Solutions to Math 300 2012WT1 Solutions

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1. (a) $z = i\log(-1+i) = i\big[\log\sqrt{2} + i(\frac{3\pi}{4} + 2n\pi)\big]$, so $\cos z = (e^{iz} + e^{-iz})/2 = -\frac{3}{4} + \frac{i}{4}$.

   (b) $z = \frac{\sqrt{3}+i}{\sqrt{2}(1+i)} = \frac{(\sqrt{3}+1-i(\sqrt{3}-1)}{2\sqrt{2}}$. Since $|z| = 1$ and z lies in the fourth quadrant, $\mathrm{Log}(z) = -i\arctan(\frac{\sqrt{3}-1}{\sqrt{3}+1}) = -i\arctan(2-\sqrt{3})$, where arctan denotes the inverse tangent function with range in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

   (c) $\cosh z = (e^z + e^{-z})/2$, so $\cosh z = \frac{1}{2}$ implies that $e^z + e^{-z} = 1$ or $e^z = \frac{1\pm i\sqrt{3}}{2}$. Therefore the solutions are of the form $z = \log\big(\frac{1\pm i\sqrt{3}}{2}\big) = i(\pm\frac{\pi}{3} + 2n\pi)$ where n is any integer.

2. (a) Since f has continuous first partial derivatives at all points, it is differentiable at all points where Cauchy-Riemann equations hold. Since $u_x = 1$, $u_y = 2$, $v_x = 4(2x-y)$ and $v_y = -2(2x-y)$, we find that the CR-equations hold if and only if $2x - y = -\frac{1}{2}$. Thus f is differentiable only at the points lying on this line.

   (b) Since the line does not contain any open set, f is analytic nowhere.

   (c) Suppose that $g = u + iw$ is an entire function. By CR equations, $w_x = -u_y = -2$ and $w_y = u_x = 1$. Therefore, $w = y - 2x + C$ where C is any constant. Hence $g = (1-2i)z + C$ for any arbitrary constant C.

3. The domain of f indicates that a branch could be defined as follows:

$$f(z) = \exp\Big[-\frac{1}{2}\mathcal{L}_{-\frac{\pi}{2}}(z-1)\Big]$$

where $\mathcal{L}_{-\frac{\pi}{2}}$ denotes the branch of the complex logarithm with the cut along the nonpositive imaginary axis. In other words, $\mathcal{L}_{-\frac{\pi}{2}}(z) = \ln|z| + i\arg(z)$, with $\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

Parametrize Γ as $z(t) = e^{it}$, $0 \le t \le \pi$. Therefore $\mathcal{L}_{-\frac{\pi}{2}}(z(t)) = it$, hence

$$\int_\Gamma f(z)\,dz = \int_0^\pi e^{-\frac{it}{2}} ie^{it}\,dt = i\int_0^\pi e^{\frac{it}{2}}\,dt = 2(i-1).$$

4. Use the residue theorem to evaluate all the integrals in this problem.
   (a) $2\pi i$
   (b) $-\pi i$
   (c) $200\pi i e^{-i}$
   (d) $-\frac{\pi^2 i}{4}$.

5. For any $K > R$, let C_K denote the circle centred at $z_0 = 0$ with radius K. We make use the inequality for derivatives of analytic functions: for any $r \ge 1$,

$$\big|f^{(n+r)}(0)\big| \le (n+r)!\frac{M_K}{K^{n+r}},$$

where $M_K = \sup_{z\in C_K}|f(z)|$. By the hypothesis of this problem, $M_K \le CK^n$. Therefore for every $K > R$, we obtain the estimate

$$\big|f^{(n+r)}(0)\big| \le (n+r)!\frac{CK^n}{K^{n+r}} = \frac{(n+1)!C}{K^r} \to 0 \text{ as } K \to \infty.$$

Thus $f^{(n+r)}(0) = 0$ for all $r \geq 1$. Now it follows from the Taylor expansion of f that

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}(z - z_0)^j,$$

in other words, f is a polynomial of degree at most n.

6. By partial fraction expansion, we find that

$$\frac{1}{(3z-1)(z+2)} = \frac{3}{7(3z-1)} - \frac{1}{7(z+2)}. \tag{1}$$

(a) For large $|z|$, both of the following inequalities $|1/3z| < 1$ and $|2/z| < 1$. We therefore arrange the expressions above so that the geometric series expansion can be used:

$$\frac{3}{7(3z-1)} = \frac{1}{7z\left(1 - \frac{1}{3z}\right)} = \frac{1}{7z}\sum_{k=0}^{\infty}\left(\frac{1}{3z}\right)^k$$
$$\frac{1}{7(z+2)} = \frac{1}{7z\left(1 + \frac{2}{z}\right)} = \frac{1}{7z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k (-1)^k$$

Therefore for large $|z|$,

$$f(z) = \frac{1}{7z}\sum_{k=0}^{\infty}(3^{-k} - (-2)^k)\, z^{-k}.$$

(b) Here the annular region must be of the form $\{z : r < |z| < R\}$ where $\frac{1}{3} < r < 1 < R < 2$. Thus now $|1/3z| < 1$ and $|z|/2 < 1$, so the second term in (1) has to be arranged differently for the geometric series formula to be applied.

$$\frac{1}{7(z+2)} = \frac{1}{14(1 + \frac{z}{2})} = \frac{1}{14}\sum_{k=0}^{\infty}\left(-\frac{z}{2}\right)^k.$$

In this region the Laurent series takes the form

$$f(z) = \frac{1}{7z}\sum_{k=0}^{\infty}\left(\frac{1}{3z}\right)^k - \frac{1}{14}\sum_{k=0}^{\infty}\left(-\frac{z}{2}\right)^k.$$

(c) The function f has two simple poles, at $z = \frac{1}{3}$ and $z = -2$ respectively, with $\mathrm{Res}_f(\frac{1}{3}) = 1/7$ and $\mathrm{Res}_f(-2) = -\frac{1}{7}$.

(d) $\int_\Gamma f(z)\,dz = 2\pi i \mathrm{Res}_f(\frac{1}{3}) - 2\pi i \mathrm{Res}_f(-2) = \frac{4\pi i}{7}$.

7. Expanding $e^{1/z}$ and $1/(1-z)$ in their Taylor expansions we find that

$$e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!z^k} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots, \quad \text{so}$$

$$\begin{aligned}\mathrm{Res}(e^{\frac{1}{z}}\frac{1}{1-z}) &= \text{ coefficient of } \frac{1}{z} \text{ in the product of the two Laurent series} \\ &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \\ &= e - 1.\end{aligned}$$