COSC 290 Discrete Structures

Lecture 33: Counting, IV

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Plan for today

1. Four types of counting problems
2. Counting when order matters (2 ways)
3. Counting when order is irrelevant (2 ways)

Four types of counting problems

Examples

How many ways can you choose...

- A starting basketball lineup of $k = 5$ players at five positions (C, PF, SF, PG, SG) from among $n = 12$ players.
- A starting volleyball lineup of $k = 6$ players from among $n = 13$ players. (Assume position irrelevant in volleyball because players rotate.)
- A team of $k = 4$ runners to compete in a single cross-country race from among $n = 10$ available runners.
- A runner for each of $k = 5$ track races from among a team of $n = 12$ available runners.
- An ideal team of $k = 4$ athelete to compete in a decathlon assembled from $n = 3$ atheletic types: sprinter, marathoner, jumper.
- A selection of $k = 12$ donuts from $n = 3$ donut types (jelly, chocolate, glazed).

Four types of common counting problem

Given a set S with n elements, let us consider counting the number of ways to choose k elements from S. To formalize this counting problem, we must answer two questions:

- Does the order in which elements are selected matter?
- Can the same element be chosen more than once?

The answers to these questions leads to four distinct counting problems:

	order matters	order irrelevant
repetition forbidden		
repetition allowed		

Example shown on board: let $S = \{A, B, C\}$ and $k = 2$. Write out solutions to all four versions of the problem.

Goal for today: fill in this table.

Counting when order matters (2 ways)

Order matters, repetition allowed

How many ways to choose a sequence of k (not necessarily distinct) elements from a set of n elements?

Example: one runner for each of $k = 5$ track races from among a team of $n = 12$ available runners. (Same runner can compete in multiple races.)

How many ways? (Use the product rule.)

$$\underbrace{n}_{\text{choices for first element}} \cdot \underbrace{n}_{\text{choices for second element}} \cdot \cdots \cdot \underbrace{n}_{\text{choices for } k^{th} \text{ element}} = n^k$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden		
repetition allowed	n^k	

Order matters, repetition forbidden

How many ways to choose a sequence of k distinct elements from a set of n elements?

Example: a starting basketball lineup of $k = 5$ players at five positions (C, PF, SF, PG, SG) from among $n = 12$ players.

How many ways? (Use the generalized product rule.)

$$\underbrace{(n)}_{\text{choices for first element}} \cdot \underbrace{(n-1)}_{\text{choices for second element}} \cdot \cdots \cdot \underbrace{(n-k+1)}_{\text{choices for } k^{th} \text{ element}} = \frac{n!}{(n-k)!}$$

Alernative derivation: using division rule

Let B be the set we are trying to count.

Let A be the set of all permutations of S. (Recall that a permutation of set S is an $|S|$-length sequence of elements of S with no repetitions.)

Let $f : A \to B$ map a permutation into k-element sequence by simply keeping first k elements of the permutation.

Example
$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	B
$\langle a, b, c, d, e\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, c, e, d\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, d, c, e\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, d, e, c\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, e, c, d\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, e, d, c\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, c, b, d, e\rangle$	$\to$	$\langle a, c\rangle$
$\langle a, c, b, e, d\rangle$	$\to$	$\langle a, c\rangle$
$\langle a, c, d, b, e\rangle$	$\to$	$\langle a, c\rangle$
	$\dots$	

Alernative derivation continued...

Example
$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	B
$\langle a, b, c, d, e\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, c, e, d\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, d, c, e\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, d, e, c\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, e, c, d\rangle$	$\to$	$\langle a, b\rangle$
$\langle a, b, e, d, c\rangle$	$\to$	$\langle a, b\rangle$
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$\langle a, c, b, e, d\rangle$	$\to$	$\langle a, c\rangle$
$\langle a, c, d, b, e\rangle$	$\to$	$\langle a, c\rangle$
	$\dots$	

How many permutations map to *same* k sequence?

Permutation maps to $\langle a, b\rangle$ iff it starts with $\langle a, b\rangle$ followed by remaining $n - k$ elements in *any order*.

There are $(n - k)!$ ways to order remaining $(n - k)$ elements.

f is a $(n - k)!$-to-1 function, so...

$$|B| = \frac{|A|}{(n-k)!} = \frac{n!}{(n-k)!}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	
repetition allowed	n^k	

Counting when order is irrelevant (2 ways)

Order irrelevant, repetition forbidden

How many ways to choose a set of k elements from a set of n elements?

Example: A starting volleyball lineup of $k = 6$ players from among $n = 13$ players. (Assume position irrelevant in volleyball because players rotate.)

This counting problem comes up so often it has a special name/notation:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

The binomial coefficient, denoted $\binom{n}{k}$, is the number of combinations of k elements chosen from n candidate elements.

Example: Counting bitstrings with k ones

How many length n bitstrings contain exactly k ones?

A length n bitstring has n bit positions $b_1 b_2 \dots b_n$.

Must choose a *set* of k positions where the ones will be placed. (The order in which the bit positions are chosen is irrelevant.)

