COSC 290 Discrete Structures

Lecture 31: Counting, II

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Wednesday, Nov. 15, 2017

Colgate University

Plan for today

Claim

Somewhere in *your* recent family tree, you have an ancestor B whose parents were blood relatives—i.e., the father of B and the mother of B have a common ancestor A.

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We will prove this using a technique introduced later in lecture.

Mapping rule

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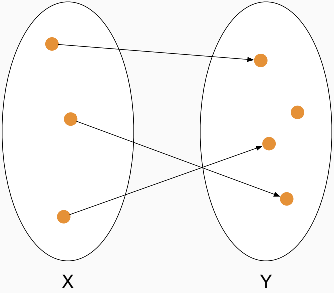
Theorem (Mapping rule)

Let $f : X \to Y$.

- $|X| \geq |Y|$ if f is onto.
- $|X| \leq |Y|$ if f is one-to-one.
- $|X| = |Y|$ if f is bijective (onto *and* one-to-one).

One-to-one



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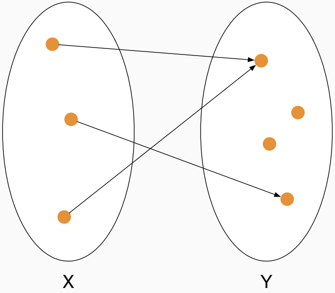

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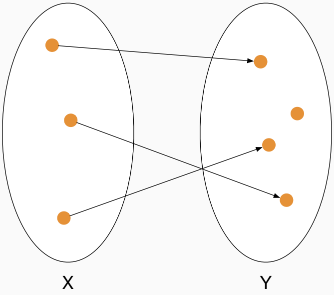
One-to-one



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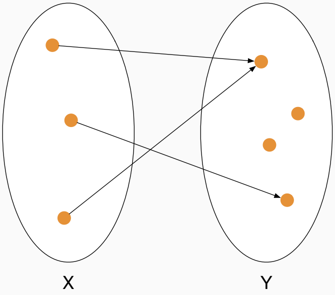
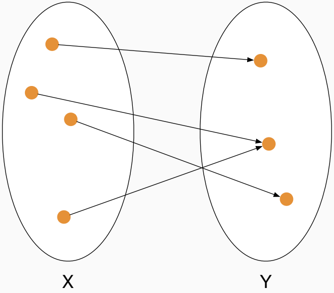


No.

One-to-one: for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.

$$f \text{ is one-to-one} \implies |X| \leq |Y|$$

Onto



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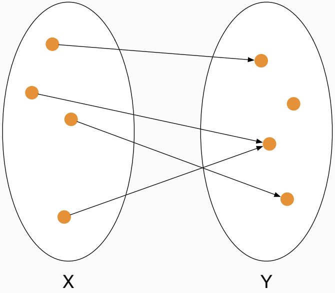

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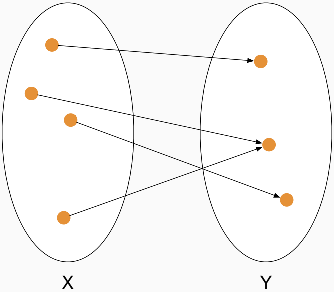
Onto



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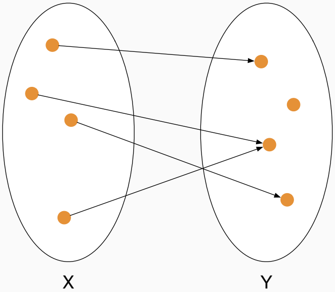
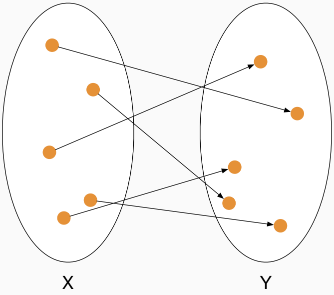


No.

