COSC 290 Discrete Structures

Lecture 10: Proofs by contradiction and cases

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Wednesday, Feb. 14, 2018

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Plan for today

1. Proof by contradiction
2. Example of proof by contradiction
3. Proof by contradiction vs. proof by contrapositive
4. Proof by cases

Logistics

- First midterm exam: 1 week from this Friday!
- Lab 1: continue to work on it this week

Proof by contradiction

Proof by contradiction

To prove that proposition φ is true,

you can assume φ is false (i.e, $\neg\varphi$ is true) and show that this assumption leads to a contradiction.

Procedure for proof by contradiction

Goal: prove that φ is true.

Process:

1. Negate the proposition, resulting in $\neg\varphi$.
   (*Note: you typically want to* ***simplify*** *this expression, pushing the negation down.*)

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2. Assume $\neg\varphi$ is true.
3. Show that this leads to a contradiction, i.e.,
   - Show that $\neg\varphi$ implies some ψ
   - Show that $\neg\varphi$ also implies $\neg\psi$
   - But $\psi \wedge \neg\psi \equiv$ *False* (a contradiction)

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   - But $\psi \wedge \neg\psi \equiv$ *False* (a contradiction)
4. Since $\neg\varphi \implies$ *False*, we can conclude φ must be true.

Truth table for contradiction

Exercise: Work in small groups to show that

$$(\neg p \implies False) \equiv p$$

Hint: recall that $p \implies q \equiv \neg p \vee q$.

p	q	$p \implies q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Proof by Contradiction Template

- **Claim**: *Write the theorem/claim to be proved, "φ is true."*
- **Proof by contradiction**: *"Assume the claim is false. In other words, [state negated form of φ]"*
  It's critical that you (a) explicitly state this is a proof by a contradiction and (b) state the assumption that will lead to the contradiction! Why?

Proof by Contradiction Template

- **Claim**: *Write the theorem/claim to be proved, "φ is true."*
- **Proof by contradiction**: *"Assume the claim is false. In other words, [state negated form of φ]"*
  - *Write main body of proof...*
  - *... establish some ψ must be true.*
  - *... establish some $\neg\psi$ must also be true.*
  - *"But [state ψ and $\neg\psi$] is a contradiction."* Be sure to clearly identify the contradiction!

Proof by Contradiction Template

- **Claim**: *Write the theorem/claim to be proved, “φ is true.”*
- **Proof by contradiction**: *“Assume the claim is false. In other words, [state negated form of φ]”*
  - *Write main body of proof...*
  - *... establish some ψ must be true.*
  - *... establish some $\neg\psi$ must also be true.*
  - *“But [state ψ and $\neg\psi$] is a contradiction.”* Be sure to clearly identify the contradiction!
  - **Conclusion**: “Therefore the original assumption that [restate $\neg\varphi$] is false, and we can conclude that [restate theorem].”

Example of proof by contradiction

Poll: proof by contradiction 1

- **Claim**: There is no integer that is both even and odd.
- **Proof by contradiction**: "Assume the claim is false. In other words, … " what goes here?

A) All integers are both odd and even.

B) All integers are not odd or not even.

C) There is an integer n that is both odd and even.

D) There is an integer n that is neither odd nor even.

E) None of above / More than one of above

Exercise: complete the proof

- **Claim**: There is no integer that is both even and odd.
- **Proof by contradiction**: Assume the claim is false. In other words, suppose there exists an integer *n* that is both even and odd.
- Work in small groups to find a contradiction!
- Useful tools:
  - $\mathbb{Z}$ is the set of all integers
  - $\mathit{Even}(x) := \exists k \in \mathbb{Z} : x = 2k$
  - $\mathit{Odd}(x) := \exists \ell \in \mathbb{Z} : x = 2\ell + 1$
  - Sum/Difference of two integers is an integer.
  - Algebra, logic.

Rational numbers

Recall: a rational number is a real number that can be expressed as the ratio of two integers.

$$Rational(y) \coloneqq \exists n \in \mathbb{Z} : \exists d \in \mathbb{Z}^{\neq 0} : y = n/d$$

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We will consider the following claim: if x^2 is irrational, then x is irrational.

Poll: from English to Predicate Logic

Consider the claim,

"If x^2 is irrational, then x is irrational."

Formulate this claim in predicate logic:

A) $\exists x \in \mathbb{R} : \neg Rational(x^2) \wedge \neg Rational(x)$

B) $\exists x \in \mathbb{R} : \neg Rational(x^2) \implies \neg Rational(x)$

C) $\forall x \in \mathbb{R} : \neg Rational(x^2) \wedge \neg Rational(x)$

D) $\forall x \in \mathbb{R} : \neg Rational(x^2) \implies \neg Rational(x)$

E) None of above / More than one of above

Poll: proof by contradiction 2

- **Claim**: If x^2 is irrational, then x is irrational.
- **Proof by contradiction**: "Assume the claim is false. In other words, ... " what goes here? be careful with negating an implication!

A) There exists x where both x and x^2 are rational.

B) There exists x where both x and x^2 are irrational.

C) There exists x where x is rational and x^2 is irrational.

D) There exists x where x is irrational and x^2 is rational.

E) None of above / More than one of above

Exercise: complete the proof

- **Claim**: If x^2 is irrational, then x is irrational.
- **Proof by contradiction**: Assume the claim is false. In other words, suppose there exists an x such that x is rational but x^2 is irrational.
- Work in small groups to find a contradiction!
- Useful tools:
    - $\mathbb{R}$ is the set of all real numbers
    - $\mathbb{Z}$ is the set of all integers
    - $Rational(y) \coloneqq \exists n \in \mathbb{Z} : \exists d \in \mathbb{Z}^{\neq 0} : y = n/d$
    - Product of two integers is an integer.
    - Algebra, logic.

Proof by contradiction vs. proof by contrapositive

Contradiction vs. contrapositive

- **Claim**: If x^2 is irrational, then x is irrational.
- **Proof by contradiction**: Assume the claim is false. In other words, suppose there exists an x such that x is rational but x^2 is irrational. We will show this leads to a contradiction...
- **Proof by contrapositive**: Assume that x is rational. We will show that x^2 is rational.

Contradiction vs. contrapositive

- Contradiction can be used for *any* proposition φ. Contrapositive only applies to φ of the form $p \implies q$.

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- How do they compare when $\varphi \coloneqq p \implies q$?
- Contrapositive: given $\neg q$, show $\neg p$.
- Contradiction: ??
  Let's look at $\neg(p \implies q)$ on the board.

Contradiction vs. contrapositive

- Contradiction can be used for *any* proposition φ. Contrapositive only applies to φ of the form $p \implies q$.
- How do they compare when $\varphi \coloneqq p \implies q$?
- Contrapositive: given $\neg q$, show $\neg p$.
- Contradiction: given $\neg(p \implies q) \equiv \neg q \wedge p$, show *some* contradiction.
  For example, you could assume $\neg q \wedge p$ and show the contrapositive (i.e. $\neg q \implies \neg p$) and then you have a contradiction $p \wedge \neg p$.

When to use proof by contradiction?

There isn't an easy answer.[1]

Try other techniques first.

Sometimes useful when trying to prove a "negative": $\sqrt{2}$ is irrational (i.e., *not* rational).

[1] https://gowers.wordpress.com/2010/03/28/when-is-proof-by-contradiction-necessary/

Proof by cases

Example

[[MH: do ramsey theory example...]]