Category Theory

Randall R. Holmes

October 8, 2019

0 Introduction

There are numerous examples of mathematical constructs (objects) and structure-preserving maps (morphisms) between them. For instance,

objects	morphisms
sets	functions
groups	group homomorphisms
rings	ring homomorphisms
modules	module homomorphisms
vector spaces	linear transformations
topological spaces	continuous maps
smooth manifolds	smooth maps
partially ordered sets	order-preserving functions

A "category" is an abstraction based on this idea of objects and morphisms. When one studies groups, rings, topological spaces, and so forth, one usually focuses on elements of these objects. Category theory shifts the focus away from the elements of the objects and toward the morphisms between the objects. In fact, the axioms of a category do not require that the objects actually be sets, so that in general it does not even make sense to speak of the elements of an object.

One strategy in category theory is to take a standard definition expressed in terms of elements and reformulate that definition using only morphisms so that it will make sense in any category. For instance, the trivial group $\{e\}$ is by definition the group with the single element e, but it can also be characterized (up to isomorphism) by the property that for each group there is a unique homomorphism from that group to the trivial group. This generalizes to the notion of "terminal object" in an arbitrary category. We will see many examples where a standard construction can be characterized as a terminal object (or as an initial object, which is the dual notion) in a suitable category.

The structure-preserving maps between categories are called "functors." They provide a means for studying the ways various categories relate to each other. In algebraic topology, for instance, one studies the functor from the category of (pointed) topological spaces to the category of groups that sends a topological space to its fundamental group and sends a continuous map between topological spaces to a homomorphism between the corresponding fundamental groups. This correspondence reveals a good deal about the structure of topological spaces. The information provided in this case illustrates the usefulness of functors in general.

1 Definitions and Examples

1.1 Class

In our discussions, we would like to consider such things as the collection C of all sets. Unfortunately, it causes problems to call this collection a set. If it were a set, then we could form the subset s of C consisting of all those sets that are not elements of themselves: $s = \{c \in C \mid c \notin c\}$. But then we would end up with the paradox that $s \in s \Leftrightarrow s \notin s$ (Russell's paradox). To avoid this, we call C something else, namely, a *class*.

For our purposes it will be enough to think of a class as a generalization of a set. So every set is a class, but there are classes, such as C, that are not sets. Such classes, called *proper classes*, can be thought of as large sets (and then regular sets as small sets). It is taken as a postulate that a proper class cannot be an element of any class, so the above paradox is avoided.

1.2 Definition of category

A **category** consists of the following:

- a class C, the elements of which are called **objects**,
- for each pair $x, y \in C$, a set $C(x, y)$, the elements of which are called **morphisms** from x to y (these sets are assumed to be pairwise disjoint),
- for each triple $x, y, z \in C$, a map $C(x, y) \times C(y, z) \to C(x, z)$, called **composition** and denoted $(\alpha, \beta) \mapsto \beta\alpha$, such that
  - (i) (Associativity) $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ for all morphisms α, β, and γ for which the indicated compositions are defined,
  - (ii) (Identity) for each $x \in C$ there exists $1_x \in C(x, x)$ such that $1_x\alpha = \alpha$ and $\beta 1_x = \beta$ for all morphisms α and β for which the indicated compositions are defined.

Let C be a category (we refer to a category by using the name of its object class).

For $x, y \in C$, we often express the fact that α is an element of $C(x, y)$ by writing $\alpha : x \to y$ or $x \xrightarrow{\alpha} y$, and we call x and y the **source** and **target**, respectively, of α. We say "α is a morphism in C" to mean that $\alpha \in C(x, y)$ for some $x, y \in C$.

Let $x \in C$. The morphism 1_x of (ii) is unique (for, if $1'_x$ also satisfies the condition, then $1'_x = 1'_x 1_x = 1_x$). This morphism is called the **identity morphism** on x.

1.3 Examples

1.3.1 *Example* Many familiar categories have as object class a collection of sets with a specified structure, and as morphisms the functions between the sets that preserve the structure, with morphism composition being usual composition of functions. Such a category is called a "concrete category." (This informal notion will be made rigorous later on.)

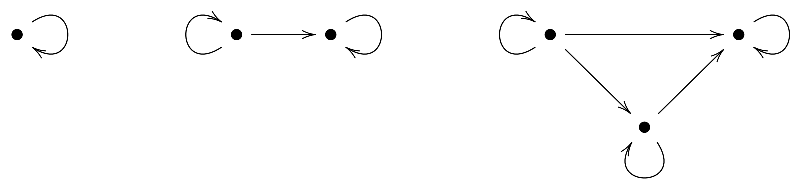
The following table lists a few common concrete categories. In general, function composition is associative, so part (i) of the definition is satisfied. In each case the role of the identity morphism is played by the identity function, so part (ii) of the definition is satisfied. The only thing that needs to be checked in each case is that a composition of morphisms is also a morphism.

Name	Objects	Morphisms
Set	sets	functions
Grp	groups	homomorphisms
Ab	abelian groups	homomorphisms
Ring	rings with identity	homomorphisms sending identity to identity
Rng	rings (no assumed identity)	homomorphisms
$\mathbf{Vect}_F$	vector spaces over the field F	linear transformations
${}_R\mathbf{Mod}$	(left) modules over the ring R (assumed to be unitary if R has an identity)	R-homomorphisms
Top	topological spaces	continuous maps

PTop	pointed topological spaces	continuous maps sending distinguished point to distinguished point
Met	metric spaces	continuous maps
Man	smooth manifolds	smooth maps

$\square$

1.3.2 *Example* One can use graphs to represent categories, with vertices representing objects and arrows representing morphisms. The graphs below represent the categories denoted **1**, **2**, and **3**, respectively. In each case, there is only one way to define composition of morphisms.



$\square$

1.3.3 *Example* Let P be a set and let $\leq$ be a preorder on P (so $\leq$ is reflexive and transitive). We can regard P as a category with object class P and with $P(x, y)$ $(x, y \in P)$ defined to be the singleton set $\{x \to y\}$ if $x \leq y$ and the empty set otherwise. For morphisms $\alpha : x \to y$ and $\beta : y \to z$, define $\beta\alpha$ to be the unique morphism $\beta\alpha : x \to z$ (well-defined by transitivity of the preorder). For $x \in P$, the unique morphism $1_x : x \to x$ (which exists since the preorder is reflexive) is an identity morphism on x. $\square$

1.3.4 *Example* Let n be a nonnegative integer. The set $\mathbf{n} = \{0, 1, 2, \ldots, n-1\}$ with the usual order can be regarded as a category as in Example 1.3.3. The categories **1**, **2**, and **3** are illustrated in Example 1.3.2. $\square$

1.3.5 *Example* Let R be a ring with identity. The category $\mathbf{Mat}_R$ has as object class the set of nonnegative integers. For objects n and m the set $\mathbf{Mat}_R(n, m)$ consists of all $m \times n$ matrices over R. Composition of morphisms is given by matrix multiplication. For an object n, the $n \times n$ identity matrix is an identity morphism on n. (Convention: If either integer m or n is zero, there is a single $m \times n$ matrix called a "zero matrix." This extends the definition of a zero matrix (all entries zero). Using the natural extension of matrix multiplication to this case one finds that a product of two matrices is zero if either factor is zero. The 0×0 identity matrix is, by definition, the 0×0 zero matrix.) $\square$

1.3.6 *Example* Let M be a monoid (a set with an associative binary operation having an identity element). The category C_M has a single object $\bullet$ and $C_M(\bullet, \bullet)$ is defined to be M. Composition of morphisms is given by the multiplication in M. The identity morphism $1_\bullet$ is the identity element of M. $\square$

1.3.7 *Example* Let C be a category and let c be a fixed object of C. The category $(c \downarrow C)$ has as objects all morphisms $c \to x$ with $x \in C$. For objects $\alpha : c \to x$ and $\beta : c \to y$ the set $(c \downarrow C)(\alpha, \beta)$ consists of all morphisms $\gamma : x \to y$ in C such that the following diagram is commutative:

$$\begin{tikzcd} & c \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ x \arrow[rr, "\gamma"] & & y \end{tikzcd} \quad .$$

(To say that a diagram such as this is commutative is to say that the compositions of morphisms along every path from one object to another yield the same result. In this case this simply means that $\beta = \gamma\alpha$.) Composition of morphisms in this category is given by the composition in the category C. $\square$

1.4 Subcategory

Let B and C be a categories. The category B is a **subcategory** of C if

- $B \subseteq C$, that is, every object of B is an object of C,
- for each $x, y \in B$ we have $B(x, y) \subseteq C(x, y)$,
- composition in B is the same as composition in C,
- for each $x \in B$ the identity morphism 1_x in B equals the identity morphism 1_x in C.

A subcategory B of a category C is **full** if for each $x, y \in B$ we have $B(x, y) = C(x, y)$. Any subcollection of objects of a category C are the objects of a uniquely defined full subcategory of C.

1.4.1 *Example* **Ab** is a full subcategory of **Grp**. $\square$

1.4.2 *Example* **Ring** is a subcategory of **Rng**. However, this subcategory is not full. Indeed, the zero map $\mathbf{Z} \to \mathbf{Z}$ is a morphism in **Rng** but not in **Ring** since it does not send 1 to 1. $\square$

1.4.3 *Example* In the category $\mathbf{Vect}_F$ (F a field), we can form the full subcategory $\mathbf{FVect}_F$ of all finite-dimensional vector spaces (with all possible linear maps between them as morphisms). $\square$

1 – Exercises

1–1 Let X and Y be sets. A **relation** from X to Y is a subset R of $X \times Y$. If R is a relation from X to Y, then for $x \in X$ and $y \in Y$ we write xRy to mean $(x, y) \in R$.

Let C be the class of all sets. For sets X and Y, let $C(X, Y)$ be the set of all relations from X to Y. For sets X, Y, Z, and relations $R \in C(X, Y)$, $S \in C(Y, Z)$, define a relation $SR \in C(X, Z)$ by putting $x(SR)z$ if and only if xRy and ySz for some $y \in Y$. Prove that C with morphisms and composition so defined is a category.

2 Generalizations of Injective and Surjective

Let $f : X \to Y$ be a function.

- The function f is **injective** if $f(x) = f(x') \;\Rightarrow\; x = x'$ $(x, x' \in X)$.
- The function f is **surjective** if for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
- The function f is a **bijection** if it is both injective and surjective.

These definitions are given in terms of elements, so they cannot be applied to a morphism in an arbitrary category. We begin by introducing the notions monic morphism and split monic morphism. The definitions involve only objects and morphisms and hence make sense in any category. We will see that they are equivalent to injective function in the category **Set**, or rather just nearly so in the case of split monic morphism (see Theorem 2.2.5).

Similarly, the notions epic morphism and split epic morphism defined below make sense in any category. They are equivalent to surjective function in the category **Set**.

Finally, we study the notions bimorphism (monic and epic) and isomorphism (split monic and split epic) and see that they coincide with bijection in the category **Set**.

2.1 Monic

Let C be a category. A morphism $\alpha : x \to y$ in C is **monic** if, in the situation

$$z \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} x \xrightarrow{\alpha} y \ ,$$

$\alpha\beta_1 = \alpha\beta_2$ implies $\beta_1 = \beta_2$. In other words, a morphism is monic if it can always be canceled on the left.

A morphism in a concrete category is a function between underlying sets, so the notion of injective morphism makes sense in any concrete category.

2.1.1 Theorem. *In a concrete category, every injective morphism is monic.*

Proof. Let C be a concrete category and let $\alpha : X \to Y$ be an injective morphism in C. Let $\beta_1, \beta_2 : Z \to X$ be morphisms in C and assume that $\alpha\beta_1 = \alpha\beta_2$. For each $z \in Z$ we have $\alpha(\beta_1(z)) = \alpha\beta_1(z) = \alpha\beta_2(z) = \alpha(\beta_2(z))$ so that $\beta_1(z) = \beta_2(z)$ since α is injective. Therefore $\beta_1 = \beta_2$ and we conclude that α is monic. □

The converse of the preceding theorem does not hold as the following example shows.

2.1.2 *Example* (Monic but not injective in **Div**) An (additive) abelian group A is **divisible** if for each $a \in A$ and nonzero $n \in \mathbf{Z}$ there exists $b \in A$ such that $a = nb$. Let $C = \mathbf{Div}$ be the full subcategory of **Grp** having the divisible groups as objects. The groups $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are objects of C and the canonical epimorphism $\alpha : \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ is not injective, but we claim that α is monic.

Let $\beta_1, \beta_2 : A \to \mathbf{Q}$ be morphisms in C and assume that $\beta_1 \neq \beta_2$. Then $\beta_1(a) - \beta_2(a) = r/s$ for some $a \in A$ and $0 \neq r, s \in \mathbf{Z}$. Since A is divisible, there exists $b \in A$ such that $a = nb$, where $n = 2r$. Then

$$n[\beta_1(b) - \beta_2(b)] = \beta_1(nb) - \beta_2(nb) = \frac{r}{s},$$

so $\beta_1(b) - \beta_2(b) = 1/(2s) \notin \mathbf{Z}$. Therefore, $\alpha\beta_1 \neq \alpha\beta_2$ and the claim follows. □

However, the following theorem gives a few common concrete categories in which the notions injective and monic coincide.

2.1.3 Theorem. *In the categories* **Set**, **Top**, **Grp**, *and* **Rng** *a morphism is injective if and only if it is monic.*

Proof. Theorem 2.1.1 gives one direction in each case.

(**Set**) Let $\alpha : X \to Y$ be a monic morphism in **Set**. Let $x_1, x_2 \in X$ and assume that $\alpha(x_1) = \alpha(x_2)$. Set $z = 0 \in \mathbf{Z}$ and put $Z = \{z\}$ (or, in fact, Z could be any singleton set) and define $\beta_1, \beta_2 : Z \to X$ by $\beta_i(z) = x_i$. Then

$$\alpha\beta_1(z) = \alpha(\beta_1(z)) = \alpha(x_1) = \alpha(x_2) = \alpha(\beta_2(z)) = \alpha\beta_2(z),$$

implying $\alpha\beta_1 = \alpha\beta_2$. Since α is monic, we get $\beta_1 = \beta_2$ so that $x_1 = \beta_1(z) = \beta_2(z) = x_2$. Therefore, α is injective.

(**Top**) The proof just given works in this case as well. One just needs to note that $Z = \{z\}$ can be given the discrete topology (which is in fact the only possible topology) and then the functions β_i are continuous and hence morphisms.

(**Grp**) Let $\alpha : G \to H$ be a monic morphism in **Grp**. Let K be the kernel of α and let $\beta_1 : K \to G$ and $\beta_2 : K \to G$ be the inclusion map and the trivial map, respectively. Then $\alpha\beta_1 = \alpha\beta_2$ since each side equals the trivial map $K \to H$. Since α is monic, it follows that $\beta_1 = \beta_2$, which is to say that the inclusion map $K \to G$ is trivial. This forces K to be trivial, which says that α is injective.

(**Rng**) The proof just given works in this case as well. □

2.2 Split monic

Let C be a category. A morphism $\alpha : x \to y$ in C is **split monic** if there exists a morphism $\beta : y \to x$ such that $\beta\alpha = 1_x$. This can be depicted as follows:

$$\begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, shift right, "\alpha"'] & y \arrow[l, shift right, dotted, "\exists\,\beta"'] \end{tikzcd} .$$

The solid arrows are given and the definition asserts the existence of a dotted arrow as indicated such that the resulting diagram is commutative. (A split monic morphism is sometimes called a "section.")

Put more succinctly, to say that a morphism is split monic is to say that it has a left inverse.

The following theorem justifies the terminology.

2.2.1 Theorem. *Every split monic morphism is monic.*

Proof. Let $\alpha : x \to y$ be a split monic morphism in C, so that there exists a morphism $\beta : y \to x$ in C such that $\beta\alpha = 1_x$. Let $\beta_1, \beta_2 : z \to x$ be morphisms in C and assume that $\alpha\beta_1 = \alpha\beta_2$. Then

$$\beta_1 = 1_x\beta_1 = \beta\alpha\beta_1 = \beta\alpha\beta_2 = 1_x\beta_2 = \beta_2,$$

implying that α is monic. □

2.2.2 Theorem. *In a concrete category every split monic morphism is injective.*

Proof. Let C be a concrete category and let $\alpha : X \to Y$ be a split monic morphism in C, so that there exists a morphism $\beta : Y \to X$ such that $\beta\alpha = 1_X$. Let $x_1, x_2 \in X$ and assume that $\alpha(x_1) = \alpha(x_2)$. Then

$$x_1 = 1_X(x_1) = \beta\alpha(x_1) = \beta(\alpha(x_1)) = \beta(\alpha(x_2)) = \beta\alpha(x_2) = 1_X(x_2) = x_2,$$

implying that α is injective. $\square$

The converse of the preceding theorem does not hold.

2.2.3 *Example* (Injective but not split monic in **Grp** and **Rng**) The inclusion map $\alpha : 2\mathbf{Z} \to \mathbf{Z}$ is an injective morphism in the category **Grp**. We claim that α is not split monic. Suppose to the contrary that there exists a morphism $\beta : \mathbf{Z} \to 2\mathbf{Z}$ such that $\beta\alpha = 1_{2\mathbf{Z}}$. Then $2\beta(1) = \beta(2) = \beta(\alpha(2)) = \beta\alpha(2) = 2$, so that $\beta(1) = 1$ contradicting that β maps into $2\mathbf{Z}$. Therefore, α is injective but not split monic.

The same argument shows that α is injective but not split monic in the category **Rng**. $\square$

2.2.4 *Example* (Injective but not split monic in **Top**) Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the identity map, where the domain has the discrete topology and the codomain has the usual topology. Then α is an injective morphism in the category **Top**. Suppose that there exists a morphism $\beta : \mathbf{R} \to \mathbf{R}$ such that $\beta\alpha = 1_{\mathbf{R}}$. Then $\beta = \beta 1_{\mathbf{R}} = \beta\alpha = 1_{\mathbf{R}}$. However, the set $\{0\}$ is open in $\mathbf{R}$ with the discrete topology, but its inverse image under β, which is also $\{0\}$, is not open in $\mathbf{R}$ with the usual topology. This contradicts that β is continuous. We conclude that α is injective but not split monic. $\square$

The following theorem shows that the notions split monic morphism and injective morphism nearly coincide in the category **Set**. It says that the only maps for which the two notions do not coincide are the rather trivial empty maps $\emptyset \to Y$ with Y nonempty, which are injective but not split monic.

2.2.5 Theorem. *Let $\alpha : X \to Y$ be a morphism in the category* **Set**. *The following are equivalent:*

(i) *α is split monic,*

(ii) *α is injective and $X = \emptyset$ implies $Y = \emptyset$.*

Proof. (i $\Rightarrow$ ii) Assume that (i) holds. Then α is injective by Theorem 2.2.2. By assumption, there exists $\beta : Y \to X$ such that $\beta\alpha = 1_X$. If $X = \emptyset$, then $Y = \emptyset$ as well, since β is a function.

(ii $\Rightarrow$ i) Assume that (ii) holds. If $X = \emptyset$, then $Y = \emptyset$ as well and the empty function $\beta : Y \to X$ satisfies $\beta\alpha = 1_X$. Assume that $X \neq \emptyset$, so that there exists some $x_0 \in X$. Since α is injective, for each $y \in \operatorname{im} \alpha$ there exists a unique element $\beta(y) \in X$ such that $\alpha(\beta(y)) = y$. This defines a function $\beta : \operatorname{im} \alpha \to X$, which we extend to a function $\beta : Y \to X$ by putting $\beta(y) = x_0$ for $y \notin \operatorname{im} \alpha$. For $x \in X$, we have $\beta\alpha(x) = \beta(\alpha(x)) = x = 1_X(x)$, so $\beta\alpha = 1_X$. Therefore, α is split monic. $\square$

In summary, according to Theorems 2.2.2 and 2.1.1 we have the following relationships in a *concrete* category:

$$\text{split monic} \ \Rightarrow \ \textit{injective} \ \Rightarrow \ \text{monic.}$$

Moreover, Examples 2.2.3 and 2.1.2 show that no two of these notions coincide in general. In an arbitrary category the middle term is no longer defined (which is what the italics are intended to indicate), but if it is deleted the resulting implication is still valid by Theorem 2.2.1.

In the full subcategory of **Set** consisting of all *nonempty* sets the three notions coincide by Theorems 2.1.3 and 2.2.5.

2.3 Epic

Let C be a category. A morphism $\alpha : x \to y$ in C is **epic** if, in the situation

$$\begin{tikzcd} x \arrow[r, "\alpha"] & y \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & z \end{tikzcd},$$

$\beta_1\alpha = \beta_2\alpha$ implies $\beta_1 = \beta_2$. In other words, a morphism is epic if it can always be canceled on the right.

This definition is "dual" to that of monic morphism, meaning that it is the same except with arrows reversed and the order of the morphisms in each composition reversed.

2.3.1 Theorem. *In a concrete category, every surjective morphism is epic.*

Proof. Let C be a concrete category and let $\alpha : X \to Y$ be a surjective morphism in C. Let $\beta_1, \beta_2 : Y \to Z$ be morphisms in C and assume that $\beta_1\alpha = \beta_2\alpha$. Let $y \in Y$. Since α is surjective, we have $y = \alpha(x)$ for some $x \in X$. Then $\beta_1(y) = \beta_1(\alpha(x)) = \beta_1\alpha(x) = \beta_2\alpha(x) = \beta_2(\alpha(x)) = \beta_2(y)$. Therefore $\beta_1 = \beta_2$ and we conclude that α is epic. □

The converse of the preceding theorem does not hold as the following example shows.

2.3.2 *Example* (Epic but not surjective in **Rng**) The inclusion map $\alpha : \mathbf{Z} \to \mathbf{Q}$ is a morphism in **Rng** that is not surjective. We claim that α is epic. Let $\beta_1, \beta_2 : \mathbf{Q} \to R$ be morphisms in **Rng** and assume that $\beta_1\alpha = \beta_2\alpha$. Then $\beta_1(n) = \beta_2(n)$ for every integer n. For $n \in \mathbf{Z}$ with $n \neq 0$ we have

$$\begin{aligned}\beta_1(n^{-1}) &= \beta_1(n^{-1} \cdot 1) = \beta_1(n^{-1})\beta_1(1) = \beta_1(n^{-1})\beta_2(1)\\ &= \beta_1(n^{-1})\beta_2(n)\beta_2(n^{-1}) = \beta_1(n^{-1})\beta_1(n)\beta_2(n^{-1}) = \beta_1(1)\beta_2(n^{-1})\\ &= \beta_2(1)\beta_2(n^{-1}) = \beta_2(1 \cdot n^{-1}) = \beta_2(n^{-1}),\end{aligned}$$

so for $m, n \in \mathbf{Z}$ with $n \neq 0$ we have

$$\beta_1(m/n) = \beta_1(m)\beta_1(n^{-1}) = \beta_2(m)\beta_2(n^{-1}) = \beta_2(m/n).$$

Hence, $\beta_1 = \beta_2$, implying that α is epic. So α is an epic morphism in the category **Rng**, but it is not surjective. □

However, the following theorem gives a few common concrete categories in which the notions surjective and epic coincide. (Note that by the preceding example we cannot include **Rng** here.)

2.3.3 Theorem. *In the categories* **Set**, **Top**, *and* **Grp** *a morphism is surjective if and only if it is epic.*

Proof. In each case, Theorem 2.3.1 gives one direction.

(**Set**) Let $\alpha : X \to Y$ be a function and assume that α is epic. Let $\beta_1 : Y \to \{0, 1\}$ be the characteristic (indicator) function of $\operatorname{im}\alpha$ (so $\beta_1(y)$ is 1 if $y \in \operatorname{im}\alpha$ and 0 otherwise) and let $\beta_2 : Y \to \{0, 1\}$ be constantly 1. Then $\beta_1\alpha = \beta_2\alpha$ (both sides are constantly 1), so that $\beta_1 = \beta_2$ since α is epic. It follows that $\operatorname{im}\alpha = Y$, that is, α is surjective.

(**Top**) We can proceed as in the proof for the case **Set**, assuming here that $\alpha : X \to Y$ is a continuous map that is epic, and giving $\{0, 1\}$ the indiscrete topology so that β_1 and β_2 are continuous.

(**Grp**) Let $\alpha : G \to H$ be a group homomorphism that is not surjective. It suffices to show that α is not epic. Put $K = \operatorname{im}\alpha$. Since α is not surjective, we have $|H : K| > 1$.

First assume that $|H : K| = 2$. Let $\beta_1 : H \to H/K$ be the canonical epimorphism and let $\beta_2 : H \to H/K$ be the trivial homomorphism. Then $\beta_1\alpha = \beta_2\alpha$ (since both sides are trivial), but $\beta_1 \neq \beta_2$ (since $K \neq H$). Therefore, α is not epic.

Now assume that $|H : K| > 2$. Then there exist two distinct right cosets $K_1 = Kh_1$ and $K_2 = Kh_2$ of K in H with $K_1, K_2 \neq K$. Put $b = h_1^{-1}h_2$ and note that $K_1b = K_2$ and $K_2b^{-1} = K_1$. Let S_H denote the symmetric group on H and let $\sigma \in S_H$ be given by

$$\sigma(h) = \begin{cases} hb, & h \in K_1, \\ hb^{-1}, & h \in K_2, \\ h, & \text{otherwise.} \end{cases}$$

Note that $\sigma^2 = 1_H$ and also that $\sigma(kh) = k\sigma(h)$ for all $k \in K$ and $h \in H$. For $h \in H$, let λ_h be the element of S_H given by $\lambda_h(x) = hx$ $(x \in H)$. Then the observation above gives $\sigma\lambda_k = \lambda_k\sigma$ for all $k \in K$. Define $\beta_1, \beta_2 : H \to S_H$ by $\beta_1(h) = \lambda_h$ and $\beta_2(h) = \sigma\lambda_h\sigma$ and note that both of these maps are group homomorphisms. For $k \in K$, we have $\beta_2(k) = \sigma\lambda_k\sigma = \lambda_k\sigma^2 = \lambda_k = \beta_1(k)$, so $\beta_1\alpha = \beta_2\alpha$. On the other hand, letting e denote the identity element of H, we have

$$\beta_2(h_1)(e) = \sigma\lambda_{h_1}\sigma(e) = \sigma(h_1) = h_2 \neq h_1 = \lambda_{h_1}(e) = \beta_1(h_1)(e),$$

so that $\beta_1 \neq \beta_2$. Therefore, α is not epic and the proof is complete. □

2.4 Split epic

Let C be a category. A morphism $\alpha : x \to y$ in C is **split epic** if there exists a morphism $\beta : y \to x$ such that $\alpha\beta = 1_y$. This can be depicted as follows:

$$\begin{tikzcd} x \arrow[r, shift right, "\alpha"'] & y \arrow[l, dotted, shift right, "\exists\,\beta"'] \arrow[loop right, "1_y"] \end{tikzcd} .$$

(A split epic morphism is sometimes called a "retraction.")

Put more succinctly, a morphism is split epic if it has a right inverse.

This definition is dual to that of split monic.

