

# **COSC 290 Discrete Structures**

## Lecture 30: Combinations and permutations

---

Prof. Michael Hay

Monday, Apr. 16, 2018

Colgate University

# Plan for today

1. Four types of counting problems
2. Counting when order matters (2 ways)
3. Counting when order is irrelevant (2 ways)

## **Four types of counting problems**

---

# Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

# Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

- A starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.

# Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

- A starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.
- A team of  $k = 4$  runners to compete in a single cross-country race from among  $n = 10$  available runners.

## Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

- A starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.
- A team of  $k = 4$  runners to compete in a single cross-country race from among  $n = 10$  available runners.
- A runner for each of  $k = 5$  track races from among a team of  $n = 12$  available runners.

# Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

- A starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.
- A team of  $k = 4$  runners to compete in a single cross-country race from among  $n = 10$  available runners.
- A runner for each of  $k = 5$  track races from among a team of  $n = 12$  available runners.
- The composition of a basketball team ( $k = 5$  players) where each player is one of  $n = 3$  types: perimeterShooter, blocker, ballHandler.



# Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

- A starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.
- A team of  $k = 4$  runners to compete in a single cross-country race from among  $n = 10$  available runners.
- A runner for each of  $k = 5$  track races from among a team of  $n = 12$  available runners.
- The composition of a basketball team ( $k = 5$  players) where each player is one of  $n = 3$  types: perimeterShooter, blocker, ballHandler.
- A selection of  $k = 12$  donuts from  $n = 3$  donut types (jelly, chocolate, glazed).

## Four types of common counting problem

Given a set  $S$  with  $n$  elements, let us consider counting the number of ways to choose  $k$  elements from  $S$ . To formalize this counting problem, we must answer two questions:

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

## Four types of common counting problem

Given a set  $S$  with  $n$  elements, let us consider counting the number of ways to choose  $k$  elements from  $S$ . To formalize this counting problem, we must answer two questions:

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

The answers to these questions leads to four distinct counting problems:

	order matters	order irrelevant
repetition forbidden		
repetition allowed		

**Example shown on board:** let  $S = \{A, B, C\}$  and  $k = 2$ . Write out solutions to all four versions of the problem.

## Four types of common counting problem

Given a set  $S$  with  $n$  elements, let us consider counting the number of ways to choose  $k$  elements from  $S$ . To formalize this counting problem, we must answer two questions:

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

The answers to these questions leads to four distinct counting problems:

	order matters	order irrelevant
repetition forbidden		
repetition allowed		

**Example shown on board:** let  $S = \{A, B, C\}$  and  $k = 2$ . Write out solutions to all four versions of the problem.

**Goal for today:** fill in this table.

## Counting when order matters (2 ways)

---

## Order matters, repetition allowed

How many ways to choose a sequence of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: one runner for each of  $k = 5$  track races from among a team of  $n = 12$  available runners. (Same runner can compete in multiple races.)

How many ways?

## Order matters, repetition allowed

How many ways to choose a sequence of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: one runner for each of  $k = 5$  track races from among a team of  $n = 12$  available runners. (Same runner can compete in multiple races.)

How many ways? (Use the product rule.)

## Order matters, repetition allowed

How many ways to choose a sequence of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: one runner for each of  $k = 5$  track races from among a team of  $n = 12$  available runners. (Same runner can compete in multiple races.)

How many ways? (Use the product rule.)

$$\underbrace{n}_{\text{choices for first element}} \cdot \underbrace{n}_{\text{choices for second element}} \cdot \dots \cdot \underbrace{n}_{\text{choices for } k^{\text{th}} \text{ element}} = n^k$$



## Four counting problems

The number of ways to choose  $k$  items from a set  $S$  of  $n$  items when...

	order matters	order irrelevant
repetition forbidden		
repetition allowed	$n^k$	

## Order matters, repetition forbidden

How many ways to choose a sequence of  $k$  **distinct** elements from a set of  $n$  elements?

Example: a starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.

How many ways?

## Order matters, repetition forbidden

How many ways to choose a sequence of  $k$  **distinct** elements from a set of  $n$  elements?

Example: a starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.

How many ways? (Use the generalized product rule.)

# Order matters, repetition forbidden

How many ways to choose a sequence of  $k$  **distinct** elements from a set of  $n$  elements?

Example: a starting basketball lineup of  $k = 5$  players at five positions (C, PF, SF, PG, SG) from among  $n = 12$  players.

