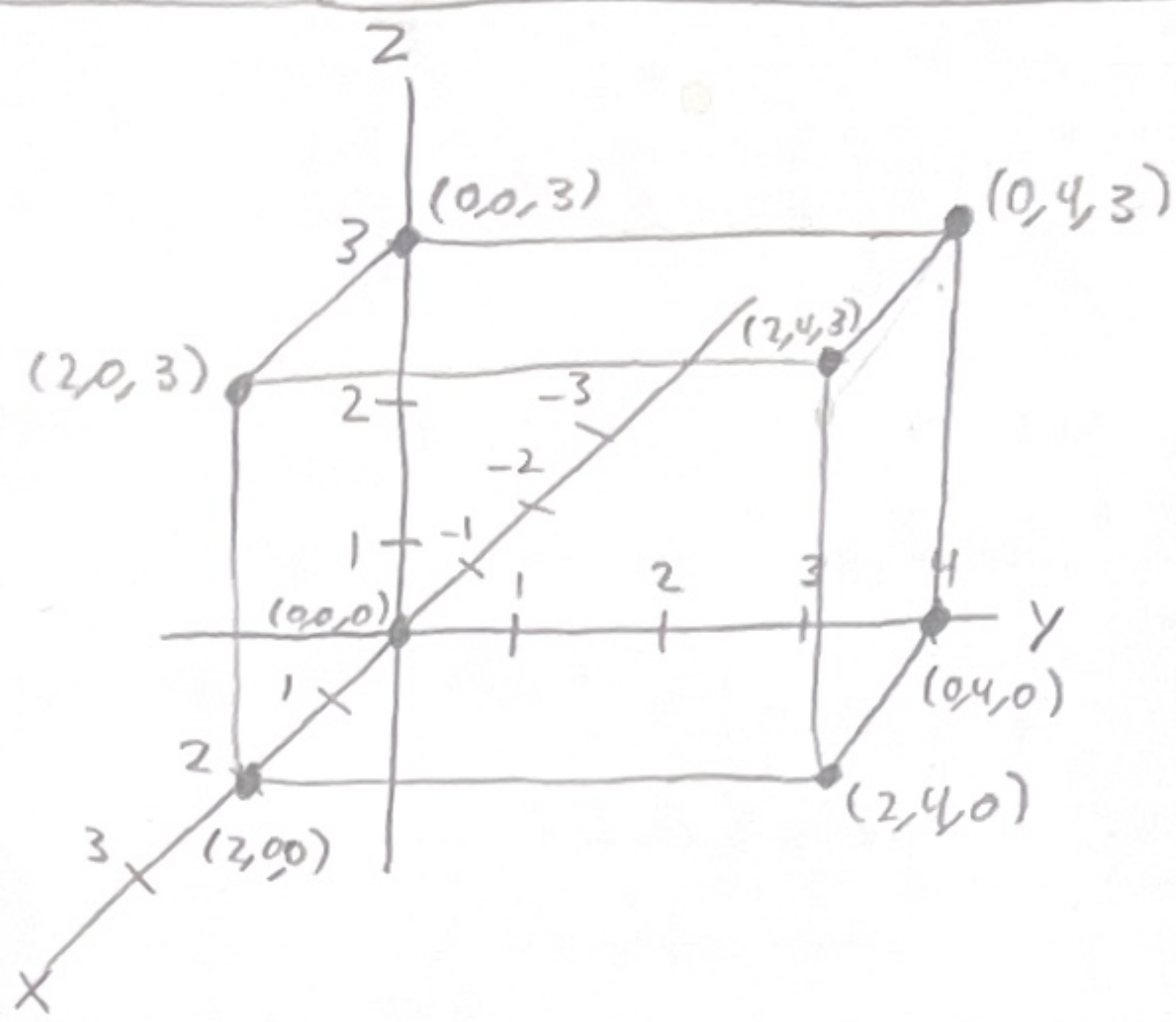
Vectors in 3D

R^3

(x, y, z)



