Starting today, the device you use to vote on polls during class will need to be registered at poll everywhere. (See problem set 0 for details.)

COSC 290 Discrete Structures

Lecture 6: Predicate Logic

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Monday, Feb. 5, 2018
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Plan for today

1. Normal forms: CNF and DNF
2. Predicate Logic
3. Quantification of variables
4. Expressing statements in predicate logic

Normal forms: CNF and DNF

Literal

Definition (Literal)

A literal is an atomic proposition or the negation of an atomic proposition (i.e. it's either p or $\neg p$ for some variable p).

Example

Let $p :=$ "Alice earns an A." and $q :=$ "Pigs can fly."

Literals: $p, \neg p, q, \neg q$.

Not literals: $p \vee q$, $q \wedge \neg p$, etc.

Conjunctive Normal Form

Definition (CNF)

A proposition is in conjunctive normal form (CNF) if it consists of:

- a single *clause*, or
- a conjunction of two or more *clauses*

where a clause is

- a single *literal*, or
- a disjunction of two or more literals

Example

These propositions are in CNF:

- $(p \vee q \vee s) \wedge (\neg p \vee r \vee \neg q)$
- $(\neg q \vee s)$
- $(\neg q \vee s) \wedge \neg q$

These propositions are *not* in CNF:

- $(p \vee q) \implies (\neg p \vee r)$
- $(\neg q \wedge s) \wedge (\neg p \vee r)$

Disjunctive Normal Form

Disjunctive Normal Form (DNF) is the same idea, just swap the role of ANDs and ORs. See textbook for definition and examples.

Informally,

- conjunctive-normal form (CNF) is an "AND" of a bunch of "ORs".
- disjunctive-normal form (DNF) is an "OR" of a bunch of "ANDs".

Poll: is this proposition in CNF?

Which of these propositions is *not* in CNF?

A) $\neg p$

B) $p \vee q$

C) $(p \vee q) \wedge (r \vee s)$

D) $(p \wedge q) \vee (r \wedge \neg p)$

E) More than one is *not* in CNF

(Definitions restated here for reference)

A proposition is in CNF if it is a single clause or the conjunction of two or more clauses where each clause is a single literal or the disjunction of two or more literals.

A literal is an atomic proposition or the negation of an atomic proposition (i.e. it's either p or $\neg p$ for some variable p).

Logical equivalence and CNF/DNF

Two important results:

1. Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in conjunctive-normal form (CNF).
2. Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in disjunctive-normal form (DNF).

Why might these results be useful?

Checking a CNF sentence for tautology

If φ is a proposition in CNF. Then checking for a tautology is easy.

- φ is a tautology if and only if each clause is a tautology.
- A clause from a CNF is a tautology if and only if it contains a literal and its opposite.

Poll: is this CNF a tautology?

Consider

$$\varphi := (p \vee q \vee \neg p) \wedge (r \vee q \vee p \vee \neg q) \wedge (\neg r \vee p)$$

Is φ in CNF? Is φ a tautology?

A) CNF: yes, tautology: yes
B) CNF: yes, tautology: no
C) CNF: no, tautology: yes
D) CNF: no, tautology: no

Predicate Logic

Predicate

An atomic proposition p is a Boolean variable. It is either true or false.

A predicate $P(x)$ is a Boolean *function*. Its truth value depends on what arguments are passed in.

Example

- $\mathit{isPrime}(x)$ returns true if x is a prime number and false otherwise.
- $\mathit{isDivisibleBy}(x, y)$ returns true if x is evenly divisible by y.

Propositions that include predicates

We can form a proposition by supplying arguments to each predicate.

$$\varphi := \mathit{isPrime}(8) \vee \mathit{isDivisibleBy}(8, 2)$$

The truth of this proposition requires *interpreting* the predicates:

- $\mathit{isPrime}(8)$ is false, according to definition of $\mathit{isPrime}$
- $\mathit{isDivisibleBy}(8, 2)$ is true, according to definition of $\mathit{isDivisibleBy}$
- thus, φ is true

Understanding terminology: predicate vs. proposition

Consider the following two expressions:

$$\varphi := \mathit{isPrime}(8) \vee \mathit{isDivisibleBy}(8, 2)$$

and

$$\psi := \mathit{isPrime}(x) \vee \mathit{isDivisibleBy}(x, 2)$$

The first one, φ, is a proposition. Why?

The second one, ψ, is *not* a proposition. Why not?

Free variables

The expression φ,

$$\psi := \mathit{isPrime}(x) \vee \mathit{isDivisibleBy}(x, 2)$$

is *not* a proposition because...

the truth value of ψ depends on the free variable x.

Thus, ψ is a *predicate*, not a proposition and we should write it like this:

$$\psi(x) := \mathit{isPrime}(x) \vee \mathit{isDivisibleBy}(x, 2)$$

Quantification of variables

Quantification

Let $S := \{2, 3, 4\}$. The proposition φ,

$$\varphi := \forall x \in S : \mathit{isDivisibleBy}(x, 2)$$

is equivalent to:

$$\mathit{isDivisibleBy}(2, 2) \wedge \mathit{isDivisibleBy}(3, 2) \wedge \mathit{isDivisibleBy}(4, 2)$$

Whereas the proposition ψ,

$$\psi := \exists x \in S : \mathit{isDivisibleBy}(x, 2)$$

is equivalent to:

$$\mathit{isDivisibleBy}(2, 2) \vee \mathit{isDivisibleBy}(3, 2) \vee \mathit{isDivisibleBy}(4, 2)$$

Quantification over set expressions

Let $S := \{2, 3, 4\}$. The proposition φ,

$$\varphi := \forall x \in (S - \{3\}) : \mathit{isDivisibleBy}(x, 2)$$

is equivalent to:

$$\mathit{isDivisibleBy}(2, 2) \wedge \mathit{isDivisibleBy}(4, 2)$$

Bound vs. free variables

Contrast expressions ψ and θ

$$\psi := \exists x \in S : \mathit{isDivisibleBy}(x, 2)$$

with

$$\theta := \exists x \in S : \mathit{isDivisibleBy}(x, y)$$

While ψ is a proposition, θ is *not* a proposition. Why?