How many ways? (Use the generalized product rule.)

$$\underbrace{(n)}_{\text{choices for first element}} \cdot \underbrace{(n-1)}_{\text{choices for second element}} \cdot \dots \cdot \underbrace{(n-k+1)}_{\text{choices for } k^{\text{th}} \text{ element}} = \frac{n!}{(n-k)!}$$

## Example: sequences of a certain size

Let  $S := \{a, b, c, d, e\}$ . Let  $n := |S|$ . How many sequences of  $k = 2$  distinct elements can be constructed from  $S$ ?

## Example: sequences of a certain size

Let  $S := \{a, b, c, d, e\}$ . Let  $n := |S|$ . How many sequences of  $k = 2$  distinct elements can be constructed from  $S$ ?

There are  $n \cdot (n - 1) = 20$  ways:

$\{\langle a, b \rangle, \langle a, c \rangle, \langle a, d \rangle, \langle a, e \rangle,$   
 $\langle b, a \rangle, \langle b, c \rangle, \langle b, d \rangle, \langle b, e \rangle,$   
 $\langle c, a \rangle, \langle c, b \rangle, \langle c, d \rangle, \langle c, e \rangle,$   
 $\langle d, a \rangle, \langle d, b \rangle, \langle d, c \rangle, \langle d, e \rangle,$   
 $\langle e, a \rangle, \langle e, b \rangle, \langle e, c \rangle, \langle e, d \rangle\}.$

## Alternative derivation: using division rule

Let  $B$  be the set we are trying to count: sequences of  $k = 2$  distinct elements from  $S := \{a, b, c, d, e\}$ .

Let  $A$  be the set of all permutations of  $S$ . (Recall that a permutation of set  $S$  is an  $|S|$ -length sequence of elements of  $S$  with no repetitions.)

Let  $f : A \rightarrow B$  map a permutation into  $k$ -element sequence by simply keeping first  $k$  elements of the permutation.

Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

$A$	$\rightarrow$	$B$
$\langle a, b, c, d, e \rangle$	$\rightarrow$	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	$\rightarrow$	$\langle a, b \rangle$
$\langle a, b, d, c, e \rangle$	$\rightarrow$	$\langle a, b \rangle$
$\langle a, b, d, e, c \rangle$	$\rightarrow$	$\langle a, b \rangle$
$\langle a, b, e, c, d \rangle$	$\rightarrow$	$\langle a, b \rangle$
$\langle a, b, e, d, c \rangle$	$\rightarrow$	$\langle a, b \rangle$
$\langle a, c, b, d, e \rangle$	$\rightarrow$	$\langle a, c \rangle$
$\langle a, c, b, e, d \rangle$	$\rightarrow$	$\langle a, c \rangle$
$\langle a, c, d, b, e \rangle$	$\rightarrow$	$\langle a, c \rangle$
		$\dots$

## Alternative derivation continued...

### Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	B
$\langle a, b, c, d, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, c, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, e, c \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, c, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, d, c \rangle$	→	$\langle a, b \rangle$
$\langle a, c, b, d, e \rangle$	→	$\langle a, c \rangle$
$\langle a, c, b, e, d \rangle$	→	$\langle a, c \rangle$
$\langle a, c, d, b, e \rangle$	→	$\langle a, c \rangle$
...		

How many permutations map to *same*  $k$  sequence?



## Alternative derivation continued...

### Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	B
$\langle a, b, c, d, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, c, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, e, c \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, c, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, d, c \rangle$	→	$\langle a, b \rangle$
$\langle a, c, b, d, e \rangle$	→	$\langle a, c \rangle$
$\langle a, c, b, e, d \rangle$	→	$\langle a, c \rangle$
$\langle a, c, d, b, e \rangle$	→	$\langle a, c \rangle$
...		

How many permutations map to *same*  $k$  sequence?

Permutation maps to  $\langle a, b \rangle$  iff it starts with  $\langle a, b \rangle$  followed by remaining  $n - k$  elements in *any order*.

## Alternative derivation continued...

### Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	B
$\langle a, b, c, d, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, c, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, e, c \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, c, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, d, c \rangle$	→	$\langle a, b \rangle$
$\langle a, c, b, d, e \rangle$	→	$\langle a, c \rangle$
$\langle a, c, b, e, d \rangle$	→	$\langle a, c \rangle$
$\langle a, c, d, b, e \rangle$	→	$\langle a, c \rangle$
...		

