COSC 290 Discrete Structures

Lecture 6: Predicate Logic

Prof. Michael Hay

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Colgate University

Plan for today

1. Normal forms: CNF and DNF
2. Predicate Logic
3. Quantification of variables
4. Theorem

Normal forms: CNF and DNF

Literal

Definition (Literal)

A literal is an atomic proposition or the negation of an atomic proposition (i.e. it's either p or $\neg p$ for some variable p).

$\oplus$

Example

Let $p :=$ "Alice earns an A." and $q :=$ "Pigs can fly."

Literals: p, $\neg p$, q, $\neg q$.

Not literals: $p \vee q$, $q \implies p$, etc.

Conjunctive Normal Form

Definition (CNF)

A proposition is in conjunctive normal form (CNF) if it consists of:

- a single *clause*, or
- a conjunction of two or more *clauses*

where a clause is

- a single *literal*, or
- a disjunction of two or more literals

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Example

These propositions are in CNF:

- $(p \vee q \vee s) \wedge (\neg p \vee r \vee \neg q)$
- $(\neg q \vee s)$

These propositions are *not* in CNF:

- $(p \vee q) \implies (\neg p \vee r)$
- $(\neg q \wedge s) \wedge (\neg p \vee r)$

Disjunctive Normal Form

Disjunctive Normal Form (DNF) is the same idea, just swap the role of ANDs and ORs. See textbook for defintion and examples.

Informally,

- conjunctive-normal form (CNF) is an “AND” of a bunch of “ORs”.
- disjunctive-normal form (DNF) is an “OR” of a bunch of “ANDs”.

Poll: is proposition in CNF?

Which of these propositions is *not* in CNF?

A) $\neg p$

B) $p \vee q$

C) $(p \vee q) \wedge (r \vee s)$

D) $(p \wedge q) \vee (r \wedge \neg p)$

E) More than one is *not* CNF / All are in CNF

(Definition restated here)

A proposition is in CNF if it is a single clause or the conjunction of two or more clauses where each clause is a single literal or the disjunction of two or more literals.

A literal is an atomic proposition or the negation of an atomic proposition (i.e. it's either p or $\neg p$ for some variable p).

Logical equivalence and CNF/DNF

Two important results:

1. Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in conjunctive-normal form (CNF).
2. Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in disjunctive-normal form (DNF).

Why might these results be useful?

Checking a CNF sentence for tautology

If φ is a proposition in CNF. Then checking for a tautology is easy.

- φ is a tautology if and only if each clause is a tautology.
- A clause from a CNF is a tautology if and only if it contains a literal and its opposite.

Poll: is this CNF a tautology?

Consider

$$\varphi := (p \lor q \lor \neg p) \land (r \lor p \lor q \lor \neg q) \land (\neg r \lor p \lor r)$$

Is φ in CNF? Is φ a tautology?

A) CNF: yes, tautology: yes

B) CNF: yes, tautology: no

C) CNF: no, tautology: yes

D) CNF: no, tautology: no

Predicate Logic

Predicate

An atomic proposition p is a Boolean variable.

A predicate $P(x)$ is a Boolean function. A predicate can take one or more arguments.

Examples:

- $isPrime(x)$ returns true if x is a prime number and false otherwise.
- $isDivisibleBy(x, y)$ returns true if x is evenly divisible by y.

Propositions that include predicates

We can form a proposition by supplying arguments to each predicate.

$$\varphi \coloneqq \mathit{isPrime}(8) \vee \mathit{isDivisibleBy}(8, 2)$$

The truth of this proposition requires *interpreting* the predicates:

Propositions that include predicates

We can form a proposition by supplying arguments to each predicate.

$$\varphi \coloneqq \mathit{isPrime}(8) \vee \mathit{isDivisibleBy}(8, 2)$$

The truth of this proposition requires *interpreting* the predicates: $\mathit{isPrime}(8)$ is false whereas $\mathit{isDivisibleBy}(8, 2)$ is true according to definitions of these predicates.

Free variables

[[MH: perhaps talk about free variables here; example – this is not a proposition because it has a free variable – in effect it's an unamed predicate]]

Quantification of variables

Quantification

[[MH: perhaps I need some more basic examples; followed by discussion of free and bound variables; then my colgate/bucknell examples are good]]

[[MH: define a set of integers, then use isDivisibleBy$(x, 2)$ *have the forall be false, then there exists be true, then perhaps have another one that is isDivisibleBy*$(x, 2) \vee$ *isDivisibleBy*$(x, 3)$ *which is true]]*

[[MH: poll: free vs. bound variables; or perhaps is this a proposition?]]

