Handout 7 (Writing): Mathematical Writing Exercises

SOLUTIONS

Most of the exercises below are taken from the following manuscripts:

- Vivaldi, Franco. Mathematical Writing. Springer, 2014.
- Gillman, Leonard. Writing mathematics well: a manual for authors. Mathematical Assn of Amer, 1987.
- Halmos, Paul R. How to write mathematics. Enseign. Math 16, no. 2 (1970): 123-152.
- Knuth, Donald Ervin, Tracy Larrabee, Paul M. Roberts, and Paul M. Roberts. Mathematical writing. Vol. 14. Washington, DC: Mathematical Association of America, 1989.
- Lee, Kevin P. A guide to writing mathematics. Retrieved September 12 (2010): 2010.

Exercises

Identify problems with the use of mathematics in the following excepts of text, and rewrite them to fix the problems.

1. Let $f(t)$ be Gryffindor's score t minutes into a game against Slytherin. f is globberfluxible at $t = 3$.

   Solution: *Don't start a sentence with a formula or a mathematical symbol.*

   The function f is globberfluxible at $x = 3$.

2. The function $z^2 + 1$ is even.

   Solution: *Distinguish functions and their values.*

   - The function f defined by $f(z) = z^2 + 1$ is even.
   - The function $z \rightarrow z^2 + 1$ is even.

3. On a compact space every real-valued continuous function f is bounded.

   Solution: *Avoid using symbols if you don't need them. Halmos: "use no letter only once".*

   - On a compact space every real-valued continuous function is bounded.
   - (possibly) If f is a real-valued continuous function on a compact space, then f is bounded.

4. If $0 \leq \lim_{n\to\infty} \alpha_n^{1/n} = \rho \leq 1$, then $\lim_{n\to\infty} \alpha_n = 0$.

   Solution: *Avoid using symbols if you don't need them.* Here, "ρ" does not contribute. If it is a preparation for a proof, then introduce it in a separate equation at the beginning of the proof, e.g. "Write $\rho = \lim_{n\to\infty} \alpha_n^{1/n}$."

5. The union of a sequence of measurable sets is measurable.

   Solution: *"Sequence" is irrelevant, so don't draw reader's attention to this concept.*

   - The union of a countable set of measurable sets is measurable.
   - The union of a countably infinite set of measurable sets is measurable.

6. For invertible X, X^* is also invertible.

   Solution: *Be careful with commas; they can lead to ambiguity.*

   For invertible X, the adjoint X^* is also invertible.

7. Let $\delta = \frac{3}{4}\epsilon > 0$. Then ....

   Solution: *Make sure your mathematical notation reads grammatically, and don't use notation to suppress conditional words like "which", "then", etc.*

   Let $\delta = \frac{3}{4}\epsilon$. Then $\delta > 0$, and ....

8. If $x > 0$, then Euler proved in 1756 that ....

   Solution: *As Gillman (1982) says: "Don't ask a historical fact to depend on a mathematical hypothesis".*

   Euler proved in 1756 that if $x > 0$, then ....

9. Assume $x = 3$. Therefore $2x = 6$.

   Solution: *Use "then", not "therefore", after an assumption.*

   Assume $x = 3$. Then $2x = 6$.

10. And when $x = -1$ instead, we can see that...

    Solution: *When you use the word "instead", be clear about the contrast you are drawing. The reader should immediately understand what you are referring to.*

    And when $x = -1$ instead of $+1$, we can see that...

11. Let the angles of the triangle be δ, a_1, and t.

    Solution: *Use consistent notation.* The angles could be $\delta_1, \delta_2, \delta_3$, or α, β, δ, or a_1, a_2, a_3, etc.

12. ...Consider the quantity $a_1x + a_2y$.

    Solution: *Use consistent notation.* Plan your notation so you end up with either $ax + by$, or $a_1x_1 + a_2x_2$.

13. Let the number of elephants in the zoo in year n with $n \in [0, \infty)$ be $e(n)$. Suppose the growth rate is $\frac{de}{dn} = 2.5$.

