COSC 290 Discrete Structures

Lecture 7: Nested quantifiers

Prof. Michael Hay
Wednesday, Feb. 7, 2018
Colgate University

Plan for today

1. Expressing statements in predicate logic (continued...)
2. Nested Quantifiers
3. Negating quantifiers

Expressing statements in predicate logic (continued...)

Review of fastest person question

Let P be the set of all people. Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Which of the following is the correct definition for $fastest(x)$?

This works:

$$fastest(x) := \forall y \in P - \{ x \} : faster(x, y)$$

So does this:

$$fastest(x) := \neg(\exists y \in P : faster(y, x))$$

however, with this version, it is possible for two or more persons to be tied for fastest.

Poll: fastest lacrosse player

As before, let P be the set of all people. Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Let $lax(x)$ be true if x plays lacrosse and false otherwise.

Which of the following is the correct definition for $fastestLax(x)$, the fastest lacrosse player?

A) $fastestLax(x) := \forall y \in P - \{x\} : (lax(y) \wedge faster(x, y))$

B) $fastestLax(x) := \forall y \in P - \{x\} : (lax(y) \implies faster(x, y))$

C) $fastestLax(x) := lax(x) \wedge \forall y \in P - \{x\} : faster(x, y)$

D) $fastestLax(x) := lax(x) \wedge \forall y \in P - \{x\} : (lax(y) \wedge faster(x, y))$

E) $fastestLax(x) := lax(x) \wedge \forall y \in P - \{x\} : (lax(y) \implies faster(x, y))$

Poll: not the slowest lacrosse player

As before, let P be the set of all people and $faster(x, y)$ be true if x runs faster than y and false otherwise; and $lax(x)$ be true if x plays lacrosse and false otherwise.

Which of the following is the correct definition for $notSlowestLax(x)$, which is true when x is *not* the slowest lacrosse player?

A) $notSlowestLax(x) := \exists y \in P : (lax(y) \wedge faster(x, y))$

B) $notSlowestLax(x) := \exists y \in P : (lax(y) \implies faster(x, y))$

C) $notSlowestLax(x) := lax(x) \wedge \exists y \in P : faster(x, y)$

D) $notSlowestLax(x) := lax(x) \wedge \exists y \in P : (lax(y) \wedge faster(x, y))$

E) $notSlowestLax(x) := lax(x) \wedge \exists y \in P : (lax(y) \implies faster(x, y))$

Nested Quantifiers

Nested quantifiers

Let $S := \{2, 3, 6\}$. Consider the following two propositions:

$$\varphi_1 := \forall x \in S \exists y \in S : isDivisibleBy(x, y)$$

$$\varphi_2 := \exists x \in S \forall y \in S : isDivisibleBy(x, y)$$

What are the truth values of these two propositions?

(Write out on board.)

φ_1 is logically equivalent to:

$(\exists y_1 \in S : isDivisibleBy(2, y_1)) \wedge (\exists y_2 \in S : isDivisibleBy(3, y_2)) \wedge (\exists y_3 \in S : isDivisibleBy(6, y_3))$

φ_2 is logically equivalent to:

$(\forall y_1 \in S : isDivisibleBy(2, y_1)) \vee (\forall y_2 \in S : isDivisibleBy(3, y_2)) \vee (\forall y_3 \in S : isDivisibleBy(6, y_3))$

Both φ_1 and φ_2 are true.

Order of quantifiers matters

Let $S := \{ 2, 3, 6 \}$.

Contrast the following two propositions:

$$\varphi_1 := \forall x \in S \exists y \in S : \mathit{isDivisibleBy}(x, y)$$

$$\varphi_3 := \exists y \in S \forall x \in S : \mathit{isDivisibleBy}(x, y)$$

We already saw that φ_1 is true. What about φ_3?

φ_3 is logically equivalent to:

$(\forall x_1 \in S : \mathit{isDivisibleBy}(x_1, 2)) \vee (\forall x_2 \in S : \mathit{isDivisibleBy}(x_2, 3)) \vee (\forall x_3 \in S : \mathit{isDivisibleBy}(x_3, 6))$

φ_3 is false.

Another example

Consider a set of professors P, students S, and courses C.

- Let $\mathit{takes}(s, c)$ be true when student s takes course c.
- Let $\mathit{teaches}(p, c)$ be true when professor p teaches course c.

