# COSC 290 Discrete Structures

## Lecture 37: Probability III


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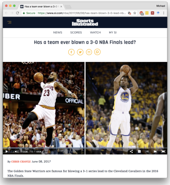
## Plan for today

1. Conditional Probability



# Conditional Probability

## Motivating example




Suppose it's June 8, 2017. What information might be relevant for estimating likelihood of Cavs coming back to beat the Warriors 4-3? The number of times...

- ... a team has won the NBA finals
- ... a team has won the NBA finals after being down 1-2
- ... a team has lost the NBA finals after being down 1-2
- ... a team has won the NBA finals after being down 0-3
- ... a team has lost the NBA finals after being down 0-3

## Poll: Three prisoner's problem

Three prisoners A, B, and C are on death row. The governor decides to pardon one of the prisoners and chooses one at random. He informs the warden of his choice but requests that the name be kept secret.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks which of B and C will be executed. The warden thinks for a while and then tells A that B is to be executed.

Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. What is the correct answer? If unsure how to calculate, take your best guess.

A) Prisoner A, B) Both wrong, C) Prisoner C

Source: adapted from Casella and Berger and Wikipedia



## Conditional Probability

**Definition (Conditional probability)**
The conditional probability of event A given event B is

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

where it is required that $Pr(B) \neq 0$.

Intuition: it is the probability of $A$ in light of the information that event $B$ has occurred.

Examples:

- Probability that Cavs win finals *given* that they lost first three games.
- Probability that A is pardoned *given* that Warden says B to be executed.



## Boy or girl paradox

Recall this two part question from previous lecture:

- Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?
- Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

Let's revisit this problem using the concept of conditional probability.

- Let A be event Mr. Jones has two girls.
- Let B be event Mr. Jones oldest child is a girl.
- Let C be event Mr. Smith has two boys.
- Let D be event Mr. Smith has at least one boy.

Source: https://en.wikipedia.org/wiki/Boy_or_Girl_paradox



## Four steps

1. Find the sample space
2. Define events of interest
3. Determine outcome probabilities (possibly using tree diagrams)
4. Compute event probabilities

## Another perspective on conditional probability

Observe that $B = (A \cap B) \cup (\bar{A} \cap B)$ where $\bar{A}$ is the event that $A$ does *not* happen. (In other words, $\bar{A} := S - A$ where $S$ is sample space.)

The conditional probability of event A given event B is

$$\begin{aligned} Pr(A|B) &= \frac{Pr(A \cap B)}{Pr(B)} && \text{definition} \\ &= \frac{Pr(A \cap B)}{Pr((A \cap B) \cup (\bar{A} \cap B))} && \text{rewrite } B \\ &= \frac{Pr(A \cap B)}{Pr(A \cap B) + Pr(\bar{A} \cap B)} && \text{events } A \cap B \text{ and } \bar{A} \cap B \text{ are disjoint.} \end{aligned}$$

Only look at events where $B$ happens. Split into two groups based on whether or not $A$ happens too. Sum the probability of outcomes in the two groups. $Pr(A|B)$ is a simple function of these two probabilities.



## Three Prisoners Problem

Let's analyze this problem using conditional probability and tree diagram:

Three prisoners A, B, and C are on death row. The governor decides to pardon one of the prisoners and chooses one at random. He informs the warden of his choice but requests that the name be kept secret.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks which of B and C will be executed. The warden thinks for a while and then tells A that B is to be executed.

Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. What is the correct answer?

Events: $A$, $B$, $C$, $WB$, $WC$. Event $A$ means A is pardoned. $WB$ means Warden says B is to be executed.



## Chain rule

The chain rule is the observation that:

$$Pr(A \cap B) = Pr(A|B)Pr(B)$$

(This follows directly from definition of conditional probability, but it's very useful!)

You can apply rule times:

$$\begin{aligned} Pr(A \cap B \cap C \cap D) &= Pr(A|B \cap C \cap D)Pr(B \cap C \cap D) \\ &= Pr(A|B \cap C \cap D)Pr(B|C \cap D)Pr(C \cap D) \\ &= Pr(A|B \cap C \cap D)Pr(B|C \cap D)Pr(C|D)Pr(D) \end{aligned}$$



## Tree diagrams and conditional probability

In tree diagram, ...

- the edge probability is a *conditional* probability of next event *given* previous events along path to root.
- the outcome probability is the product of these conditional probabilities

Example: in three prisoners problem, what the Warden says depends on who has been pardoned! (And implicitly on who is asking.)

- $Pr(\text{Warden says C}|\text{A pardoned}) = \frac{1}{2}$
- $Pr(\text{Warden says C}|\text{B pardoned}) = 1$