170E Week 3 Discussion Notes

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This week for discussion section we will do two homework problems and two additional fun problems.

Exercise. (1.4-12 in the book) Flip an unbiased coin five independent times. Compute the probability of the following events:

(a) $HTHHT$

(b) $HHHTH$

(c) $HTHTH$

(d) Getting (at least) three heads.

Solution. For (a), the short answer is that it is $(\frac{1}{2})^5$ because each flip is independent. More precisely, we consider the results $A_1, \dots, A_5$ of the individual flips. Then

$$\begin{aligned}\mathbb{P}(HTHHT) &= \mathbb{P}(A_1 = H, A_2 = T, A_3 = H, A_4 = H, A_5 = T) \\ &= \mathbb{P}(A_1 = H)\mathbb{P}(A_2 = T)\mathbb{P}(A_3 = H)\mathbb{P}(A_4 = H)\mathbb{P}(A_5 = T) \quad \text{(independence)} \\ &= \left(\frac{1}{2}\right)^5.\end{aligned}$$

Parts (b) and (c) are pretty much exactly the same.

For part (d), we compute the probability by counting. The sample space is the set of sequences of five flips, which has size 2^5. Having at least 3 heads means having exactly 3, exactly 4, or exactly 5, and there are of these

$$\binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 16.$$

So our answer is 1/2. (Alternatively, we can argue by symmetry.) □

Exercise. (1.5-5 in the book) At a hospital emergency room, 20% of the patients are critical, 30% are serious, and 50% are stable. Of the critical ones, 30% die; of the serious, 10% die; and of the stable, 1% die. Given that a patient dies, what is the probability that the patient was classified as critical?

Solution. This is a simple application of Bayes':

$$\begin{aligned}\mathbb{P}(\text{critical} \mid \text{dies}) &= \frac{\mathbb{P}(\text{dies} \mid \text{critical})\mathbb{P}(\text{critical})}{\mathbb{P}(\text{dies})} \\ &= \frac{0.3 \cdot 0.2}{\mathbb{P}(\text{dies})}.\end{aligned}$$

To compute the denominator, we condition on the type of patient:

$$\begin{aligned}\mathbb{P}(\text{dies}) = &\ \mathbb{P}(\text{dies} \mid \text{critical})\mathbb{P}(\text{critical}) \\ &+\mathbb{P}(\text{dies} \mid \text{serious})\mathbb{P}(\text{serious}) \\ &+\mathbb{P}(\text{dies} \mid \text{stable})\mathbb{P}(\text{stable}) \\ = &\ 0.3 \cdot 0.2 + 0.1 \cdot 0.3 + 0.01 \cdot 0.5.\end{aligned}$$

Our answer is thus $6/95$. □

Exercise. Choose n points uniformly randomly on a circle (the boundary of a disk). What is the probability that the convex polygon connecting the n points contains the center of the circle?

Solution. The polygon contains the center if and only if the points do not all lie on a single semicircle. We proceed by computing the probability that the points all lie on a single semicircle.

Here is one approach that does not work. Let A_i denote the event that the other $n-1$ points lie on the semicircle centered at point i. Then the event $A_1 \cup \cdots \cup A_n$ is not what we are looking for. (Why?)

We instead define B_i to be the event that the other $n-1$ points lie on the semicircle that starts at point i and extends clockwise. Then the event $B_1 \cup \cdots \cup B_n$ is the event that all points lie on a single semicircle. Moreover, the events $B_1, \ldots, B_n$ are pairwise disjoint (assuming the n points are distinct)! Thus

$$\mathbb{P}(B_1 \cup \cdots \cup B_n) = \mathbb{P}(B_1) + \cdots + \mathbb{P}(B_n) = n\left(\frac{1}{2}\right)^{n-1}.$$

Our answer is therefore $1 - n(\frac{1}{2})^{n-1}$. □

Exercise. Suppose we are playing Russian roulette with a 6-chamber revolver with one bullet against a gracious opponent.

(*i*) Suppose we take turns shooting, and we do not stop until someone loses. Our opponent graciously offers to let us to go first. Should we?

(*ii*) Suppose that before each trigger pull, we spin the barrel. Our opponent again graciously offers to let us to go first. Should we?

(*iii*) Suppose we now have two bullets in the revolver (in random spots) and that our opponent went first and survived. Our opponent graciously offers to let us to spin the barrel. Should we? (For simplicity, let us just maximize our probability of surviving the next pull.)

(*iv*) Suppose the two bullets are consecutive and that our opponent again went first and survived. Our opponent again graciously offers to let us spin the barrel. Should we? (For simplicity, let us just maximize our probability of surviving the next pull.)

Solution. For (*i*), the result is determined by whether the bullet is in an even or odd location, so it does not matter!

For (*ii*), let x denote the probability that we win if we go first. If we go first and we survive, then our opponent wins with probability x. Thus by conditioning on the result of our first shot, we have

$$\begin{aligned} x &= \mathbb{P}(\text{we win} \mid \text{we die})\mathbb{P}(\text{we die}) + \mathbb{P}(\text{we win} \mid \text{we survive})\mathbb{P}(\text{we survive}) \\ &= 0 \cdot \frac{1}{6} + (1-x) \cdot \frac{5}{6}, \end{aligned}$$

so $x = 5/11$. Thus we should go second and have a $6/11$ probability of winning.

For (*iii*), if we do not spin the barrel, then there are 5 remaining slots, 2 of which have bullets, so we survive the next pull with probability $3/5$. If we do spin the barrel, then we survive the next pull with probability $2/3$. Thus, to maximize our probability of surviving the next pull, we should spin the barrel.

For (*iv*), if we do not spin the barrel, then (by drawing a picture) we survive the next shot with probability $3/4$. If we do spin the barrel, then we survive the next shot with probability $2/3$. So we should not spin the barrel! $\square$