COSC 290 Discrete Structures

Lecture 9: Proof by contrapositive

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Plan for today

1. Proofs: recap XXX
2. Example of direct proofs
3. Proof by contrapositive
4. Example of proof by contrapositive
5. Proving “if and only if” statements

Proofs: recap XXX

Proofs

[[MH: recap from last time...]]

[[MH: maybe a little song and dance about why proof – learning to argue]]

Example of direct proofs

Poll: setting up a proof

Claim: The sum of any two odd numbers is even.

In the proof, what is "Given" and what do you "Want to Show" (WTS)?

A) Given: Assume the sum of two odd numbers is even.
   WTS: This follows from axioms of algebra.

B) Given: Assume x and y are odd numbers.
   WTS: x + y is even.

C) Given: Assume 3 and 5[1] are odd.
   WTS: sum of 3 and 5 is even.

D) You cannot use direct proof template, because claim is not of the form "if ... then ..."

[1] You can pick something else, but we chose 3 and 5

Formalizing claim

Background:

- $\mathbb{Z}$ is the set of all integers
- $\mathit{Even}(x) := \exists k \in \mathbb{Z} : x = 2k$
- $\mathit{Odd}(x) := \exists \ell \in \mathbb{Z} : x = 2\ell + 1$

Let's formalize the claim:

Claim: $\mathit{Odd}(x) \wedge \mathit{Odd}(y) \implies \mathit{Even}(x + y)$

Formalizing claim

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- $Even(x) := \exists k \in \mathbb{Z} : x = 2k$
- $Odd(x) := \exists \ell \in \mathbb{Z} : x = 2\ell + 1$

Let's formalize the claim:

Claim: $Odd(x) \wedge Odd(y) \implies Even(x + y)$

Let's be even more formal:

Claim: $\forall x \in \mathbb{Z} : \forall y \in \mathbb{Z} : (Odd(x) \wedge Odd(y) \implies Even(x + y))$

[[MH: add the proof – do it on the board?]]

Proof by contrapositive

Proof by contrapositive

To prove a proposition of the form

$$\forall x : P(x) \implies Q(x)$$

you can equivalently prove its contrapositive form

$$\forall x : \neg Q(x) \implies \neg P(x)$$

[[MH: revise to match notation from Friday or vice versa...]]

Procedure for proof by contrapositive

1. Derive contrapositive form $\neg q \implies \neg p$.
2. Assume q is false (take it as "given").
3. Show that $\neg p$ logically follows.

Truth table for implication

p	q	$\neg q$	$\neg p$	$\neg q \implies \neg p$	
T	T	F	F	T	
T	F	T	F	F	*Rule this row out!*
F	T	F	T	T	
F	F	T	T	T	

[[MH: have this match Friday's slide]]

Proof by Contrapositive Template

- **Claim**: *Write the theorem/claim to be proved, "If p, then q"*
- **Proof**: *"We will prove the contrapositive: [state claim in contrapositive form]"* It's important to say this! Why?

Proof by Contrapositive Template

- **Claim**: *Write the theorem/claim to be proved, "If p, then q"*
- **Proof**: *"We will prove the contrapositive: [state claim in contrapositive form]"*
  - **Given**: *Assume that* $\neg q$ *is true*
  - **Want to show**: $\neg p$ *is true*
  - *Write main body of proof...*
  - *End the body with... "[restate* $\neg p$*], which is what was to be shown."*
  - **Conclusion**: "Therefore *by proving its contrapositive*, we have shown [restate theorem]."

Example of proof by contrapositive

Example

- **Claim**: "Let x, y be numbers such that $x \neq 0$. Then either $x + y \neq 0$ or $x - y \neq 0$.
- **Proof**: "We will prove the contrapositive"
    - **Given**: Assume that ...
    - **Want to show**: ...
    - *[Proof details]*
    - **Conclusion**: "Therefore *by proving its contrapositive*, we have shown ..."

Poll: What is given?

- **Claim**: "Let x, y be numbers such that $x \neq 0$. Then either $x + y \neq 0$ or $x - y \neq 0$.
- **Proof**: "We will prove the contrapositive"
    - **Given**: Assume that … what goes here?

A) $x + y \neq 0$ or $x - y \neq 0$

B) $x + y = 0$ or $x - y = 0$

C) $x + y = 0$ and $x - y = 0$

D) $x = 0$

E) None of above / More than one

Poll: What do we want to show?

- **Claim**: "Let x, y be numbers such that $x \neq 0$. Then either $x + y \neq 0$ or $x - y \neq 0$.
- **Proof**: "We will prove the contrapositive"
    - **Given**: Assume that $x + y = 0$ and $x - y = 0$.
    - **Want to show**: ... what goes here?

A) $x \neq 0$

B) $x = 0$

C) $x = 0$ and $y = 0$

D) $x + y \neq 0$ or $x - y \neq 0$

E) None of above / More than one

When to use proof by contrapositive?

Since $p \implies q$ is logically equivalent to $\neg q \implies \neg p$, it shouldn't matter whether you use direct proof or proof by contrapositive.

In practice, can try both and see which one gives you a better starting place (e.g., more information).

When to use proof by contrapositive?

Since $p \implies q$ is logically equivalent to $\neg q \implies \neg p$, it shouldn't matter whether you use direct proof or proof by contrapositive.

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Common use case: proving $p \iff q$

- $p \iff q \equiv (p \implies q) \wedge (q \implies p)$
- $q \implies p \equiv \neg p \implies \neg q$
- So prove $p \iff q$ by proving $p \implies q$ and then $\neg p \implies \neg q$. In both cases you get to start with p and work towards q.

Proving “if and only if” statements

[[MH: state in natural language]]

A number is even if and only if its square is even. *[[MH: this is ex 4.19 from book! let's change to: n div by 3 iff n^2 div by 3.]]*

[[MH: use contrapos here; what are steps to proving this? etc. hmm... we kind of need the fact that not even means odd.]]

Poll: what is the contrapositive?

If n^2 is divisible by 3, then n is divisible by 3. *[[MH: actually frame this in context of proof... given, what to prove]]*

What is the contrapositive statement?

1. If n is divisible by 3, then n^2 is divisible by 3.
2. If n^2 is *not* divisible by 3, then n is *not* divisible by 3.
3. If n is *not* divisible by 3, then n^2 is *not* divisible by 3.