COSC 290 Discrete Structures

Lecture 7: Nested quantifiers

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Plan for today

1. Expressing statements in predicate logic (continued...)
2. Nested Quantifiers
3. Negating quantifiers
4. Practice: nested and negated quantifiers
5. Logistics

Expressing statements in predicate logic (continued...)

Review of fastest person question

Let P be the set of all people. Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Which of the following is the correct definition for $fastest(x)$?

This works:

$$fastest(x) \coloneqq \forall y \in P - \{ x \} : faster(x, y)$$

So does this:

$$fastest(x) \coloneqq \neg (\exists y \in P : faster(y, x))$$

however, with this version, it is possible for two or more persons to be tied for fastest.

Poll: fastest lacrosse player

As before, let P be the set of all people. Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Let $lax(x)$ be true if x plays lacrosse and false otherwise.

Which of the following is the correct definition for $fastestLax(x)$, the fastest lacrosse player?

A) $fastestLax(x) \coloneqq \forall y \in P - \{ x \} : (lax(y) \wedge faster(x, y))$

B) $fastestLax(x) \coloneqq \forall y \in P - \{ x \} : (lax(y) \implies faster(x, y))$

C) $fastestLax(x) \coloneqq lax(x) \wedge \forall y \in P - \{ x \} : faster(x, y)$

D) $fastestLax(x) \coloneqq lax(x) \wedge \forall y \in P - \{ x \} : (lax(y) \wedge faster(x, y))$

E) $fastestLax(x) \coloneqq lax(x) \wedge \forall y \in P - \{ x \} : (lax(y) \implies faster(x, y))$

Poll: not the slowest lacrosse player

As before, let P be the set of all people and $faster(x, y)$ be true if x runs faster than y and false otherwise; and $lax(x)$ be true if x plays lacrosse and false otherwise.

Which of the following is the correct definition for $notSlowestLax(x)$, which is true when x is *not* the slowest lacrosse player?

A) $notSlowestLax(x) := \exists y \in P : (lax(y) \wedge faster(x, y))$

B) $notSlowestLax(x) := \exists y \in P : (lax(y) \implies faster(x, y))$

C) $notSlowestLax(x) := lax(x) \wedge \exists y \in P : faster(x, y)$

D) $notSlowestLax(x) := lax(x) \wedge \exists y \in P : (lax(y) \wedge faster(x, y))$

E) $notSlowestLax(x) := lax(x) \wedge \exists y \in P : (lax(y) \implies faster(x, y))$

Nested Quantifiers

Nested quantifiers

Let $S := \{2, 3, 6\}$. Consider the following two propositions:

$$\varphi_1 := \forall x \in S \exists y \in S : \mathit{isDivisibleBy}(x, y)$$

$$\varphi_2 := \exists x \in S \forall y \in S : \mathit{isDivisibleBy}(x, y)$$

What are the truth values of these two propositions?

(Write out on board.)

Nested quantifiers

Let $S \coloneqq \{2, 3, 6\}$. Consider the following two propositions:

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What are the truth values of these two propositions?

(Write out on board.)

φ_1 is logically equivalent to:

$(\exists y_1 \in S : \mathit{isDivisibleBy}(2, y_1)) \wedge (\exists y_2 \in S : \mathit{isDivisibleBy}(3, y_2)) \wedge (\exists y_3 \in S : \mathit{isDivisibleBy}(6, y_3))$

φ_2 is logically equivalent to:

$(\forall y_1 \in S : \mathit{isDivisibleBy}(2, y_1)) \vee (\forall y_2 \in S : \mathit{isDivisibleBy}(3, y_2)) \vee (\forall y_3 \in S : \mathit{isDivisibleBy}(6, y_3))$

Both φ_1 and φ_2 are true.

Order of quantifiers matters

Let $S \coloneqq \{2, 3, 6\}$.

Contrast the following two propositions:

$$\varphi_1 \coloneqq \forall x \in S \exists y \in S : \mathit{isDivisibleBy}(x, y)$$

$$\varphi_3 \coloneqq \exists y \in S \forall x \in S : \mathit{isDivisibleBy}(x, y)$$

We already saw that φ_1 is true. What about φ_3?

Order of quantifiers matters

Let $S \coloneqq \{2, 3, 6\}$.

Contrast the following two propositions:

$$\varphi_1 \coloneqq \forall x \in S \exists y \in S : \mathit{isDivisibleBy}(x, y)$$

$$\varphi_3 \coloneqq \exists y \in S \forall x \in S : \mathit{isDivisibleBy}(x, y)$$

We already saw that φ_1 is true. What about φ_3?

φ_3 is logically equivalent to:

$(\forall x_1 \in S : \mathit{isDivisibleBy}(x_1, 2)) \vee (\forall x_2 \in S : \mathit{isDivisibleBy}(x_2, 3)) \vee (\forall x_3 \in S : \mathit{isDivisibleBy}(x_3, 6))$

φ_3 is false.

