COSC 290 Discrete Structures

Lecture 35: Random Variables and Expectation

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Plan for today

1. Random Variables
2. Expectation

Logistics

- Lab due tonight. Lab is happening this afternoon. Please come with questions!
- Today at 11:30: talk by a candidate for visiting position.
- Last pset is out and due next Tuesday.

The Ferengi

Figure 1: An alien species in Star Trek notorious for extreme sexism.

http://memory-alpha.wikia.com/wiki/The_Magnificent_Ferengi_(episode)

Credit: Slide adapted with permission from Ashwin Machanavajjhalla and Jun Yang, Duke University

Tech interview question

Ferengi want boys, so every family keeps on having children until a boy is born.

- If the newborn is a girl, have another child
- If the newborn is a boy, stop

Can their strategy influence the composition of their population?

Let's think about the sample space, outcomes, and probability.

Let's draw tree diagram, or at least part of it, on board.

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Random Variables

Random variable

A random variable assigns a numerical value to every outcome in sample space.

Despite being called a variable, random variable X is formally a function $X : S \to \mathbb{R}$.

Example: possible outcomes in Ferengi family, and random variables B (# boys) and G (# girls).

Outcome	Probability	B	G
boy	$\frac{1}{2}$	1	0
girl,boy	$\frac{1}{4}$	1	1
girl,girl,boy	$\frac{1}{8}$	1	2
girl,girl,girl,boy	$\frac{1}{16}$	1	3
...	...	...	...

Probability mass function

We can associate a probability with a random variable as follows,

$$Pr(X = x) := Pr(\{ s \in S : X(s) = x \})$$

In other words, we can define an event as the set of outcomes s where random variable X maps s to x.

Example: probability of G and B:

Outcome	Probability	B	G
boy	$\frac{1}{2}$	1	0
girl,boy	$\frac{1}{4}$	1	1
girl,girl,boy	$\frac{1}{8}$	1	2
girl,girl,girl,boy	$\frac{1}{16}$	1	3
...	...	...	...

G	Prob.
0	$\frac{1}{2}$
1	$\frac{1}{4}$
2	$\frac{1}{8}$
3	$\frac{1}{16}$
...	...

B	Prob.
0	0
1	1
2	0
3	0
...	...

Poll: the pmf for F

Let's define a new random variable F which is equal to the fraction of boys in a Ferengi family. (In other words, $F = B/(B+G)$.)

What is $Pr(F \geq \frac{1}{3})$?

A) $\frac{1}{2}$
B) $\frac{2}{3}$
C) $\frac{3}{4}$
D) $\frac{7}{8}$
E) $\frac{15}{16}$

Expectation

Expectation

The expected value of a random variable X, denoted $\mathbb{E}[X]$ is the average value of X, defined as

$$\mathbb{E}[X] := \sum_{s \in S} X(s) \cdot Pr(s)$$

Alternatively, let $Range(X)$ denote the range of values that X can take. The expected value can also be calculated as,

$$\mathbb{E}[X] = \sum_{x \in Range(X)} x \cdot Pr(X = x)$$

Back to the Ferengi...

Expected number of boys?

$$\mathbb{E}[B] = 1$$

Expected number of girls?

$$\mathbb{E}[G] = \sum_{g=0}^{\infty} g \cdot Pr(G = g) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \cdots = ???$$

(We can also use Theorem 10.5 (p. 1046) from textbook.)

Poll: expected value of T

Let's define a new random variable, $T := B + G$ (total number of children). What is $\mathbb{E}[T]$?

$$
\begin{aligned}
\mathbb{E}[T] &= \sum_{s \in S} Pr(s) \cdot T(s) && \text{definition of expectation} \\
&= \sum_{s \in S} Pr(s) \cdot (B(s) + G(s)) && \text{definition of } T \\
&= \sum_{s \in S} Pr(s) \cdot B(s) + Pr(s) \cdot G(s) && \text{distribute mult. over addition} \\
&= \sum_{s \in S} Pr(s) \cdot B(s) + \sum_{s \in S} Pr(s) \cdot G(s) && \text{split into two summations} \\
&= ??? && \text{Can you simplify last line and solve?}
\end{aligned}
$$

Therefore, $\mathbb{E}[T]$ is (a) 0, (b) 1, (c) 1.5, (d) 2, (e) 2.5.

Linearity of expectations

Let X_1 and X_2 be any two random variables.

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

More generally, let $X_1, X_2, \ldots, X_n$ be any n random variables.

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

Finally, it's not hard to show that for any a that is *constant*,

$$\mathbb{E}[aX] = a\mathbb{E}[X]$$

More expectations

Recall $F = B/(B + G)$. What is $\mathbb{E}[F]$? Write out an expression.

- $\mathbb{E}[F] = \mathbb{E}[B/(B + G)] \neq \mathbb{E}[B]\,/\mathbb{E}[B + G]$!
- $\mathbb{E}[B]\,/\mathbb{E}[B + G] = \frac{1}{2}$ but $\mathbb{E}[F]$ is considerably more than half!

```
expectedValue = 0
boys = 1
for girls in range(0, 100):
    f = boys / (boys + girls)
    p = 1 / 2**(girls+1)
    expectedValue += f * p

print("Estimate of E[F]", expectedValue)
```

Expected fraction of boys in Ferengi family: $\approx 70\%$!