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Problem Set 2

Section 2.5

Q1 $f(x) = ax^2 + bxy + cy^2$

- we need $\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = 0$

$\frac{df}{dx} = 2ax + by, \quad \frac{d^2 f}{dx} = 2a$

$\frac{df}{dy} = bx + 2cy, \quad \frac{d^2 f}{dx} = 2c$

$2a + 2c = 0$

$2a = -2c$

$a = -c$

$\therefore \boxed{ax^2 + bxy - ay^2}$

Q2

- We know that if u, v are harmonic, then $f(z) = u(x,y) + iv(x,y)$ is analytic. We also know that for u, we can find a set of harmonic conjugates v with different constants attached

$f(z) = -u(x,y) + iv(x,y)$ is also analytic

$\therefore -u$ is a harmonic conjugate for v

Q3

- If v is a harmonic conjugate

$f(z) = u(x,y) + iv(x,y)$ is Analytic. So uv is Harmonic in D

Section 3.1

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_0$$

$$a_n (z-z_1)^{d_1} (z-z_2)^{d_2} \dots (z-z_r)^{d_r}$$

$$= a_n (z^{d_1} - d_1 z^{d_1-1} z_1 + \dots (-1)^n z_1^{d_1})(z^{d_2} - d_2 z^{d_2-1} z_2 + \dots (-1)^n z_2^{d_2})$$

$$\dots (z^{d_r} - d_r z^{d_r-1} z_r + \dots (-1)^n z_r^{d_r})$$

$$= a_n z^{d_1+d_2+d_3+\dots d_r} - a_n (d_1 z_1 + d_2 z_2 + \dots d_r z_r) z^{d_1+d_2+d_3 \dots d_r-1)}$$

$$\dots a_n (-1)^{d_1+d_2+d_3 \dots d_r} z_1^{d_1} z_2^{d_2} \dots z_r^{d_r}$$

$\therefore n = d_1 + d_2 \dots d_r$, $a_0 = a_n (-1)^n z_1^{d_1} z_2^{d_2} \dots z_r^{d_r}$

$n \geq 1$, $|a_0| > 1$

$|a_0| = |z_1|^{d_1} |z_2|^{d_2} \dots |z_r|^{d_r} > 1$

$|z_i| > 1$ for $i \dots$

$\therefore$ $\boxed{p(z) \text{ has } 1 \text{ or more zeros outside the unit circle}}$

$\boxed{Q2}$

$$R_{m,n}(z) = \frac{P(z)}{Q(z)}, \quad r_{m,n}(z) = \frac{p(z)}{q(z)}$$

$\boxed{Q3}$

$$R(z) = \frac{r(z)}{(z-z_0)^m}$$

$$R'(z) = \frac{r'(z)(z-z_0)^m - mr(z)(z-z_0)^{m-1}}{(z-z_0)^{2m}}$$

$$= \frac{r'(z)(z-z_0) - mr(z)}{(z-z_0)^{m+1}} = \frac{h(z)}{(z-z_0)^{m+1}}$$

$\boxed{Q4}$

$$R(z) = \frac{d_1}{z-z_1} + \frac{d_2}{z-z_2} + \cdots \frac{d_r}{z-z_r}, \quad Im(z)>0$$

Hint: $$\overline{R(z)} = \frac{d_1(z-z_1)}{|z-z_1|^2} + \frac{d_2(z-z_2)}{|z-z_2|^2} + \cdots \frac{d_1(z-z_r)}{|z-z_r|^2}$$

Section 3.2

$\boxed{Q_1}$

(a) $$\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{\sin z - \sin(0)}{z} = (\sin z)'\Big|_0$$

$$= \cos(z)\Big|_0 = \boxed{1}$$

(b) $$\lim_{z\to 0} \frac{\cos z - 1}{z} = \lim_{z\to 0} \frac{\cos z - \cos(0)}{z} = [\cos(z)]'\Big|_0 = \sin(z)\Big|_0 = \boxed{0}$$