2.4.1 Theorem. *Every split epic morphism is epic.*

Proof. Let $\alpha : x \to y$ be a split epic morphism in C, so that there exists a morphism $\beta : y \to x$ in C such that $\alpha\beta = 1_y$. Let $\beta_1, \beta_2 : y \to z$ be morphisms in C and assume that $\beta_1\alpha = \beta_2\alpha$. Then

$$\beta_1 = \beta_1 1_y = \beta_1\alpha\beta = \beta_2\alpha\beta = \beta_2 1_y = \beta_2,$$

implying that α is epic. □

The statement of the preceding theorem is dual to that of Theorem 2.2.1. This implies that a proof can be obtained by simply dualizing the earlier proof, meaning, reversing arrows and reversing the order of the morphisms in every composition (even in definitions, so that, for instance, "monic" becomes "epic"). Proofs of dual statements are usually omitted. (The dual nature of the proof just given is somewhat obscured by the choice of notation. For instance, we wrote $\alpha : x \to y$ as in the definition of split epic instead of $\alpha : y \to x$, which is how the earlier proof begins with arrow reversed.)

2.4.2 Theorem. *In a concrete category every split epic morphism is surjective.*

Proof. Let C be a concrete category and let $\alpha : X \to Y$ be a split epic morphism in C, so that there exists a morphism $\beta : Y \to X$ such that $\alpha\beta = 1_Y$. Let $y \in Y$. Then $x := \beta(y) \in X$ and

$$\alpha(x) = \alpha(\beta(y)) = \alpha\beta(y) = 1_Y(y) = y,$$

implying that α is surjective. □

The converse of this theorem does not hold.

2.4.3 *Example* (Surjective but not split epic in **Grp** and **Rng**) The map $\alpha : \mathbf{Z}_4 \to \mathbf{Z}_2$ given by

$$\alpha : \begin{cases} 0, 2 \mapsto 0, \\ 1, 3 \mapsto 1 \end{cases}$$

is a surjective group homomorphism. Since $1 \in \mathbf{Z}_2$ has order 2, any homomorphism $\beta : \mathbf{Z}_2 \to \mathbf{Z}_4$ must send 1 to either 0 or 2 so that $\alpha\beta \neq 1_{\mathbf{Z}_2}$. Therefore α is a surjective morphism that is not split epic in the category **Grp**. (This α is also a surjective morphism that is not split epic in the category **Rng**, and for the same reason.) □

2.4.4 *Example* (Surjective but not split epic in **Top**) Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the identity map, where the domain has the discrete topology and the codomain has the usual topology. Then α is a surjective morphism in the category **Top**. An argument very similar to that given in Example 2.2.4 shows that α is not split epic. □

2.4.5 Theorem. *In the category* **Set** *a morphism is split epic if and only if it is surjective.*

Proof. Let $\alpha : X \to Y$ be a morphism in **Set**. If α is split epic, then it is surjective by Theorem 2.4.2. Assume that α is surjective. For each $y \in Y$ there exists $\beta(y) \in X$ such that $\alpha(\beta(y)) = y$. This defines a function $\beta : Y \to X$, which satisfies $\alpha\beta = 1_Y$. Therefore, α is split epic. □

In summary, according to Theorems 2.4.2 and 2.3.1 we have the following relationships in a *concrete* category:

$$\text{split epic} \;\Rightarrow\; \textit{surjective} \;\Rightarrow\; \text{epic}.$$

Moreover, Examples 2.4.3 and 2.3.2 show that no two of these notions coincide in general. In an arbitrary category the middle term is no longer defined (which is what the italics are intended to indicate), but if it is deleted the resulting implication is still valid by Theorem 2.4.1.

In the category **Set** all three notions coincide by Theorems 2.4.5 and 2.3.3.

2.5 Bimorphism and Isomorphism

Let C be a category and let $\alpha : x \to y$ be a morphism in C.

- α is a **bimorphism** if it is both monic and epic.
- α is an **isomorphism** if there exists $\beta : y \to x$ such that $\beta\alpha = 1_x$ and $\alpha\beta = 1_y$.

If α is an isomorphism, then the morphism β in the definition is unique and is denoted α^{-1}.

Two objects x and y of C are **isomorphic** if there exists an isomorphism from x to y. In this case, we write $x \cong y$.

The following characterization of isomorphism is useful.

2.5.1 Theorem. *A morphism is an isomorphism if and only if it is both split monic and split epic.*

Proof. It follows immediately from the definitions that an isomorphism is both split monic and split epic. Let $\alpha : x \to y$ be a morphism in C and assume that α is both split monic and split epic, so that there exist morphisms $\beta, \gamma : y \to x$ such that $\beta\alpha = 1_x$ and $\alpha\gamma = 1_y$. Then $\beta = \beta 1_y = \beta\alpha\gamma = 1_x\gamma = \gamma$, so α is an isomorphism. $\square$

Piecing together results from the previous sections, we get the following relationships:

$$
\begin{tikzcd}
\text{monic} & & \text{epic} \\
& \text{bimorphism} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \\
\textit{injective} \arrow[uu, Rightarrow] & & \textit{surjective} \arrow[uu, Rightarrow] \\
& \textit{bijection} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[uu, Rightarrow] & \\
\text{split monic} \arrow[uu, Rightarrow] & & \text{split epic} \arrow[uu, Rightarrow] \\
& \text{isomorphism} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[uu, Rightarrow] &
\end{tikzcd}
$$

(the italicized entries are to be included only if the category in question is *concrete*).

2.5.2 Theorem. *In the categories* **Set** *and* **Grp** *the notions bimorphism, bijection, and isomorphism coincide.*

Proof. Let C be either **Set** or **Grp** and let $\alpha : X \to Y$ be a bimorphism in C. In view of the preceding diagram, it suffices to show that α is an isomorphism. By Theorems 2.1.3 and 2.3.3 α is a bijection. The inverse α^{-1} of α is a morphism (this is not immediate in the case $C = \mathbf{Grp}$ but it is elementary to show) and since $\alpha^{-1}\alpha = 1_X$ and $\alpha\alpha^{-1} = 1_Y$ it follows that α is an isomorphism. □

The category **Rng** cannot be included in this theorem since the inclusion map $\mathbf{Z} \hookrightarrow \mathbf{Q}$ is a bimorphism (Theorem 2.1.1 and Example 2.3.2) but not a bijection.

Nor can the category **Top** be included since the identity map $\mathbf{R} \to \mathbf{R}$ with domain and codomain given the discrete and usual topologies, respectively, is a bijection but not an isomorphism (= homeomorphism) (Example 2.2.4 and Theorem 2.5.1).

2 – Exercises

2–1 Let **Haus** be the full subcategory of **Top** consisting of all Hausdorff spaces. Prove the following: If $\alpha : X \to Y$ is a morphism in **Haus** and $\operatorname{im} \alpha$ is dense in Y (i.e., $\overline{\operatorname{im} \alpha} = Y$), then α is epic. (The converse also happens to be true.) Use this result to give an example of a morphism in **Haus** that is epic but not surjective.

2–2 Let C be a category.

(a) Prove that a morphism in C is monic and split epic if and only if it is epic and split monic.

(b) Prove that if every monic morphism in C is split monic, then every bimorphism in C is an isomorphism. (Note: The dual statement with "epic" replacing "monic" also holds.)

3 Basic Constructions

3.1 Initial object and terminal object

Let C be a category and let $c \in C$.

The object c is **initial** if for each object x of C there exists a unique morphism from c to x.

The object c is **terminal** if for each object x of C there exists a unique morphism from x to c.

3.1.1 *Examples*

- In **Set** and **Top** the empty set $\emptyset$ is initial (since for each object X the empty function is the unique morphism from $\emptyset$ to X), and each one-point set is terminal.
- In **Grp**, **Rng**, and ${}_R\mathbf{Mod}$ (R a ring) there is a one-element (trivial) object, and any such is both initial and terminal.
- In **Ring** (rings with identity), the ring $\mathbf{Z}$ is initial (since for each object R the map $\alpha : \mathbf{Z} \to R$ given by $\alpha(n) = n1$ is the unique morphism from $\mathbf{Z}$ to R), and the trivial ring (with $1 = 0$) is a terminal object.
- In a preordered set, regarded as a category as in Example 1.3.3, a least element (if one exists) is initial and a greatest element (if one exists) is terminal.

$\square$

3.1.2 **Theorem**.

(i) *Any two initial objects of C are isomorphic.*

(ii) *Any two terminal objects of C are isomorphic.*

Proof. (i) Let c and c' be two initial objects of C. Since c is initial there exists a morphism $\alpha : c \to c'$. Similarly, since c' is initial there exists a morphism

$\beta : c' \to c$. Now $\beta\alpha$ is a morphism from c to itself, and so is the morphism 1_c. By the uniqueness assumption in the definition of initial object, $\beta\alpha = 1_c$. Similarly, $\alpha\beta = 1_{c'}$. Therefore, $\alpha : c \to c'$ is an isomorphism, implying $c \cong c'$.

(ii) The proof here is similar. $\square$

3.2 Motivation for product

Let X_1 and X_2 be sets and let P be the Cartesian product of X_1 and X_2:

$$P = X_1 \times X_2 = \{(x_1, x_2) \,|\, x_i \in X_i\}.$$

For $i \in \{1, 2\}$ let $\pi_i : P \to X_i$ be the projection map: $\pi_i((x_1, x_2)) = x_i$. The pair $(P, \{\pi_i\})$ has the following "universal mapping property": If X is a set and $\alpha_i : X \to X_i$ $(i = 1, 2)$ are functions, then there exists a unique function $\gamma : X \to P$ such that $\pi_i\gamma = \alpha_i$ for each i, i.e., such the following diagram is commutative for each i:

$$\begin{tikzcd} X \arrow[rr, dotted, "\exists!\,\gamma"] \arrow[dr, "\alpha_i"'] & & P \arrow[dl, "\pi_i"] \\ & X_i & \end{tikzcd} \quad . \tag{1}$$

(Proof: Assume the hypothesis. The map $\gamma : X \to P$ given by $\gamma(x) = (\alpha_1(x), \alpha_2(x))$ makes the diagram commutative for each i, and if $\gamma' : X \to P$ also makes the diagram commutative for each i, then for each $x \in X$ we have

$$\gamma'(x) = (\pi_1\gamma'(x), \pi_2\gamma'(x)) = (\alpha_1(x), \alpha_2(x)) = \gamma(x),$$

so that $\gamma' = \gamma$, establishing uniqueness.)

Let D be the category having as objects pairs $(X, \{\alpha_i\})$ with X a set and $\alpha_i : X \to X_i$ $(i = 1, 2)$ functions, and with morphisms from the object $(X, \{\alpha_i\})$ to the object $(Y, \{\beta_i\})$ being those functions $\gamma : X \to Y$ making the following diagram commutative for each i:

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "\alpha_i"'] & & Y \arrow[dl, "\beta_i"] \\ & X_i & \end{tikzcd} \quad ,$$

and with composition of morphisms being composition of functions. Then according to the previous paragraph the pair $(P, \{\pi_i\})$, with $P = X_1 \times X_2$, is a terminal object of D.

We refer to D as an "auxiliary" category; it encapsulates in a single object the items of relevance in this situation, namely, a set and maps from it to the fixed sets X_1 and X_2.

One thing to be gained from this is that we now know from Theorem 3.1.2 that the pair $(P, \{\pi_i\})$ is unique up to an isomorphism in D. Such an isomorphism is seen to be an isomorphism in **Set** (i.e., a bijection) but it is a special isomorphism in that, being a morphism in D, it is compatible with the maps to the sets X_1 and X_2. More precisely, if $\gamma : (X, \{\alpha_i\}) \to (P, \{\pi_i\})$ is an isomorphism in D, then $\gamma : X \to P$ is an isomorphism in **Set** and $\alpha_i = \pi_i\gamma$, so that α_i is the same as π_i once γ is used to identify X and P.

So the Cartesian product $P = X_1 \times X_2$ is special in how it relates to X_1 and X_2 through the projection maps π_i in that it has the universal mapping property (1) and it is essentially the only set with this property.

There are several other common constructions that possess similar universal mapping properties. Each can be viewed as an initial or terminal object of a suitable category. We consider some such constructions in the following sections.

3.3 Product

Here, we generalize the idea of the preceding section.

Let C be a category and let $\{c_i\}_{i \in I}$ be a family of objects of C. Form an auxiliary category $D = D_{\text{pr}}$ as follows: Take as objects pairs $(x, \{\alpha_i\})$, where x is an object of C and $\alpha_i : x \to c_i$ is a morphism in C for each $i \in I$; take as morphisms from the object $(x, \{\alpha_i\})$ to the object $(y, \{\beta_i\})$ all morphisms $\gamma : x \to y$ in C such that $\beta_i\gamma = \alpha_i$ for each i, that is, such that the following diagram is commutative for each i:

$$\begin{tikzcd} x \arrow[rr, "\gamma"] \arrow[dr, "\alpha_i"'] & & y \arrow[dl, "\beta_i"] \\ & c_i & \end{tikzcd} ;$$

and define composition of morphisms in D to be the composition in C.

A terminal object $(p, \{\pi_i\})$ of D is called a **product** of the family $\{c_i\}$. By definition, such an object has the property that for any object $(x, \{\alpha_i\})$ of D there exists a unique morphism $\gamma : x \to p$ in C such that $\pi_i\gamma = \alpha_i$ for each i, that is, such that the following diagram is commutative for each i:

$$\begin{tikzcd} x \arrow[rr, dotted, "\exists!\,\gamma"] \arrow[dr, "\alpha_i"'] & & p \arrow[dl, "\pi_i"] \\ & c_i & \end{tikzcd} .$$

According to Theorem 3.1.2, a product, if one exists, is unique up to an isomorphism in D, which amounts to an isomorphism in C that is compatible with the given morphisms to the objects c_i.

We say that "products exist" in the category C if a product exists for each family of objects of C.

3.3.1 *Example* (Products exist in **Set**) Let $\{X_i\}_{i\in I}$ be a family of sets. The natural generalization of the special case $I = \{1, 2\}$ discussed in Section 3.2 gives rise to a product here, but we need a definition of Cartesian product that does not use tuples (since I need not be finite, or even countable). Put

$$P = \prod_i X_i := \{x : \underset{i \longmapsto x_i}{I \to \textstyle\bigcup_i X_i} \mid x_i \in X_i \, \forall i\}$$

and for each $i \in I$ define $\pi_i : P \to X_i$ by $\pi_i(x) = x_i$.

We claim that $(P, \{\pi_i\})$ is a product of the family $\{X_i\}$. To prove this, we need to show that this pair is a terminal object of the auxiliary category $D = D_{\mathrm{pr}}$ defined above (with $C = \mathbf{Set}$). Let $(X, \{\alpha_i\})$ be an object of D. Define $\gamma : X \to P$ by $\gamma(x)_i = \alpha_i(x)$. For each $x \in X$ and each $i \in I$ we have $\pi_i\gamma(x) = \pi_i(\gamma(x)) = \gamma(x)_i = \alpha_i(x)$, so $\pi_i\gamma = \alpha_i$ for each i. Therefore, $\gamma : (X, \{\alpha_i\}) \to (P, \{\pi_i\})$ is a morphism in D.

Finally, let $\gamma' : (X, \{\alpha_i\}) \to (P, \{\pi_i\})$ be a morphism in D. Then for each $x \in X$ and each $i \in I$ we have

$$\gamma'(x)_i = \pi_i(\gamma'(x)) = \pi_i\gamma'(x) = \alpha_i(x) = \gamma(x)_i,$$

so $\gamma' = \gamma$ establishing uniqueness. Therefore $(P, \{\pi_i\})$ is terminal in D and hence a product of the family $\{X_i\}$. $\square$

3.3.2 *Example* (Products exist in **Grp**, **Rng**, ${}_R$**Mod**) Take the category **Grp** first. We proceed as in the preceding example, except that here we assume that each X_i is a group. Then the Cartesian product $P = \prod_i X_i$ is a group with componentwise operation: $(xy)_i = x_i y_i$ $(x, y \in P,\ i \in I)$. This group P is the **direct product** of the family $\{X_i\}$. The maps π_i $(i \in I)$ and γ defined above are seen to be homomorphisms, and the rest of the verifications are just as above, so $(P, \{\pi_i\})$ is a product of the family $\{X_i\}$.

The categories **Rng** and ${}_R$**Mod** are handled similarly: the Cartesian product is given the appropriate structure by defining operations componentwise and the maps are verified to be homomorphisms. □

3.3.3 *Example* (Products exist in **Top**) Again, we proceed as in Example 3.3.1. Here, each X_i is assumed to be a topological space. For a topology on the Cartesian product $P = \prod_i X_i$ we take the product topology, which has as subbase the collection of all $\pi_i^{-1}(U)$ with $i \in I$ and U open in X_i. In particular, each one of these sets is open in P, so that each projection map π_i is continuous.

To check that the function $\gamma : X \to P$ defined in Example 3.3.1 is continuous, it is sufficient to show that the inverse image of each subbase element is open. We have $\gamma^{-1}(\pi_i^{-1}(U)) = (\pi_i\gamma)^{-1}(U) = \alpha_i^{-1}(U)$, which is open for each $i \in I$ and U open in X_i since α_i is assumed to be continuous. The rest of the proof that the pair $(P, \{\pi_i\})$ is a product for the family $\{X_i\}$ is just as above. □

3.4 Coproduct

If we take the definition of product and turn arrows around we get the dual notion of "coproduct." The terminology follows the usual naming convention of prepending "co" to the name of a notion to get the name of its dual notion. (This convention is not consistently applied; for instance, epic morphism is the dual of monic morphism.)

Let C be a category and let $\{c_i\}_{i \in I}$ be a family of objects of C. Form a category $D = D_{\text{copr}}$ as follows: take as objects pairs $(x, \{\alpha_i\})$, where x is an object of C and $\alpha_i : c_i \to x$ is a morphism in C for each $i \in I$; take as morphisms from the object $(x, \{\alpha_i\})$ to the object $(y, \{\beta_i\})$ all morphisms $\gamma : x \to y$ in C such that $\gamma\alpha_i = \beta_i$ for each i, that is, such that the following diagram is commutative for each i:

$$\begin{tikzcd} x \arrow[rr, "\gamma"] & & y \\ & c_i \arrow[ul, "\alpha_i"] \arrow[ur, "\beta_i"'] & \end{tikzcd} ;$$

and define composition of morphisms in D to be the composition in C.

An initial object $(q, \{\iota_i\})$ of D is called a **coproduct** of the family $\{c_i\}$. By definition, such an object has the property that for any object $(x, \{\alpha_i\})$ of D there is a unique morphism $\gamma : q \to x$ in C such that $\gamma\iota_i = \alpha_i$ for each i, that is, such that the following diagram is commutative for each i:

$$\begin{tikzcd} q \arrow[rr, dotted, "\exists!\,\gamma"] & & x \\ & c_i \arrow[ul, "\iota_i"] \arrow[ur, "\alpha_i"'] & \end{tikzcd} .$$

As with products, we say that "coproducts exist" in the category C if a coproduct exists for each family of objects of C.

3.4.1 *Example* (Coproducts exist in **Set**) Let $\{X_i\}_{i\in I}$ be a family of sets. Let Q be the disjoint union of the X_i. Thus, $Q = \bigcup_i X_i'$, where $X_i' = \{(x,i) \,|\, x \in X_i\}$ for each i. (We have $X_i' \cong X_i$ in **Set** for each i and the X_i' are pairwise disjoint even if the X_i are not.) For each $i \in I$ define $\iota_i : X_i \to Q$ by $\iota_i(x) = (x,i)$.

Claim: The pair $(Q, \{\iota_i\})$ is a coproduct of the family $\{X_i\}$ in **Set**. To prove this, we need to show that it is an initial object of the auxiliary category $D = D_{\mathrm{copr}}$ defined above (with $C = \mathbf{Set}$). Let $(X, \{\alpha_i\})$ be an object of D. Define $\gamma : Q \to X$ by $\gamma((x,i)) = \alpha_i(x)$. For each $i \in I$ and each $x \in X_i$, we have $\gamma\iota_i(x) = \gamma(\iota_i(x)) = \gamma((x,i)) = \alpha_i(x)$, so $\gamma\iota_i = \alpha_i$ for each i. Therefore, γ is a morphism in D. Let $\gamma' : (Q, \{\iota_i\}) \to (X, \{\alpha_i\})$ be a morphism in D. For each $(x,i) \in Q$ we have

$$\gamma'((x,i)) = \gamma'\iota_i(x) = \alpha_i(x) = \gamma((x,i)),$$

so $\gamma' = \gamma$ establishing uniqueness and the claim. $\square$

3.4.2 *Example* (Coproducts exist in **Top**) Let $\{X_i\}_{i\in I}$ be a family of topological spaces and let Q be the disjoint union of the X_i as in Example

3.4.1. A topology on Q is obtained by declaring a subset U of Q to be open if and only if $\iota_i^{-1}(U)$ is open in X_i for each i. Then the functions $\iota_i : X_i \to Q$ ($i \in I$) and γ defined in Example 3.4.1 are continuous, and the rest of that example shows that $(Q, \{\iota_i\})$ is a coproduct of the family $\{X_i\}$. □

3.4.3 *Example* (Coproducts exist in ${}_R\mathbf{Mod}$) Let R be a ring and let $\{M_i\}_{i\in I}$ be a family of R-modules. Let $M = \bigoplus_i M_i$ be the direct sum of the M_i. Thus, M is the submodule of the direct product $\prod_i M_i$ (see Example 3.3.2) consisting of those functions that are 0 for all but finitely many i:

$$M = \bigoplus_i M_i = \{m \in \prod_i M_i \,|\, m_i = 0 \text{ for all but finitely many } i\}.$$

For each $i \in I$ define $\iota_i : M_i \to M$ by $\iota_i(m)_j = \delta_{ij}(m)$, where $\delta_{ij} : M_i \to M_j$ is the identity map if $i = j$ and zero otherwise. Then ι_i is an R-homomorphism for each i.

Claim: The pair $(M, \{\iota_i\})$ is a coproduct of the family $\{M_i\}$ in ${}_R\mathbf{Mod}$, that is, an initial object of the auxiliary category $D = D_{\text{copr}}$. Let $(N, \{\alpha_i\})$ be an object of D. Define $\gamma : M \to N$ by $\gamma(m) = \sum_i \alpha_i(m_i)$ (well-defined since only finitely many terms of this sum are nonzero). It is straightforward to check that γ is an R-homomorphism. For each $i \in I$ and $m \in M_i$ we have

$$\gamma\iota_i(m) = \sum_j \alpha_j(\iota_i(m)_j) = \sum_j \alpha_j(\delta_{ij}(m)) = \alpha_i(m),$$

so $\gamma\iota_i = \alpha_i$ for each i. Therefore, γ is a morphism in D.

Let $\gamma' : (M, \{\iota_i\}) \to (N, \{\alpha_i\})$ be a morphism in D. Let $m \in M$. For each $j \in I$ we have $m_j = \sum_i \delta_{ij}(m_i) = \sum_i \iota_i(m_i)_j = (\sum_i \iota_i(m_i))_j$, so $m = \sum_i \iota_i(m_i)$. Therefore,

$$\gamma'(m) = \gamma'\big(\sum_i \iota_i(m_i)\big) = \sum_i \gamma'\iota_i(m_i) = \sum_i \alpha_i(m_i) = \gamma(m),$$

so that $\gamma' = \gamma$ establishing uniqueness and the claim. □

Letting $R = \mathbf{Z}$ in the preceding example we get that coproducts exist in $\mathbf{Ab}$. For a field F letting $R = F$ we get that coproducts exist in $\mathbf{Vect}_F$.

3.4.4 *Example* (Coproducts exist in $\mathbf{Grp}$) Let $\{G_i\}_{i\in I}$ be a family of groups. Let Q be the "free product" of the family. The elements of this group are reduced words on the (disjoint) union U of the G_i, meaning strings $x_1x_2\cdots x_n$ ($n \geq 0$, $x_j \in U$) in which no factor is an identity and adjacent factors do not lie in the same group. The operation is juxtaposition followed by any necessary reductions to achieve a reduced word (i.e., multiplication of adjacent factors if in the same group and removal of any instances of the identity). The identity of X is the empty word. The inverse of the reduced word $x_1x_2\cdots x_n$ is $x_n^{-1}x_{n-1}^{-1}\cdots x_1^{-1}$. We omit the check that the operation is associative since it is long and tedious. For each $i \in I$ define $\iota_i : G_i \to Q$ by letting $\iota_i(a)$ be the word a if a is not the identity and the empty word otherwise. Then ι_i is a homomorphism for each i.

Claim: The pair $(Q, \{\iota_i\})$ is a coproduct of the family $\{G_i\}$ in **Grp**, that is, an initial object of the auxiliary category $D = D_{\text{copr}}$. Let $(X, \{\alpha_i\})$ be an object of D. Define $\gamma : Q \to X$ by $\gamma(x_1x_2\cdots x_n) = \alpha_{j_1}(x_1)\alpha_{j_2}(x_2)\cdots\alpha_{j_n}(x_n)$, where $x_i \in G_{j_i}$. Then γ is the unique morphism in D from $(Q, \{\iota_i\})$ to $(X, \{\alpha_i\})$ (check omitted), and this establishes the claim. $\square$

The examples show the dependence of a coproduct of a family on the ambient category. The coproduct in **Ab** of a family of abelian groups is isomorphic to the direct sum of the groups with the natural injections (see remark after Example 3.4.3). But a coproduct of this same family in **Grp** is isomorphic to the free product with the natural injections. The copies of the given groups in the direct sum commute elementwise, but not so in the free product. (Cf. also Exercise 3–1.)

3.5 Equalizer

Let C be a category and let $\lambda_1, \lambda_2 : a \to b$ be two morphisms in C:

$$a \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} b \ .$$

Form an auxiliary category $D = D_{\text{eq}}$ as follows: Take as objects pairs (x, α) with x an object of C and $\alpha : x \to a$ a morphism in C such that $\lambda_1\alpha = \lambda_2\alpha$:

$$x \xrightarrow{\alpha} a \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} b \ ;$$

take as morphisms from the object (x, α) to the object (y, β) those morphisms $\gamma : x \to y$ in C such that $\beta\gamma = \alpha$, that is, such that the triangle in the following diagram is commutative:

$$\begin{tikzcd} x \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & & \\ & a \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & b\,; \\ y \arrow[ur, "\beta"'] & & \end{tikzcd}$$

and define composition of morphisms to be the composition in C.

A terminal object (p, ι) of D is called an **equalizer** of λ_1 and λ_2. By definition, such an object has the property that for any object (x, α) of D there exists a unique morphism $\gamma : x \to p$ in C such that $\iota\gamma = \alpha$, that is, such that the triangle in the following diagram is commutative:

$$\begin{tikzcd} x \arrow[d, dotted, "\exists!\,\gamma"'] \arrow[dr, "\alpha"] & & \\ & a \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & b\ . \\ p \arrow[ur, "\iota"'] & & \end{tikzcd}$$

We say that "equalizers exist" in the category C if an equalizer exists for each pair of "parallel" morphisms.

