COSC 290 Discrete Structures

Lecture 25: Partial orders and Warshall relations

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Plan for today

1. Review: Transitive Closure
2. Warshall relations
3. Partial orders
4. Hasse diagram

Review: Transitive Closure

Review: Transitive closures

A transitive closure of a relation R on A is a smallest $R' \supseteq R$ that satisfies transitivity.

Review: Computing the transitive closure

Input: Relation $R \subseteq A \times A$.
Output: smallest $R' \supseteq R$ that is *transitive*

```
R' := R
repeat
    new := (R ∘ R') − R'
    R' := R' ∪ new
until |new| = 0
return R'
```


Warshall relations

Warshall relation

Warshall relations are a more efficient way of computing transitive closure.

Warshall relations are a sequence of relations $W_0, W_1, \ldots, W_n$. Each one can be computed with a "small" update from the previous one.

In the end, W_n is the transitive closure of R.

Warshall relation

Let $A := \{ a_1, a_2, \ldots, a_n \}$, a finite set.

Let R be a relation on A.

For $k = 0$ to n, let W_k denote the k^{th} Warshall relation for R where W_k is defined as...

- $W_0 := R$
- For $k \geq 1$, W_k is a relation on A such that $\langle a_i, a_j \rangle \in W_k$ iff there is a sequence of relationships in R connecting a_i to a_j using any subset of the elements $\{a_1, a_2, \ldots, a_k\}$ as intermediates.

(Example shown on board.)

Summary of example shown on board

W_0 (i.e., this is the relation R)

F F F T
T F F F
F T F F
F T F F

W_1

F F F T
T F F T
F T F F
F T F F

W_2

F F F T
T F F T
T T F T
T T F T

W_3

F F F T
T F F T
T T F T
T T F T

W_4

T T F T
T T F T
T T F T
T T F T

Poll: what pairs are in W_k?

Background: for relation R on A where $|A| = n$, the Warshall relations are a sequence of relations $W_0, W_1, \ldots W_{n-1}, W_n$. Relation W_k is a relation on A such that $\langle a_i, a_j \rangle \in W_k$ iff there is a sequence of relationships in R connecting a_i to a_j using any subset of the elements $\{a_1, a_2, \ldots, a_k\}$ as intermediates.

Question: Consider two Warshall relations W_{k-1} and W_k and the difference between them $W_k - W_{k-1}$. Consider some $\langle x, y \rangle \in W_k - W_{k-1}$. Which of the following statements could be true?

A) No such pair exists (implying $W_{k-1} \supseteq W_k$)
B) $\langle x, y \rangle \in R$
C) There is a path from x to y using only $\{ a_1, \ldots, a_{k-1} \}$
D) $\langle x, a_k \rangle \in R$ and $\langle a_k, y \rangle \in R$
E) $\langle x, a_k \rangle \in W_{k-1}$ and $\langle a_k, y \rangle \in W_{k-1}$
F) More than one / None of the above

Warshall relations: key ideas

If you are *adding* a pair $\langle x, y \rangle$ to W_k that is not already in W_{k-1}, the following must be true:

1. There is a path from x to y that uses a subset of $\{ a_1, \ldots, a_k \}$ as intermediates.
2. Element a_k appears on that path
3. The path from x to a_k must only require a subset of $\{ a_1, \ldots, a_{k-1} \}$. Similarly for the path from a_k to y.

Thus, to go from W_{k-1} to W_k, you can focus on paths through a_k.

Partial orders

Special relation: partial order

Relation R on A is a partial order if it is reflexive, antisymmetric, transitive.

Conventions: use $\preceq$ as the "name" of the relation (as opposed to a letter like R) and use *infix* notation: $a \preceq b$ instead of $\langle a, b\rangle \in \ \preceq$.

Intuition: partial order relations behave like $\leq$ except that some pairs may be *incomparable*.

Example (Partial order)

The *prefixOf* relation is a partial order:

- "a" $\preceq$ "aa"
- "aa" $\preceq$ "aardvark"

Note: not all pairs comparable: "a" $\npreceq$ "b" and "b" $\npreceq$ "a"

Special relation: strict partial order

Relation R on A is a strict partial order if it is irreflexive, (antisymmetric), transitive.

Conventions: use $\prec$ as the "name" of the relation (as opposed to a letter like R) and use *infix* notation: $a \prec b$ instead of $\langle a, b\rangle \in \ \prec$.

Intuition: strict partial order relations behave like $<$ except that some pairs may be *incomparable*.

Example (Strict partial order)

The *ancestorOf* relation (ancestor is parent or (recursively) parent of ancestor):

- "DT" $\prec$ "Don Jr"
- "Hanns Drumpf" $\prec$ "DT" (#makedonalddrumpfagain)
- not all pairs comparable: "Harry Potter" $\nprec$ "Aunt Petunia" and "Aunt Petunia" $\nprec$ "Harry Potter"

Poll: partial order

Relation $\preceq$ is a partial order if it is reflexive, antisymmetric, transitive.

Consider two relations on a set of track runners:

- $a \preceq_1 b$ if the number of races in which a competed is no more than the number in which b competed.
- $a \preceq_2 b$ if the total amount of time (measured in nanoseconds with laser precision so that ties are impossible) that a ran is no more than the total amount of time that b ran.

Is $\preceq_1$ a partial order? Is $\preceq_2$ a partial order?

A) Yes, Yes
B) Yes, No
C) No, Yes
D) No, No

Hasse diagram

Hasse diagram

A partial order $\preceq$ on A can be drawn using a Hasse diagram.

- Draw nodes: one node for each A
- Draw edges: edge from a to b if $a \preceq b$, except...
- ... *omit* edges that can be inferred by reflexivity
- ... *omit* edges that can be inferred by transitivity
- ... and *layout* nodes "by level" if $a \preceq b$ for $a \neq b$, then a is placed *lower* than b

Example: isSubstringOf relation on the strings $\{ a, b, c, ab, bc, abc, cd \}$.

Exercise: draw Hasse diagram

Complete the following exercise: on a piece of paper, draw a Hasse diagram for the relation on $A := \{ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \}$ for the relation $R \subseteq A \times A$ where

$$R := \{\langle x, y\rangle \in A \times A : y \bmod x = 0\}$$

- Draw nodes: one node for each A
- Draw edges: edge from a to b if $a \preceq b$, except...
- ... *omit* edges that can be inferred by reflexivity
- ... *omit* edges that can be inferred by transitivity
- ... and *layout* nodes "by level" if $a \preceq b$ for $a \neq b$, then a is placed *lower* than b

Example partial order

A to do list,

$$[attendClass, sleep, borrowBook, eat, brushTeeth, study]$$

with constraints:

- $borrowBook \preceq study$
- $study \preceq attendClass$
- $sleep \preceq attendClass$
- $eat \preceq brushTeeth$
- $brushTeeth \preceq sleep$

What should you do *first*? Brush teeth? Eat? Borrow book?

Total order

Relation R is a total order if it is a partial order where every pair is comparable (either $\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$).

A total order can be written succinctly as an ordered list.

Is previous example a total order?

Topological ordering

Given a partial order $\preceq$, a topological ordering is a total order $\preceq_{total}$ that is *consistent* with $\preceq$.

(See book for formal definition of consistent; see earlier lectures for algorithms for topological sort.)