Mathematical Writing

Franco Vivaldi
Queen Mary, University of London

Mathematical Reading Writing Translating

Franco Vivaldi
Queen Mary, University of London

Why mathematical writing?

Why mathematical writing?

3. Writing is an essential skill, and many students are not good at it.

Why mathematical writing?

3. Writing is an essential skill, and many students are not good at it.

2. A preparation for --or an alternative to-- a final year project.

Why mathematical writing?

3. Writing is an essential skill, and many students are not good at it.

2. A preparation for --or an alternative to-- a final year project.

1. Writing <u>forces</u> one to think about structures.

Why mathematical writing?

3. Writing is an essential skill, and many students are not good at it.

2. A preparation for --or an alternative to-- a final year project.

1. Writing <u>forces</u> one to think about structures.

0. Teaching MW gives you X-rays into the students' thoughts.

Why mathematical writing?

3. Writing is an essential skill, and many students are not good at it.

2. A preparation for --or an alternative to-- a final year project.

1. Writing <u>forces</u> one to think about structures.

0. Teaching MW gives you X-rays into the students' thoughts.

- L. Alcock & A. Simpson, *Ideas from mathematics education*, MSOR (2009).

  "We have found it illuminating to ask students to think aloud ... as they work on mathematical tasks. This often uncovers fragility in the knowledge of even the strongest students."

Why mathematical writing?

3. Writing is an essential skill, and many students are not good at it.

2. A preparation for --or an alternative to-- a final year project.

1. Writing <u>forces</u> one to think about structures.

0. Teaching MW gives you X-rays into the students' thoughts.

- L. Alcock & A. Simpson, *Ideas from mathematics education*, MSOR (2009).

  "We have found it illuminating to ask students to think aloud ... as they work on mathematical tasks. This often uncovers fragility in the knowledge of even the strongest students."

- MW students commented on the "unexpected depth" required of their thinking, when asked to offer verbal explanations.

An exercise

An exercise

On the plane, I have a circle and a point outside it, and I must find the lines through this point which are tangent to the circle. What shall I do?

An exercise

On the plane, I have a circle and a point outside it, and I must find the lines through this point which are tangent to the circle. What shall I do?

Answer this question **over the phone**.

An exercise

On the plane, I have a circle and a point outside it, and I must find the lines through this point which are tangent to the circle. What shall I do?

Answer this question **over the phone**.

- This task requires grasp of structure and organisation;

An exercise

> On the plane, I have a circle and a point outside it, and I must find the lines through this point which are tangent to the circle. What shall I do?

Answer this question **over the phone**.

- This task requires grasp of structure and organisation;
- it gives the students an opportunity to express their knowledge, intelligence, and individuality;

An exercise

> On the plane, I have a circle and a point outside it, and I must find the lines through this point which are tangent to the circle. What shall I do?

Answer this question **over the phone**.

- This task requires grasp of structure and organisation;
- it gives the students an opportunity to express their knowledge, intelligence, and individuality;
- but it also exposes logical faults, immaturity, incompetence.

~100 words

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

Adjoin the equation of the line to that of the circle and eliminate one of the unknowns. After a substitution, you'll end up with a quadratic equation in one unknown, whose coefficients depend on the parameter.

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

Adjoin the equation of the line to that of the circle and eliminate one of the unknowns. After a substitution, you'll end up with a quadratic equation in one unknown, whose coefficients depend on the parameter.

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

Adjoin the equation of the line to that of the circle and eliminate one of the unknowns. After a substitution, you'll end up with a quadratic equation in one unknown, whose coefficients depend on the parameter.

Equate the discriminant of the quadratic equation to zero, to obtain an equation -also quadratic- for the slope. Its two solutions are the desired slopes of the tangent lines. Any configuration involving vertical lines (infinite slope) will require a variant of the above procedure.

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

Adjoin the equation of the line to that of the circle and eliminate one of the unknowns. After a substitution, you'll end up with a quadratic equation in one unknown, whose coefficients depend on the parameter.

Equate the discriminant of the quadratic equation to zero, to obtain an equation -also quadratic- for the slope. Its two solutions are the desired slopes of the tangent lines. Any configuration involving vertical lines (infinite slope) will require a variant of the above procedure.

~100 words

Write down the equation of a line passing trough the point. This equation will depend on one parameter, the line's slope, which is the quantity to be determined.

Adjoin the equation of the line to that of the circle and eliminate one of the unknowns. After a substitution, you'll end up with a quadratic equation in one unknown, whose coefficients depend on the parameter.

Equate the discriminant of the quadratic equation to zero, to obtain an equation -also quadratic- for the slope. Its two solutions are the desired slopes of the tangent lines. Any configuration involving vertical lines (infinite slope) will require a variant of the above procedure.

