COSC 290 Discrete Structures

Lecture 31: Probability: Sample Space and Events

Prof. Michael Hay

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Colgate University

Plan for today

1. Probability problems
2. Key concepts: sample space, probability, event

Probability problems

Monty Hall Problem

Described in book (p. 1012). Should you switch doors?

Yes!

Interesting backstory: Marilyn vos Savant (`http://www.nytimes.com/1991/07/21/us/behind-monty-hall-s-doors-puzzle-debate-and-answer.html`)

Poll: boy or girl paradox

The following two part question is similar to a question posed to Ms. vos Savant (a math savant who used to write the ‘Ask Marilyn’ puzzler column in Parade magazine).

- Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?
- Ms. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

A) $\frac{1}{2}$ and $\frac{1}{2}$ respectively
B) $\frac{1}{3}$ and $\frac{1}{2}$ respectively
C) $\frac{1}{2}$ and $\frac{1}{3}$ respectively
D) $\frac{1}{3}$ and $\frac{1}{3}$ respectively
E) None of the above

Source: https://en.wikipedia.org/wiki/Boy_or_Girl_paradox

Key concepts: sample space, probability, event

Strange dice

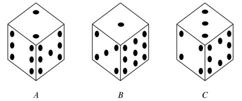

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each 1/3.

A \$100 bet. You choose a die, other player chooses die from among remaining two dice. Each player rolls his/her die. Highest roll wins. Should you take the bet? If so, which die do you choose?

Source for strange dice image: Mathematics for Computer Science, 2017, Eric Lehman, F Tom Leighton, Albert R Meyer.

Four steps

1. Find the sample space
2. Define events of interest
3. Determine outcome probabilities (possibly using tree diagrams)
4. Compute event probabilities

Sample space

An outcome of a probabilistic process is the sequence of results for all randomly determined quantities.

The sample space, denoted S, is the set of all outcomes.

Examples:

1. Flipping a coin. $S = \{ \text{H}, \text{T} \}$
2. Flipping two coins. $S = \{ \text{HH}, \text{HT}, \text{TH}, \text{TT} \}$
3. The birth month of a randomly chosen student.
   $S = \{ \text{Jan}, \text{Feb}, \ldots, \text{Dec} \}$
4. Drawing a poker hand from a 52 card deck.
   S contains $\binom{52}{5}$ possible outcomes, here is *one* of those outcomes: $\{ J\heartsuit, 4\clubsuit, 3\clubsuit, 10\diamondsuit, 2\heartsuit \}$

Four steps

1. Find the sample space
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Events

Let S be a sample space.

An event E is a subset of S.

Examples:

1. When flipping a coin: getting heads $E_1 := \{ \text{H} \}$
2. When flipping two coins: getting at least one tails
   $E_2 := \{ \text{HT}, \text{TH}, \text{TT} \}$
3. When selecting birth month: the month ends in 'y'
   $E_3 := \{ \text{Jan}, \text{Feb}, \text{May}, \text{Jul} \}$
4. When drawing a poker hand from a 52 card deck: getting four of a kind.
   E_4 is a subset containing $\binom{13}{1} \cdot \binom{48}{1}$ of the $\binom{52}{5}$ possible outcomes.
   This is *one* of the outcomes in E_4: $\{ 10\heartsuit, 10\clubsuit, 10\spadesuit, 10\diamondsuit, 2\heartsuit \}$

Four steps

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Probability

A probability function on sample space S is a function $Pr : S \to \mathbb{R}$ such that

$$\sum_{s \in S} Pr(s) = 1$$

and $\forall s \in S : Pr(s) \geq 0$.

One (but not only!) interpretation: Pr represents fraction of the time outcome would occur if random process was repeated many times.

Examples:

1. Flipping a coin. $s \coloneqq \text{H}$ and $Pr(s) = \frac{1}{2}$
2. Flipping two coins. $s \coloneqq \text{HT}$ and $Pr(s) = \frac{1}{4}$
3. The birth month of a randomly chosen student. $s \coloneqq \text{Feb}$ and $Pr(s) = \frac{28}{365}$
4. Drawing a poker hand from a 52 card deck. Let $s \coloneqq \{ 10\heartsuit, 10\clubsuit, 10\spadesuit, 10\diamondsuit, 2\heartsuit \}$, then $Pr(s) = \frac{1}{\binom{52}{5}}$

Tree diagrams

When the outcome is determined by a *sequence* of randomly determined quantities, it can be helpful to represent the process using a tree diagram.