Another example

Consider a set of professors P, students S, and courses C.

- Let $takes(s, c)$ be true when student s takes course c.
- Let $teaches(p, c)$ be true when professor p teaches course c.

Let's define predicate $favCourse(c)$ that is true when c is a course taken by all students.

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$$favCourse(c) \coloneqq \forall s \in S : takes(s, c)$$

Example continued...

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Let's define $profOfFav(p)$ that is true when professor p teaches a course that is taken by all students.

Example continued...

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- Let $takes(s, c)$ be true when student s takes course c.
- Let $teaches(p, c)$ be true when professor p teaches course c.
- Let $favCourse(c)$ be true when c is a course taken by all students.

Let's define $profOfFav(p)$ that is true when professor p teaches a course that is taken by all students.

$$profOfFav(c) := \exists c \in C : (teaches(p, c) \wedge favCourse(c))$$

Unpacking a complex predicate can reveal nested quantifiers

Let's "unpack" the predicate *profOfFav*:

$$\begin{aligned} \mathit{profOfFav}(p) &\coloneqq \exists c \in C : (\mathit{teaches}(p, c) \land \mathit{favCourse}(c)) \\ &\equiv \exists c \in C : (\mathit{teaches}(p, c) \land (\forall s \in S : \mathit{takes}(s, c))) \end{aligned}$$

Order of quantification

The order the of the quantifiers matters.

$$Q_1(p) \coloneqq \exists c \in C : \forall s \in S : (teaches(p, c) \land takes(s, c))$$

$Q_1(p)$: "Prof. who teaches a course every student takes"

vs.

$$Q_2(p) \coloneqq \forall s \in S : \exists c \in C : (teaches(p, c) \land takes(s, c)))$$

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vs.

$$Q_2(p) \coloneqq \forall s \in S : \exists c \in C : (teaches(p, c) \wedge takes(s, c)))$$

$Q_2(p)$: “Prof. who teaches every student”
(but not necessarily in the same course).

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$$\beta := \exists x \in S : Q(x)$$

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$$\beta := \exists x \in S : Q(x)$$

$$\neg\beta \equiv \neg(\exists x \in S : Q(x)) \equiv \forall x \in S : \neg Q(x)$$

Negating nested quantifiers

Let $S \coloneqq \{ 2, 3, 6 \}$. Recall the definition of φ_2,

$$\varphi_2 \coloneqq \exists x \in S \forall y \in S : \mathit{isDivisibleBy}(x, y)$$

What is $\neg\varphi_2$?

Negating nested quantifiers

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$$\varphi_2 \coloneqq \exists x \in S \forall y \in S : \mathit{isDivisibleBy}(x, y)$$

What is $\neg\varphi_2$?

Let's take this part "$\forall y \in S : \mathit{isDivisibleBy}(x, y)$" and define it as a predicate: $\mathit{divisibleByAll}(x) \coloneqq \forall y \in S : \mathit{isDivisibleBy}(x, y)$.

Thus, $\varphi_2 \equiv \exists x \in S : \mathit{divisibleByAll}(x)$

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Thus, $\varphi_2 \equiv \exists x \in S : \mathit{divisibleByAll}(x)$

$$\begin{aligned}
\neg\varphi_2 &\equiv \neg(\exists x \in S : \mathit{divisibleByAll}(x)) \\
&\equiv \forall x \in S : \neg\mathit{divisibleByAll}(x) \\
&\equiv \forall x \in S : \neg(\forall y \in S : \mathit{isDivisibleBy}(x, y)) \\
&\equiv \forall x \in S : \exists y \in S : \neg\mathit{isDivisibleBy}(x, y)
\end{aligned}$$

Negating quantifiers

What about a course that *isn't* a favorite? In other words, a course for which $\neg favCourse(c)$ is true?

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$$\begin{aligned} \neg \mathit{favCourse}(c) &\equiv \neg(\forall s \in S : \mathit{takes}(s, c)) \\ &\equiv \exists s \in S : \neg \mathit{takes}(s, c) \end{aligned}$$

Negating quantifiers

What about a course that *isn't* a favorite? In other words, a course for which $\neg favCourse(c)$ is true?

$$\begin{aligned} \neg favCourse(c) &\equiv \neg(\forall s \in S : takes(s, c)) \\ &\equiv \exists s \in S : \neg takes(s, c) \end{aligned}$$

If the course isn't a favorite, then there must be one student who doesn't take it.

Negating nested quantifiers

What about a professor that *isn't* a teacher of a favorite course? In other words, a professor for which $\neg \mathit{profOfFav}(p)$ is true?