Classroom schizophrenia

Classroom schizophrenia

Teacher	Student

Classroom schizophrenia

Teacher

Student

definitions
theorems

Classroom schizophrenia

Teacher

Student

definitions
theorems

↑

examples

Classroom schizophrenia

Teacher

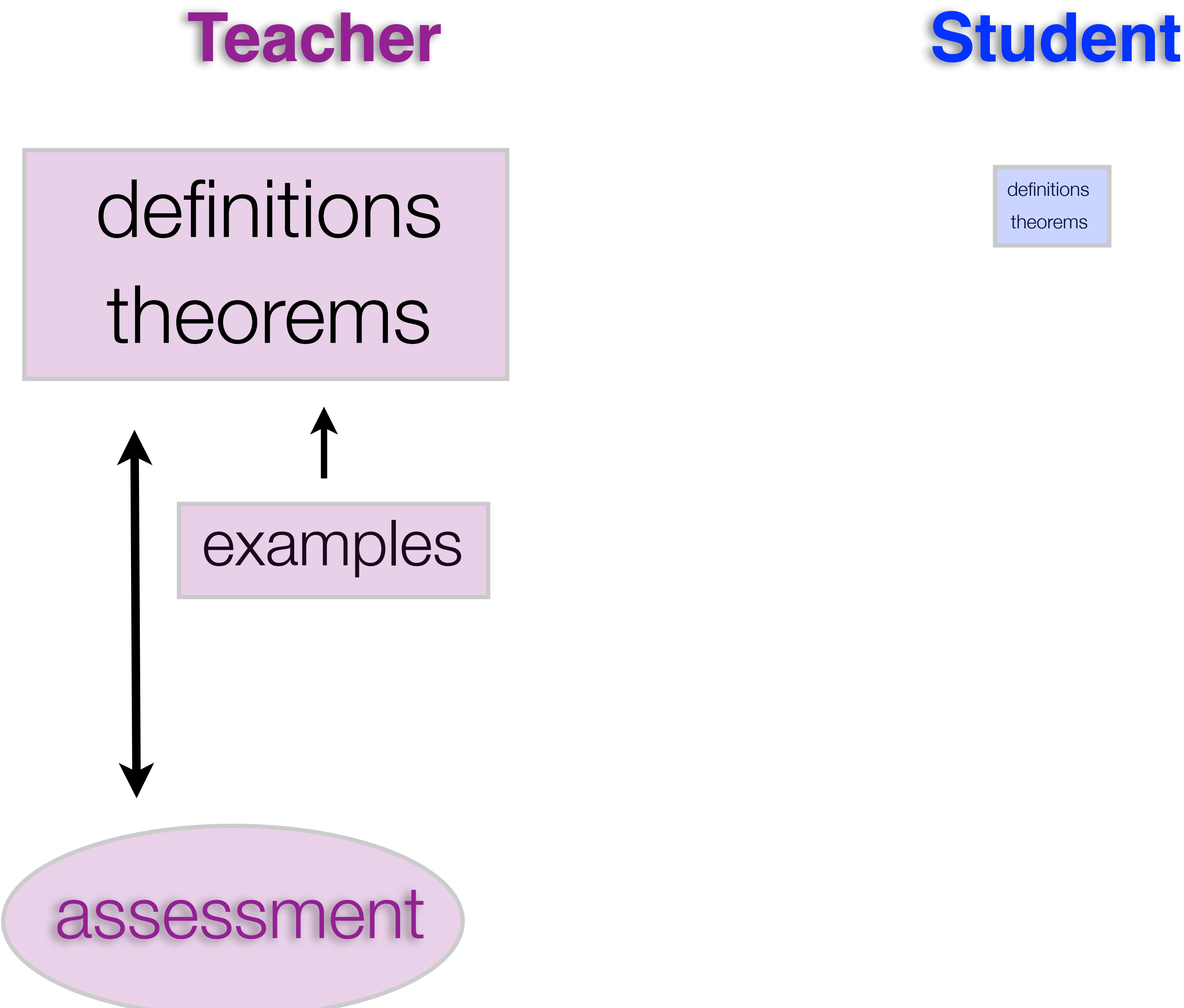
Student

definitions
theorems

examples

assessment

Classroom schizophrenia



Classroom schizophrenia

Teacher

definitions
theorems

examples

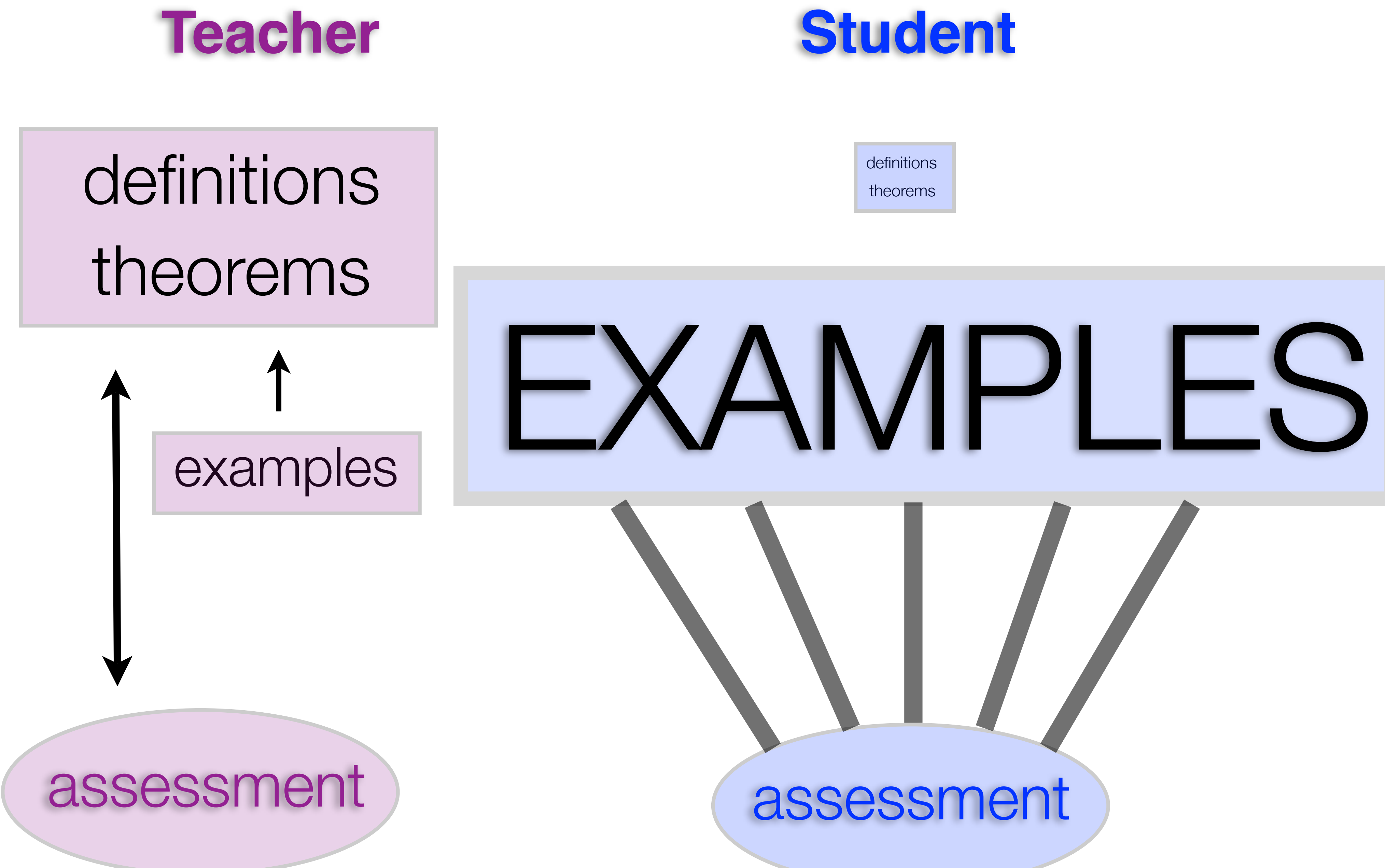
assessment

Student

definitions
theorems

EXAMPLES

Classroom schizophrenia



Many mathematics students are not prepared for the mathematical language.