3.5.1 *Example* (Equalizers exist in **Set**) Let $\lambda_1, \lambda_2 : A \to B$ be two functions. Put

$$P = \{a \in A \,|\, \lambda_1(a) = \lambda_2(a)\}$$

and let $\iota : P \to A$ be the inclusion map. Then $\lambda_1\iota = \lambda_2\iota$, so (P, ι) is an object of the auxiliary category $D = D_{\mathrm{eq}}$ defined above (with $C = \mathbf{Set}$).

Claim: (P, ι) is an equalizer of λ_1 and λ_2, that is, a terminal object of D. Let (X, α) be an object of D. For every $x \in X$ we have $\lambda_1(\alpha(x)) = \lambda_1\alpha(x) = \lambda_2\alpha(x) = \lambda_2(\alpha(x))$, so $\operatorname{im}\alpha \subseteq P$. Therefore, letting $\gamma : X \to P$ be α with codomain restricted to P we get $\iota\gamma = \alpha$, so γ is a morphism in D from (X, α) to (P, ι):

$$\begin{tikzcd} X \arrow[d, dotted, "\gamma"'] \arrow[dr, "\alpha"] & & \\ & A \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & B\ . \\ P \arrow[ur, "\iota"'] & & \end{tikzcd}$$

Finally, ι is injective and hence monic by Theorem 2.1.1, so γ is the unique such morphism and the claim follows. $\square$

3.5.2 *Example* (Equalizers exist in **Top**) If $\lambda_1, \lambda_2 : A \to B$ are two continuous maps, and P, as defined in Example 3.5.1, is given the subspace topology, then the inclusion map $\iota : P \to A$ is continuous and (P, ι) is an equalizer of λ_1 and λ_2, the verification being just as in that example. $\square$

3.5.3 *Example* (Equalizers exist in **Grp**, **Rng**, and ${}_R$**Mod**) Take the category **Grp** first. If $\lambda_1\lambda_2 : A \to B$ are two group homomorphisms, then P, as defined in Example 3.5.1, is a subgroup of A and the inclusion map $\iota : P \to A$ is a homomorphism, so (P, ι) is an equalizer of λ_1 and λ_2, the verification being just as in that example.

A similar argument works for the categories **Rng** and ${}_R$**Mod** for any ring R. $\square$

3.5.4 Theorem. *If $\lambda_1, \lambda_2 : a \to b$ are two morphisms in C and (p, ι) is an equalizer of λ_1 and λ_2, then ι is monic.*

Proof. Assume the hypothesis and let $\beta_1, \beta_2 : x \to p$ be morphisms in C with

$$\iota\beta_1 = \iota\beta_2 =: \alpha. \qquad (2)$$

Since (p, ι) is an object of the auxiliary category $D = D_{\mathrm{eq}}$, we have

$$\lambda_1\alpha = \lambda_1\iota\beta_1 = \lambda_2\iota\beta_1 = \lambda_2\alpha,$$

so (x, α) is an object of D as well:

$$\begin{tikzcd}
x \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] \arrow[dr, "\alpha"] & & \\
 & a \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & b \,. \\
p \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

In the diagram above, the triangle with β_1 and the triangle with β_2 are both commutative by Equation (2), so by the uniqueness assumption in the definition of equalizer we get $\beta_1 = \beta_2$. Therefore, ι is monic. $\square$

3.6 Coequalizer

In this section we dualize the definition of equalizer to obtain the definition of "coequalizer."

Let C be a category and let $\lambda_1, \lambda_2 : a \to b$ be two morphisms in C:

$$a \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} b\ .$$

Form an auxiliary category $D = D_{\mathrm{coeq}}$ as follows: Take as objects pairs (x, α) with x an object of C and $\alpha : b \to x$ a morphism in C such that $\alpha\lambda_1 = \alpha\lambda_2$:

$$a \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} b \xrightarrow{\alpha} x\ ;$$

take as morphisms from the object (x, α) to the object (y, β) those morphisms $\gamma : x \to y$ in C such that $\gamma\alpha = \beta$, that is, such that the triangle in the following diagram is commutative:

$$\begin{tikzcd} & & y \\ a \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & b \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & \\ & & x \arrow[uu, "\gamma"'] \end{tikzcd}\ ;$$

and define composition of morphisms to be the composition in C.

An initial object (q, π) of D is called a **coequalizer** of λ_1 and λ_2. By definition, such an object has the property that for any object (x, α) of D there exists a unique morphism $\gamma : q \to x$ in C such that $\gamma\pi = \alpha$, that is, such that the triangle in the following diagram is commutative:

$$\begin{tikzcd} & & x \\ a \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & b \arrow[ur, "\alpha"] \arrow[dr, "\pi"'] & \\ & & q \arrow[uu, dotted, "\exists!\,\gamma"'] \end{tikzcd}\ .$$

We say that "coequalizers exist" in the category C if a coequalizer exists for each pair of "parallel" morphisms.

3.6.1 *Example* (Coequalizers exist in **Set**) Let $\lambda_1, \lambda_2 : A \to B$ be functions. Put $R = \{(\lambda_1(a), \lambda_2(a)) \mid a \in A\} \subseteq B \times B$ and let $\sim$ be the equivalence relation on B generated by R. Thus, $b \sim c$ if and only if there exists a nonnegative integer n and $x_0, x_1, x_2, \ldots, x_n \in B$ such that $b = x_0$, $c = x_n$, and for each $1 \le i \le n$ either $(x_{i-1}, x_i) \in R$ or $(x_i, x_{i-1}) \in R$. Let Q be the quotient $B/{\sim} = \{\bar{b} \mid b \in B\}$ (= set of equivalence classes of B relative to $\sim$) and define $\pi : B \to Q$ by $\pi(b) = \bar{b}$.

Claim: The pair (Q, π) is a coequalizer of λ_1 and λ_2 in the category **Set**. For each $a \in A$, $\pi\lambda_1(a) = \overline{\lambda_1(a)} = \overline{\lambda_2(a)} = \pi\lambda_2(a)$ since $\lambda_1(a) \sim \lambda_2(a)$. Therefore, $\pi\lambda_1 = \pi\lambda_2$, implying that (Q, π) is an object of the auxiliary category $D = D_{\text{coeq}}$ defined above (with $C = \mathbf{Set}$).

We need to show that (Q, π) is an initial object. Let (X, α) be an object of D. Define $\gamma : Q \to X$ by $\gamma(\bar{b}) = \alpha(b)$. Note that for $(x_1, x_2) \in R$ we have $(x_1, x_2) = (\lambda_1(a), \lambda_2(a))$ for some $a \in A$, implying $\alpha(x_1) = \alpha\lambda_1(a) = \alpha\lambda_2(a) = \alpha(x_2)$. Therefore, if $\bar{b} = \bar{c}$, then $b \sim c$ and, with notation as above, $\alpha(b) = \alpha(x_0) = \alpha(x_1) = \cdots = \alpha(x_n) = \alpha(c)$ and γ is well-defined. Also, $\gamma\pi = \alpha$, so γ is a morphism in D from (Q, π) to (X, α):

$$\begin{tikzcd} & & X \\ A \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & B \arrow[ur, "\alpha"] \arrow[dr, "\pi"'] & \\ & & Q \arrow[uu, "\gamma"'] \end{tikzcd} .$$

Finally, π is surjective and hence epic by Theorem 2.3.1, so γ is the unique such morphism and the claim follows. □

3.6.2 *Example* (Coequalizers exist in **Top**) Let $\lambda_1, \lambda_2 : A \to B$ be continuous maps. Let Q and $\pi : B \to Q$ be as in Example 3.6.1. Endow Q with the quotient topology (so a subset U of Q is open if and only if $\pi^{-1}(U)$ is open in B). Then $\pi : B \to Q$ is continuous. Arguing that (Q, π) is a coequalizer of λ_1 and λ_2 as in Example 3.6.1 we are given an object (X, α) of D_{coeq} (defined with $C = \mathbf{Top}$) and we need only show that the map $\gamma : Q \to X$ given by $\gamma(\bar{b}) = \alpha(b)$ is continuous. Let U be an open subset of X. Since $\gamma\pi = \alpha$, we have $\pi^{-1}(\gamma^{-1}(U)) = \alpha^{-1}(U)$, which is open in B since α is continuous. Hence, $\gamma^{-1}(U)$ is open in Q. We conclude that γ is continuous and the argument is complete. □

3.6.3 *Example* (Coequalizers exist in **Grp**) Let $\lambda_1, \lambda_2 : A \to B$ be two group homomorphisms. Let N be the normal closure in B of the set

$$S = \{\lambda_1(a)\lambda_2(a)^{-1} \mid a \in A\}.$$

This means that N is the intersection of all normal subgroups of B containing S. Put $Q = B/N$ and let $\pi : B \to Q$ be the canonical epimorphism. For $a \in A$ we have $\lambda_1(a)\lambda_2(a)^{-1} \in S \subseteq N$, so $\pi\lambda_1(a) = \lambda_1(a)N = \lambda_2(a)N = \pi\lambda_2(a)$. Therefore, $\pi\lambda_1 = \pi\lambda_2$, implying that the pair (Q, π) is an object of the auxiliary category $D = D_{\text{coeq}}$ defined above (with $C = \mathbf{Grp}$).

Claim: The pair (Q, π) is a coequalizer of λ_1 and λ_2. Let (X, α) be an object of D. Then $S \subseteq \ker \alpha$ and, since $\ker \alpha$ is normal, $N \subseteq \ker \alpha$. By the fundamental homomorphism theorem there is a unique homomorphism $\gamma : Q \to X$ such that $\gamma\pi = \alpha$. In other words, there is a unique morphism γ in D from (Q, π) to (X, α), which is to say that (Q, π) is an initial object of D, as desired. □

In the preceding example, if A and B are (additive) *abelian* groups, then the difference $\lambda_1 - \lambda_2$ is a homomorphism $A \to B$, and S, as defined in the example, is simply the image of this homomorphism and is therefore a subgroup of B (automatically normal). So in this case $N = S$, which yields a simpler description of the coequalizer. An analogous argument gives a coequalizer in the category ${}_R\mathbf{Mod}$ for any ring R.

3.6.4 Theorem. *If $\lambda_1, \lambda_2 : a \to b$ are morphisms in C and (q, π) is a coequalizer of λ_1 and λ_2, then π is epic.*

Proof. The proof is the same as the proof of Theorem 3.5.4 with arrows reversed. □

3.7 Pullback

Let C be a category and let $\lambda_i : a_i \to b$ $(i = 1, 2)$ be two morphisms in C:

$$\begin{tikzcd} & a_2 \arrow[d, "\lambda_2"] \\ a_1 \arrow[r, "\lambda_1"'] & b\,. \end{tikzcd}$$

Form an auxiliary category $D = D_{\mathrm{pb}}$ as follows: Take for objects pairs $(x, (\alpha_1, \alpha_2))$, where x is an object of C and $\alpha_i : x \to a_i$ $(i = 1, 2)$ are morphisms in C such that $\lambda_1\alpha_1 = \lambda_2\alpha_2$, that is, such that the following diagram is commutative:

$$\begin{tikzcd} x \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & a_2 \arrow[d, "\lambda_2"] \\ a_1 \arrow[r, "\lambda_1"'] & b\,; \end{tikzcd}$$

take as morphisms from the object $(x, (\alpha_1, \alpha_2))$ to the object $(y, (\beta_1, \beta_2))$ all morphisms $\gamma : x \to y$ in C such that $\beta_i \gamma = \alpha_i$ $(i = 1, 2)$, that is, such that the following diagram is commutative:

$$\begin{tikzcd}
x \arrow[dr, "\gamma"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\
& y \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & a_2 \arrow[d, "\lambda_2"] \\
& a_1 \arrow[r, "\lambda_1"'] & b\,;
\end{tikzcd}$$

and define composition of morphisms to be the composition in C.

A terminal object $(p, (\sigma_1, \sigma_2))$ of D is called a **pullback** of the pair (λ_1, λ_2). By definition, such an object has the property that for any object $(x, (\alpha_1, \alpha_2))$ of D there exists a unique morphism $\gamma : x \to p$ in C such that the following diagram is commutative:

$$\begin{tikzcd}
x \arrow[dr, dotted, "\exists!\,\gamma"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\
& p \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"'] & a_2 \arrow[d, "\lambda_2"] \\
& a_1 \arrow[r, "\lambda_1"'] & b\ .
\end{tikzcd}$$

We say that "pullbacks exist" in the category C if a pullback exists for each pair of morphisms in C with the same target.

3.7.1 *Example* (Pullbacks exist in **Set**) Let $\lambda_i : A_i \to B$ $(i = 1, 2)$ be two functions. Define

$$A_1 \times_B A_2 = \{(a_1, a_2) \,|\, a_i \in A_i \text{ and } \lambda_1(a_1) = \lambda_2(a_2)\} \subseteq A_1 \times A_2,$$

called the **fibered product** of λ_1 and λ_2. Put $P = A_1 \times_B A_2$ and for $i = 1, 2$ define $\sigma_i : P \to A_i$ by $\sigma_i((a_1, a_2)) = a_i$. For $a = (a_1, a_2) \in P$ we have $\lambda_1\sigma_1(a) = \lambda_1(a_1) = \lambda_2(a_2) = \lambda_2\sigma_2(a)$. Therefore, $\lambda_1\sigma_1 = \lambda_2\sigma_2$, implying that the pair $(P, (\sigma_1, \sigma_2))$ is an object of the auxiliary category $D = D_{\mathrm{pb}}$ defined as above (with $C = \mathbf{Set}$).

Claim: The pair $(P, (\sigma_1, \sigma_2))$ is a pullback of the pair (λ_1, λ_2). Let $(X, (\alpha_1, \alpha_2))$ be an object of the category D. Define $\gamma : X \to P$ by $\gamma(x) =$

$(\alpha_1(x), \alpha_2(x))$. Since $\lambda_1\alpha_1(x) = \lambda_2\alpha_2(x)$ for each $x \in X$, γ maps into P as indicated. Moreover, $\sigma_i\gamma = \alpha_i$ $(i = 1, 2)$, so γ is a morphism in D from $(X, (\alpha_1, \alpha_2))$ to $(P, (\sigma_1, \sigma_2))$:

$$\begin{tikzcd}
X \arrow[dr, "\gamma"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\
& P \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"'] & A_2 \arrow[d, "\lambda_2"] \\
& A_1 \arrow[r, "\lambda_1"'] & B\,.
\end{tikzcd}$$

Finally, let γ' be a morphism from $(X, (\alpha_1, \alpha_2))$ to $(P, (\sigma_1, \sigma_2))$. For every $x \in X$ we have

$$\gamma'(x) = (\sigma_1\gamma'(x), \sigma_2\gamma'(x)) = (\alpha_1(x), \alpha_2(x)) = \gamma(x),$$

so $\gamma' = \gamma$ establishing uniqueness and the claim. $\square$

3.7.2 *Example* (Pullbacks exist in **Top**) Let $\lambda_i : A_i \to B$ $(i = 1, 2)$ be two continuous maps, give $A_1 \times A_2$ the product topology, and let $P = A_1 \times_B A_2$ be as in Example 3.7.1 endowed with the subspace topology. In that example, the maps $\sigma_i : P \to A_i$ $(i = 1, 2)$ are the restrictions to P of the usual projection maps from $A_1 \times A_2$ and are therefore continuous. For an object $(X, (\alpha_1, \alpha_2))$ of the category D_{pb} defined above (with $C = \mathbf{Top}$), the map $\gamma : X \to P \subseteq A_1 \times A_2$ defined as before has continuous component maps α_1 and α_2 and is therefore continuous. Therefore, the pair $(P, (\sigma_1, \sigma_2))$ is a pullback of the pair (λ_1, λ_2), the remaining arguments being just as before. $\square$

3.7.3 *Example* (Pullbacks exist in **Grp**, **Rng**, ${}_R$**Mod**) Take the category **Grp** first. Let $\lambda_i : A_i \to B$ $(i = 1, 2)$ be two group homomorphisms and let $P = A_1 \times_B A_2$ and $\sigma_i : P \to A_i$ $(i = 1, 2)$ be as in Example 3.7.1. Then P is a subgroup of $A_1 \times A_2$ and the maps σ_i $(i = 1, 2)$ as well as the map γ defined in that example are homomorphisms, so $(P, (\sigma_1, \sigma_2))$ is a pullback of the pair (λ_1, λ_2).

A similar argument works for the categories **Rng** and ${}_R$**Mod** for any ring R. $\square$

We say that "finite products exist" in the category C if a product exists for each finite family of objects of C (including the empty family).

3.7.4 Theorem. *The following are equivalent for the category C:*

(i) *Equalizers and finite products exist in C,*

(ii) *Pullbacks and a terminal object exist in C.*

Proof. (i $\Rightarrow$ ii) Assume that equalizers and finite products exist in C. A product of the empty family of objects is a terminal object of C, so it remains to show that pullbacks exist. Let $\lambda_i : a_i \to b$ $(i = 1, 2)$ be two morphisms in C. By assumption there exists a product $(a_1 \times a_2, \{\pi_i\})$ of the family $\{a_i\}$, that is, a terminal object of the corresponding auxiliary category D_{pr}. In turn, by assumption there exists an equalizer (p, ι) of $\lambda_1\pi_1$ and $\lambda_2\pi_2$, that is, a terminal object of the corresponding auxiliary category D_{eq} (see diagrams). Put $\sigma_i = \pi_i\iota$ $(i = 1, 2)$. We have $\lambda_1\sigma_1 = \lambda_1\pi_1\iota = \lambda_2\pi_2\iota = \lambda_2\sigma_2$, so $(p, (\sigma_1, \sigma_2))$ is an object of the auxiliary category D_{pb} in the definition of pullback for the pair (λ_1, λ_2).

$$\begin{tikzcd}
x \arrow[drrr, bend left, "\alpha_2"] \arrow[dddr, bend right, "\alpha_1"'] \arrow[dr, dotted, "\gamma"] \arrow[ddrr, dotted, "\delta"'] & & & \\
& p \arrow[rr, "\sigma_2"] \arrow[dd, "\sigma_1"'] \arrow[dr, dotted, "\iota"] & & a_2 \arrow[dd, "\lambda_2"] \\
& & a_1 \times a_2 \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
& a_1 \arrow[rr, "\lambda_1"'] & & b\ .
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[dr, "\delta"] \arrow[dd, dotted, "\gamma"'] & & \\
& a_1 \times a_2 \arrow[r, shift left, "\lambda_1\pi_1"] \arrow[r, shift right, "\lambda_2\pi_2"'] & b\ . \\
p \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

Claim: The pair $(p, (\sigma_1, \sigma_2))$ is a pullback of the pair (λ_1, λ_2). Let $(x, (\alpha_1, \alpha_2)) \in D_{\mathrm{pb}}$, so that $\lambda_1\alpha_1 = \lambda_2\alpha_2$. The universal mapping property of the product yields a morphism $\delta : x \to a_1 \times a_2$ such that $\pi_i\delta = \alpha_i$ $(i = 1, 2)$. We have $\lambda_1\pi_1\delta = \lambda_1\alpha_1 = \lambda_2\alpha_2 = \lambda_2\pi_2\delta$, so $(x, \delta) \in D_{\mathrm{eq}}$. The universal mapping property of the equalizer yields a morphism $\gamma : x \to p$ such that $\iota\gamma = \delta$. Now $\sigma_i\gamma = \pi_i\iota\gamma = \pi_i\delta = \alpha_i$ $(i = 1, 2)$, so γ is a morphism in D_{pb} from $(x, (\alpha_1, \alpha_2))$ to $(p, (\sigma_1, \sigma_2))$. Finally, let γ' be another such morphism. Then $\pi_i\iota\gamma' = \sigma_i\gamma' = \alpha_i$ and similarly $\pi_i\iota\gamma = \alpha_i$ $(i = 1, 2)$, so the uniqueness assumption in the definition of product gives $\iota\gamma' = \iota\gamma$. By Theorem 3.5.4, ι is monic, so $\gamma' = \gamma$ and the claim is established.

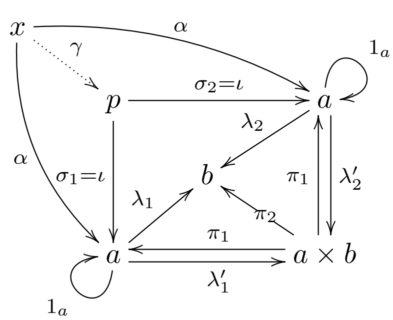
(ii $\Rightarrow$ i) Assume that pullbacks exist in C and that C has a terminal object t. To prove that finite products exist in C it suffices to show that a product exists for a family $\{a_1, a_2\}$ of two objects of C (Exercise 3–2). Since t is terminal, there exist morphisms $\lambda_i : a_i \to t$ $(i = 1, 2)$ in C. By assumption, there exists a pullback $(p, (\pi_1, \pi_2))$ of the pair (λ_1, λ_2), that is, a terminal object of the corresponding auxiliary category D_{pb}. The square in the following diagram is commutative:

$$\begin{tikzcd}
x \arrow[dr, dotted, "\gamma"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\
& p \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & a_2 \arrow[d, "\lambda_2"] \\
& a_1 \arrow[r, "\lambda_1"'] & t\,.
\end{tikzcd}$$

Claim: The pair $(p, \{\pi_i\})$ is a product of the family $\{a_i\}$. Let $(x, \{\alpha_i\})$ be an object of the auxiliary category D_{pr} in the definition of product of the family $\{a_i\}$. Then $\lambda_i\alpha_i : x \to t$ $(i = 1, 2)$ are two morphisms to the terminal object t, so $\lambda_1\alpha_1 = \lambda_2\alpha_2$ by uniqueness. Therefore, $(x, (\alpha_1, \alpha_2)) \in D_{\mathrm{pb}}$ giving rise to a unique morphism $\gamma : x \to p$ in C such that $\pi_i\gamma = \alpha_i$ $(i = 1, 2)$. The claim follows.

It remains to show that equalizers exist in C. Let $\lambda_i : a \to b$ $(i = 1, 2)$ be two parallel morphisms in C. By what we have shown already, there exists a product $(a \times b, \{\pi_i\})$ of the family $\{a, b\}$, that is, a terminal object of the corresponding auxiliary category D_{pr}. Using the universal mapping property of product twice (see left diagram) we get morphisms $\lambda_1', \lambda_2' : a \to a \times b$ such that

$$\begin{aligned} \pi_1\lambda_1' &= 1_a, & \qquad \pi_1\lambda_2' &= 1_a, \\ \pi_2\lambda_1' &= \lambda_1, & \qquad \pi_2\lambda_2' &= \lambda_2 \ : \end{aligned} \tag{3}$$



$$\begin{tikzcd}
x \arrow[dr, "\alpha"] \arrow[dd, dotted, "\gamma"'] & & \\
& a \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & b\,. \\
p \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

By assumption, there exists a pullback $(p, (\sigma_1, \sigma_2))$ of the pair (λ_1', λ_2'), that is, a terminal object of the corresponding auxiliary category D_{pb}. Using the first line of (3), we find that

$$\sigma_1 = 1_a\sigma_1 = \pi_1\lambda_1'\sigma_1 = \pi_1\lambda_2'\sigma_2 = 1_a\sigma_2 = \sigma_2.$$

Put $\iota = \sigma_1 = \sigma_2$. Using the second line of (3), we find that

$$\lambda_1\iota = \pi_2\lambda_1'\sigma_1 = \pi_2\lambda_2'\sigma_2 = \lambda_2\iota,$$

so (p, ι) is an object of the auxiliary category D_{eq} in the definition of equalizer of λ_1 and λ_2 (see right diagram above).

Claim: The pair (p, ι) is an equalizer of λ_1 and λ_2. Let $(x, \alpha) \in D_{\mathrm{eq}}$. In particular, $\lambda_1\alpha = \lambda_2\alpha =: \beta$. Now $\lambda_i'\alpha$ $(i = 1, 2)$ are both morphisms in D_{pr} from $(x, \{\alpha, \beta\})$ to $(a \times b, \{\pi_1, \pi_2\})$. Indeed, from (3) we have

$$\pi_1\lambda_i'\alpha = 1_a\alpha = \alpha, \qquad \pi_2\lambda_i'\alpha = \lambda_i\alpha = \beta$$

$(i = 1, 2)$. So by uniqueness, $\lambda_1'\alpha = \lambda_2'\alpha$. Therefore, $(x, (\alpha, \alpha)) \in D_{\mathrm{pb}}$. This yields a unique morphism $\gamma : x \to p$ such that $\sigma_i\gamma = \alpha$ $(i = 1, 2)$ or, equivalently, such that $\iota\gamma = \alpha$. The claim follows and the proof is complete. □

3.8 Pushout

In this section, we dualize the definition of pullback to obtain the definition of "pushout."

Let C be a category and let $\lambda_1 : a \to b_1$ and $\lambda_2 : a \to b_2$ be two morphisms in C:

$$\begin{tikzcd} a \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & b_2 . \\ b_1 & \end{tikzcd}$$

Form an auxiliary category $D = D_{\mathrm{po}}$ as follows: Take for objects pairs $(x, (\alpha_1, \alpha_2))$, where x is an object of C and $\alpha_i : b_i \to x$ $(i = 1, 2)$ are morphisms in C such that $\alpha_1\lambda_1 = \alpha_2\lambda_2$, that is, such that the following diagram is commutative:

$$\begin{tikzcd} a \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & b_2 \arrow[d, "\alpha_2"] \\ b_1 \arrow[r, "\alpha_1"'] & x ; \end{tikzcd}$$

take as morphisms from the object $(x, (\alpha_1, \alpha_2))$ to the object $(y, (\beta_1, \beta_2))$ all morphisms $\gamma : x \to y$ in C such that $\gamma\alpha_i = \beta_i$ $(i = 1, 2)$, that is, such that the following diagram is commutative:

$$\begin{tikzcd}
a \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & b_2 \arrow[d, "\alpha_2"] \arrow[ddr, bend left, "\beta_2"] & \\
b_1 \arrow[r, "\alpha_1"'] \arrow[drr, bend right, "\beta_1"'] & x \arrow[dr, "\gamma"] & \\
& & y\,;
\end{tikzcd}$$

and define composition of morphisms to be the composition in C.

An initial object $(q, (\tau_1, \tau_2))$ of D is called a **pushout** of the pair (λ_1, λ_2). By definition, such an object has the property that for any object $(x, (\alpha_1, \alpha_2))$ of D there exists a unique morphism $\gamma : q \to x$ in C such that the following diagram is commutative:

$$\begin{tikzcd}
a \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & b_2 \arrow[d, "\tau_2"] \arrow[ddr, bend left, "\alpha_2"] & \\
b_1 \arrow[r, "\tau_1"'] \arrow[drr, bend right, "\alpha_1"'] & q \arrow[dr, dotted, "\exists!\,\gamma"] & \\
& & x\,.
\end{tikzcd}$$

We say that "pushouts exist" in the category C if a pushout exists for each pair of morphisms in C with the same source.

