COSC 290 Discrete Structures

Lecture 32: Counting, III

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Colgate University

Plan for today

1. Division rule
2. Pigeonhole principle

Division rule

k-to-1 functions and division rule

Definition (k-to-1 functions)
Let $f : X \to Y$. We say that f is k-to-1 if for all $y \in Y$, there are k distinct elements of X such that $f(x) = y$. In other words,

$$\forall y \in Y : |\{ x \in X : f(x) = y \}| = k$$

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Theorem (Division rule)
Let $f : X \to Y$. If f is k-to-1, then $|X| = k \cdot |Y|$.

(Draw picture on board)

Example: division rule

Revisit rectangular range query counting problem.

Assume city is divided into $n \times n$ discrete points, $\{1, \ldots, n\}^2$.

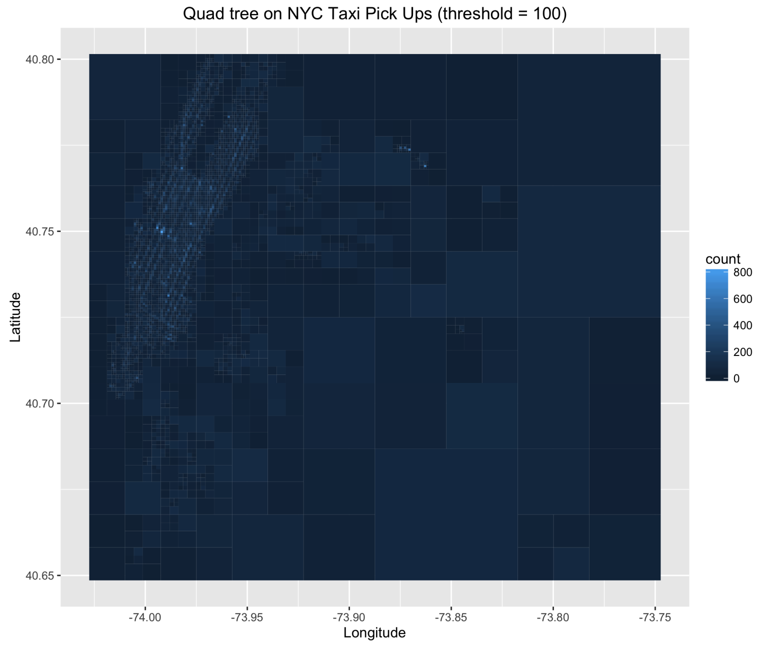
A *rectangle query* is a 4-tuple $\langle x_1, y_1, x_2, y_2 \rangle$ where $x_1 \leq x_2$ and $y_1 \leq y_2$.

(Think of $\langle x_1, y_1 \rangle$ as defining lower left corner of rectangle and $\langle x_2, y_2 \rangle$ as defining upper right corner.)

The *answer* to a query is the total number of items located at any point $\langle x, y \rangle$ such that $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$.

How many rectangle queries are possible? (Solution on board.)


40.80
40.75
40.70
40.65
lat
-74.00
-73.95
-73.90
-73.85
-73.80
-73.75
long


Quad tree on NYC Taxi Pick Ups (threshold = 100)
Latitude
40.80
40.75
40.70
40.65
Longitude
-74.00
-73.95
-73.90
-73.85
-73.80
-73.75
count
800
600
400
200
0

Exercise: Knights at a round table

How many ways can you *arrange* n knights at a round table?

A *seating* defines who sits where. Two seatings are considered same *arrangement* if each knight sits between the same two knights in both seatings.

Example: here are two distinct *seatings*, but they represent the same *arrangement*.

Images taken and problem adapted from Lehman et al. *Mathematics for Computer Science*, 2017.

Poll: division rule

Peer review. Suppose there are P papers submitted to a conference and the conference organizers must find a set R of reviewers. Each paper must be read by k reviewers. Each reviewer will be assigned ℓ papers to review.

If $|P| = 100$ and $k = 3$ and $\ell = 9$, then how big must R be? Hint: it might help to think about Q, the set of reviews written by the reviewers, and apply the division rule *twice*.

A) 33
B) 34
C) 100
D) 300
E) 2700

Pigeonhole principle

Claim

Somewhere in *your* recent family tree, you have an ancestor B whose parents were blood relatives—i.e., the father of B and the mother of B have a common ancestor A.

"Recent" means last 4000 years.

We will prove this using the pigeonhole principle.

Pigeonhole principle

(Informal) If you have more pigeons than pigeonholes, there is at least one pigeonhole with at least two pigeons.

Theorem (Pigeonhole principle)

Let X and Y be sets such that $|X| > |Y|$. Let f be any function $f : X \to Y$. Then f is *not* one-to-one.

Back to family tree claim

Claim: In last 4000 years, there exists an ancestor B in your family tree such that the father of B and the mother of B have a common ancestor A.

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Proof: Proof makes a few (reasonable) assumptions.

- Everyone has a biological mother and father.
- No one lives to be more than 100
- At most 1 trillion people have ever lived.

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Sketch of proof: 40 generations. All lived within last 4000 years.

At least 2^{40} distinct ancestor *roles*. $2^{40} >$ trillion. Pigeonhole principle: more roles than people!

Some ancestor played two roles. Call this person A. There must be two distinct paths from A to you. Eventually paths meet at some B.

Adapted from Kleinberg, `https://www.edge.org/response-detail/11067`

Poll: pigeons prefer matching socks

Suppose among your roommates, their earliest class is a 1:20 and they do not *not* like being disturbed when you rise in the wee hours to make it on time to COSC 290. Thus, you dress in the dark.

Suppose you have 4 different colors of socks, **black**, **gray**, **blue**, and **red**. You want a matching pair, but you can't see anything in the dark. How many socks should you take to ensure you have a matching pair?

A) 2
B) 3
C) 4
D) 5
E) 6