COSC 290 Discrete Structures

Lecture 24: Relations: XXXX

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Plan for today

1. Closures
2. Warshall relations
3. Equivalence relations and partial orders
4. Hasse diagram
5. Topological sort

Closures

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  (*hint*: what does $R \circ R$ give you?)

Poll: towards transitive closure

Consider the *parentOf* relation on persons where $\langle p, c \rangle \in \mathit{parentOf}$ if p is the parent of c. What is $\mathit{parentOf} \circ \mathit{parentOf}$?

A) ancestorOf
B) grandParentOf
C) parentOf
D) childOf
E) grandChildOf
F) descendantOf

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- transitive closure:

$$R' = R \cup (R \circ R) \cup ((R \circ R) \circ R) \cup \cdots$$

Warshall relations

Warshall relation

Let $A := \{ a_1, a_2, \ldots, a_n \}$, a finite set.

Let R be a relation on A.

For $k = 0$ to n, let W_k denote the k^{th} Warshall relation for R where W_k is defined as...

- $W_0 := R$
- For $k \geq 1$, W_k is a relation on A such that $\langle a_i, a_j \rangle \in W_k$ iff there is a sequence of relationships in R connecting a_i to a_j using any subset of the elements $\{a_1, a_2, \ldots, a_k\}$ as intermediates.

Example

W_0 (i.e., this is the relation R)

$$\begin{matrix} F & F & F & T \\ T & F & F & F \\ F & T & F & F \\ F & T & F & F \end{matrix}$$

W_1

$$\begin{matrix} F & F & F & T \\ T & F & F & T \\ F & T & F & F \\ F & T & F & F \end{matrix}$$

W_2

$$\begin{matrix} F & F & F & T \\ T & F & F & T \\ T & T & F & T \\ T & T & F & T \end{matrix}$$

W_3

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W_4

$$\begin{matrix} T & T & F & T \\ T & T & F & T \\ T & T & F & T \\ T & T & F & T \end{matrix}$$

Equivalence relations and partial orders

Recall: relation properties

For relation R on $A \times A$.

R *reflexive*: for every $a \in A$, $\langle a, a\rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a\rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b\rangle \in R$, then $\langle b, a\rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b\rangle \in R$, then $\langle b, a\rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$, then $\langle a, c\rangle \in R$.

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Intuition: equivalence relations behave like $=$.

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Example: the prefixOf relation:

- "a" $\preceq$ "aa"
- "aa" $\preceq$ "aardvark"
- not all pairs comparable: "a" $\not\preceq$ "b" and "b" $\not\preceq$ "a"

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Example: the ancestorOf relation (ancestor is parent or (recursively) parent of ancestor):

- "DT" $\prec$ "Don Jr"
- "Hanns Drumpf" $\prec$ "DT" (#makedonalddrumpfagain)
- not all pairs comparable: "Harry Potter" $\not\prec$ "Aunt Petunia" and "Aunt Petunia" $\not\prec$ "Harry Potter"

Poll: partial order

Relation $\preceq$ is a partial order if it is reflexive, antisymmetric, transitive.

Consider two relations on a set of track runners:

- $a \preceq_1 b$ if the number of races in which a competed is no more than the number in which b competed.
- $a \preceq_2 b$ if the total amount of time (measured in nanoseconds with laser precision) that a ran is no more than the total amount of time that b ran.

Is $\preceq_1$ a partial order? Is $\preceq_2$ a partial order?

A) Yes, Yes
B) Yes, No
C) No, Yes
D) No, No

Hasse diagram

Hasse diagram

A partial order $\preceq$ on A can be drawn using a Hasse diagram.

- Draw nodes: one node for each A
- Draw edges: edge from a to b if $a \preceq b$, except...
- ... *omit* edges that can be inferred by reflexivity
- ... *omit* edges that can be inferred by transitivity
- ... and *layout* nodes “by level“ if $a \preceq b$ for $a \neq b$, then a is placed *lower* than b

Example: isSubstringOf relation on the strings $\{ a, b, c, ab, bc, abc, cd \}$.

Exercise: draw Hasse diagram

Complete the following exercise: on a piece of paper, draw a Hasse diagram for the relation on $A := \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$ for the relation $R \subseteq A \times A$ where

$$R := \{\langle x, y\rangle \in A \times A : y \bmod x = 0\}$$

- Draw nodes: one node for each A
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Topological sort

Example

A to do list,

$$[attendClass, sleep, borrowBook, eat, brushTeeth, study]$$

with constraints:

- $borrowBook \preceq study$
- $study \preceq attendClass$
- $sleep \preceq attendClass$
- $eat \preceq brushTeeth$
- $brushTeeth \preceq sleep$

What should you do *first*? Brush teeth? Eat? Borrow book?

Topological ordering

Given a partial order $\preceq$, a topological ordering is a total order $\preceq_{total}$ that is *consistent* with $\preceq$.

Total order

Relation R is a total order if it is a partial order where every pair is comparable (either $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$).

A total order can be written succinctly as an ordered list.

Exercise

Suppose you have a findMinimal(S) method that finds a minimal element among S.

x is minimal in S if $\forall y \in S - \{ x \} : y \not\preceq x$

How could you use this to compute the topological sort of a partial order?

Suppose findMinimal(S) had cost $f(n)$ where n is the size of the set. What is the runtime of your algorithm?