COSC 290 Discrete Structures

Lecture 30: Combinations and permutations

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Monday, Apr. 16, 2018

Colgate University

Plan for today

1. Four types of counting problems

2. Counting when order matters (2 ways)

3. Counting when order is irrelevant (2 ways)

4. Pigeonhole principle

Four types of counting problems

Examples

Consider the following counting problems. How are they similar/different?

How many ways can you choose...

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- A runner for each of $k = 5$ track races from among a team of $n = 12$ available runners.

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- A team of $k = 4$ runners to compete in a single cross-country race from among $n = 10$ available runners.
- A runner for each of $k = 5$ track races from among a team of $n = 12$ available runners.
- The composition of a basketball team ($k = 5$ players) where each player is one of $n = 3$ types: perimeterShooter, blocker, ballHandler.

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- A team of $k = 4$ runners to compete in a single cross-country race from among $n = 10$ available runners.
- A runner for each of $k = 5$ track races from among a team of $n = 12$ available runners.
- The composition of a basketball team ($k = 5$ players) where each player is one of $n = 3$ types: perimeterShooter, blocker, ballHandler.
- A selection of $k = 12$ donuts from $n = 3$ donut types (jelly, chocolate, glazed).

Four types of common counting problem

Given a set S with n elements, let us consider counting the number of ways to choose k elements from S. To formalize this counting problem, we must answer two questions:

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

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The answers to these questions leads to four distinct counting problems:

	order matters	order irrelevant
repetition forbidden		
repetition allowed		

Example shown on board: let $S = \{ A, B, C \}$ and $k = 2$. Write out solutions to all four versions of the problem.

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Example shown on board: let $S = \{A, B, C\}$ and $k = 2$. Write out solutions to all four versions of the problem.

Goal for today: fill in this table.

Counting when order matters (2 ways)

Order matters, repetition allowed

How many ways to choose a sequence of k (not necessarily distinct) elements from a set of n elements?

Example: one runner for each of $k = 5$ track races from among a team of $n = 12$ available runners. (Same runner can compete in multiple races.)

How many ways?

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$$\underbrace{n}_{\text{choices for first element}} \cdot \underbrace{n}_{\text{choices for second element}} \cdot \cdots \cdot \underbrace{n}_{\text{choices for } k^{th} \text{ element}} = n^k$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden		
repetition allowed	n^k	

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How many ways? (Use the generalized product rule.)

$$\underbrace{(n)}_{\text{choices for first element}} \cdot \underbrace{(n-1)}_{\text{choices for second element}} \cdots \cdots \underbrace{(n-k+1)}_{\text{choices for } k^{th} \text{ element}} = \frac{n!}{(n-k)!}$$

Example: sequences of a certain size

Let $S := \{ a, b, c, d, e \}$. Let $n := |S|$. How many sequences of $k = 2$ distinct elements can be constructed from S?

Example: sequences of a certain size

Let $S \coloneqq \{ a, b, c, d, e \}$. Let $n \coloneqq |S|$. How many sequences of $k = 2$ distinct elements can be constructed from S?

There are $n \cdot (n-1) = 20$ ways:

$$
\begin{aligned}
&\{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle a, e\rangle, \\
&\langle b, a\rangle, \langle b, c\rangle, \langle b, d\rangle, \langle b, e\rangle, \\
&\langle c, a\rangle, \langle c, b\rangle, \langle c, d\rangle, \langle c, e\rangle, \\
&\langle d, a\rangle, \langle d, b\rangle, \langle d, c\rangle, \langle d, e\rangle, \\
&\langle e, a\rangle, \langle e, b\rangle, \langle e, c\rangle, \langle e, d\rangle\}.
\end{aligned}
$$

Alternative derivation: using division rule

Let B be the set we are trying to count: sequences of $k = 2$ distinct elements from $S := \{ a, b, c, d, e \}$.

Let A be the set of all permutations of S. (Recall that a permutation of set S is an $|S|$-length sequence of elements of S with no repetitions.)