Section 3.3

Q1

(a) $\log^{z_1 z_2} = \log^{|z_1 z_2|} + i\arg(z_1 z_2)$

$= \log^{|z_1|} + \log^{|z_2|} + i\arg(z_1) + i\arg(z_2)$

$= \log^{|z_1|} + i\arg(z_1) + \log^{|z_2|} + i\arg(z_2)$

$= \log^{z_1} + \log^{z_2}$

(b) $\log^{z_1/z_2} = \log^{z_1 z_2^{-1}} = \log^{|z_1 z_2^{-1}|} + i\arg(z_1 z_2^{-1})$

$= \log^{|z_1|} - \log^{|z_2|} + i\arg(z_1) - i\arg(z_2)$

$= \log^{|z_1|} + i\arg(z_1) - \log^{|z_2|} - i\arg(z_2)$

$= \log^{z_1} - \log^{z_2}$

Q2

$\log i(i-1) = \log(i^2 - i)$

$= \log(-1-i)$

$= \log\sqrt{2}\left(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \log\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$= \log\sqrt{2}\, e^{i\frac{-3\pi}{4}}$

$= \log\sqrt{2} - \frac{3\pi}{4}i$

Continued.... $\operatorname{Log} z_1 z_2 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

~~Log(i) + Log~~ $\operatorname{Log} i + \operatorname{Log}(i-1) = \operatorname{Log}(e^{i\pi/2}) + \operatorname{Log}(\sqrt{2}e^{i3\pi/4})$

$$= i\frac{\pi}{2} + i\frac{3\pi}{4} + \operatorname{Log}\sqrt{2}$$

$$= i\frac{5\pi}{4} + \operatorname{Log}\sqrt{2}$$

$\therefore \operatorname{Log} z_1 z_2 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$ if $z_1 = i$, $z_2 = i-1$

$\boxed{Q3}$

$z = -z$, since $z^2 = (-z)^2$

$2\operatorname{Log}(z) = 2\operatorname{Log}(-z)$, $\operatorname{Log}(z) = \operatorname{Log}(-z)$ $\therefore z = e^{\operatorname{Log}(z)} = e^{\operatorname{Log}(-z)} = -z$

$z = -i$

$\operatorname{Log} z^2 = 2\operatorname{Log}|z| + i\operatorname{Arg}(z^2)$ $\rightarrow$ $\operatorname{Arg}(z) = -\pi/2$

$= 2\operatorname{Log}|z| + i\pi$

$2\operatorname{Log}|z| + i\pi \neq 2\operatorname{Log}|z| - i\pi$

Section 3.5

Q1 $\boxed{\text{NO}}$

$$z^{\alpha} = e^{\alpha \cos(z)}$$

if $z=1$

$$1^{\alpha} = e^{\alpha \log(1)} = \cancel{e^{\alpha C \log}}$$

$$= \cancel{e^{\alpha C \log(1)} + k}$$

$$= e^{\alpha (\log(1) + i \arg(C_1) \cdot C + 2\pi k))}$$

$$= e^{2k\pi\alpha i}$$

let $k=1$, $\alpha = 1/2$, $e^{2(1)(\pi)(1/2)i} = e^{i\pi} = \boxed{-1 \neq 1}$

Q2 (a) $z^{-\alpha} = e^{-\alpha \log z} = e^{-(\alpha \log z)}$

$$= \left(e^{\alpha \log(z)}\right)^{-1} = \frac{1}{z^{\alpha}} \rightarrow z^{-\alpha} = \frac{1}{z^{\alpha}}$$

(b) $z^{\alpha} z^{\beta} = e^{\alpha \log z} e^{\beta \log z} = e^{\alpha \log z + \beta \log z}$

$$= e^{(\alpha+\beta)\log z} = z^{\alpha+\beta} \rightarrow z^{\alpha} \cdot z^{\beta} = z^{\alpha+\beta}$$

(c) $\dfrac{z^{\alpha}}{z^{\beta}} = \dfrac{e^{\alpha \log(z)}}{e^{\beta \log(z)}} = e^{\alpha \log z - \beta \log z} = e^{(\alpha-\beta)\log z}$

$$= z^{\alpha-\beta}$$

$$\rightarrow \frac{z^{\alpha}}{z^{\beta}} = z^{\alpha-\beta}$$

Section 4.1

$$\gamma_1(t) = z_1 + t(z_2 - z_1)$$
$$= -2 + 2i + t(-1-(-2+2i))$$
$$= -2+2i + t(1-2i)$$
$$= t - 2 + i(2-2t)$$

$$\gamma_2(t) = \cancel{z_0} + (1)e^{-it}$$
$$= e^{-it}, \quad -\pi \le t \le 0$$
$$= e^{-it}$$

$$\boxed{\Gamma = \begin{cases} t-2+i(2-2t), & 0 \le t \le 1 \\ e^{-it}, & -\pi \le t \le 0 \end{cases}}$$

