Math. 300. Solutions to practice problems for Quiz 3. Summer 2021

Problem 1. For each function $f(z)$ below evaluate

$$\int_\Gamma f(z)\, dz\,,$$

where Γ is the positively oriented circle of radius 2 centered at the origin.

(a) $f(z) = \dfrac{\sin(z)}{2z}$, (b) $f(z) = e^z \cos(z)$, (c) $f(z) = \overline{z}$, (d) $f(z) = \dfrac{e^{iz}}{(z+i)^3}$.

Solution: (a) Let $g(z) = \sin(z)/2$. By Cauchy's Integral Formula

$$\int_\Gamma f(z)\, dz = \int_\Gamma \frac{g(z)}{z}\, dz = 2\pi i g(0) = 0\,.$$

(b) Here $f(z)$ is analytic in the entire plane. Hence, by the Cauchy Integral Theorem,

$$\int_\Gamma f(z)\, dz = 0\,.$$

(c) Note that on Γ, $|z| = 2$. Thus $f(z) = \overline{z} = \dfrac{|z|^2}{z} = \dfrac{4}{z}$ and

$$\int_\Gamma f(z)\, dz = \int_\Gamma \frac{4}{z}\, dz = 4\int_\Gamma \frac{1}{z}\, dz = 4 \cdot (2\pi i) = 8\pi i\,.$$

(d) Let $g(z) = e^{iz}$. Then $g'(z) = ie^{iz}$ and $g''(z) = -e^{iz}$. By the Cauchy Integral Formula,

$$\int_\Gamma f(z)\, dz = \frac{2\pi i}{2!} g''(-i) = \pi i \cdot (-e^1) = -\pi e i\,.$$

Problem 2. For which complex numbers z do the following series converge?

(a) $\sum_{j=0}^\infty e^{zj}$? (b) $\sum_{j=0}^\infty (j+i) z^j$?

Solution: The series in part (a) is a geometric series $\displaystyle\sum_{j=0}^\infty q^n$, where $q = e^z$. It converges if and only if $|q| < 1$, i.e., $|e^z| < 1$. This is equivalent to $e^{\mathrm{Re}(z)} < 1$ or $\mathrm{Re}(z) < 0$.

In summary, the series in part (a) converges if and only if z is of the form $x + yi$, where x and y are real numbers and $x < 0$.

In part (b), I claim that (i) the series converges if $|z| < 1$ and (ii) the series diverges if $|z| \geq 1$.

To prove (i), use the ratio test. Here the jth term of the series is $a_j = (j+i)z^j$, so

$$|\frac{a_{j+1}}{a_j}| = |\frac{j+i+1}{j+i}| \cdot |\frac{z^{j+1}}{z^j}| = |\frac{1+\frac{i+1}{j}}{1+\frac{i}{j}}| \cdot |z| \to |z|,$$

as $j \to \infty$. If $|z| < 1$, the ratio test (Lecture 18 or Theorem 2, p. 237 in the text) tells us that the series converges.

To prove (ii), note that when $|z| \geq 1$, $|a_j| = |j+i| \cdot |z^j| \geq (j-1) \cdot 1 = j-1$ does not converge to 0, so the series diverges by the nth term test (Lecture 18, p.2 or Exercise 5.1.5, p. 240 in the text).

Problem 3. Suppose $f(z)$ is an entire function such that $|f(z)| < 2|z|^2 + 5$ for all complex numbers z. Show that $f(z)$ is a polynomial of degree ≤ 2.

Hint: Use the Cauchy estimate for $|f'''(z)|$ to show that the third derivative $f'''(z)$ is identically zero.

Solution: Let z_0 be a complex number and C_R be a circle of radius R centered at z_0. Then for any z on C_R,

$$|z| \leq |z - z_0| + |z_0| = R + |z_0|$$

and thus $|f(z)| < 2|z|^2 + 5 \leq 2(R + |z_0|)^2 + 5$. By Cauchy's estimate (Theorem 20 on p. 215 in the book, with $n = 3$),

$$|f'''(z_0)| < \frac{3! \cdot (2(R + |z_0|)^2 + 5)}{R^3}.$$

Letting $R \to \infty$, we obtain $|f'''(z_0)| = 0$ and thus $f'''(z_0) = 0$. We conclude that $f'''(z)$ is identically zero. Since the derivative of the zero function is again zero everywhere, we see that $f^{(n)}(z)$ is identically 0 for every $n \geq 3$. Expanding $f(z)$ into a Taylor series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

at $z_0 = 0$, we see that $a_j = \dfrac{f^{(j)}(0)}{j!} = 0$ for every $j \geq 3$ and thus $f(z) = a_0 + a_1 z + a_2 z^2$ is a polynomial of degree ≤ 2.

Problem 4. Let $f(z)$ and $g(z)$ be two analytic functions defined in the unit disk $|z| < 1$.

(a) If $f(\frac{1}{n}) = g(\frac{1}{n})$ for every integer $n \geq 2$, then $f(z) = g(z)$ for every complex number z.

(b) Give an example of two functions $f(z)$ and $g(z)$ analytic in the unit disk $|z| < 1$ such that $f(-1 + \frac{1}{n}) = g(-1 + \frac{1}{n})$ for every integer $n \geq 2$ but $f \neq g$.

(c) Can you explain the discrepancy between (a) and (b)?