Let's define predicate $\mathit{favCourse}(c)$ that is true when c is a course taken by all students.

$$\mathit{favCourse}(c) := \forall s \in S : \mathit{takes}(s, c)$$

Example continued...

Consider a set of professors P, students S, and courses C.

- Let $\mathit{takes}(s, c)$ be true when student s takes course c.
- Let $\mathit{teaches}(p, c)$ be true when professor p teaches course c.
- Let $\mathit{favCourse}(c)$ be true when c is a course taken by all students.

Let's define $\mathit{profOfFav}(p)$ that is true when professor p teaches a course that is taken by all students.

$$\mathit{profOfFav}(c) := \exists c \in C : (\mathit{teaches}(p, c) \wedge \mathit{favCourse}(c))$$

Unpacking a complex predicate can reveal nested quantifiers

Let's "unpack" the predicate *profOfFav*:

$$\begin{aligned} \mathit{profOfFav}(p) &:= \exists c \in C : (\mathit{teaches}(p, c) \wedge \mathit{favCourse}(c)) \\ &\equiv \exists c \in C : (\mathit{teaches}(p, c) \wedge (\forall s \in S : \mathit{takes}(s, c))) \end{aligned}$$

Order of quantification

The order the of the quantifiers matters.

$$Q_1(p) := \exists c \in C : \forall s \in S : (\mathit{teaches}(p, c) \land \mathit{takes}(s, c))$$

$Q_1(p)$: “Prof. who teaches a course every student takes”

vs.

$$Q_2(p) := \forall s \in S : \exists c \in C : (\mathit{teaches}(p, c) \land \mathit{takes}(s, c)))$$

$Q_2(p)$: “Prof. who teaches every student”
(but not necessarily in the same course).

Negating quantifiers

Negating quantifier

Negating a quantifier “flips” it:

$$\alpha := \forall x \in S : P(x)$$

$$\neg\alpha \equiv \neg(\forall x \in S : P(x)) \equiv \exists x \in S : \neg P(x)$$

$$\beta := \exists x \in S : Q(x)$$

$$\neg\beta \equiv \neg(\exists x \in S : Q(x)) \equiv \forall x \in S : \neg Q(x)$$

Negating nested quantifiers

Let $S := \{2, 3, 6\}$. Recall the definition of φ_2,

$$\varphi_2 := \exists x \in S \forall y \in S : \mathit{isDivisibleBy}(x, y)$$

What is $\neg\varphi_2$?

Let’s take this part “$\forall y \in S : \mathit{isDivisibleBy}(x, y)$” and define it as a predicate: $\mathit{divisibleByAll}(x) := \forall y \in S : \mathit{isDivisibleBy}(x, y)$.

Thus, $\varphi_2 \equiv \exists x \in S : \mathit{divisibleByAll}(x)$

$$\begin{aligned}\neg\varphi_2 &\equiv \neg(\exists x \in S : \mathit{divisibleByAll}(x))\\ &\equiv \forall x \in S : \neg\mathit{divisibleByAll}(x)\\ &\equiv \forall x \in S : \neg(\forall y \in S : \mathit{isDivisibleBy}(x, y))\\ &\equiv \forall x \in S : \exists y \in S : \neg\mathit{isDivisibleBy}(x, y)\end{aligned}$$

Negating quantifiers

What about a course that *isn't* a favorite? In other words, a course for which $\neg favCourse(c)$ is true?

$$\begin{aligned}\neg favCourse(c) &\equiv \neg(\forall s \in S : takes(s, c)) \\ &\equiv \exists s \in S : \neg takes(s, c)\end{aligned}$$

If the course isn't a favorite, then there must be one student who doesn't take it.

Negating nested quantifiers

What about a professor that *isn't* a teacher of a favorite course? In other words, a professor for which $\neg profOfFav(p)$ is true?

$$\begin{aligned}\neg profOfFav(p) &\equiv \neg(\exists c \in C : (teaches(p, c) \land favCourse(c))) \\ &\equiv \forall c \in C : \neg(teaches(p, c) \land favCourse(c)) \\ &\equiv \forall c \in C : (\neg teaches(p, c) \lor \neg favCourse(c)) \\ &\equiv \forall c \in C : (teaches(p, c) \implies \neg favCourse(c)) \\ &\equiv \forall c \in C : (teaches(p, c) \implies (\exists s \in S : \neg takes(s, c)))\end{aligned}$$

If the professor isn't a favorite, then for every course that this professor teaches, there must be at least one student not taking the course.