For $-\Gamma$

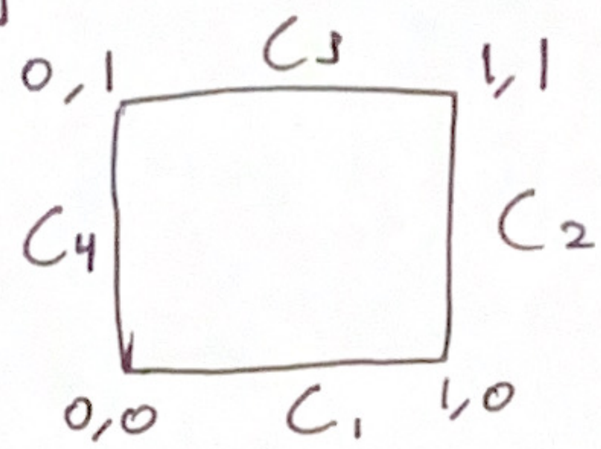
$$\gamma_1(t) = e^{it}$$

$$\gamma_2(t) = -1 + t(-2+2i-(-1))$$
$$= -t + 2it - 1$$

$$\boxed{-\Gamma = \begin{cases} e^{it}, & 0 \le t \le \pi \\ -t+2it-1, & 0 \le t \le 1 \end{cases}}$$

Section 4.2

Q1 $\int_C \bar{z}^2 \, dz$



$C_1 \rightarrow z(t) = x + iy = \cancel{t, 0 \le t \le 1}$
$= t, 0 \le t \le 1$

$C_2 \rightarrow z(t) = x + iy = 1 + it, 0 \le t \le 1$

$C_3 \rightarrow z(t) = x + iy = 1 - t + i, 0 \le t \le 1$

$C_4 \rightarrow z(t) = x + iy = i - it, 0 \le t \le 1$

$$\int_C \bar{z}^2 dz = \int_{C_1} \bar{z}^2 dz + \int_{C_2} \bar{z}^2 dz + \int_{C_3} \bar{z}^2 dz + \int_{C_4} \bar{z}^2 dz$$

$$= \int_0^1 (t)^2 dt + \int_0^1 (1+it)^2 \, d(1+it) + \int_0^1 ((1-t)+i)^2 \, d(1-t+i) + \int_0^1 (i-it)^2 \, d(i-it)$$

$$= \frac{1}{3}(1) - \frac{1}{3}\left[(1-i)^3 - 1\right] + \frac{1}{3}\left[(-i)^3 - (1-i)^3\right] - \frac{i}{3}(0-1)$$

$$= \frac{1}{3} - \frac{1}{3}(1-i)^3 + \frac{1}{3} + \frac{1}{3}(-i)^3 - \frac{1}{3}(1-i)^3 + i/3$$

$$= \frac{2}{3} - \frac{2}{3}(1-i)^3 + \frac{i}{3} + \frac{i}{3}$$

$$= \frac{2}{3} - \frac{2}{3}(-2-2i) + \frac{2}{3}i$$

$$= 2 + 2i$$

$$\boxed{\therefore \int_C \bar{z}^2 dz = 2 + 2i}$$

$\boxed{Q2}$

(a) $|z^2 - i| \geq |z^2| - |i|$

$= |z|^2 - 1$

$= 9 - 1$

$= 8$

$\frac{1}{|z^2 - i|} \leq \frac{1}{8}$ if $|z| = 3$

$$\left| \int_C \frac{dz}{z^2 - i} \right| \leq \max \left| \frac{1}{z^2 - i} \right| \cdot l(|z| = 3)$$