Magnitude (norm)

$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$

For all vectors $\|\vec{v}\| \geq 0$

Unit Vectors

"has magnitude = 1"

"normalizing the vector"

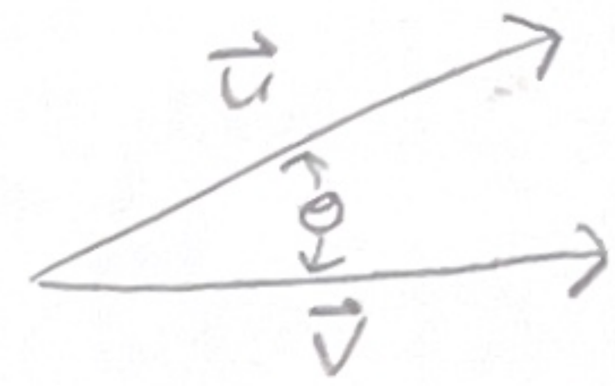
$\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Dot Product

① $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$

"Scalar as an answer"

② $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$ *smaller angle



$0 \leq \theta \leq 180°$

Standard Position

a vector is in standard position when its tail is placed at the origin
This gives its head a unique set of points

A vector $\vec{v}$, placed in standard position with the location of its head at $[x_1, y_1, z_1]$ has COMPONENT FORM; $\vec{v} = [x, y, z]$

Standard Unit Vectors

$\vec{e}_1 = [1, 0, 0] = \hat{i}$

$\vec{e}_2 = [0, 1, 0] = \hat{j}$

$\vec{e}_3 = [0, 0, 1] = \hat{k}$

Dot Product Properties

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

③ $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v}$ "or" $(k\vec{v}) \cdot \vec{u}$

*④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Orthogonal (Perpendicular)

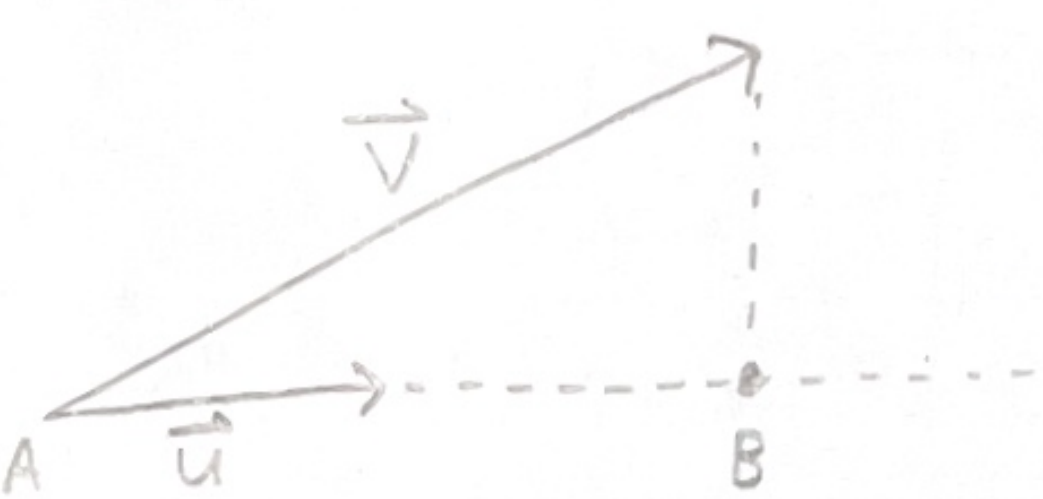
$\vec{u} \cdot \vec{v} = \vec{0}$ (they are orthogonal)

*This means they are perpendicular $\vec{u} \perp \vec{v}$, $\vec{0}$ is orthogonal to all vectors

Vector Projections

$\text{proj}_{\vec{u}} \vec{v} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$

"projection of $\vec{v}$ onto $\vec{u}$"



$\text{proj}_{\vec{u}} \vec{v} = \overrightarrow{AB}$

Cross Product

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

Note: if $\vec{u}$ and $\vec{v}$ are parallel
$\vec{u} \times \vec{v} = \vec{0}$

Gives the area of a parallelogram

Parrellepiped Volume

$|(\vec{u} \times \vec{v}) \cdot \vec{w}|$

Formula:

$\vec{u} \times \vec{v} = [y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2]$

Why?