    Solution: *Don't use standard symbols in nonstandard ways.* In this case, e usually represents the number $2.71828\ldots$, and n usually represents an integer, not a real number. *Furthermore, display inline fractions and derivatives with slashes, when possible.*

    Let the number of elephants in the zoo in year t be $x(t)$. Suppose the growth rate is $dx/dt = 2.5$.

14. Looking at the graph, we can see that the result is true.

    Solution: *Make sure you fully explain how the graph fits into your mathematical argument.*

    The graph increases sharply at $t = 3$, confirming our earlier prediction that the robots will begin a homicidal rampage three years from now.

15. (Hypotheses: f is continuous, f is differentiable, G is abelian.) ...
    By hypothesis, we have ...

    Solution: *When you invoke a key hypothesis, identify it as such and tell which one it is.*

    By hypothesis, the function f is continuous, hence we have ...

16. Compute $argmax(sin(x))$.

    Solution: *Use upright roman text for operators. Otherwise they will be confused with variables.* To do this in LaTeX, you can use `DeclareMathOperator` at the beginning of the document, or you can define a one-off operator using `operatorname`.

    Compute $\operatorname{argmax} \sin x$.

17. Isaac Newton once said "If I have seen further than others, it is by standing upon the shoulders of giants".

    Solution: *Do not use double quotes in LaTex. Instead, use the symbols are* ` `` ` *at the beginning of the quote (upper left corner of keyboard), and* `''` *at the end of the quote (two single quotation marks, just to the left of the ";" key.)*

    Isaac Newton once said "If I have seen further than others, it is by standing upon the shoulders of giants."

    You may ask whether the punctuation, here a period, goes inside or outside the quote. Often it is inside, but this is a convention that depends on the country, discipline, and even journal. In American English, periods and commas go inside the quotes; dashes, colons, and semicolons usually go outside; question marks or exclamation marks go inside or outside depending on whether they are part of the quote or not. In British English, periods and commas follow the same conventions as question marks and exclamation marks. You should check what the convention is in the publication you are writing for.

18. Let's derive the formula

    $$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}, \qquad r \neq 1. \tag{1}$$

    Define the LHS of the above equation to be $S_{r,n} = 1 + r + \cdots + r^n$. We compute

    $$rS_{r,n} - S_{r,n} = r^{n+1} - 1.$$

    If $r \neq 1$, divide both sides by $r - 1$ to obtain

    $$S_{r,n} = \frac{r^{n+1} - 1}{r - 1}$$

    which is equivalent to (1).

    Solution: *Suppress useless information. You do not need to routinely announce all the variables that a temporary symbol depends on.* In this case, we never need to know that $S_{r,n}$ depends on r or n, so omit these in the notation, and call the sum S. Try this instead:

    Define the LHS of (1) to be

    $$S = 1 + r + \cdots + r^n .$$

We compute

$$rS - S = r^{n+1} - 1.$$

If $r \neq 1$, divide both sides by $r - 1$ to obtain (1). □

Alternatively, you could skip introducing a symbol for $1 + r + \cdots + r^n$ altogether, and just repeat this expression when needed.

19. Let P be the escaped wombat population (in thousands) t years after 1990 and suppose that

$$P = 0.5(1.12)^t.$$

The wombat population in 1992 is approximately 672. We can see this by setting $t = 2$ and observing that

$$P = 0.5(1.12)^2 = 0.6272 \text{ thousand wombats.}$$

If we want to predict when the wombat population will reach 2000, we set $P = 2$ and solve for t using logarithms.

$$\begin{aligned} 2 &= 0.5(1.12)^t \\ \log 2 &= \log 0.5 + t \log 1.12 \\ t &= \frac{\log 2 - \log 0.5}{\log 1.12} \approx 12.23 \text{ years.} \end{aligned}$$

The wombat population will reach 2000 in the year 2002.