$$= \frac{1}{8} \cdot 2\pi(3)$$

$$= \frac{3\pi}{4}$$

$$\therefore \boxed{\left| \int_C \frac{dz}{z^2 - i} \right| \leq 3\pi/4}$$

(b) $|e^{3z}| = |e^{3(x+iy)}|$

$= |e^{3x}| |e^{i(3y)}|$

$|e^{3z}| = |e^{3x}| \longrightarrow |e^{3z}| = e^{3R}$

$\cancel{|e^{3z}| = e^{3R}}$

$|e^z + 1| \geq |e^z| - 1$

$\geq |e^{x+iy}| - 1$

$\geq |e^x| - 1$

$\therefore \left| \frac{1}{1+e^z} \right| \leq \frac{1}{e^R - 1}$

$$\left| \int_\gamma \frac{e^{3z}}{1+e^z} dz \right| \leq \max \left| \frac{e^{3z}}{1+e^z} \right| l(\gamma)$$

$$\leq \frac{2\pi e^{3R}}{e^R - 1}$$

$$\boxed{\therefore \left| \int_\gamma \frac{e^{3z}}{1+e^z} dz \right| \leq \frac{2\pi e^{3R}}{e^R - 1}}$$

(c) $l(\Gamma) = \frac{1}{4} \cdot 2\pi = \pi/2$

$$\text{Log}\, z = \log|z| + i\,\text{Arg}\, z$$
$$= 0 + i\,\text{Arg}\, z$$
$$= i\,\text{Arg}\, z$$

$$|\text{Log}\, z| = |i\,\text{Arg}\, z| \leq \pi/2$$

$$\left| \int_\Gamma \text{Log}\, z\, dz \right| \leq \max |\text{Log}\, z|\, l(\Gamma)$$
$$\leq \pi^2/4$$

$$\boxed{\therefore \left| \int_\Gamma \text{Log}\, z\, dz \right| \leq \pi^2/4}$$

(d) $f(z) = e^{\sin z} = e^{\sin(x+iy)} = e^{\sin x \cosh y} \cdot e^{i(\cos x \sinh y)}$

$$|f(z)| = |e^{\sin x \cosh y}\, e^{i(\cos x \sinh y)}|$$

$$|e^{\sin z}| = e^{|\sin x \cosh y|}$$

$$|e^{\sin z}| = |e^{\sin(0)\cosh(y)}|$$
$$= 1$$

$$\left| \int_\gamma e^{\sin z}\, dz \right| \leq \max |e^{\sin z}|\, l(\gamma)$$
$$\leq 1$$

$$\cancel{\boxed{\therefore \left| \int_\gamma e^{\sin z}\, dz \right| \leq 1}}$$

$$\boxed{\therefore \left| \int_\gamma e^{\sin z}\, dz \right| \leq 1}$$

Section 4.3

Q1 $\boxed{\text{FALSE}}$

$$\int_C f(z)\,dz = 2\pi i\,[R_1 + R_2 \cdots + R_n), \quad f(z) = \frac{1}{z}$$

$$\int_{|z|=2} \frac{1}{z}\,dz = 2\pi i \cdot R_1, \quad R_1 \neq 0 \rightarrow \int_{|z|=2} \frac{1}{z}\,dz \neq 0$$

Q2

$$\int_C \frac{1}{z-z_0}\,dz \rightarrow \lim_{\alpha,\beta \to r} \int_C \frac{1}{z-z_0}\,dz$$

$$= \lim_{\beta,\alpha \to r} \left(\operatorname{Log}(\beta - z_0) - \operatorname{Log}(\alpha - z_0)\right)$$

$$= \operatorname{Log}(r - z_0) + i\pi - (\operatorname{Log}(r - z_0) - i\pi)$$

$$= \boxed{i \cdot 2\pi}$$

Q3

$$\int_\gamma f'(z)\,dz = f(z_2) - f(z_1)$$

$$|f(z_2) - f(z_1)| = \left|\int_\gamma f'(z)\,dz\right|$$

$$\leq M\,l(\gamma)$$

$$\leq \boxed{M|z_2 - z_1|}$$

$$\downarrow$$

$\therefore\ |f(z_2) - f(z_1)| \leq M|z_2 - z_1|$ for any z_1, z_2 in D