COSC 290 Discrete Structures

Lecture 38: Probability IV

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Plan for today

1. Independence
2. Conditional Probability
3. Bayes' rule

Independence

Poll: independent events

Consider the following game: you roll a six-sided die twice. If it comes up 6 on either roll, you win. Arnauld reasons that the probability of winning is 2/6. Here is his flawed reasoning. At which step does he err?

Let WIN denote the event of winning the game. Let A_1 be the event that the first roll is a six; let A_2 be the even that the second roll is a six.

A) $Pr(A_1) = \frac{1}{6}$ because each side is equally likely.
B) $Pr(A_2) = \frac{1}{6}$ for the same reason.
C) $Pr(\text{WIN}) = Pr(A_1 \cup A_2)$
D) $Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2)$
E) $Pr(A_1) + Pr(A_2) = \frac{2}{6}$

Vote F) if you think Arnauld's reasoning is correct.

Independence

Definition (Independence)

Two events A_1 and A_2 are independent if and only if

$$Pr(A_1 \cap A_2) = Pr(A_1)Pr(A_2)$$

Revisiting the dice game

Let's calculate the probability of WIN two ways:

1. $Pr(\text{WIN}) = Pr(A_1 \cup A_2)$
2. $Pr(\text{WIN}) = 1 - Pr(\text{LOSE})$

Shown on board.

Poll: Independence

Flip three fair coins.

- Let A_1 be event that coins 1 and 2 match.
- Let A_2 be event that coins 2 and 3 match.
- Let A_3 be event that coins 1 and 3 match.

Are A_1 and A_2 independent? Are A_2 and A_3 independent? Are A_1 and A_3 independent?

A) Yes for all

B) Yes for some but not all

C) No for all

D) I'm not sure

Mutual Independence

The book does not include mutual independence, but it's important!

Definition (Mutual Independence)

Events $A_1, A_2, \ldots, A_k$ are mutually independent if and only if for every subset of $J \subseteq \{1, \ldots, k\}$,

$$Pr\left(\bigcap_{i \in J} A_i\right) = \prod_{i=1}^{k} Pr(A_i)$$

Independence vs. Mutual independence

Flip three fair coins.

- Let A_1 be event that coins 1 and 2 match.
- Let A_2 be event that coins 2 and 3 match.
- Let A_3 be event that coins 1 and 3 match.

Are A_1, A_2, A_3 *mutually* independent?

Independence vs. Mutual independence

Flip three fair coins.

- Let A_1 be event that coins 1 and 2 match.
- Let A_2 be event that coins 2 and 3 match.
- Let A_3 be event that coins 1 and 3 match.

Are A_1, A_2, A_3 *mutually* independent? No!

Independence vs. Mutual independence

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Are A_1, A_2, A_3 *mutually* independent? No!

Morals of the story:

- Pairwise independence does *not* imply mutual independence!
- Before you simplify $Pr(B_1 \cap B_2 \cap \cdots \cap B_k)$ as $\prod_{i=1}^{k} Pr(B_i)$ check for *mutual* independence.
- Mutual independence occurs when the random process that determines event B_i does not depend on the outcomes of B_j for $j \neq i$. Example: flipping k coins, B_i is i^{th} coin being heads.

Conditional Probability

Recall: Conditional Probability

Definition (Conditional probability)

The conditional probability of event A given event B is

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

where it is required that $Pr(B) \neq 0$.

Recall: Conditional Probability

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Intuition: it is the probability of A in light of the information that event B has occurred.

Examples:

- Probability that Cavs win finals *given* that they lost first three games.
- Probability that A is pardoned *given* that Warden says B to be executed.

Poll: conditional probability

Teams A and B compete in a best of 3 series. The teams are equally matched (probability of A winning is 1/2).

