Mathematics 300. Solutions to practice problems for Quiz 2

Problem 1: Let $v(x, y) = 5x - xy + 4$.
(a) Show that $v(x, y)$ is harmonic in the entire plane.
(b) Construct an entire function $f(z)$ such that $\text{Im}(f(z)) = v(x, y)$.

Solution: (a) One readily checks that $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial y^2} = 0$. Hence, v satisfies Laplace's equation, i.e., v is harmonic.

(b) Suppose $f(z) = u(x, y) + v(x, y)i$. We want to solve the Cauchy-Riemann equations for $u(x, y)$. The first Cauchy-Riemann equation tells us that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -x.$$

Anti-differentiating with respect to x, we obtain

$$u(x, y) = -\frac{x^2}{2} + \phi(y)$$

for some function $\phi(y)$. Note that $\phi(y)$ depends only on y, not on x. The second Cauchy-Riemann equation tells us that

$$\phi'(y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -(5 - y).$$

Thus $\phi'(y) = y - 5$, and $\phi(y) = \dfrac{y^2}{2} - 5y + C$, where C is a real constant.

In summary, $u(x, y) = -\dfrac{x^2}{2} + \dfrac{y^2}{2} - 5y + C$ and $f(z) = -\dfrac{x^2}{2} + \dfrac{y^2}{2} - 5y + C + (5x - xy + 4)i$.

Problem 2: Find the partial fraction decomposition of

$$R(z) = \frac{2}{z(1-z)^2}.$$

Solution: The partial fraction decomposition of $R(z)$ is of the form

$$\frac{2}{z(1-z)^2} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}.$$

To solve for A, B, and C, clear denominators:

$$2 = A(1-z)^2 + Bz(1-z) + Cz.$$

Setting $z = 0$, we obtain $A = 2$. Setting $z = 1$, we obtain $C = 2$. Comparing the coefficients of z^2 on both sides, we obtain $0 = A - B$. Thus $B = A = 2$. The final answer is

$$\frac{2}{z(1-z)^2} = \frac{2}{z} + \frac{2}{1-z} + \frac{2}{(1-z)^2}.$$

Problem 3: Show that the function $f(z) = \text{Log}(-z) + i\pi$ is a branch of $\log(z)$ that is analytic in the open subset D of the complex plane, where D is the entire complex plane with the non-negative real axis removed.

Solution: To show that $f(z)$ is a branch of $\log(z)$, we need to check that $e^{f(z)} = z$. Indeed,

$$e^{\mathrm{Log}(-z)+\pi i} = e^{\mathrm{Log}(-z)}e^{\pi i} = (-z)(-1) = z.$$

To show that $f(z)$ is analytic in D, note that since D is an open subset of $\mathbb{C}$ it is enough to show that $f(z)$ is differentiable at every z_0 in D. Recall that $\mathrm{Log}(z)$ is differentiable at any z_0 away from the non-positive real axis, and the constant function $i\pi$ is analytic in the entire complex plane. Hence, using the sum rule and the Chain rule for complex derivatives, we see that $f(z)$ is differentiable at z_0 whenever $-z_0$ lies away from the non-positive real axis, i.e., for every z_0 in D.

Problem 4: Find all complex solutions to the equation $\sinh(z) = i$. Here, as usual, $\sinh(z)$ denotes the hyperbolic sine function, $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$.

Solution: In the calculation below $\Longleftrightarrow$ stands for "if and only if".

$\sinh(z) = i \Longleftrightarrow \dfrac{e^z - e^{-z}}{2} = i \Longleftrightarrow e^z - e^{-z} = 2i \Longleftrightarrow e^{2z} - 1 = 2ie^z$

$\Longleftrightarrow e^{2z} - 2ie^z - 1 = 0 \Longleftrightarrow (e^z - i)^2 = 0 \Longleftrightarrow e^z = i \Longleftrightarrow e^z = e^{\frac{\pi}{2}i}$

$\Longleftrightarrow z = (\dfrac{\pi}{2} + 2\pi n)i$, where n is an integer.

