Mathematics 300, Practice problems for Quiz 1

Problem 1: Find the absolute value and (the principal value of) the argument for each of the complex numbers below. Recall that the principal value of the argument is the unique value in the interval $[-\pi, \pi)$.

(a) $\dfrac{2}{i} + \dfrac{i}{5}$. (b) $\left(\dfrac{1+i\sqrt{3}}{2}\right)^{3000}$.

Solution: (a) Note that $\dfrac{2}{i} = \dfrac{2i}{i^2} = -2i$. Thus $z = -\dfrac{9}{5}i$. We conclude that $|z| = \dfrac{9}{5}$ and $\mathrm{Arg}(z) = -\dfrac{\pi}{2}$.

(b) Here $\dfrac{1+i\sqrt{3}}{2} = e^{\pi i/3}$ and thus $w = e^{1000\pi i} = 1$. Thus $|w| = 1$ and $\mathrm{Arg}(w) = 0$.

Problem 2: Find all solutions of the equation

$$z^4 = 8iz$$

and express them in the Cartesian form, $z = a + ib$, where a and b are real numbers.

Solution: The equation can be rewritten as

$$z(z^3 - 8i) = 0.$$

Thus there are four solutions, $z = 0$ and the three cube roots of $8i = 8e^{\pi/2}$, namely

$z = 2e^{\pi i/6} = \sqrt{3} + i$,

$z = 2e^{\pi i/6 + 2\pi i/3} = 2e^{5\pi i/6} = -\sqrt{3} + i$, and

$z = 2e^{\pi i/6 + 4\pi i/3} = 2e^{9\pi i/6} = 2e^{3\pi/2} = 2e^{2\pi i - (\pi i/2)} = 2 \cdot e^{2\pi i} \cdot e^{-\pi i/2} = 2 \cdot 1 \cdot (-i) = -2i$.

Problem 3: Is it true that $|e^z| \leq e^{|z|}$ for every complex number z? Give a proof (if true) or a counterexample (if false).

Solution: True. Write $z = x + yi$ in Cartesian form. Here x and y are real numbers. Then by definition of e^z, $|e^z| = e^x$. Since

$$x \leq |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z|,$$

we conclude that $|e^z| = e^x \leq e^{|z|}$.

Problem 4: Show that if $f(z)$ and $\overline{f(z)}$ are both analytic in a domain D, then $f(z)$ is constant in D.

Solution: If both are analytic, then so are $g(z) = f(z) + \overline{f(z)}$ and $h(z) = f(z) - \overline{f(z)}$. As we showed in class, the only real-valued analytic function in a domain D, is a constant function. Thus $g(z) = c_1$ is constant. Similarly, since $h(z)$ is analytic in D and assumes only pure imaginary values, $h(z) = c_2$ is constant.

Now $f(z) = \dfrac{1}{2}(g(z) + h(z)) = \dfrac{1}{2}(c_1 + c_2)$ is also constant.

Problem 5: Show that the image of the circle $|z-1| = 1$ under the invertion map $w = f(z) = 1/z$ is the line $\text{Re}(w) = \frac{1}{2}$.

Solution: We need to show

(i) if $|z-1| = 1$ then $\text{Re}(f(z)) = \frac{1}{2}$. In other words, f takes points on the circle $|z-1| = 1$ to the line $\text{Re}(w) = \frac{1}{2}$.

(ii) if $\text{Re}(w) = \dfrac{1}{2}$ then $w = f(z)$ for some z such that $|z-1| = 1$. In other words, the image of the circle $|z-1| = 1$ under f fills the entire line, i.e., f is onto.

(i) was proved in Problem 1.2.16 from Assignment 1. To apply Problem 1.2.16, replace z by $1 - z$.

Proof of (ii): Set $z = \dfrac{1}{w}$. We want to show that $|z-1| = 1$. Write $w = \dfrac{1}{2} + yi$, where y is a real number. Then

$$|z-1| = |\frac{1}{1/2 + yi} - 1| = |\frac{1 - 1/2 - yi}{1/2 + yi}| = |\frac{1/2 - yi}{1/2 + yi}| = |\frac{\overline{w}}{w}| = \frac{|\overline{w}|}{|w|} = 1\,.$$

Problem 6: Is the function $f(z) = (x^3 + 3xy^2 - 3x) + (y^3 + 3x^2y - 3y)i$ differentiable at $z = i$? Is it analytic at $z = i$? Explain your answer. Here, as usual, $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

Solution: Write $f(z) = u(x, y) + v(x, y)i$, where $u(x, y) = x^3 + 3xy^2 - 3x$ and $v(x, y) = y^3 + 3x^2y - 3y$. Let us compute the partial derivatives of $u(x, y)$ and $v(x, y)$:

$$\begin{array}{rr} u_x = 3x^2 + 3y^2 - 3 & v_x = 6xy \\ u_y = 6xy & v_y = 3x^2 + 3y^2 - 3 \end{array}$$

Note that the first Cauchy-Riemann $u_x = v_y$ is satisfied for every z. The second Cauchy-Riemann equation, $u_y - = v_x$ or equivalently $6xy = -6yx$, is satisfied if and only if $x = 0$ or $y = 0$. In particular, this tells us that $f(z)$ is differentiable at $z = i$. On the other hand, since $f(z)$ is not differentiable in any open ball centered at $z = i$, it is not analytic at $z = i$.