Solution: (a) The function $h(z) = f(z) - g(z)$ is analytic in D. Note that $h(\frac{1}{n}) = 0$ for every $n \geq 2$. We want to show that $h(z)$ is identically 0 in D. Since $h(z)$ is analytic in D, it is continuous and thus $h(0) = \lim_{n \to \infty} h(\frac{1}{n}) = \lim_{n \to \infty} 0 = 0$. As we showed at the beginning of Lecture 20 (see also Corollary 3 on p. 278), either

(i) $h(z) \neq 0$ for any $z \neq 0$ in some open disk centered at 0, or

(ii) $h(z) = 0$ for every z in some disk centered at 0.

Since $h(\frac{1}{n}) = 0$ for every n, (i) is impossible, so (ii) holds. It follows that $h^{(j)}(0) = 0$ for every $j \geq 1$, so the Taylor series of $h(z)$ at 0 is identically 0. By uniqueness of the Taylor series, $h(z) = 0$ in all of D, as claimed.

(b) Set $f(z) = e^{\frac{2\pi i}{z+1}} - 1$ and $g(z) = 0$ in D. Then $f(-1 + \frac{1}{n}) = g(-1 + \frac{1}{n}) = 0$ for every $n \geq 2$ but

$$f(-\frac{1}{3}) = e^{\frac{2\pi i}{2/3}} - 1 = e^{3\pi i} - 1 = -2 \neq 0 = g(-\frac{1}{3}).$$

(c) The sequence of $z_n = \frac{1}{n}$ of zeros of $h = f - g$ in part (a) converges to 0, which is within D. In particular, $h(z)$ is analytic at 0. On the other hand, the sequence $w_n = -1 + \frac{1}{n}$ of zeros of $h = f - g$ in part (a) converges to -1, where h is not analytic. (Note that in part (b), $g = 0$, so $h(z) = f(z)$.)

Problem 5. Let $f(z)$ be an analytic function in the unit disk. Denote its Taylor expansion at the origin by $\sum_{j=0}^{\infty} a_j z^j$. Show that if $f(z)$ is even, i.e., $f(-z) = f(z)$ for every z in D, then $a_{2k-1} = 0$ for $k = 1, 2, 3, \ldots$.

Solution: $f(z) = f(-z) = \sum_{j=0}^{\infty} a_j(-z)^j = \sum_{j=0}^{\infty} (-1)^j a_j z^j$ is another power series expansion for $f(z)$. By uniqueness of the power series representing $f(z)$ (Theorem 11 on p. 256), the two series have to have the same coefficients. In other words, $(-1)^j a_j = a_j$ for every $j \geq 0$. For even j this condition is vacuous. For odd $j = 2k-1$ it yields $a_{2k-1} = -a_{2k-1}$ or equivalently, $a_{2k-1} = 0$.

Problem 6. Let $\sum_{j=0}^{\infty} a_j z^j$ be the Taylor series for $\sec(z)$ at $z_0 = 0$. Find a_0, a_1, a_2, a_3 and a_4.

Solution: We know that $f(z) = \sec(z) = \frac{1}{\cos(z)}$ is an even function of z, so by Problem 5, $a_j = 0$ when j is odd. In particular, $a_1 = a_3 = 0$. On the other hand, $a_0 = f(0) = \sec(0) = 1$. To determine a_3 and a_5, we expand $\cos(z) \cdot \sec(z) = 1$ as

$$(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \ldots)(1 + a_2 z^2 + a_4 z^4 + \ldots) = 1.$$

Equating the z^2 terms on both sides, we obtain $a_2 - \frac{1}{2} = 0$, so $a_2 = \frac{1}{2}$. Equating the z^4 terms, we obtain $a_4 - \frac{1}{2}a_2 + \frac{1}{4!} = 0$. Thus

$$a_4 = \frac{1}{2}a_2 - \frac{1}{24} = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}.$$

In summary, $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 0$, and $a_4 = \frac{5}{24}$.

Problem 7. Expand $f(z) = \frac{z}{z^2+1}$ into a Laurent series in the annulus $0 < |z-i| < 2$.

Solution: First we find the partial fraction decomposition of $f(z)$:

$$f(z) = \frac{z}{(z-i)(z+i)} = \frac{1/2}{z-i} + \frac{1/2}{z+i}.$$

I am leaving out some steps here, but they are standard (we have done this many times earlier in the course) and easy (because $z^2+1 = (z-i)(z+i)$ has only two roots, each with multiplicity 1). Please check the details.

Note that our Laurent series is centered at $z_0 = i$, so it will be a series in the (positive and negative) powers of $z-i$. In particular, $\frac{1/2}{z-i} = \frac{1}{2}(z-i)^{-1}$ will be one of the terms in this series. On the other hand, the second partial fraction $\frac{1/2}{z+i}$, is analytic at $z_0 = i$, so it will only contribute non-negative powers of $z-i$. To find the coefficient, we make use of the formula for the geometric series:

$$\frac{1}{z+i} = \frac{1}{2i + (z-i)} = \frac{1}{2i}\frac{1}{1-(-\frac{z-i}{2i})} = \frac{1}{2i}\frac{1}{1-(\frac{i(z-i)}{2})} = \frac{1}{2i}\sum_{j=0}^{\infty}(\frac{i}{2})^j(z-i)^j.$$

Note that this series converges because $|\frac{z-i}{2i}| = |\frac{z-i}{2}| < 1$.

In summary, the Laurent expansion of $f(z) = \frac{z}{z^2+1}$ in the annulus $0 < |z-i| < 2$ is

$$f(z) = \frac{1}{2}(z-i)^{-1} + \frac{1}{2}\sum_{j=0}^{\infty}\frac{i^{j-1}}{2^{j+1}}(z-i)^j = \frac{1}{2}(z-i)^{-1} + \sum_{j=0}^{\infty}\frac{i^{j-1}}{2^{j+2}}(z-i)^j.$$