3.8.1 *Example* (Pushouts exist in **Set**) Let $\lambda_i : A \to B_i$ $(i = 1, 2)$ be two functions. Let B be the disjoint union of B_1 and B_2 (see Example 3.4.1). We use the injections $B_1, B_2 \hookrightarrow B$ to identify B_1 and B_2 with their images and consequently regard these sets as being disjoint. Put $R = \{(\lambda_1(a), \lambda_2(a)) \mid a \in A\}$ and let $\sim$ be the equivalence relation on B generated by R (see Example 3.6.1). Let Q be the quotient $B/{\sim} = \{\bar{b} \mid b \in B\}$, define $\pi : B \to Q$ by $\pi(b) = \bar{b}$, and put $\tau_i = \pi|_{B_i} : B_i \to Q$ $(i = 1, 2)$.

Claim: The pair $(Q, (\tau_1, \tau_2))$ is a pushout of the pair (λ_1, λ_2). For each $a \in A$, $\tau_1\lambda_1(a) = \overline{\lambda_1(a)} = \overline{\lambda_2(a)} = \tau_2\lambda_2(a)$ (using that $\lambda_1(a) \sim \lambda_2(a)$), so $\tau_1\lambda_1 = \tau_2\lambda_2$ implying that $(Q, (\tau_1, \tau_2))$ is an object of the auxiliary category

$D = D_{\text{po}}$ defined above (with $C = \textbf{Set}$). Let $(X, (\alpha_1, \alpha_2))$ be an object of D. Define $\gamma_0 : B \to X$ by

$$\gamma_0(b) = \begin{cases} \alpha_1(b), & b \in B_1, \\ \alpha_2(b), & b \in B_2 \end{cases}$$

(well-defined since B_1 and B_2 are disjoint). If $(r_1, r_2) \in R$, then $r_1 = \lambda_1(a)$ and $r_2 = \lambda_2(a)$ for some $a \in A$, so $\gamma_0(r_1) = \alpha_1\lambda_1(a) = \alpha_2\lambda_2(a) = \gamma_0(r_2)$. It follows that γ_0 is constant on the equivalence classes of B and therefore induces a unique map $\gamma : Q \to X$ such that $\gamma\pi = \gamma_0$. We have $\gamma\tau_i = \alpha_i$ $(i = 1, 2)$, so γ is a morphism in D from $(Q, (\lambda_1, \lambda_2))$ to $(X, (\alpha_1, \alpha_2))$:

$$\begin{tikzcd}
A \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & B_2 \arrow[d, "\tau_2"] \arrow[ddr, bend left, "\alpha_2"] & \\
B_1 \arrow[r, "\tau_1"'] \arrow[drr, bend right, "\alpha_1"'] & Q \arrow[dr, "\gamma"] & \\
 & & X\,.
\end{tikzcd}$$

Let γ' be another such morphism, and let $b \in B$. Then $b \in B_i$ for some $i \in \{1, 2\}$ and

$$\gamma'(\bar{b}) = \gamma'\pi(b) = \gamma'\tau_i(b) = \alpha_i(b) = \gamma(\bar{b}),$$

whence $\gamma' = \gamma$ and the claim follows. $\square$

3.8.2 *Example* (Pushouts exist in **Top**) Let $\lambda_i : A \to B_i$ $(i = 1, 2)$ be two continuous maps. If Q, as defined in Example 3.8.1, is endowed with the quotient topology, then all of the maps defined in that example are continuous and $(Q, (\tau_1, \tau_2))$ is a pushout of the pair (λ_1, λ_2). $\square$

3.8.3 *Example* (Pushouts exist in ${}_R\textbf{Mod}$) Let R be a ring and let $\lambda_i : A \to B_i$ $(i = 1, 2)$ be two R-homomorphisms. Put $B = B_1 \oplus B_2$ and define $S = \{(\lambda_1(a), -\lambda_2(a)) \,|\, a \in A\}$, which is seen to be a submodule of B. Put $Q = B/S$ and define $\tau_i = \pi\iota_i : B_i \to Q$ $(i = 1, 2)$, where the maps $\iota_i : B_i \to B$ are the natural injections and $\pi : B \to Q$ is the canonical epimorphism. Then the pair $(Q, (\tau_1, \tau_2))$ is a pushout of the pair (λ_1, λ_2). $\square$

3.8.4 *Example* (Pushouts exist in **Grp**) Let $\lambda_i : A \to B_i$ ($i = 1, 2$) be group homomorphisms. Regard B_1 and B_2 as subgroups of the free product $B = B_1 * B_2$ of the family $\{B_1, B_2\}$ (see Example 3.4.4) and let N be the normal closure of the set $\{\lambda_1(a)\lambda_2(a)^{-1} \mid a \in A\}$. Put $Q = B/N$ and define $\tau_i = \pi|_{B_i} : B_i \to Q$ ($i = 1, 2$), where $\pi : B \to Q$ is the canonical epimorphism. Then the pair $(Q, (\tau_1, \tau_2))$ is a pushout of the pair (λ_1, λ_2). The group Q is the **amalgamated free product** of B_1 and B_2 relative to the pair (λ_1, λ_2) and is denoted $B_1 *_A B_2$. The free product $B_1 * B_2$ is the special case where A is trivial. □

3.8.5 *Example* (Pushouts exist in **CRing**) Let **CRing** denote the category of commutative rings (with identity). Let $\lambda_i : A \to B_i$ ($i = 1, 2$) be two homomorphisms of commutative rings. The underlying abelian group of B_1 becomes a (right) A-module by defining $b_1 \cdot a = b_1\lambda_1(a)$ ($b_1 \in B_1, a \in A$), and similarly B_2 is a (left) A-module with action defined by $a \cdot b_2 = \lambda_2(a)b_2$ ($b_2 \in B_2, a \in A$), so the tensor product $Q = B_1 \otimes_A B_2$ is defined. Moreover, Q is a commutative ring with multiplication defined on generators by $(b_1 \otimes b_2)(c_1 \otimes c_2) = (b_1c_1) \otimes (b_2c_2)$ and extended linearly. Define $\tau_i : B_i \to Q$ by $\tau_1(b_1) = b_1 \otimes 1$ and $\tau_2(b_2) = 1 \otimes b_2$. Then the pair $(Q, (\tau_1, \tau_2))$ is a pushout of the pair (λ_1, λ_2). □

3.8.6 Theorem. *The following are equivalent for the category C:*

(i) *Coqualizers and finite coproducts exist in C,*

(ii) *Pushouts and an initial object exist in C.*

Proof. The proof is dual to that of Theorem 3.7.4. □

3 – Exercises

3–1 For groups G_1 and G_2 let $\iota_i : G_i \to G_1 \times G_2$ be the injections defined by $\iota_1(g_1) = (g_1, e_2)$ and $\iota_2(g_2) = (e_1, g_2)$, where e_i is the identity of G_i. Prove that $(G_1 \times G_2, \{\iota_i\})$ is *not*, in general, a coproduct of the family $\{G_i\}$ in the category **Grp**.

Hint: Let G be a nonabelian group and take $G_i = G$ ($i = 1, 2$). In the definition of coproduct, let $\alpha_i = 1_G : G_i \to G$ ($i = 1, 2$).

3–2 Let C be a category. Prove that the following are equivalent:

(i) Finite products exist in C. (See definition before Theorem 3.7.4.)

(ii) C has a terminal object, and for each pair $c_1, c_2 \in C$ there exists a product of the family $\{c_i\}_{i \in I}$ in C, where $I = \{1, 2\}$.

3–3 Let K be a field. Prove that coequalizers exist in the category $\mathbf{Mat}_K$ (see Example 1.3.5).

HINT: First assume that one of the two given morphisms is the zero matrix.

4 Functor

4.1 Definition and examples

Let C and D be categories. A **(covariant) functor** from C to D, written $F : C \to D$, is a function that sends each object x of C to an object $F(x)$ of D and sends each morphism α in C to a morphism $F(\alpha)$ in D such that the following hold:

(i) If $\alpha : x \to y$ is a morphism in C, then $F(\alpha) : F(x) \to F(y)$,

(ii) $F(\beta\alpha) = F(\beta)F(\alpha)$ for all morphisms α and β in C for which $\beta\alpha$ is defined,

(iii) $F(1_c) = 1_{F(c)}$ for each $c \in C$.

Let $F : C \to D$ be a functor. For any two objects x and y of C, the functor F restricts to a function

$$C(x, y) \to D(F(x), F(y)).$$

- F is **faithful** if this restricted function is injective for each pair of objects x and y.
- F is **full** if this restricted function is surjective for each pair of objects x and y.

(Cf. Example 4.2.6.)

4.1.1 *Example* (Forgetful functor) The functor **Grp** $\to$ **Set** that maps each group to its underlying set and each morphism to itself is an example of a "forgetful functor." In general, a forgetful functor is a functor that arises by forgetting some structure. This notion is not rigorously defined but is used to describe certain naturally occurring functors. For example, there are forgetful functors **Rng** $\to$ **Ab** (forget the multiplication), **PTop** $\to$ **Top** (forget the point), **Met** $\to$ **Top** (forget the metric). □

Earlier, we informally used the term "concrete category" to refer to a category in which the objects are sets with (possible) structure, and the morphisms are functions that preserve the structure. We can now give a rigorous definition. A **concrete category** is a pair (C, F), where C is a category and $F : C \to \mathbf{Set}$ is a faithful functor. For example, $(\mathbf{Grp}, F)$ is a concrete category with $F : \mathbf{Grp} \to \mathbf{Set}$ the forgetful functor.

Many familiar categories, such as **Set**, **Grp**, **Rng**, and **Top**, have a natural (faithful) forgetful functor to **Set** and thereby acquire the structure of concrete category. But even for categories having no obvious forgetful functor to **Set** (such as those in Examples 1.3.3, 1.3.5, and 1.3.6) it is sometimes possible to define a faithful functor to **Set** in order to give them the structure of concrete category. For example, if P is a preordered set regarded as a category (1.3.3) and $F : P \to \mathbf{Set}$ is the functor that sends every object of P to the set $\{0\}$ and every morphism of P to the identity map on $\{0\}$, then F is faithful, so that (P, F) is concrete.

4.1.2 *Example* (Hom functor) Let C be a category and let c be a fixed object of C. We get a functor $F : C \to \mathbf{Set}$ as follows: For an object x of C we put $F(x) = C(c, x)$ and for a morphism $\alpha : x \to y$ in C we define $F(\alpha) : F(x) \to F(y)$ (that is, $F(\alpha) : C(c, x) \to C(c, y)$) by $F(\alpha)(\beta) = \alpha\beta$. This functor is the **(covariant) Hom functor** corresponding to c (cf. Example 4.4.1). The terminology is due to the common alternate notation $\mathrm{Hom}_C(c, x)$ for $C(c, x)$.

This functor is often denoted by $C(c, \_) : C \to \mathbf{Set}$. When this notation is used, the image of the morphism $\alpha : x \to y$ is denoted by $C(c, \alpha)$, or often more simply by α_* (so that $\alpha_*(\beta) = \alpha\beta$ for a morphism $\beta : c \to x$). □

4.1.3 *Example* (Monoid homomorphism as functor) Let M and N be monoids and let $\varphi : M \to N$ be a monoid homomorphism, meaning $\varphi(xy) = \varphi(x)\varphi(y)$ $(x, y \in M)$ and $\varphi(1_M) = 1_N$. We get a functor $F : C_M \to C_N$ between the associated categories (Example 1.3.6) by letting $F(\bullet) = \bullet$ and $F(x) = \varphi(x)$ $(x \in M)$. Conversely, the restriction of any functor $F : C_M \to C_N$ to morphisms yields a homomorphism $M \to N$. □

4.1.4 *Example* (Power set functor) The **power set functor** $P : \mathbf{Set} \to \mathbf{Set}$ is defined as follows: For an object X, put

$$P(X) = \{\, S |\, S \subseteq X\}$$

(the power set of X), and for a morphism $\alpha : X \to Y$ define $P(\alpha) : P(X) \to P(Y)$ by $P(\alpha)(S) = \alpha[S]$. (Cf. Example 4.4.3.) □

4.1.5 *Example* (Fundamental group functor) Let X be a topological space and let p be a fixed point of X. A "loop at p" is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = p$. Two loops γ and δ at p are "homotopic," written $\gamma \sim \delta$, if γ can be continuously deformed to δ, that is, if there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that $H(t,0) = \gamma(t)$, $H(t,1) = \delta(t)$, $H(0,t) = p$, and $H(1,t) = p$ for all $t \in [0,1]$ (intuitively, H is the loop γ at the bottom of the unit square and the loop δ at the top, with intermediate horizontal lines corresponding to intermediate deformations from γ to δ). The relation $\sim$ is an equivalence relation on the set of all loops at p. The equivalence class of γ is denoted $[\gamma]$. Put $\pi_1(X,p) = \{[\gamma] \,|\, \gamma \text{ is a loop at } p\}$.

The product $\gamma_1\gamma_2$ of two loops γ_1 and γ_2 at p is the loop at p defined by

$$(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq 1/2, \\ \gamma_2(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

(i.e., γ_1 followed by γ_2, both traversed twice as fast as usual). The induced product on $\pi_1(X,p)$, given by $[\gamma_1][\gamma_2] = [\gamma_1\gamma_2]$, makes $\pi_1(X,p)$ a group, the **fundamental group** of X at the base point p. For example, one has $\pi_1(S^1,p) \cong \mathbf{Z}$ and $\pi_1(S^2,p) \cong \{e\}$ for any base point p, where S^1 is the circle and S^2 is the sphere.

A morphism $\alpha : (X,p) \to (X',p')$ in the category **PTop** of pointed topological spaces (see Example 1.3.1) induces a group homomorphism $\pi_1(\alpha) : \pi_1(X,p) \to \pi_1(X',p')$ defined by $\pi_1(\alpha)([\gamma]) = [\alpha\gamma]$, where $\alpha\gamma : [1,0] \xrightarrow{\gamma} X \xrightarrow{\alpha} Y$ is the composition. This defines a functor $\pi_1 : \mathbf{PTop} \to \mathbf{Grp}$ called the **fundamental group functor**. $\square$

4.1.6 *Example* (Constant functor) Let C and D be categories and let d be a fixed object of D. The **constant functor** from C to D determined by d is the functor that sends every object of C to the object d and sends every morphism in C to the identity morphism on d. $\square$

4.2 Elementary properties

Let $F : C \to D$ be a functor.

We say that F "preserves isomorphisms" if whenever the morphism $\alpha : x \to y$ in C is an isomorphism, the morphism $F(\alpha) : F(x) \to F(y)$ in D is also an isomorphism.

On the other hand, we say that F "reflects isomorphisms" if whenever a morphism $\alpha : x \to y$ in C has the property that $F(\alpha) : F(x) \to F(y)$ is an isomorphism, then $\alpha : x \to y$ is also an isomorphism.

We generalize these definitions: Let P be a property involving objects and/or morphism (e.g., isomorphism, equalizer, product, initial object, and so forth). For a family X of objects and/or morphisms in C put $F(X) = \{F(x) \,|\, x \in X\}$.

- F **preserves** the property P if, whenever a family X of objects and/or morphisms in C satisfies P, then so does the family $F(X)$ in D.
- F **reflects** the property P if, whenever a family X of objects and/or morphisms in C has the property that the family $F(X)$ in D satisfies P, then so does the family X.

4.2.1 Theorem.

(i) *Every functor preserves commutative diagrams.*

(ii) *Every faithful functor reflects commutative diagrams.*

Proof. Let $F : C \to D$ be a functor. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ be morphisms in C such that $\alpha, \beta : x \to y$ for some objects x and y of C, where $\alpha = \prod \alpha_i$ and $\beta = \prod \beta_i$. If $\alpha = \beta$, then

$$\prod F(\alpha_i) = F(\prod \alpha_i) = F(\alpha) = F(\beta) = F(\prod \beta_i) = \prod F(\beta_i).$$

This proves (i).

Now assuming that F is faithful and $\prod F(\alpha_i) = \prod F(\beta_i)$, we get (rearranging the string of equalities above) $F(\alpha) = F(\beta)$, so that $\alpha = \beta$. This proves (ii). $\square$

4.2.2 Theorem.

(i) *Every functor preserves split monics, split epics, and isomorphisms.*

(ii) *Every faithful and full functor reflects split monics, split epics, and isomorphisms.*

Proof. Let $F : C \to D$ be a functor. If $\alpha : x \to y$ is split monic in C, then there exists $\beta : y \to x$ such that $\beta\alpha = 1_x$, whence $F(\beta)F(\alpha) = F(\beta\alpha) = F(1_x) = 1_{F(x)}$ showing that $F(\alpha) : F(x) \to F(y)$ is split monic. An analogous proof shows that F preserves split epics and hence isomorphisms. This proves (i).

Now assume that F is faithful and full. Let $\alpha : x \to y$ be a morphism in C and assume that $F(\alpha) : F(x) \to F(y)$ is split monic. Then there exists $\beta' : F(y) \to F(x)$ such that $\beta' F(\alpha) = 1_{F(x)}$. Since F is full, we have $\beta' = F(\beta)$ for some $\beta : y \to x$. Therefore, $F(\beta\alpha) = F(\beta)F(\alpha) = \beta' F(\alpha) = 1_{F(x)} = F(1_x)$ and, since F is faithful, we get $\beta\alpha = 1_x$, so that α is split monic. An analogous proof shows that F reflects split monics and hence isomorphisms. This proves (ii). □

4.2.3 *Example* (Circle is not homeomorphic to sphere) We use the preceding theorem to show that the circle S^1 is not homeomorphic to the sphere S^2. Suppose, to the contrary, that $\alpha : S^1 \to S^2$ is an isomorphism in **Top**. Fix a point p_1 in S^1 and put $p_2 = \alpha(p_1)$. Then $\alpha : (S^1, p_1) \to (S^2, p_2)$ is an isomorphism in **PTop**. Let $\pi_1 : \mathbf{PTop} \to \mathbf{Grp}$ be the fundamental group functor (Example 4.1.5). By Theorem 4.2.2, π_1 preserves isomorphisms, so we get the contradiction

$$\mathbf{Z} \cong \pi_1(S^1, p_1) \cong \pi_1(S^2, p_2) \cong \{e\}.$$

Therefore, S^1 is not homeomorphic to S^2. □

A functor $F : C \to D$ is **essentially surjective** if each object of D is isomorphic to $F(x)$ for some $x \in C$.

4.2.4 Theorem.

(i) *Every faithful, full, and essentially surjective functor preserves monics, epics, and bimorphisms.*

(ii) *Every faithful functor reflects monics, epics, and bimorphisms.*

Proof. Let $F : C \to D$ be a functor. Assume that F is faithful, full, and essentially surjective and let $\alpha : a \to b$ be a monic morphism in C. We claim that $F(\alpha) : F(a) \to F(b)$ is monic. Let $\mu_1, \mu_2 : y \to F(a)$ be two morphisms in D and assume that $F(\alpha)\mu_1 = F(\alpha)\mu_2$. Since F is essentially surjective, there exists an isomorphism $\beta : F(x) \to y$ for some $x \in C$. Since F is full, there exist morphisms $\lambda_1, \lambda_2 : x \to a$ such that $F(\lambda_i) = \mu_i\beta$ $(i = 1, 2)$:

$$x \overset{\lambda_1}{\underset{\lambda_2}{\rightrightarrows}} a \xrightarrow{\alpha} b\ , \qquad\qquad F(x) \xrightarrow{\beta} y \overset{\mu_1}{\underset{\mu_2}{\rightrightarrows}} F(a) \xrightarrow{F(\alpha)} F(b)\ .$$

Now $F(\alpha\lambda_i) = F(\alpha)F(\lambda_i) = F(\alpha)\mu_i\beta$ $(i = 1, 2)$, so $F(\alpha\lambda_1) = F(\alpha)\mu_1\beta = F(\alpha)\mu_2\beta = F(\alpha\lambda_2)$. Since F is faithful, we get $\alpha\lambda_1 = \alpha\lambda_2$ so that the monic property of α yields $\lambda_1 = \lambda_2$. Consequently, $\mu_1\beta = F(\lambda_1) = F(\lambda_2) = \mu_2\beta$ and, since β is epic (Theorems 2.5.1 and 2.4.1) we get $\mu_1 = \mu_2$. Therefore, $F(\alpha)$ is monic. An analogous proof shows that F preserves epics and hence bimorphisms. This proves (i).

Now assume that F is faithful and let $\alpha : a \to b$ be a morphism in C such that $F(\alpha) : F(a) \to F(b)$ is monic. We claim that α is monic. Let $\lambda_1, \lambda_2 : x \to a$ be morphisms in C and assume that $\alpha\lambda_1 = \alpha\lambda_2$. Then $F(\alpha)F(\lambda_1) = F(\alpha\lambda_1) = F(\alpha\lambda_2) = F(\alpha)F(\lambda_2)$ so that the monic property of $F(\alpha)$ yields $F(\lambda_1) = F(\lambda_2)$. Since F is faithful, we get $\lambda_1 = \lambda_2$. Therefore, α is monic. An analogous proof shows that F reflects epics and hence bimorphisms. This proves (ii). □

We end this section by giving examples to show that functors might not behave as expected.

4.2.5 *Example* (Image of functor not a subcategory) Let C and D be the categories depicted on the left and right, respectively (identity morphisms not shown):

$$\begin{tikzcd} & b_1 & \\ a \arrow[ur] & & c \\ & b_2 \arrow[ur] & \end{tikzcd} \qquad\qquad \begin{tikzcd} x \arrow[r] \arrow[rr, bend left] & y \arrow[r] & z \end{tikzcd}$$

Let $F : C \to D$ be the functor determined by the object map $a \mapsto x$, $b_1, b_2 \mapsto y$, $c \mapsto z$. The morphisms $x \to y$ and $y \to z$ are in the image of F, but their composition is not. Therefore, the image of F is not a subcategory of D. □

The following example shows that a faithful functor is not necessarily injective as an object map, nor as a morphism map. It also shows that a full functor is not necessarily surjective as an object map, nor as a morphism map.

4.2.6 *Example* Let C be the following category:

$$\begin{tikzcd} x \arrow[loop left] \arrow[r, shift left] & y \arrow[l, shift left] \arrow[loop right] \end{tikzcd} .$$

Let $F : C \to C$ be the functor determined by the object map $x, y \mapsto x$. Then F is faithful and full, but as a map of objects or a map of morphisms, it is neither injective nor surjective.

Here are less trivial examples: If (C, F) is a concrete category, then $F : C \to$ **Set** is faithful, but not necessarily injective on objects or morphisms (e.g., F : **Top** $\to$ **Set**, forgetful functor). If C is a full subcategory of a category D, then the inclusion functor $C \to D$ is full but not necessarily surjective on objects or morphisms (e.g., **Ab** $\to$ **Grp**). □

4.3 Category of small categories

A category is **small** if its class of objects is actually a set. For example, the category associated with a preordered set (Example 1.3.3) is small, as is the category $\mathbf{Mat}_R$ (Example 1.3.5). On the other hand, the categories **Set**, **Grp**, **Ring**, ${}_R\mathbf{Mod}$, **Top** are not small.

We get a category **Cat** by taking as objects all small categories, as morphisms between the objects C and D all functors from C to D, and as composition of morphisms composition of functors. For an object C, the identity morphism 1_C is the functor that sends each object to itself and each morphism to itself.

It is not possible to form a category by taking as objects *all* categories because this collection is not a class as required by the definition of category.

Notions defined for categories in general can be studied in the case of **Cat**. For instance, the following example shows the existence of a product of any two objects of this category. The construction does not require that the given categories be small.

4.3.1 *Example* (Product of categories) Let C_1 and C_2 be categories. The **Cartesian product** of C_1 and C_2 is the category $C = C_1 \times C_2$ having as objects all pairs (x_1, x_2) with $x_i \in C_i$ $(i = 1, 2)$ and as morphisms from (x_1, x_2) to (y_1, y_2) all pairs (α_1, α_2) with $\alpha_i : x_i \to y_i$ $(i = 1, 2)$ morphisms, and with composition given by $(\beta_1, \beta_2)(\alpha_1, \alpha_2) = (\beta_1\alpha_1, \beta_2\alpha_2)$.

For each i, let $F_i : C \to C_i$ be the functor defined on objects by $F_i((x_1, x_2)) = x_i$ and on morphisms by $F_i((\alpha_1, \alpha_2)) = \alpha_i$.

If C_1 and C_2 are small, and therefore objects of **Cat**, their Cartesian product $C = C_1 \times C_2$ is also small and the pair $(C, \{F_i\})$ is a product of the family $\{C_1, C_2\}$ in the category **Cat**.

(This construction extends to the case of an arbitrary family $\{C_i\}_{i \in I}$ of categories in a fashion analogous to the construction in Example 3.3.1.)

$\square$

4.4 Contravariant functor

Let C and D be categories. A **contravariant functor** from C to D is a function that sends each object x of C to an object $F(x)$ of D and sends each morphism α in C to a morphism $F(\alpha)$ in D such that the following hold:

(i) If $\alpha : x \to y$ is a morphism in C, then $F(\alpha) : F(y) \to F(x)$,

(ii) $F(\beta\alpha) = F(\alpha)F(\beta)$ for all morphisms α and β in C for which $\beta\alpha$ is defined.

(iii) $F(1_c) = 1_{F(c)}$ for each $c \in C$.

(Note the reversals in (i) and (ii).)

The functor of Section 4.1 is sometimes referred to as a "covariant functor" to distinguish it from a contravariant functor. The notation $F : C \to D$ is used for both types of functors.

4.4.1 *Example* (Contravariant Hom functor) Let C be a category and let c be a fixed object of C. We get a contravariant functor $F : C \to \mathbf{Set}$ as follows: For an object x of C we put $F(x) = C(x, c)$ and for a morphism $\alpha : x \to y$ in C we define $F(\alpha) : F(y) \to F(x)$ (that is, $F(\alpha) : C(y, c) \to C(x, c)$) by $F(\alpha)(\beta) = \beta\alpha$. This contravariant functor is the **contravariant Hom functor** corresponding to c (cf. Example 4.1.2).

This contravariant functor is often denoted by $C(\_, c) : C \to \mathbf{Set}$. When this notation is used, the image of the morphism $\alpha : x \to y$ is denoted by $C(\alpha, c)$, or often more simply by α^* (so that $\alpha^*(\beta) = \beta\alpha$ for a morphism $\beta : y \to c$). □

4.4.2 *Example* (Dual space contravariant functor) Let K be a field and recall that $\mathbf{Vect}_K$ denotes the category of vector spaces over K. Since K is an object of $\mathbf{Vect}_K$ we can form the contravariant Hom functor corresponding to K, namely, $\mathbf{Vect}_K(\_, K) : \mathbf{Vect}_K \to \mathbf{Set}$ (see Example 4.4.1). This contravariant functor is usually denoted by $^* : \mathbf{Vect}_K \to \mathbf{Set}$ with object map $V \mapsto V^*$ and morphism map $\alpha \mapsto \alpha^*$. Note that $(\beta\alpha)^* = \alpha^*\beta^*$ for any morphisms α and β for which the composition on the left is defined.