$\vec{u} \times \vec{v} = \text{determinant} \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$

$= \hat{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \hat{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \hat{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} =$

$\hat{i}(y_1 z_2 - z_1 y_2) - \hat{j}(x_1 z_2 - z_1 x_2) + \hat{k}(x_1 y_2 - y_1 x_2)$

Lines and Planes

Direction Vector

a non zero vector $\vec{v}$ that is parallel to the line

Equation of a line

① The line contains the point $P(x_0, y_0, z_0)$

② we know a direction vector $\vec{v}[A, B, C]$ that is parallel to the line

The the line is

$[x, y, z] = [x_0, y_0, z_0] + t[A, B, C]$

Parametric

$x = x_0 + tA$

$y = y_0 + tB$

$z = z_0 + tC$

Point-Normal Equation

A plane can be determined from

① a point $P(x, y, z)$ in the plane

② a vector $\vec{n} = [A, B, C]$ that is normal (Perpendicular) to the plane

Equation of A Plane

① The plane contains a point $P(x_0, y_0, z_0)$

② The plane contains two vectors $\vec{v}[V_1, V_2, V_3]$ and $\vec{u}[U_1, U_2, U_3]$ that are parallel to the plane and not parallel to each other

Equation for the plane is

$[x, y, z] = [x_0, y_0, z_0] + t[U_1, U_2, U_3] + s[V_1, V_2, V_3]$

General Equation (for o

$Ax + By + Cz = D$

where A, B, C, and D are consta

x, y, z are coordinates

Equation of the plane is

$\vec{n} = [A, B, C]$

$P(x_0, y_0, z_0)$

*constant

$Ax_0 + By_0 + Cz_0 = D$

* plug in coordinate points and for D to obtain general

*orthogonal vectors be independent but all independent v are orthogonal

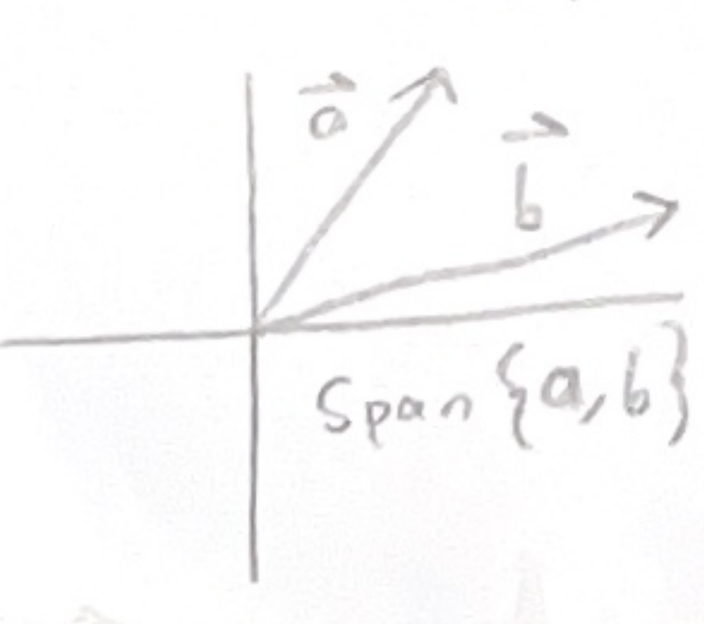
Spanning sets and Linear Independence

Linear Dependent

$c_1[V_1] + c_2[V_2] + c_3[V_3] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

If the only solution to this is $c_1 = c_2 = c_3 = 0$; Then V_1, V_2, and V_3 are independent

If there is another solution the set is dependent



$\text{Span}\{a, b\} = R^2$

Subspaces

V is a subspace in R^n if

Conditions:

① V contains $\vec{0}$

② closed under multiplication:

if $\vec{x}$ is in V; $c\vec{x}$ is also in V for any value C

③ closed under addition:

if $\vec{a}$ is in V and $\vec{b}$ is in V

the $(\vec{a} + \vec{b})$ is also in V

$(\vec{a} + \vec{b}) \in V$

Basis

The minimum set o vectors that spans subspace

if S is a subspac of R^n. For $T = \{V_1, V_2, V_3 \dots V_k\}$ to be a basis for S...