How many permutations map to *same*  $k$  sequence?

Permutation maps to  $\langle a, b \rangle$  iff it starts with  $\langle a, b \rangle$  followed by remaining  $n - k$  elements in *any order*.

There are  $(n - k)!$  ways to order remaining  $(n - k)$  elements.

## Alternative derivation continued...

### Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	B
$\langle a, b, c, d, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, c, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, e, c \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, c, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, d, c \rangle$	→	$\langle a, b \rangle$
$\langle a, c, b, d, e \rangle$	→	$\langle a, c \rangle$
$\langle a, c, b, e, d \rangle$	→	$\langle a, c \rangle$
$\langle a, c, d, b, e \rangle$	→	$\langle a, c \rangle$
...		

How many permutations map to *same*  $k$  sequence?

Permutation maps to  $\langle a, b \rangle$  iff it starts with  $\langle a, b \rangle$  followed by remaining  $n - k$  elements in *any order*.

There are  $(n - k)!$  ways to order remaining  $(n - k)$  elements.

$f$  is a  $(n - k)!$ -to-1 function, so...

## Alternative derivation continued...

### Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	B
$\langle a, b, c, d, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, c, e \rangle$	→	$\langle a, b \rangle$
$\langle a, b, d, e, c \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, c, d \rangle$	→	$\langle a, b \rangle$
$\langle a, b, e, d, c \rangle$	→	$\langle a, b \rangle$
$\langle a, c, b, d, e \rangle$	→	$\langle a, c \rangle$
$\langle a, c, b, e, d \rangle$	→	$\langle a, c \rangle$
$\langle a, c, d, b, e \rangle$	→	$\langle a, c \rangle$
...		

How many permutations map to *same*  $k$  sequence?

Permutation maps to  $\langle a, b \rangle$  iff it starts with  $\langle a, b \rangle$  followed by remaining  $n - k$  elements in *any order*.

There are  $(n - k)!$  ways to order remaining  $(n - k)$  elements.

$f$  is a  $(n - k)!$ -to-1 function, so...

$$|B| = \frac{|A|}{(n - k)!} = \frac{n!}{(n - k)!}$$

## Four counting problems

The number of ways to choose  $k$  items from a set  $S$  of  $n$  items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	
repetition allowed	$n^k$	

## **Counting when order is irrelevant (2 ways)**

---

## Order irrelevant, repetition forbidden

How many ways to choose a **set** of  $k$  elements from a set of  $n$  elements?

Example: A starting volleyball lineup of  $k = 6$  players from among  $n = 13$  players. (Assume position irrelevant in volleyball because players rotate.)

## Order irrelevant, repetition forbidden

How many ways to choose a **set** of  $k$  elements from a set of  $n$  elements?

Example: A starting volleyball lineup of  $k = 6$  players from among  $n = 13$  players. (Assume position irrelevant in volleyball because players rotate.)

This counting problem comes up so often it has a special name/notation:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

The **binomial coefficient**, denoted  $\binom{n}{k}$ , is the number of **combinations** of  $k$  elements chosen from  $n$  candidate elements.



## Example: Counting bitstrings with $k$ ones

How many length  $n$  bitstrings contain exactly  $k$  ones?

## Example: Counting bitstrings with $k$ ones

How many length  $n$  bitstrings contain exactly  $k$  ones?

A length  $n$  bitstring has  $n$  bit positions  $b_1 b_2 \dots b_n$ .

## Example: Counting bitstrings with $k$ ones

How many length  $n$  bitstrings contain exactly  $k$  ones?

A length  $n$  bitstring has  $n$  bit positions  $b_1b_2 \dots b_n$ .

Must choose a set of  $k$  positions where the ones will be placed. (The order in which the bit positions are chosen is irrelevant.)

Thus, the number of length  $n$  bitstrings that contain exactly  $k$  ones is  $\binom{n}{k}$ .

## Example: subsets of a certain size

Let  $S := \{a, b, c, d, e\}$ . Let  $n := |S|$ . How many subsets of size  $k = 2$  can be constructed from  $S$ ?

## Example: subsets of a certain size

Let  $S := \{a, b, c, d, e\}$ . Let  $n := |S|$ . How many subsets of size  $k = 2$  can be constructed from  $S$ ?

There are  $\binom{n}{k} = \binom{5}{2} = 10$ :

$\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\},$   
 $\{b, c\}, \{b, d\}, \{b, e\},$   
 $\{c, d\}, \{c, e\},$   
 $\{d, e\}.$

# Derivation of $\binom{n}{k}$ using division rule

Let  $C$  be the set we are trying to count.

