MATH 300. Summer 2021.
Final exam practice
Answer key to practice problems for week 6

6.1.4. If the Laurent series for $f(z)$ in a small punctured neighborhood of z_0 is

$$f(z) = \sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j} + \sum_{k=0}^{\infty} c_k(z-z_0)^k$$

then

$$f'(z) = \sum_{j=1}^{\infty} -jc_{-j}(z-z_0)^{-j-1} + \sum_{k=0}^{\infty} kc_k(z-z_0)^{k-1}\,.$$

The residue of $f'(z)$ at z_0 is the coefficient of $(z-z_0)^{-1}$ in this Laurent series. This coeficient is $0 \cdot c_0 = 0$.

6.1.6. Since $f(z)$ has a zero of order m at z_0, we can write it as $f(z) = (z-z_0)^m g(z)$ in some punctured disk centered at z_0. Here $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Now

$$f'(z) = m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z) = (z-z_0)^{m-1}h(z),$$

where $h(z) = mg(z) + (z-z_0)g'(z)$. In particular, $h(z)$ is analytic at z_0 and $h(z_0) = mg(z_0) \neq 0$. Now

$$\frac{f'(z)}{f(z)} = \frac{(z-z_0)^{m-1}h(z)}{(z-z_0)^m g(z)} = \frac{k(z)}{(z-z_0)},$$

where $k(z) = \dfrac{h(z)}{g(z)}$ is analytic at z_0 and $k(z_0) = \dfrac{h(z_0)}{g(z_0)} = m \neq 0$. Thus the Taylor series for $k(z)$ has the form

$$k(z) = m + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

for some complex numbers $a_1, a_2, \dots$. Dividing both sides by $(z-z_0)$, we obtain

$$\frac{f'(z)}{f(z)} = \frac{k(z)}{z-z_0} = \frac{m}{z-z_0} + a_1 + a_2(z-z_0) + \dots$$

This shows that $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_0, and that the residue is m.

6.2.2. First note that since $\cos(2\pi - \theta) = \cos(\theta)$, we have

$$\int_{\pi}^{2\pi} \frac{8}{5+2\cos(\theta)}d\theta = \int_0^{\pi} \frac{8}{5+2\cos(\theta)}d\theta$$

and thus

$$\int_0^{\pi} \frac{8}{5+2\cos(\theta)}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{8}{5+2\cos(\theta)}d\theta = \int_0^{2\pi} \frac{4}{5+2\cos(\theta)}d\theta.$$

(The same argument was used in Example 2 on p. 316.) To compute the last integral, we use the substitution described on pp. 314-315.

$$\int_0^{2\pi} \frac{4}{5+2\cos(\theta)} d\theta = \int_\Gamma \frac{4}{5+(z+1/z)} \cdot \frac{dz}{iz} = \frac{4}{i} \int_\Gamma \frac{1}{5z+z^2+1} dz,$$

where Γ is the positively oriented unit circle. The quadratic equation $z^2+5z+1=0$ has two roots, $z_1 = \dfrac{-5+\sqrt{21}}{2}$ and $z_2 = \dfrac{-5-\sqrt{21}}{2}$. Of the two of them only z_1 lies inside the unit circle. By the Residue Theorem,

$$\int_\Gamma \frac{1}{z^2+5z+1} dz = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{z^2+5z+1}, z_1\right).$$

The function $\dfrac{1}{z^2+5z+1} = \dfrac{1}{(z-z_1)(z-z_2)}$ has a simple pole at z_1. To find the residue at z_1, we multiply by $z-z_1$ and substitute z_1 for z (see Theorem 1 on p. 310, with $m=1$):

$$\operatorname{Res}\left(\frac{1}{z^2+5z+1}, z_1\right) = \frac{1}{z_1-z_2} = \frac{1}{\sqrt{21}}.$$

In summary,

$$\int_0^{2\pi} \frac{4}{5+2\cos(\theta)} d\theta = \frac{4}{i} \int_\Gamma \frac{1}{5z+z^2+1} dz = \frac{4}{i} \cdot 2\pi i \cdot \frac{1}{\sqrt{21}} = \frac{8\pi}{\sqrt{21}}.$$