Onto: for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.

$$f \text{ is onto} \implies |X| \geq |Y|$$

Bijective



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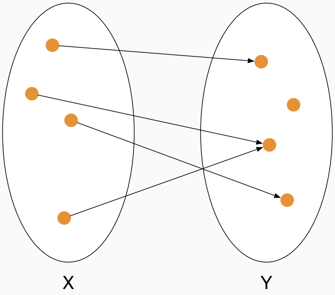

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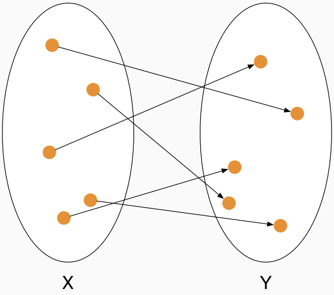
Bijective



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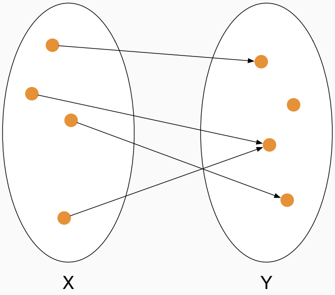

No.

Bijective: for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.

$$f \text{ is bijective} \implies |X| = |Y|$$

Example

Let $S = \{ s_1, s_2, \ldots, s_n \}$ be a set where $|S| = n$.

Claim: $|\mathcal{P}(S)| = 2^n$.

Proof uses a bijection between $\{ 0, 1 \}^n$ and $\mathcal{P}(S)$.

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Let $f : \{ 0, 1 \}^n \to \mathcal{P}(S)$ where

$$f(\langle b_1, b_2, \ldots, b_n \rangle) = \{ s_i \in S : b_i = 1 \}$$

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Need to show:

- $|\{ 0, 1 \}^n| = 2^n$ (use the product rule)
- f is onto: let $X \subseteq S$, show how to construct bit string b_X, show that $f(b_X)$ maps to X.
- f is one-to-one: suppose two distinct bitstrings b, b' mapped to same $X \subseteq S$, show this leads to a contradiction.

Example

Let $B_n \subset \{\,], [\, \}^n$ be a set of strings consisting of *balanced brackets*.

Examples:

$$B_4 = \{\, [[]], [][], \}$$

$$B_6 = \{\, [[[]]], [[][]], [[]][], [][[]], [][][], \}$$

Claim: $2^{n/4} \leq |B_n| \leq 2^n$.

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Claim: $2^{n/4} \leq |B_n| \leq 2^n$.

We will prove this using the mapping rule (repeated below):

Let $f : X \to Y$.

- $|X| \geq |Y|$ if and only if f is onto.
- $|X| \leq |Y|$ if and only if f is one-to-one.
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Poll: applying the mapping rule

Let $f : Friends \to \{-1, 102, 201, 290, 301, 302, 304\}$ be a function that maps each person in *Friends* to the highest numbered required COSC course they have taken (-1 if none).

When applied to your friends, suppose the maximum value is 302. Therefore you have less than 7 friends. What's wrong with this claim?

A) f is not necessarily bijective
B) f is not necessarily onto
C) f is not necessarily one-to-one
D) The logic is correct, but claim is false because one of my friends took 304!

Mapping rule: Let $f : X \to Y$.

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Division rule

Division rule

Definition (k-to-1 functions)
Let $f : X \to Y$. We say that f is k-to-1 if for all $y \in Y$, there are k distinct elements of X such that $f(x) = y$. In other words,

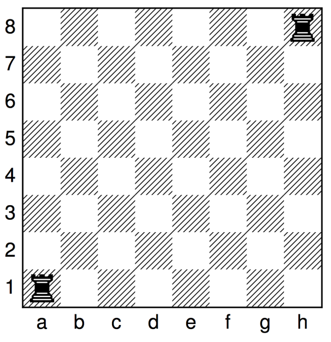
$$\forall y \in Y : |\{ x \in X : f(x) = y \}| = k$$

Theorem (Division rule)
Let $f : X \to Y$. If f is k-to-1, then $|X| = k \cdot |Y|$.

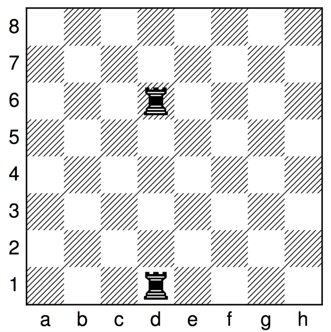
(Draw picture on board)

Example: division rule

How many ways to place two knights on a chess board such that they don't *threaten* each other? (Solution on board.)



Not threatening.



Threatening.

Images taken and problem adapted from Lehman et al. *Mathematics for Computer Science*, 2017.

