COSC 290 Discrete Structures

Lecture 5: Logical Equivalence

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Plan for today

1. Evaluating propositions
2. Logical equivalence
3. Normal forms: CNF and DNF

Evaluating propositions

Evaluating compound proposition

Question: How do we know whether a given proposition is true?

- If the answer is, “Well, we have to read the proposition and decide if it seems true on a case-by-case basis,” then logic FAIL.
- Truth should be determined based on the *form* of the proposition, the semantics of logical operators, and the truth of each variable.

Truth table for negation

Truth table for negation:

p	$\neg p$
T	F
F	T

Suppose that $p :=$ "There are 15 minutes left in class." If p is true, then what do we know about $\neg p$?

It *does not matter what the proposition is*: the negation of a proposition always has the opposite truth value.

Truth table for conjunction

Truth table for conjunction ("p and q"):

p	q	$p \wedge q$
T	T	? T
T	F	? F
F	T	? F
F	F	? F

Poll: exclusive vs. inclusive or

There two "or" operators, the "inclusive or" and the "exclusive or."

Inclusive or

p	q	$p \vee q$
T	T	?
T	F	?
F	T	?
F	F	?

Exclusive or

p	q	$p \oplus q$
T	T	?
T	F	?
F	T	?
F	F	?

In which rows do their truth tables differ?

A) The T T row
B) The T F row
C) The F T row
D) The F F row
E) None of the above / More than one of the above

Exclusive or vs. inclusive or

Inclusive or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example

"Bob likes chicken or fish." (He might like both.)

Example

"Alice is either in her office or exercising at the gym." (She can't be in both places at once.)

Truth table for implication

$p \implies q$ is true unless p is true and q is false.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

The restaurant owner assures the police officer, "If a person is drinking beer, then they are at least 21."

What evidence does the police officer need to show that this proposition, $p \implies q$, is false?

Poll: evaluating propositions

Let p, q, r be the following *atomic* propositions.

- $p :=$ "In COSC 290, Alice earned an A on each lab, the take-home midterm, and the final." (assume this is true)
- $q :=$ "Alice's final grade for COSC 290 was an A." (assume this is true)
- $r :=$ "7 is prime." (this is true)

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Which of the following *compound* propositions are true?

A) $p \implies q$

B) $q \implies p$

C) $r \implies q$

D) A and B only

E) A, B, and C

Implication and causality

In logic, we are looking at the *form* of the arguments.

To know if $p \implies q$, it is not necessary for p to *cause* q. (This might seem counter-intuitive.)

To determine truth of $p \implies q$, we need to know the truth values of p and q and then consult the truth table.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

"If 7 is prime, then Alice's final grade for COSC 290 was an A" is true because both statements are given as true.

Counter-intuitive nature of implication

A second counter-intuitive aspect is that $p \implies q$ is true whenever p is false.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

- Consider this sentence: "If pigs can fly, then Alice will earn the highest grade in COSC 290."
  We can write this as $p \implies q$.
- Pigs can't fly. So, $p \implies q$ is true!

(Contract analogy)

Quick aside: Notation

We will often use letters to represent atomic propositions: p, q, r, etc.

To represent *compound* propositions, we will often use Greek letters: φ, ψ, α, β, etc.

Example

Let $\varphi := (p \vee q) \implies (\neg p)$.

Truthiness of a sentence

Consider the proposition

$$\varphi := (p \vee q \vee \neg r) \wedge (p \vee \neg q) \wedge (p \vee r)$$

How do we evaluate whether this sentence is true?

Assign truth value to each variable.

Follow order of operations.

Logical equivalence

Logical equivalence

Two sentences ψ and φ are logically equivalent, written $\psi \equiv \varphi$, if they have identical truth tables.

Example

Let $\psi := p \implies q$.

Let $\varphi := \neg p \vee q$.

ψ is logically equivalent to φ, i.e., $\psi \equiv \varphi$, because they have their truth tables are the same.

p	q	$p \implies q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example

Recall earlier sentence,

$$\varphi := (p \vee q \vee \neg r) \wedge (p \vee \neg q) \wedge (p \vee r)$$

This sentence is logically equivalent to simply p.

In other words, $\varphi \equiv p$.

Poll: logical equivalence

Recall that two sentences ψ and φ are logically equivalent, written $\psi \equiv \varphi$, if they have identical truth tables.

Consider the following two propositions:

- $\varphi := \neg(p \wedge q)$
- $\psi := \neg p \vee \neg q$

Are φ and ψ logically equivalent? In other words, is $\varphi \equiv \psi$?

A) Yes

B) No

C) Only when both p and q are True

D) Only when both p and q are False

Important equivalence relationships

$(\alpha \wedge \beta)$	$\equiv$	$(\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta)$	$\equiv$	$(\beta \vee \alpha)$	commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma)$	$\equiv$	$(\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma)$	$\equiv$	$(\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$\neg(\neg\alpha)$	$\equiv$	α	double-negation elimination
$(\alpha \implies \beta)$	$\equiv$	$(\neg\beta \implies \neg\alpha)$	contraposition
$(\alpha \implies \beta)$	$\equiv$	$(\neg\alpha \vee \beta)$	implication elimination
$(\alpha \iff \beta)$	$\equiv$	$((\alpha \implies \beta) \wedge (\beta \implies \alpha))$	biconditional elimination
$\neg(\alpha \wedge \beta)$	$\equiv$	$(\neg\alpha \vee \neg\beta)$	De Morgan's law #1
$\neg(\alpha \vee \beta)$	$\equiv$	$(\neg\alpha \wedge \neg\beta)$	De Morgan's law #2
$(\alpha \wedge (\beta \vee \gamma))$	$\equiv$	$((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma))$	$\equiv$	$((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$

Operator substitution: can we replace $\oplus$?

