COSC 290 Discrete Structures

Lecture 26: Hasse Diagrams and Basic Rules of Counting

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Friday, Apr. 6, 2018
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Plan for today

1. Partial orders & Hasse Diagrams
2. Counting
3. Sum and Product rules

Partial orders & Hasse Diagrams

Recall: partial order

Relation $\preceq$ is a partial order if it is reflexive, antisymmetric, transitive.

Convention: we use *infix* notation: $a \preceq b$ instead of $\langle a, b \rangle \in \ \preceq$.

Intuition: partial order relations behave like $\leq$ except that some pairs may be *incomparable*.

Example (Partial order)

The *prefixOf* relation is a partial order:

- "a" $\preceq$ "aa"
- "aa" $\preceq$ "aardvark"

Note: not all pairs comparable: "a" $\not\preceq$ "b" and "b" $\not\preceq$ "a"

Recall: strict partial order

Relation $\prec$ is a strict partial order if it is irreflexive, (antisymmetric), transitive.

Intuition: strict partial order relations behave like $<$ except that some pairs may be *incomparable*.

Example (Strict partial order)

The *ancestorOf* relation (ancestor is parent or (recursively) parent of ancestor):

- "DT" $\prec$ "Don Jr"
- "Hanns Drumpf" $\prec$ "DT" (#makedonalddrumpfagain)
- not all pairs comparable: "Harry Potter" $\not\prec$ "Aunt Petunia" and "Aunt Petunia" $\not\prec$ "Harry Potter"

Review example from last time

Definition: Relation $\preceq$ is a partial order if it is reflexive, antisymmetric, transitive.

Example: Let $\preceq_1$ be a relation on a set of track runners where $a \preceq_1 b$ if the number of races in which a competed is no more than the number in which b competed.

This is *not* a partial order. Why?

Because two runners, say *Al* and *Bob* may have competed in the *same* number of races, and therefore,

$Al \preceq_1 Bob$ and $Bob \preceq_1 Al$ but $Al \neq Bob$

What property does this violate? Antisymmetry!

Hasse diagram

A partial order $\preceq$ on A can be drawn using a Hasse diagram.

- Draw nodes: one node for each A
- Draw edges: edge from a to b if $a \preceq b$, except...
- ... *omit* edges that can be inferred by reflexivity
- ... *omit* edges that can be inferred by transitivity
- ... and *layout* nodes "by level" if $a \preceq b$ for $a \neq b$, then a is placed *lower* than b

Example: isSubstringOf relation on the strings $\{ a, b, c, ab, bc, abc, cd \}$.

Exercise: draw Hasse diagram

Complete the following exercise: on a piece of paper, draw a Hasse diagram for the relation on $A := \{ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \}$ for the relation $R \subseteq A \times A$ where

$$R := \{\langle x, y\rangle \in A \times A : y \bmod x = 0\}$$

- Draw nodes: one node for each A
- Draw edges: edge from a to b if $a \preceq b$, except...
- ... *omit* edges that can be inferred by reflexivity
- ... *omit* edges that can be inferred by transitivity
- ... and *layout* nodes "by level" if $a \preceq b$ for $a \neq b$, then a is placed *lower* than b

Example partial order

A to do list,

$$[attendClass, sleep, borrowBook, eat, brushTeeth, study]$$

with constraints:

- $borrowBook \preceq study$
- $study \preceq attendClass$
- $sleep \preceq attendClass$
- $eat \preceq brushTeeth$
- $brushTeeth \preceq sleep$

What should you do *first*? Brush teeth? Eat? Borrow book?

Total order

Relation R is a total order if it is a partial order where every pair is comparable (either $\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$).

A total order can be written succinctly as an ordered list.

Is previous example a total order?

Topological ordering

Given a partial order $\preceq$, a topological ordering is a total order $\preceq_{total}$ that is *consistent* with $\preceq$.

(See book for formal definition of consistent; see earlier lectures for algorithms for topological sort.)

Counting

Short URLs

`http://bit.ly/2AF3U9c`

Bitly is a URL shortening service.

- Input: regular URL; output: short url.
- Short url is a string of 6 or 7 characters from { 0-9, A-Z, a-z }.
- If url is 7 characters, first character must be 1 or 2.[1]

Bitly claims to have shortened $34,033,678,000$ urls. How many short urls does Bitly have left?

[1]This claim based on empirical observation.

Counting

The essence of counting:

Define some set S that is of practical or theoretical interest.

Determine $|S|$.

Sum and Product rules

Sum Rule

If A and B are *disjoint*, then $|A \cup B| = |A| + |B|$.

Example (Short urls)

$shortUrls = sixCharUrls \cup sevenCharUrls$.
$sixCharUrls \cap sevenCharUrls = \emptyset$. Thus,
$|shortUrls| = |sixCharUrls| + |sevenCharUrls|$.

Product Rule

Let $S = A_1 \times A_2 \times \cdots \times A_k$.

Then $|S| = \prod_{i=1}^{k} |A_i|$.

Example (Six character urls)

$|sixCharUrls| = ?$

Let $A = \{$ 0-9, A-Z, a-z $\}$. We have $|A| = 10 + 26 + 26 = 62$. There are 62 choices for each character and six characters total.

Thus $|sixCharUrls| = |A|^6 = 62^6 \approx 56.8$ billion

Example (Seven character urls)

$|sevenCharUrls| = ?$

Thus $|sevenCharUrls| = |\{1, 2\}| \cdot |A|^6 = 2 \cdot 62^6 \approx 113.6$ billion

How many short urls left?

$|shortUrls| = |sixCharUrls| + |sevenCharUrls| \approx 170.4$ billion.

Bitly has used roughly $34/170 \approx 20\%$ of available URLs.

Short urls used by cloud services to support collaboration (Google map directions, online documents, etc.) Any potential concerns?

`http://bit.ly/2AF3U9c`

Poll: using the sum and product rule

How many 3 digit numbers are divisible by 5? *Hint: what values can be the first digit take? The middle digit? The last digit?*

A) 180
B) 190
C) 199
D) 200
E) None of above

(Mis)applying the product rule

Suppose the Colgate Coders club must choose three officers—President, Secretary, Treasurer—from among the four leaders of the club: Alice, Bob, Chen, and Divesh.

- Bob doesn't want to be president.
- Only Chen or Divesh can be Treasurer.
- A person can serve in at most one role.

How many distinct officer combinations?

- $P = \{$ A, C, D $\}$ (no Bob)
- $S = \{$ A, B, C, D $\}$
- $T = \{$ C, D $\}$ (no Alice, no Bob)

Let S denote the set of all possible officer assignments. So, $|S| = |P| \cdot |S| \cdot |T| = 3 \cdot 4 \cdot 2 = 24$. What's wrong with this?