Department of Mathematics

University of British Columbia

MATH 300

Quiz 1 Solutions

July 15, 2021, 7:30-8:45pm

Problem 1. Find real numbers x and y such that $\dfrac{(1-i)^9}{1+i} = x + iy$.

Solution: Work in polar form: $1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$ and $1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$. Thus

$$\frac{(1-i)^9}{1+i} = \frac{(\sqrt{2})^9\, e^{-\frac{9\pi}{4}i}}{\sqrt{2}\, e^{\frac{\pi}{4}i}} = (\sqrt{2})^8 e^{\frac{-10\pi i}{4}} = 2^4 e^{-\frac{5\pi}{2}i} = 16e^{-\frac{\pi}{2}i} = -16i\ .$$

Thus $\boxed{x = 0 \text{ and } y = -16}$.

Problem 2. Find all complex solutions z to the equation $\dfrac{e^z - e^{-z}}{e^z + e^{-z}} = i$.

Solution:

$$\begin{aligned}
\frac{e^z - e^{-z}}{e^z + e^{-z}} = i \qquad & \text{if and only if} \qquad & e^z - e^{-z} &= ie^z + ie^{-z} \\
& \text{if and only if} \qquad & e^z(1-i) &= e^{-z}(1+i) \\
& \text{if and only if} \qquad & e^{2z} &= \frac{1+i}{1-i} = i = e^{i\frac{\pi}{2}} \\
& \text{if and only if} \qquad & 2z &= \frac{\pi}{2}i + 2k\pi i \quad \text{for some integer } k \\
& \text{if and only if} \qquad & z &= \frac{\pi}{4}i + k\pi i \quad \text{for some integer } k.
\end{aligned}$$

Problem 3. Find all complex numbers z satisfying $z^8 - 3z^4 - 4 = 0$.

Solution: Let $w = z^4$. Then w satisfied the quadratic equation $w^2 - 3w - 4 = 0$. Using the quadratic formula, we see that this equation has two solutions, $w = 4$ and $w = -1$. Thus z is either a fourth root of 4 or a fourth root of -1. To find these roots, we use the formula in Section 1.5.

$z^4 = 4 = 4e^{0i}$. Here $z = \sqrt[4]{4}e^{(0/4+2\pi k/4)i}$, where $k = 0, 1, 2, 3$. This way we obtain four roots,

$$z_1 = \sqrt{2}, \quad z_2 = \sqrt{2}i, \quad z_3 = -\sqrt{2}, \quad z_4 = -\sqrt{2}i.$$

$z^4 = -1 = e^{\pi i}$. Here $z = e^{\pi i/4 + 2\pi ik/4}$, where $k = 0, 1, 2, 3$. This way we obtain four additional roots,

$$z_5 = e^{\pi i/4} = \frac{\sqrt{2}}{2}(1+i), \quad z_6 = e^{3\pi i/4} = \frac{\sqrt{2}}{2}(-1+i), \quad z_7 = e^{5\pi i/4} = \frac{\sqrt{2}}{2}(-1-i), \quad z_8 = e^{7\pi i/4} = \frac{\sqrt{2}}{2}(1-i).$$

Answer: The equation $z^8 - 3z^4 - 4 = 0$ has exactly 8 complex solutions, $z_1, \ldots, z_8$ listed above.

Problem 4: For each of the statements below, indicate whether they are true or false. If true, give a proof. If false, give a counter example.

(a) The equation $e^z = 0$ has no complex solution.

(b) $\lim_{z\to 1} \dfrac{z^2-1}{z^3-1}$ does not exist.

Solution: (a) $\boxed{\text{True.}}$ If $z = x+yi$, where x and y are real numbers, then by definition $|e^z| = e^x > 0$ for any complex numbers z.

(b) $\boxed{\text{False.}}$ Factoring z^2-1 as $(z-1)(z+1)$ and $z^3-1 = (z-1)(z^2+z+1)$, and cancelling $z-1$, we see that

$$\lim_{z\to 1} \frac{z^2-1}{z^3-1} = \lim_{z\to 1} \frac{z+1}{z^2+z+1} = \frac{2}{3}.$$

Problem 5. (a) Find all complex numbers $z_0 = x_0 + iy_0$ such that $f(x+iy) = y^2 - x^2 + \dfrac{2i}{xy}$ is differentiable at z_0?

(b) For every z_0 where $f(z)$ is differentiable, find the complex derivative $f'(z_0)$.

(c) Find all complex numbers z_0 such that $f(z)$ is analytic at $z = z_0$?

Solution: (a) $f(z)$ is not defined on the coordinate axes $y = 0$ and $x = 0$. So, there is no hope of f being differentiable there. From now on I will assume that $x_0 \neq 0$ and $y_0 \neq 0$.

Let $u(x,y) = y^2 - x^2$ and $v(x,y) = \dfrac{2i}{xy}$ be the real and imaginary parts of $f(z)$, respectively. The partial derivatives

$$\begin{aligned} u_x &= -2x \quad & v_x &= -\frac{2}{x^2y} \\ u_y &= 2y \quad & v_y &= -\frac{2}{xy^2} \end{aligned}$$

are continuous in the entire complex plane. The Cauchy-Riemann equations, $v_y = u_x$ and $v_x = -u_y$, translate to

$$x = \frac{1}{xy^2} \text{ and } y = \frac{1}{x^2y}.$$

Both are satisfied if and only if $x^2y^2 = 1$, i.e., $xy = 1$ or $xy = -1$, In other words, $f(z)$ is differentiable at z_0 if and only if z_0 lies on the hyperbola $xy = 1$ or on the hyperbola $xy = -1$. i.e., z_0 is of the form $\boxed{z_0 = t \pm \frac{i}{t} \text{ for some real number } t \neq 0.}$

(b) As we showed in class, when $f(z)$ is differentiable at $z_0 = x_0 + y_0 i$, the derivative is given by the formula

$$f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0)i = u_x(x_0, y_0) - u_y(x_0, y_0)i$$

If z_0 is of the form $t + \dfrac{i}{t}$ for some real number t, then $u_x = -2t$ and $u_y = 2y$, so $\boxed{f'(z_0) = -2t - \frac{2i}{t}}$.

Similarly, if z_0 is of the form $t - \dfrac{i}{t}$, then $\boxed{f'(z_0) = -2t + \frac{2i}{t}}$.

(c) By part (a), $f(z)$ is not differentiable in any disk, hence, $\boxed{f(z) \text{ is not analytic at any } z_0}$.