For a vector space V, the set $V^* = \mathbf{Vect}_K(V, K)$ of linear transformations from V to K is a vector space: For $\alpha, \beta \in V^*$ and $s \in K$ the map $\alpha + \beta$ is defined by $(\alpha+\beta)(v) = \alpha(v) + \beta(v)$ and the map $s\alpha$ is defined by $(s\alpha)(v) = s\alpha(v)$ $(v \in V)$. Moreover, if $\alpha : V \to W$ is a linear transformation, then $\alpha^* : W^* \to V^*$ (given by $\alpha^*(\beta) = \beta\alpha$) is linear. Therefore, the contravariant functor * actually maps into the category $\mathbf{Vect}_K$, that is, $^* : \mathbf{Vect}_K \to \mathbf{Vect}_K$. This is the **dual space contravariant functor**. □

4.4.3 *Example* (Contravariant power set functor) The **contravariant power set functor** $P' : \mathbf{Set} \to \mathbf{Set}$ is defined as follows: For an object X, put

$$P'(X) = \{\, S |\, S \subseteq X\}$$

(the power set of X), and for a morphism $\alpha : X \to Y$ define $P'(\alpha) : P'(Y) \to P'(X)$ by $P'(\alpha)(T) = \alpha^{-1}[T]$. (Cf. Example 4.1.4.) □

We introduce the notion of "opposite category" in order to give another interpretation of a contravariant functor.

The **opposite category** of the category C is the category C^{op} with objects the same as the objects of C, but with arrows and composition reversed:

(i) $C^{\mathrm{op}} = C$;

(ii) For $x, y \in C^{\mathrm{op}}$, $C^{\mathrm{op}}(x, y) = C(y, x)$;

(iii) For morphisms α and β in C^{op}, the composition $\beta\alpha$ in C^{op} is the composition $\alpha\beta$ in C (when defined).

A contravariant functor $F : C \to D$ can be regarded as a (covariant) functor $F : C^{\mathrm{op}} \to D$ (same object map and same morphism map), and vice versa.

Finally, if $\alpha : X \to Y$ is a morphism in the category **Set**, then the morphism $\alpha : Y \to X$ in $\mathbf{Set}^{\mathrm{op}}$ is not a function from Y to X in general. Because of this, the following result might be unexpected.

4.4.4 Theorem. *If (C, F) is a concrete category, then (C^{op}, F') is a concrete category for some (faithful) functor $F' : C^{\mathrm{op}} \to \mathbf{Set}$.*

Proof. Let (C, F) be a concrete category. The functor $F : C \to \mathbf{Set}$ induces a faithful (covariant) functor $F : C^{\mathrm{op}} \to \mathbf{Set}^{\mathrm{op}}$ (same object map and same morphism map). The contravariant power set functor P' of Example 4.4.3 can be regarded as a covariant functor $P' : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$.

Claim: P' is faithful. Let $\alpha, \beta : Y \to X$ be morphisms in $\mathbf{Set}^{\mathrm{op}}$ and assume that $\alpha \neq \beta$. Then the functions $\alpha, \beta : X \to Y$ are not equal, implying that $\alpha(x) \neq \beta(x)$ for some $x \in X$. Putting $T = Y \backslash \{\alpha(x)\} \in P'(Y)$ we have $x \notin \alpha^{-1}[T] = P'(\alpha)(T)$, but $x \in \beta^{-1}[T] = P'(\beta)(T)$. Hence $P'(\alpha)(T) \neq P'(\beta)(T)$, yielding $P'(\alpha) \neq P'(\beta)$. The claim follows.

Finally, the composition $F' = P'F : C^{\mathrm{op}} \to \mathbf{Set}$ is a faithful functor as desired. $\square$

4.5 Isomorphic categories

Although it is not possible, due to Russell's paradox, to form the "category of all categories" (with functors as morphisms), such a visualization at least suggests useful notions, such as the following.

A functor $F : C \to D$ is an **isomorphism** if there exists a functor $G : D \to C$ such that $GF = 1_C$ and $FG = 1_D$. Two categories C and D are **isomorphic**, written $C \cong D$, if there exists an isomorphism $C \to D$. In the category **Cat** of small categories, this definition coincides with the definition of isomorphic objects (cf. Section 2.5).

4.5.1 *Example* (${}_{\mathbf{Z}}\mathbf{Mod} \cong \mathbf{Ab}$) The forgetful functor $F : {}_{\mathbf{Z}}\mathbf{Mod} \to \mathbf{Ab}$ that sends each **Z**-module to its underlying abelian group and each morphism to itself is an isomorphism. Indeed, in the definition we can take $G : \mathbf{Ab} \to {}_{\mathbf{Z}}\mathbf{Mod}$ to be the functor that sends each abelian group to itself viewed as a $\mathbf{Z}$-module in the natural way and each morphism to itself. $\square$

An isomorphism of categories is seen to be a bijection when viewed just as a function on objects. In particular, the two categories

$$\begin{tikzcd} x \arrow[loop left, "1_x"] \end{tikzcd} \qquad\qquad \begin{tikzcd} y \arrow[loop left, "1_y"] \arrow[r, shift left, "\alpha"] & z \arrow[l, shift left, "\beta"] \arrow[loop right, "1_z"] \end{tikzcd}$$

are not isomorphic. However, the two objects y and z in the category on the right are isomorphic since $\beta\alpha = 1_y$ and $\alpha\beta = 1_z$, so in a sense these categories are the same (one object, one morphism). There is a relation weaker than isomorphic that two categories satisfy if they are the same except for differences due to isomorphisms, such as in this example. A rigorous definition of this weaker relation, "equivalence of categories," requires the notion "natural transformation," which we turn to next.

4 – Exercises

4–1 Let C be a small category (see Section 4.3). Prove that there exists a faithful functor $F : C \to \mathbf{Set}$ (so that the pair (C, F) is concrete).

4–2 Let C be a category and let c be a fixed object of C. Prove that the (covariant) Hom functor $C(c, \_) : C \to \mathbf{Set}$ preserves pullbacks. (Cf. Example 4.1.2.)

4–3 Prove the following statements:

(a) A full and essentially surjective functor preserves terminal objects.

(b) A faithful and full functor reflects terminal objects.

(Note: The same statements hold with "initial" replacing "terminal.")

5 Natural transformation

5.1 Definition and examples

Let $F, G : C \to D$ be two functors. A **natural transformation** η from the functor F to the functor G, denoted $\eta : F \to G$, consists of a family of morphisms $\eta_x : F(x) \to G(x)$, one for each object x of C, such that for each morphism $\alpha : x \to y$ in C the diagram on the right is commutative:

$$\begin{tikzcd} x \arrow[d, "\alpha"'] \\ y \end{tikzcd} \qquad \begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(\alpha)"'] & G(x) \arrow[d, "G(\alpha)"] \\ F(y) \arrow[r, "\eta_y"'] & G(y) \end{tikzcd} .$$

5.1.1 *Example* (Determinant as natural transformation) Let $\alpha : R \to S$ be a morphism in the category **CRing** of commutative rings with identity (so in particular $\alpha(1) = 1$). For any positive integer n we get an induced group homomorphism $\varphi_\alpha : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ given by $\varphi_\alpha([a_{ij}]) = [\alpha(a_{ij})]$ (i.e., apply α to each matrix entry). (Here, $\mathrm{GL}_n(R)$ denotes the group of all invertible $n \times n$ matrices over R, which is the same as the group of all $n \times n$ matrices over R having determinant a unit in R.)

Fix a positive integer n. Define a functor $F : \mathbf{CRing} \to \mathbf{Grp}$ by $F(R) = \mathrm{GL}_n(R)$, $F(\alpha) = \varphi_\alpha$. Also, define a functor $G : \mathbf{CRing} \to \mathbf{Grp}$ by $G(R) = R^\times$, $G(\alpha) = \alpha|_{R^\times}$, where $R^\times$ denotes the group of units of R under multiplication and $\alpha : R \to S$ is a morphism in **CRing**. For an object R of **CRing** let $\eta_R = \det_R : \mathrm{GL}_n(R) \to R^\times$ be the determinant function.

We claim that $\eta : F \to G$ is a natural transformation. Let $\alpha : R \to S$ be a morphism in **CRing**. The diagram in the definition is

$$\begin{tikzcd} R \arrow[d, "\alpha"'] \\ S \end{tikzcd} \qquad \begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\det_R"] \arrow[d, "\varphi_\alpha"'] & R^\times \arrow[d, "\alpha|_{R^\times}"] \\ \mathrm{GL}_n(S) \arrow[r, "\det_S"'] & S^\times \end{tikzcd} .$$

For any $A = [a_{ij}] \in \mathrm{GL}_n(R)$ we have, using that $\alpha(\pm 1) = \pm 1$,

$$\begin{aligned}\alpha|_{R^\times} \det_R(A) &= \alpha\Big(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}\Big) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \alpha(a_{i\sigma(i)}) \\ &= \det_S \varphi_\alpha(A),\end{aligned}$$

where S_n is the symmetric group on $\{1, \dots, n\}$ and $\operatorname{sgn}(\sigma)$ is 1 or -1 according as σ is even or odd. Therefore, $\alpha|_{R^\times} \det_R = \det_S \varphi_\alpha$ and the diagram on the right is commutative as desired. $\square$

5.1.2 *Example* (Double dual vector space) Let K be a field. For a vector space V over K, denote by V^* the dual space of V (cf. Example 4.4.2). Thus, V^* is the vector space of all linear maps $V \to K$. For a linear map $\alpha : V \to W$, the dual map $\alpha^* : W^* \to V^*$ is defined by $\alpha^*(f) = f\alpha$. Note that for such an α we have $\alpha^{**} : V^{**} \to W^{**}$.

Define a functor $G : \mathbf{Vect}_K \to \mathbf{Vect}_K$ by $G(V) = V^{**}$, $G(\alpha) = \alpha^{**}$, and let $F : \mathbf{Vect}_K \to \mathbf{Vect}_K$ be the identity functor. For a vector space V let $\eta_V : V \to V^{**}$ be the linear map given by $[\eta_V(v)](f) = f(v)$.

We claim that $\eta : F \to G$ is a natural transformation. Let $\alpha : V \to W$ be a linear map. The diagram in the definition is

$$\begin{tikzcd} V \arrow[d, "\alpha"'] & V \arrow[r, "\eta_V"] \arrow[d, "\alpha"'] & V^{**} \arrow[d, "\alpha^{**}"] \\ W & W \arrow[r, "\eta_W"'] & W^{**} . \end{tikzcd}$$

For every $v \in V$ and $f \in W^*$ we have

$$\begin{aligned}[\alpha^{**}\eta_V(v)](f) &= [\eta_V(v)\alpha^*](f) = [\eta_V(v)](\alpha^*(f)) = \alpha^*(f)(v) \\ &= (f\alpha)(v) = f(\alpha(v)) = [\eta_W(\alpha(v))](f) = [\eta_W\alpha(v)](f).\end{aligned}$$

Therefore, $\alpha^{**}\eta_V = \eta_W\alpha$ and the diagram on the right is commutative as desired. $\square$

5.2 Functor category

Let C and D be categories. Let $F, G, H : C \to D$ be three functors from C to D and let $\eta : F \to G$, $\theta : G \to H$ be natural transformations. We define the composition $\theta\eta : F \to H$ by $(\theta\eta)_x = \theta_x\eta_x$ $(x \in C)$. This composition is a natural transformation since for any morphism $\alpha : x \to y$ in C the large rectangle in the diagram on the right is commutative:

$$\begin{tikzcd} x \arrow[d, "\alpha"'] \\ y \end{tikzcd} \qquad \begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(\alpha)"'] & G(x) \arrow[r, "\theta_x"] \arrow[d, "G(\alpha)"] & H(x) \arrow[d, "H(\alpha)"] \\ F(y) \arrow[r, "\eta_y"'] & G(y) \arrow[r, "\theta_y"'] & H(y) \end{tikzcd} .$$

Composition of natural transformations defined this way is associative (since it is associative objectwise).

For any functor $F : C \to D$ we get a natural transformation $1_F : F \to F$ by defining $(1_F)_x = 1_{F(x)} : F(x) \to F(x)$ $(x \in C)$; this natural transformation is the **identity natural transformation** on F.

Assume that C is small, meaning that its class of objects is a set. The **functor category** D^C from C to D is the category with objects all functors from C to D and morphisms from the object F to the object G the set of all natural transformations from F to G (our assumption that C is small guarantees that this collection is actually a set). Composition of morphisms is as defined above. In particular, identity morphisms exist as required.

5.2.1 *Example* In this example, we define the category of linear representations of a group and show that it can be regarded as a functor category.

Let G be a group and let K be a field. A homomorphism $\rho : G \to \mathrm{GL}(V)$, with V a vector space over K, is called a K-linear representation of G. The collection of all such is the class of objects of a category $\mathbf{Rep}_K(G)$. A morphism $\varphi : \rho \to \rho'$ from the object $\rho : G \to \mathrm{GL}(V)$ to the object $\rho' : G \to \mathrm{GL}(V')$ in this category is a linear map $\varphi : V \to V'$ such that $\varphi\rho(g) = \rho'(g)\varphi$ for all $g \in G$. (One can regard V as a G-set with action $gv = \rho(g)(v)$ $(g \in G,\ v \in V)$ and similarly for V'. Then the morphism φ satisfies $\varphi(gv) = g\varphi(v)$ for all $g \in G$ and $v \in V$, that is, it is a G-homomorphism.)

Let $C = C_G$ be the category associated with the group G as in Example 1.3.6. This category has a single object $\bullet$ and $C(\bullet, \bullet) = G$. Put $D = \mathbf{Vect}_K$. Let F be an object of the functor category D^C, that is, a functor from C to D. Put $V = F(\bullet)$ (= a vector space over K). Since F is a functor, it follows that its restriction to morphisms is a group homomorphism $F : G \to \mathrm{GL}(V)$, that is, an object of $\mathbf{Rep}_K(G)$.

Let F' be another object of D^C and put $V' = F'(\bullet)$. Let $\eta : F \to F'$ be a morphism in D^C, i.e., a natural transformation from F to F', and put $\varphi = \eta_\bullet$. For each $g \in G$, commutativity of the diagram on the right

$$\begin{tikzcd} \bullet \arrow[d, "g"'] \\ \bullet \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "F(g)"'] & V' \arrow[d, "F'(g)"] \\ V \arrow[r, "\varphi"'] & V' \end{tikzcd}$$

says that φ is a morphism in $\mathbf{Rep}_K(G)$ from F to F'. In short, the functor category D^C is essentially the category of K-linear representations of G. □

5.3 Natural isomorphism

Let $F, G : C \to D$ be two functors and let $\eta : F \to G$ be a natural transformation. We say that η is a **natural isomorphism** if there exists a natural transformation $\theta : G \to F$ such that $\theta\eta = 1_F$ and $\eta\theta = 1_G$. (In the case that C is small, η is a natural isomorphism if and only if it is an isomorphism in the functor category D^C.) Equivalently, η is a natural isomorphism if and only if each η_x ($x \in C$) is an isomorphism.

The functors $F, G : C \to D$ are **isomorphic**, written $F \cong G$, if there exists a natural isomorphism from F to G (which, in the case C is small, is the same as saying F and G are isomorphic objects of the functor category D^C).

5.3.1 *Example* The map $\eta_V : V \to V^{**}$ of Example 5.1.2 is an isomorphism if and only if V is finite-dimensional, so η is not a natural isomorphism as is. However, if the category $\mathbf{Vect}_K$ in that example is replaced by the category of *finite-dimensional* vector spaces over K, then the resulting η will be a natural isomorphism. □

5.3.2 *Example* Let G be a group. The "opposite group" of G is the group G^{op} having the same underlying set as that of G but with operation $a \cdot b = ba$ ($a, b \in G^{\mathrm{op}}$), where the product on the right is that in G. Let $F : \mathbf{Grp} \to \mathbf{Grp}$ be the functor given by $F(G) = G^{\mathrm{op}}$, $F(\varphi) = \varphi$. For a group G, define $\eta_G : G \to G^{\mathrm{op}}$ by $\eta_G(a) = a^{-1}$. Then η is a natural isomorphism from the identity functor on **Grp** to the functor F. $\square$

5 – Exercises

5–1 The **center** $Z(C)$ of a category C is the class of all natural transformations $\eta : 1_C \to 1_C$, where 1_C is the identity functor on C. Let R be a ring with identity and put $C = {}_R\mathbf{Mod}$. Prove that there is a bijection $Z(R) \to Z(C)$, where $Z(R)$ is the center of R, that is, $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$.

5–2 Let

$$B \xrightarrow{J} C \xrightarrow{F,G} D \xrightarrow{K} E$$

be functors and let $\eta : F \to G$ be a natural transformation.

(a) Prove that $\eta J : FJ \to GJ$ given by $(\eta J)_b = \eta_{J(b)}$ is a natural transformation.

(b) Prove that $K\eta : KF \to KG$ given by $(K\eta)_c = K(\eta_c)$ is a natural transformation.

6 Equivalence of categories

6.1 Definition and examples

Let C and D be categories. A functor $F : C \to D$ is an **equivalence** if there exists a functor $G : D \to C$ such that $GF \cong 1_C$ and $FG \cong 1_D$.

The categories C and D are **equivalent** if there exists an equivalence from C to D. This defines an equivalence relation on the collection of categories.

The notion of equivalent categories is weaker than that of isomorphic categories (i.e., isomorphic categories are equivalent, but not conversely).

6.1.1 *Example* ($\mathbf{Mat}_K$ is equivalent to $\mathbf{FVect}_K$) Let K be a field. In this example, we show that the category $C = \mathbf{Mat}_K$ of Example 1.3.5 and the category $D = \mathbf{FVect}_K$ of finite-dimensional vector spaces over K are equivalent.

We get a functor $F : C \to D$ by defining $F(n) = K^n$ (= space of n-dimensional column vectors over K) for each object n of C and $F(A) = \mu_A$ for each morphism $A : n \to n'$ in C, where $\mu_A : K^n \to K^{n'}$ is the linear map given by $\mu_A(v) = Av$.

Choose a basis for each finite-dimensional vector space, with the choice for each vector space K^n ($n \in \mathbf{N}$) being the standard basis. We get a functor $G : D \to C$ by defining $G(V) = \dim V$ for each object V of D and $G(\alpha) = M_\alpha$ for each morphism $\alpha : V \to V'$ in D, where M_α is the matrix of α relative to the chosen bases of V and V'.

For a finite-dimensional vector space V, define $\eta_V : V \to K^{\dim V}$ by $\eta_V(v) = [v]$, where $[v]$ is the coordinate vector of v relative to the chosen basis of V. We claim that $\eta : 1_D \to FG$ is a natural isomorphism. Since η_V is an isomorphism for each V it suffices to check the naturality condition. Let $\alpha : V \to V'$ be a morphism in D and put $n = \dim V$, $n' = \dim V'$. We check commutativity of the diagram on the right:

$$\begin{tikzcd} V \arrow[d, "\alpha"'] & V \arrow[r, "\eta_V"] \arrow[d, "\alpha"'] & K^n \arrow[d, "FG(\alpha)"] \\ V' & V' \arrow[r, "\eta_{V'}"'] & K^{n'} . \end{tikzcd}$$

For $v \in V$ we have

$$(\eta_{V'}\alpha)(v) = [\alpha(v)] = M_\alpha[v] = \mu_{M_\alpha}([v]) = (FG(\alpha)\eta_V)(v),$$

where the second equality uses the definition of M_α. Therefore $\eta_{V'}\alpha = FG(\alpha)\eta_V$ as desired. We conclude that $FG \cong 1_D$.

On the other hand, we have $GF = 1_C$ (implying $GF \cong 1_C$). Indeed, for $n \in C$, we have $GF(n) = \dim K^n = n = 1_C(n)$ and for a morphism $A : n \to n'$ in C we have $GF(A) = M_{\mu_A} = A = 1_C(A)$, the second equality using our assumption that the chosen basis for K^n (resp., $K^{n'}$) is the standard basis.

We have shown that $F : C \to D$ is an equivalence, so $\mathbf{Mat}_K$ is equivalent to $\mathbf{FVect}_K$. $\square$

6.1.2 *Example* (**Met** equivalent to **MTop**) Here we show that the category $C = \mathbf{Met}$ of metric spaces is equivalent to the category $D = \mathbf{MTop}$ of metrizable topological spaces, both with continuous maps as morphisms.

We get a functor $F : C \to D$ by letting $F((X, d)) = (X, \tau_d)$ for each object (X, d) of C, where τ_d denotes the topology on X induced by the metric d, and by sending each morphism in C to itself.

We get a functor $G : D \to C$ by letting $G((X, \tau)) = (X, d_\tau)$ for each object (X, τ) of D, where d_τ is a choice of metric on X that induces the topology τ (such a metric exists since (X, τ) is metrizable), and by sending each morphism in D to itself.

We claim that $GF \cong 1_C$. For each object (X, d) of C, the metric d_{τ_d} induces the topology τ_d, which is the topology induced by d. In other words, the identity map on X is a homeomorphism $\eta_{(X,d)} : (X, d) \to (X, d_{\tau_d})$. This gives a natural isomorphism $\eta : 1_C \to GF$, the naturality following at once from the fact that each component map $\eta_{(X,d)}$ is the identity. We conclude that $GF \cong 1_C$.

On the other hand, we have $FG = 1_D$ (so that $FG \cong 1_D$). Indeed, for an object (X, τ) of D, the topology τ_{d_τ} equals τ by the definitions, so $FG((X, \tau)) = (X, \tau_{d_\tau}) = (X, \tau) = 1_D((X, \tau))$.

We have shown that F is an equivalence, so **Met** and **MTop** are equivalent. $\square$

6.2 Characterization of Equivalence

Recall (Section 4.2) that a functor $F : C \to D$ is **essentially surjective** if each object of D is isomorphic to $F(x)$ for some $x \in C$.

6.2.1 Theorem. *A functor is an equivalence if and only if it is faithful, full, and essentially surjective.*

Proof. Let $F : C \to D$ be a functor.

($\Rightarrow$) Assume that F is an equivalence, so that there exists a functor $G : D \to C$ and natural isomorphisms $\eta : GF \to 1_C$ and $\theta : FG \to 1_D$.

We first prove that F is faithful. Let $\alpha_1, \alpha_2 : x \to y$ be morphisms in C and assume that $F(\alpha_1) = F(\alpha_2)$. The diagram on the right is commutative for each $i = 1, 2$:

$$\begin{tikzcd} x \arrow[d, "\alpha_i"'] \\ y \end{tikzcd} \qquad \begin{tikzcd} GF(x) \arrow[r, "\eta_x"] \arrow[d, "GF(\alpha_i)"'] & x \arrow[d, "\alpha_i"] \\ GF(y) \arrow[r, "\eta_y"'] & y \end{tikzcd}.$$

We have $\alpha_1\eta_x = \eta_y GF(\alpha_1) = \eta_y GF(\alpha_2) = \alpha_2\eta_x$. Since η_x is an isomorphism and hence epic (Theorems 2.5.1 and 2.4.1), we get $\alpha_1 = \alpha_2$, so F is faithful. For future reference, we observe that a similar proof shows that G is faithful as well.

Now we show that F is full. Let $x, y \in C$ and let $\beta : F(x) \to F(y)$ be a morphism in D. Put $\alpha = \eta_y G(\beta)\eta_x^{-1}$. From the commutativity of the diagram on the right

$$\begin{tikzcd} x \arrow[d, "\alpha"'] \\ y \end{tikzcd} \qquad \begin{tikzcd} GF(x) \arrow[r, "\eta_x"] \arrow[d, "GF(\alpha)"'] & x \arrow[d, "\alpha"] \\ GF(y) \arrow[r, "\eta_y"'] & y \end{tikzcd}$$

we get $\eta_y G(\beta) = \alpha\eta_x = \eta_y GF(\alpha)$. Since η_y is an isomorphism, it is monic, giving $G(\beta) = G(F(\alpha))$. As observed above, the functor G is faithful, so $\beta = F(\alpha)$, and thus F is full.

Finally, if d is an object of D, then $c := G(d)$ is an object of C and $d = 1_D(d) \cong F(G(d)) = F(c)$, the isomorphism being given by θ_d^{-1}. Hence, F is essentially surjective.

($\Leftarrow$) Assume that F is faithful, full, and essentially surjective. Due to this last property, for every object d of D there exists an object $G(d)$ of C and an isomorphism $\theta_d : F(G(d)) \to d$. Let $\beta : x \to y$ be a morphism in D. We have $\theta_y^{-1}\beta\theta_x : F(G(x)) \to F(G(y))$, so there exists ($F$ is full) a unique (F is faithful) morphism $G(\beta) : G(x) \to G(y)$ such that $F(G(\beta)) = \theta_y^{-1}\beta\theta_x$. This gives commutativity of the diagram on the right:

$$
\begin{tikzcd}
x \arrow[d, "\beta"'] \\
y
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(x) \arrow[r, "\theta_x"] \arrow[d, "FG(\beta)"'] & x \arrow[d, "\beta"] \\
FG(y) \arrow[r, "\theta_y"'] & y\,.
\end{tikzcd}
$$

So far we have a map $G : D \to C$ of categories. We claim that G is a functor. Given morphisms $\alpha : x \to y$ and $\beta : y \to z$ in D, we have

$$
\begin{aligned}
F(G(\beta)G(\alpha)) = FG(\beta)FG(\alpha) &= \theta_z^{-1}\beta\theta_y\,\theta_y^{-1}\alpha\theta_x \\
&= \theta_z^{-1}\beta\alpha\theta_x = F(G(\beta\alpha)),
\end{aligned}
$$

so $G(\beta\alpha) = G(\beta)G(\alpha)$, since F is faithful. Also, for every object d of D we have

$$F(G(1_d)) = \theta_d^{-1}1_d\theta_d = 1_{FG(d)} = F(1_{G(d)}),$$

so $G(1_d) = 1_{G(d)}$, using again that F is faithful. Therefore, G is a functor, and the forgoing discussion shows that $\theta : FG \to 1_D$ is a natural isomorphism, giving $FG \cong 1_D$.

Finally, we show that $GF \cong 1_C$. For an object c of C we have $\theta_{F(c)} : FGF(c) \to F(c)$, so there exists a unique morphism $\eta_c : GF(c) \to c$ such that $F(\eta_c) = \theta_{F(c)}$ since F is faithful and full. Note that for each $c \in C$ the morphism η_c is an isomorphism since $\theta_{F(c)}$ is an isomorphism and F, being faithful and full, reflects isomorphisms (Theorem 4.2.2). For a morphism $\alpha : x \to y$ in C, if we apply the functor F to the diagram on the right

$$
\begin{tikzcd}
x \arrow[d, "\alpha"'] \\
y
\end{tikzcd}
\qquad
\begin{tikzcd}
GF(x) \arrow[r, "\eta_x"] \arrow[d, "GF(\alpha)"'] & x \arrow[d, "\alpha"] \\
GF(y) \arrow[r, "\eta_y"'] & y
\end{tikzcd}
$$

we get a commutative diagram since $\theta : FG \to 1_D$ is a natural transformation. But F is faithful so it reflects commutative diagrams (Theorem 4.2.1), so the diagram on the right above is commutative as well. We conclude that $\eta : GF \to 1_C$ is a natural isomorphism, so that $GF \cong 1_C$.

