170E Week 10 Discussion Notes

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Here are some review problems for the final exam. I've included most of the solutions.

Problem 1. Recall that the moment generating function $M_X(t)$ of a random variable X in general may not exist or may only defined on a neighborhood of 0. In this problem, for simplicity we will assume that our random variables have mgf's that are defined everywhere.

(i) Show that $M_{X+Y}(t) = M_X(t)M_Y(t)$ for independent random variables X and Y.

(ii) Show that $M_{aX+b}(t) = e^{bt}M_X(at)$ for a random variable X and scalars $a, b \in \mathbb{R}$.

(iii) Deduce that if $U \sim \text{Unif}(0, 1)$, then $5U + 3 \sim \text{Unif}(3, 8)$.

(iv) Is true that if $U, V \sim \text{Unif}(0, 1)$ are independent, then $U + V \sim \text{Unif}(0, 2)$?

(v) Prove the central limit theorem, using that Taylor's theorem gives $\exp(tX) = 1 + tX + \frac{1}{2}t^2X^2 + t^2h(t)$ where $\lim_{t\to 0} h(t) = 0$.

Solution. For (i), recall that since X and Y are independent, $f(X)$ and $g(Y)$ are independent for any functions f and g. Thus

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}(\exp(t(X + Y))) \\ &= \mathbb{E}(\exp(tX) \exp(tY)) \\ &= \mathbb{E}(\exp(tX))\mathbb{E}(\exp(tY)) \\ &= M_X(t)M_Y(t). \end{aligned}$$

For (ii), observe that

$$M_{aX+b}(t) = \mathbb{E}(\exp(t(aX + b))) = \exp(bt)\mathbb{E}(\exp(atX)) = \exp(bt)M_X(at).$$

For (iii), recall that

$$M_{\text{Unif}(a,b)}(t) = \frac{e^{bt} - e^{at}}{t(b - a)}$$

(for $t \neq 0$ and 1 for $t = 0$). Thus since

$$M_{5U+3}(t) = e^{3t} M_U(5t) = e^{3t} \cdot \frac{e^{5t-1}}{5t} = \frac{e^{8t} - e^{3t}}{5t} = M_{\mathrm{Unif}(3,8)}(t),$$

we conclude that $5U + 3 \sim \mathrm{Unit}(3, 8)$ because their mgf's agree.

For (iv), it is not true because

$$M_{U+V}(t) = M_U(t) M_V(t) = \frac{(e^t - 1)^2}{t^2} \neq \frac{e^{2t} - 1}{2t} = M_{\mathrm{Unif}(0,2)}(t).$$

For (v), we want to show that if $X_1, \ldots, X_n \sim X$ are i.i.d. random variables with mean μ and variance σ^2, then

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to $\mathrm{Norm}(0, 1)$ as $n \to \infty$. By standardizing, we can assume $\mu = 0$ and $\sigma^2 = 1$, so

$$M_X(t) = \mathbb{E}(\exp(tX)) = \mathbb{E}\left(1 + tX + \frac{1}{2}t^2X^2 + t^2h(t)\right) = 1 + \frac{1}{2}t^2 + t^2h(t)$$

using Taylor's theorem as described in the problem. By (i) and (ii), we have

$$M_{Z_n}(t) = M_X\left(\frac{t}{\sqrt{n}}\right)^n = \left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)^n.$$

Thus

$$\lim_{n\to\infty} M_{Z_n}(t) = \lim_{n\to\infty}\left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)^n = \exp\left(\frac{t^2}{2}\right) = M_{\mathrm{Norm}(0,1)}(t),$$

where (if we want to be extra careful) we evaluated the limit by Taylor expanding $\log(1 + x) = x + xg(x)$ where $g(x) \to 0$ as $x \to 0$:

$$\begin{aligned}
\lim_{n\to\infty}\left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)^n &= \exp\left(\lim_{n\to\infty} n\log\left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)\right) \\
&= \exp\left(\lim_{n\to\infty} n\left(\frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)\right) \\
&= \exp\left(\frac{t^2}{2}\right).
\end{aligned}$$

□

Problem 2. Before doing this problem, please review Exercises 4.2(ii) and 4.3 in the lecture notes, which we did in discussion section in weeks 8 and 9. Let (X, Y) be bivariate normal with correlation ρ.