Thus, the number of length n bitstrings that contain exactly k ones is $\binom{n}{k}$.

Example: subsets of a certain size

Let $S := \{a, b, c, d, e\}$. Let $n := |S|$. How many subsets of size $k = 2$ can be constructed from S?

There are $\binom{n}{k} = \binom{5}{2} = 10$:

$\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\},$
$\{b, c\}, \{b, d\}, \{b, e\},$
$\{c, d\}, \{c, e\},$
$\{d, e\}$.

Derivation of $\binom{n}{k}$ using division rule

Let C be the set we are trying to count.

Let A be the set of all permutations of S.

Let $g : A \to C$ map a permutation into k-element sequence by simply taking first k elements of the permutation and putting them in a set.

Example

$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	C
$\langle a, b, c, d, e \rangle$	$\to$	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	$\to$	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	$\to$	$\{a, c\}$
	...	
$\langle a, e, d, c, b \rangle$	$\to$	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	$\to$	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	$\to$	$\{a, b\}$
$\langle b, a, d, c, e \rangle$	$\to$	$\{a, b\}$
$\langle b, a, d, e, c \rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	$\to$	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	$\to$	$\{b, c\}$
	...	

Derivation of $\binom{n}{k}$ continued...

Example

$S = \{a, b, c, d, e\}$ and $k = 2$.

A	$\to$	C
$\langle a, b, c, d, e \rangle$	$\to$	$\{a, b\}$
$\langle a, b, c, e, d \rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, c, e \rangle$	$\to$	$\{a, b\}$
$\langle a, b, d, e, c \rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, c, d \rangle$	$\to$	$\{a, b\}$
$\langle a, b, e, d, c \rangle$	$\to$	$\{a, b\}$
$\langle a, c, b, d, e \rangle$	$\to$	$\{a, c\}$
	...	
$\langle a, e, d, c, b \rangle$	$\to$	$\{a, e\}$
$\langle b, a, c, d, e \rangle$	$\to$	$\{a, b\}$
$\langle b, a, c, e, d \rangle$	$\to$	$\{a, b\}$
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$\langle b, a, d, e, c \rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, c, d \rangle$	$\to$	$\{a, b\}$
$\langle b, a, e, d, c \rangle$	$\to$	$\{a, b\}$
$\langle b, c, a, d, e \rangle$	$\to$	$\{b, c\}$
	...	

How many permutations map to *same* set?

Permutation maps to $\{a, b\}$ iff it starts with the elements in $\{a, b\}$ in *any order* followed by remaining $n-k$ elements in *any order*.

There are $k!$ ways to order the first k elements. There are $(n-k)!$ ways to order remaining $(n-k)$ elements.

g is a $k!(n-k)!$-to-1 function, so...

$$|C| = \frac{|A|}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$$

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
repetition allowed	n^k	

Poll: Counting number of ways to select lineups

Suppose your hockey team has only one goalie and 12 other players. You want to organize your time into a starting line (the goalie + 5 players), a backup line (the goalie + 5 other players), and benchwarmers (remaining 2 players).

Let's assume position does not matter.

How many different ways can you select the starting line, backup line, and benchwarmers?

A) 792

B) 16632

C) 95040

D) 3991680

E) I'm stumped

Order irrelevant, repetition allowed

How many ways to choose an unordered collection of k (not necessarily distinct) elements from a set of n elements?

Example: how many ways to choose a dozen donuts from three donut types (jelly, chocolate, glazed)?

Let's reduce this to a problem we already know how to solve...

$$\underbrace{00\ldots0}_{\text{jelly}}\ 1\ \underbrace{00\ldots0}_{\text{chocolate}}\ 1\ \underbrace{00\ldots0}_{\text{glazed}}$$

A bit-string with k zeroes and $n-1$ ones. (Total length of bit string is $n+k-1$.)

$$\binom{n+k-1}{k}$$

Derivation, a little more formally

Use mapping rule. Represent a choice of k elements from set of n candidates as sequence $\langle x_1, x_2, \ldots, x_n\rangle$ where x_i is the number of times we chose candidate element i.

Since we choose a total of k elements, we require:

$$\sum_{i=1}^{n} x_i = k$$

Bijective mapping to bit-strings:

$$f(\langle x_1, x_2, \ldots, x_n\rangle) = \underbrace{00\ldots0}_{x_1\text{ times}}\ 1\ \underbrace{00\ldots0}_{x_2\text{ times}}\ 1\ \ldots\ 1\ \underbrace{00\ldots0}_{x_n\text{ times}}$$

(Bit string is always length $n+k-1$ because there are $n-1$ ones and the total number of zeros must add up to k.)

Four counting problems

The number of ways to choose k items from a set S of n items when...

	order matters	order irrelevant
repetition forbidden	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
repetition allowed	n^k	$\binom{n+k-1}{k}$

Poll

How many different solutions are there to the equation $a+b+c=8$ where a, b, c must be non-negative integers?

A) 45
B) 56
C) 120
D) 165
E) I'm stumped