Universal Quantification

Let $P \coloneqq \{ p_1, p_2, \ldots, \}$ be the (infinite) set of all persons.

$$\forall p \in P : At(p, Colgate) \implies BrushesTeeth(p)$$

means "Every person at Colgate brushes their teeth."

The above is *roughly* equivalent to

$$\begin{aligned}
&(At(p_1, Colgate) \implies BrushesTeeth(p_1)) \\
\land\ &(At(p_2, Colgate) \implies BrushesTeeth(p_2)) \\
\land\ &(At(p_3, Colgate) \implies BrushesTeeth(p_3)) \\
\land\ &\ldots
\end{aligned}$$

Common mistake with universal quantification

Typically, $\implies$ is the main connective with $\forall$.

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall p \in P : At(p, Colgate) \wedge BrushesTeeth(p)$$

means “Every person is at Colgate and everyone brushes their teeth.”

Existential Quantification

Let $P\{\,p_1, p_2, \ldots,\}$ be the (infinite) set of all persons.

$$\exists p \in P : At(p, Bucknell) \wedge BrushesTeeth(p)$$

means "Some person at Bucknell brushes their teeth."

The above is *roughly* equivalent to

$$\begin{aligned}
&(At(p_1, Bucknell) \wedge BrushesTeeth(p_1)) \\
\vee\ &(At(p_2, Bucknell) \wedge BrushesTeeth(p_2)) \\
\vee\ &(At(p_3, Bucknell) \wedge BrushesTeeth(p_3)) \\
\vee\ &\ldots
\end{aligned}$$

Common mistake with existential quantification

Typically, $\wedge$ is the main connective with $\exists$.

Common mistake: using $\implies$ as the main connective with $\exists$:

$$\exists p \in P : At(p, Bucknell) \implies BrushesTeeth(p)$$

is true provided that there is some person who is *not* at Bucknell!

Credit: Slide adapted from Russell & Norvig, AI: A Modern Approach

Constructing Predicates

In programming, we can define functions that call other functions.

In predicate logic, we can define predicate in terms of other predicates.

Examples:

- $\mathit{follows}(x, y)$ means that x follows the tweets of y
- $\mathit{TrumpFollower}(x) := \mathit{follows}(x, \text{@realDonaldTrump})$
- $\mathit{popularTweeter}(y) := \forall x \in P : \mathit{follows}(x, y)$

Poll: fastest person

Let $\mathit{faster}(x, y)$ be true if x runs faster than y or x and y run the same speed and false otherwise.

Which of the following is the correct definition for $\mathit{fastest}(x)$?

A) $\exists y \in P : \mathit{faster}(x, y)$

B) $\neg(\exists y \in P : \mathit{faster}(y, x))$

C) $\forall y \in P : \mathit{faster}(x, y)$

D) $\neg(\forall y \in P : \mathit{faster}(y, x))$

E) None of the above / More than one of the above

Poll: fastest lacrosse player

Let $\textit{faster}(x, y)$ be true if x runs faster than y or x and y run the same speed and false otherwise.

Let $\textit{lax}(x)$ be true if x plays lacrosse.

Which of the following is the correct definition for $\textit{fastestLacrossePlayer}(x)$?

A) $\forall y \in P : \textit{lax}(y) \land \textit{faster}(x, y)$

B) $\forall y \in P : \textit{lax}(y) \implies \textit{faster}(x, y)$

C) $\textit{lax}(x) \land \forall y \in P : \textit{lax}(y) \land \textit{faster}(x, y)$

D) $\textit{lax}(x) \land \forall y \in P : \textit{lax}(y) \implies \textit{faster}(x, y)$

E) None of the above / More than one of the above

Theorem

Theorems

[[MH: need to be careful here... distinction between theorem (true for all P) and proposition (true for specific P)]]

[[MH: we could do this, but there could be a fair bit to this...]]

[[MH: take one theorem from fig. 3.23 and prove it by assuming antecedent; disprove one of the implications that only goes one way. could use the same claim for both things... exercise 3.130 and its converse]]