COSC 290 Discrete Structures

Lecture 30: Counting, I

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Plan for today

1. Sum and Product rules
2. Generalized product rule
3. Difference Rule
4. Inclusion-Exclusion

Sum and Product rules

Short URLs

`http://bit.ly/2AF3U9c`

Bitly is a URL shortening service.

- Input: regular URL; output: short url.
- Short url is a string of 6 or 7 characters from { 0-9, A-Z, a-z }.
- If url is 7 characters, first character must be 1 or 2.[1]

Bitly claims to have shortened $34,033,678,000$ urls. How many short urls does Bitly have left?

[1]This claim based on empirical observation.

Sum Rule

If A and B are *disjoint*, then $|A \cup B| = |A| + |B|$.

Example (Short urls)

$shortUrls = sixCharUrls \cup sevenCharUrls$.
$sixCharUrls \cap sevenCharUrls = \emptyset$. Thus,
$|shortUrls| = |sixCharUrls| + |sevenCharUrls|$.

Product Rule

Let $S = A_1 \times A_2 \times \cdots \times A_k$.

Then $|S| = \prod_{i=1}^{k} |A_i|$.

Example (Six character urls)

$|sixCharUrls| = ?$

Let $A = \{$ 0-9, A-Z, a-z $\}$. We have $|A| = 10 + 26 + 26 = 62$ There are 62 choices for each character and six characters total.

Thus $|sixCharUrls| = |A|^6 = 62^6 \approx$ 56.8 billion

Example (Seven character urls)

$|sevenCharUrls| = ?$

Thus $|sevenCharUrls| = |\{1, 2\}| \cdot |A|^6 = 2 \cdot 62^6 \approx$ 113.6 billion

How many short urls left?

$|shortUrls| = |sixCharUrls| + |sevenCharUrls| \approx 170.4$ billion.

Bitly has used roughly $34/170 \approx 20\%$ of available URLs.

Short urls used by cloud services to support collaboration (Google map directions, online documents, etc.) Any potential concerns?

`http://bit.ly/2AF3U9c`

Poll: using the sum and product rule

How many 3 digit numbers are divisible by 5? *Hint: what values can be the first digit take? The middle digit? The last digit?*

A) 180
B) 190
C) 199
D) 200
E) None of above

(Mis)applying the product rule

Suppose the Colgate Coders club must choose three officers—President, Secretary, Treasurer—from among the four leaders of the club: Alice, Bob, Chen, and Divesh.

- Bob doesn't want to be president.
- Only Chen or Divesh can be Treasurer.
- A person can serve in at most one role.

How many distinct officer combinations?

- $P = \{ \text{A, C, D} \}$ (no Bob)
- $S = \{ \text{A, B, C, D} \}$
- $T = \{ \text{C, D} \}$ (no Alice, no Bob)

Let S denote the set of all possible officer assignments. So, $|S| = |P| \cdot |S| \cdot |T| = 3 \cdot 4 \cdot 2 = 24$. What's wrong with this?

Generalized product rule

Generalized product rule

Let S denote a set of length-k sequences such that following condition holds:

For each $i \in \{ 1, \ldots, k \}$ and for each choice of first $i - 1$ components, there are n_i choices for the i^{th} component.

Then $|S| = \prod_{i=1}^{k} n_i$.

(Important: the value n_i does not depend on *what* was chosen for first $i - 1$ components.)

Example (PINs without repetition)

An ATM PIN consists of four digits. A certain bank prohibits a PIN to use a digit more than once. How many PINs are possible?

$10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

Poll: ATM PINs

A certain bank requires four digit PINs where repetition is disallowed and the last digit must be either a 0 or a 1. How many valid PINs are there?

A) $1,008$

B) $1,440$

C) $5,040$

D) None of above / Not sure because product rule doesn't apply

Difference Rule

Difference rule

Let $B \subseteq A$. Then $|A - B| = |A| - |B|$.

This rule follows from the sum rule.

$$\begin{aligned} |A| &= |(A - B) \cup B| \\ &= |(A - B)| + |B| \qquad \text{disjoint sets, apply sum rule} \end{aligned}$$

Example (PINs with repeats)
How many PINs have at least one repeated digit?

$$\begin{aligned} |pinWithRepeats| &= |allPINs| - |pinWithoutRepeats| \\ &= 10^4 - (10 \cdot 9 \cdot 8 \cdot 7) \\ &= 10,000 - 5,040 = 4,960 \end{aligned}$$

(Mis)applying sum rule

For Spring 2018, there are 20 students on the cosc290 wait list and 15 students on the cosc201 wait list. Therefore there are 35 students waiting to take a 200-level course in Computer Science.

What's wrong with this?

Inclusion-Exclusion

Inclusion-exclusion

Two sets: $|A \cup B| = |A| + |B| - |A \cap B|$

Three sets:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Intuition: adjust for elements that are double/triple counted.

Example (Numbers divisible by 3 or 5)

How many numbers in $\{1, \ldots, 1000\}$ are divisble by 3 or 5?

- $|divBy3| = 333$
- $|divBy5| = 200$
- $|divBy15| = 66$
- $|divBy3or5| = 333 + 200 - 66 = 467$

Poll: lower and upper bounds based on inclusion-exclusion

In an introductory COSC course, instructor conducts survey. Results show 20 students have programmed in Python, 15 in Java, and 12 in C. Plus, 11 students have experience with both Python and Java.

How many students have programmed in *at least one* of the three languages? Choose the *narrowest* range that contains the true value.

A) between 0 and 47
B) between 12 and 47
C) between 12 and 36
D) between 24 and 36
E) We are not given enough information