170E Week 2 Discussion Notes

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This week for discussion section we will do a quick review of Bayes' theorem and then discuss two problems that involve conditioning on things in a somewhat creative way.

Bayes' theorem

Warmup. What is the definition of the conditional probability $\mathbb{P}(A \mid B)$ of A given B? Draw a picture illustrating this definition. Derive Bayes' theorem

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

from this definition.

The typical Bayes' theorem problem looks like this:

Exercise. Suppose a Covid test has a true positive rate (TPR) of 0.99 and a true negative rate (TNR) of 0.80, and suppose the proportion of people who have Covid is 0.05. Given that someone tests positive for Covid, what is the probability that they actually have Covid?

Solution. We have

$$\begin{aligned}
\mathbb{P}(\text{Covid} \mid \text{pos}) &= \frac{\mathbb{P}(\text{pos} \mid \text{Covid})\mathbb{P}(\text{Covid})}{\mathbb{P}(\text{pos})} && \text{(Bayes' theorem)} \\
&= \frac{0.99 \cdot 0.05}{\mathbb{P}(\text{pos} \mid \text{Covid})\mathbb{P}(\text{Covid}) + \mathbb{P}(\text{pos} \mid \text{no Covid})\mathbb{P}(\text{no Covid})} \\
& && \text{(condition on whether they have Covid)} \\
&= \frac{0.99 \cdot 0.05}{0.99 \cdot 0.05 + (1-0.8)(1-0.05)},
\end{aligned}$$

which is $\approx 21\%$. □

Remark. More generally, the probability is

$$p = \frac{\text{TPR} \cdot \text{prop}}{\text{TPR} \cdot \text{prop} + (1-\text{TNR})(1-\text{prop})}.$$

Let us analyze this qualitatively. We have $p = 0$ when $\text{prop} = 0$ and $p = 1$ when $\text{prop} = 1$, as expected. When $\text{TNR} = 1$, we get $p = 1$, and when $\text{TPR} = 1$, we get

$$p = \frac{1}{1 + \text{FPR}(\frac{1}{\text{prop}} - 1)}.$$

Conditioning

Recall. The law of total probability says that if $B_1, B_2, \ldots$ are mutually exclusive and exhaustive events, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i).$$

We say that we are *conditioning* on the B_i. (We can think of this as doing case-work on the B_i.)

Exercise. (Conditioning on previous successes) Let A, B, C be events. Condition on $A \cap B$ and A to show that

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(C \mid A \cap B)\mathbb{P}(B \mid A)\mathbb{P}(A).$$

Generalize this to events $A_1, \ldots, A_n$.

Solution. We have

$$\begin{aligned}
\mathbb{P}(A \cap B \cap C) = &\mathbb{P}(A \cap B \cap C \mid A \cap B)\mathbb{P}(A \cap B) \\
&+ \mathbb{P}(A \cap B \cap C \mid (A \cap B)')\mathbb{P}((A \cap B)') \quad \text{(condition on } A \cap B\text{)} \\
=&\mathbb{P}(C \mid A \cap B)\mathbb{P}(A \cap B) \quad (A \cap B \cap C \text{ is false if } (A \cap B)' \text{ is true}) \\
=&\mathbb{P}(C \mid A \cap B)(\mathbb{P}(A \cap B \mid A)\mathbb{P}(A) + \mathbb{P}(A \cap B \mid A')\mathbb{P}(A')) \\
&\qquad\qquad \text{(condition on } A \text{ to compute } \mathbb{P}(A \cap B)\text{)} \\
=&\mathbb{P}(C \mid A \cap B)\mathbb{P}(B \mid A)\mathbb{P}(A).
\end{aligned}$$

Alternatively, we can directly use the definition of conditional probability:

$$\begin{aligned}
\mathbb{P}(C \mid A \cap B)\mathbb{P}(B \mid A)\mathbb{P}(A) &= \frac{\mathbb{P}(C \cap (A \cap B))}{\mathbb{P}(A \cap B)} \cdot \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \cdot \mathbb{P}(A) \\
&= \mathbb{P}(A \cap B \cap C).
\end{aligned}$$

In general, for events $A_1, \ldots, A_n$, we have

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \prod_{k=1}^{n} \mathbb{P}(A_k \,| A_1 \cap \cdots \cap A_{k-1})\mathbb{P}(A_1 \cap \cdots \cap A_{k-1})$$

by induction. $\square$

Often, the hardest part of a problem is figuring out what we should condition on. Let us keep this in mind as we do the following two problems.