Tree:

- Internal node: a single random quantity (edges are possible choices for that random quantity)
- Leaf: an outcome

Probabilities:

- Assign a probability to each edge in tree.
- Compute probability of each outcome (leaf): equal to product of edge probabilities on path from root to leaf.

Examples:

- Flipping two coins
- Strange dice game

Sample space, events, probabilities for a game of strange dice

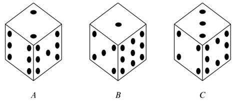

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each 1/3.

Suppose you choose B and other player chooses A. What is sample space? What is the event that you win? What is probability of an outcome?

Source for strange dice image: Mathematics for Computer Science, 2017, Eric Lehman, F Tom Leighton, Albert R Meyer.

Four steps

1. Find the sample space
2. Define events of interest
3. Determine outcome probabilities (possibly using tree diagrams)
4. Compute event probabilities

Probability of an event

The probability of event E is

$$Pr(E) = \sum_{s \in E} Pr(s)$$

Examples: what is the probability of...

1. When flipping a coin: getting heads $E_1 := \{ \text{H} \}$ and $Pr(E_1) = Pr(\text{H}) = \frac{1}{2}$
2. When flipping two coins: getting at least one tails $E_2 := \{ \text{HT}, \text{TH}, \text{TT} \}$ and $Pr(E_2) = Pr(\text{HT}) + Pr(\text{TH}) + Pr(\text{TT}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$
3. When selecting birth month: the month ends in 'y' $E_3 := \{ \text{Jan}, \text{Feb}, \text{May}, \text{Jul} \}$ and $Pr(E) = \frac{31}{365} + \frac{28}{365} + \frac{31}{365} + \frac{31}{365} = \frac{121}{365}$
4. When drawing a poker hand from a 52 card deck: getting four of a kind. $Pr(E_4) = \frac{\binom{13}{1} \cdot \binom{48}{1}}{\binom{52}{5}}$

Probability that you win game of strange dice

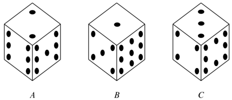

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each 1/3.

Suppose you choose B and other player chooses A. What is probability that you win?

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Poll: strange dice

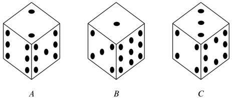

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each 1/3.

Suppose you choose A and other player chooses C. What is the probability that you win (roll higher)?

A) $\frac{3}{9}$ B) $\frac{4}{9}$ C) $\frac{5}{9}$ D) $\frac{6}{9}$

Source for strange dice image: Mathematics for Computer Science, 2017, Eric Lehman, F Tom Leighton, Albert R Meyer.

Sample space, events, probabilities for a game of strange dice

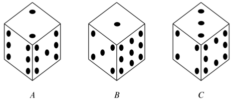

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each 1/3.

This time you get to choose first. Since A beats B more often than not and C beats A more often than not, then you pick C. Other player chooses B. Are you more likely to win? No! ("beats more often" relation is not transitive!)

Source for strange dice image: Mathematics for Computer Science, 2017, Eric Lehman, F Tom Leighton, Albert R Meyer.

Strange dice: new game

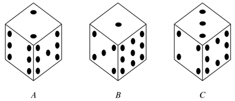

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each 1/3.

New game: each player rolls twice and highest sum wins. Other player gets to pick first, and chooses B. You choose A. What is probability you win?

Source for strange dice image: Mathematics for Computer Science, 2017, Eric Lehman, F Tom Leighton, Albert R Meyer.

Poll: Tree diagram for new strange dice game

The game: Each player rolls twice and highest sum wins. Other player gets to pick first, and chooses B. You choose A.

Suppose we represent an outcome as a tuple $\langle A_1, A_2, B_1, B_2 \rangle$ where A_1 represents the outcome of the *first* roll of die A, and A_2 represents the *second* roll, and B_1 and B_2 are defined similarly.

How many leaves would the corresponding tree diagram contain?

1. 9
2. 16
3. 27
4. 81