MATH 300, Summer 2021.
Solutions to Quiz 3, 1pm sitting

Problem 1. Evaluate
$$\int_\Gamma \frac{\sin(z)}{z^2(z-3)}\,dz\,,$$
where Γ is the positively oriented unit circle, $|z|=1$.

Solution: Write $\dfrac{\sin(z)}{z^2(z-3)}$ as $\dfrac{f(z)}{z^2}$, where $f(z)=\sin(z)(z-3)^{-1}$ is analytic in the domain $|z|<3$, containing Γ. Hence, by Cauchy's integral formula,
$$\int_\Gamma \frac{\sin(z)}{z^2(z-3)}\,dz = \int_\Gamma \frac{f(z)}{z^2} = 2\pi i f'(0)\,. \tag{1}$$
By the product rule,
$$f'(z)=\sin(z)(-1)(z-3)^{-2}+\cos(z)(z-3)^{-1}$$
and thus $f'(0)=0+1\cdot(-3)^{-1}=-\dfrac{1}{3}$. Substituting this into (1), we obtain
$$\int_\Gamma \frac{\sin(z)}{z^2(z-3)}\,dz = -\frac{2\pi}{3}i\,.$$

Problem 2. Let $f(z)$ be an entire function, i.e., a complex-valued function analytic in the entire complex plane. Suppose there exists a disk D of radius $r>0$ in the complex plane such that $f(z)$ does not assume any values in D. In other words, $|f(z)-w|\geqslant r$ for every complex number z, where w denotes the centre of D. Show that $f(z)$ is constant in the entire complex plane.

Solution: Under the assumptions of this problem, $g(z)=\dfrac{1}{f(z)-w}$ is entire. Moreover, $g(z)$ is bounded because
$$|g(z)|\le \frac{1}{|f(z)-w|}\le \frac{1}{r}$$
for every z in the complex plane. By Louiville's Theorem, $g(z)$ is a constant function, $g(z)=c$. Hence,
$$f(z)=\frac{1}{g(z)}+w=c+w$$
is also constant, as claimed.

Problem 3. Suppose $f(z)$ is an analytic function in the open unit disc, $|z| < 1$, and $f^{(n)}(0) = n!$ for every integer $n \geq 0$. Find $f(\frac{i}{2})$. Express your answer in the form $a + bi$, where a and b are real numbers.
Here $f^{(n)}$ denotes the nth derivative of f at the origin. In particular, $f^{(0)} = f$.

Solution: The Taylor expansion of $f(z)$ at the origin is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!} = 1 \text{ for every } n.$$

In other words, the Taylor series for $f(z)$ is the geometric series $1 + z + z^2 + z^3 + \ldots$ and thus

$$f(z) = 1 + z + z^2 + z^3 + \ldots = \frac{1}{1-z}.$$

In particular,

$$\frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2(2+i)}{5} = \frac{4}{5} + \frac{2}{5}i.$$

Problem 4. Suppose $f(z)$ is analytic at $z = 0$, and its Taylor series at $z = 0$ is of the form

$$f(z) = 2z^2 + 3z^3 + 5z^4 + \text{higher order terms}$$

Assume further that the Laurent expansion of $\frac{1}{f(z)}$ at $z = 0$ is

$$\frac{1}{f(z)} = az^{-2} + bz^{-1} + c + dz + \ldots$$

Find a, b and c.

Solution: Writing $f(z) = z^2(2+3z+5z^2+\ldots)$ and $\frac{1}{f(z)} = z^{-2}(a+bz+cz^2+dz^3+\ldots)$, we obtain

$$(2 + 3z + 5z^2 + \ldots)(a + bz + cz^2 + dz^3 + \ldots) = 1.$$

Multiplying term by term, we obtain

$$(2a) + (2b + 3a)z + (2c + 3b + 5a)z^2 + \ldots = 1.$$

The left hand side of this equation is the Taylor expansion of the constant function 1 at 0. By uniqueness of the Taylor expansion, the constant term on the left hand side is 1, and all other coefficients are 0. In particular,

$$\begin{aligned} 2a &= 1 \\ 2b + 3a &= 0 \\ 2c + 3b + 5a &= 0. \end{aligned}$$

Solving for a, b, and c, we obtain

$$a = \frac{1}{2}, \quad b = -\frac{3}{2}a = -\frac{3}{4}, \quad c = -\frac{1}{2}(3b + 5a) = -\frac{1}{2}(-\frac{9}{4} + \frac{5}{2}) = -\frac{1}{8}.$$