Exercise. Suppose we shuffle a standard deck of 52 cards and deal 13 cards to each of four people. What is the probability that each person receives an ace?

Remark. Conditioning on the absolute position of the left-most ace, then the next one, and so on, is quite messy.

Solution. Think of our four aces as occupying four random spots in $1, \dots, 52$, and consider four equal-sized regions of our shuffled deck, *e.g.* $[1, 13]$, $[14, 26]$, $[27, 39]$, $[40, 52]$. Let A_{ij} be the event that the ith ace is in a different region than the jth ace. We want the probability

$$\mathbb{P}\left(\bigcap_{i \neq j} A_{ij}\right) = \mathbb{P}\left(\bigcap_{i < j} A_{ij}\right).$$

Note that these events A_{ij} are not independent, so we cannot blindly write this as a product of individual probabilities. However, we can consider the aces sequentially and condition on previous successes (see the above exercise) and write

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i<j} A_{ij}\right) &= \mathbb{P}(A_{12} \cap (A_{13} \cap A_{23}) \cap (A_{14} \cap A_{24} \cap A_{34})) \\
&= \mathbb{P}(A_{12})\mathbb{P}(A_{13} \cap A_{23} \mid A_{12})\mathbb{P}(A_{14} \cap A_{24} \cap A_{34} \mid A_{12} \cap A_{13} \cap A_{23}) \\
&= \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} \\
&\approx 10.5\%.
\end{aligned}$$

Alternatively, we can count. There are $52!/(13!)^4$ ways to deal the cards (up to permutation of each person's cards) and for $4! \cdot 48!/(12!)^4$ of these, each person has an ace. So the probability is

$$\frac{4! \cdot 48!/(12!)^4}{52!/(13!)^4} = \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49},$$

which agrees with above. □

Remark. In my opinion, the alternative solution is kind of lame because it does not really explain why the answer is what it is.

Exercise. Suppose we have a jar with 10 red balls, 20 orange balls, and 30 yellow balls. We randomly take out balls one-by-one until we take out all of the yellow balls. What is the probability that there is still at least one red ball and one orange ball remaining in the jar?

Remark. Conditioning on the position of the last yellow ball is extremely messy. Conditioning on the number of red and orange balls before (or after) the last yellow ball is also extremely messy.

Solution. Condition on the color of the last ball L! If L is yellow, then we have lost. If L is red, then we want the probability that the last orange ball comes after the last yellow ball. This is equivalent to the last orange or yellow ball being orange, which happens with probability $2/5$. Similarly, if L is orange, then we win with probability $1/4$. Thus our answer is

$$\begin{aligned}
\mathbb{P}(\text{win}) &= \mathbb{P}(\text{win} \mid L \text{ red})\mathbb{P}(L \text{ red}) \\
&\quad + \mathbb{P}(\text{win} \mid L \text{ orange})\mathbb{P}(L \text{ orange}) \\
&\quad + \mathbb{P}(\text{win} \mid L \text{ yellow})\mathbb{P}(L \text{ yellow}) \\
&= \frac{2}{5} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} \\
&= \frac{3}{20} \\
&= 15\%.
\end{aligned}$$

□

Problem for TA office hours

Exercise. Choose n points uniformly randomly on a circle (the boundary of a disk). What is the probability that the convex polygon connecting the n points contains the center of the circle?

Remark. Here are some hints / subproblems to think about. Let P_i be the event that all the points are on the semicircle centered at the ith point. Is $P_1 \cup \cdots \cup P_n$ equivalent to the event that we are interested in? Let Q_i be the event that all the points are on the semicircle starting at the ith point (clockwise, say). Is $Q_1 \cup \cdots \cup Q_n$ equivalent to the event that we are interested in? Are they independent? Mutually exclusive?