The *conditional probability* of event A given event B is

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

However team A is easily demoralized and so in the game following a loss, the probability that A wins drops to 1/3. (In a game following a win, the teams remain evenly matched.) What is the probability that A wins the series given that they lose *at least* one game? (Hint: Use tree diagram and conditional probability.)

1. $\frac{1}{6}$
2. $\frac{1}{12}$
3. $\frac{1}{3}$
4. $\frac{2}{9}$
5. None of the above

Bayes' rule

A probability puzzle

1. Rate of breast cancer among women 40 to 50 years old with no symptoms nor genetic history is 0.8%.
2. Among breast cancer patients, mammogram test reports positive 90% of time.
3. Among cancer-free patients, mammogram test reports positive 7% of time.
4. Imagine woman matching above criteria has a positive mammogram. What is the probability that she has breast cancer?

What does the doc say?

Previous question was posed to doctors as part of a study on risk estimation. Here is the reported reaction of one study participant, a department chief at teaching hospital with more than 30 years experience:

> *"[He] was visibly nervous while trying to figure out what he would tell the woman. After mulling the numbers over, he finally estimated the woman's probability of having breast cancer, given that she has a positive mammogram, to be 90 percent. Nervously, he added, 'Oh, what nonsense. I can't do this. You should test my daughter; she is studying medicine.'"*

http://opinionator.blogs.nytimes.com/2010/04/25/chances-are/

Revisiting example

1. Rate of breast cancer among women 40 to 50 years old with no symptoms nor genetic history is 0.8%.
2. Among breast cancer patients, mammogram test reports positive 90% of time.
3. Among cancer-free patients, mammogram test reports positive 7% of time.
4. Imagine woman matching above criteria has a positive mammogram. What is the probability that she has breast cancer?

Let's formalize the problem:

- Let A be the event the woman has breast cancer.
- Let B be the event that the mammogram test is positive.
- What probability are we being asked to calcuate?
- What information are we given?

Bayes' Rule

For any two events A and B,

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)}$$

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Proof:

$$\begin{aligned}
Pr(A|B) &= \frac{Pr(A \cap B)}{Pr(B)} && \text{definition of conditional probability} \\
&= \frac{Pr(B \cap A)}{Pr(B)} && A \cap B = B \cap A \\
&= \frac{Pr(B|A)Pr(A)}{Pr(B)} && \text{chain rule}
\end{aligned}$$

Total law of probability

Theorem (Total law of probability)

$$Pr(B) = Pr(B|A)Pr(A) + Pr(B|\bar{A})Pr(\bar{A})$$

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Proof:

$$\begin{aligned}
Pr(B) &= Pr((B \cap A) \cup (B \cap \bar{A})) && B = (B \cap A) \cup (B \cap \bar{A}) \\
&= Pr(B \cap A) + Pr(B \cap \bar{A}) && \text{events } B \cap A \text{ and } B \cap \bar{A} \text{ are disjoint} \\
&= Pr(B|A)Pr(A) + Pr(B|\bar{A})Pr(\bar{A}) && \text{chain rule}
\end{aligned}$$

Answer to probability puzzle

Answer worked out on board, around 10%.

Survey says

In a survey of American doctors, 95 out of 100 estimated the woman’s probability of having breast cancer to be somewhere around 75 percent.

http://opinionator.blogs.nytimes.com/2010/04/25/chances-are/

Poll: Bayes' rule

Complete the following exercise in groups. When you are done, raise your hand. I will give you several minutes to work on it, so take the time to work out the calculation.

Bayes' rule:

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)}$$

Total law:

$$Pr(B) = Pr(B|A)Pr(A) + Pr(B|\bar{A})Pr(\bar{A})$$

Students from upstate NY attend either Colgate or Cornell and they are either happy or sad. Students prefer Colgate to Cornell at a rate of 75%. Among Colgate students, 75% are happy; among Cornell students, 50% are happy.

1. What is the probability that a student is happy?

2. Given that a student is happy, what is the probability they attend Colgate?