Let  $A$  be the set of all permutations of  $S$ .

Let  $g : A \rightarrow C$  map a permutation into  $k$ -element sequence by simply taking first  $k$  elements of the permutation and putting them in a set.

Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

$A$	$\rightarrow$	$C$
$\langle a, b, c, d, e \rangle$	$\rightarrow$	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	$\rightarrow$	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	$\rightarrow$	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	$\rightarrow$	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	$\rightarrow$	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	$\rightarrow$	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	$\rightarrow$	$\{a, c\}$
...		
$\langle a, e, d, c, b \rangle$	$\rightarrow$	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	$\rightarrow$	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	$\rightarrow$	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	$\rightarrow$	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	$\rightarrow$	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	$\rightarrow$	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	$\rightarrow$	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	$\rightarrow$	$\{b, c\}$
...		

# Derivation of $\binom{n}{k}$ continued...

## Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	C
$\langle a, b, c, d, e \rangle$	→	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	→	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	→	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	→	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	→	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	→	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	→	$\{a, c\}$
...		
$\langle a, e, d, c, b \rangle$	→	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	→	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	→	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	→	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	→	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	→	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	→	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	→	$\{b, c\}$
...		

How many permutations map to *same* set?

# Derivation of $\binom{n}{k}$ continued...

## Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	C
$\langle a, b, c, d, e \rangle$	→	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	→	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	→	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	→	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	→	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	→	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	→	$\{a, c\}$
...		
$\langle a, e, d, c, b \rangle$	→	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	→	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	→	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	→	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	→	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	→	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	→	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	→	$\{b, c\}$
...		

How many permutations map to *same* set?

Permutation maps to  $\{a, b\}$  iff it starts with the elements in  $\{a, b\}$  in *any order* followed by remaining  $n - k$  elements in *any order*.



# Derivation of $\binom{n}{k}$ continued...

## Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	C
$\langle a, b, c, d, e \rangle$	→	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	→	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	→	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	→	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	→	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	→	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	→	$\{a, c\}$
...		
$\langle a, e, d, c, b \rangle$	→	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	→	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	→	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	→	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	→	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	→	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	→	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	→	$\{b, c\}$
...		

How many permutations map to *same* set?

Permutation maps to  $\{a, b\}$  iff it starts with the elements in  $\{a, b\}$  in *any order* followed by remaining  $n - k$  elements in *any order*.

There are  $k!$  ways to order the first  $k$  elements. There are  $(n - k)!$  ways to order remaining  $(n - k)$  elements.

## Derivation of $\binom{n}{k}$ continued...

### Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	C
$\langle a, b, c, d, e \rangle$	→	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	→	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	→	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	→	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	→	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	→	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	→	$\{a, c\}$
...		
$\langle a, e, d, c, b \rangle$	→	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	→	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	→	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	→	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	→	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	→	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	→	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	→	$\{b, c\}$
...		

How many permutations map to *same* set?

Permutation maps to  $\{a, b\}$  iff it starts with the elements in  $\{a, b\}$  in *any order* followed by remaining  $n - k$  elements in *any order*.

There are  $k!$  ways to order the first  $k$  elements. There are  $(n - k)!$  ways to order remaining  $(n - k)$  elements.

$g$  is a  $k!(n - k)!$ -to-1 function, so...

# Derivation of $\binom{n}{k}$ continued...

## Example

$S = \{a, b, c, d, e\}$  and  $k = 2$ .

A	→	C
$\langle a, b, c, d, e \rangle$	→	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	→	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	→	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	→	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	→	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	→	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	→	$\{a, c\}$
...		
$\langle a, e, d, c, b \rangle$	→	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	→	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	→	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	→	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	→	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	→	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	→	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	→	$\{b, c\}$
...		

How many permutations map to *same* set?

Permutation maps to  $\{a, b\}$  iff it starts with the elements in  $\{a, b\}$  in *any order* followed by remaining  $n - k$  elements in *any order*.

There are  $k!$  ways to order the first  $k$  elements. There are  $(n - k)!$  ways to order remaining  $(n - k)$  elements.

