# COSC 290 Discrete Structures

## Lecture 30: Combinations and permutations

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# Plan for today

1. Combinations and Permutations
2. Pigeonhole principle

# Combinations and Permutations

# Recall: Four types of common counting problem

Given a set $S$ with $n$ elements, let us consider counting the number of ways to choose $k$ elements from $S$.

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

# Recall: Four types of common counting problem

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Four distinct counting problems:

| | order matters | order irrelevant |
|---|---|---|
| repetition forbidden | $\frac{n!}{(n-k)!}$ | $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ |
| repetition allowed | $n^k$ | ?? |

Today we will fill in the last cell...

# Order irrelevant, repetition forbidden

Suppose your hockey team 13 players, only one of whom is a goalie. You want to organize your time into a starting line (goalie + 5 players), a backup line (goalie + 5 other players), and benchwarmers (remaining 2 players).

Let's assume that, except for the goalie, position does not matter.

How many different ways can you select the starting line, backup line, and benchwarmers?

Answer is $\binom{12}{5} \cdot \binom{7}{5}$, but why?

# Derivation of number of lineups

Let $P$ denote 12 players (minus goalie). Let $\mathbf{P} := \mathcal{P}(P)$, the powerset of $P$.

Let $S$ be set where each element is a 3-tuple $\langle a, b, c \rangle$ where $a$ is a starting line, $b$ is a backup line, and $c$ is benchwarmers.

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- How many choices for $a$? Let $A$ be set of all choices, $|A| = \binom{12}{5}$
- Choice for $b$ depends on which $a \in A$ is chosen. Let $B(a)$ be the set of all choices of $b$ for a given $a$. What is $|B(a)|$?

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- Choice for $c$ depends on prior choices. Let $C(a, b)$ be the set of all choices of $c$ for a given $a$ and $b$. What is $|C(a, b)|$?

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- Choice for $c$ depends on prior choices. Let $C(a, b)$ be the set of all choices of $c$ for a given $a$ and $b$. What is $|C(a, b)|$? $\binom{2}{2}$.

So, how big is $S$? Apply *generalized product rule*!

Let $n_1 := |A|$, $n_2 := |B(a)|$, $n_3 := |C(a, b)|$. Then
$|S| = n_1 \cdot n_2 \cdot n_3 = \binom{12}{5} \cdot \binom{7}{5} \cdot \binom{2}{2}$.

# Generalized product rule

Let $S$ denote a set of length-$k$ sequences such that following condition holds:

For each $i \in \{1, \ldots, k\}$ and for each choice of first $i-1$ components, there are $n_i$ choices for the $i^{th}$ component.

Then $|S| = \prod_{i=1}^{k} n_i$.

(Important: the value $n_i$ does not depend on *what* was chosen for first $i-1$ components.)

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How many ways to choose an unordered collection of $k$ (not necessarily distinct) elements from a set of $n$ elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

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Examples: $k = 6$ donuts from $n = 3$ types (jelly, chocolate, glazed):

- 1 jelly, 2 chocolate, 3 glazed
- 1 jelly, 4 chocolate, 1 glazed
- 0 jelly, 0 chocolate, 6 glazed

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This equals the number of bit strings of length $n+k-1$ with $n-1$ ones:

$$\binom{n+k-1}{n-1}$$

# Four counting problems

The number of ways to choose $k$ items from a set $S$ of $n$ items when...

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# Poll: how many b-ball teams?

How many ways can you choose the composition of a basketball team ($k = 5$ players) where each player is one of $n = 3$ types: perimeterShooter, blocker, ballHandler.

A) $\binom{5}{3}$
B) $\binom{3}{5}$
C) $\binom{8}{5}$
D) $\binom{8}{3}$
E) $\binom{7}{5}$

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# Derivation, a little more formally

Use mapping rule. Represent a choice of $k$ elements from set of $n$ candidates as sequence $\langle x_1, x_2, \ldots, x_n \rangle$ where $x_i$ is the number of times we chose candidate element $i$.

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Since we choose a total of $k$ elements, we require:

$$\sum_{i=1}^{n} x_i = k$$

Bijective mapping to bit-strings:

$$f(\langle x_1, x_2, \ldots, x_n \rangle) = \underbrace{00\ldots0}_{x_1 \text{ times}}\ 1\ \underbrace{00\ldots0}_{x_2 \text{ times}}\ 1\ \ldots\ 1\ \underbrace{00\ldots0}_{x_n \text{ times}}$$

(Bit string is always length $n + k - 1$ because there are $n - 1$ ones and the total number of zeros must add up to $k$.)

# Poll: ways to sum

How many different solutions are there to the equation $a + b + c = 8$ where $a, b, c$ must be non-negative integers? *(You have to do some calculations but with a little bit of algebra, you can do this by hand or with a basic calculator.)*

A) 45

B) 56

C) 120

D) 165

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# Pigeonhole principle

# Claim

Somewhere in *your* family tree, you have an ancestor $B$ whose parents were blood relatives—i.e., the father of $B$ and the mother of $B$ have a common ancestor $A$.

"Somewhere" means sometime in last 4000 years.

We will prove this using the pigeonhole principle.

# Pigeonhole principle

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**Theorem (Pigeonhole principle)**

Let $X$ and $Y$ be sets such that $|X| > |Y|$. Let $f$ be any function $f : X \to Y$. Then $f$ is *not* one-to-one.

# Back to family tree claim

**Claim**: In last 4000 years, there exists an ancestor $B$ in your family tree such that the father of $B$ and the mother of $B$ have a common ancestor $A$.

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**Proof**: Proof makes a few (reasonable) assumptions.

- Everyone has a biological mother and father.
- No one lives to be more than 100
- At most 1 trillion people have ever lived.

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Sketch of proof: 40 generations. All lived within last 4000 years.

At least $2^{40}$ distinct ancestor *roles*. $2^{40} >$ trillion.

Pigeonhole principle: more roles than people means someone played two roles!!

Call this person $A$. There must be two distinct paths from $A$ to you. Eventually paths meet at some $B$.

Adapted from Kleinberg, `https://www.edge.org/response-detail/11067`

# Poll: pigeons prefer matching socks

Suppose among your roommates, their earliest class is a 1:20 and they do not *not* like being disturbed when you rise in the wee hours to make it on time to COSC 290. Thus, you dress in the dark.

Suppose you have 4 different colors of socks, **black**, **gray**, **blue**, and **red**. You want a matching pair, but you can't see anything in the dark. How many socks should you take to ensure you have a matching pair?

A) 2
B) 3
C) 4
D) 5
E) 6