# COSC 290 Discrete Structures

Lecture 27: Rules of Counting

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## Plan for today

1. Counting basics
2. Generalized product rule
3. Difference rule
4. Inclusion-exclusion rule
5. Mapping rule



# Counting basics

## Counting basics

The essence of counting:

Define some set $S$ that is interest.

Determine $|S|$.

## Recall: Sum and product rules

**Sum Rule:** If $A$ and $B$ are *disjoint*, then $|A \cup B| = |A| + |B|$.

**Product Rule:** Let $S \coloneqq A_1 \times A_2 \times \cdots \times A_k$. Then $|S| = \prod_{i=1}^{k} |A_i|$.



## Recall: (Mis)applying the product rule

Suppose the Colgate Coders club must choose three officers—President, Secretary, Treasurer—from among the four leaders of the club: Alice, Bob, Chen, and Divesh.

- Bob doesn't want to be president.
- Only Chen or Divesh can be Treasurer.
- A person can serve in at most one role.

Let $U$ be distinct officer combinations. What is $|U|$?

- $P = \{ \text{A, C, D} \}$ (no Bob)
- $S = \{ \text{A, B, C, D} \}$
- $T = \{ \text{C, D} \}$ (no Alice, no Bob)

So, $|U| = |P| \cdot |S| \cdot |T| = 3 \cdot 4 \cdot 2 = 24$. What's wrong with this? Tree of outcomes drawn on board.



# Generalized product rule

## Generalized product rule

Let $S$ denote a set of length-$k$ sequences such that following condition holds:

For each $i \in \{ 1, \ldots, k \}$ and for each choice of first $i - 1$ components, there are $n_i$ choices for the $i^{th}$ component.

Then $|S| = \prod_{i=1}^{k} n_i$.

(Important: the value $n_i$ does not depend on *what* was chosen for first $i - 1$ components.)

**Example (PINs without repetition)**

An ATM PIN consists of four digits. A certain bank prohibits a PIN to use a digit more than once. How many PINs are possible?

$10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

## Applying generalized product rule

Can we apply the generalized product rule to the example of choosing officers for Colgate Coders?

(Shown on board).



## Poll: ATM PINs

A certain bank requires four digit PINs where repetition is disallowed and the last digit must be either a 0 or a 1. How many valid PINs are there?

A) $1,008$
B) $1,440$
C) $5,040$
D) None of above
E) Not sure because product rule doesn't apply



# Difference rule

## Difference rule

Let $B \subseteq A$. Then $|A - B| = |A| - |B|$.

This rule follows from the sum rule.

$$\begin{aligned} |A| &= |(A - B) \cup B| \\ &= |(A - B)| + |B| && \text{disjoint sets, apply sum rule} \end{aligned}$$

**Example (PINs with repeats)**

How many PINs have at least one repeated digit?

$$\begin{aligned} |pinWithRepeats| &= |allPINs| - |pinWithoutRepeats| \\ &= 10^4 - (10 \cdot 9 \cdot 8 \cdot 7) \\ &= 10,000 - 5,040 = 4,960 \end{aligned}$$

## (Mis)applying sum rule

For Fall 2018, there are (already!) 20 students on the cosc290 wait list and 15 students on the cosc201 wait list. Therefore there are 35 students waiting to take a 200-level course in Computer Science.

What's wrong with this?



# Inclusion-exclusion rule

## Inclusion-exclusion for two sets

$|A \cup B| = |A| + |B| - |A \cap B|$

*Intuition*: adjust for elements that are double counted.

**Example (Waitlists)**

With 20 students on the cosc290 wait list and 15 students on the cosc201 wait list, how many students waiting in total for a 200-level course? If exact answer not possible, can we calculate upper/lower bounds?

$|cosc290 \cup cosc201| = |cosc290| + |cosc201| - |cosc290 \cap cosc201|$
Cases:

- $|cosc290 \cap cosc201| = 15$
- $|cosc290 \cap cosc201| = 0$

So $20 \leq |cosc290 \cup cosc201| \leq 35$



## Inclusion-exclusion for three sets

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

*Intuition*: adjust for elements that are double/triple counted.

**Example (Numbers divisible by 2, 3 or 5)**

How many numbers in $\{1, \ldots, 1000\}$ are divisble by 2, 3 or 5?

- $|divBy2| = 500$
- $|divBy3| = 333$
- $|divBy5| = 200$
- $|divBy6| = 166$ ($divBy6 = divBy2 \cap divBy3$)
- $|divBy10| = 100$
- $|divBy15| = 66$
- $|divBy30| = 33$ ($divBy30 = divBy2 \cap divBy3 \cap divBy5$)
- $|divBy2or3or5| = 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$

## Poll: lower and upper bounds based on inclusion-exclusion

Inclusion-exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

In an introductory COSC course, instructor conducts survey. Results show 20 students have programmed in Python, 15 in Java, and 12 in C. Plus, 11 students have experience with both Python and Java.

How many students have programmed in *at least one* of the three languages? Choose the *narrowest* range that is guaranteed to contain the true value.

A) between 0 and 47
B) between 12 and 47
C) between 12 and 36
D) between 23 and 32
E) We are not given enough information



## Mapping rule

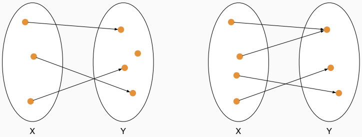
## Mapping rule

**Theorem (Mapping rule)**
Let $f : X \to Y$.

- $|X| \geq |Y|$ if $f$ is onto.
- $|X| \leq |Y|$ if $f$ is one-to-one.
- $|X| = |Y|$ if $f$ is bijective (onto *and* one-to-one).



## One-to-one

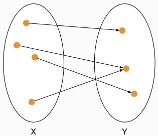



Yes. No.

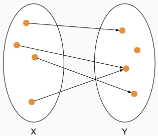
One-to-one: for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.

$f$ is one-to-one $\implies |X| \leq |Y|$

## Onto




Yes.



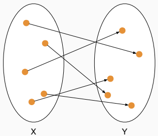

No.

Onto: for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.

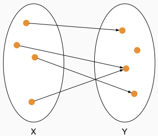
$$f \text{ is onto} \implies |X| \geq |Y|$$



## Bijective




Yes.




No.

Bijective: for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.

$$f \text{ is bijective} \implies |X| = |Y|$$



## Example: applying the mapping rule

Let $S := \{ s_1, s_2, \ldots, s_n \}$ be a set where $|S| = n$.

**Claim**: $|\mathcal{P}(S)| = 2^n$.

Proof uses a bijection between $\{ 0, 1 \}^n$ and $\mathcal{P}(S)$.

Let $f : \{ 0, 1 \}^n \to \mathcal{P}(S)$ where

$$f(\langle b_1, b_2, \ldots, b_n \rangle) := \{ s_i \in S : b_i = 1 \}$$

Need to show:

- $|\{ 0, 1 \}^n| = 2^n$ (use the product rule)
- $f$ is onto: let $X \subseteq S$, show how to construct bit string $b_X$ such that $f(b_X)$ maps to $X$.
- $f$ is one-to-one: suppose two distinct bitstrings $b, b'$ mapped to same $X \subseteq S$, show this leads to a contradiction.