COSC 290 Discrete Structures

Lecture 17: Structural induction on logic

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Plan for today

1. Practice with structural induction on trees
2. Structural induction on propositions

Practice with structural induction on trees

Recall: structural induction

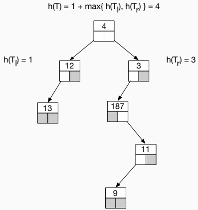
If you have a recursively defined structure – a structure defined in terms of one or more *base* cases and one or more *inductive* cases – you can prove properties about it using structural induction.

With structural induction, proof components should align with components of recursive definition.

Height of a tree, defined recursively

We can also define height *recursively*: let $h(T)$ denote the height of tree T.

- Base case: tree T is empty, $h(T) = -1$.
- Inductive case: T is non-empty, thus it consists a root node x, a left subtree T_ℓ, and a right subtree T_r. Then, $h(T) = 1 + \max\{\, h(T_\ell), h(T_r) \,\}$.



Recall: exercise from last class

Claim: $nodes(T) \le 2^{h(T)+1} - 1$

For reference: let $h(T)$ denote the height of tree T.

- Base case: tree T is empty, $h(T) = -1$.
- Inductive case: T is non-empty, thus it consists a root node x, a left subtree T_ℓ, and a right subtree T_r. Then, $h(T) = 1 + \max\{\, h(T_\ell), h(T_r) \,\}$.

Proof of base case

Claim: $nodes(T) \le 2^{h(T)+1} - 1$

Base case: T is empty so $nodes(T) = 0$ and $h(T) = -1$.

Indeed $nodes(T) \le 2^{h(T)+1} - 1 = 2^{-1+1} - 1 = 0$.

Proof of inductive case

Claim: $nodes(T) \le 2^{h(T)+1} - 1$

Inductive case: T is a non-empty tree of height $h(T)$, consisting of node x and left and right subtrees T_ℓ and T_r.

$$\begin{aligned}
nodes(T) &= 1 + nodes(T_\ell) + nodes(T_r) && \text{(b. +1 for root)}\\
&\le 1 + (2^{h(T_\ell)+1} - 1) + (2^{h(T_r)+1} - 1) && \text{(c. ind. hypothesis)}\\
&\le 1 + (2^{(h(T)-1)+1} - 1) + (2^{(h(T)-1)+1} - 1) && \text{(d. why?)}\\
&= 2^{h+1} - 1 = 2^{h(T)+1} - 1 && \text{(e. algebra)}
\end{aligned}$$

Explanation for line d:

Recall definition of height: $h(T) \coloneqq 1 + \max\{\, h(T_\ell), h(T_r) \,\}$.

So $h(T_\ell) \le h(T) - 1$ and $2^{h(T_\ell)+1} \le 2^{(h(T)-1)+1}$.

Same idea for T_r.

Poll: Lower bound?

- **False Claim**: $nodes(T) \geq 2^{h(T)+1} - 1$
- **Faulty proof by structural induction**:
  - **Base cases:** T is empty, height is -1 and $nodes(T) \geq 2^{-1+1} - 1 = 0$.
  - **Inductive case:** T is a non-empty tree of height $h(T)$, consisting of node x and left and right subtrees T_ℓ and T_r.

$$\begin{aligned} nodes(T) &= 1 + nodes(T_\ell) + nodes(T_r) && \text{(b. +1 for root)} \\ &\geq 1 + (2^{h(T_\ell)+1} - 1) + (2^{h(T_r)+1} - 1) && \text{(c. ind. hypothesis)} \\ &\geq 1 + (2^{(h(T)-1)+1} - 1) + (2^{(h(T)-1)+1} - 1) && \text{(d. subtree heights)} \\ &= 2^{h+1} - 1 = 2^{h(T)+1} - 1 && \text{(e. algebra)} \end{aligned}$$

Where's the flaw? A) first sentence of inductive case; B) line b; C) line c; D) line d; E) line e.

