170E Week 0 Discussion Notes

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This week we will review some basic set theory and discuss some basic examples of probability involving dice. The goal of the examples is to give you some intuition about events and probability, before going into the first lecture on Friday.

Sets

Warmup. Let $A, B \subset \Omega$ be subsets of a set Ω. Draw the following things:

- their union $A \cup B$
- the intersection $A \cap B$
- $A \setminus B$, the set A minus the set B
- $B \setminus A$, the set B minus the set A
- the complement A' of A in Ω
- the complement B' of B in Ω.

What does it mean intuitively for $A \cup B = \Omega$? What about for $A \cap B = \varnothing$?

Let us prove two basic properties about sets. Such properties of sets are basically translations of properties of logic.

Exercise. Show that intersections distribute over unions: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, where A, B, C are sets.

Solution. We have $x \in A \cap (B \cup C)$ if and only if $x \in A$ and $x \in B \cup C$. This means $x \in A$ and also $x \in B$ or $x \in C$. Equivalently, we have $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. Thus $x \in (A \cap B) \cup (A \cap B)$. $\square$

Exercise. Prove De Morgan's laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

Solution. For the first law, observe that $x \in A \cup B$ if and only if $x \in A$ or $x \in B$. Taking negations shows that $x \in (A \cup B)'$ if and only if $x \notin A$ and $x \notin B$ if and only if $x \in A'$ and $x \in B'$ if and only if $x \in A' \cap B'$. For the second law, $x \in A' \cup B'$ if and only if $x \in A'$ or $x \in B'$ if and only if $x \notin A$ or $x \notin B$. This is equivalent to the negation of $x \in A$ and $x \in B$, or in other words $x \notin A \cap B$, *i.e.* $x \in (A \cap B)'$. $\square$

Events and probability

Let us now work through some examples of events and probability involving dice. Suppose we roll two 6-sided dice, one after the other. We can enumerate the possible outcomes as follows:

$$\Omega = \begin{Bmatrix} 11, & 12, & 13, & 14, & 15, & 16, \\ 21, & 22, & 23, & 24, & 25, & 26, \\ 31, & 32, & 33, & 34, & 35, & 36, \\ 41, & 42, & 43, & 44, & 45, & 46, \\ 51, & 52, & 53, & 54, & 55, & 56, \\ 61, & 62, & 63, & 64, & 65, & 66 \end{Bmatrix}.$$

An *event* is a subset of these outcomes. We think of an event $A \subset \Omega$ as getting any one of the outcomes in A.

Example. For example, the event that we roll equal numbers is

$$E = \{11, 22, 33, 44, 55, 66\} \subset \Omega,$$

and the event that we roll two different numbers is $E' \subset \Omega$. Similarly, the event that we roll exactly a 4 and then a 5 is the event $\{45\}$. The event that our first roll is a 5 is the event $F_5 = \{51, 52, 53, 54, 55, 56\}$, and the event that our second roll is a 5 is the event $S_5 = \{15, 25, 35, 45, 55, 65\}$.

Exercise. What is the event that our two numbers sum to 7? To 6? 1?

Solution. They are $\{16, 25, 34, 43, 52, 61\}$, $\{15, 24, 33, 42, 51\}$, and $\varnothing$. $\square$

We denote by $\mathbb{P}(A)$ the *probability of an event* A.

Example. Since the outcomes are equally likely to occur and since we have 36 possibly outcomes, we have $\mathbb{P}(\{11\}) = \mathbb{P}(\{12\}) = \cdots = \mathbb{P}(\{66\}) = \frac{1}{36}$.

Exercise. The probabilities of disjoint events should add in the sense that $\mathbb{P}(A \sqcup B) = \mathbb{P}(A) + \mathbb{P}(B)$. In fact, we will see that this is part of the definition of probability. Use this idea to give an argument for why, in this situation, $\mathbb{P}(A) = |A|/36$.

Solution. We can write A as a disjoint union $A = \bigsqcup_{a \in A} \{a\}$, and since $\mathbb{P}(\{a\}) = 1/36$, we have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigsqcup_{a \in A} \{a\}\right) = \sum_{a \in A} \mathbb{P}(\{a\}) = \sum_{a \in A} \frac{1}{36} = \frac{|A|}{36},$$

as desired. $\square$

When events are not disjoint, their probabilities generally do not add.

Example. For example, consider the event F_5 that our first roll is a 5 and the event S_5 that our second roll is a 5. We have $\mathbb{P}(F_5) = |F_5|/36 = 1/6$ and $\mathbb{P}(S_5) = |S_5|/36 = 1/6$, but

$$\mathbb{P}(F_5 \cup S_5) = \mathbb{P}(\{51, 52, 53, 54, 55, 56, 15, 25, 35, 45, 65\}) = 11/36.$$

The correction is given by the *inclusion-exclusion principle*:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Example. The reason why $\mathbb{P}(F_5 \cup S_5) \neq \mathbb{P}(F_5) + \mathbb{P}(S_5)$ is that $\mathbb{P}(F_5 \cap S_5) = \mathbb{P}(\{55\}) = 1/36 \neq 0$. Indeed, we can see that

$$\mathbb{P}(F_5 \cup S_5) = \mathbb{P}(F_5) + \mathbb{P}(S_5) - \mathbb{P}(F_5 \cap S_5) = 1/6 + 1/6 - 1/36 = 11/36,$$

as we computed earlier.