COSC 290 Discrete Structures

Lecture 22: Relations & Composition

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Plan for today

1. Relations
2. Practice with composition
3. Representations of relations

Relations

Relations

Let A and B be sets.

A (binary) relation on $A \times B$ is a subset of $A \times B$.

A binary relation on $A \times A$ is a subset of $A \times A$ and is simply called a relation on A.

Examples

- *EnrolledAt* is a relation on *Persons* $\times$ *College*: $\langle p, c\rangle \in \mathit{EnrolledAt}$ if person p attends college c.
- *FacebookFriends* is a relation on *FacebookUsers*: $\langle u, v\rangle \in \mathit{FacebookFriends}$ if u has friended v on Facebook.
- $\leq$ is a relation on $\mathbb{R}$: $\langle x, y\rangle \in \leq$ if x is less than or equal to y. (We often write using *infix* notation: $x \leq y$.)
- *abs* is a relation on $\mathbb{R} \times \mathbb{R}^{\geq 0}$: $\langle x, y\rangle \in \mathit{abs}$ if $|x| = y$.

Inverse and composition

Let R be a relation on $A \times B$ and S be a relation on $B \times C$.

Definition (Inverse)

Let R be a relation on $A \times B$. The inverse R^{-1} of R is a relation on $B \times A$ defined by $R^{-1} := \{\, \langle b, a\rangle \in B \times A : \langle a, b\rangle \in R \,\}$

Definition (Composition)

The composition of R and S is a relation on $A \times C$, denoted $S \circ R$, where $\langle a, c\rangle \in S \circ R$ iff there exists a $b \in B$ such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S$.

(write on the board for later use)

Practice with composition

Examples of composition

Let us define sets *Persons*, *Clubs*, *Animals*.

Let *Members* be a relation on *Persons* $\times$ *Clubs* such that $\langle p, c\rangle \in \mathit{Members}$ means that person p is a member of club c.

Let *Mascots* be a relation on *Clubs* $\times$ *Animals* such that $\langle c, a\rangle \in \mathit{Mascots}$ means that club c has animal a for a mascot.

Let *Leaders* be a relation on *Clubs* $\times$ *Persons* such that $\langle c, p\rangle \in \mathit{Leaders}$ means that club c is led by person p.

Consider these compositions:

1. $\mathit{Mascots} \circ \mathit{Members}$?
2. $\mathit{Leaders} \circ \mathit{Members}$?
3. $\mathit{Members} \circ \mathit{Leaders}$?

Another example of composition

Suppose we have the following relations:

- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$
- $teaching \subseteq Professors \times Classes$
- $doing \subseteq Students \times Performances$

Let's derive a new relation from the above relations plus the inverse and composition operators. Let $R \subseteq Students \times Times$ where $\langle s, t \rangle \in R$ indicates that student s is taking a class at time t.

How do we express R?

$$R = at \circ taking$$

Poll: deriving new relations, 1

Suppose we have the following relations:

- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$
- $teaching \subseteq Professors \times Classes$
- $doing \subseteq Students \times Performances$

Let's derive a new relation from the above relations plus the inverse and composition operators. Let $R \subseteq Students \times Professors$ where $\langle s, p \rangle \in R$ indicates that student s is taking at least one class with professor p.

A) $taking \circ teaching$
B) $teaching \circ taking$
C) $taking \circ teaching^{-1}$
D) $teaching^{-1} \circ taking$
E) None of the above / More than one

Poll: deriving new relations, 2

Suppose we have the following relations:

- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$
- $teaching \subseteq Professors \times Classes$
- $doing \subseteq Students \times Performances$

Let's derive a new relation from the above relations plus the inverse and composition operators. Let $R \subseteq Professors \times Performances$ where $\langle p, p' \rangle \in R$ means professor p should attend performance p' to see at least one of the professor's students perform.

A) $doing \circ (taking^{-1} \circ teaching)$
B) $doing \circ (teaching \circ taking^{-1})$
C) $(taking^{-1} \circ teaching) \circ doing$
D) $(teaching \circ taking^{-1}) \circ doing$
E) None of the above / More than one

Poll: Minimum cardinality

Suppose that sets A, B, C have cardinalities n_A, n_B, n_C respectively. Further, $\min\{n_A, n_B, n_C\} > 0$. Let R be a relation on $A \times B$ and S a relation on $B \times C$. What is the minimum cardinality of $S \circ R$? (In discussion, justify your answer.)

A) 0
B) n_B
C) $n_A + n_C$
D) $n_A \cdot n_C$
E) $\min\{n_A, n_C\}$
F) $\min\{n_A, n_B, n_C\}$

Poll: Maximum cardinality

Suppose that sets A, B, C have cardinalities n_A, n_B, n_C respectively. Let R be a relation on $A \times B$ and S a relation on $B \times C$. What is the maximum cardinality of $S \circ R$? (In discussion, justify your answer.)

A) n_B
B) $n_A + n_C$
C) $n_A \cdot n_C$
D) $\max \{ n_A, n_C \}$
E) $\max \{ n_A, n_B, n_C \}$

Representations of relations

Graphical representations of relations

A binary relation on $A \times B$ can be represented visually in a couple of ways:

- A table (Fig. 8.2 from book, left side)
- A mapping (Fig. 8.2 from book, right side)

A binary relation on A can be conveniently represented as a graph.

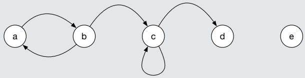
Graphical representation of relation on A

Example

Let $A := \{ a, b, c, d, e \}$. And consider relation R on A defined as

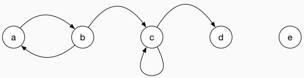
$$R := \{ \langle a, b \rangle , \langle b, c \rangle , \langle b, a \rangle , \langle c, c \rangle , \langle c, d \rangle \}$$

R can be represented as a graph:



Exercise

Consider this relation R:



On a piece of paper,

1. Draw R^{-1}
2. Draw $R \circ R$.