Department of Mathematics

University of British Columbia
MATH 300
Solutions to Quiz 1
July 15, 2021, 1-2:15pm

Problem 1. Find real numbers x and y such that $\dfrac{1+i}{(1-i)^{11}} = x + iy$.

Solution: Work in polar form: $1+i = \sqrt{2}e^{\frac{\pi}{4}i}$ and $1-i = \sqrt{2}e^{-\frac{\pi}{4}i}$. Thus

$$\frac{1+i}{(1-i)^{11}} = \frac{\sqrt{2}\,e^{\frac{\pi i}{4}}}{(\sqrt{2})^{11}\,e^{-\frac{11\pi i}{4}}} = \frac{e^{\frac{12\pi i}{4}}}{(\sqrt{2})^{10}} = \frac{e^{3\pi i}}{2^5} = \frac{-1}{32}\ .$$

Answer: $\boxed{x = -\dfrac{1}{32} \text{ and } y = 0}$.

Problem 2. Find all complex solutions z to the equation $|e^{\frac{1}{z-1}}| = 1$.

Solution: The equality $|e^{\frac{1}{z-1}}| = 1$ is equivalent to $\mathrm{Re}(\frac{1}{z-1}) = 0$. In other words, $\dfrac{1}{z-1} = si$, where s is a real number. Equivalently, $z - 1 = \dfrac{1}{si} = -\dfrac{1}{s}i = ti$ for some real number $t \neq 0$ (why?). In summary, z satisfies the identity in question if and only if $z \neq 1$ but $\mathrm{Re}(z) = 1$, i.e., $\boxed{z = 1 + ti}$ for some real number $t \neq 0$.

Problem 3: For each of the statements below, indicate whether they are true or false. If true, give a proof. If false, give a counterexample.

(a) Let z and w be complex numbers. If $z^3 = w^3$, then $z = w$.

(b) The function $f(z) = \dfrac{\overline{z}-1}{\overline{z}+1}$ is differentiable at $z = 0$.

Solution: (a) $\boxed{\text{False.}}$ Counterexample: $z = e^{\frac{2\pi i}{3}}$ and $w = 1$. Here $z^3 = w^3 = 1$ but $z \neq w$.

(b) $\boxed{\text{False.}}$ Solve for $\overline{z}$ in terms of $f(z)$:

$$f(z)(\overline{z}+1) = \overline{z} - 1 \implies f(z) + 1 = \overline{z}(1 - f(z)) \implies \overline{z} = \frac{f(z)+1}{1-f(z)}.$$

Note that $f(0) = -1$. By the quotient rule for differentiation, if $f(z)$ were differentiable at 0, then $\overline{z}$ would also be differentiable at 0, and we know that $\overline{z}$ is not differentiable anywhere.

Problem 4. Let $f(z) = x + 2y + (2x - y)^2\, i$, where $z = x + yi$.

(a) Find all complex numbers z_0 such that $f(z)$ is differentiable at z_0?

(b) For every z_0 where $f(z)$ is differentiable, find the complex derivative $f'(z_0)$.

(c) Find all complex numbers z_0 such that $f(z)$ is analytic at $z = z_0$?

Solution: (a) Let $u(x,y) = x + 2y$ and $v(x,y) = (2x-y)^2$ be the real and imaginary parts of $f(z)$, respectively. The partial derivatives

$$\begin{array}{ll} u_x = 1 & v_x = 2(2x-y)\cdot 2 \\ u_y = 2 & v_y = 2(2x-y)\cdot(-1) \end{array}$$

are continuous in the entire complex plane. The Cauchy-Riemann equations, $v_y = u_x$ and $v_x = -u_y$, translate to

$$-2(2x-y) = 1 \text{ and } 4(2x-y) = -2.$$

Both are satisfied if and only if $2x - y = -\dfrac{1}{2}$. In other words, $f(z)$ is differentiable at z_0 if and only if z_0 lies on the line $2x - y = -\dfrac{1}{2}$, i.e., $\boxed{z_0 = t + (2t + \frac{1}{2})i}$ for some real number t.

(b) As we showed in class, when $f(z)$ is differentiable at $z_0 = x_0 + y_0 i$, the derivative is given by the formula

$$f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0)i = u_x(x_0, y_0) - u_y(x_0, y_0)i.$$

If z_0 is of the form $t + (2t + \frac{1}{2})i$ for some real number t, then $u_x = 1$ and $u_y = 0$, so $\boxed{f'(z_0) = 1 - 2i}$.

(c) By part (a), $f(z)$ is not differentiable in any disk, hence, $\boxed{f(z) \text{ is not analytic at any } z_0}$.

Problem 5: Do the following limits exist?

(a) $\lim\limits_{z\to 1} \dfrac{\overline{z} - 1}{z - 1}$, (b) $\lim\limits_{z\to i} \dfrac{e^{\pi z} + 1}{z - i}$.

Justify your answers. If the answer is "yes", compute the limit.

Solution: The limit in part (a) is the derivative of $f(z) = \overline{z}$ at $z_0 = 1$. We showed in class that $f(z)$ is not differentiable at any z_0. Hence, $\boxed{\text{the limit in part (a) does not exist}}$.

The limit in part (b) is the derivative of $g(z) = e^{\pi z}$ at $z_0 = i$. We showed in class that e^z is differentiable and $\dfrac{d}{dz}(e^z) = e^z$. Hence, by the Chain Rule, $g'(z) = e^{\pi z} \cdot \pi$. We conclude that $\boxed{\text{the limit in part (b) exists and equals } e^{\pi i} \cdot \pi = -\pi}$.