<u>Solution:</u> *Don't use the same symbol for multiple things. Here, P is the wombat population in general, then in 1992, then $P = 2$.*

Let P be the escaped wombat population (in thousands) t years after 1990 and suppose that

$$P = 0.5(1.12)^t.$$

By substituting 2 for t in the above equation, we can see that in 1992, the wombat population is approximately 672.

$$0.5(1.12)^2 = 0.6272 \text{ thousand wombats.}$$

Let t_{2000} be the year when the wombat population reaches 2000. Then,

$$\begin{aligned} 2 &= 0.5(1.12)^{t_{2000}} \\ \log 2 &= \log 0.5 + t_{2000} \log 1.12 \\ t_{2000} &= \frac{\log 2 - \log 0.5}{\log 1.12} \approx 12.23 \text{ years.} \end{aligned}$$

The wombat population will reach 2000 in the year 2002.

20. (Proof that the functions e^x and e^{2x} are linearly independent over the reals.)
Assume on the contrary that e^x and e^{2x} are linearly dependent. Then there exist constants c_1 and c_2, not both zero, such that

$$c_1 e^x + c_2 e^{2x} = 0 \qquad \text{for all } x.$$

Then $c_1 + c_2 e^x = 0$. Differentiating, we get $c_2 e^x = 0$, so $c_2 = 0$. But then $c_1 = 0$. Thus, $c_1 = c_2 = 0$.

This contradicts the fact that c_1 and c_2 are not both zero; therefore we must reject the assumption that the functions are linearly dependent. Consequently, they are linearly independent.

Solution: *Avoid spurious proofs by contradiction; use direct proofs when you can.*
Here, no use is ever made of the fact that c_1 and c_2 are not both zero. The proof can be made direct by deleting extraneous assumptions, as follows:

Assume that there exist real numbers c_1, c_2 such that

$$c_1 e^x + c_2 e^{2x} = 0 \qquad \text{for all } x.$$

Then $c_1 + c_2 e^x = 0$. Differentiating, we get $c_2 e^x = 0$, whence $c_2 = 0$. Then $c_1 = 0$. Thus, $c_1 = c_2 = 0$. Therefore, e^x and e^{2x} are linearly independent. □

21. (Proof of the Mean Value Theorem, from Rolle's Theorem.)
Consider a function f which is continuous on $[a, b]$ and differentiable on its interior. Let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)\,. \tag{2}$$

Then $F(a) = 0$ and $F(b) = 0$ and F is continuous on $[a, b]$ and differentiable in (a, b). Therefore F satisfies the hypotheses of Rolle's theorem, so there is a point z in (a, b) such that $F'(z) = 0$. Taking the derivative of (2) shows that $f'(z) = (f(b) - f(a))/(b - a)$.

Solution: *Don't use math notation if you don't have to.* Here, most people believe that two points determine a line, so we don't need to bring in the detailed formula for F. In addition, this proof gives no insight into where F comes from. Instead, try:

Consider a function f which is continuous on $[a, b]$ and differentiable on its interior. Let g be a straight line that agrees with f at a and at b. Then $f - g$ satisfies the hypotheses of Rolle's Theorem, so there is a point z in (a, b) such that $f'(z) = g'(z)$. Since the slope of g is $(f(b) - f(a))/(b - a)$, the result follows. □

For an even more horrendous proof, see the Wikipedia article on the Mean Value Theorem: `https://en.wikipedia.org/wiki/Mean_value_theorem#Proof`.

22. Consider a set of n isolated vertices, and let P be the probability measure that is uniform on the set of all undirected graphs. Let $M > 0$ be a positive integer, and consider a sequence of graphs $G_1, G_2, \ldots, G_M$ chosen independently from distribution P. Consider the subsequence $G_{\alpha_1}, \ldots, G_{\alpha_k}$ of graphs with at least $2n$ edges, and suppose they have $n_{\alpha_1}, \ldots, n_{\alpha_k}$ edges respectively. What is the probability that the total number of edges in this subsequence is at least $2Mn$?