Many mathematics students are not prepared for the mathematical language.

Why?

Many mathematics students are not prepared for the mathematical language.

Why?

- Insufficient exposure to formal logical systems (grammar and syntax, Latin, Euclidean geometry, programming, philosophy, foreign languages).

Many mathematics students are not prepared for the mathematical language.

Why?

- Insufficient exposure to formal logical systems (grammar and syntax, Latin, Euclidean geometry, programming, philosophy, foreign languages).

  Words and symbols don’t have **exact meaning.**

Many mathematics students are not prepared for the mathematical language.

Why?

- Insufficient exposure to formal logical systems (grammar and syntax, Latin, Euclidean geometry, programming, philosophy, foreign languages).

  Words and symbols don't have **exact meaning.**

- League tables.

Many mathematics students are not prepared for the mathematical language.

Why?

- Insufficient exposure to formal logical systems (grammar and syntax, Latin, Euclidean geometry, programming, philosophy, foreign languages).

  Words and symbols don't have **exact meaning.**

- League tables.

  Concepts are replaced by **processes**.

Many mathematics students are not prepared for the mathematical language.

Why?

- Insufficient exposure to formal logical systems (grammar and syntax, Latin, Euclidean geometry, programming, philosophy, foreign languages).

  Words and symbols don’t have **exact meaning.**

- League tables.

  Concepts are replaced by **processes**.

lack of conceptual accuracy

Many mathematics students are not prepared for the mathematical language.

Why?

- Insufficient exposure to formal logical systems (grammar and syntax, Latin, Euclidean geometry, programming, philosophy, foreign languages).

  Words and symbols don't have **exact meaning.**

- League tables.

  Concepts are replaced by **processes**.

lack of conceptual accuracy ⟶ inadequate reading
ineffective learning
difficulties with abstraction
difficulties with reasoning
poor writing

The language of processes

The language of processes

Compute the value of the following expression:

$$\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$$

The language of processes

Compute the value of the following expression:

$$\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$$

Great complexity, limited conceptual content.

The language of processes

Compute the value of the following expression:

$$\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$$

Great complexity, limited conceptual content.

Compute the derivative of the following function:

$$f(x)=\log(2+\sin(x^2)).$$

The language of processes

Compute the value of the following expression:

$$\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$$

Great complexity, limited conceptual content.

Compute the derivative of the following function:

$$f(x)=\log(2+\sin(x^2)).$$

The language of processes

Compute the value of the following expression:

$$\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$$

Great complexity, limited conceptual content.

Compute the derivative of the following function:

$$f(x)=\log(2+\sin(x^2)).$$

To complete the task, knowledge of the exact meaning of words and symbols is irrelevant.

The language of concepts: reading symbols

The language of concepts: reading symbols

Chinese:

九	Nine
力	Power
刀	Knife

The language of concepts: reading symbols

Chinese:

九 Nine

力 Power

刀 Knife

Mathematics:

$$f^{-1}(x)$$

$$f^{-1}(\{x\})$$

$$f(x)^{-1}$$

The language of concepts: reading symbols

Chinese:

九 Nine

力 Power

刀 Knife

Mathematics:

$$f^{\boxed{-1}}(x)$$

$$f^{\boxed{-1}}(\{x\})$$

$$f(x)^{\boxed{-1}}$$

One symbol, three meanings:

The language of concepts: reading symbols

Chinese:

九 Nine

力 Power

刀 Knife

Mathematics:

$f^{\boxed{-1}}(x)$	the value of an inverse function at a point
$f^{\boxed{-1}}(\{x\})$	the inverse image of a set under a function
$f(x)^{\boxed{-1}}$	the reciprocal of the value of a function at a point

One symbol, three meanings:

The language of concepts: reading symbols

Chinese:

九	Nine
力	Power
刀	Knife

Mathematics:

$f^{\boxed{-1}}(x)$	the value of an inverse function at a point
$f^{\boxed{-1}}(\{x\})$	the inverse image of a set under a function
$f(x)^{\boxed{-1}}$	the reciprocal of the value of a function at a point

One symbol, three meanings:

$$(x \boxed{+} I) \boxed{+} (y \boxed{+} I) := (x \boxed{+} y) \boxed{+} I$$

... within a single expression!

The language of concepts: reading symbols

Chinese:

九 Nine

力 Power

刀 Knife

Mathematics:

$f^{-1}(x)$ the value of an inverse function at a point

$f^{-1}(\{x\})$ the inverse image of a set under a function

$f(x)^{-1}$ the reciprocal of the value of a function at a point

One symbol, three meanings:

$$(x+I)+(y+I) := (x+y)+I$$

... within a single expression!

$$A \cup (B \cap C) \qquad A \subset (B \cap C) \qquad (A \subset B) \cap C$$

The language of concepts: reading symbols

Chinese: 九 Nine 力 Power 刀 Knife

Mathematics:

$f^{-1}(x)$	the value of an inverse function at a point
$f^{-1}(\{x\})$	the inverse image of a set under a function
$f(x)^{-1}$	the reciprocal of the value of a function at a point

One symbol, three meanings:

$$(x+I)+(y+I) := (x+y)+I$$

... within a single expression!

$A \cup (B \cap C)$	$A \subset (B \cap C)$	$(A \subset B) \cap C$
set	sentence	nonsense

The language of concepts: reading symbols

Chinese:

九 Nine

力 Power

刀 Knife

Mathematics:

$f^{-1}(x)$	the value of an inverse function at a point
$f^{-1}(\{x\})$	the inverse image of a set under a function
$f(x)^{-1}$	the reciprocal of the value of a function at a point

One symbol, three meanings:

$$(x + I) + (y + I) := (x + y) + I$$

... within a single expression!

$A \cup (B \cap C)$	$A \subset (B \cap C)$	$(A \subset B) \cap C$
set	sentence	nonsense

The dog is black. The black is dog.