Therefore, F is an equivalence. $\square$

6.2.2 Corollary. *An equivalence of categories preserves and reflects the following: monics, epics, bimorphisms, split monics, split epics, isomorphisms, and commutative diagrams.*

6.3 Skeleton of category

A category C is **skeletal** if a morphism $\alpha : x \to y$ in C is an isomorphism only if $x = y$, i.e., if no two distinct objects of C are isomorphic.

6.3.1 Theorem. *Two skeletal categories are equivalent if and only if they are isomorphic.*

Proof. Let C and D be skeletal categories.

Assume that C and D are equivalent. Then there exist functors $F : C \to D$ and $G : D \to C$ and natural isomorphisms $\eta : GF \to 1_C$ and $\theta : FG \to 1_D$. For every object c of C the morphism $\eta_c : GF(c) \to c$ is an isomorphism, so $GF(c) = c$ since C is skeletal. Similarly, using θ we get $FG(d) = d$ for every object d of D. Therefore, as a map of objects F is a bijection and G is its inverse.

Let $G' : D \to C$ be the same as G on objects and for a morphism $\beta : x \to y$ in D let $G'(\beta) = \eta_{G(y)} G(\beta) \eta^{-1}_{G(x)}$. In this definition, the codomain of $\eta^{-1}_{G(x)}$ is $GFG(x) = G(x)$ and the domain of $\eta_{G(y)}$ is $GFG(y) = G(y)$, so the indicated composition is defined. A routine check shows that G' is a functor. For any morphism $\alpha : a \to b$ in C we have

$$G'F(\alpha) = G'(F(\alpha)) = \eta_b GF(\alpha)\eta_a^{-1} = \alpha,$$

using that $\eta : GF \to 1_C$ is a natural transformation for the last equality. Let $\beta : x \to y$ be a morphism in D. Since $x = F(G(x))$ and $y = F(G(y))$ and the functor F is full (Theorem 6.2.1), there exists a morphism $\alpha : G(x) \to G(y)$ in C such that $F(\alpha) = \beta$. Using the equation above we get

$$FG'(\beta) = FG'F(\alpha) = F(\alpha) = \beta.$$

We conclude that $G'F = 1_C$ and $FG' = 1_D$, so C is isomorphic to D.

The converse is immediate. □

A **skeleton** of a category C is a full subcategory C_0 of C that is skeletal and has the property that each object of C is isomorphic to some object of C_0.

6.3.2 Theorem. *Let C be a category.*

(i) *C has a skeleton.*

(ii) *C is equivalent to each of its skeletons.*

Proof. (i) Form a class C_0 by choosing an object from each isomorphism class of objects in C. (This requires the axiom of choice.) The full subcategory of C with object class C_0 is a skeleton of C.

(ii) Let C_0 be a skeleton of C. The inclusion functor $F : C_0 \to C$ is faithful (as is any inclusion functor) and it is full and essentially surjective by the definition of skeleton. By Theorem 6.2.1, F is an equivalence so that C_0 and C are equivalent. □

The following corollary says roughly that two categories are equivalent if and only if they are identical except possibly for differences due to isomorphisms between objects.

6.3.3 Corollary. *Two categories are equivalent if and only if they have isomorphic skeletons.*

Proof. This follows immediately from Theorems 6.3.2 and 6.3.1 using transitivity of equivalence. □

6 – Exercises

6–1 Prove that a composition of equivalences is an equivalence.

6–2 For a functor $F : C \to D$ and a small category A define a functor $F^A : C^A \to D^A$ by $F^A(f) = Ff$ for $f \in C^A$ and $F^A(\nu)_a = F(\nu_a)$ for ν a morphism in C^A. Prove that a functor F is an equivalence if and only if F^A is an equivalence for each small category A.

7 Universal

7.1 Motivation

Let V be a vector space over a field K and let X be a basis of V. A fact from elementary linear algebra is that if W is a vector space and $\alpha : X \to W$ is a function, then there exists a unique linear map $\gamma : V \to W$ that extends α. The map γ is given by $\gamma(\sum_{x \in X} a_x x) = \sum_{x \in X} a_x \alpha(x)$, for any scalars $a_x \in K$ with $a_x = 0$ for all but finitely many x.

We can state this fact using category terms. Let $F : \mathbf{Vect}_K \to \mathbf{Set}$ be the forgetful functor and let $\iota : X \to F(V)$ be the inclusion map. The fact just stated can be expressed as follows: For the object X of **Set**, the pair (V, ι), with V an object of $\mathbf{Vect}_K$ and $\iota : X \to F(V)$ a morphism in **Set**, has the property that for any object W of $\mathbf{Vect}_K$ and any morphism $\alpha : X \to F(W)$ in **Set**, there exists a unique morphism $\gamma : V \to W$ in $\mathbf{Vect}_K$ such that the diagram on the right

$$\begin{tikzcd} V \arrow[r, dotted, "\exists!\,\gamma"] & W \end{tikzcd} \qquad \begin{tikzcd} F(V) \arrow[rr, dotted, "F(\gamma)"] & & F(W) \\ & X \arrow[ul, "\iota"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

is commutative, that is, $F(\gamma)\iota = \alpha$.

7.2 Universal from object to functor

Let $F : C \to D$ be a functor and let d be an object of D. Let $(d \downarrow F)$ be the auxiliary category having as objects all pairs (x, α) with $x \in C$ and $\alpha : d \to F(x)$ a morphism in D, and with morphisms from the object (x, α) to the object (y, β) being all morphisms $\gamma : x \to y$ in C such that the diagram on the right

$$\begin{tikzcd} x \arrow[r, "\gamma"] & y \end{tikzcd} \qquad \begin{tikzcd} F(x) \arrow[rr, "F(\gamma)"] & & F(y) \\ & d \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

is commutative, that is, such that $F(\gamma)\alpha = \beta$. This category $(d \downarrow F)$ is called a **comma category** (terminology due to a notation that is no longer in use).

An initial object (u, ι) of $(d \downarrow F)$ is called a **universal from** d **to** F. By definition such an object has the property that for any object (c, α) of $(d \downarrow F)$ there exists a unique morphism $\gamma : u \to c$ in C such that the diagram on the right

$$
\begin{tikzcd}
u \arrow[r, dotted, "\exists!\,\gamma"] & c
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(u) \arrow[rr, dotted, "F(\gamma)"] & & F(c) \\
& d \arrow[ul, "\iota"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
$$

is commutative, that is, such that $F(\gamma)\iota = \alpha$.

By Theorem 3.1.2 a universal from d to F is unique up to isomorphism in $(d \downarrow F)$.

7.2.1 *Examples* (Free objects) When C is a concrete category and $F : C \to \mathbf{Set}$ is the forgetful functor, a universal (U, ι) from a set X to F is said to be a **free object** on the set X. Here are a few examples:

- ($C = \mathbf{Set}$) Let X be a set. Let $U = X$ and let $\iota : X \to F(U)$ be the identity map. Then (U, ι) is a free object on the set X in the category **Set**.

- ($C = {}_R\mathbf{Mod}$) Let R be a ring with identity and let X be a set. Let $U = \bigoplus_{x \in X} R_x$, where $R_x = R$ for each x (see Example 3.4.3). Define $\iota : X \to F(U)$ by $\iota(x)_y = \delta_{xy}$ (Kronecker delta). Then (U, ι) is a free object on the set X in the category ${}_R\mathbf{Mod}$. For an R-module M and a function $\alpha : X \to F(M)$ the required homomorphism $\gamma : U \to M$ is given by $\gamma(r) = \sum_{x \in X} r_x \alpha(x)$. (In the case R is a field, this is the usual construction of a vector space having the given set X as a basis and the universal mapping property described in Section 7.1.)

- ($C = \mathbf{Top}$) Let X be a set. Let $U = X$ endowed with the discrete topology and let $\iota : X \to F(U)$ be the identity map. Then (U, ι) is a free object on the set X in the category **Top**.

- ($C = \mathbf{Grp}$) Let X be a set. Let $X^{-1} = \{x^{-1} \,|\, x \in X\}$ be a set in one-to-one correspondence with X, the correspondence being given by $x \leftrightarrow x^{-1}$. Let U be the set of all reduced words $u_1 u_2 \cdots u_n$ with $n \in \mathbf{N} \cup \{0\}$, $u_i \in X \cup X^{-1}$, where "reduced" means that x is never adjacent to x^{-1}. The product on U given by juxtaposition followed by reduction (i.e., removal of factors x and x^{-1} if side by side) makes U a group (the associativity property is nontrivial to check). Let $\iota : X \to F(U)$ be the map $\iota(x) = x$. Then (U, ι) is a free object on the set X in the category **Grp**. The group U is called the **free group** on X.

- ($C =$ **CRing**) Let X be a set. Let $U = \mathbf{Z}[X]$ be the polynomial ring over $\mathbf{Z}$ in the set of indeterminates X and let $\iota : X \to F(U)$ be given by $\iota(x) = x$. Then (U, ι) is a free object on the set X in the category **CRing**. (More generally, if R is any commutative ring with identity and $U = R[X]$, then (U, ι) is a free object on the set X in the category of commutative R-algebras.)

□

7.2.2 *Example* (Field of Quotients) Let R be an integral domain and let $Q(R)$ be the field of quotients of R. Thus, $Q(R) = \{r/s \,|\, r, s \in R,\ s \neq 0\}$ with $r/s = t/u$ if and only if $ru = st$ and with usual addition and multiplication of fractions. Define $\iota : R \to Q(R)$ by $\iota(r) = r/1$.

The pair $(Q(R), \iota)$ satisfies the following property: If K is a field and $\alpha : R \to K$ is an injective ring homomorphism (sending identity to identity), then there is a unique field homomorphism $\gamma : Q(R) \to K$, given by $\gamma(r/s) = \alpha(r)/\alpha(s)$, such that $\gamma\iota = \alpha$. In other words, the pair $(Q(R), \iota)$ is a universal from R to the forgetful functor from the category of fields to the category of integral domains with injective homomorphisms as morphisms. □

7.3 Universal from functor to object

Let $G : D \to C$ be a functor and let c be an object of C. Let $(G \downarrow c)$ be the auxiliary category having as objects all pairs (x, α) with $x \in D$ and $\alpha : G(x) \to c$ a morphism in C, and with morphisms from the object (x, α) to the object (y, β) being all morphisms $\gamma : x \to y$ in D such that the diagram on the left

$$\begin{tikzcd} G(x) \arrow[rr, "G(\gamma)"] \arrow[dr, "\alpha"'] & & G(y) \arrow[dl, "\beta"] \\ & c & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\gamma"] & y \end{tikzcd}$$

is commutative, that is, such that $\beta G(\gamma) = \alpha$.

A terminal object (u, π) of $(G \downarrow c)$ is called a **universal from** G **to** c. By definition such an object has the property that for any object (d, α) of $(G \downarrow c)$ there exists a unique morphism $\gamma : d \to u$ in D such that the diagram on the left

$$\begin{tikzcd} G(d) \arrow[rr, dotted, "G(\gamma)"] \arrow[dr, "\alpha"'] & & G(u) \arrow[dl, "\pi"] \\ & c & \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, dotted, "\exists!\,\gamma"] & u \end{tikzcd}$$

is commutative, that is, such that $\pi G(\gamma) = \alpha$.

By Theorem 3.1.2 a universal from G to c is unique up to isomorphism in $(G \downarrow c)$.

7.3.1 *Example* (Cartesian product as universal) Let D be a category. The product category $D \times D$ was defined in Section 4.3.1. The "diagonal" functor $\triangle : D \to D \times D$ is defined on objects by $\triangle(x) = (x, x)$ and on morphisms by $\triangle(\alpha) = (\alpha, \alpha)$.

Now assume $D = \mathbf{Set}$. Let X_1 and X_2 be sets, put $X = X_1 \times X_2$, and let $\pi_i : X \to X_i$ $(i = 1, 2)$ be the usual projections. We have a morphism $\pi = (\pi_1, \pi_2) : (X, X) \to (X_1, X_2)$ in $D \times D$.

Claim: The pair (X, π) is a universal from the functor $\triangle$ to the object (X_1, X_2) of $D \times D$. First note that π is indeed a morphism from $\triangle(X)$ to the object (X_1, X_2) as required in the definition. Let Y be a set and let $\alpha : \triangle(Y) \to (X_1, X_2)$ be a morphism in $D \times D$. Then $\alpha = (\alpha_1, \alpha_2)$ for some maps $\alpha_i : Y \to X_i$ $(i = 1, 2)$. Now using the fact that the pair $(X, \{\pi_i\})$ is a product of the family $\{X_1, X_2\}$ (see Example 3.3.1), we get a unique map $\gamma : Y \to X$ such that $\pi_i \gamma = \alpha_i$ $(i = 1, 2)$, in other words, such that $\pi \triangle(\gamma) = (\pi_1, \pi_2)(\gamma, \gamma) = (\pi_1 \gamma, \pi_2 \gamma) = (\alpha_1, \alpha_2) = \alpha$. This establishes the claim.

(An analogous argument shows that a Cartesian product in any of the categories **Grp**, **Rng**, ${}_R$**Mod**, and **Top** can be regarded as a universal.) $\square$

7 – Exercises

7–1 Prove that for every group G there exists a universal from G to the inclusion functor $F : \mathbf{Ab} \to \mathbf{Grp}$.

8 Limit and colimit

8.1 Limit

Let $F : I \to C$ be a fixed functor. In this section, we use the notation $i \mapsto F_i$ and $\alpha \mapsto F_\alpha$ for the object and morphism maps, respectively, of F and regard $\{F_i\}_{i \in I}$ as a family of objects of C indexed by the objects of I. In the applications, the class of objects of I is usually a set (i.e., I is small) and this set is often even finite.

A **cone to** F is a pair (x, η) with x an object of C and η a family of morphisms $\eta_i : x \to F_i$, one for each object i of I, such that for each morphism $\tau : i \to j$ in I the diagram on the right

$$\begin{tikzcd} i \arrow[d, "\tau"] \\ j \end{tikzcd} \qquad \begin{tikzcd} & F_i \arrow[d, "F_\tau"] \\ x \arrow[ur, "\eta_i"] \arrow[dr, "\eta_j"'] & \\ & F_j \end{tikzcd}$$

is commutative, that is, $F_\tau \eta_i = \eta_j$.

Let $D = D_{\lim}$ be the auxiliary category having as objects all cones to F and as morphisms from the cone (x, η) to the cone (y, θ) all morphisms $\alpha : x \to y$ in C such that for each $i \in I$ the diagram

$$\begin{tikzcd} x \arrow[rr, "\alpha"] \arrow[dr, "\eta_i"'] & & y \arrow[dl, "\theta_i"] \\ & F_i & \end{tikzcd}$$

is commutative, that is, $\theta_i \alpha = \eta_i$.

A terminal object (u, π) of the category D is a **limit** of the functor F. By definition, such an object has the property that for any cone (x, η) to F there exists a unique morphism $\gamma : x \to u$ such that the diagram

$$\begin{tikzcd} x \arrow[rr, dotted, "\exists!\,\gamma"] \arrow[dr, "\eta_i"'] & & u \arrow[dl, "\pi_i"] \\ & F_i & \end{tikzcd}$$

is commutative, that is, $\pi_i \gamma = \eta_i$.

The object portion u of a limit (u, π) of F is often written $\varprojlim F$. The following visualization of a limit (u, π) of F is sometimes useful:

$$\begin{tikzcd}
 & & F_i \arrow[dd, "F_\tau"] & i \arrow[dd, "\tau"] \\
x \arrow[r, dotted, "\exists!\,\gamma"] \arrow[urr, bend left, "\eta_i"] \arrow[drr, bend right, "\eta_j"'] & u \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & & \\
 & & F_j & j
\end{tikzcd}$$

8.1.1 *Example* (Product as limit) Let $\{c_i\}_{i \in I}$ be a family of objects in the category C. Regard the set I as a discrete category (i.e., no nonidentity morphisms). We get a functor $F : I \to C$ with object map $i \mapsto c_i$ and with morphism map sending identities to identities. An object (x, η) in D_{lim} (i.e., a cone to F) is also an object of the auxiliary category D_{pr} in the definition of product of the family, and vice versa. Moreover, a morphism in C is a morphism in D_{lim} if and only if it is a morphism in D_{pr}. It follows that these two auxiliary categories are identical so that an object is a limit of F if and only if it is a product of the family $\{c_i\}$. □

8.1.2 *Example* (Pullback as limit) Let λ_1 and λ_2 be morphisms in the category C as shown on the left below. Let I be the category with object class $\{1, 2, 3\}$ and morphisms $\tau_1 : 1 \to 3$ and $\tau_2 : 2 \to 3$ (along with identity morphisms) as shown on the right:

$$\begin{tikzcd}
 & & & & \arrow[ll, bend right, "F"'] & & \\
 & a_2 \arrow[d, "\lambda_2"] & & & & & 2 \arrow[d, "\tau_2"] \\
a_1 \arrow[r, "\lambda_1"'] & a_3 & & & & 1 \arrow[r, "\tau_1"'] & 3\,.
\end{tikzcd}$$

We get a functor $F : I \to C$ with object map $i \mapsto a_i$ and morphism map $\tau_j \mapsto \lambda_j$ $(j = 1, 2)$ (sending identities to identities). If (x, η) is an object of D_{lim} (i.e., a cone to F),

$$\begin{tikzcd}
x \arrow[r, "\eta_2"] \arrow[d, "\eta_1"'] \arrow[dr, "\eta_3"] & a_2 \arrow[d, "\lambda_2"] \\
a_1 \arrow[r, "\lambda_1"'] & a_3
\end{tikzcd}$$

then $\lambda_1\eta_1 = \eta_3 = \lambda_2\eta_2$, so $(x, (\eta_1, \eta_2))$ is an object of the auxiliary category D_{pb} in the definition of pullback of the pair (λ_1, λ_2).

Conversely, an object $(x, (\eta_1, \eta_2))$ of D_{pb} determines an object (x, η) of D_{lim} with $\eta_3 = \lambda_1\eta_1$ $(= \lambda_2\eta_2)$. Moreover, a morphism in C is a morphism in D_{lim} if and only if it is a morphism in D_{pb}. This shows that the categories D_{lim} and D_{pb} are isomorphic and hence can be identified. Under this identification, an object is a limit of F if and only if it is a pullback of the pair (λ_1, λ_2). $\square$

A similar argument (Exercise 8–1) shows that an equalizer can be construed as a limit.

We say that the category C is **small complete** if for each functor $F : I \to C$ with I small there exists a limit of F.

8.1.3 *Example* (**Set** is small complete) Let I be a small category and let $F : I \to \mathbf{Set}$ be a functor. Put $P = \prod_{i \in I} F_i$. Define

$$U = \{x \in P \mid F_\tau \pi_i(x) = \pi_j(x) \text{ for every morphism } \tau : i \to j \text{ in } I\},$$

where $\pi_i : P \to F_i$ $(i \in I)$ is the projection map. For each $i \in I$, let $\hat{\pi}_i : U \to F_i$ be the restriction of π_i to U.

Claim: The pair $(U, \hat{\pi})$ is a limit of F. It is immediate from the definition of U that $(U, \hat{\pi})$ is a cone to F. Let (X, η) be a cone to F. Since the pair $(P, \{\pi_i\})$ is a product in the category **Set** (see Example 3.3.1) there exists a unique map $\alpha : X \to P$ such that $\pi_i\alpha = \eta_i$ for each i. For each $x \in X$ and morphism $\tau : i \to j$ in I, we have

$$F_\tau \pi_i(\alpha(x)) = F_\tau \eta_i(x) = \eta_j(x) = \pi_j(\alpha(x)),$$

so α maps into U and therefore provides the unique morphism in D_{lim} from (X, η) to (U, π) required in the definition. This establishes the claim. $\square$

8.1.4 *Example* (**Grp**, **Rng**, ${}_R\mathbf{Mod}$, and **Top** are small complete) Let $F : I \to C$ be a functor with I small and C one of the indicated categories. A limit of F is constructed just as in Example 8.1.3 with the aid of either Example 3.3.2 or Example 3.3.3, which shows the existence of products in C. All that needs to be checked is that U is a subobject of P and that the restrictions $\hat{\pi}_i$ of the projections are morphisms, and these checks are routine. (In the case $C = \mathbf{Top}$, the subset U is given the subspace topology and this guarantees that the maps $\hat{\pi}_i$ are continuous.) $\square$

8.1.5 *Example* (Ring of r-adic integers) Let I be the set $\mathbf{N} = \{1, 2, \dots\}$ of natural numbers viewed as a category as in Example 1.3.3 using the usual order. Fix $r \in \mathbf{N}$. We get a functor $F : I^{\mathrm{op}} \to \mathbf{Rng}$ by sending an object i of I to the ring $\mathbf{Z}/r^i\mathbf{Z}$ and sending a morphism $j \to i$ of I^{op} (so $i \leq j$) to the homomorphism $\kappa_{ij} : \mathbf{Z}/r^j\mathbf{Z} \to \mathbf{Z}/r^i\mathbf{Z}$ induced by the canonical epimorphism $\mathbf{Z} \to \mathbf{Z}/r^i\mathbf{Z}$.

As in 8.1.3, put $P = \prod_i \mathbf{Z}/r^i\mathbf{Z}$. The set U of that example is

$$\hat{\mathbf{Z}}_r := \big\{ x \in P \mid \kappa_{ij}\pi_j(x) = \pi_i(x) \text{ for all } i \leq j \big\}.$$

It is immediate from the fact that the maps π_i and κ_{ij} are homomorphisms that $\hat{\mathbf{Z}}_r$ is a subring of P, and the argument in Example 8.1.3 shows that $(\hat{\mathbf{Z}}_r, \hat{\pi})$ is a limit of F. The ring $\hat{\mathbf{Z}}_r$ is called the **ring of r-adic integers**.

We briefly describe a common notation used for the elements of $\hat{\mathbf{Z}}_r$, which facilitates computations: For each i, the ring $\mathbf{Z}/r^i\mathbf{Z}$ identifies naturally with the ring $\{0, 1, 2, \dots, r^i - 1\}$ under addition and multiplication modulo r^i. Each element of this latter set can be written in the form $\sum_{k=0}^{i-1} d_k r^k$ for unique integers $0 \leq d_k < r$, and abbreviated as $d_{i-1}d_{i-2}\cdots d_1 d_0$. With these identifications it follows from the definition that $\hat{\mathbf{Z}}_r$ is the set of all sequences of the form $(d_0, d_1d_0, d_2d_1d_0, \dots)$ $(0 \leq d_i < r)$. The "r-adic representation" of such a sequence is $\cdots d_i \cdots d_2d_1d_0$; the d_i are referred to as "digits." An example of a 5-adic representation is $\cdots 4242301$.

The map $\mathbf{Z} \to P$ induced by the canonical maps $\mathbf{Z} \to \mathbf{Z}/r^i\mathbf{Z}$ $(i \geq 1)$ is an injective ring homomorphism into $\hat{\mathbf{Z}}_r$. We use it to identify $\mathbf{Z}$ as a subring of $\hat{\mathbf{Z}}_r$. For instance, we have the 3-adic representations $17 = \cdots 00122$ and $-2 = \cdots 2221$. The nonnegative integers are precisely those elements of $\hat{\mathbf{Z}}_r$ having r-adic representations in which all but finitely many digits equal 0, while the negative integers are precisely those having r-adic representations in which all but finitely many digits equal $r - 1$.

Addition in $\hat{\mathbf{Z}}_r$ follows the usual rule involving carrying. For instance, adding the 3-adic representations of 17 and -2 given above, we get $\cdots 00120$, which is the 3-adic representation of 15. Multiplication is also carried out using the usual algorithm. $\square$

8.2 Limit as universal

In this section we show that a limit (u, π) of a functor $F : I \to C$ with I small can be construed as a universal from a functor to an object.

Let I and C be categories with I small. Recall (Section 5.2) that C^I denotes the functor category, with functors $F : I \to C$ as objects and natural transformations between functors as morphisms.

We define a functor $\triangle : C \to C^I$, called the "diagonal functor" as follows: As a map of objects $\triangle$ sends an object c of C to the constant functor from I to C determined by c (see Example 4.1.6). Thus, for each object c of C we have $\triangle(c)_i = c$ for all $i \in I$ and $\triangle(c)_\tau = 1_c$ for each morphism $\tau : i \to j$ in I. As a map of morphisms, $\triangle$ sends each morphism $\alpha : c \to d$ in C to the constant natural transformation $\triangle(\alpha) : \triangle(c) \to \triangle(d)$ given by $\triangle(\alpha)_i = \alpha$ for all $i \in I$.

(When I is the discrete category with two objects, the functor category C^I is isomorphic to the Cartesian product $C \times C$ and the diagonal functor here coincides with the diagonal functor $\triangle : C \to C \times C$ of Example 7.3.1.)

8.2.1 Theorem. *A pair (u, π) with u an object of C and $\pi_i : u \to F_i$ $(i \in I)$ morphisms in C is a limit of F if and only if it is a universal from the functor $\triangle : C \to C^I$ to the object F of C^I.*

Proof. Let D_{univ} be the comma category $(\triangle \downarrow F)$, which is the auxiliary category in the definition of universal from the functor $\triangle : C \to C^I$ to the object F (see Section 7.3). Let D_{lim} be the auxiliary category in the definition of limit of the functor F (see Section 8.1). We will show that these categories are identical. Since a limit of F is a terminal object of D_{lim} and a universal from the functor $\triangle$ to the object F is a terminal object of D_{univ}, the claim will follow.

Let (c, η) be an object of D_{lim}, that is, a cone to F. Then for every morphism $\tau : i \to j$ in I the diagram on the right

$$\begin{tikzcd} i \arrow[d, "\tau"'] & c \arrow[r, "\eta_i"] \arrow[d, "1_c"'] & F_i \arrow[d, "F_\tau"] \\ j & c \arrow[r, "\eta_j"'] & F_j \end{tikzcd} .$$

is commutative since it is essentially the commutative triangle in the definition of cone. Rewriting this diagram we get a commutative diagram on the right

$$
\begin{tikzcd}
i \arrow[d, "\tau"'] & \triangle(c)_i \arrow[r, "\eta_i"] \arrow[d, "\triangle(c)_\tau"'] & F_i \arrow[d, "F_\tau"] \\
j & \triangle(c)_j \arrow[r, "\eta_j"'] & F_j
\end{tikzcd}
$$

for each morphism $\tau : i \to j$ in I, so $\eta : \triangle(c) \to F$ is a natural transformation, i.e., a morphism in the functor category C^I. Therefore, the pair (c, η) is an object of the category D_{univ}. Conversely, these steps can be reversed to show that an object (c, η) of the category D_{univ} is a cone to F and hence an object of the category D_{lim}. This shows that the objects of the categories D_{lim} and D_{univ} coincide.

