MATH 300, July 2021. Solutions to Quiz 2, 1pm sitting

Problem 1. (5 marks) Is there a function $f(z) = u(x, y) + v(x, y)i$, analytic in the entire complex plane, for which

(a) $u(x, y) = y^2$?

(b) $u(x, y) = xy$?

Explain your answer in each part. If the answer is "yes", find $v(x, y)$. Here as usual, $z = x + iy$.

Solution: (a) No, because u is not harmonic: $u_{xx} + u_{yy} = 2 \neq 0$.

(b) Here u is harmonic because $u_{xx} = u_{yy} = 0$. Solving the Cauchy–Riemann equations,

$$v_x = -u_y = -x,$$
$$v_y = v_x = y$$

we see that $\boxed{v(x, y) = \frac{1}{2}(-x^2 + y^2) + C}$, where C is a real constant.

Side remark: Note that $f(z) = xy + \frac{1}{2}(y^2 - x^2)i + Ci = -\frac{i}{2}z^2 + Ci$.

Problem 2. (5 marks) Consider a polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ of degree $\leqslant n$ with complex coefficients. Let $z_0 = 1 + i$ and suppose that $p(z_0) = p'(z_0) = \ldots = p^{(j)}(z_0) = \ldots = p^{(n)}(z_0) = 0$. Here $p^{(j)}(z_0)$ denotes the jth derivative of $p(z)$ at z_0. Show that $p(z)$ is the zero polynomial, i.e., $p(z) = 0$ for every complex number z.

Solution: Write $p(z)$ in Taylor form centered at z_0:

$$p(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \ldots + b_n(z - z_0)^n.$$

Here $b_0 = p(z_0) = 0$ and $b_j = \dfrac{p^{(j)}(z_0)}{j!} = 0$ for every $j = 1, 2, \ldots, n$. Hence, $p(z)$ is the zero polynomial, as claimed.

Problem 3. (5 marks) Determine for which nonzero complex numbers z

$$\mathrm{Log}(z^3) = 3\,\mathrm{Log}(z)$$

and for which non-zero complex numbers z

$$\mathrm{Log}(z^3) \neq 3\,\mathrm{Log}(z).$$

Here as usual, $\mathrm{Log}(z)$ denotes the principal branch of $\log(z)$.

Solution: Note that $z \neq 0$; otherwise neither side is defined. Write z in polar form, as $z = re^{i\theta}$, where $r > 0$ and the argument θ is chosen in the interval $(-\pi, \pi]$. In other words, $\theta = \mathrm{Arg}(z)$. Then

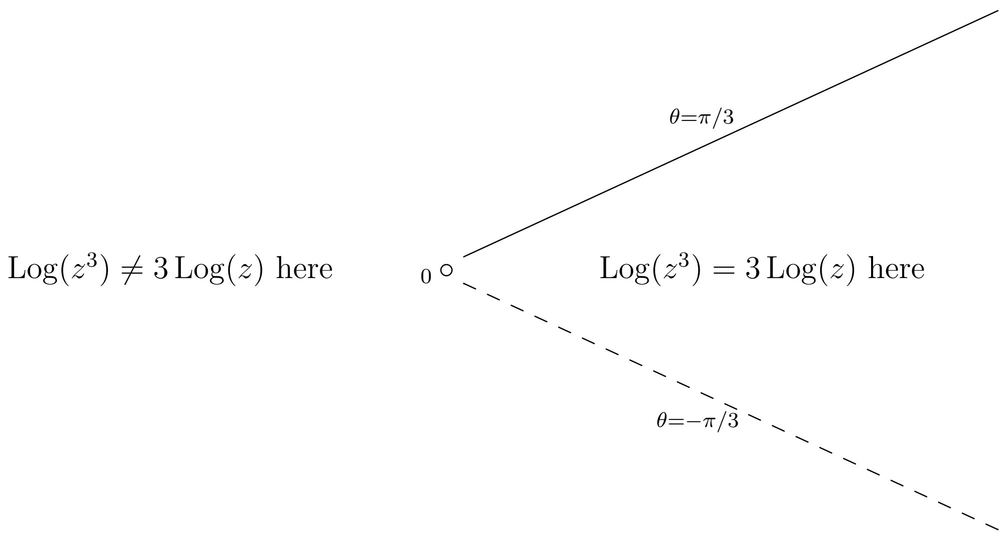
$$\mathrm{Log}(z^3) = \ln(r^3) + i\,\mathrm{Arg}(z^3) = 3\ln(r) + i\,\mathrm{Arg}(z^3)$$