6.2.4. Since $\sin(\theta+\pi) = -\sin(\theta)$, we can use the substitution $\theta \mapsto \theta+\pi$ to change the limits from $-\pi$ and π to 0 and 2π

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2(\theta)} = \int_0^{2\pi} \frac{d\theta}{1+\sin^2(\theta)}.$$

Now we can apply the substitution described on pp. 314-315 once more:

$$\int_0^{2\pi} \frac{d\theta}{1+\sin^2(\theta)} = \int_\Gamma \frac{1}{1+(\frac{1}{2i}(z-1/z))^2} \cdot \frac{dz}{iz} = \frac{1}{i} \cdot \int_\Gamma \frac{-4z\,dz}{z^4-6z^2+1}.$$

We would like to compute this integral using the Residue Theorem. The first order of business is to find the singularities. They occur when $z^4-6z^2+1=0$. To solve this equation, we substitute w for z^2. Then $w^2-6w+1=0$, and $w = 3+2\sqrt{2}$ by the quadratic formula. Set $w_1 = 3-2\sqrt{2}$ and $w_2 = 3+2\sqrt{2}$. There are thus four singular points, z_1, $-z_1$, z_2 and $-z_2$, where $z_1^2 = w_1$ and $w_1^2 = w_2$. Note that $|w_1| < 1$ and $|w_2| > 1$ and thus $|z_1| < 1$ and $|z_2| > 1$. Thus the only singularities inside the unit circle are z_1 and $-z_1$. By the Residue Theorem,

$$\int_\Gamma \frac{-4z\,dz}{z^4-6z^2+1} = 2\pi i\Big(\operatorname{Res}\left(\frac{-4z}{z^4-6z^2+1}, z_1\right) + \operatorname{Res}\left(\frac{-4z}{z^4-6z^2+1}, -z_1\right)\Big).$$

A quick way to compute these residues is to use the formula in Example 2 on p. 309, with $P(z) = 4z$ and $Q(z) = z^4-6z^2+1$. (Check that the conditions of Example 2 are satisfied here.)

$$\operatorname{Res}\left(\frac{-4z}{z^4-6z^2+1}, z_1\right) = \frac{P(z_1)}{Q'(z_1)} = \frac{-4z_1}{4z_1^3-12z_1} = \frac{-1}{z_1^2-3} = \frac{-1}{w_1-3} = \frac{1}{2\sqrt{2}}.$$

Similarly,

$$\text{Res}(\frac{-4z}{z^4-6z^2+1}, -z_1) = \frac{P(-z_1)}{Q'(-z_1)} = \frac{4z_1}{-4z_1^3+12z_1} = \frac{1}{-z_1^2+3} = \frac{1}{-w_1+3} = \frac{1}{2\sqrt{2}}.$$

Putting it all together,

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2(\theta)} = \frac{1}{i} \cdot \int_\Gamma \frac{-4z\,dz}{z^4-6z^2+1} = \frac{1}{i} \cdot 2\pi i(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}) = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}.$$

6.3.1. Let us integrate $f(z) = \dfrac{1}{z^2+2z+2}$ along the countour Γ_ρ shown in Figure 6.4 on page 321 in the text. The contour Γ_ρ consists of two curves: a straight line segment along the x-axis, from $-\rho$ to ρ, and a semi-circle of radius ρ in the upper half plane. Note that

$$z^2+2z+2 = (z+1)^2+1 = (z+1+i)(z+1-i)\,,$$

so $f(z)$ has only two singularities, $-1+i$ and $-1-i$. Lemma 1 on page 322 tells us that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{\Gamma_\rho} f(z)dz$$

for any $\rho > \sqrt{2}$. To evaluate the latter integral, we apply the Residue Theorem. The only singularity of $f(z)$ inside Γ_ρ is $-1+i$. The residue of $f(z)$ at this point is

$$\lim_{z\to-1+i} (z+1-i)f(z) = \lim_{z\to-1+i} \frac{1}{z+1+i} = \frac{1}{2i}\,.$$

Thus by the Residue Theorem,

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} &= \int_{\Gamma_{rho}} f(z)dz \\ &= 2\pi i \cdot \text{Res}\left(\frac{z}{z^2+2z+2}; -1+i\right) = 2\pi i \frac{1}{2i} = \pi\,. \end{aligned}$$

