COSC 290 Discrete Structures

Lecture 13: Weak induction

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Monday, Feb. 25, 2018
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Plan for today

1. Proof by weak induction
2. Example: exact change theorem

Proof by weak induction

Proof by weak induction

Suppose you want to prove that predicate $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 1}$.

$P(n)$ is some statement that is "parameterized" by an integer n.

Examples: n could represent...

- the size of an array
- the n^{th} iteration through a loop
- the height of a binary tree
- the number of variables in a proposition
- etc.

Example

Claim: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

This mathematical identity can be expressed as a predicate on n.

Define predicate P as

$$P(n) \coloneqq \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

The claim can be formally stated as

$$\forall n \in \mathbb{Z}^{\geq 1} : P(n)$$

We can prove this using induction.

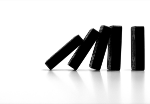
Procedure for proof by induction

Claim: $\forall n \in \mathbb{Z}^{\geq 1} : P(n)$

Process:

1. Base case: show that $P(1)$ is true.
2. Inductive case: show that $\forall n \in \mathbb{Z}^{\geq 2} : P(n-1) \implies P(n)$
    - Typically shown using direct proof by *assuming the antecedent*.
    - Let n be arbitrary value from $\mathbb{Z}^{\geq 2}$.
    - *Assume* that $P(n-1)$ is true, and
    - *Show* that $P(n)$ must also be true.

Intuition



(a) Base case: show first one falls

(b) Inductive case: show this can't happen

Show that the first domino falls and if the $(n-1)^{th}$ domino falls, the n^{th} domino falls too.

Example continued...

Claim: Let $P(n) \coloneqq \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The claim is that $\forall n \in \mathbb{Z}^{\geq 1} : P(n)$.

Proof by induction: We will prove by induction on n.

Base case: $n = 1$.

In this case, the left hand side is $\sum_{i=1}^{n} i = \sum_{i=1}^{1} i = 1$.

The right hand side is $\frac{n(n+1)}{2} = \frac{1 \times (1+1)}{2} = 1$.

Thus $P(1)$ is true.

Proof continued...

(Proof shown on board. And on next slide.)

Example continued...

Inductive case:

- **Given**: Let $n \geq 2$. Assume the claim is true for $P(n-1)$.
- **Want to show:** Claim is true for $P(n)$.

Since $P(n-1)$ is true, we have $\sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2}$

We will use this fact to show $P(n)$:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \left(\sum_{i=1}^{n-1} i\right) + n && \text{definition of summation} \\
&= \frac{(n-1)((n-1)+1)}{2} + n && \text{inductive hypothesis} \\
&= \frac{(n-1)n + 2n}{2} && \text{rearranging/simplifying terms} \\
&= \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} && \text{algebra}
\end{aligned}$$

$\square$

Starting point

You don't necessarily have to start at one. The proper starting point depends on the claim you are trying to prove.

Suppose $P(n)$ only holds for $n \geq 3$.

- Base case: show $P(3)$ is true.
- Inductive case: Show $\forall n \geq 4 : P(n-1) \implies P(n)$ is true.

Proof by Induction Template

- **Claim**: *State the claim. One way: define $P(n)$ and then the claim is something simple like "$\forall n \in \mathbb{Z}^{\geq 0} : P(n)$"*
- **Proof by induction**:
  - **Base case** ($n = 0$)**:** *Prove that $P(0)$ is true*
    - *Usually the base case is easy.*
  - **Inductive case** ($n \geq 1$)**:** *Show that $P(n-1) \implies P(n)$*
    - **Assume:** $P(n-1)$ is true. *Write out what $P(n-1)$ is in full!*
    - **Want to show:** $P(n)$ *is true.*
    - *Prove the inductive case...* "Suppose that *[$P(n-1)$ is true]...* "
    - *... body of proof for inductive case...*
    - "... therefore the inductive step holds."
  - **Conclusion:** "By induction, the claim has been shown."

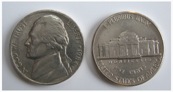
Proof by Induction (Alternative) Template

- **Claim**: *State the claim. One way: define $P(n)$ and then the claim is something simple like "$\forall n \in \mathbb{Z}^{\geq 0} : P(n)$"*
- **Proof by induction**:
  - **Base case** ($n = 0$)**:** *Prove that $P(0)$ is true*
    - *Usually the base case is easy.*
  - **Inductive case** ($n \geq 0$)**:** *Show that $P(n) \implies P(n+1)$*
    - **Assume:** $P(n)$ is true. *Write out what $P(n)$ is in full!*
    - **Want to show:** $P(n+1)$ *is true.*
    - *Prove the inductive case...* "Suppose that *[$P(n)$ is true]...* "
    - *... body of proof for inductive case...*
    - "... therefore the inductive step holds."
  - **Conclusion:** "By induction, the claim has been shown."

