COSC 290 Discrete Structures

Lecture 30: Combinations and permutations

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Plan for today

1. Combinations and Permutations
2. Pigeonhole principle

Combinations and Permutations

Recall: Four types of common counting problem

Given a set S with n elements, let us consider counting the number of ways to choose k elements from S.

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

Four distinct counting problems:

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
repetition allowed	n^k	??

Today we will fill in the last cell...

Order irrelevant, repetition forbidden

Suppose your hockey team 13 players, only one of whom is a goalie. You want to organize your time into a starting line (goalie + 5 players), a backup line (goalie + 5 other players), and benchwarmers (remaining 2 players).

Let's assume that, except for the goalie, position does not matter.

How many different ways can you select the starting line, backup line, and benchwarmers?

Answer is $\binom{12}{5} \cdot \binom{7}{5}$, but why?

Derivation of number of lineups

Let P denote 12 players (minus goalie). Let $\mathbf{P} := \mathcal{P}(P)$, the powerset of P.

Let S be set where each element is a 3-tuple $\langle a, b, c \rangle$ where a is a starting line, b is a backup line, and c is benchwarmers. Formally,

$$S := \{ \langle a, b, c \rangle \in \mathbf{P}^3 : \ a \cup b \cup c = P \text{ and } \ |a| = |b| = 5 \text{ and } |c| = 2 \}$$

- How many choices for a? Let A be set of all choices, $|A| = \binom{12}{5}$
- Choice for b depends on which $a \in A$ is chosen. Let $B(a)$ be the set of all choices of b for a given a. What is $|B(a)|$? $\binom{7}{5}$
- Choice for c depends on prior choices. Let $C(a, b)$ be the set of all choices of c for a given a and b. What is $|C(a, b)|$? $\binom{2}{2}$.

So, how big is S? Apply *generalized product rule*!

Let $n_1 := |A|$, $n_2 := |B(a)|$, $n_3 := |C(a, b)|$. Then $|S| = n_1 \cdot n_2 \cdot n_3 = \binom{12}{5} \cdot \binom{7}{5} \cdot \binom{2}{2}$.

Generalized product rule

Let S denote a set of length-k sequences such that following condition holds:

For each $i \in \{ 1, \ldots, k \}$ and for each choice of first $i - 1$ components, there are n_i choices for the i^{th} component.

Then $|S| = \prod_{i=1}^{k} n_i$.

(Important: the value n_i does not depend on *what* was chosen for first $i - 1$ components.)

Order irrelevant, repetition allowed

How many ways to choose an unordered collection of k (not necessarily distinct) elements from a set of n elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

Let's translate this to a problem we already know how to solve...

$$\underbrace{00 \ldots 0}_{\text{jelly}} \ 1 \ \underbrace{00 \ldots 0}_{\text{chocolate}} \ 1 \ \underbrace{00 \ldots 0}_{\text{glazed}}$$

Examples: $k = 6$ donuts from $n = 3$ types (jelly, chocolate, glazed):

- 1 jelly, 2 chocolate, 3 glazed
- 1 jelly, 4 chocolate, 1 glazed
- 0 jelly, 0 chocolate, 6 glazed

Translating to a bit-string counting problem

How many ways to choose an unordered collection of k (not necessarily distinct) elements from a set of n elements?

Each choice can be mapped to a bit-string with k zeroes and $n-1$ ones. (Total length of bit string is $n+k-1$.)

How many bit strings of length $n+k-1$ with k zeroes?

$$\binom{n+k-1}{k}$$

This equals the number of bit strings of length $n+k-1$ with $n-1$ ones:

$$\binom{n+k-1}{n-1}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
repetition allowed	n^k	$\binom{n+k-1}{k}$

Poll: how many b-ball teams?

How many ways can you choose the composition of a basketball team ($k=5$ players) where each player is one of $n=3$ types: perimeterShooter, blocker, ballHandler.

A) $\binom{5}{3}$
B) $\binom{3}{5}$
C) $\binom{8}{5}$
D) $\binom{8}{3}$
E) $\binom{7}{5}$

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
repetition allowed	n^k	$\binom{n+k-1}{k}$

Derivation, a little more formally

Use mapping rule. Represent a choice of k elements from set of n candidates as sequence $\langle x_1, x_2, \ldots, x_n \rangle$ where x_i is the number of times we chose candidate element i.

Since we choose a total of k elements, we require:

$$\sum_{i=1}^{n} x_i = k$$

Bijective mapping to bit-strings:

$$f(\langle x_1, x_2, \ldots, x_n \rangle) = \underbrace{00\ldots0}_{x_1 \text{ times}}\ 1\ \underbrace{00\ldots0}_{x_2 \text{ times}}\ 1\ \ldots\ 1\ \underbrace{00\ldots0}_{x_n \text{ times}}$$

(Bit string is always length $n+k-1$ because there are $n-1$ ones and the total number of zeros must add up to k.)

Poll: ways to sum

How many different solutions are there to the equation $a + b + c = 8$ where a, b, c must be non-negative integers? *(You have to do some calculations but with a little bit of algebra, you can do this by hand or with a basic calculator.)*

A) 45
B) 56
C) 120
D) 165

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
repetition allowed	n^k	$\binom{n+k-1}{k}$

Pigeonhole principle

Claim

Somewhere in *your* family tree, you have an ancestor B whose parents were blood relatives—i.e., the father of B and the mother of B have a common ancestor A.

"Somewhere" means sometime in last 4000 years.

We will prove this using the pigeonhole principle.

Pigeonhole principle

(Informal) If you have more pigeons than pigeonholes, there is at least one pigeonhole with at least two pigeons.

Theorem (Pigeonhole principle)

Let X and Y be sets such that $|X| > |Y|$. Let f be any function $f : X \to Y$. Then f is *not* one-to-one.

Back to family tree claim

Claim: In last 4000 years, there exists an ancestor B in your family tree such that the father of B and the mother of B have a common ancestor A.

Proof: Proof makes a few (reasonable) assumptions.

- Everyone has a biological mother and father.
- No one lives to be more than 100
- At most 1 trillion people have ever lived.

Sketch of proof: 40 generations. All lived within last 4000 years.

At least 2^{40} distinct ancestor *roles*. $2^{40} >$ trillion.

Pigeonhole principle: more roles than people means someone played two roles!!

Call this person A. There must be two distinct paths from A to you. Eventually paths meet at some B.

Adapted from Kleinberg, `https://www.edge.org/response-detail/11067`

Poll: pigeons prefer matching socks

Suppose among your roommates, their earliest class is a 1:20 and they do not *not* like being disturbed when you rise in the wee hours to make it on time to COSC 290. Thus, you dress in the dark.

Suppose you have 4 different colors of socks, **black**, **gray**, **blue**, and **red**. You want a matching pair, but you can't see anything in the dark. How many socks should you take to ensure you have a matching pair?

A) 2
B) 3
C) 4
D) 5
E) 6