COSC 290 Discrete Structures

Lecture 28: Division rule

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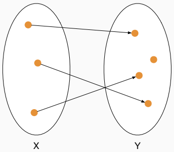
Plan for today

1. Mapping rule
2. Division rule

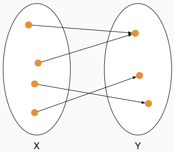
Mapping rule

One-to-one

One-to-one: for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.



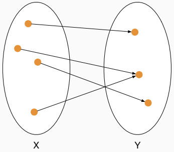
f is one-to-one $\implies |X| \leq |Y|$



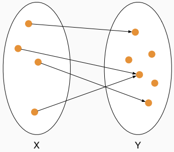
f is *not* one-to-one and $|X| \not\leq |Y|$

Onto

Onto: for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.



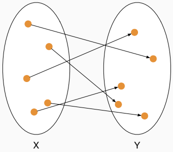
f is onto $\implies |X| \geq |Y|$



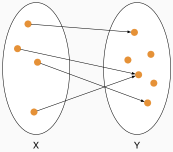
f is *not* onto and $|X| \not\geq |Y|$

Bijective

Bijective: for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.



f is bijective $\implies |X| = |Y|$



f is *not* bijective and $|X| \neq |Y|$

Example: applying the mapping rule

Let $B_n \subset \{\,], [\, \}^n$ be a set of strings consisting of *balanced brackets*.

Examples:

$$B_4 = \{\, [[]], [][] \,\}$$

$$B_6 = \{\, [[[]]], [[][]], [[]][], [][[]], [][][] \,\}$$

Claim: $2^{n/4} \leq |B_n| \leq 2^n$.

We will prove this using the mapping rule (repeated below):

Let $f : X \to Y$.

- $|X| \geq |Y|$ if and only if f is onto.
- $|X| \leq |Y|$ if and only if f is one-to-one.
- $|X| = |Y|$ if and only if f is bijective (onto *and* one-to-one).

Poll: only so many friends

Let f : *Friends* $\to \{\, -1, 102, 201, 290, 301, 302, 304, 460, 480 \,\}$ be a function that maps each person in *Friends* to the highest numbered required COSC course they have taken (-1 if none).

When applied to your friends, suppose the maximum value is 301. Therefore you have at most 5 friends. What's wrong with this claim?

A) f is not necessarily bijective
B) f is not necessarily onto
C) f is not necessarily one-to-one
D) The logic is correct, but claim is false because one of my friends took 304!

Mapping rule: Let $f : X \to Y$.

- $|X| \geq |Y|$ if and only if f is onto.
- $|X| \leq |Y|$ if and only if f is one-to-one.
- $|X| = |Y|$ if and only if f is bijective (onto *and* one-to-one).

Division rule

k-to-1 functions and division rule

Definition (k-to-1 functions)

Let $f : X \to Y$. We say that f is k-to-1 if for all $y \in Y$, there are k distinct elements of X such that $f(x) = y$. In other words,

$$\forall y \in Y : |\{x \in X : f(x) = y\}| = k$$

Theorem (Division rule)

Let $f : X \to Y$. If f is k-to-1, then $|X| = k \cdot |Y|$.

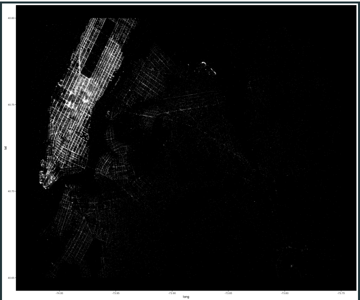
(Draw picture on board)

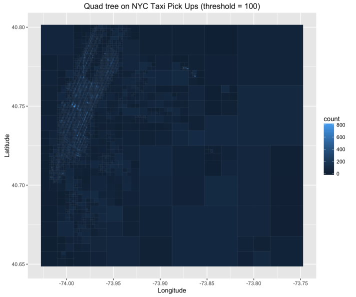
Example: division rule

Spatial indexing (next two slides). To efficiently support map services (Google maps, Yelp "nearby"), data stored in spatial indexes.