① T is independent

② T spans S

Determinants

$\begin{bmatrix} + & - & \\ - & + & \\ + & - & \\ & - & + \end{bmatrix}$

Triangular Matrices

$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ upper $\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$ Lower $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ diagonal

det(A) = the terms along the diagonal multiplied

ie: $(a_{11})(a_{22})(a_{33}) = \det(A)$

Fundomental Thereom of Invertible Matrices

① A is not invertible
② A does not RREF to I
③ $\det(A) = 0$
④ $A\vec{x} = \vec{0}$ has ∞ sol'ns
*if one is true all are true

row operations affect

…apping 2 rows or Columns | $\det(B) = -\det(A)$
…ltiplying rows or colums by a scalar k | $\det(B) = k\det(A)$
…ding rows or colums together | $\det(B) = \det(A)$ ie: nothing

…erties (A and B are nxn matrices)

$\det(AB) = \det(A) \cdot \det(B)$
$\det(kA) = k^n \cdot \det(A)$
$\det(A^T) = \det(A)$
…f A is invertible $\det(A^{-1}) = \frac{1}{\det(A)}$

Proof of ④

let A be an nxn matrix, A is invertible
$AA^{-1} = I$
$\det(AA^{-1}) = \det(I)$
$\det(A) \cdot \det(A^{-1}) = 1$
$\det(A^{-1}) = \frac{1}{\det(A)}$; $\det(A) \neq 0$

Eigenvalues and Eigenvectors

$A\vec{x} = \lambda\vec{x}$

…eigenvector has …ssociated eigenvalue
…y eigenvalue has an …nite set of eigenvectors

Eigenspace E_λ

The set of all $\vec{x}$ vectors that satisfy $A\vec{x} = \lambda\vec{x}$ (Subspace of $\mathbb{R}^n$)

* $\vec{0}$ is not an eigenvector

Determinant

The determinant of A is the multiplication of the eigenvalues $\prod_{i=1}^{n} \lambda_i = \det(A)$

Trace

The trace of A is the sum of the diagonal entries or the sum of the eigenvalues

$\sum_{i=1}^{n} \lambda_i = tr(A)$

…aracteristic Equation

$C_A(\lambda) = \det(A - \lambda I) = 0$
…e eigenvalues are the roots … $C_A(\lambda)$
…he eigenvectors are the solution $(A - \lambda I)\vec{x} = 0$ for a specific λ
Note $C_A(0) = \det(A)$

F.T.I.M

① A is not invertible
② $\det(A) = 0$
③ $\lambda = 0$ is an eigenvalue of A
*if one is true all are true

…xample

$A = \begin{bmatrix} 2 & 0 & 0 \\ 9 & -3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ "Find eigenvalues and associated eigenvectors"

$\det(A - \lambda I) = \begin{bmatrix} (2-\lambda) & 0 & 0 \\ 9 & (-3-\lambda) & 6 \\ 0 & 0 & (2-\lambda) \end{bmatrix} = (2-\lambda)(-3-\lambda)(2-\lambda)$
$(2-\lambda)^2(-3-\lambda)$

let $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 2-2 & 0 & 0 \\ 9 & -3-2 & 6 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 9 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{array}{c} \begin{array}{ccc} & t & s \end{array} \\ \begin{bmatrix} 1 & -5/9 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} x = \frac{5}{9}t - \frac{2}{3}s \\ y = t \\ z = s \end{array}$$