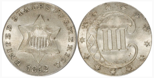
Example: exact change theorem

Exact Change Theorem

- **Claim**: For any price $n \geq 8$, the price n can be paid using only 5-cent coins and 3-cent coins.



(a) A five cent piece (nickel)



(b) A three cent piece (circa 1865-1873)

Poll: base case

- **Claim**: For any price $n \geq 8$, the price n can be paid using only 5-cent coins and 3-cent coins.
- **Proof by induction**:
  - **Base case:** Show the claim holds for $n = ??$

A) 0 cents

B) 1 cent

C) 2 cents

D) 3 cents

E) None of the above / More than one

Poll: inductive case

- **Claim**: For any price $n \geq 8$, the price n can be paid using only 5-cent coins and 3-cent coins.
- **Proof by induction**:
  - **Base case:** Show the claim holds for $n = 8$
  - **Inductive case:**
    - **Assume:** Assume that ... what goes here?
    - **Want to show:** We will show that ... and here?

A) ...the claim is true for $n = 8$... the claim holds for $n = 9$.

B) ...the claim is true for any $n \geq 8$... the claim holds for n.

C) ...the claim is true for any $n > 8$... the claim holds for n.

D) ...the claim is true for any $n \geq 8$... the claim holds for $n + 1$.

E) ...the claim is true for any $n > 8$... the claim holds for $n + 1$.

Inductive step

Assume that when price is n, we can make change using only three-cent and five-cent coins.

Show that this implies you can make change for price $n + 1$.

Poll: making change

Suppose you used $k = 10$ five-cent coins and $\ell = 10$ three-cent coins to pay the price of $n = \$0.80$. What is the *smallest* amount by which we must change k and ℓ to pay the price $n + 1 = \$0.81$?

A) $k = 0$ five-cent coins and $\ell = 27$ three-cent coins.
B) $k = 9$ five-cent coins and $\ell = 12$ three-cent coins.
C) $m = 12$ five-cent coins and $\ell = 7$ three-cent coins.
D) None of the above / More than one

Poll: a solution to the inductive case?

Suppose you used k five-cent coins and ℓ three-cent coins to pay the price of $n = 5k + 3\ell$. How can we modify the number of coins so that we can pay the price $n + 1$?

You ask a student who took 290 last spring, they suggest the following approach: use $k - 1$ five-cent coins and $\ell + 2$ three-cent coins. Will this work?

A) Yes, I think this always works.
B) Hmm... I'm not convinced it always works but I don't see a specific problem.
C) It won't always work and I know why.

Poll: without a nickel to your name

Suppose you used $k = 0$ five-cent coins and ℓ three-cent coins to pay the price of $n = 5k + 3\ell = 0 + 3\ell$. How can we modify the number of coins so that we can pay the price $n + 1$?

You ask a student who took 290 last spring, and they say you can't.

A) The student is right. [Explain why during discussion.]
B) Hmm... I'm not convinced either way.
C) The student is wrong. [Explain why during discussion.]

Poll: three is the magic number?

If you used $k = 0$ five-cent coins and ℓ three-cent coins to pay the price of $n = 5k + 3\ell$, you can pay price $n + 1$ provided that $\ell \geq 3$.

Student A asks: "What if we don't have that many three-cent coins?"

Student B says: "Don't worry about it, just pay with your 'Gate Card."

Choose the best answer:

A) Darn... student A just ruined our proof.

B) Student B is kinda right: we don't need to worry about it.

C) Student A raises a good point, we need to add this case to our proof.

D) None of the above / More than one

Final proof

Claim: For any price $n \geq 8$, the price n can be paid using only 5-cent coins and 3-cent coins. In other words, for any $n \geq 8$, there exists k and ℓ such that $n = 5k + 3\ell$.

Proof by induction:

- **Base case:** Show the claim holds for $n = 8$
  - One five-cent coin ($k = 1$) and one three-cent coin ($\ell = 1$) works.
- **Inductive case:**
  - **Assume:** Assume that for any $n \geq 8$, one can make change. That is, there is some k and ℓ such that $n = 5k + 3\ell$.
  - **Want to show:** One can make change for $n + 1$.
  - Cases:
    - $k > 0$: Then $5(k-1)$ and $3(\ell+2) = n + 1$
    - $k = 0$ and $\ell \geq 3$: Then $5(k+2)$ and $3(\ell-3) = n + 1$
    - The case $k = 0$ and $\ell < 3$ is not possible because then $n = 3\ell < 8$, which would violate our assumption that $n \geq 8$.
  - Therefore, the inductive step holds.