

COSC 290 Discrete Structures

Lecture 22: Relations & Composition

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Relations

Plan for today

1. Relations
2. Practice with composition
3. Representations of relations

1

Relations

Let A and B be sets.

A (binary) relation on $A \times B$ is a subset of $A \times B$.

A binary relation on $A \times A$ is a subset of $A \times A$ and is simply called a relation on A .

2

Examples

- *EnrolledAt* is a relation on *Persons* \times *College*: $\langle p, c \rangle \in \text{EnrolledAt}$ if person p attends college c .
- *FacebookFriends* is a relation on *FacebookUsers*: $\langle u, v \rangle \in \text{FacebookFriends}$ if u has friended v on Facebook.
- \leq is a relation on \mathbb{R} : $\langle x, y \rangle \in \leq$ if x is less than or equal to y . (We often write using *infix* notation: $x \leq y$.)
- *abs* is a relation on $\mathbb{R} \times \mathbb{R}^{\geq 0}$: $\langle x, y \rangle \in \text{abs}$ if $|x| = y$.

3

Inverse and composition

Let R be a relation on $A \times B$ and S be a relation on $B \times C$.

Definition (Inverse)

Let R be a relation on $A \times B$. The **inverse** R^{-1} of R is a relation on $B \times A$ defined by $R^{-1} := \{ \langle b, a \rangle \in B \times A : \langle a, b \rangle \in R \}$

Definition (Composition)

The **composition** of R and S is a relation on $A \times C$, denoted $S \circ R$, where $\langle a, c \rangle \in S \circ R$ iff there exists a $b \in B$ such that $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in S$.

(write on the board for later use)

4

Practice with composition

Examples of composition

Let us define sets *Persons*, *Clubs*, *Animals*.

Let **Members** be a relation on *Persons* \times *Clubs* such that $\langle p, c \rangle \in \text{Members}$ means that person p is a member of club c .

Let **Mascots** be a relation on *Clubs* \times *Animals* such that $\langle c, a \rangle \in \text{Mascots}$ means that club c has animal a for a mascot.

Let **Leaders** be a relation on *Clubs* \times *Persons* such that $\langle c, p \rangle \in \text{Leaders}$ means that club c is led by person p .

Consider these compositions:

1. $\text{Mascots} \circ \text{Members}$?
2. $\text{Leaders} \circ \text{Members}$?
3. $\text{Members} \circ \text{Leaders}$?

5

Another example of composition

Suppose we have the following relations:

- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$
- $teaching \subseteq Professors \times Classes$
- $doing \subseteq Students \times Performances$

Let's derive a new relation from the above relations plus the inverse and composition operators. Let $R \subseteq Students \times Times$ where $\langle s, t \rangle \in R$ indicates that student s is taking a class at time t .

How do we express R ?

$$R = at \circ taking$$

6

Poll: deriving new relations, 1

Suppose we have the following relations:

- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$
- $teaching \subseteq Professors \times Classes$
- $doing \subseteq Students \times Performances$

Let's derive a new relation from the above relations plus the inverse and composition operators. Let $R \subseteq Students \times Professors$ where $\langle s, p \rangle \in R$ indicates that student s is taking at least one class with professor p .

- A) $taking \circ teaching$
- B) $teaching \circ taking$
- C) $taking \circ teaching^{-1}$
- D) $teaching^{-1} \circ taking$
- E) None of the above / More than one

7

Poll: deriving new relations, 2

Suppose we have the following relations:

- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$
- $teaching \subseteq Professors \times Classes$
- $doing \subseteq Students \times Performances$

Let's derive a new relation from the above relations plus the inverse and composition operators. Let $R \subseteq Professors \times Performances$ where $\langle p, p' \rangle \in R$ means professor p should attend performance p' to see at least one of the professor's students perform.

- A) $doing \circ (taking^{-1} \circ teaching)$
- B) $doing \circ (teaching \circ taking^{-1})$
- C) $(taking^{-1} \circ teaching) \circ doing$
- D) $(teaching \circ taking^{-1}) \circ doing$
- E) None of the above / More than one

8

Poll: Minimum cardinality

Suppose that sets A, B, C have cardinalities n_A, n_B, n_C respectively. Further, $\min \{ n_A, n_B, n_C \} > 0$. Let R be a relation on $A \times B$ and S a relation on $B \times C$. What is the **minimum** cardinality of $S \circ R$? (In discussion, justify your answer.)

- A) 0
- B) n_B
- C) $n_A + n_C$
- D) $n_A \cdot n_C$
- E) $\min \{ n_A, n_C \}$
- F) $\min \{ n_A, n_B, n_C \}$

9

Poll: Maximum cardinality

Suppose that sets A, B, C have cardinalities n_A, n_B, n_C respectively. Let R be a relation on $A \times B$ and S a relation on $B \times C$. What is the **maximum** cardinality of $S \circ R$? (In discussion, justify your answer.)

- A) n_B
- B) $n_A + n_C$
- C) $n_A \cdot n_C$
- D) $\max \{ n_A, n_C \}$
- E) $\max \{ n_A, n_B, n_C \}$

10

Representations of relations

Graphical representations of relations

A binary relation on $A \times B$ can be represented visually in a couple of ways:

- A table (Fig. 8.2 from book, left side)
- A mapping (Fig. 8.2 from book, right side)

A binary relation on A can be conveniently represented as a **graph**.

11

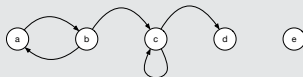
Graphical representation of relation on A

Example

Let $A := \{ a, b, c, d, e \}$. And consider relation R on A defined as

$$R := \{ \langle a, b \rangle, \langle b, c \rangle, \langle b, a \rangle, \langle c, c \rangle, \langle c, d \rangle \}$$

R can be represented as a graph:



12

Exercise

Consider this relation R :



On a piece of paper,

1. Draw R^{-1}
2. Draw $R \circ R$.