A rectangular range query specifies a type of item (e.g., taqueria) and a *rectangle* and returns all such items that overlap with the given rectangle.

Assume city is divided into $n \times n$ discrete points. Rectangle must include at least one point. How many rectangle queries are possible?





Example: division rule

The rectangular range query counting problem.

Assume city is divided into $n \times n$ discrete points, $\{0, \ldots, n-1\}^2$.

A *rectangle query* is a 4-tuple $\langle x_1, y_1, x_2, y_2 \rangle$ where $x_1 \leq x_2$ and $y_1 \leq y_2$.

(Think of $\langle x_1, y_1 \rangle$ as defining lower left corner of rectangle and $\langle x_2, y_2 \rangle$ as defining upper right corner.)

The answer to a query is the total number of items located at any point $\langle x, y \rangle$ such that $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$.

How many rectangle queries are possible?

Counting rectangular range queries

A *rectangle query* is a 4-tuple $\langle x_1, y_1, x_2, y_2 \rangle$ where $x_1 \leq x_2$ and $y_1 \leq y_2$.

Cases:

1. $x_1 < x_2$ and $y_1 < y_2$
2. $x_1 = x_2$ and $y_1 < y_2$
3. $x_1 < x_2$ and $y_1 = y_2$
4. $x_1 = x_2$ and $y_1 = y_2$

These cases are disjoint and cover all possibilities, so we can count each one separately and apply the sum rule to get the total number of rectangle queries.

Case 1: $x_1 < x_2$ and $y_1 < y_2$

How many ways to choose two distinct x values? Let $N := \{0, \ldots, n-1\}$.

Let $S := \{ \langle x_a, x_b \rangle \in N \times N : x_a \neq x_b \}$ What is $|S|$?

$|S| = n \cdot (n-1)$

But in rectangle query, x_1 must be less than x_2. So for each pair in S, apply this function to swap out-of-order pairs:

$$f(\langle x_a, x_b \rangle) := \begin{cases} \langle x_a, x_b \rangle & \text{when } x_a < x_b \\ \langle x_b, x_a \rangle & \text{when } x_a > x_b \end{cases}$$

Let $T := \{ f(\langle x_a, x_b \rangle) : \langle x_a, x_b \rangle \in S \}$. What is $|T|$?

This function f is 2-to-1, so by division rule, $|T| = |S|/2 = n \cdot (n-1)/2$

Apply same reasoning to $\langle y_1, y_2 \rangle$. By product rule, total 4-tuples $\langle x_1, x_2, y_1, y_2 \rangle$ is $(n \cdot (n-1)/2)^2$

Counting rectangular range queries continued...

A *rectangle query* is a 4-tuple $\langle x_1, y_1, x_2, y_2 \rangle$ where $x_1 \leq x_2$ and $y_1 \leq y_2$.

Cases:

1. $x_1 < x_2$ and $y_1 < y_2$: $(n \cdot (n-1)/2)^2$ tuples
2. $x_1 = x_2$ and $y_1 < y_2$: $n^2 \cdot (n-1)/2$ tuples
3. $x_1 < x_2$ and $y_1 = y_2$: $n^2 \cdot (n-1)/2$ tuples
4. $x_1 = x_2$ and $y_1 = y_2$: n^2 tuples

Other cases are left as exercises... but answers are shown above.

Exercise: Knights at a round table

How many ways can you *arrange* n knights at a round table?

A *seating* defines who sits where. Two seatings are considered same *arrangement* if each knight has the same knight on its left and right in both seatings.

Example: here are two distinct *seatings*, but they represent the same *arrangement*.

Images taken and problem adapted from Lehman et al. *Mathematics for Computer Science*, 2017.