Next, we show that the morphisms in these two categories coincide. Let $\alpha : c \to u$ be a morphism in C. Then $\alpha : (c, \eta) \to (u, \pi)$ is a morphism in the category D_{lim} if and only if the diagram

$$
\begin{tikzcd}
c \arrow[rr, "\alpha"] \arrow[dr, "\eta_i"'] & & u \arrow[dl, "\pi_i"] \\
& F_i &
\end{tikzcd}
$$

is commutative for each $i \in I$. But saying this diagram is commutative for each $i \in I$ is the same as saying that the diagram

$$
\begin{tikzcd}
\triangle(c) \arrow[rr, "\triangle(\alpha)"] \arrow[dr, "\eta"'] & & \triangle(u) \arrow[dl, "\pi"] \\
& F &
\end{tikzcd}
$$

is commutative, which is the same as saying that $\alpha : (c, \eta) \to (u, \pi)$ is a morphism in the category D_{univ}.

This shows that the categories D_{lim} and D_{univ} are identical. The theorem follows. $\square$

8.3 Limit from products and equalizers

8.3.1 Theorem. *A category C is small complete if and only if products and equalizers exist in C.*

Proof. Let C be a category. If C is small complete, then products exist in C by Example 8.1.1 and equalizers exist in C by Exercise 8–1.

Assume that products and equalizers exist in C. Let I be a small category and let $F : I \to C$ be a functor. Denote by $\mathrm{Mor}(I)$ the set of all morphisms in I (this is indeed a set since I is small) and for $\tau : i \to j$ in $\mathrm{Mor}(I)$ put $T_\tau = F_j$. By assumption, there exists a product $(\prod T_\tau, \{\pi_\tau\})$ of the family $\{T_\tau\}_{\tau \in \mathrm{Mor}(I)}$, and also a product $(\prod F_i, \{\pi_i\})$ of the family $\{F_i\}_{i \in I}$.

For each $\tau : i \to j$ in $\mathrm{Mor}(I)$ we have a morphism π_j as indicated in the upper triangle in the diagram below, so using the definition of product there exists λ_1 as indicated making the triangle commutative. Similarly, for each $\tau : i \to j$ in $\mathrm{Mor}(I)$ we have morphisms π_i and F_τ in the lower right-hand square and this yields λ_2 making the square commutative:

$$
\begin{tikzcd}
 & & F_j = T_\tau & & & \\
u \arrow[r, "\iota"] \arrow[ddr, "\rho_i"] & \prod F_i \arrow[ur, "\pi_j"] \arrow[rr, shift left, "\lambda_1"] \arrow[rr, shift right, "\lambda_2"'] \arrow[dd, "\pi_i"] & & \prod T_\tau \arrow[ul, "\pi_\tau"'] \arrow[dd, "\pi_\tau"] & & \\
 & & & & i \arrow[r, "\tau"] & j \\
c \arrow[uu, dotted, "\exists!\,\gamma"] \arrow[uur, "\alpha"] \arrow[r, "\eta_i"'] & F_i \arrow[rr, "F_\tau"'] & & T_\tau = F_j & &
\end{tikzcd}
$$

By assumption, there exists an equalizer (u, ι) of λ_1 and λ_2. For each $i \in I$ put $\rho_i = \pi_i \iota$.

Claim: (u, ρ) is a limit of F. First, for each $\tau : i \to j$ in $\mathrm{Mor}(I)$ we have

$$F_\tau \rho_i = F_\tau \pi_i \iota = \pi_\tau \lambda_2 \iota = \pi_\tau \lambda_1 \iota = \pi_j \iota = \rho_j,$$

so (u, ρ) is a cone to F. Let (c, η) be a cone to F. From the definition of product, we get a morphism $\alpha : c \to \prod F_i$ such that $\pi_i \alpha = \eta_i$ for each $i \in I$. For each $\tau : i \to j$ in $\mathrm{Mor}(I)$ we have

$$\pi_\tau \lambda_1 \alpha = \pi_j \alpha = \eta_j = F_\tau \eta_i = F_\tau \pi_i \alpha = \pi_\tau \lambda_2 \alpha,$$

so by the uniqueness statement in the definition of product we get $\lambda_1 \alpha = \lambda_2 \alpha$. Therefore, from the definition of equalizer we get a unique morphism

$\gamma : c \to u$ such that $\iota\gamma = \alpha$. For each $i \in I$,

$$\rho_i\gamma = \pi_i\iota\gamma = \pi_i\alpha = \eta_i,$$

so γ is a morphism in the auxiliary category D_{lim}. Let $\gamma' : (c, \eta) \to (u, \rho)$ be a morphism in D_{lim}. For each $i \in I$,

$$\pi_i\iota\gamma' = \rho_i\gamma' = \eta_i = \pi_i\alpha,$$

so the uniqueness statement in the definition of product gives $\iota\gamma' = \alpha$. But then uniqueness of γ yields $\gamma' = \gamma$. This establishes the claim and completes the proof. □

8.4 Colimit

In this section, we discuss the notion of "colimit," which is dual to the notion of limit.

Let $F : I \to C$ be a fixed functor. A **cone from** F is a pair (x, η) with x an object of C and η a family of morphisms $\eta_i : F_i \to x$, one for each object i of I, such that for each morphism $\tau : i \to j$ in I the diagram on the right

$$\begin{tikzcd} i \arrow[d, "\tau"] & & F_i \arrow[d, "F_\tau"] \arrow[dr, "\eta_i"] & \\ j & & F_j \arrow[r, "\eta_j"'] & x \end{tikzcd}$$

is commutative, that is, $\eta_j F_\tau = \eta_i$.

Let $D = D_{\mathrm{colim}}$ be the auxiliary category having as objects all cones from F and as morphisms from the cone (x, η) to the cone (y, θ) all morphisms $\alpha : x \to y$ in C such that for each $i \in I$ the diagram

$$\begin{tikzcd} x \arrow[rr, "\alpha"] & & y \\ & F_i \arrow[ul, "\eta_i"] \arrow[ur, "\theta_i"'] & \end{tikzcd}$$

is commutative, that is, $\alpha\eta_i = \theta_i$.

An initial object (u, ι) of the category D is a **colimit** of the functor F. By definition, such an object has the property that for any cone (x, η) from F

there exists a unique morphism $\gamma : u \to x$ such that the diagram

$$\begin{tikzcd}
u \arrow[rr, dotted, "\exists!\,\gamma"] & & x \\
& F_i \arrow[ul, "\iota_i"] \arrow[ur, "\eta_i"'] &
\end{tikzcd}$$

is commutative, that is, $\gamma\iota_i = \eta_i$.

The object portion u of a colimit (u, ι) of F is often written $\varinjlim F$. The following visualization of a colimit (u, ι) of F is sometimes useful:

$$\begin{tikzcd}
i \arrow[dd, "\tau"] & F_i \arrow[dd, "F_\tau"] \arrow[dr, "\iota_i"] \arrow[drr, bend left, "\eta_i"] & & \\
& & u \arrow[r, dotted, "\exists!\,\gamma"] & x \\
j & F_j \arrow[ur, "\iota_j"'] \arrow[urr, bend right, "\eta_j"'] & &
\end{tikzcd}$$

The following characterization of a colimit as a universal is the dual of Theorem 8.2.1.

8.4.1 Theorem. *A pair (u, ι) with u an object of C and $\iota_i : F_i \to u$ $(i \in I)$ morphisms in C is a colimit of F if and only if it is a universal from the object F of C^I to the functor $\triangle : C \to C^I$.*

8.4.2 *Example* (Coproduct, Pushout, Coequalizer as colimits) A coproduct can be construed as a colimit, the argument being the dual to that in Example 8.1.1. The same is true for a pushout (cf. Example 8.1.2) and a coequalizer (cf. Exercise 8–1). □

We say that the category C is **small cocomplete** if for each functor $F : I \to C$ with I small there exists a colimit of F. The following theorem is dual to Theorem 8.3.1.

8.4.3 Theorem. *A category C is small cocomplete if and only if coproducts and coequalizers exist in C.*

8.4.4 *Example* (**Set**, **Top**, **Grp**, and ${}_R$**Mod** are small cocomplete) Coproducts and coequalizers exist in each of the categories **Set**, **Top**, **Grp**, and ${}_R$**Mod** (see Sections 3.4 and 3.6), so each of these categories is small cocomplete according to Theorem 8.4.3. With C denoting any one of these categories, the object portion of a colimit of a functor $F : I \to C$ (I small) is obtained as a suitable quotient of a coproduct of the family $\{F_i\}_{i \in I}$. $\square$

Sometimes there is an easier description of a colimit than that given in the preceding example. This is the case in the following example.

8.4.5 *Example* (Abelian group as colimit of finitely generated subgroups) Let A be an abelian group. A subgroup B of A is **finitely generated** if there exist $b_1, b_2, \dots, b_n \in B$ such that $B = \langle b_1, b_2, \dots, b_n \rangle$ (= intersection of all subgroups of A containing the set $\{b_1, b_2, \dots, b_n\}$).

Let $\{F_i\}_{i \in I}$ be the family of all finitely generated subgroups of A. Regard the set I as a category by letting $I(i, j)$ be the singleton set $\{i \to j\}$ if $F_i \subseteq F_j$ and the empty set otherwise. Then we get a functor $F : I \to \mathbf{Ab}$ by $i \mapsto F_i$ and $(i \to j) \mapsto (F_i \to F_j)$ (inclusion map). For each $i \in I$ let $\iota_i : F_i \to A$ be the inclusion map.

Claim: (A, ι) is a colimit of the functor F. First, for a morphism $\tau : i \to j$ in I we have $F_i \subseteq F_j \subseteq A$, so $\iota_j F_\tau = \iota_i$ (composition of inclusion maps equals inclusion map). Therefore, (A, ι) is a cone from F.

Let (B, η) be a cone from F. Define $\gamma : A \to B$ by $\gamma(a) = \eta_i(a)$, where i is any object of I for which $a \in F_i$. For $a \in A$ we have $a \in \langle a \rangle = F_i$ for some $i \in I$, and if also $a \in F_j$ with $j \in I$, then $F_i = \langle a \rangle \subseteq F_j$ so $\eta_j(a) = \eta_j F_\tau(a) = \eta_i(a)$, where $\tau : i \to j$. This shows that γ is well-defined.

For $a, a' \in A$ we have $\langle a, a' \rangle = F_i$ for some $i \in I$, so, since $a, a', a + a' \in F_i$, we have

$$\gamma(a + a') = \eta_i(a + a') = \eta_i(a) + \eta_i(a') = \gamma(a) + \gamma(a'),$$

implying that γ is a morphism in **Ab**.

For $i \in I$ and $a \in F_i$, we have $\gamma\iota_i(a) = \gamma(a) = \eta_i(a)$. Therefore, $\gamma\iota_i = \eta_i$ for each $i \in I$, which shows that $\gamma : (A, \iota) \to (B, \eta)$ is a morphism in D_{colim}.

Finally, let $\gamma' : (A, \iota) \to (B, \eta)$ be a morphism in D_{colim}. Then for $a \in A$ we have $a \in \langle a \rangle = F_i$ for some $i \in I$ and

$$\gamma'(a) = \gamma' \iota_i(a) = \eta_i(a) = \gamma(a),$$

so that $\gamma' = \gamma$. This establishes the claim and completes the proof. $\square$

8 – Exercises

8–1 Let C be a category and let $\lambda_1, \lambda_2 : a \to b$ be two morphisms in C. Prove that an equalizer (p, ι) of λ_1 and λ_2 can be construed as a limit of a suitable functor F.

9 Set-valued functor

The focus in this section is the case of a set-valued functor $F : C \to \mathbf{Set}$. We study, for instance, conditions under which such a functor is isomorphic to a covariant Hom functor (see Section 9.2).

9.1 Universal element

Let $F : C \to \mathbf{Set}$ be a functor. Let $D = D_{\mathrm{ue}}$ be the auxiliary category having as objects all pairs (x, s) with $x \in C$ and $s \in F(x)$, and with morphisms from the object (x, s) to the object (y, t) being all morphisms $\gamma : x \to y$ in C such that $F(\gamma)(s) = t$:

$$x \xrightarrow{\gamma} y \qquad F(x) \xrightarrow{F(\gamma)} F(y)$$
$$s \longmapsto t\,.$$

An initial object (u, e) of D is called a **universal element** of the functor F. By definition such an object has the property that for any object (x, s) of D there exists a unique morphism $\gamma : u \to x$ in C such that $F(\gamma)(e) = s$:

$$u \xrightarrow{\exists!\,\gamma} x \qquad F(u) \xrightarrow{F(\gamma)} F(x)$$
$$e \longmapsto s\,.$$

Sometimes the object u in the definition can be inferred from the element e (as is the case in the following example). When this happens the element e is often referred to as a universal element of F, this being the more natural usage of the term.

9.1.1 *Example* (Canonical epimorphism as universal element) Let H be a fixed group and let N be a normal subgroup of H. For a group G put

$$F(G) = \{\psi : H \to G \,|\, \psi \text{ is a homomorphism with } \ker\psi \supseteq N\}$$

and for a group homomorphism $\varphi : G \to G'$ define $F(\varphi) : F(G) \to F(G')$ by $F(\varphi)(\psi) = \varphi\psi$. This defines a functor $F : \mathbf{Grp} \to \mathbf{Set}$ as is routine to check. Let $\pi : H \to H/N$ be the canonical epimorphism.

Claim: $(H/N, \pi)$ is a universal element of the functor F. First, $\ker \pi = N$ so $\pi \in F(H/N)$ implying that $(H/N, \pi)$ is an object of the auxiliary category

$D = D_{\mathrm{ue}}$ in the definition of universal element. Let (G, ψ) be an object of D. We have $\ker\psi \supseteq N$ so by the fundamental homomorphism theorem there exists a unique homomorphism $\gamma : H/N \to G$ such that $\gamma\pi = \psi$:

$$\begin{tikzcd} H/N \arrow[r, dotted, "\exists!\,\gamma"] & G \end{tikzcd} \qquad \begin{tikzcd} F(H/N) \arrow[r, dotted, "F(\gamma)"] & F(G) \\ \pi \arrow[r, dotted, mapsto] & \psi\,. \end{tikzcd}$$

Since $F(\gamma)(\pi) = \gamma\pi = \psi$ it follows that $\gamma : (H/N, \pi) \to (G, \psi)$ is a morphism in the auxiliary category D and it is the unique such. The claim follows. $\square$

The next theorem shows that a universal element is (essentially) a special case of a universal from an object to a functor.

9.1.2 Theorem. *Let $F : C \to$ **Set** be a functor. If (u, e) is a universal element of F, then (u, ι) is a universal from the singleton set $\{\bullet\}$ to F, where $\iota : \{\bullet\} \to F(u)$ is given by $\bullet \mapsto e$.*

Proof. Let (u, e) be a universal element of F and let ι be as described. Then (u, ι) is an object of the auxiliary category $D_{\mathrm{univ}} = (\{\bullet\} \downarrow F)$ in the definition of universal from $\{\bullet\}$ to F. Let $(c, \alpha) \in D_{\mathrm{univ}}$ and put $a = \alpha(\bullet)$. Then $(c, a) \in D_{\mathrm{ue}}$. For an arbitrary morphism $\gamma : u \to c$ in C we have

$$\begin{aligned} \gamma : (u, e) \to (c, a) \text{ in } D_{\mathrm{ue}} \quad &\iff \quad F(\gamma)(e) = a \\ &\iff \quad F(\gamma)\iota(\bullet) = \alpha(\bullet) \\ &\iff \quad F(\gamma)\iota = \alpha \\ &\iff \quad \gamma : (u, \iota) \to (c, \alpha) \text{ in } D_{\mathrm{univ}} \end{aligned}$$

with corresponding diagrams

$$\begin{tikzcd} u \arrow[r, "\gamma"] & c \end{tikzcd} \qquad \begin{tikzcd} F(u) \arrow[r, "F(\gamma)"] & F(c) \\ e \arrow[r, mapsto] & a \end{tikzcd} \qquad \begin{tikzcd} F(u) \arrow[rr, "F(\gamma)"] & & F(c)\,. \\ & \{\bullet\} \arrow[ul, "\iota"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Since (u, e) is initial in D_{ue} it follows that (u, ι) is initial in D_{univ} and the proof is complete. $\square$

Conversely, a universal from an object to a functor is a special case of a universal element, as the next theorem shows.

9.1.3 Theorem. *Let $F : C \to D$ be a functor. Let $u \in C$, $d \in D$, and let ι be a morphism in D. The pair (u, ι) is a universal from d to F if and only if it is a universal element of the functor $D(d, F(\_)) : C \to \mathbf{Set}$.*

Proof. For simplicity, put $G = D(d, F(\_)) : C \to \mathbf{Set}$. Let $D_{\text{univ}} = (d \downarrow F)$ be the auxiliary category in the definition of universal from d to F, and let D_{ue} be the auxiliary category in the definition of universal element of F. For $c \in C$, we have $\alpha \in G(c)$ if and only if $\alpha : d \to F(c)$, so the objects (c, α) of the categories D_{ue} and D_{univ} coincide.

Let $(c, \alpha) \in D_{\text{ue}}(= D_{\text{univ}})$. For an arbitrary morphism $\gamma : u \to c$ in C we have

$$
\begin{aligned}
\gamma : (u, \iota) \to (c, \alpha) \text{ in } D_{\text{univ}} \quad &\iff \quad F(\gamma)\iota = \alpha \\
&\iff \quad D(d, F(\gamma))(\iota) = \alpha \\
&\iff \quad G(\gamma)(\iota) = \alpha \\
&\iff \quad \gamma : (u, \iota) \to (c, \alpha) \text{ in } D_{\text{ue}}
\end{aligned}
$$

with corresponding diagrams

$$
u \xrightarrow{\gamma} c \qquad
\begin{tikzcd}
F(u) \arrow[rr, "F(\gamma)"] & & F(c) \\
& d \arrow[ul, "\iota"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{array}{c}
G(u) \xrightarrow{G(\gamma)} G(c) \\
\iota \longmapsto \alpha\,.
\end{array}
$$

Therefore, (u, ι) is initial in D_{univ} if and only if it is initial in D_{ue}. The claim follows. $\square$

9.2 Yoneda's lemma

For functors F and G, we denote by $\text{Nat}(F, G)$ the class of all natural transformations from F to G.

Let $F : C \to \mathbf{Set}$ be a functor and fix $c \in C$. Define maps

$$
\text{Nat}(C(c, \_), F) \underset{z}{\overset{y}{\rightleftarrows}} F(c)
$$

by

$$y(\eta) = \eta_c(1_c), \qquad\qquad z(e) = \eta^e,$$

where η^e is the natural transformation with component map corresponding to $x \in C$ given by

$$\eta^e_x : C(c, x) \to F(x), \qquad\qquad \gamma \mapsto F(\gamma)(e).$$

9.2.1 Theorem (Yoneda's Lemma). *Let the notation be as above.*

(i) *The maps*

$$\mathrm{Nat}(C(c, \_), F) \underset{z}{\overset{y}{\rightleftarrows}} F(c)$$

are well-defined and they are inverses of each other. In particular, each is a bijection.

(ii) *For $e \in F(c)$ the corresponding natural transformation $z(e) = \eta^e$ is a natural isomorphism if and only if the pair (c, e) is a universal element of F.*

Proof. (i) That y is well-defined follows immediately from the definition so we turn to z. Let $e \in F(c)$. For any morphism $\alpha : x \to y$ in C the diagram on the right is commutative

$$\begin{array}{ccc} x & \qquad & \begin{tikzcd} C(c,x) \arrow[r, "\eta^e_x"] \arrow[d, "\alpha_*"'] & F(x) \arrow[d, "F(\alpha)"] \\ C(c,y) \arrow[r, "\eta^e_y"'] & F(y) \end{tikzcd} \\ \downarrow \alpha & & \\ y & & \end{array}$$

since for every $\gamma \in C(c, x)$ we have

$$\begin{aligned} F(\alpha)\eta^e_x(\gamma) &= F(\alpha)(F(\gamma)(e)) = F(\alpha)F(\gamma)(e) = F(\alpha\gamma)(e) = \eta^e_y(\alpha\gamma) \\ &= \eta^e_y(\alpha_*(\gamma)) = \eta^e_y\alpha_*(\gamma) \end{aligned}$$

(see Example 4.1.2 for notation). This shows that η^e is a natural transformation and therefore an element of $N = \mathrm{Nat}(C(c, \_), F)$, so z is well-defined.

For $\eta \in N$, $x \in C$, and $\gamma : c \to x$ we have

$$\begin{aligned} (zy(\eta))_x(\gamma) &= \eta^{y(\eta)}_x(\gamma) = F(\gamma)(y(\eta)) = F(\gamma)(\eta_c(1_c)) = F(\gamma)\eta_c(1_c) \\ &= \eta_x\gamma_*(1_c) = \eta_x(\gamma), \end{aligned}$$

where the penultimate equality uses naturality of η. Therefore, $zy = 1_N$. On the other hand, for $e \in F(c)$ we have

$$(yz)(e) = y(\eta^e) = \eta^e_c(1_c) = F(1_c)(e) = e,$$

so $yz = 1_{F(c)}$. This completes the proof of (i).

(ii) Let $e \in F(c)$. The natural transformation η^e is a natural isomorphism if and only if $\eta^e_x : C(c,x) \to F(x)$ is bijective for each $x \in C$. This latter is the case if and only if for each $x \in C$ and $t \in F(x)$ there exists a unique morphism $\gamma : c \to x$ such that $t = \eta^e_x(\gamma) = F(\gamma)(e)$, which is to say that (c, e) is a universal element of F. This completes the proof. $\square$

9.3 Universality via Hom functors

Here, we discuss a special case of Yoneda's lemma, which we will have use for later. It provides a characterization of universal from object to functor much like Yoneda's lemma characterizes universal element.

Let $F : C \to D$ be a functor and fix $c \in C$ and $d \in D$. Define maps

$$\mathrm{Nat}(C(c, \_), D(d, F(\_))) \underset{z}{\overset{y}{\rightleftarrows}} D(d, F(c))$$

by

$$y(\eta) = \eta_c(1_c), \qquad z(\beta) = \eta^\beta,$$

where η^β is the natural transformation with component map corresponding to $x \in C$ given by

$$\eta^\beta_x : C(c,x) \to D(d, F(x)), \qquad \gamma \mapsto F(\gamma)\beta.$$

9.3.1 Corollary. *Let the notation be as above.*

(i) *The maps*

$$\mathrm{Nat}(C(c, \_), D(d, F(\_))) \underset{z}{\overset{y}{\rightleftarrows}} D(d, F(c))$$

are well-defined and they are inverses of each other. In particular, each is a bijection.

(ii) *For $\beta : d \to F(c)$ the corresponding natural transformation $z(\beta) = \eta^\beta$ is a natural isomorphism if and only if the pair (c, β) is a universal from d to F.*

Proof. (i) Take for F in Yoneda's lemma (9.2.1) the functor $D(d, F(\_))$. The map η^β_x in that lemma sends γ to $D(d, F(\gamma))(\beta)$, which equals $F(\gamma)\beta$, so the two definitions of η^β_x coincide. Part (i) of this corollary now follows immediately from part (i) of Yoneda's lemma.

(ii) Let $\beta : d \to F(c)$. The natural transformation η^β is a natural isomorphism if and only if $\eta^\beta_x : C(c, x) \to D(d, F(x))$ is bijective for each $x \in C$. This latter is the case if and only if for each $x \in C$ and morphism $\delta : d \to F(x)$ there exists a unique morphism $\gamma : c \to x$ such that $\delta = \eta^\beta_x(\gamma) = F(\gamma)\beta$, which is to say that (c, β) is a universal from d to F. This completes the proof. □

Note that part (ii) also follows from part (ii) of Yoneda's Lemma and Theorem 9.1.3.

9.4 Yoneda Embedding

9.4.1 Corollary (Yoneda Embedding). *If C is a small category, then C is isomorphic to a full subcategory of the functor category $(\mathbf{Set}^C)^{\mathrm{op}}$.*

Proof. Let C be a small category. We define a functor $Y : C \to (\mathbf{Set}^C)^{\mathrm{op}}$ with object map given by $Y(c) = C(c, \_)$.

Fix $c, d \in C$ and put $F = C(c, \_) : C \to \mathbf{Set}$. The map z in Yoneda's Lemma (9.2.1) gives a bijection

$$C(c, d) = F(d) \xrightarrow{z} \mathbf{Set}^C(C(d, \_), F) = (\mathbf{Set}^C)^{\mathrm{op}}(Y(c), Y(d)). \qquad (5)$$

The restriction of Y to $C(c, d)$ is taken to be this map, and since c and d are arbitrary, this defines Y as a map of morphisms. For $\alpha : c \to d$ we have $Y(\alpha) = z(\alpha) = \eta^\alpha : C(d, \_) \to C(c, \_)$ (in $\mathbf{Set}^C$), where

$$\eta^\alpha_x(\gamma) = F(\gamma)(\alpha) = C(c, \gamma)(\alpha) = \gamma\alpha$$

for $x \in C$ and $\gamma : d \to x$.

We claim that Y is a functor. Let $\alpha : c \to d$ and $\beta : d \to e$ be morphisms in C. For $x \in C$ and $\gamma : e \to x$ we have

$$\eta_x^{\beta\alpha}(\gamma) = \gamma\beta\alpha = \eta_x^{\alpha}\eta_x^{\beta}(\gamma) = (\eta^{\alpha}\eta^{\beta})_x(\gamma),$$

so $Y(\beta\alpha) = \eta^{\beta\alpha} = \eta^{\alpha}\eta^{\beta} = Y(\beta)Y(\alpha)$, this last composition taken in $(\mathbf{Set}^C)^{\mathrm{op}}$. For $c, x \in C$ and $\gamma : c \to x$ we have

$$\eta_x^{1_c}(\gamma) = \gamma 1_c = \gamma = 1_{C(c,x)}(\gamma) = (1_{C(c,\_)})_x(\gamma),$$

so $Y(1_c) = \eta^{1_c} = 1_{C(c,\_)} = 1_{Y(c)}$. Therefore, Y is a functor as claimed.

Since the map given in (5) is a bijection, Y is faithful and full. Moreover, Y is injective as a map of objects, since, if $c, d \in C$ and $Y(c) = Y(d)$, then $C(c, c) = C(d, c)$, implying $c = d$ (morphism sets are assumed to be pairwise disjoint). Therefore, restricting the codomain of Y to $\operatorname{im} Y$, we get an isomorphism $C \to \operatorname{im} Y$. Since $\operatorname{im} Y$ is a full subcategory of $(\mathbf{Set}^C)^{\mathrm{op}}$ the proof is complete. $\square$

9.5 Representation

A **representation** of a set-valued functor $F : C \to \mathbf{Set}$ is a pair (c, η), where c is an object of C and $\eta : C(c, \_) \to F$ is a natural isomorphism. A functor F is **representable** if there exists a representation of F.

The following corollary of Yoneda's lemma completely classifies all representations of a set-valued functor.