Problem 5: Let Γ be the piece of the parabola $y = x^2$ from 0 to $2 + 4i$. Find

$$\int_\Gamma |z|^2 dz.$$

Solution: Parametrize the parabola as follows: $z(t) = t + t^2 i$, where $0 \le t \le 2$. Now

$$\int_\Gamma |z|^2 dz = \int_0^2 |z(t)|^2 z'(t)dt = \int_0^2 (t^2 + t^4)(1 + 2ti)dt =$$

$$\int_0^2 (t^2 + t^4)dt + 2i(\int_0^2 (t^3 + t^5)dt) = (\frac{t^3}{3} + \frac{t^5}{5})|_0^2 + 2i(\frac{t^4}{4} + \frac{t^6}{6})|_0^2 =$$

$$(\frac{8}{3} + \frac{32}{5}) + 2i(\frac{16}{4} + \frac{64}{6}) = 9\frac{1}{15} + 29\frac{1}{3}i.$$

Problem 6: Compute

$$\int_\Gamma \frac{dz}{(z-1)(z+1)},$$

where Γ is the circle $|z| = 2$ traversed once in the counterclockwise direction.

Hint: Use partial fractions.

Solution: Following the hint, we decompose $\dfrac{1}{(z-1)(z+1)}$ as a sum of partial fractions. To obtain the partial fraction decomposition, we set

$$\frac{1}{(z-1)(z+1)} = \frac{a}{z-1} + \frac{b}{z+1}.$$

To solve for a and b, we first multiply both sides by $(z-1)(z+1)$:

$$1 = a(z+1) + b(z-1).$$

Substituting $z = 1$, we obtain $1 = 2a$. Thus $a = \frac{1}{2}$. Similarly, substituting -1 for z, we obtain $b = -\frac{1}{2}$. We have thus decomposed $\frac{1}{(z-1)(z+1)}$ as a sum of partial fractions:

$$\frac{1}{(z-1)(z+1)} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1},$$

Integrating both sides over Γ, we obtain

$$\int_\Gamma \frac{dz}{(z-1)(z+1)} = \frac{1}{2}\int_\Gamma \frac{dz}{z-1} - \frac{1}{2}\int_\Gamma \frac{dz}{z+1},$$

The two integrals on the right will turn out to be easier to evaluate than the integral on the left. The reason is that $\frac{1}{(z-1)(z+1)}$ is non-analytic at two points inside Γ, namely, -1 and 1, where as each of the partial fractions $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ non-analytic at only one point. This gives us greater freedom to deform Γ into a simpler contour.

To evaluate the first integral, we deform Γ to Γ_1, where Γ_1 is a positively oriented circle of radius 1 centered at 1. This can be done within the open set $\mathbb{C} \setminus \{1\}$, where $\frac{1}{z-1}$ is analytic. By Theorem 2 from lecture 14,

$$\int_\Gamma \frac{dz}{z-1} = \int_{\Gamma_1} \frac{dz}{z-1},$$

and we know that the latter integral is $2\pi i$. Similarly,

$$\int_\Gamma \frac{dz}{z+1} = \int_{\Gamma_2} \frac{dz}{z+1} = 2\pi i\,,$$

where Γ_2 is a positively oriented circle of radius 1 centered at -1. Note that Γ can be deformed to Γ_2 within the open set $\mathbb{C} \setminus \{-1\}$, where $\frac{1}{z+1}$ is analytic. We conclude that

$$\int_\Gamma \frac{dz}{(z-1)(z+1)} = \int_\Gamma \frac{1}{2}\frac{dz}{z-1} - \int_\Gamma \frac{1}{2}\frac{dz}{z+1} = \frac{1}{2}\cdot 2\pi i - \frac{1}{2}\cdot 2\pi i = 0.$$