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$$\begin{aligned}
\neg \mathit{profOfFav}(p) &\equiv \neg(\exists c \in C : (\mathit{teaches}(p, c) \wedge \mathit{favCourse}(c))) \\
&\equiv \forall c \in C : \neg(\mathit{teaches}(p, c) \wedge \mathit{favCourse}(c)) \\
&\equiv \forall c \in C : (\neg \mathit{teaches}(p, c) \vee \neg \mathit{favCourse}(c)) \\
&\equiv \forall c \in C : (\mathit{teaches}(p, c) \implies \neg \mathit{favCourse}(c)) \\
&\equiv \forall c \in C : (\mathit{teaches}(p, c) \implies (\exists s \in S : \neg \mathit{takes}(s, c)))
\end{aligned}$$

Negating nested quantifiers

What about a professor that *isn't* a teacher of a favorite course? In other words, a professor for which $\neg \mathit{profOfFav}(p)$ is true?

$$\begin{aligned}
\neg \mathit{profOfFav}(p) &\equiv \neg(\exists c \in C : (\mathit{teaches}(p, c) \wedge \mathit{favCourse}(c))) \\
&\equiv \forall c \in C : \neg(\mathit{teaches}(p, c) \wedge \mathit{favCourse}(c)) \\
&\equiv \forall c \in C : (\neg \mathit{teaches}(p, c) \vee \neg \mathit{favCourse}(c)) \\
&\equiv \forall c \in C : (\mathit{teaches}(p, c) \implies \neg \mathit{favCourse}(c)) \\
&\equiv \forall c \in C : (\mathit{teaches}(p, c) \implies (\exists s \in S : \neg \mathit{takes}(s, c)))
\end{aligned}$$

If the professor isn't a favorite, then for every course that this professor teaches, there must be at least one student not taking the course.

Practice: nested and negated quantifiers

True love, expressed mathematically

Predicate $loves(p_1, p_2)$ means "p_1 loves p_2." We can express the loves predicate visually using a graph.

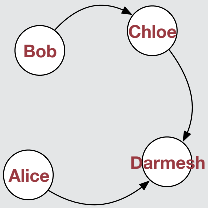
Nodes are individuals. Edge from p_1 to p_2 indicates $loves(u, v)$.

Example

The proposition

$$loves(Alice, Darmesh) \wedge loves(Bob, Chloe) \wedge loves(Chloe, Darmesh)$$

can be shown as.

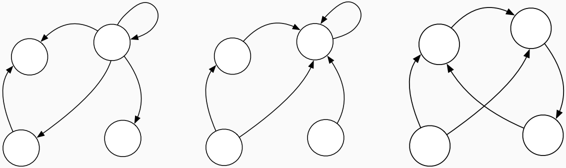


Poll: nested quantifiers, part 1

Predicate *loves*(p_1, p_2) means "p_1 loves p_2," shown by an arrow from p_1 to p_2.

For which figure(s) is the following proposition true:

$$\forall p_1 \in P : \exists p_2 \in P : loves(p_1, p_2)$$



(a) (b) (c)

More than one/ None of the above

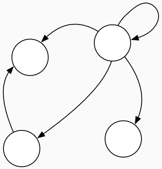
(d)

Poll: nested quantifiers, part 2

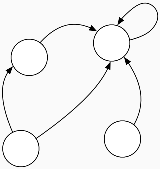
Predicate *loves*(p_1, p_2) means "p_1 loves p_2," shown by an arrow from p_1 to p_2.

For which figure(s) is the following proposition true:

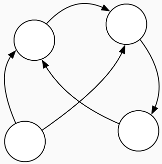
$$\exists p_2 \in P : \forall p_1 \in P : loves(p_1, p_2)$$



(a)



(b)



(c)

More than one/ None of the above

(d)

Poll: showing a proposition is false

Consider the following proposition

$$\forall p_1 \in P : \exists p_2 \in P : loves(p_1, p_2)$$

How could we show that this proposition is false?

A) Show there is a person who loves everyone
B) Show there is a person who loves no one
C) Show there is a person who nobody loves
D) Show there is a person who everyone loves
E) Other/more/none

Poll: negating nested quantifiers

Consider the following proposition

$$\varphi := \exists p_2 \in P : \forall p_1 \in P : \mathit{loves}(p_1, p_2)$$

What is the correct negation of φ? In other words, which of the following is logically equivalent to $\neg\varphi$?

A) $\forall p_2 \forall p_1 \neg \mathit{loves}(p_1, p_2)$

B) $\forall p_2 \exists p_1 \neg \mathit{loves}(p_1, p_2)$

C) $\exists p_2 \forall p_1 \neg \mathit{loves}(p_1, p_2)$

D) $\forall p_2 \exists p_1 \neg \mathit{loves}(p_2, p_1)$

E) Other/more/none

Logistics

Logistics

Next topic on schedule: proofs!

Reading does not align neatly with lecture plan.

We will spend some time (roughly 1 lecture each) on several proof techniques:

- direct proof
- proof by contrapositive
- proof by contradiction
- proof by cases

We will *return* to Ch. 4.2 to apply these techniques to Hamming codes.