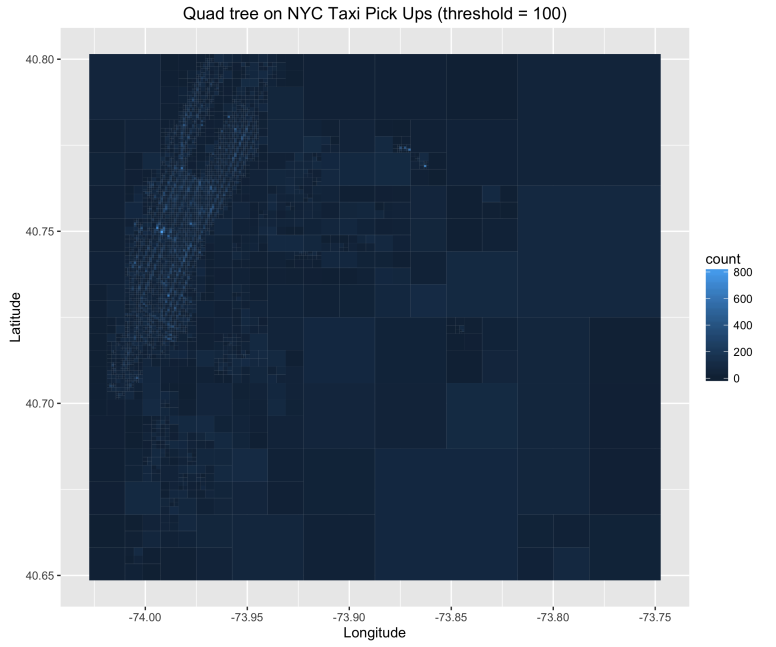
Example: division rule

Spatial indexing (next two slides). To efficiently support map services (Google maps, Yelp “nearby”), data stored in spatial indexes.

A rectangular range query specifies a type of item (e.g., taqueria) and a *rectangle* and returns all such items that overlap with the given rectangle.

Assume city is divided into $n \times n$ discrete points. Rectangle must include at least one point. How many rectangle queries are possible? (Solution on board.)

40.80
40.75
40.70
40.65
lat
-74.00
-73.95
-73.90
-73.85
-73.80
-73.75
long


Quad tree on NYC Taxi Pick Ups (threshold = 100)
Latitude
40.80
40.75
40.70
40.65
Longitude
-74.00
-73.95
-73.90
-73.85
-73.80
-73.75
count
800
600
400
200
0

Knights at a round table

How many ways can you place n knights at a round table?

Poll: division rule

Peer review. Suppose there are P papers submitted to a conference and the conference organizers must find a set R of reviewers. Each paper must be read by k reviewers. Each reviewer will be assigned ℓ papers to review.

If $|P| = 100$ and $k = 3$ and $\ell = 9$, then how big must R be? Hint: it might help to think about Q, the set of reviews written by the reviewers, and apply the division rule *twice*.

A) 33
B) 34
C) 100
D) 300
E) 2700

Pigeonhole principle

Pigeonhole principle

(Informal) If you have more pigeons than pigeonholes, there is at least one pigeonhole with at least two pigeons.

Theorem (Pigeonhole principle)

Let X and Y be sets such that $|X| > |Y|$. Let f be any function $f : X \to Y$. Then f is *not* one-to-one.

Back to family tree claim

Claim: In last 4000 years, there exists an ancestor B in your family tree such that the father of B and the mother of B have a common ancestor A.

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Proof: Proof makes a few (reasonable) assumptions.

- Everyone has a biological mother and father.
- No one lives to be more than 100
- At most 1 trillion people have ever lived.

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Sketch of proof: 40 generations. All lived within last 4000 years.

At least 2^{40} distinct ancestor *roles*. $2^{40} >$ trillion. Pigeonhole principle: more roles than people!

Some ancestor played two roles. Call this person *A*. There must be two distinct paths from *A* to you. Eventually paths meet at some *B*.

Adapted from Kleinberg, `https://www.edge.org/response-detail/11067`

Poll: pigeons prefer matching socks

Suppose among your roommates, their earliest class is a 1:20 and they do not *not* like being disturbed when you rise in the wee hours to make it on time to COSC 290. Thus, you dress in the dark.

Suppose you have 4 different colors of socks, **black**, **gray**, **blue**, and **red**. You want a matching pair, but you can't see anything in the dark. How many socks should you take to ensure you have a matching pair?

A) 2
B) 3
C) 4
D) 5
E) 6