MATH 300, July 2021. Solutions to Quiz 2, 7:30pm sitting

Problem 1. (5 marks) Show that $u(x, y) = xy^3 - x^3y$ is a harmonic function in the complex plane, and find a harmonic conjugate function $v(x, y)$,

Solution: To check that $u(x, y)$ is harmonic, we compute its partial derivatives:

$$u_x = y^3 - 3x^2y,\ u_{xx} = -6xy, \text{ and } u_y = 3xy^2 - x^3,\ u_{yy} = 6xy.$$

Thus $u_{xx} + u_{yy} = -6xy + 6xy = 0$, i.e., $u(x, y)$ is harmonic, as desired.

To find $v(x, y)$, we solve the Cauchy-Riemann equations:

$$v_x = -u_y = -3xy^2 + x^3 \text{ and } v_y = u_x = y^3 - 3x^2y.$$

Anti-differentiating the first equation with respect to x, we obtain

$$v(x, y) = -\frac{3}{2}x^2y^2 + \frac{1}{4}x^4 + h(y)\,,$$

where $h(y)$ is a function of y. Substituting this into $v_y = y^3 - 3x^2y$, we obtain

$$-3x^2y + h'(y) = y^3 - 3x^2y$$

or equivalently, $h'(y) = y^3$. Solving for $h(y)$, we obtain $h(y) = \dfrac{1}{4}y^4 + C$, where C is a constant.

We thus arrive at the following answer: $v(x, y) = -\frac{3}{2}x^2y^2 + \frac{1}{4}x^4 + h(y) = -\frac{3}{2}x^2y^2 + \frac{1}{4}x^4 + \frac{1}{4}y^4 + C$.

Problem 2. (5 marks) Find the partial fraction decomposition of the function

$$R(z) = \frac{5z^3 + 3z^2 + z + 1}{z^2(z^2 + 1)}\,.$$

Solution: The denominator factors as $z^2(z + i)(z - i)$. The partial fraction decomposition has the form

$$\frac{5z^3 + 3z^2 + z + 1}{z^2(z^2 + 1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z - i} + \frac{D}{z + i}\,;$$

see Theorem 2 on p. 105 in the text. One of the things I was testing in this problem is whether or not you know this general form. If your partial fractions had z^2+1 or $z(z^2+1)$ in the denominator, I usually took off 2 marks.

To compute A, B, C and D, let us multiply both sides by $z^2(z^2 + 1)$. This yields

(1) $$5z^3 + 3z^2 + z + 1 = Az(z^2 + 1) + B(z^2 + 1) + Cz^2(z + i) + Dz^2(z - i)\,.$$

Setting $z = 0$, we obtain $\boxed{B = 1.}$

Setting $z = i$, we obtain $-5i - 3 + i + 1 = (-1)C(2i)$.

Equivalently, $-4i - 2 = (-2i)C$ or $\boxed{C = 2 - i.}$

Similarly, setting $z = -i$, we obtain $5i - 3 - i + 1 = D(2i)$.

This simplifies to $4i - 2 = (2i) \cdot D$ or equivalently, to $\boxed{D = 2 + i.}$

To find A, we can proceed in one of several ways. One is to equate the coefficient of z^3 on both sides of equation (1) This yields $5 = A + C + D$. Since $C + D = 4$, we obtain $\boxed{A = 1.}$

Answer:

$$\frac{5z^3 + 3z^2 + z + 1}{z^2(z^2 + 1)} = \frac{1}{z} + \frac{1}{z^2} + \frac{2 - i}{z - i} + \frac{2 + i}{z + i}.$$

Problem 3. (5 marks) Is there a complex number z such that $\cosh(z) = 0$? If the answer is "no", give a proof. If the answer is "yes", find all complex numbers z satisfying the above equation. Here as usual, $\cosh(z)$ denotes the hyperbolic cosine of z.

Solution: Since $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$, our equation translates to $e^z = -e^{-z}$ or equivalently (after multiplying both sides by e^z) to $e^{2z} = -1$. Using Theorem 3 from Section 3.2 and remembering Euler's Formula, $e^{\pi i} = -1$, we obtain $2z = \pi i + 2\pi i k$, or equivalently, $z = (k + \dfrac{1}{2})\pi i$, where k ranges over the integers.