The language of concepts: reading symbols

Chinese:

九 Nine

力 Power

刀 Knife

Mathematics:

$f^{-1}(x)$	the value of an inverse function at a point
$f^{-1}(\{x\})$	the inverse image of a set under a function
$f(x)^{-1}$	the reciprocal of the value of a function at a point

One symbol, three meanings:

$$(x + I) + (y + I) := (x + y) + I$$

... within a single expression!

relational operator

set operator

$A \cup (B \cap C)$	$A \subset (B \cap C)$	$(A \subset B) \cap C$
set	sentence	nonsense

The dog is black. The black is dog.

The language of concepts: reading words

The language of concepts: reading words

Every detail matters:

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

definite

indefinite

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

definite

indefinite

The dictionary is constantly needed:

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

definite

indefinite

The dictionary is constantly needed:

Let the restriction of *f* to the open unit interval be surjective.

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

definite

indefinite

The dictionary is constantly needed:

Let the restriction of f to the open unit interval be surjective.

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

definite

indefinite

The dictionary is constantly needed:

Let the restriction of f to the open unit interval be surjective.

Verba volant, scripta manent.

Heb je geen paard, gebruik dan een ezel.

The language of concepts: reading words

Every detail matters:

The set of even integers.
A set of even integers.
The set of the divisors of a large integer.
A set of divisors of a multiple of 24.

definite

indefinite

The dictionary is constantly needed:

EXAM

EXAMPLES

EXA

EXAMPLES

The mathematical language is full of idiomatic expressions:

The mathematical language is full of idiomatic expressions:

$$\sin(\mathbb{Q}) \qquad \mathbb{P}(A|B) \qquad \mathbb{R}/\pi\mathbb{Z} \qquad |\{1,\ldots,f(n)\}|$$

The mathematical language is full of idiomatic expressions:

$$\sin(\mathbb{Q}) \qquad \mathbb{P}(A|B) \qquad \mathbb{R}/\pi\mathbb{Z} \qquad |\{1,\ldots,f(n)\}|$$

There is pathological reluctance to standardise basic symbols:

The mathematical language is full of idiomatic expressions:

$$\sin(\mathbb{Q}) \qquad \mathbb{P}(A|B) \qquad \mathbb{R}/\pi\mathbb{Z} \qquad |\{1,\ldots,f(n)\}|$$

There is pathological reluctance to standardise basic symbols:

$$\mathbb{N} = \{1,2,3,\ldots\} \qquad \mathbb{N} = \{0,1,2,3,\ldots\}$$

$$\mathbb{Z} \subset \mathbb{Z} \qquad \mathbb{Z} \subseteq \mathbb{Z}$$

The mathematical language is full of idiomatic expressions:

$$\sin(\mathbb{Q}) \qquad \mathbb{P}(A|B) \qquad \mathbb{R}/\pi\mathbb{Z} \qquad |\{1,\ldots,f(n)\}|$$

There is pathological reluctance to standardise basic symbols:

$$\mathbb{N} = \{1,2,3,\ldots\} \qquad \mathbb{N} = \{0,1,2,3,\ldots\}$$

$$\mathbb{Z} \subset \mathbb{Z} \qquad \mathbb{Z} \subseteq \mathbb{Z}$$

...and words:

The mathematical language is full of idiomatic expressions:

$$\sin(\mathbb{Q}) \qquad \mathbb{P}(A|B) \qquad \mathbb{R}/\pi\mathbb{Z} \qquad |\{1,\dots,f(n)\}|$$

There is pathological reluctance to standardise basic symbols:

$$\mathbb{N} = \{1,2,3,\dots\} \qquad \mathbb{N} = \{0,1,2,3,\dots\}$$

$$\mathbb{Z} \subset \mathbb{Z} \qquad \mathbb{Z} \subseteq \mathbb{Z}$$

...and words:

By a **triangle** we mean a metric space of cardinality three.

By a **segment** we mean a maximal subpath of P that contains only light or only heavy edges.

By a **circle** we mean an affinoid isomorphic to max $\mathbf{C}_p(T,T\text{-}1)$.

Oblivious teaching

Oblivious teaching

Conditional probability:

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Oblivious teaching

Conditional probability:

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of A given B

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of A, given B

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of A, given B

$$\mathbb{P}(A \cap B) \qquad \mathbb{P}(A \cup B) \qquad \mathbb{P}(A \setminus B) \qquad \mathbb{P}(A|B)$$

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of A, given B

$$\mathbb{P}(A \cap B) \qquad \mathbb{P}(A \cup B) \qquad \mathbb{P}(A \setminus B) \qquad \mathbb{P}(A|B)$$

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of A, given B

$\mathbb{P}(A \cap B)$ $\mathbb{P}(A \cup B)$ $\mathbb{P}(A \setminus B)$ $\mathbb{P}(A|B)$

$\mathbb{P}_c(A, B)$

Oblivious teaching

Conditional probability:

The probability of A intersection B

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of A, given B

$\mathbb{P}(A \cap B)$ $\mathbb{P}(A \cup B)$ $\mathbb{P}(A \setminus B)$ $\mathbb{P}(A|B)$

$\mathbb{P}_c(A, B)$

Grammar and syntax should take precedence over semantics.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

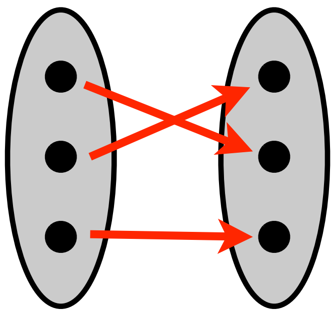
Injectivity: A function is injective if distinct elements of the domain have distinct images.

icons

metaphors

Injectivity: A function is injective if distinct elements of the domain have distinct images.

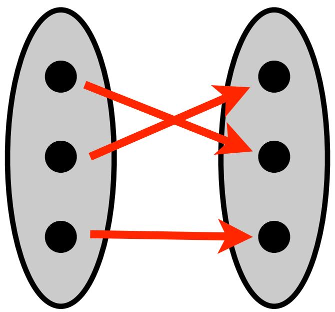
icons



metaphors

Injectivity: A function is injective if distinct elements of the domain have distinct images.

icons

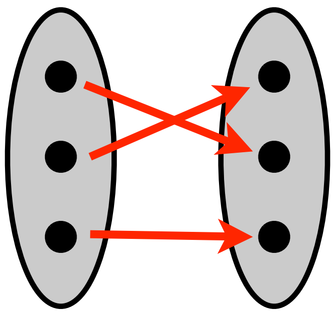


metaphors

We have a group of archers (the elements of the domain), each with one arrow (the function). If all enemies get killed, the function is surjective, if nobody is hit twice, the function is injective.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

icons



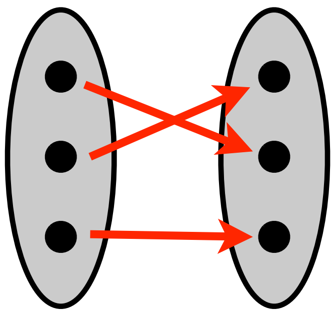
metaphors

We have a group of archers (the elements of the domain), each with one arrow (the function). If all enemies get killed, the function is surjective, if nobody is hit twice, the function is injective.

