'(z)| \le \frac{1}{2\pi} \int_{|w-z|=r} \frac{|f(w)|}{|w-z|^2}\, dw$$

$$\le \frac{1}{2\pi} \int_{|w-z|=r} \frac{1 - \sin(w)}{r^2}\, dw$$

$|w-z| = r$

Triangle Equality:

$|w| \le r + |z|$

Taylor Series

$$\left| f(z_0) + \frac{f'(z_0)(z-z_0)}{1} \ldots\ldots - z + \frac{z^3}{3!} - \frac{z^5}{5!} \ldots\ldots \right| > 1$$

Q5 $f(z) = \frac{1+z}{1+z+z^2}$; $z_0 = 0$

$$\sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)(z-z_0)^2}{2!} \ldots\ldots$$

Geometric Series :

$$\frac{1}{1-z} = 1+z+z^2+z^3 \ldots\ldots$$

$$\therefore \frac{1}{1-z^3} = 1+z^3+z^5+z^6 \ldots$$

$$f(z) = \frac{(1+z)(1-z)}{(1+z+z^2)(1-z)} = \frac{(1-z^2)}{(1-z^3)}$$

$$f(z) = \frac{(1-z^2)}{(1-z^3)} \qquad f(0) = 1$$

$$f'(z) = \frac{(-2z)(1-z^3) - (-3z)(1-z^2)}{(1-z^3)} \qquad f'(0) = 0$$

$$f(z) = 1-z^2 \cdot \frac{1}{1-z^3} = (1-z^2)(1+z^3+z^5+z^6 \ldots\ldots)$$

* Terms Cancel

$$= (1+z^3+\cancel{z^5} \ldots)(\cancel{1}-z^2-\cancel{z^5}-\cancel{z^7})$$

Cauchy Integral Formula : $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$

$\cancel{f^{(2)}(0) = \frac{2!}{2\pi i}\int}$

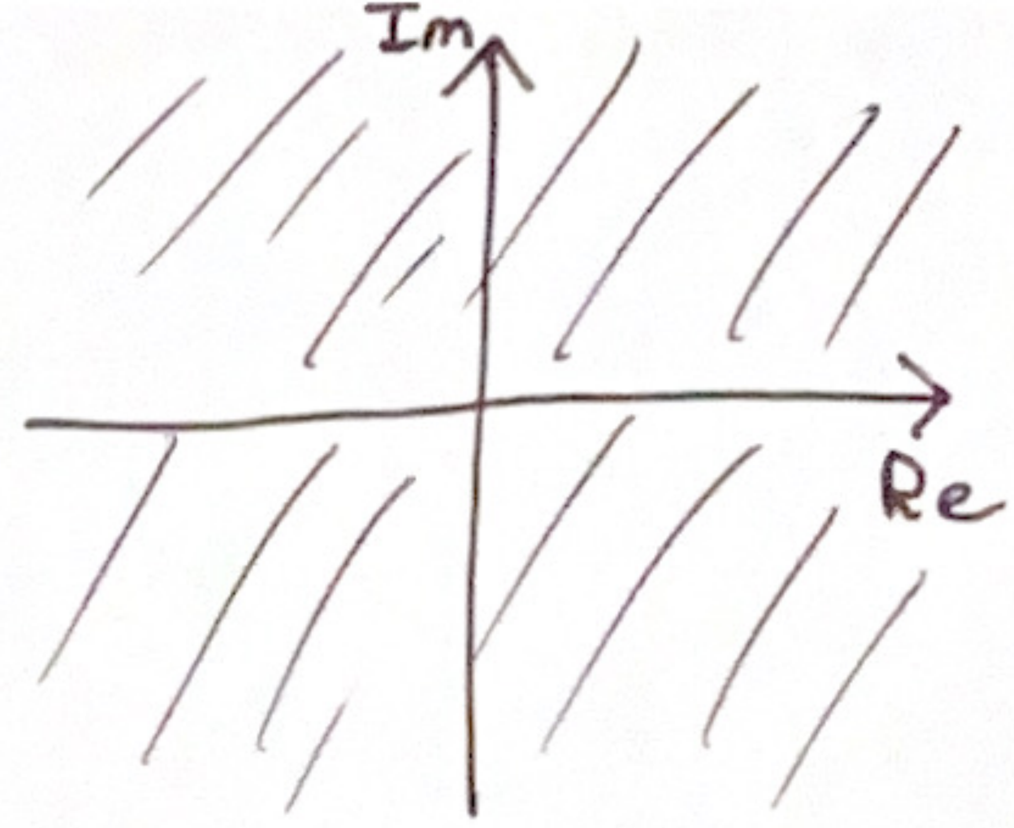
$$f(z) = 1+z^3-z^2 = z^3-z^2+1$$

(a) $f(z) = \boxed{z^3-z^2+1}$

(b)

Q6

- $f(z)$ is entire. Analytic for $z \in \mathbb{C}$
- $f(w)$ is real for all real numbers



$f(z)$ analytic $z \in \mathbb{C}$

Assertion: $\overline{f(z)} = f(\bar{z})$ for $z \in \mathbb{C}$

① We know $f(z) = \bar{z}$ is analytic nowhere

Maclarin Series $(z_0 = 0)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} \cdot z^n = f(0) + \frac{f'(0) \cdot z}{1} + \frac{f''(0)(z)^2}{2} \ldots\ldots$$

- $f(w)$ is real for all real inputs w. Meaning it lies along the Real line. The conjugate of a real number is simply the same number. ie: $Im(w) = 0$

- $f(z)$ being analytic means we can represent it as a power series and that derivatives of all orders exist. ∴ Because it's on the real line for w