Structural induction on propositions

Propositions, recursively defined

A proposition φ is a well-formed formula (wff) over the variables in the set $P := \{ p_1, \ldots, p_n \}$, is one of the following:

- (base case) $\varphi := p$ for some $p \in P$
- (inductive cases)
  - $\varphi := \alpha \lor \beta$
  - $\varphi := \alpha \land \beta$
  - $\varphi := \alpha \implies \beta$
  - $\varphi := \neg\alpha$

  where α and β are well-formed formulas.

Negation Normal Form

Definition (Negation Normal Form (NNF))

A proposition φ is in negation normal form if the negation connective is applied only to variables and not to more complex expressions, and furthermore, the only connectives allowed are in the set $\{\land, \lor, \neg\}$.

Exercise: Negation Normal Form

Given

$$\varphi := \neg(p \land (\neg q \lor r)) \lor s$$

let's write an equivalent proposition that is in NNF by "pushing negations down."

Hint: double negation and De Morgan's laws are useful.

$$\begin{array}{rcll} \neg(\neg\alpha) & \equiv & \alpha & \text{double-negation elimination} \\ \neg(\alpha \land \beta) & \equiv & (\neg\alpha \lor \neg\beta) & \text{De Morgan's law \#1} \\ \neg(\alpha \lor \beta) & \equiv & (\neg\alpha \land \neg\beta) & \text{De Morgan's law \#2} \end{array}$$

All propositions can be expressed in NNF

Claim: For any wff φ, there exists a proposition φ' that is in NNF and is logically equivalent to φ.

Restating claim

Claim: For any well-formed formula φ, there exists a proposition φ' that is in negation normal form and is logically equivalent to φ.

Notation:

- $\mathit{isNNF}(\varphi)$ denotes the predicate: φ is in NNF.
- $\mathit{hasNNF}(\varphi)$ denotes the predicate: there exists a proposition φ' that is in NNF and $\varphi' \equiv \varphi$.
- $\mathcal{W}$ denotes the set of all well-formed formulas.

Thus, our claim can be restated as $\forall \varphi \in \mathcal{W} : \mathit{hasNNF}(\varphi)$.

Proof

Claim A: $\forall \varphi \in \mathcal{W} : \mathit{hasNNF}(\varphi)$.

We will instead prove the *stronger* claim:

Claim B: $\forall \varphi \in \mathcal{W} : \mathit{hasNNF}(\varphi) \land \mathit{hasNNF}(\neg\varphi)$.

How is this "stronger?" How does this help?

(See also book discussion on p. 540.)

Poll: what is the base case?

Claim B: $\forall \varphi \in \mathcal{W} : hasNNF(\varphi) \wedge hasNNF(\neg\varphi)$.

We will do a proof by structural induction.

How should we structure the base case(s)?

A) Two base cases: $\varphi := p$ and $\varphi := \neg p$. In each, want to show $hasNNF(\varphi)$.
B) One base cases: $\varphi := p$, want to show: $hasNNF(p) \wedge hasNNF(\neg p)$
C) Either of above is acceptable.
D) Structural induction proofs do not have base cases

Inductive cases

Claim B: $\forall \varphi \in \mathcal{W} : hasNNF(\varphi) \wedge hasNNF(\neg\varphi)$.

We will do a proof by structural induction. How many inductive cases? One case for each case in the recursive definition of WFF:

1. AND: $\varphi := \alpha \wedge \beta$
2. OR: $\varphi := \alpha \vee \beta$
3. NOT: $\varphi := \neg\alpha$
4. IMPLIES: $\varphi := \alpha \implies \beta$.

Poll: Inductive case 1, what to show?

Claim B: $\forall \varphi \in \mathcal{W} : hasNNF(\varphi) \wedge hasNNF(\neg\varphi)$.

Proof continued...

Inductive cases: We focus on case 1: $\varphi := \alpha \wedge \beta$. What do we want to show?

A) $hasNNF(\alpha)$
B) $hasNNF(\alpha \wedge \beta)$
C) $hasNNF(\neg(\alpha \wedge \beta))$
D) $hasNNF(\neg\alpha \vee \neg\beta)$
E) More than one / None of the above

Poll: Inductive case 1, what can we assume?