In absence of a definition, icons and metaphors may only illustrate themselves.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

icons



metaphors

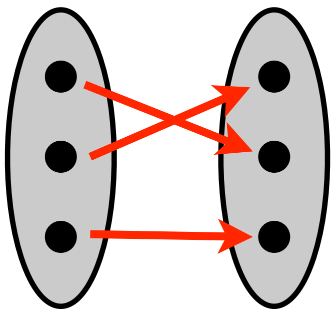
We have a group of archers (the elements of the domain), each with one arrow (the function). If all enemies get killed, the function is surjective, if nobody is hit twice, the function is injective.

In absence of a definition, icons and metaphors may only illustrate themselves.

Attention should shift to the defining sentence.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

icons



metaphors

We have a group of archers (the elements of the domain), each with one arrow (the function). If all enemies get killed, the function is surjective, if nobody is hit twice, the function is injective.

In absence of a definition, icons and metaphors may only illustrate themselves.

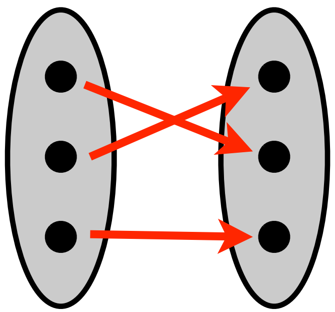
Attention should shift to the defining sentence.

Change words:

A diet is varied if distinct days of the week have distinct menus.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

icons



metaphors

We have a group of archers (the elements of the domain), each with one arrow (the function). If all enemies get killed, the function is surjective, if nobody is hit twice, the function is injective.

In absence of a definition, icons and metaphors may only illustrate themselves.

Attention should shift to the defining sentence.

Change words:

A diet is varied if distinct days of the week have distinct menus.

Introduce symbols:

Let D be a diet and let x and y be two days of the week...

Encourage logical analysis:

Encourage logical analysis:

A function f is injective if distinct elements of the domain of f have distinct images.

Encourage logical analysis:

A function f is injective if distinct elements of the domain of f have distinct images.

(f is a dummy variable)

Encourage logical analysis:

A function f is injective if distinct elements of the domain of f have distinct images.

(f is a dummy variable)

The function f is injective because distinct elements of the domain of f have distinct images.

Encourage logical analysis:

A function f is injective if distinct elements of the domain of f have distinct images.

(f is a dummy variable)

The function f is injective because distinct elements of the domain of f have distinct images.

(f is not a dummy variable)

Encourage logical analysis:

A function f is injective if distinct elements of the domain of f have distinct images.

(f is a dummy variable)

The function f is injective because distinct elements of the domain of f have distinct images.

(f is not a dummy variable)

A function f is injective if any two elements of the domain of f have distinct images.

Encourage logical analysis:

A function f is injective if distinct elements of the domain of f have distinct images.

(f is a dummy variable)

The function f is injective because distinct elements of the domain of f have distinct images.

(f is not a dummy variable)

A function f is injective if any two elements of the domain of f have distinct images.

(empty definition)

The Mathematical Writing course: syllabus

The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.

The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.
- Formal mathematical sentences: predicate calculus, quantifiers.

The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.

- Formal mathematical sentences: predicate calculus, quantifiers.

- Describing real functions: the language of analysis.

The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.
- Formal mathematical sentences: predicate calculus, quantifiers.
- Describing real functions: the language of analysis.
- Writing effectively: choosing notation; some techniques; writing a short summary (150 words).

The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.
- Formal mathematical sentences: predicate calculus, quantifiers.
- Describing real functions: the language of analysis.
- Writing effectively: choosing notation; some techniques; writing a short summary (150 words).
- Forms of arguments: methods of proof; spotting and correcting bad arguments.

The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.
- Formal mathematical sentences: predicate calculus, quantifiers.
- Describing real functions: the language of analysis.
- Writing effectively: choosing notation; some techniques; writing a short summary (150 words).
- Forms of arguments: methods of proof; spotting and correcting bad arguments.
- Existence and definitions: existence proofs, unique existence.

Most mathematics students require <u>explicit</u> instructions on how to read and analyse mathematical expressions.

Most mathematics students require <u>explicit</u> instructions on how to read and analyse mathematical expressions.

Exercise 6. Some of these expressions are grammatically or logically incorrect. Identify them and explain what is the fault. (In what follows, $f : \mathbb{R} \to \mathbb{R}$ is a real function and $A, B, C \subset \mathbb{R}$.)

$\{1+1\}$	$\{3\} \setminus \{\{3\}\}$	$1+1 \Rightarrow 2$
$\{1,2\} \Leftrightarrow \{2,1\}$	$\sqrt{2} \Rightarrow \notin \mathbb{Q}$	$\mathbb{Z} \setminus (\mathbb{Z} \setminus \mathbb{N})$
$\mathbb{Z} \Rightarrow \mathbb{Q}$	$(\lvert x\rvert \in \mathbb{Z}) \Rightarrow (\lvert x\rvert \in \mathbb{Q})$	$(x \in A) \cup (x \in B)$
$(3 < 1) \Rightarrow \emptyset$	$A \leqslant (A \setminus B)$	$f(A) \in \{f(A)\}$
$(A \subset B) \cap C$	$A \subset (B \cap C)$	$A \subset B \subset A$
$(2,4,6,\ldots) \subset (1,2,3,\ldots)$	$\{A, \mathbb{Z}\}$	$\{\emptyset\} \cap \emptyset$
$f(1) \in \{2,3\}$	$f(\{1,2\}) \in \mathbb{N}$	$f(\mathbb{Q}) \subset \mathbb{Q}$
$\{x \in \mathbb{N} : -x\}$	$\{-x : x \in \mathbb{N}\}$	$\{x : x \Leftrightarrow 2\}$
$\{x \in \mathbb{Z} : x \notin \mathbb{Z}\}$	$\{\{x : \lvert x\rvert < 2\}\}$	$\{x \in \mathbb{Q} : 1 = 0\}$
$\{x \in \mathbb{Q} : x^2 \notin \mathbb{Z}\}$	$\{\{f(x)\} : x \in \mathbb{Q}\}$	$\{x : f(x) \in \mathbb{Q}\}$

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.
- An infinite sequence.