Problem 4. (5 marks) Evaluate the integral

$$\int_\gamma \frac{1}{z}\, dz,$$

where γ is the directed line segment starting at $-1 + i$ and ending at $-1 - i$.

Hint: Remember that the formula $\dfrac{d}{dz}\operatorname{Log}(z) = \dfrac{1}{z}$ is only valid away from the negative real axis. Here as usual, $\operatorname{Log}(z)$ denotes the principal branch of $\log(z)$.

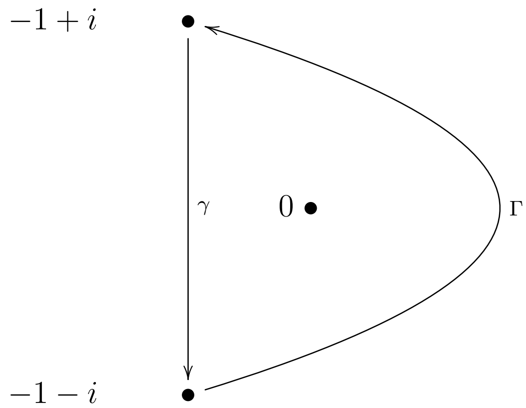
Solution: By the Fundamental Theorem of Calculus the answer is $F(-1-i) - F(-1+i)$, where $F(z)$ is an anti-derivative of $f(z) = \dfrac{1}{z}$ in some open subset D of the complex plane containing γ.

Note that setting $F(z) = \operatorname{Log}(z)$, where Log is the principal branch of the complex logarithm will not work, because $\operatorname{Log}(z)$ is not differentiable or even continuous on the negative real axis (and in particular, at the point -1, which lies on γ).

One way to get around this difficulty is to take $F(z) = \operatorname{Log}_0(z)$, where $\operatorname{Log}_0(z) = \ln|z| + \operatorname{Arg}_0(z)i$, where $\operatorname{Arg}_0(z)$ is the value of $\arg(z)$ in the interval $(0, 2\pi]$. This function is differentiable in the half-plane to the left of the y-axis, and its derivative is $\dfrac{1}{z}$. Hence,

$$\int_\gamma \frac{1}{z}\, dz = \operatorname{Log}_0(-1 - i) - \operatorname{Log}_0(-1 + i) = (\ln(\sqrt{2}) + \frac{5\pi}{4}i) - (\ln(\sqrt{2}) + \frac{3\pi}{4}i) = \frac{\pi}{2}i.$$

Another way to compute the same integral is to introduce a new path Γ, which a a segment of the circle $|z| = \sqrt{2}$ traversed in the clockwise direction, starting at $-1 - i$ and ending at $-1 + i$; see the diagram below.



Deforming the closed contour $\gamma+\Gamma$ to a positively oriented circle C centered at the origin, we see that

$$\int_\gamma \frac{1}{z}\,dz + \int_\Gamma \frac{1}{z}\,dz = \int_C \frac{1}{z}\,dz = 2\pi i\,.$$

On the other hand, the second term can be evaluated using $F(z) = \operatorname{Log}(z) =$ the principal branch of $\log(z)$. That is,

$$\int_\Gamma \frac{1}{z}\,dz = \operatorname{Log}(-1+i) - \operatorname{Log}(-1-i) = (\ln(\sqrt{2}) + \frac{3\pi}{4}i) - (\ln(\sqrt{2} - \frac{3\pi}{4}i) = \frac{3\pi}{2}i.$$

(Why?) Consequently,

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i - \int_\Gamma \frac{1}{z}\,dz = 2\pi i - \frac{3\pi}{2}i = \frac{\pi}{2}i\,.$$

Yet another possible approach is to remove a small segment from γ between $-1+\epsilon$ and $-1-\epsilon$, compute the integral of $\frac{1}{z}$ over each of the remaining two line segments (again, we can use the Fundamental Theorem of Calculus with $F(z) = \operatorname{Log}(z)$ for both computations), then take the limit as $\epsilon \to 0$. This is similar to the example at the beginning of Lecture 13. I will leave it to you to check that this leads to the same answer, $\frac{\pi}{2}i$.