6.3.2. Here we integrate $f(z) = \dfrac{z^2}{(z^2+9)^2}$ along the same countour Γ_ρ as in Problem 6.3.1, with $\rho > 3$. Note that $f(z)$ has only two singularities, $z_1 = 3i$ and $z_2 = -3i$. Both are poles of order 2. Of these two singularities, only z_1 lies inside Γ_ρ. By the Residue Theorem

$$\int_{\Gamma_\rho} f(z)dz = 2\pi i\,\text{Res}(f(z), 3i).$$

Once again, Lemma 1 on page 322 tells that as $\rho \longrightarrow \infty$,

$$\int_{\Gamma_\rho} f(z)dz \longrightarrow \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2} dx,$$

On the other hand,

$$\int_{\Gamma_\rho} f(z)dz = 2\pi i\,\text{Res}(f(z), 3i)$$

does not depend on ρ. We conclude that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2} = 2\pi i \operatorname{Res}(f(z), 3i).$$

To find the residue, we use the formula in Theorem 1 on p. 310:

$$\begin{aligned}
\operatorname{Res}(f(z), 3i) &= \lim_{z\to 3i} \frac{d}{dz}\left((z-3i)^2 f(z)\right)\\
&= \lim_{z\to 3i} \frac{d}{dz}\, z^2(z+3i)^{-2}\\
&= \lim_{z\to 3i} \left(2z(z+3i)^{-2} - 2z^2(z+3i)^3\right)\\
&= \frac{2\cdot(3i)}{(6i)^2} - 2\frac{(3i)^2}{(6i)^3}\\
&= \frac{1}{6i} - \frac{1}{12i} = \frac{1}{12i}.
\end{aligned}$$

In summary,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2} = 2\pi i \operatorname{Res}(f(z), 3i) 2\pi i \frac{1}{12i} = \frac{\pi}{6}.$$

6.3.3. Since $f(z) = \dfrac{x^2+1}{x^4+1}$ is an even function,

$$\int_0^{\infty} \frac{x^2+1}{x^4+1} dx = \frac{1}{2}\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} dx.$$

To compute the latter integral, we integrate $f(z) = \dfrac{z^2+1}{z^4+1}$ along the countour Γ_ρ, as in Problem 6.3.1, this time with $\rho > 1$. Note that $f(z)$ has four singularities, $z_j = e^{(2j-1)\pi i/4}$, where $j = 1, 2, 3, 4$. Each is a pole of $f(z)$ of order 1. Two of these poles, $z_1 = e^{\pi i/4} = \dfrac{\sqrt{2}}{2}(1+i)$ and $z_2 = e^{3\pi i/4} = \dfrac{\sqrt{2}}{2}(-1+i)$ are inside Γ_ρ. Using the formulas

$$z_1^2 = e^{\pi i/2} = i, \quad z_2^2 = e^{3\pi i/2} = -i, \quad z_3 = -z_1 \quad \text{and} \quad z_4 = -z_2,$$

we can compute the residues of $f(z)$ at z_1 and z_2 as follows:

$$\operatorname{Res}(f(z), z_1) = \lim_{z\to z_1} \frac{z^2+1}{(z-z_3)(z^2+i)} = \frac{1+i}{(2z_1)(2i)} == \frac{\sqrt{2}z_1}{4z_1 i} = \frac{\sqrt{2}}{4i}$$

and

$$\operatorname{Res}(f(z), z_2) = \lim_{z\to z_2} \frac{z^2+1}{(z-z_4)(z^2-i)} = \frac{1-i}{(2z_2)(-2i)} = \frac{-\sqrt{2}z_2}{(2z_2)(-2i)} = \frac{\sqrt{2}}{4i}.$$

Arguing as in Exercise 6.3.1, we obtain

$$\text{p.v.} \int_0^{\infty} \frac{x^2+1}{x^4+1} dx = \frac{1}{2}(2\pi i \left(\operatorname{Res}(e^{\pi i/4}) + \operatorname{Res}(e^{3\pi i/4})\right) = \frac{1}{2}\cdot 2\pi i \cdot \frac{\sqrt{2}}{2i} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}.$$