COSC 290 Discrete Structures

Lecture 29: Combinations and permutations

Prof. Michael Hay
Friday, Apr. 13, 2018
Colgate University

Plan for today

1. Finish up division rule
2. Four types of counting problems
3. Counting when order matters (2 ways)
4. Counting when order is irrelevant (2 ways)

Finish up division rule

Recap: k-to-1 functions and division rule

Definition (k-to-1 functions)

Let $f : X \to Y$. We say that f is k-to-1 if for all $y \in Y$, there are k distinct elements of X such that $f(x) = y$. In other words,

$$\forall y \in Y : |\{ x \in X : f(x) = y \}| = k$$

Theorem (Division rule)

Let $f : X \to Y$. If f is k-to-1, then $|X| = k \cdot |Y|$.

Recall example: Knights at a round table

How many ways can you *arrange* n knights at a round table?

A *seating* defines who sits where. Two seatings are considered same *arrangement* if each knight has the same knight on its left and right in both seatings.

Example: here are two distinct *seatings*, but they represent the same *arrangement*.

Images taken and problem adapted from Lehman et al. *Mathematics for Computer Science*, 2017.

Poll: division rule

Peer review. Suppose there are P papers submitted to a conference and the conference organizers must find a set R of reviewers. Each paper must be read by k reviewers. Each reviewer will be assigned ℓ papers to review.

If $|P| = 100$ and $k = 3$ and $\ell = 9$, then how big must R be? Hint: it might help to think about Q, the set of reviews written by the reviewers, and apply the division rule *twice*. You will probably want to do some work on a piece of paper.

A) 33
B) 34
C) 100
D) 300
E) 2700

Four types of counting problems

Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

- A starting basketball lineup of $k = 5$ players at five positions (C, PF, SF, PG, SG) from among $n = 12$ players.
- A team of $k = 4$ runners to compete in a single cross-country race from among $n = 10$ available runners.
- A runner for each of $k = 5$ track races from among a team of $n = 12$ available runners.
- The composition of a basketball team ($k = 5$ players) where each player is one of $n = 3$ types: perimeterShooter, blocker, ballHandler.
- A selection of $k = 12$ donuts from $n = 3$ donut types (jelly, chocolate, glazed).

Four types of common counting problem

Given a set S with n elements, let us consider counting the number of ways to choose k elements from S. To formalize this counting problem, we must answer two questions:

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

The answers to these questions leads to four distinct counting problems:

	order matters	order irrelevant
repetition forbidden		
repetition allowed		

Example shown on board: let $S = \{A, B, C\}$ and $k = 2$. Write out solutions to all four versions of the problem.

Goal for today: fill in this table.

Counting when order matters (2 ways)

Order matters, repetition allowed

How many ways to choose a sequence of k (not necessarily distinct) elements from a set of n elements?

Example: one runner for each of $k = 5$ track races from among a team of $n = 12$ available runners. (Same runner can compete in multiple races.)

How many ways? (Use the product rule.)

$$\underbrace{n}_{\text{choices for first element}} \cdot \underbrace{n}_{\text{choices for second element}} \cdot \cdots \cdot \underbrace{n}_{\text{choices for } k^{th} \text{ element}} = n^k$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden		
repetition allowed	n^k	

Order matters, repetition forbidden

How many ways to choose a sequence of k distinct elements from a set of n elements?

Example: a starting basketball lineup of $k = 5$ players at five positions (C, PF, SF, PG, SG) from among $n = 12$ players.

How many ways? (Use the generalized product rule.)

$$\underbrace{(n)}_{\text{choices for first element}} \cdot \underbrace{(n-1)}_{\text{choices for second element}} \cdot \cdots \cdot \underbrace{(n-k+1)}_{\text{choices for } k^{th} \text{ element}} = \frac{n!}{(n-k)!}$$

Example: sequences of a certain size

Let $S := \{a, b, c, d, e\}$. Let $n := |S|$. How many sequences of $k = 2$ distinct elements can be constructed from S?

There are $n \cdot (n-1) = 20$ ways:

$$\{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle a,e\rangle,$$
$$\langle b,a\rangle, \langle b,c\rangle, \langle b,d\rangle, \langle b,e\rangle,$$
$$\langle c,a\rangle, \langle c,b\rangle, \langle c,d\rangle, \langle c,e\rangle,$$
$$\langle d,a\rangle, \langle d,b\rangle, \langle d,c\rangle, \langle d,e\rangle,$$
$$\langle e,a\rangle, \langle e,b\rangle, \langle e,c\rangle, \langle e,d\rangle\}.$$

Alternative derivation: using division rule

Let B be the set we are trying to count: sequences of $k = 2$ distinct elements from $S := \{a, b, c, d, e\}$.

Let A be the set of all permutations of S. (Recall that a permutation of set S is an $|S|$-length sequence of elements of S with no repetitions.)

Let $f : A \to B$ map a permutation into k-element sequence by simply keeping first k elements of the permutation.

Example
$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	B
$\langle a,b,c,d,e\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,c,e,d\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,d,c,e\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,d,e,c\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,e,c,d\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,e,d,c\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,c,b,d,e\rangle$	$\to$	$\langle a,c\rangle$
$\langle a,c,b,e,d\rangle$	$\to$	$\langle a,c\rangle$
$\langle a,c,d,b,e\rangle$	$\to$	$\langle a,c\rangle$
	...	

Alternative derivation continued...