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.
- An infinite sequence.
- An infinite sequence of polynomials.

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.
- An infinite sequence.
- An infinite sequence of polynomials.
- An infinite sequence of polynomials in one indeterminate.

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

- A sequence.
- An infinite sequence.
- An infinite sequence of polynomials.
- An infinite sequence of polynomials in one indeterminate.

put symbols in a context:

with integer coefficients.

with increasing degree.

with bounded coefficients.

$\ldots$

Structure of expressions

Structure of expressions

$(\cdots)^2$	a square
$\sum \cdots$	a sum

Structure of expressions

$(\cdots)^2$	*a square*
$\sum\cdots$	*a sum*

$(\sum\cdots)^2$	*the square of a sum*
$\sum(\cdots)^2$	*a sum of squares*

Structure of expressions

$(\cdots)^2$	*a square*
$\sum \cdots$	*a sum*
$(\sum \cdots)^2$	*the square of a sum*
$\sum (\cdots)^2$	*a sum of squares*
$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$	*the sum of the square of the reciprocal of the natural numbers*

Structure of expressions

$(\cdots)^2$	a square
$\sum \cdots$	a sum
$(\sum \cdots)^2$	the square of a sum
$\sum (\cdots)^2$	a sum of squares
$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$	the sum of the square of the reciprocal of the natural numbers
$\displaystyle\left(\sum_{n=1}^{\infty} a_n\right)^2 \quad a_n \in \mathbb{Q}$	the square of the sum of the elements of a rational sequence,

Exercise 5. For each expression, provide two levels of description: [$\notin$]

> *i*) a coarse description, which only identifies the object's type (set, function, equation, statement, etc.);
> *ii*) a finer description, which defines the object in question or characterises its structure.

1. $x^3 - x - 2$
2. $x^3 - x - 2 = 0$
3. $3^3 + 4^3 + 5^3 = 6^3$
4. $x - y > 0$
5. $x = x + 1$
6. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
7. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
8. $2\mathbb{Z} \supset 4\mathbb{Z}$
9. $(\mathbb{Q} \setminus \mathbb{Z})^2$
10. $(a_1, a_3, a_5, \ldots)$
11. $((x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots)$
12. $\sin \circ \cos$

Exercise 5. For each expression, provide two levels of description: [$\notin$]

i) a coarse description, which only identifies the object's type (set, function, equation, statement, etc.);
ii) a finer description, which defines the object in question or characterises its structure.

1. $x^3 - x - 2$ polynomial
2. $x^3 - x - 2 = 0$ equation
3. $3^3 + 4^3 + 5^3 = 6^3$ identity
4. $x - y > 0$ inequality
5. $x = x + 1$
6. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
7. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
8. $2\mathbb{Z} \supset 4\mathbb{Z}$ sentence
9. $(\mathbb{Q} \setminus \mathbb{Z})^2$ set
10. $(a_1, a_3, a_5, \ldots)$ sequence
11. $((x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots)$
12. $\sin \circ \cos$ function

From symbols to words: synthesis

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$. Rewrite each symbolic sentence without symbols, apart from f.

1.	$f(0) \in \mathbb{Q}$	2.	$f(\mathbb{R}) = \mathbb{R}$
3.	$\# f(\mathbb{R}) = 1$	4.	$f(\mathbb{Z}) = \{0\}$
5.	$0 \in f(\mathbb{Z})$	6.	$f^{-1}(\{0\}) = \mathbb{Z}$
7.	$f(\mathbb{R}) \subset \mathbb{Q}$	8.	$f(\mathbb{R}) \supset \mathbb{Z}$
9.	$f(\mathbb{Z}) = f(\mathbb{N})$	10.	$f(\mathbb{Q}) \cap \mathbb{Q} = \emptyset$
11.	$f^{-1}(\mathbb{Q}) = \emptyset$	12.	$\# f^{-1}(\mathbb{Z}) < \infty$.

From symbols to words: synthesis

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$. Rewrite each symbolic sentence without symbols, apart from f.

1.	$f(0) \in \mathbb{Q}$	2.	$f(\mathbb{R}) = \mathbb{R}$
3.	$\# f(\mathbb{R}) = 1$	4.	$\boxed{f(\mathbb{Z}) = \{0\}}$
5.	$0 \in f(\mathbb{Z})$	6.	$f^{-1}(\{0\}) = \mathbb{Z}$
7.	$f(\mathbb{R}) \subset \mathbb{Q}$	8.	$f(\mathbb{R}) \supset \mathbb{Z}$
9.	$f(\mathbb{Z}) = f(\mathbb{N})$	10.	$f(\mathbb{Q}) \cap \mathbb{Q} = \emptyset$
11.	$f^{-1}(\mathbb{Q}) = \emptyset$	12.	$\# f^{-1}(\mathbb{Z}) < \infty$.

From symbols to words: synthesis

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$. Rewrite each symbolic sentence without symbols, apart from f.

1.	$f(0) \in \mathbb{Q}$	2.	$f(\mathbb{R}) = \mathbb{R}$
3.	$\# f(\mathbb{R}) = 1$	4.	$\boxed{f(\mathbb{Z}) = \{0\}}$
5.	$0 \in f(\mathbb{Z})$	6.	$f^{-1}(\{0\}) = \mathbb{Z}$
7.	$f(\mathbb{R}) \subset \mathbb{Q}$	8.	$f(\mathbb{R}) \supset \mathbb{Z}$
9.	$f(\mathbb{Z}) = f(\mathbb{N})$	10.	$f(\mathbb{Q}) \cap \mathbb{Q} = \emptyset$
11.	$f^{-1}(\mathbb{Q}) = \emptyset$	12.	$\# f^{-1}(\mathbb{Z}) < \infty.$

The image of the set of integers under the function f is the set consisting of the integer 0.

[Robotic, no understanding]

From symbols to words: synthesis

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$. Rewrite each symbolic sentence without symbols, apart from f.