$(A - 2\lambda)\vec{x} = 0$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 5/9 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix}$ $\dim(E_2) = 2$

let $\lambda = -3$:

$$\det(A + 3I) = \begin{bmatrix} 5 & 0 & 6 \\ 9 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{array}{c} \begin{array}{ccc} & t & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} x = 0 \\ y = t \\ z = 0 \end{array}$$

$(A + 3I)\vec{x} = 0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\dim(E_{-3}) = 1$

Eigenvalue λ	Algabraic Multiplicity	Geometric Multiplicity
2	2	2
-3	1	1

Matrices

$$A = \begin{bmatrix} 1 & -2 & 5 & 7 \\ 2 & 0 & 1 & 4 \\ -3 & -8 & 6 & 11 \end{bmatrix}$$

Dimension: 3×4

$a_{13} = 5$ | $row_2(A) = [2, 0, 1, 4]$

$a_{34} = 11$ | $Col_3(A) = [5, 1, 6]$

$a_{23} = 1$

Multiplication:

$$AB = \underset{2\times 3}{\begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}} \underset{3 \times 4}{\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & 3 & -2 \\ -1 & 1 & 4 & -3 \end{bmatrix}} = \underset{2 \times 4}{\begin{bmatrix} 1 & 3 & 11 & - \\ 1 & 3 & 8 & - \end{bmatrix}}$$

$$= \begin{bmatrix} (3)(1)+(-1)(0)+(2)(-1), & (3)(1)+(-1)(2)+(2)(1) \ldots \\ (2)(1)+(0)(0)+(1)(-1), & (2)(1)+(0)(2)+(1)(1) \ldots \end{bmatrix}$$

Solving Linear Systems:

$2x - y + z = 4$

$x + 2y - z = 1$

$4x - 7y + 2z = 5$

Augmented matrix $\rightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 1 & 2 & -1 & 1 \\ 4 & -7 & 2 & 5 \end{array}\right]$

Matrix eqn of the form $A\vec{x} = \vec{b}$ $\rightarrow \overset{A}{\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & -1 \\ 4 & -7 & 2 \end{bmatrix}}$

Null Space

$$A = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix} \end{array} \xrightarrow{RREF} \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

let $x_3 = s$; $x_1 = s - 4t$

$x_5 = t$; $x_2 = -2s - t$

$x_4 = 3t$

1 $row(A) = \{y_1, y_2, y_3, y_4\}$

Basis for $row(A) = \{a_1, a_2, a_3\}$

2 $col(A) = \{x_1, x_2, x_3, x_4, x_5\}$

Basis for $col(A) = \{x_1, x_2, x_4\}$

3 $Null(A) = \left\{ s\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, t\begin{bmatrix} -4 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

2 dimensional subspace in R^5

$\perp$ to the rowspace

$Rank(A) + dim(Null(A)) = n$ / $3 + 2 = 5$ ✓

Four Fundamental Subspaces of a matrix:

1 Rowspace of A: $row(A)$

2 Columnspace of A: $Col(A)$

3 Nullspace of A: $Null(A)$

4 Nullspace of transpose: Null(

Linear Transformations

ain
et of
input vectors

Range
The set of all $T(\vec{x})$ output vectors

Codomain
Space all the $T(\vec{x})$'s occupy

perty

$C_1\vec{V}_1 + C_2\vec{V}_2 + C_3\vec{V}_3 \ldots\ldots C_K\vec{V}_K)$

$) = T(C_1\vec{V}_1 + C_2\vec{V}_2 + C_3\vec{V}_3 \ldots\ldots C_K\vec{V}_K)$

T's that aren't linear won't have a [T] but still have a range and Kernel

ubspaces of [T]

T is Linear | so [T] exists

rnel of T: ker(T)

set of vectors that satisfy $T(\vec{u}) = 0$

e Nullspace of [T] | $[T][\vec{x}] = 0$ | Null([T])

ange of T: range(T)

The columnspace of [T] Col([T])