Example
$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	B
$\langle a,b,c,d,e\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,c,e,d\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,d,c,e\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,d,e,c\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,e,c,d\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,b,e,d,c\rangle$	$\to$	$\langle a,b\rangle$
$\langle a,c,b,d,e\rangle$	$\to$	$\langle a,c\rangle$
$\langle a,c,b,e,d\rangle$	$\to$	$\langle a,c\rangle$
$\langle a,c,d,b,e\rangle$	$\to$	$\langle a,c\rangle$
	...	

How many permutations map to *same* k sequence?

Permutation maps to $\langle a,b\rangle$ iff it starts with $\langle a,b\rangle$ followed by remaining $n-k$ elements in *any order*.

There are $(n-k)!$ ways to order remaining $(n-k)$ elements.

f is a $(n-k)!$-to-1 function, so...

$$|B| = \frac{|A|}{(n-k)!} = \frac{n!}{(n-k)!}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	
repetition allowed	n^k	

Counting when order is irrelevant (2 ways)

Order irrelevant, repetition forbidden

How many ways to choose a set of k elements from a set of n elements?

Example: A starting volleyball lineup of $k = 6$ players from among $n = 13$ players. (Assume position irrelevant in volleyball because players rotate.)

This counting problem comes up so often it has a special name/notation:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

The binomial coefficient, denoted $\binom{n}{k}$, is the number of combinations of k elements chosen from n candidate elements.

Example: Counting bitstrings with k ones

How many length n bitstrings contain exactly k ones?

A length n bitstring has n bit positions $b_1 b_2 \ldots b_n$.

Must choose a *set* of k positions where the ones will be placed. (The order in which the bit positions are chosen is irrelevant.)

Thus, the number of length n bitstrings that contain exactly k ones is $\binom{n}{k}$.

Example: subsets of a certain size

Let $S := \{a, b, c, d, e\}$. Let $n := |S|$. How many subsets of size $k = 2$ can be constructed from S?

There are $\binom{n}{k} = \binom{5}{2} = 10$:

$\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\},$
$\{b, c\}, \{b, d\}, \{b, e\},$
$\{c, d\}, \{c, e\},$
$\{d, e\}.$

Derivation of $\binom{n}{k}$ using division rule

Let C be the set we are trying to count.

Let A be the set of all permutations of S.

Let $g : A \to C$ map a permutation into k-element sequence by simply taking first k elements of the permutation and putting them in a set.

Example
$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	C
$\langle a, b, c, d, e\rangle$	$\to$	$\{a, b\}$
$\langle a, b, c, e, d\rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, c, e\rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, e, c\rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, c, d\rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, d, c\rangle$	$\to$	$\{a, b\}$
$\langle a, c, b, d, e\rangle$	$\to$	$\{a, c\}$
	$\dots$	
$\langle a, e, d, c, b\rangle$	$\to$	$\{a, e\}$
$\langle b, a, c, d, e\rangle$	$\to$	$\{a, b\}$
$\langle b, a, c, e, d\rangle$	$\to$	$\{a, b\}$
$\langle b, a, d, c, e\rangle$	$\to$	$\{a, b\}$
$\langle b, a, d, e, c\rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, c, d\rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, d, c\rangle$	$\to$	$\{a, b\}$
$\langle b, c, a, d, e\rangle$	$\to$	$\{b, c\}$
	$\dots$	

Derivation of $\binom{n}{k}$ continued...

Example
$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	C
$\langle a, b, c, d, e\rangle$	$\to$	$\{a, b\}$
$\langle a, b, c, e, d\rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, c, e\rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, e, c\rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, c, d\rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, d, c\rangle$	$\to$	$\{a, b\}$
$\langle a, c, b, d, e\rangle$	$\to$	$\{a, c\}$
	$\dots$	
$\langle a, e, d, c, b\rangle$	$\to$	$\{a, e\}$
$\langle b, a, c, d, e\rangle$	$\to$	$\{a, b\}$
$\langle b, a, c, e, d\rangle$	$\to$	$\{a, b\}$
$\langle b, a, d, c, e\rangle$	$\to$	$\{a, b\}$
$\langle b, a, d, e, c\rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, c, d\rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, d, c\rangle$	$\to$	$\{a, b\}$
$\langle b, c, a, d, e\rangle$	$\to$	$\{b, c\}$
	$\dots$	

How many permutations map to *same* set?

Permutation maps to $\{a, b\}$ iff it starts with the elements in $\{a, b\}$ in *any order* followed by remaining $n - k$ elements in *any order*.

There are $k!$ ways to order the first k elements. There are $(n - k)!$ ways to order remaining $(n - k)$ elements.

g is a $k!(n - k)!$-to-1 function, so...

$$|C| = \frac{|A|}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
repetition allowed	n^k	

Poll: Counting number of ways to select lineups

Suppose your hockey team 13 players, only one of whom is a goalie. You want to organize your time into a starting line (goalie + 5 players), a backup line (goalie + 5 other players), and benchwarmers (remaining 2 players).

Let's assume that, except for the goalie, position does not matter.

How many different ways can you select the starting line, backup line, and benchwarmers?

A) $\binom{13}{6} \cdot \binom{13}{6} \cdot \binom{13}{3}$
B) $\binom{13}{6} \cdot \binom{7}{6}$
C) $\binom{12}{5} + \binom{7}{5}$
D) $\binom{12}{5} \cdot \binom{7}{5}$
E) More than one / None of the above