COSC 290 Discrete Structures

Lecture 23: Relations: Reflexivity, Symmetry, Transitivity

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Plan for today

1. Relations
2. Review: Representations of relations
3. Properties of relations on a single set
4. Closures

Relations

Relations

Let A and B be sets.

A (binary) relation on $A \times B$ is a subset of $A \times B$.

A binary relation on $A \times A$ is a subset of $A \times A$ and is simply called a relation on A.

Review: Representations of relations

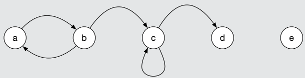
Recall: Graphical representation of relation on A

Example

Let $A := \{ a, b, c, d, e \}$. And consider relation R on A defined as

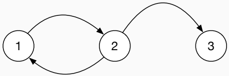
$$R := \{ \langle a, b \rangle, \langle b, c \rangle, \langle b, a \rangle, \langle c, c \rangle, \langle c, d \rangle \}$$

R can be represented as a graph:

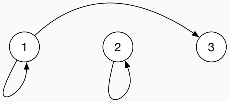


Review: composition as an operation on graphs

Consider this relation R:



What is $R \circ R$?



Properties of relations on a single set

Properties of relations

Let's now consider relations on a single set (i.e., relation $R \subseteq A \times A$ for some set A).

Such relations may exhibit certain properties:

- reflexivity
- irreflexivity
- symmetry
- antisymmetry
- asymmetry
- transitivity

Reflexivity

A relation R on A is reflexive if for every $a \in A$, $\langle a, a \rangle \in R$.

A relation R on A is irreflexive if for every $a \in A$, $\langle a, a \rangle \notin R$.

A relation can be reflexive, irreflexive, or neither.

Example (Reflexive relation)

Let *simNetflix* be a relation on *Persons* where $\langle p, p' \rangle \in \mathit{simNetflix}$ if the Netflix playlist for p has at least 10 shows in common with the playlist for p'.

Example (Irreflexive relation)

The *sibling* relation on *Persons*.

Symmetry

A relation R on A is symmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$ too.

A relation R on A is asymmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

A relation R on A is antisymmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

A relation can be none of the above, or more than one of the above.

Example (Symmetric relation)

sibling relation on *Persons*.

Example (Asymmetric relation)

prereq relation on *Courses*.

Example (Antisymmetric relation)

Let S be some set. The $\subseteq$ relation on $\mathcal{P}(S)$.

Transitive

A relation R on A is transitive if for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$ too.

A relation can be transitive, or not.

Example (Transitive relation)

sibling relation on *Persons*.

Poll: ancestorOf

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the *ancestorOf* relation on persons where $\langle a, p \rangle \in$ *ancestorOf* if person a is an ancestor of person p. Which properties does this relation have?

A) R, antiS, AS, T

B) IR, AS, T

C) IR, antiS, AS, T

D) IR, antiS, AS

E) None of the above

Poll: Implies

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the *implies* relation on all possible propositions expressed in the English language where $\langle p, q \rangle \in$ *implies* if $p \implies q$ is true. (Discuss this.)Which properties does this relation have?

A) R, T

B) R, S, T

C) R, antiS, T

D) IR, AS, T

E) None of the above

Poll: unequal sets

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Let X be an arbitrary set. Consider the relation *diffSize* on $\mathcal{P}(X)$ where $\langle S_1, S_2 \rangle \in$ *diffSize* if $|S_1| \neq |S_2|$. Which properties does this relation have?

A) R, S, T

B) IR, S

C) IR, antiS

D) IR, S, T

E) None of the above

Poll: Even divider

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the relation R on $\mathbb{Z}$ where $\langle x, y \rangle \in R$ if $x \mod 2 = 0$ and $y \mod x = 0$. (Look at examples.) Which properties does this relation have?

A) R, antiS, T

B) IR, antiS, T

C) antiS, T

D) antiS

E) None of the above

Closures

Closures

A closure of a relation R on A is a smallest $R' \supseteq R$ that satisfies a desired property.

- reflexive closure:

$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:
  (*hint*: what does $R \circ R$ give you?)

Poll: towards transitive closure

Consider the *parentOf* relation on persons where $\langle p, c \rangle \in \mathit{parentOf}$ if p is the parent of c. What is $\mathit{parentOf} \circ \mathit{parentOf}$?

A) ancestorOf
B) grandParentOf
C) parentOf
D) childOf
E) grandChildOf
F) descendantOf

Closures

A closure of a relation R on A is a smallest $R' \supseteq R$ that satisfies a desired property.

- reflexive closure:

$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:

$$R' = R \cup (R \circ R) \cup ((R \circ R) \circ R) \cup \cdots$$