Geometric Transformations

Horizontal compression or expansion: $[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ $k > 1$ Expansion, $k < 1$ Compression

Vertical compression or expansion: $[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ $k > 1$ Expansion, $k < 1$ Compression

Horizontal Shear: $[T] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ $k > 0$ Shear Right, $k < 0$ Shear Left

Vertical Shear: $[T] = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ $k > 0$ Shear up, $k < 0$ Shear down

Invertible

T is invertible if there is an S such that

$(T \circ S)\vec{x} = \vec{x}$ $(S \circ T)\vec{x} = \vec{x}$

So $S = T^{-1}$ (reverses the effects of T)

Thereom

if T is linear and invertible.....

$[T^{-1}] = [T]^{-1}$

Linear

1. Additivity | $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. Homogeneity | $T(k\vec{v}) = kT(\vec{v})$
3. $T(\vec{0}) = \vec{0}$

Standard Matrix

if T is linear and $T(\vec{x}) = A\vec{x}$ then there is a matrix [T] such that $[T]\vec{x} = A\vec{x}$

Example $R^4 \rightarrow R^3$

$$T\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + y \\ 2y - x - w \\ 2y + z \end{bmatrix}$$

Find [T] (Standard matrix)

Solution:

$$[T] = \begin{array}{cccc} x & y & z & w \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$T(\hat{\imath}) = T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$T(\hat{\jmath}) = T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$T(\hat{k}) = T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(\hat{\omega}) = T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Column vectors
*plug in the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$, etc

Compositions

$(T \circ S)(\vec{x}) = T(S(\vec{x}))$

if T and S are both linear so [T] and [S] then

$(T \circ S) = [T][S]$

Fundamental Thereom $T: R^n - R^n$

*if one is true, all are true

1. T is invertible
2. T is one-to-one
3. [T] is invertible
4. $\det([T]) \neq 0$
5. $\ker(T) = \vec{0}$
6. $\text{range}(T) = r^n$

Proof

if $S: R^m - R^n$ and $T: R^n \rightarrow R^p$ are linear then $(T \circ S): R^m \rightarrow R^p$ is linear

1. **Additivity:** $(S \circ T)(\vec{v} + \vec{w}) = S(T(v + w))$
   $= S(T(v) + T(w)) = S(T(v)) + S(T(w))$
   $= (S \circ T)v + (S \circ T)\vec{w}$ ✓ *closed under additivity
2. **Homogeneity:**
   $(S \circ T)(kv) = S(T(kv)) = S(kT(v))$
   $= kS(T(v)) = k(S \circ T)v$ ✓ closed under scalar multiplication

Similarity and Diagonalization

Col(A) Independent = Invertible
Eigenvectors(A) Independent = Diagonalizable

$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$A^3 = \begin{bmatrix} a^3 & 0 & 0 \\ 0 & b^3 & 0 \\ 0 & 0 & c^3 \end{bmatrix}$

Example

$A = \begin{bmatrix} -3 & 2 \\ 10 & 5 \end{bmatrix}$

Find A^{1000}

Solution:

eigenvalues of A are $\lambda_1 = -5$, $\lambda_2 = 7$

eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Matrix P has column vectors as eigenvectors of A

$P = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 1/6 \end{bmatrix}$

$D = P^{-1}AP$

$= \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 1/6 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$

$= \begin{bmatrix} -5 & 0 \\ 0 & 7 \end{bmatrix}$ *diagonal entries are the eigenvalues of P

$A = PDP^{-1}$

$A^{1000} = (PDP^{-1})^{1000}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} (-5)^{1000} & 0 \\ 0 & 7^{1000} \end{bmatrix}\begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 1/6 \end{bmatrix}$

To solve:

