COSC 290 Discrete Structures

Lecture 32: Conditional Probability

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Plan for today

1. Review basic concepts
2. Conditional Probability

Review basic concepts

Four steps

1. Find the sample space
2. Define events of interest
3. Determine outcome probabilities (possibly using tree diagrams)
4. Compute event probabilities

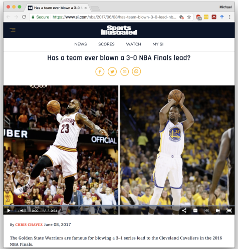
Exercise: Cavs vs. Pacers

Suppose Cleveland Cavaliers play Indiana Pacers in a best of 3 series (first team to win two of three games wins). Suppose probability of Cavs winning a game is 3/5, regardless of results of previous games.

- What is the probability that three games are played? (Event A)
- What is the probability that the winning team loses the first game? (Event B)
- What is the probability the Cavs win? (Event C)

Conditional Probability

Motivating example



Suppose it's June 8, 2017. What statistical information might be relevant for estimating probability of a team coming back from an 0-3 deficit to win 4-3?

Poll: Three prisoner's problem

Three prisoners A, B, and C are on death row. The governor decides to pardon one of the prisoners and chooses one at random. He informs the warden of his choice but requests that the name be kept secret.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks which of B and C will be executed. The warden thinks for a while and then tells A that B is to be executed.

Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. What is the correct answer? If unsure how to calculate, take your best guess.

A) Prisoner A, B) Both wrong, C) Prisoner C

Source: adapted from Casella and Berger and Wikipedia

Conditional Probability

Definition (Conditional probability)

The conditional probability of event A given event B is

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

where it is required that $Pr(B) \neq 0$.

Intuition: it is the probability of A in light of the information that event B has occurred.

Examples:

- Probability that Cavs win finals *given* that they lost first three games.
- Probability that A is pardoned *given* that Warden says B to be executed.

Boy or girl paradox

Recall this two part question from previous lecture:

- Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?
- Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

Let's revisit this problem using the concept of conditional probability.

- Let A be event Mr. Jones has two girls.
- Let B be event Mr. Jones oldest child is a girl.
- Let C be event Mr. Smith has two boys.
- Let D be event Mr. Smith has at least one boy.

Source: https://en.wikipedia.org/wiki/Boy`or`Girl`paradox

Four steps

1. Find the sample space
2. Define events of interest
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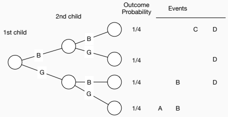
Boy or Girl pardox explained



$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}} = \frac{1}{2}$$

$$Pr(C|D) = \frac{Pr(C \cap D)}{Pr(D)} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{1}{3}$$

Poll: conditional probability

Let S be sample space and let B and C be events.

Suppose $B \subset C$ and $C \subseteq S$. What can you say about $Pr(C|B)$?

A) it's equal to 0
B) it's equal to 1
C) it lies somewhere between 0 and 1 but cannot be exactly 0 or 1
D) it can be anything: 0, 1, or in between 0 and 1

Another perspective on conditional probability

Observe that $B = (A \cap B) \cup (\bar{A} \cap B)$ where $\bar{A}$ is the event that A does *not* happen. (In other words, $\bar{A} := S - A$ where S is sample space.)

The conditional probability of event A given event B is

$$
\begin{aligned}
Pr(A|B) &:= \frac{Pr(A \cap B)}{Pr(B)} && \text{definition} \\
&= \frac{Pr(A \cap B)}{Pr((A \cap B) \cup (\bar{A} \cap B))} && \text{rewrite } B \\
&= \frac{Pr(A \cap B)}{Pr(A \cap B) + Pr(\bar{A} \cap B)} && \text{events } A \cap B \text{ and } \bar{A} \cap B \text{ are disjoint.}
\end{aligned}
$$

Intuition: Only look at outcomes where B happens. Split into two groups based on whether or not A happens too. Conditional probability $P(A|B)$ depends on which of these groups is more likely.

Three Prisoners Problem

Let's analyze using conditional probability and a tree diagram:

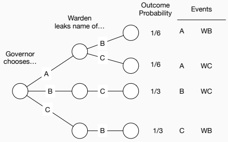
Three prisoners A, B, and C are on death row. The governor decides to pardon one of the prisoners and chooses one at random. He informs the warden of his choice but requests that the name be kept secret.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks which of B and C will be executed. The warden thinks for a while and then tells A that B is to be executed.

Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. What is the correct answer?

Events: A, B, C, WB, WC. Event A means A is pardoned. WB means Warden says B is to be executed.

Three Prisoners Problem



$$Pr(A|WB) = \frac{Pr(A \cap WB)}{Pr(WB)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{3}$$

$$Pr(C|WB) = \frac{Pr(C \cap WB)}{Pr(WB)} = \frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = \frac{2}{3}$$