Claim B: $\forall \varphi \in \mathcal{W} : hasNNF(\varphi) \wedge hasNNF(\neg\varphi)$.

Proof continued...

Inductive cases: We focus on case 1: $\varphi := \alpha \wedge \beta$.

Want to show: $hasNNF(\alpha \wedge \beta) \wedge hasNNF(\neg(\alpha \wedge \beta))$. Which of the following can we assume is true (by the inductive hypothesis)?

A) $hasNNF(\alpha)$ ... recall this means that α is logically equivalent to some NNF proposition.
B) $hasNNF(\neg\alpha)$
C) $isNNF(\alpha)$ ... recall this means that α is an NNF.
D) A and B
E) A, B, and C

Proof for inductive case 1

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Inductive cases: We focus on case 1: $\varphi \coloneqq \alpha \wedge \beta$.

Want to show: *hasNNF*$(\alpha \wedge \beta) \wedge$ *hasNNF*$(\neg(\alpha \wedge \beta))$.

Assume by inductive hypothesis:

- *hasNNF*(α), *hasNNF*(β), *hasNNF*$(\neg\alpha)$, *hasNNF*$(\neg\beta)$

Part 1: Since *hasNNF*(α), there exists α' such that $\alpha' \equiv \alpha$ and *isNNF*(α'). Similarly for β. Let $\varphi' \coloneqq \alpha' \wedge \beta'$. We have *isNNF*$(\varphi')$ and $\varphi' \equiv \alpha \wedge \beta$. Thus *hasNNF*$(\alpha \wedge \beta)$.

Part 2: $\neg\varphi = \neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$ by DeMorgan's law. Since *hasNNF*$(\neg\alpha)$, there exists $\bar{\alpha}$ such that $\bar{\alpha} \equiv \neg\alpha$ and *isNNF*$(\bar{\alpha})$. Similarly for β. Thus, let $\bar{\varphi} \coloneqq \bar{\alpha} \vee \bar{\beta}$. We have *isNNF*$(\bar{\varphi})$ and $\bar{\varphi} \equiv \neg(\alpha \wedge \beta)$. Thus *hasNNF*$(\neg(\alpha \wedge \beta))$.

Proof for inductive case 2

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Inductive cases: Case 2: $\varphi \coloneqq \alpha \vee \beta$.

Proof is identical to case 1, just replace ANDs with ORs and vice versa.

Poll: Inductive case 3, what to show?

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Inductive cases: Case 3: $\varphi \coloneqq \neg\alpha$.

What do we want to show?

A) *hasNNF*(α)
B) *hasNNF*$(\neg\alpha)$
C) *hasNNF*$(\neg\neg\alpha)$
D) B and C
E) A, B, and C

Proof for inductive case 3

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Inductive cases: Case 3: $\varphi \coloneqq \neg\alpha$.

Want to show: *hasNNF*$(\neg\alpha) \wedge$ *hasNNF*$(\neg\neg\alpha)$.

Assume by inductive hypothesis:

- *hasNNF*(α), *hasNNF*$(\neg\alpha)$

Still need to show: *hasNNF*$(\neg\neg\alpha)$.

Since $\neg\neg\alpha \equiv \alpha$ and *hasNNF*(α), then let α' be such that $\alpha' \equiv \alpha$ and *isNNF*(α'). Let $\bar{\varphi} \coloneqq \alpha'$. Since $\bar{\varphi} \equiv \neg\neg\alpha$ and *isNNF*$(\bar{\varphi})$, thus *hasNNF*$(\neg\neg\alpha)$.

Proof for inductive case 4

Claim B: $\forall \varphi \in \mathcal{W} : \mathit{hasNNF}(\varphi) \land \mathit{hasNNF}(\neg\varphi)$.

Inductive cases: Case 2: $\varphi \coloneqq \alpha \implies \beta$.

The cleanest way to handle this is to have a lemma that states for any φ there is an equivalent φ' that contains only the connectives $\{\neg, \vee, \wedge\}$. This lemma can be proven using structural induction.