In both, the variable x is a bound variable. The "$\exists x \in S$" part *binds* variable x to *each* element in S.

In θ, the variable y is a free variable, thus θ is really a *predicate*, not a proposition.

$$\theta(y) := \exists x \in S : \mathit{isDivisibleBy}(x, y)$$

The predicate is true when S contains something divisible by y.

Poll: quantification and free/bound variables

Let $S \coloneqq \{2, 3, 4\}$. Consider this expression φ,

$$\varphi \coloneqq \forall x \in S : \mathit{isDivisibleBy}(x, 2) \vee \mathit{isDivisibleBy}(x, 3)$$

Which of the following statements is accurate?

A) φ is a true proposition and x is a bound variable.
B) φ is a true proposition and x is a free variable.
C) φ is a false proposition and x is a bound variable.
D) φ is a false proposition and x is a free variable.
E) φ is *not* a proposition

Expressing statements in predicate logic

Universal Quantification

Let $P \coloneqq \{p_1, p_2, \ldots,\}$ be the (infinite) set of all persons.

$$\forall p \in P : \mathit{At}(p, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p)$$

means "Every person at Colgate brushes their teeth."

The above is *roughly* equivalent to

$$\begin{aligned} &(\mathit{At}(p_1, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p_1)) \\ &\wedge (\mathit{At}(p_2, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p_2)) \\ &\wedge (\mathit{At}(p_3, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p_3)) \\ &\wedge \ldots \end{aligned}$$

Common mistake with universal quantification

Typically, $\implies$ is the main connective with $\forall$.

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall p \in P : \mathit{At}(p, \mathit{Colgate}) \wedge \mathit{BrushesTeeth}(p)$$

means "Every person is at Colgate and everyone brushes their teeth."

This statement is false as long as there is one person who does not attend Colgate.

Credit: Slide adapted from Russell & Norvig, AI: A Modern Approach

Existential Quantification

Let $P \{ p_1, p_2, \ldots, \}$ be the (infinite) set of all persons.

$$\exists p \in P : At(p, Bucknell) \land BrushesTeeth(p)$$

means "Some person at Bucknell brushes their teeth."

The above is *roughly* equivalent to

$$\begin{aligned}
&(At(p_1, Bucknell) \land BrushesTeeth(p_1)) \\
&\lor (At(p_2, Bucknell) \land BrushesTeeth(p_2)) \\
&\lor (At(p_3, Bucknell) \land BrushesTeeth(p_3)) \\
&\lor \ldots
\end{aligned}$$

Common mistake with existential quantification

Typically, $\land$ is the main connective with $\exists$.

Common mistake: using $\implies$ as the main connective with $\exists$:

$$\exists p \in P : At(p, Bucknell) \implies BrushesTeeth(p)$$

is true provided that there is some person who is *not* at Bucknell!

Credit: Slide adapted from Russell & Norvig, AI: A Modern Approach

Constructing Predicates

In programming, we can define functions that call other functions.

In predicate logic, we can define predicate in terms of other predicates.

Examples:

- $follows(x, y)$ means that x follows the tweets of y
- $TrumpFollower(x) := follows(x, \text{@realDonaldTrump})$
- $popularTweeter(y) := \forall x \in P : follows(x, y)$
- $hasFollower(y) :=$??? $\exists x \in P : follows(x, y)$
- $followsEveryone(x) :=$??? $\forall y \in P : follows(x, y)$

Poll: fastest person

Let P be the set of all people. Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Which of the following is the correct definition for $fastest(x)$?

A) $fastest(x) := \exists y \in P : faster(x, y)$
B) $fastest(x) := \forall y \in P : faster(x, y)$
C) $fastest(x) := \forall y \in P - \{ x \} : faster(x, y)$
D) $fastest(x) := \neg(\exists y \in P : faster(y, x))$
E) C and D

Poll: fastest lacrosse player

As before, let P be the set of all people. Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Let $lax(x)$ be true if x plays lacrosse and false otherwise.

Which of the following is the correct definition for $fastestLax(x)$, the fastest lacrosse player?

A) $fastestLax(x) := \forall y \in P - \{x\} : lax(y) \wedge faster(x, y)$

B) $fastestLax(x) := \forall y \in P - \{x\} : lax(y) \implies faster(x, y)$

C) $fastestLax(x) := lax(x) \wedge \forall y \in P - \{x\} : faster(x, y)$

D) $fastestLax(x) := lax(x) \wedge \forall y \in P - \{x\} : lax(y) \wedge faster(x, y)$

E) $fastestLax(x) := lax(x) \wedge \forall y \in P - \{x\} : lax(y) \implies faster(x, y)$