Solution: *Lots of poor notation here: (1) don't use n_i for the number of edges, since n was used for the number of vertices; (2) we could avoid double subscripts by not explicitly labelling the first sequence of graphs; (2) saying "$M > 0$ is a positive integer" is redundant, and is not grammatical. Other issues include: saying "sequence" when order is not important, and not saying which kinds of graphs P is a measure for.* We could rewrite this as:

Let P be the probability measure that is uniform on the set of all undirected graphs with n vertices. Let M be a positive integer, and consider a set of M graphs chosen independently from distribution P. Suppose that K of these graphs have at least $2n$ edges; call these graphs $G_1, G_2, \ldots, G_K$. What is the probability that the total number of edges in the graphs $G_1, G_2, \ldots, G_K$ is at least $2Mn$?

Or, with less notation:

Let P be the probability measure that is uniform on the set of all undirected graphs with n vertices. Let M be a positive integer, and consider a set of M graphs chosen independently from distribution P. What is the probability that, among all the graphs with at least $2n$ edges, the total number of edges is at least $2Mn$?

23. Simplify this definition:

$$A = \left\{ y \in \mathbb{Q} \;:\; y = \frac{x}{x^2+1}, \quad x \in \mathbb{Z}, \quad x < 0 \right\}.$$

Solution: *There is too much notation here, to define a relatively simple set. The inequality adds further complexity.*

Instead, try:

$$A = \left\{ \frac{-n}{n^2+1} \;:\; n \in \mathbb{N} \right\}.$$

24. Simplify / improve this definition:

$$z(y_1, y_2, \ldots) = \sum_{i=1}^{\infty} \sum_{y=0}^{y_i - 1} (y+1) x^{i-1}.$$

Solution: *Meaning of z is not at all clear. Furthermore, this quantity depends on x, but this dependence does not appear explicitly. Finally, parameters y_i are integers, so it's more clear to use a standard symbol for an integer, such as n_i.*

To simplify, start by splitting up the sum, as

$$z(x, n_1, n_2, \ldots) = \sum_{i=1}^{\infty} d_i x^{i-1}, \qquad d_i = \sum_{k=0}^{n_i - 1} (k+1).$$

We now see the sums defining the d_i can be evaluated explicitly, leading to the more transparent definition

$$z(x, \mathbf{n}) = \frac{1}{2} \sum_{i=1}^{\infty} n_i (n_i + 1) x^{i-1}, \qquad \mathbf{n} = (n_1, n_2, \ldots).$$

Quotes and Notes

Halmos

- My advice about the use of words can be summed up as follows. (1) Avoid technical terms, and especially the creation of new ones, whenever possible. (2) Think hard about the new ones that you must create; consult Roget; and make them as appropriate as possible. (3) Use the old ones correctly and consistently, but with a minimum of obtrusive pedantry.

- Everything said about words applies, mutatis mutandis, to the even smaller units of mathematical writing, the mathematical symbols. The best notation is no notation; whenever it is possible to avoid the use of a complicated alphabetic apparatus, avoid it. A good attitude to the preparation of written mathematical exposition is to pretend that it is spoken. Pretend that you are explaining the subject to a friend on a long walk in the woods, with no paper available; fall back on symbolism only when it is really necessary.

- One place where cumbersome notation quite often enters is in mathematical induction. Sometimes it is unavoidable. More often, however, I think that indicating the step from 1 to 2 and following it by an airy "and so on" is as rigorously unexceptionable as the detailed computation, and much more understandable and convincing. Similarly, a general statement about n X n matrices is frequently best proved not by the exhibition of many a_{ij}'s, accompanied by triples of dots laid out in rows and columns and diagonals, but by the proof of a typical (say 3x3) special case.

Gillman

- I. Use an uncomplicated symbol in place of an elaborate one. II. Discard any symbol that is just plain unnecessary. III. Simplify the mathematical argument itself.

- That's the interesting thing about this kind of editing: once you have simplified the mathematics and the notation so that the argument is easier to follow, you may see a way of simplifying the reasoning still further.

Other remarks Interesting discussion on pedantic notation

- `https://math.stackexchange.com/questions/2505777/abusing-mathematical-notation-are-these-exampl`