170E Week 6 Discussion Notes

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Exercise. (Book 3.1-4) If the mgf of X is

$$M_X(t) = \begin{cases} \dfrac{e^{5t} - e^{4t}}{t} & t \neq 0 \\ 1 & t = 0, \end{cases}$$

then find $\mathbb{E}(X)$, $\mathrm{Var}(X)$, and $\mathbb{P}(4.2 < X \leq 4.7)$.

Solution. It is not so easy to take derivatives and evaluate at $t = 0$. Instead, we determine the pdf of X (assuming we do not recognize it somehow). Observe that

$$\frac{e^{5t} - e^{4t}}{t} = \frac{e^{xt}}{t}\bigg|_4^5 = \int_4^5 e^{xt}\,\mathrm{d}x,$$

so we deduce that $f_X(x) = 1\{4 \leq x \leq 5\}$ and $X \sim \mathrm{Unif}(4, 5)$. Thus $\mathbb{E}(X) = 9/2$ and $\mathrm{Var}(X) = 1/12$ and $\mathbb{P}(4.2 < X \leq 4.7) = 1/2$. $\square$

Exercise. (Book 3.3-10) If $X \sim \mathrm{Norm}(\mu, \sigma^2)$, then show that $Y = aX + b \sim \mathrm{Norm}(a\mu + b, a^2\sigma^2)$, where $a \neq 0$.

Solution. Following the hint given in the book, it suffices to show that the cdf of Y is the same as the cdf of $\mathrm{Norm}(a\mu + b, a^2\sigma^2)$. To that effect, observe that the cdf of Y is

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(aX + b \leq y) \\ &= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) \\ &= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{\sigma}} e^{\frac{1}{2\sigma}(x-\mu)^2}\,\mathrm{d}x. \end{aligned}$$

Changing variables to $u = ax + b$, we get

$$\begin{aligned} \int_{-\infty}^{y} \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{\sigma}} e^{\frac{1}{2\sigma^2}\left(\frac{u-b}{a} - \mu\right)^2} \frac{1}{a}\mathrm{d}u &= \int_{-\infty}^{y} \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{a^2\sigma}} e^{\frac{1}{2a^2\sigma^2}(u-(a\mu+b))^2}\,\mathrm{d}u \\ &= \int_{-\infty}^{y} f_{\mathrm{Norm}(a\mu+b, a^2\sigma^2)}(u)\,\mathrm{d}u \\ &= F_{\mathrm{Norm}(a\mu+b, a^2\sigma^2)}(y), \end{aligned}$$

as required. $\square$

Alternate solution. Here is another way to do this problem. Observe that in general, the mgf of a linear transformation $aX + b$ of a random variable X is

$$M_{aX+b}(t) = \mathbb{E}(e^{t(aX+b)}) = e^{bt}\mathbb{E}(e^{atX}) = e^{bt}M_X(at).$$

For $X \sim \text{Norm}(\mu, \sigma^2)$, we have $M_X(t) = e^{\mu t + \sigma^2 t^2}$, so

$$\begin{aligned} M_{aX+b}(t) &= e^{bt}M_X(at) \\ &= e^{bt}e^{\mu(at)+\sigma^2(at)^2} \\ &= e^{(a\mu+b)t+(a^2\sigma^2)t^2} \\ &= M_{\text{Norm}(a\mu+b,a^2\sigma^2)}(t). \end{aligned}$$

It follows that $aX + b \sim \text{Norm}(a\mu + b, a^2\sigma^2)$ because their mgf's agree. $\square$

Exercise. (Lecture notes 3.2) The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,\mathrm{d}y$$

for $\alpha > 0$. It generalizes factorials in the sense that $\Gamma(n) = (n-1)!$ for any $n \in \mathbb{N}$. The Gamma distribution $\Gamma(\alpha, \theta)$ has support $\mathbb{R}_+$ and pdf

$$f_{\Gamma(\alpha,\theta)}(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-\frac{x}{\theta}}.$$

show that the *Gamma distribution* is well-defined.