Let $f : A \to B$ map a permutation into k-element sequence by simply keeping first k elements of the permutation.

Example

$S = \{ a, b, c, d, e \}$ and $k = 2$.

A	$\to$	B
$\langle a, b, c, d, e \rangle$	$\to$	$\langle a, b \rangle$
$\langle a, b, c, e, d \rangle$	$\to$	$\langle a, b \rangle$
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How many permutations map to *same* k sequence?

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Permutation maps to $\langle a, b \rangle$ iff it starts with $\langle a, b \rangle$ followed by remaining $n - k$ elements in *any order*.

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Permutation maps to $\langle a, b \rangle$ iff it starts with $\langle a, b \rangle$ followed by remaining $n - k$ elements in *any order*.

There are $(n - k)!$ ways to order remaining $(n - k)$ elements.

Alternative derivation continued...

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f is a $(n - k)!$-to-1 function, so...

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f is a $(n - k)!$-to-1 function, so...

$$|B| = \frac{|A|}{(n-k)!} = \frac{n!}{(n-k)!}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	
repetition allowed	n^k	

Counting when order is irrelevant (2 ways)

Order irrelevant, repetition forbidden

How many ways to choose a set of k elements from a set of n elements?

Example: A starting volleyball lineup of $k = 6$ players from among $n = 13$ players. (Assume position irrelevant in volleyball because players rotate.)

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This counting problem comes up so often it has a special name/notation:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

The binomial coefficient, denoted $\binom{n}{k}$, is the number of combinations of k elements chosen from n candidate elements.

Example: Counting bitstrings with k ones

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How many length n bitstrings contain exactly k ones?

A length n bitstring has n bit positions $b_1 b_2 \dots b_n$.

Must choose a *set* of k positions where the ones will be placed. (The order in which the bit positions are chosen is irrelevant.)

Thus, the number of length n bitstrings that contain exactly k ones is $\binom{n}{k}$.

Example: subsets of a certain size

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There are $\binom{n}{k} = \binom{5}{2} = 10$:

$\{ a, b \}, \{ a, c \}, \{ a, d \}, \{ a, e \},$
$\{ b, c \}, \{ b, d \}, \{ b, e \},$
$\{ c, d \}, \{ c, e \},$
$\{ d, e \}.$

Derivation of $\binom{n}{k}$ using division rule

Let C be the set we are trying to count.

Let A be the set of all permutations of S.

Let $g : A \to C$ map a permutation into k-element sequence by simply taking first k elements of the permutation and putting them in a set.

Example

$S = \{a, b, c, d, e\}$ and $k = 2$.

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Derivation of $\binom{n}{k}$ continued...

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How many permutations map to *same* set?

Derivation of $\binom{n}{k}$ continued...

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Permutation maps to $\{ a, b \}$ iff it starts with the elements in $\{ a, b \}$ in *any order* followed by remaining $n - k$ elements in *any order*.

Derivation of $\binom{n}{k}$ continued...

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Permutation maps to $\{a, b\}$ iff it starts with the elements in $\{a, b\}$ in *any order* followed by remaining $n - k$ elements in *any order*.

There are $k!$ ways to order the first k elements. There are $(n - k)!$ ways to order remaining $(n - k)$ elements.

Derivation of $\binom{n}{k}$ continued...

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Derivation of $\binom{n}{k}$ continued...

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	$\dots$	

How many permutations map to *same* set?

Permutation maps to $\{ a, b \}$ iff it starts with the elements in $\{ a, b \}$ in *any order* followed by remaining $n - k$ elements in *any order*.

There are $k!$ ways to order the first k elements. There are $(n - k)!$ ways to order remaining $(n - k)$ elements.

g is a $k!(n - k)!$-to-1 function, so...

$$|C| = \frac{|A|}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
repetition allowed	n^k	

Poll: Counting number of ways to select lineups

Suppose your hockey team 13 players, only one of whom is a goalie. You want to organize your time into a starting line (goalie + 5 players), a backup line (goalie + 5 other players), and benchwarmers (remaining 2 players).

