COSC 290 Discrete Structures

Lecture 24: Transitive closure and equivalence relations

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Monday, Apr. 2, 2018
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Plan for today

1. Closures
2. Equivalence relations and partial orders

Closures

Closures

A closure of a relation R on A is a smallest $R' \supseteq R$ that satisfies a desired property.

- reflexive closure:

$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:
  (*hint*: what does $R \circ R$ give you?)

Poll: towards transitive closure

Consider the *parentOf* relation on persons where $\langle p, c\rangle \in \mathit{parentOf}$ if p is the parent of c. What is $\mathit{parentOf} \circ \mathit{parentOf}$?

A) ancestorOf
B) grandParentOf
C) parentOf
D) childOf
E) grandChildOf
F) descendantOf

Bonus question for you to consider during the discussion period: what is $\mathit{parentOf} \cup (\mathit{parentOf} \circ \mathit{parentOf})$?

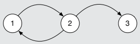
Computing the transitive closure

Input: Relation $R \subseteq A \times A$.
Output: smallest $R' \supseteq R$ that is *transitive*

$R' := R$
repeat
    $\mathit{new} := (R \circ R') - R'$
    $R' := R' \cup \mathit{new}$
until $|\mathit{new}| = 0$
return R'

Example (Applying transitive closure algorithm)

Let's apply the algorithm to this example:



Exercise

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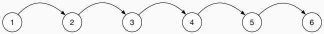
Exercise: working in groups, apply the algorithm to this graph. How many times does the loop repeat?



Closures

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- reflexive closure:

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- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:

$$R' = R \cup (R \circ R) \cup ((R \circ R) \circ R) \cup \cdots$$

Equivalence relations and partial orders

Recall: relation properties

For relation R on $A \times A$.

- **R** *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.
- **IR** *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.
- **S** *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.
- **antiS** *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.
- **AS** *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.
- **T** *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Special relation: equivalence relation

Relation R on A is an equivalence relation if it is reflexive, symmetric, transitive.

Conventions: use $\equiv$ as the "name" of the relation (as opposed to a letter like R) and use *infix* notation: $a \equiv b$ instead of $\langle a, b \rangle \in \ \equiv$.

Intuition: equivalence relations behave like $=$.

Equivalence classes

When R is an equivalence relation on A, the elements of A be can be partitioned into equivalence classes. (See book for formal definition.)

Example (Equivalence classes)

Let R denote the equivalence relation on $\{ 0, 1, 2, \ldots, 10 \}$ where $\langle a, b \rangle \in R$ if $(a \bmod 2) = (b \bmod 2)$

The equivalence classes are:

- $\{ 0, 2, 4, 6, 8, 10 \}$
- $\{ 1, 3, 5, 7, 9 \}$

Exercise

Let $S \coloneqq \{0, 1\}^3$ be the set of length 3 bitstrings. Consider the two binary relations R_1 and R_2 on S defined as follows:

1. $(x, y) \in R_1$ if x and y are identical or reverses of each other. For example, if $x = b_1 b_2 \dots b_n$, we say that $\text{reverse}(x) = b_n b_{n-1} \dots b_1$. Then, $(x, y) \in R_1$ iff $x = y$ or $x = \text{reverse}(y)$.
2. $(x, y) \in R_2$ if x and y are rearrangements/permutations of each other. For example, if $x = b_1 b_2 \dots b_n$, then $(x, y) \in R_2$ iff there exists some bijection $p : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $y = b_{p(1)} b_{p(2)} \dots b_{p(n)}$.

Working in small groups, write out the equivalence classes for R_1 and R_2.