Are all of the logical connectives really necessary? You already know that $\implies$ is unnecessary because $p \implies q \equiv \neg p \vee q$.

What about $\oplus$ (exclusive or)? Can we replace it with an expression involving $\neg$, $\vee$ and $\wedge$?

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$$

Poll: can we replace $\wedge$?

Are all of the logical connectives really necessary? You already know that $\implies$ is unnecessary because $p \implies q \equiv \neg p \vee q$.

What about $\wedge$ (and)?

A) $p \wedge q$ can be replaced with $\neg(p \vee q) \vee (\neg p \vee q)$
B) $p \wedge q$ can be replaced with $\neg(\neg p \vee \neg q)$
C) $p \wedge q$ can be replaced with $\neg(p \implies q)$
D) $p \wedge q$ can be replaced with something else
E) $\wedge$ is necessary: it cannot be replaced.

Minimal set of logical connectives

It turns out that the following operators are *not* necessary:

- if, $\implies$
- iff, $\iff$
- exclusive or, $\oplus$
- and, $\wedge$

Because we can represent all of the above using only two connectives:

- Or $\vee$ and Not $\neg$

The set of connectives $\{\vee, \neg\}$ is functionally complete, meaning that any statement we can write in propositional logic we can write with only these two connectives.

Can we get it down to just one?

Ways to show logical equivalence

There are basically two ways to show logical equivalence $\psi \equiv \varphi$:

1. Using a truth table.
   - Make a truth table with columns for ψ and φ.
   - Equivalent if and only if the T/F values in each row are identical between the two columns.
2. Using known logical equivalences.
   - Step-by-step approach, resembling a proof.
   - Equivalent if and only if one can start with ψ and gradually transform it into φ using only known logical equivalence properties.

Example

A proposition ψ is a tautology if it is true under every assignment of its variables. In other words, ψ is a tautology if $\psi \equiv True$.

Can we show that

$$\psi := p \wedge (p \implies q) \implies q$$

is a tautology?

We can use either of the two approaches to show $\psi \equiv True$:

1. Truth table approach: column should be all *True*
2. Transformation approach: manipulate φ until it equals *True*.

Shown on board

Normal forms: CNF and DNF

Literal

Definition (Literal)

A literal is an atomic proposition or the negation of an atomic proposition (i.e. it's either p or $\neg p$ for some variable p).

Example

Let $p :=$ "Alice earns an A." and $q :=$ "Pigs can fly."

Literals: p, $\neg p$, q, $\neg q$.

Not literals: $p \vee q$, $q \implies p$, etc.

Conjunctive Normal Form

Definition (CNF)

A proposition is in conjunctive normal form (CNF) if it consists of:

- a single *clause*, or
- a conjunction of two or more *clauses*

where a clause is

- a single *literal*, or
- a disjunction of two or more literals

Example

These propositions are in CNF:

- $(p \vee q \vee s) \wedge (\neg p \vee r \vee \neg q)$
- $(\neg q \vee s)$

These propositions are *not* in CNF:

- $(p \vee q) \implies (\neg p \vee r)$
- $(\neg q \wedge s) \wedge (\neg p \vee r)$

Disjunctive Normal Form

Disjunctive Normal Form (DNF) is the same idea, just swap the role of ANDs and ORs. See textbook for defintion and examples.

Informally,

- conjunctive-normal form (CNF) is an "AND" of a bunch of "ORs".
- disjunctive-normal form (DNF) is an "OR" of a bunch of "ANDs".

Poll

Which of these propositions is *not* in CNF?

A) $\neg p$

B) $p \vee q$

C) $(p \vee q) \wedge (r \vee s)$

D) $(p \wedge q) \vee (r \wedge \neg p)$

E) More than one is *not* CNF / All are in CNF

(Definition restated here)
A proposition is in CNF if it is a single clause or the conjunction of two or more clauses where each clause is a single literal or the disjunction of two or more literals.

A literal is an atomic proposition or the negation of an atomic proposition (i.e. it's either p or $\neg p$ for some variable p).

CNF and DNF

Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in disjunctive-normal form (DNF) – an "or" of a bunch of "ands".

Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in conjunctive-normal form (CNF) – an "and" of a bunch of "ors".

Why might this be useful?

Checking a CNF sentence for tautology

If φ is a proposition in CNF. Then checking for a tautology is easy.

- φ is a tautology if and only if each clause is a tautology.
- A clause from a CNF is a tautology if and only if it contains a literal and its opposite.

Poll: is this CNF a tautology?

Consider

$$\varphi := (p \vee q \vee \neg p) \wedge (r \vee p \vee q \vee \neg q) \wedge (\neg r \vee p \vee r)$$

Is φ in CNF? Is φ a tautology?

A) CNF: yes, tautology: yes

B) CNF: yes, tautology: no

C) CNF: no, tautology: yes

D) CNF: no, tautology: no