(i) Set $Z_X = (X - \mu_X)/\sigma_X$ and $Z_Y = (Y - \mu_Y)/\sigma_Y$. Show that (Z_X, Z_Y) is bivariate normal with correlation ρ and with Z_X and Z_Y unit normal.

(ii) Set $U_X = (Z_X - \rho Z_Y)/\sqrt{1-\rho^2}$ and $U_Y = Y$. Show that (U_X, U_Y) is unit bivariate normal. Deduce that U_X and U_Y are independent.

(iii) Suppose from now on that $\mu_X = -2$, $\mu_Y = -1$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and $\rho_{X,Y} = 1/2$. Find the conditional distribution of X given that $Y = y$ and the conditional distribution of Y given that $X = x$.

(iv) Draw a picture of the joint pdf of (X, Y) (*e.g.* draw contour lines) and use it to corroborate the conditional expectations found in part *(iii)*.

(v) Find the conditional distribution of $2X - Y$ given that $Y = 3X - 3$.

Solution. For *(i)* and *(ii)*, see the discussion notes from weeks 8 and 9.

For *(iii)*, we have

$$(X \mid Y = y) \sim \text{Norm}\left(-2 + \frac{1}{4}(y+1), \frac{3}{4}\right)$$

and

$$(Y \mid X = x) \sim \text{Norm}(-1 + 2(x+2), 3).$$

For *(v)*, note that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{Norm}\left(\begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\right),$$

so

$$\begin{aligned} \begin{pmatrix} 2X - Y \\ Y - 3X \end{pmatrix} &= \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &\sim \text{Norm}\left(\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}\right) \\ &= \text{Norm}\left(\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 & -5 \\ -5 & 7 \end{pmatrix}\right). \end{aligned}$$

Thus since the new correlation is $-5/2\sqrt{7}$, we have

$$(2X - Y \mid Y - 3X = -3) \sim \text{Norm}\left(\frac{19}{7}, \frac{3}{7}\right). \qquad \square$$

Problem 3. Let $X_1, \ldots, X_n \sim X$ be i.i.d. random variables. Consider the rth smallest value $X_{(r)}$ of $X_1, \ldots, X_n$, so in particular $X_{(1)} \leq \cdots \leq X_{(r)}$. This is called the rth order statistic of $X_1, \ldots, X_n$.

(i) Determine the pdf of the minimum $X_{(1)}$.

(ii) Determine the pdf of the maximum $X_{(n)}$.

(iii) In fact

$$f_{X_{(r)}}(x) = r\binom{n}{r} F_X(x)^{n-r}(1 - F_X(x))^{r-1} f_X(x)$$

(this is tricky to show). Use this to show that if $X \sim \text{Unif}(0,1)$, then $X_{(r)} \sim \text{Beta}(r, n-r+1)$.

Solution. For (i), the key is to look at the cdf, which is

$$\begin{aligned} F_{X_{(1)}}(x) &= \mathbb{P}(X_{(1)} \geq x) \\ &= \mathbb{P}(X_i \geq x \text{ for all } i = 1, \ldots, n) \\ &= \mathbb{P}(X \geq x)^n && \text{(independence)} \\ &= F_X(x)^n, \end{aligned}$$

so the pdf is

$$f_{X_{(1)}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X_{(1)}}(x) = n F_X(x)^{n-1} f_X(x).$$

For (ii), the cdf is

$$\begin{aligned} F_{X_{(n)}}(x) &= \mathbb{P}(X_{(n)} \geq x) \\ &= 1 - \mathbb{P}(X_{(n)} < x) \\ &= 1 - \mathbb{P}(X_i < x \text{ for all } i = 1, \ldots, n) \\ &= 1 - \mathbb{P}(X < x)^n && \text{(indepenence)} \\ &= 1 - (1 - F_X(x))^n, \end{aligned}$$

so the pdf is

$$F_{X_{(n)}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X_{(n)}}(x) = n(1 - F_X(x))^{n-1} f_X(x).$$

For (iii), simply compare their pdf's, recalling that $\Gamma(n) = (n-1)!$ for positive integers n. □

Problem 4. Review the following problems from previous discussion sections:

(i) The marbles problem from week 2

(ii) The convex polygon problem from week 3

(iii) The Russian roulette problem from week 3

(iv) The coupon collectors problem from week 4

(v) The slow administrator problem from week 5