COSC 290 Discrete Structures

Lecture 26: Relations, III

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Plan for today

1. Relations & Relational Operators (Review)
2. Properties of relations (Review)
3. Properties of relations (practice)
4. Closures
5. Warshall relations

Relations & Relational Operators (Review)

Recall: Relations

A (binary) relation on $A \times B$ is a subset of $A \times B$.

Sometimes interested in relations on $A \times A$ which is sometimes simply called a relation on A.

Recall: inverse of a relation

Definition (Inverse)

Let R be a relation on $A \times B$. The inverse R^{-1} of R is a relation on $B \times A$ defined by $R^{-1} := \{ \langle b, a \rangle \in B \times A : \langle a, b \rangle \in R \}$

Intuition for inverse: think of R a table with columns A, B, inverse reorders the columns B, A.

Recall: composing two relations

Definition (Composition)

The composition of R and S is a relation on $A \times C$, denoted $S \circ R$, where $\langle a, c \rangle \in S \circ R$ iff there exists a $b \in B$ such that $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in S$.

Intuition for composition: think of R a table with columns A, B and think of S a table with columns B, C. Composition creates new table with columns A,C by matching rows from R and S that have matching B values.

Poll: Cardinality

Suppose that sets A, B, C have cardinalities n_A, n_B, n_C respectively. Let R be a relation on $A \times B$ and S a relation on $B \times C$. What is the *maximum* cardinality of $S \circ R$? (In discussion, justify your answer.)

1. n_B
2. $n_A + n_C$
3. $n_A \cdot n_C$
4. $\min \{ n_A, n_C \}$
5. $\min \{ n_A, n_B, n_C \}$

Properties of relations (Review)

Reflexivity

A relation R on A is reflexive if for every $a \in A$, $\langle a, a \rangle \in R$.

A relation R on A is irreflexive if for every $a \in A$, $\langle a, a \rangle \notin R$.

A relation can be reflexive, irreflexive, or neither.

Symmetry

A relation R on A is symmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$ too.

A relation R on A is antisymmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

A relation R on A is asymmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

A relation can be none of the above, or more than one of the above.

Transitive

A relation R on A is transitive if for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$ too.

A relation can be transitive, or not.

Review poll from last time

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the *implies* relation on all possible propositions expressed in the English language where $\langle p, q \rangle \in$ *implies* if $p \implies q$ is true. Which properties does this relation have? (You can choose more than one.)

A) R

B) IR

C) S

D) antiS

E) AS

F) T

Properties of relations (practice)

Poll: unequal sets

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Let X be an arbitrary set. Consider the relation *diffSize* on $\mathcal{P}(X)$ where $\langle S_1, S_2 \rangle \in \mathit{diffSize}$ if $|S_1| \neq |S_2|$. Which properties does this relation have? (You can choose more than one.)

A) R
B) IR
C) S
D) antiS
E) AS
F) T

Poll: even divider

R *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

IR *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

S *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

antiS *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

AS *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

T *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the relation R on $\mathbb{Z}$ where $\langle x, y \rangle \in R$ if $x \mod 2 = 0$ and $y \mod x = 0$. Which properties does this relation have? (You can choose more than one.)

A) R
B) IR
C) S
D) antiS
E) AS
F) T

Closures

Closures

A closure of a relation R on A is a smallest $R' \supseteq R$ that satisfies a desired property.

- Reflexive closure:

$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- Symmetric closure:

$$R' = R \cup R^{-1}$$

- Transitive closure:
  (*hint*: what does $R \circ R$ give you?)

Poll: towards transitive closure

Consider the *parentOf* relation on persons where $\langle p, c \rangle \in \mathit{parentOf}$ if p is the parent of c. What is $\mathit{parentOf} \circ \mathit{parentOf}$?

A) ancestorOf
B) grandParentOf
C) parentOf
D) grandParentOrParentOf
E) none of the above

Closures

A closure of a relation R on A is a smallest $R' \supseteq R$ that satisfies a desired property.

- reflexive closure:

$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:

$$R' = R \cup (R \circ R) \cup ((R \circ R) \circ R) \cup \cdots$$

Poll: transitive closure

Consider the *parentOf* relation on persons where $\langle p, c \rangle \in \mathit{parentOf}$ if p is the parent of c. What is the transitive closure of $\mathit{parentOf}^{-1}$?

A) ancestorOf
B) parentOf
C) childOf
D) descendantOf
E) siblingOf

Warshall relations

Warshall relation

Let $A \coloneqq \{ a_1, a_2, \ldots, a_n \}$, a finite set.

Let R be a relation on A.

For $k = 0$ to n, let W_k denote the k^{th} Warshall relation for R where W_k is defined as...

- $W_0 \coloneqq R$
- For $k \geq 1$, W_k is a relation on A such that $\langle a_i, a_j \rangle \in W_k$ iff there is a sequence of relationships in R connecting a_i to a_j using any subset of the elements $\{a_1, a_2, \ldots, a_k\}$ as intermediates.

Example

W_0 (i.e., this is the relation R)

```
F F F T
T F F F
F T F F
F T F F
```

W_1

```
F F F T
T F F T
F T F F
F T F F
```

W_2

```
F F F T
T F F T
T T F T
T T F T
```

W_3

```
F F F T
T F F T
T T F T
T T F T
```

W_4

```
T T F T
T T F T
T T F T
T T F T
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