Solution. We need to show that the pdf integrates to 1. At once,

$$\begin{aligned} \int_0^\infty f_{\Gamma(\alpha,\theta)}(x)\,\mathrm{d}x &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-\frac{x}{\theta}}\,\mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty (\theta u)^{\alpha-1}e^{-u}\,\theta\mathrm{d}u \\ &\qquad\qquad\text{(change variables to } u = x/\theta\text{)} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}e^{-u}\,\mathrm{d}u \\ &= 1. \end{aligned}$$

$\square$

Exercise. (Lecture notes 3.3) The *chi-squared distribution* is defined as $\chi^2(r) = \Gamma(r/2, 2)$ for $r \in \mathbb{N}$. Find $\mathbb{E}(\chi^2(r))$, $\text{Var}(\chi^2(r))$, and the mgf of $\chi^2(r)$.

Solution. We may as well determine these things for the Gamma distribution and then plug in $\alpha = r/2$ and $\theta = 2$. Recall that the Gamma function satisfies the functional equation

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

Our strategy is to use the previous exercise:

$$\mathbb{E}(\Gamma(\alpha,\theta)) = \int_0^\infty x f_{\Gamma(\alpha,\theta)}(x)\,\mathrm{d}x = \frac{\Gamma(\alpha+1)\theta^{\alpha+1}}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty f_{\Gamma(\alpha+1,\theta)}(x)\,\mathrm{d}x = \alpha\theta.$$

Similarly

$$\begin{aligned}
\mathbb{E}(\Gamma(\alpha,\theta)^2) &= \int_0^\infty x^2 f_{\Gamma(\alpha,\theta)}(x)\,\mathrm{d}x \\
&= \frac{\Gamma(\alpha+2)\theta^{\alpha+2}}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty f_{\Gamma(\alpha+2,\theta)}(x)\,\mathrm{d}x \\
&= \alpha(\alpha+1)\theta^2,
\end{aligned}$$

so

$$\mathrm{Var}(\Gamma(\alpha,\theta)) = \mathbb{E}(\Gamma(\alpha,\theta)^2) - \mathbb{E}(\Gamma(\alpha,\theta))^2 = \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2.$$

Finally, the mgf is

$$\begin{aligned}
M_{\Gamma(\alpha,\theta)}(t) &= \mathbb{E}(e^{t\Gamma(\alpha,\theta)}) \\
&= \int_0^\infty e^{tx} f_{\Gamma(\alpha,\theta)}(x)\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-(\frac{1}{\theta}-t)x}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha}\left(\frac{u}{\frac{1}{\theta}-t}\right)^{\alpha-1} e^{-u}\,\frac{1}{\frac{1}{\theta}-t}\mathrm{d}u \\
&\qquad\qquad\text{(change variables to } u = (\tfrac{1}{\theta}-t)x\text{)} \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\left(\frac{1}{\frac{1}{\theta}-t}\right)^{\alpha}\int_0^\infty u^{\alpha-1}e^{-u}\,\mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\left(\frac{1}{\frac{1}{\theta}-t}\right)^{\alpha}\int_0^\infty \Gamma(\alpha) f_{\Gamma(\alpha,1)}(u)\,\mathrm{d}u \\
&= \frac{1}{\theta^\alpha}\left(\frac{1}{\frac{1}{\theta}-t}\right)^{\alpha} \\
&= \left(\frac{1}{1-\theta t}\right)^{\alpha}.
\end{aligned}$$

Thus

$$\mathbb{E}(\chi^2(r)) = r, \qquad \mathrm{Var}(\chi^2(r)) = 2r, \qquad M_{\chi^2(r)}(t) = \left(\frac{1}{1-2t}\right)^{\frac{r}{2}}. \qquad \square$$