9.5.1 Corollary. *Let $F : C \to \mathbf{Set}$ be a functor.*

(i) *The representations of F are precisely the pairs (c, η^e) such that (c, e) is a universal element of F and $\eta_x^e : C(c, x) \to F(x)$ is given by $\eta_x^e(\gamma) = F(\gamma)(e)$.*

(ii) *If (c, η^e) and (d, η^f) are two representations of F, then there exists an isomorphism $\alpha : c \to d$ in C such that $F(\alpha)(e) = f$.*

Proof. (i) Let (c, η) be a representation of F. We use the notation of Yoneda's lemma (9.2.1). Putting $e = y(\eta)$ we have $\eta^e = z(e) = zy(\eta) = \eta$, and since η is a natural isomorphism, it follows from part (ii) of that Corollary that (c, e) is a universal element of F. Conversely, if (c, e) is a universal element of F, then $\eta^e : C(c, \_) \to F$ is a natural isomorphism (again by part (ii) of that Corollary), so the pair (c, η^e) is a representation of F.

(ii) Let (c, η^e) and (d, η^f) be two representations of F. Then (c, e) and (c, f) are universal elements of F, that is, initial objects of the auxiliary category $D = D_{\text{ue}}$ in the definition of universal element of F. By Theorem 3.1.2 there exists an isomorphism $\alpha : (c, e) \to (c, f)$ in D. Then $\alpha : c \to d$ is an isomorphism in C such that $F(\alpha)(e) = f$. $\square$

9 – Exercises

9–1

(a) Find a representation of the contravariant power set functor (Example 4.4.3) regarded as a covariant functor $\mathbf{Set}^{\text{op}} \to \mathbf{Set}$ (see Section 4.4).

(b) Prove that the (covariant) power set functor (Example 4.1.4) is not representable.

10 Adjoint Pair

10.1 Motivation

Let K be a field. For each set X choose a vector space $G(X)$ having basis X (for instance, one can let $G(X) = \bigoplus_{x \in X} K_x$, where $K_x = K$ for every x as in Example 3.4.3), and let $\iota_X : X \to F(G(X))$ be the inclusion map, where $F : \mathbf{Vect}_K \to \mathbf{Set}$ is the forgetful functor.

For each set X the pair $(G(X), \iota_X)$ is a universal from X to F (see Example 7.2.1, $C = {}_R\mathbf{Mod}$), meaning that for each object V of $\mathbf{Vect}_K$ and each function $\alpha : X \to F(V)$ there exists a unique morphism $\gamma : G(X) \to V$ such that $F(\gamma)\iota_X = \alpha$:

$$
\begin{tikzcd}
G(X) \arrow[r, dotted, "\exists!\,\gamma"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
F(G(X)) \arrow[rr, dotted, "F(\gamma)"] & & F(V) \\
& X \arrow[ul, "\iota_X"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
$$

Put another way, for each $X \in \mathbf{Set}$ and $V \in \mathbf{Vect}_K$ the map

$$\varphi_{X,V} : \mathbf{Vect}_K(G(X), V) \to \mathbf{Set}(X, F(V))$$

given by $\varphi_{X,V}(\gamma) = F(\gamma)\iota_X$ is a bijection.

We can use the universal property to extend the map of objects $G : \mathbf{Set} \to \mathbf{Vect}_K$ to a functor: For a morphism $\beta : X \to Y$ in $\mathbf{Set}$, let $G(\beta) : G(X) \to G(Y)$ be the unique morphism such that $F(G(\beta))\iota_X = \iota_Y\beta$:

$$
\begin{tikzcd}
G(X) \arrow[r, dotted, "\exists!\,G(\beta)"] & G(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(G(X)) \arrow[r, dotted, "F(G(\beta))"] & F(G(Y)) \\
X \arrow[u, "\iota_X"] \arrow[r, "\beta"'] & Y \arrow[u, "\iota_Y"']
\end{tikzcd}
$$

(so let $V = G(Y)$ and $\alpha = \iota_Y\beta$ above). It is shown in Section 10.3 that G is indeed a functor and that the bijection $\varphi_{X,V}$ is natural in each variable X and V (the precise meaning of this being given below).

The functor $G : \mathbf{Set} \to \mathbf{Vect}_K$ is called a "left adjoint" of the functor $F : \mathbf{Vect}_K \to \mathbf{Set}$ (and F is called a "right adjoint" of G). The pair (G, F) is referred to as an "adjoint pair."

10.2 Hom-set adjunction

Let C and D be categories. A **hom-set adjunction** from C to D is a triple (G, F, φ), often written $\varphi : G \dashv F$, where $F : C \to D$ and $G : D \to C$ are functors and $\varphi : C(G(\_), \_) \to D(\_, F(\_))$ is a natural isomorphism, the indicated functors being regarded as functors $D^{\mathrm{op}} \times C \to \mathbf{Set}$.

In more detail, the natural isomorphism φ in the definition is a family of isomorphisms

$$\varphi = \varphi_{d,c} : C(G(d), c) \to D(d, F(c)),$$

one for each pair $d \in D$, $c \in C$, natural in each variable, meaning that for fixed $d \in D$ and morphism $\alpha : c \to c'$ in C the diagram on the right is commutative

$$\begin{tikzcd} c \arrow[d, "\alpha"'] & C(G(d), c) \arrow[r, "\varphi_{d,c}"] \arrow[d, "\alpha_*"'] & D(d, F(c)) \arrow[d, "F(\alpha)_*"] \\ c' & C(G(d), c') \arrow[r, "\varphi_{d,c'}"'] & D(d, F(c')) \end{tikzcd}$$

and for fixed $c \in C$ and morphism $\beta : d' \to d$ in D the diagram on the right is commutative

$$\begin{tikzcd} d' \arrow[d, "\beta"'] & C(G(d), c) \arrow[r, "\varphi_{d,c}"] \arrow[d, "G(\beta)^*"'] & D(d, F(c)) \arrow[d, "\beta^*"] \\ d & C(G(d'), c) \arrow[r, "\varphi_{d',c}"'] & D(d', F(c)) \,. \end{tikzcd}$$

Let $\varphi : G \dashv F$ be a hom-set adjunction from C to D. The functor G is a **left adjoint** of F and the functor F is a **right adjoint** of G. The pair (G, F) is an **adjoint pair**.

It is convenient to have formulas expressing the commutativity of the two diagrams above. Such formulas are given in the following theorem.

10.2.1 Theorem. *Let $\alpha : c \to c'$ in C and $\beta : d' \to d$ in D. The two diagrams above are commutative if and only if for every morphism $\gamma : G(d) \to c$ the following hold, respectively:*

(i) $F(\alpha)\varphi(\gamma) = \varphi(\alpha\gamma)$,

(ii) $\varphi(\gamma)\beta = \varphi(\gamma G(\beta))$.

Proof. For every morphism $\gamma : G(d) \to c$ we have

$$\begin{aligned} F(\alpha)_*\varphi(\gamma) = \varphi\alpha_*(\gamma) &\iff F(\alpha)_*(\varphi(\gamma)) = \varphi(\alpha_*(\gamma)) \\ &\iff F(\alpha)\varphi(\gamma) = \varphi(\alpha\gamma) \end{aligned}$$

and

$$\begin{aligned} \beta^*\varphi(\gamma) = \varphi G(\beta)^*(\gamma) &\iff \beta^*(\varphi(\gamma)) = \varphi(G(\beta)^*(\gamma)) \\ &\iff \varphi(\gamma)\beta = \varphi(\gamma G(\beta)), \end{aligned}$$

so the claim follows. □

In the next section we will see that adjoint pairs arise from the existence of universals. This will allow us to give several examples of adjoint pairs without even having to exhibit a natural isomorphism φ as in the definition. However, for now we give a classical example with an explicit description of φ.

10.2.2 *Example* (Adjoint associativity) Let R and S be rings and let L be an (R, S)-bimodule. We get a functor $F : {}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$ with object map $F(M) = \mathrm{Hom}_R(L, M)$ and a functor $G : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$ with object map $G(N) = L \otimes_S N$.

Let N be an S-module and let M be an R-module. Define

$$\begin{tikzcd} \mathrm{Hom}_R(L \otimes_S N, M) \arrow[r, "\varphi_{N,M}"] \arrow[d, equal] & \mathrm{Hom}_S(N, \mathrm{Hom}_R(L, M)) \arrow[d, equal] \\ \mathrm{Hom}_R(G(N), M) & \mathrm{Hom}_S(N, F(M)) \end{tikzcd}$$

by $[\varphi_{N,M}(f)(n)](l) = f(l \otimes n)$. Then $\varphi_{N,M}$ is a bijection and it is natural in the variables N and M as the reader can check. Therefore, $\varphi : G \dashv F$ is a hom-set adjunction. The inverse of $\varphi_{N,M}$ is given by $[\varphi^{-1}_{N,M}(g)](l \otimes n) = [g(n)](l)$.

This relationship between the tensor functor and the hom functor is referred to as **adjoint associativity**. □

The following theorem says that a functor having a left adjoint (resp., right adjoint) can be regarded as a weak equivalence.

10.2.3 Theorem. *Let $F : C \to D$ and $G : D \to C$ be equivalences with $GF \cong 1_C$ and $FG \cong 1_D$. Then $\varphi : G \dashv F$ is a hom-set adjunction from C to D for some φ.*

Proof. Let $d \in D$ and $c \in C$. We have maps

$$C(G(d), c) \longrightarrow D(FG(d), F(c)) \longrightarrow D(d, F(c)) .$$

The first is application of F. It is a bijection since F is faithful and full (Theorem 6.2.1). The second uses the isomorphism $FG(d) \cong d$ and is also a bijection. The composition $\varphi_{d,c}$ is a bijection and it is natural in both variables d and c (since $FG \cong 1_D$ is a natural isomorphism). This gives the desired hom-set adjunction $\varphi : G \dashv F$. □

10.3 Adjoint pair from universals

The following theorem generalizes the construction of Section 10.1.

10.3.1 Theorem. *Let $F : C \to D$ be a functor and assume that a universal $(G(d), \eta_d)$ from d to F exists for each $d \in D$. For a morphism $\beta : d' \to d$ in D let $G(\beta) : G(d') \to G(d)$ be the unique morphism for which $F(G(\beta))\eta_{d'} = \eta_d\beta$. For $d \in D$ and $c \in C$, define $\varphi_{d,c} : C(G(d), c) \to D(d, F(c))$ by $\varphi_{d,c}(\gamma) = F(\gamma)\eta_d$.*

Then $G : D \to C$ is a functor and $\varphi : G \dashv F$ is a hom-set adjunction from C to D.

Proof. For a morphism $\beta : d' \to d$ in D the existence of a unique morphism $G(\beta) : G(d') \to G(d)$ such that $F(G(\beta))\eta_{d'} = \eta_d\beta$ is guaranteed by the initial property of the pair $(G(d'), \eta_{d'})$:

$$\begin{tikzcd} G(d') \arrow[r, dotted, "\exists!\, G(\beta)"] & G(d) \end{tikzcd} \qquad \begin{tikzcd} F(G(d')) \arrow[r, dotted, "F(G(\beta))"] & F(G(d)) \\ d' \arrow[u, "\eta_{d'}"] \arrow[r, "\beta"'] & d \arrow[u, "\eta_d"'] \end{tikzcd}$$

We show that G is a functor. Let $\delta : d'' \to d'$ and $\beta : d' \to d$ be morphisms in D. Then

$$F(G(\beta)G(\delta))\eta_{d''} = F(G(\beta))F(G(\delta))\eta_{d''} = F(G(\beta))\eta_{d'}\delta = \eta_d\beta\delta,$$

so by uniqueness we get $G(\beta\delta) = G(\beta)G(\delta)$. Let $d \in D$. Then

$$F(1_{G(d)})\eta_d = 1_{FG(d)}\eta_d = \eta_d = \eta_d 1_d,$$

so by uniqueness we get $G(1_d) = 1_{G(d)}$. Therefore, G is a functor.

Fix $d \in D$. For $c \in C$ and $\gamma : G(d) \to c$ we have

$$\varphi_{d,c}(\gamma) = F(\gamma)\eta_d = \eta_c^{\eta_d}(\gamma)$$

where $\eta^{\eta_d} = z(\eta_d)$ as in Corollary 9.3.1. Therefore $\varphi_{d,-} = \eta^{\eta_d}$. Since $(G(d), \eta_d)$ is a universal from d to F, it follows from part (ii) of that theorem that $\varphi_{d,-} : C(G(d), -) \to D(d, F(-))$ is a natural isomorphism.

Fix $c \in C$ and let $\beta : d' \to d$ be a morphism in D. For every morphism $\gamma : G(d) \to c$ we have

$$\varphi(\gamma)\beta = F(\gamma)\eta_d\beta = F(\gamma)F(G(\beta))\eta_{d'} = F(\gamma G(\beta))\eta_{d'} = \varphi(\gamma G(\beta)),$$

so $\varphi_{-,c}$ is a natural transformation by Theorem 10.2.1.

Therefore, $\varphi : G \dashv F$ is a hom-set adjunction from C to D as claimed. $\square$

Similarly, if the functor $G : D \to C$ has the property that there exists a universal $(F(c), \varepsilon_c)$ from G to c for every object c of C, then F is the object map of a functor $F : C \to D$ such that F is a right adjoint of G.

10.4 Examples

10.4.1 *Example* (Left adjoints via free objects) In Example 7.2.1, we listed some concrete categories (C, F) for which a universal (U, ι) from X to F exists for every set X (such a universal is called free on X). In view of Theorem 10.3.1 we then get a left adjoint $G : \mathbf{Set} \to C$ of the forgetful functor $F : C \to \mathbf{Set}$ in each of these cases:

$C = \mathbf{Set}$,	$G(X) = X$
$C = {}_R\mathbf{Mod}$,	$G(X) = \bigoplus_{x \in X} R_x$ $(R_x = R\ \forall x)$, direct sum (R is a ring with identity)
$C = \mathbf{Top}$,	$G(X) = X$, with discrete topology
$C = \mathbf{Grp}$,	$G(X) =$ free group on X
$C = \mathbf{CRing}$,	$G(X) = \mathbf{Z}[X]$, polynomial ring over $\mathbf{Z}$ in the set of indeterminates X

$\square$

10.4.2 *Example* (Abelianization functor as left adjoint) For a group H let $H^{(1)}$ denote the commutator subgroup of H (so $H^{(1)}$ is the subgroup of H generated by all commutators $[h,k] = h^{-1}k^{-1}hk$ with $h,k \in H$). A standard fact about $H^{(1)}$ is that it is the smallest normal subgroup of H with corresponding quotient abelian (i.e., $N \triangleleft H$, H/N abelian $\Leftrightarrow H^{(1)} \subseteq N$).

Let $F : \mathbf{Ab} \to \mathbf{Grp}$ be the inclusion functor. Let H be a group. Put $G(H) = H/H^{(1)} \in \mathbf{Ab}$ and let $\pi_H : H \to H/H^{(1)} = F(G(H))$ be the canonical epimorphism.

Claim: $(G(H), \pi_H)$ is a universal from H to F. Let A be an abelian group and let $\alpha : H \to F(A)$ $(= A)$ be a homomorphism. We have $H/\ker\alpha \cong \operatorname{im}\alpha \leq A$. In particular, $H/\ker\alpha$ is abelian, so that $H^{(1)} \subseteq \ker\alpha$. By the fundamental homomorphism theorem there exists a unique homomorphism $\gamma : H/H^{(1)} \to A$ such that $\gamma\pi_H = \alpha$, that is, a unique homomorphism $\gamma : G(H) \to A$ such that $F(\gamma)\pi_H = \alpha$. This establishes the claim (cf. Exercise 7–1).

According to Theorem 10.3.1, we get a functor $G : \mathbf{Grp} \to \mathbf{Ab}$ with object map $H \mapsto H/H^{(1)}$ that is a left adjoint of F. The functor G is the **abelianization functor**. $\square$

10.4.3 *Example* (Indiscrete topology) Let $G : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. For a set X put $F(X) = X \in \mathbf{Top}$, where X is endowed with the indiscrete topology, and let $\pi_X : X \to X$ be the identity map. Then $(F(X), \pi_X)$ is a universal from G to X. By the dual of Theorem 10.3.1 (see comment following that theorem), we get a functor $F : \mathbf{Set} \to \mathbf{Top}$ with object map $X \mapsto F(X)$ that is a right adjoint of G. $\square$

10.4.4 *Example* (Stone-Cech compactification) Let **CompHaus** be the full subcategory of **Top** consisting of compact Hausdorff spaces and let $F : \mathbf{CompHaus} \to \mathbf{Top}$ be the inclusion functor. For arbitrary $X \in \mathbf{Top}$ there exists a universal $(G(X), \iota_X)$ from X to F, called the Stone-Cech compactification of X. (Here is one construction of such a pair: Let $X \in \mathbf{Top}$, let C be the set of all continuous functions $f : X \to [0,1]$, and endow $[0,1]^C$ with the product topology. Then $\iota : X \to [0,1]^C$ defined by $\iota(x)_f = f(x)$ is continuous. Let $G(X)$ be the closure of $\operatorname{im}\iota$ and restrict the codomain of ι to get $\iota_X : X \to G(X)$.)

According to Theorem 10.3.1, we get a functor $G : \mathbf{Top} \to \mathbf{CompHaus}$ with object map $X \mapsto G(X)$ that is a left adjoint of F. $\square$

10.5 Counit-unit adjunction

Let C and D be categories. A **counit-unit adjunction** from C to D is a quadruple $(G, F, \varepsilon, \eta)$, often written $(\varepsilon, \eta) : G \dashv F$, where $F : C \to D$ and $G : D \to C$ are functors and $\eta : 1_D \to FG$ and $\varepsilon : GF \to 1_C$ are natural transformations, called the **unit** and **counit**, respectively, such that the following triangles are commutative:

$$\begin{tikzcd} G \arrow[r, "G\eta"] \arrow[dr, "1_G"'] & GFG \arrow[d, "\varepsilon G"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} FGF \arrow[d, "F\varepsilon"'] & F \arrow[l, "\eta F"'] \arrow[dl, "1_F"] \\ F & \end{tikzcd} ,$$

that is, such that the following **counit-unit identities** are satisfied for all $c \in C$ and $d \in D$:

$$\varepsilon_{G(d)} G(\eta_d) = 1_{G(d)} \qquad \text{and} \qquad F(\varepsilon_c)\eta_{F(c)} = 1_{F(c)} \tag{6}$$

(cf. Exercise 5–2).

10.5.1 Theorem. *If $(\varepsilon, \eta) : G \dashv F$ is a counit-unit adjunction from C to D, then $\varphi : G \dashv F$ is a hom-set adjunction from C to D, where the maps*

$$C(G(d), c) \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} D(d, F(c))$$

are given by

$$\varphi(\alpha) = F(\alpha)\eta_d \qquad \text{and} \qquad \varphi^{-1}(\beta) = \varepsilon_c G(\beta)$$

for $c \in C$ and $d \in D$.

Proof. Let $(\varepsilon, \eta) : G \dashv F$ be a counit-unit adjunction from C to D and let φ be as indicated. For $d \in D$ and $c \in C$ define $\psi = \psi_{d,c} : D(d, F(c)) \to C(G(d), c)$ by $\psi(\beta) = \varepsilon_c G(\beta)$. We claim that φ and ψ are inverses of each other.

Let $c \in C$ and $d \in D$. For $\gamma \in C(G(d), c)$ we have

$$\begin{aligned}\psi\varphi(\gamma) &= \psi(F(\gamma)\eta_d) = \varepsilon_c G(F(\gamma)\eta_d) = \varepsilon_c GF(\gamma)G(\eta_d) \\ &= \gamma\varepsilon_{G(d)}G(\eta_d) = \gamma,\end{aligned}$$

where the last two equalities use naturality of ε and Equations (6), respectively. Therefore, $\psi\varphi$ is the identity map on $C(G(d), c)$. An analogous argument shows that $\varphi\psi$ is the identity map on $D(d, F(c))$. Therefore, φ is a bijection and φ^{-1} is as indicated.

Let $\alpha : c \to c'$ in C and $\beta : d' \to d$ in D. For $\gamma : G(d) \to c$ we have

$$F(\alpha)\varphi(\gamma) = F(\alpha)F(\gamma)\eta_d = F(\alpha\gamma)\eta_d = \varphi(\alpha\gamma)$$

and

$$\varphi(\gamma)\beta = F(\gamma)\eta_d\beta = F(\gamma)FG(\beta)\eta_{d'} = F(\gamma G(\beta))\eta_{d'} = \varphi(\gamma G(\beta)),$$

so $\varphi = \varphi_{d,c}$ is natural in each variable d and c by Theorem 10.2.1. This completes the proof. $\square$

10.6 Universals and Counit-unit adjunction from Hom-set adjunction

Let C and D be categories and let $\varphi : G \dashv F$ be a hom-set adjunction from C to D. Then

$$\varphi_{d,c} : C(G(d), c) \to D(d, F(c))$$

is a bijection for each $c \in C$ and $d \in D$.

The **unit** $\eta : 1_D \to FG$ and the **counit** $\varepsilon : GF \to 1_C$ of the adjunction are defined by

$$\eta_d = \varphi_{d,G(d)}(1_{G(d)}) \qquad \text{and} \qquad \varepsilon_c = \varphi^{-1}_{F(c),c}(1_{F(c)}).$$

The justification for the terminology is given in the following theorem.

10.6.1 Theorem. *Let $\varphi : G \dashv F$ be a hom-set adjunction from C to D and let η and ε be the unit and counit, respectively, of the adjunction.*

(i) *$(\varepsilon, \eta) : G \dashv F$ is a counit-unit adjunction from C to D.*

(ii) *For each $d \in D$ the pair $(G(d), \eta_d)$ is a universal from d to F.*

(iii) *For each $c \in C$ the pair $(F(c), \varepsilon_c)$ is a universal from G to c.*

Proof. (i) It follows from the definitions that for $d \in D$ we have $\eta_d : d \to FG(d)$. Let $\beta : d' \to d$ be a morphism in D:

$$\begin{tikzcd} d' \arrow[d, "\beta"'] \\ d \end{tikzcd} \qquad \begin{tikzcd} d' \arrow[r, "\eta_{d'}"] \arrow[d, "\beta"'] & FG(d') \arrow[d, "FG(\beta)"] \\ d \arrow[r, "\eta_d"'] & FG(d) \end{tikzcd} .$$

Using Theorem 10.2.1 we get

$$\begin{aligned} FG(\beta)\eta_{d'} &= FG(\beta)\varphi(1_{G(d')}) = \varphi(G(\beta)1_{G(d')}) = \varphi(1_{G(d)}G(\beta)) \\ &= \varphi(1_{G(d)})\beta = \eta_d\beta, \end{aligned}$$

so $\eta : 1_D \to FG$ is a natural transformation. An analogous argument shows that $\varepsilon : GF \to 1_C$ is a natural transformation as well.

For $c \in C$ we have, using Theorem 10.2.1,

$$F(\varepsilon_c)\eta_{F(c)} = F(\varepsilon_c)\varphi(1_{GF(c)}) = \varphi(\varepsilon_c 1_{GF(c)}) = \varphi(\varepsilon_c) = 1_{F(c)}.$$

Similarly, $\varepsilon_{G(d)}G(\eta_d) = 1_{G(d)}$. Therefore, ε and η satisfy the counit-unit identities (Equations (6)) so $(\varepsilon, \eta) : G \dashv F$ is a counit-unit adjunction from C to D as claimed.

(ii) Let $d \in D$. By assumption,

$$\varphi_{d,-} : C(G(d), -) \to D(d, F(-))$$

is a natural isomorphism. Its correspondent in $D(d, FG(d))$ as in Corollary 9.3.1 is

$$y(\varphi_{d,-}) = \varphi_{d,G(d)}(1_{G(d)}) = \eta_d.$$

Therefore, by part (ii) of that theorem $(G(d), \eta_d)$ is a universal from d to F.

(iii) Let $c \in C$. An argument analogous to that for part (ii), but using the dual to Corollary 9.3.1, shows that $(F(c), \varepsilon_c)$ is a universal from G to c. $\square$

10.7 Preservation of limits/colimits

A functor $F : C \to D$ **preserves limits** provided the following property holds: If $f : I \to C$ is a functor with I small and (c, η) is a limit of f, then $(F(c), F\eta)$ is a limit of the composition $Ff : I \to D$. Here, $F\eta$ is defined by $(F\eta)_i = F(\eta_i)$ (and therefore has the same interpretation as in Exercise 5–2 viewing η as a natural transformation as in Section 8.2).

10.7.1 Theorem. *If the functor $F : C \to D$ has a left adjoint, then F preserves limits.*

Proof. Let $F : C \to D$ be a functor and assume that F has a left adjoint G. Then there exists a hom-set adjunction $\varphi : G \dashv F$. Let $f : I \to C$ be a functor with I small and let (c, η) be a limit of f. Let (d, θ) be a cone to Ff:

$$
\begin{tikzcd}
i \arrow[d, "\tau"] \\
j
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & f_i \arrow[dd, "f_\tau"] \\
G(d) \arrow[r, dotted, "\gamma"] \arrow[urr, bend left, "\varphi^{-1}(\theta_i)"] \arrow[drr, bend right, "\varphi^{-1}(\theta_j)"'] & c \arrow[ur, "\eta_i"] \arrow[dr, "\eta_j"'] & \\
 & & f_j
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & F(f_i) \arrow[dd, "F(f_\tau)"] \\
d \arrow[r, dotted, "\varphi(\gamma)"] \arrow[urr, bend left, "\theta_i"] \arrow[drr, bend right, "\theta_j"'] & F(c) \arrow[ur, "F(\eta_i)"] \arrow[dr, "F(\eta_j)"'] & \\
 & & F(f_j)
\end{tikzcd}
$$

For each $\tau : i \to j$ in I we have, using Theorem 10.2.1,

$$\varphi(f_\tau \varphi^{-1}(\theta_i)) = F(f_\tau)\theta_i = \theta_j,$$

so $f_\tau \varphi^{-1}(\theta_i) = \varphi^{-1}(\theta_j)$, which says that $(G(d), \varphi^{-1}(\theta))$ is a cone to f, where $\varphi^{-1}(\theta)_i := \varphi^{-1}(\theta_i)$ for each $i \in I$. For $\gamma : G(d) \to c$ and each $i \in I$ we have, using Theorem 10.2.1 again,

$$
\begin{aligned}
\eta_i \gamma = \varphi^{-1}(\theta)_i &\iff \eta_i \gamma = \varphi^{-1}(\theta_i) \\
&\iff \varphi(\eta_i \gamma) = \theta_i \\
&\iff F(\eta_i)\varphi(\gamma) = \theta_i.
\end{aligned}
$$

Therefore, since (c, η) is a limit of f, it follows that $(F(c), F\eta)$ is a limit of Ff. $\square$

The dual statement holds as well.

10.7.2 Theorem. *If the functor $G : D \to C$ has a right adjoint, then G preserves colimits.*

10.7.3 *Example* (Product in **Grp** necessarily Cartesian product) The forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint (Example 10.4.1). Therefore, since a product can be construed as a limit, Theorem 10.7.1 implies that the underlying set of any product of a family in **Grp** must be a product of the underlying sets of the family members, which is (isomorphic to) the Cartesian product of those sets by Example 3.3.1.

In other words, the choice to use the Cartesian product in our construction of a product in **Grp** (Example 3.3.2) was in a way forced on us. □