1.	$f(0) \in \mathbb{Q}$	2.	$f(\mathbb{R}) = \mathbb{R}$
3.	$\#f(\mathbb{R}) = 1$	4.	$\boxed{f(\mathbb{Z}) = \{0\}}$
5.	$0 \in f(\mathbb{Z})$	6.	$f^{-1}(\{0\}) = \mathbb{Z}$
7.	$f(\mathbb{R}) \subset \mathbb{Q}$	8.	$f(\mathbb{R}) \supset \mathbb{Z}$
9.	$f(\mathbb{Z}) = f(\mathbb{N})$	10.	$f(\mathbb{Q}) \cap \mathbb{Q} = \emptyset$
11.	$f^{-1}(\mathbb{Q}) = \emptyset$	12.	$\#f^{-1}(\mathbb{Z}) < \infty.$

The image of the set of integers under the function f is the set consisting of the integer 0. [Robotic, no understanding]

The function f vanishes at all integers. [Good]

From words to symbols

Exercise 4. The following expressions define sets. Turn words into symbols.

1. The set of negative odd integers.
2. The set of natural numbers with three decimal digits.
3. The set of rational numbers which are the ratio of odd integers.
4. The set of rational numbers between 3 and π.
5. The set of real numbers at distance 1/4 from an integer.
6. The complement of the unit circle in the Cartesian plane.
7. The set of lines tangent to the unit circle.

From words to symbols

Exercise 4. The following expressions define sets. Turn words into symbols.

1. The set of negative odd integers.
2. The set of natural numbers with three decimal digits.
3. The set of rational numbers which are the ratio of odd integers.
4. The set of rational numbers between 3 and π.
5. The set of real numbers at distance 1/4 from an integer.
6. The complement of the unit circle in the Cartesian plane.
7. The set of lines tangent to the unit circle.

$$\{n \in \mathbb{N} : 10^2 \leqslant n < 10^3\}$$

From words to symbols

Exercise 4. The following expressions define sets. Turn words into symbols.

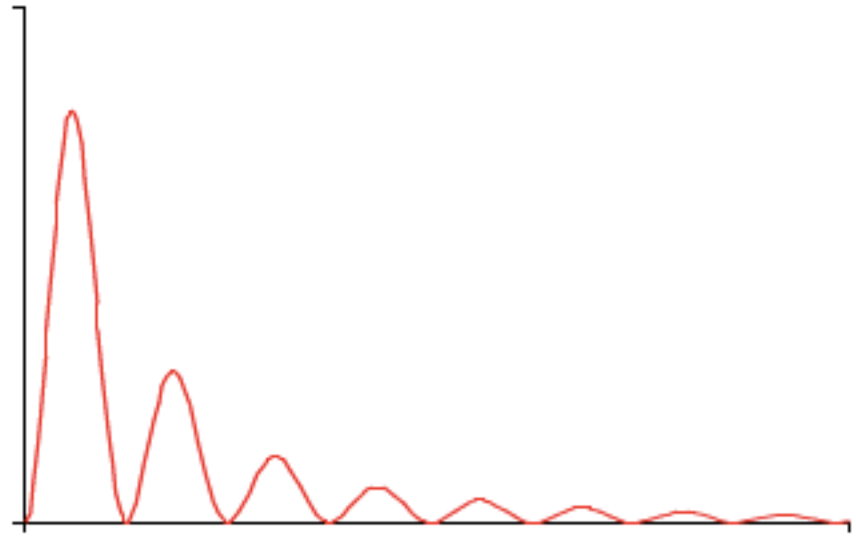
1. The set of negative odd integers.
2. The set of natural numbers with three decimal digits.
3. The set of rational numbers which are the ratio of odd integers.
4. The set of rational numbers between 3 and π.
5. The set of real numbers at distance 1/4 from an integer.
6. The complement of the unit circle in the Cartesian plane.
7. The set of lines tangent to the unit circle.

$$\{n \in \mathbb{N} : 10^2 \leqslant n < 10^3\}$$

$$\{ax + by = 1 : a^2 + b^2 = 1\}$$

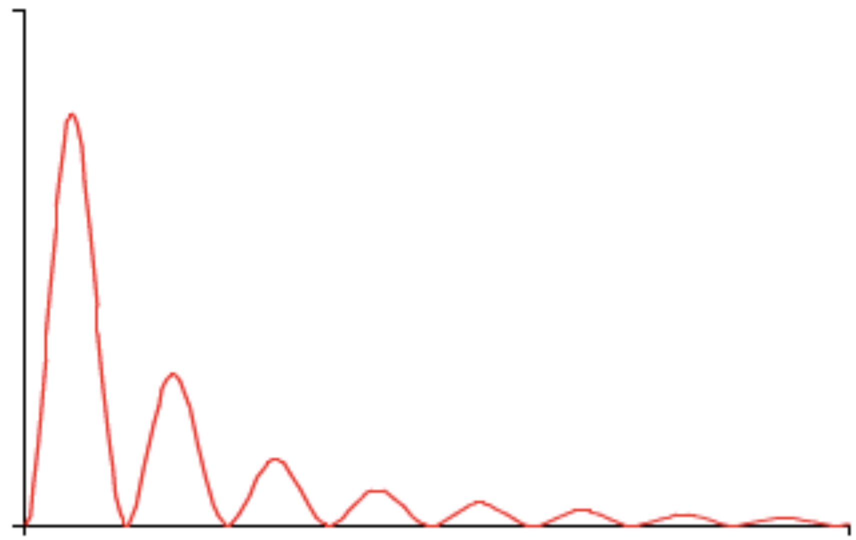
Describing functions

EXAMPLE. Describe the following function: [$\not{c}$]