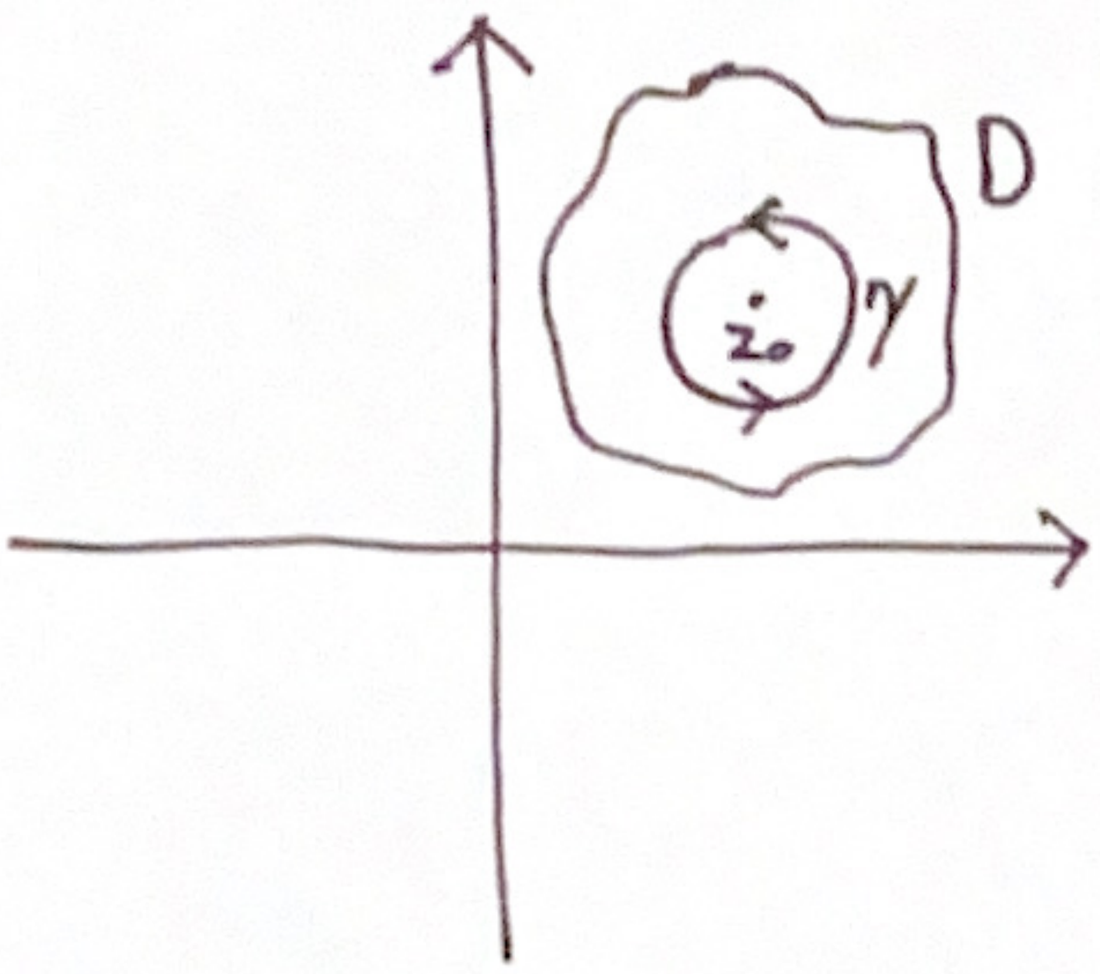
$\boxed{\text{True}}$ $\overline{f(z)} = f(\bar{z})$ for $z \in \mathbb{C}$

Q7

$$\int_\gamma \frac{1}{f(z)-f(z_0)}\,dz \quad ; \quad f'(z_0) = 16i$$

Cauchy's Integral Formula (General):

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)}\,dz$$



$$\therefore\ 16i = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)}\,dz$$

$$\int_\gamma \frac{f(z)}{(z-z_0)}\,dz = -32\pi$$

- $f(z)-f(z_0)$ is small such that the only place where $f(z)$ is not analytic is at z_0

$$\int_\gamma \frac{f(z)}{(z-z_0)}\,dz = -32\pi \longrightarrow \int_\gamma \frac{1}{f(z)-f(z_0)}\,dz = \boxed{-32\pi}$$

$\boxed{Q8}$ $\displaystyle\int_0^{\infty} \frac{1}{(1+x^2)^4}$

Since $\dfrac{1}{(1+x^2)^4}$ is an even function,

$$\int_0^{\infty} \frac{1}{(1+x^2)^4}\,dx = \frac{1}{2}\,\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^4}\,dx$$

$$\int_{\gamma} \frac{1}{(1+z^2)^4}\,dz = \int \frac{1}{\underbrace{(z+i)^4}_{\text{outside}}\underbrace{(z-i)^4}_{\text{Res}(z_i)}}\,dz$$

$$\text{Res}(i) = \lim_{z \to i} \frac{1}{(4-1)!} \cdot \frac{d^3}{dz^3}\left[(z-i)^4 \cdot \frac{1}{(z+i)^4 \cdot (z-i)^4}\right]$$

$$= \lim_{z \to i} \frac{1}{3!} \cdot \left[(-4)(-5)(-6)(z+i)^{-7}\right]$$

$$= \frac{1}{3!} \cdot \left(\frac{-15i}{16}\right) = \frac{-5i}{32} \qquad \therefore \text{Res}(i) = \frac{-5i}{32}$$

$$\therefore \int_0^{\infty} \frac{1}{(1+x^2)^4}\,dx = \left(\frac{1}{2}\right) \cdot (2\pi i) \cdot \left(\frac{-5i}{32}\right) = \boxed{\frac{5\pi}{32}}$$