and

$$3\,\mathrm{Log}(z) = 3\ln(r) + 3i\theta\,.$$

The real part of these two numbers is always the same, as long as $r \neq 0$ thus we need to find out for which z, $\mathrm{Arg}(z^3) = \mathrm{Arg}(r^3 e^{3\theta})$ equal 3θ? By the definition of Arg, this happens if and only if 3θ lies in the interval $(-\pi, \pi]$ or equivalently, θ lies in the interval $(-\pi/3, \pi/3]$.

Answer: $\mathrm{Log}(z^3) = 3\,\mathrm{Log}(z)$ if $z \neq 0$ and $\mathrm{Arg}(z) \in (-\pi/3,\, \pi/3]$, i.e., if and only if z lies in the wedge-shaped region pictured below:



Problem 4. Let Γ be the positively oriented circle of radius 3 centered at the origin. Evaluate

$$\int_\Gamma \frac{z+1}{z^2+4} dz.$$

Solution: Since $z^2 + 4 = (z - 2i)(z + 2i)$, we can decompose $\dfrac{z+1}{z^2+4}$ into a sum of partial fractions:

$$\frac{z+1}{z^2+4} = \frac{a}{z-2i} + \frac{b}{z+2i}\,.$$

Multiplying both sides by $z^2 + 4$, we obtain

$$z + 1 = a(z + 2i) + b(z - 2i).$$

Substitute $z = 2i$ and solve for a: $2i + 1 = a \cdot 4i$ and thus $\boxed{a = \frac{1}{2} - \frac{1}{4}i}$. Now substitute $z = -2i$ and solve for b: $-2i + 1 = b \cdot (-4i)$ and thus $\boxed{b = \frac{1}{2} + \frac{1}{4}i}$. Our partial fraction decomposition is thus

$$\frac{z+1}{z^2+4} = (\frac{1}{2} - \frac{1}{4}i)(z-2i)^{-1} + (\frac{1}{2} + \frac{1}{4}i)(z+2i)^{-1}\,,$$

and we can simplify our integral as follows:

$$\int_\Gamma \frac{z+1}{z^2+4} dz = (\frac{1}{2} - \frac{1}{4}i)\int_\Gamma (z-2i)^{-1} dz + (\frac{1}{2} + \frac{1}{4}i)\int_\Gamma (z+2i)^{-1} dz.$$

Now observe that

$$\int_\Gamma (z-2i)^{-1} dz = \int_{\Gamma_1} (z-2i)^{-1} dz,$$

where Γ_1 is the positively oriented circle of radius 1 centered at $2i$. This follows from deformation of path: Γ can be deformed into Γ_1 in the domain $D = \mathbb{C} \setminus \{2i\}$. The same argument shows that

$$\int_\Gamma (z-2i)^{-1} dz = \int_{\Gamma_2} (z+2i)^{-1} dz,$$

where Γ_2 is the positively oriented circle of radius 1 centered at $-2i$. As we showed in Lecture 13 (or in Example 2 on p. 166 in the text),

$$\int_{\Gamma_1} (z-2i)^{-1} dz = \int_{\Gamma_2} (z+2i)^{-1} dz = 2\pi i.$$

In summary,

$$\begin{aligned}
\int_\Gamma \frac{z+1}{z^2+4} dz &= (\frac{1}{2} - \frac{1}{4}i)\int_\Gamma (z-2i)^{-1} dz + (\frac{1}{2} + \frac{1}{4}i)\int_\Gamma (z+2i)^{-1} dz \\
&= (\frac{1}{2} - \frac{1}{4}i)\int_{\Gamma_1} (z-2i)^{-1} dz + (\frac{1}{2} + \frac{1}{4}i)\int_{\Gamma_2} (z+2i)^{-1} dz \\
&= (\frac{1}{2} - \frac{1}{4}i)\, 2\pi i + (\frac{1}{2} + \frac{1}{4}i)\, 2\pi i = 2\pi i.
\end{aligned}$$