Describing functions

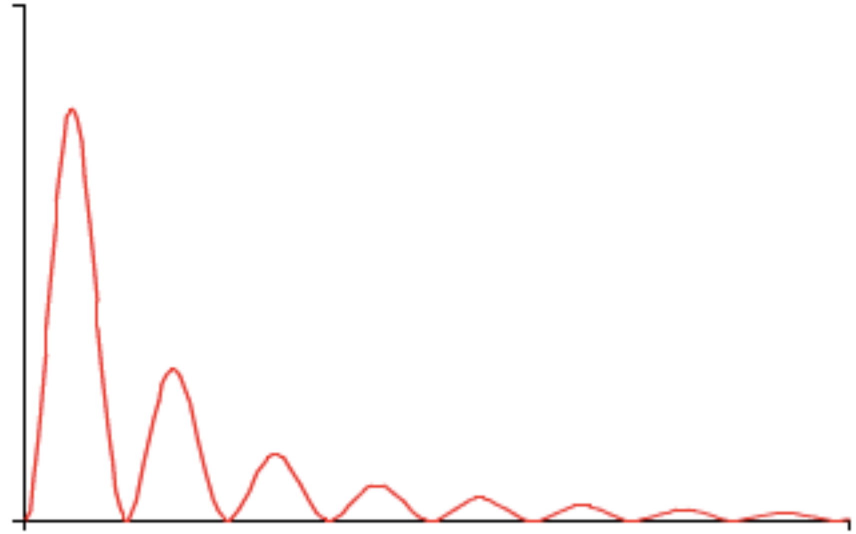
EXAMPLE. Describe the following function: [$\not e$]



This is a smooth function, which is bounded and non-negative. It features an infinite sequence of evenly spaced local maxima, whose height decreases monotonically to zero. The function has a zero between any two consecutive maxima.

Describing functions

EXAMPLE. Describe the following function: [$\not{e}$]



This is a smooth function, which is bounded and non-negative. It features an infinite sequence of evenly spaced local maxima, whose height decreases monotonically to zero. The function has a zero between any two consecutive maxima.

The bigger picture

The bigger picture

- Writing is essential for learning: students should write more.

The bigger picture

- Writing is essential for learning: students should write more.
- The development of conceptual accuracy requires small-scale writing exercises (words, symbols, phrases, short sentences).

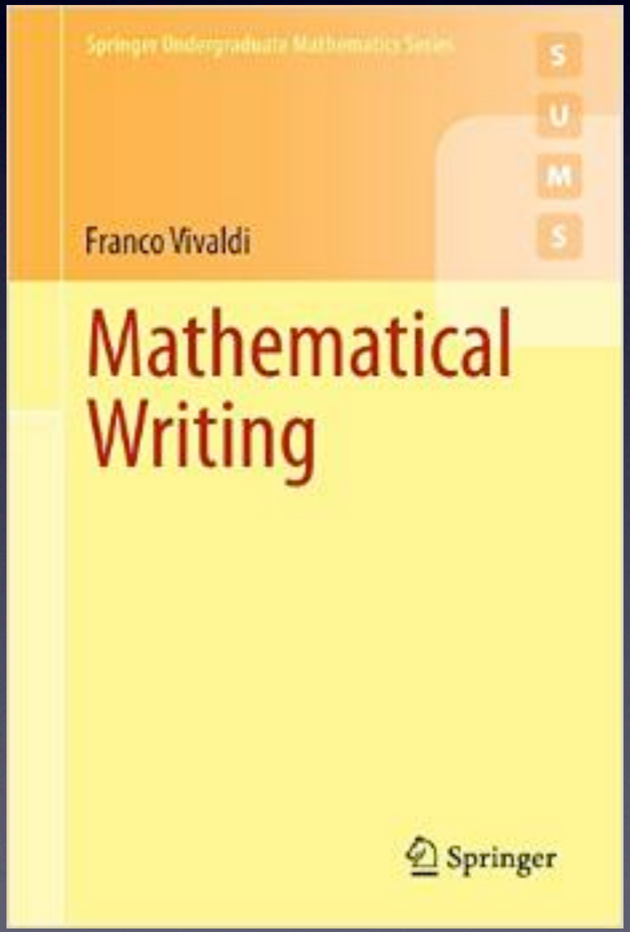
The bigger picture

- Writing is essential for learning: students should write more.
- The development of conceptual accuracy requires small-scale writing exercises (words, symbols, phrases, short sentences).
- One specialised course is insufficient: elements of writing should be embedded in most courses (as in the *Writing in the Disciplines* programme at American universities).

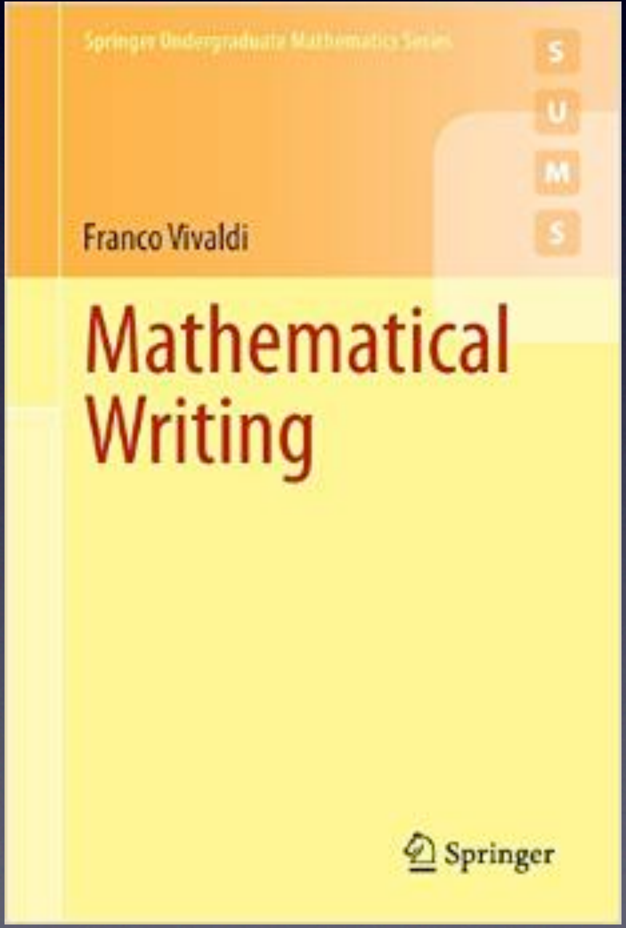
The bigger picture

- Writing is essential for learning: students should write more.
- The development of conceptual accuracy requires small-scale writing exercises (words, symbols, phrases, short sentences).
- One specialised course is insufficient: elements of writing should be embedded in most courses (as in the *Writing in the Disciplines* programme at American universities).
- Universities should develop centrally run schemes to raise the profile of writing and to support departments.

Thank you for your attention



Only £22.49


Springer Undergraduate Mathematics Series
SUMS
Franco Vivaldi
Mathematical Writing
Springer