$g$  is a  $k!(n - k)!$ -to-1 function, so...

$$|C| = \frac{|A|}{k!(n - k)!} = \frac{n!}{k!(n - k)!}$$

# Four counting problems

The number of ways to choose  $k$  items from a set  $S$  of  $n$  items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
repetition allowed	$n^k$	

## Poll: Counting number of ways to select lineups

Suppose your hockey team 13 players, only one of whom is a goalie. You want to organize your time into a starting line (goalie + 5 players), a backup line (goalie + 5 other players), and benchwarmers (remaining 2 players).

Let's assume that, except for the goalie, position does not matter.

How many different ways can you select the starting line, backup line, and benchwarmers?

- A)  $\binom{13}{6} \cdot \binom{13}{6} \cdot \binom{13}{3}$
- B)  $\binom{13}{6} \cdot \binom{7}{6}$
- C)  $\binom{12}{5} + \binom{7}{5}$
- D)  $\binom{12}{5} \cdot \binom{7}{5}$
- E) More than one / None of the above

## Order irrelevant, repetition allowed

How many ways to choose an unordered collection of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

## Order irrelevant, repetition allowed

How many ways to choose an unordered collection of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

Let's translate this to a problem we already know how to solve...

## Order irrelevant, repetition allowed

How many ways to choose an unordered collection of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

Let's translate this to a problem we already know how to solve...

$$\underbrace{00 \dots 0}_{\text{jelly}} \quad 1 \quad \underbrace{00 \dots 0}_{\text{chocolate}} \quad 1 \quad \underbrace{00 \dots 0}_{\text{glazed}}$$



## Order irrelevant, repetition allowed

How many ways to choose an unordered collection of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

Let's translate this to a problem we already know how to solve...

$$\underbrace{00 \dots 0}_{\text{jelly}} \ 1 \ \underbrace{00 \dots 0}_{\text{chocolate}} \ 1 \ \underbrace{00 \dots 0}_{\text{glazed}}$$

A bit-string with  $k$  zeroes and  $n - 1$  ones. (Total length of bit string is  $n + k - 1$ .)

## Order irrelevant, repetition allowed

How many ways to choose an unordered collection of  $k$  (not necessarily distinct) elements from a set of  $n$  elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

Let's translate this to a problem we already know how to solve...

$$\underbrace{00 \dots 0}_{\text{jelly}} \ 1 \ \underbrace{00 \dots 0}_{\text{chocolate}} \ 1 \ \underbrace{00 \dots 0}_{\text{glazed}}$$

A bit-string with  $k$  zeroes and  $n - 1$  ones. (Total length of bit string is  $n + k - 1$ .)

$$\binom{n + k - 1}{k}$$

## Poll: how many b-ball teams?

How many ways can you choose the composition of a basketball team ( $k = 5$  players) where each player is one of  $n = 3$  types: perimeterShooter, blocker, ballHandler.

- A)  $\binom{5}{3}$
- B)  $\binom{3}{5}$
- C)  $\binom{8}{5}$
- D)  $\binom{8}{3}$
- E)  $\binom{7}{5}$

## Derivation, a little more formally

Use mapping rule. Represent a choice of  $k$  elements from set of  $n$  candidates as sequence  $\langle x_1, x_2, \dots, x_n \rangle$  where  $x_i$  is the number of times we chose candidate element  $i$ .

## Derivation, a little more formally

Use mapping rule. Represent a choice of  $k$  elements from set of  $n$  candidates as sequence  $\langle x_1, x_2, \dots, x_n \rangle$  where  $x_i$  is the number of times we chose candidate element  $i$ .

Since we choose a total of  $k$  elements, we require:

$$\sum_{i=1}^n x_i = k$$

Bijjective mapping to bit-strings:

$$f(\langle x_1, x_2, \dots, x_n \rangle) = \underbrace{00 \dots 0}_{x_1 \text{ times}} 1 \underbrace{00 \dots 0}_{x_2 \text{ times}} 1 \dots 1 \underbrace{00 \dots 0}_{x_n \text{ times}}$$

(Bit string is always length  $n + k - 1$  because there are  $n - 1$  ones and the total number of zeros must add up to  $k$ .)

## Four counting problems

The number of ways to choose  $k$  items from a set  $S$  of  $n$  items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
repetition allowed	$n^k$	$\binom{n+k-1}{k}$

## Poll: ways to sum

How many different solutions are there to the equation  $a + b + c = 8$  where  $a, b, c$  must be non-negative integers? *(You have to do some calculations but with a little bit of algebra, you can do this by hand or with a basic calculator.)*

- A) 45
- B) 56
- C) 120
- D) 165