1. P *column vectors are eigenvectors of A
2. P^{-1}
3. $D = P^{-1}AP$ *D is diagonal eigenvalues of A
4. $A = PDP^{-1}$

Diagonalizable

A is diagonalizable if there exists P, P^{-1} and D such that $A = PDP^{-1}$

*An nxn matrix A must have n eigenvectors to be diagonalizable

Failure

if A has geometric < algebraic P^{-1} does not exist and A is not diagonalizable

Similarity

A and B are nxn matrices

$A \sim B$ if there is an invertible matrix P such that $A = PBP^{-1}$

Thereom

*if one is true all are true

1. A is diagonalizable
2. A has n independent eigenvectors
3. geometric multiplicity is n
4. geometric = algebraic
5. A has no defective eigenvalues

Properties

1. Reflexive: for a square matrix A; $A \sim A$
2. Symmetric: if $A \sim B$; $B \sim A$
3. Transitive: if $A \sim B$ and $B \sim C$; $A \sim C$

Thereom A and B are similar such that $A = PBP^{-1}$ for some P.....

1. $C_A(\lambda) = C_B(\lambda)$
2. $\det(A) = \det(B)$ and $tr(A) = tr(B)$
3. if λ is an eigenvalue; A and B have same dimension eigenspace
4. $Rank(A) = Rank(B)$

Complex Numbers

$z = a + bi$

where z is a complex number

a and b are real

and $i = \sqrt{-1}$

Conjugate

$z = a + bi$

$\bar{z} = a - bi$

Division

$\frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2}$

Example $\frac{2-i}{5+3i}\frac{(5-3i)}{(5-3i)}$

$= \frac{10 - 6i - 5i - 3}{25+9}$

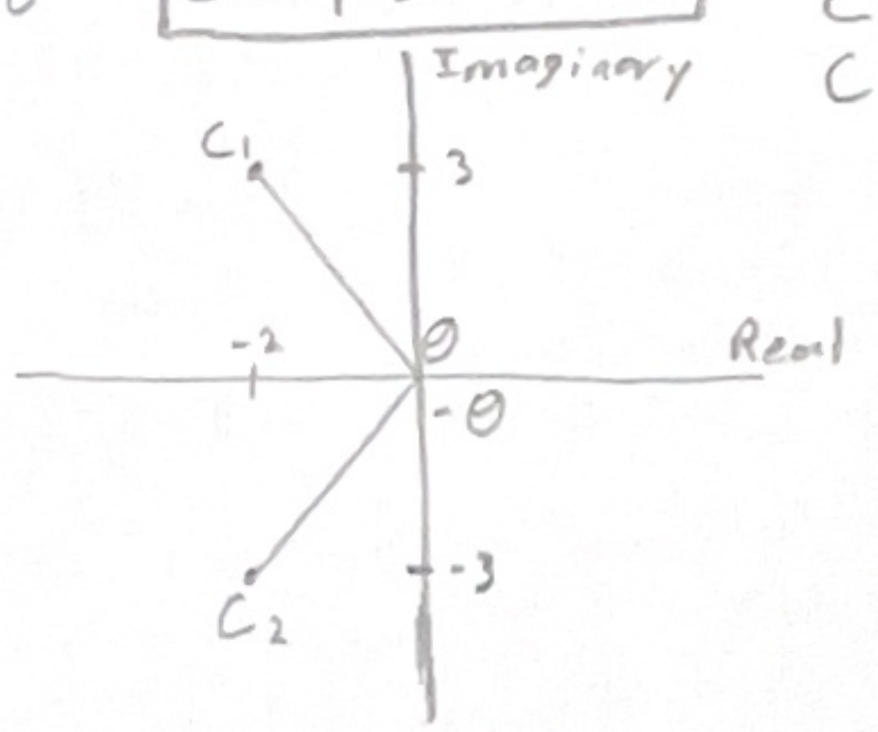
$= \frac{7-11i}{34} = \underbrace{\frac{7}{34}}_{a} - \underbrace{\frac{11i}{34}}_{bi}$

*Note: $(i)^2 = (-1)$, $(-i)^2 = (-1)$

$x^2 + 1 = 0$

$x = \pm i$

Complex Plane



$C_1 = -2 + 3i$

$C_2 = -2 - 3i$