Let's assume that, except for the goalie, position does not matter.

How many different ways can you select the starting line, backup line, and benchwarmers?

A) $\binom{13}{6} \cdot \binom{13}{6} \cdot \binom{13}{3}$
B) $\binom{13}{6} \cdot \binom{7}{6}$
C) $\binom{12}{5} + \binom{7}{5}$
D) $\binom{12}{5} \cdot \binom{7}{5}$
E) More than one / None of the above

Order irrelevant, repetition allowed

How many ways to choose an unordered collection of k (not necessarily distinct) elements from a set of n elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

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$$\binom{n+k-1}{k}$$

Poll: how many b-ball teams?

How many ways can you choose the composition of a basketball team ($k = 5$ players) where each player is one of $n = 3$ types: perimeterShooter, blocker, ballHandler.

A) $\binom{5}{3}$
B) $\binom{3}{5}$
C) $\binom{8}{5}$
D) $\binom{8}{3}$
E) $\binom{7}{5}$

Derivation, a little more formally

Use mapping rule. Represent a choice of k elements from set of n candidates as sequence $\langle x_1, x_2, \ldots, x_n \rangle$ where x_i is the number of times we chose candidate element i.

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Use mapping rule. Represent a choice of k elements from set of n candidates as sequence $\langle x_1, x_2, \ldots, x_n \rangle$ where x_i is the number of times we chose candidate element i.

Since we choose a total of k elements, we require:

$$\sum_{i=1}^{n} x_i = k$$

Bijective mapping to bit-strings:

$$f(\langle x_1, x_2, \ldots, x_n \rangle) = \underbrace{00\ldots0}_{x_1 \text{ times}} \; 1 \; \underbrace{00\ldots0}_{x_2 \text{ times}} \; 1 \; \ldots \; 1 \; \underbrace{00\ldots0}_{x_n \text{ times}}$$

(Bit string is always length $n + k - 1$ because there are $n - 1$ ones and the total number of zeros must add up to k.)

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Poll: ways to sum

How many different solutions are there to the equation $a + b + c = 8$ where a, b, c must be non-negative integers? *(You have to do some calculations but with a little bit of algebra, you can do this by hand or with a basic calculator.)*

A) 45

B) 56

C) 120

D) 165

Pigeonhole principle

Claim

Somewhere in *your* family tree, you have an ancestor B whose parents were blood relatives—i.e., the father of B and the mother of B have a common ancestor A.

“Somewhere” means sometime in last 4000 years.

We will prove this using the pigeonhole principle.

Pigeonhole principle

(Informal) If you have more pigeons than pigeonholes, there is at least one pigeonhole with at least two pigeons.

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Theorem (Pigeonhole principle)

Let X and Y be sets such that $|X| > |Y|$. Let f be any function $f : X \to Y$. Then f is *not* one-to-one.

Back to family tree claim

Claim: In last 4000 years, there exists an ancestor B in your family tree such that the father of B and the mother of B have a common ancestor A.

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Proof: Proof makes a few (reasonable) assumptions.

- Everyone has a biological mother and father.
- No one lives to be more than 100
- At most 1 trillion people have ever lived.

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Sketch of proof: 40 generations. All lived within last 4000 years.

At least 2^{40} distinct ancestor *roles*. $2^{40} >$ trillion. Pigeonhole principle: more roles than people!

Some ancestor played two roles. Call this person *A*. There must be two distinct paths from *A* to you. Eventually paths meet at some *B*.

Adapted from Kleinberg, `https://www.edge.org/response-detail/11067`

Poll: pigeons prefer matching socks

Suppose among your roommates, their earliest class is a 1:20 and they do not *not* like being disturbed when you rise in the wee hours to make it on time to COSC 290. Thus, you dress in the dark.

Suppose you have 4 different colors of socks, **black**, **gray**, **blue**, and **red**. You want a matching pair, but you can't see anything in the dark. How many socks should you take to ensure you have a matching pair?

A) 2
B) 3
C) 4
D) 5
E) 6