170E Week 1 Discussion Notes

Colin Ni

September 29, 2025

Probability practice

Warmup. Recall that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Draw a picture of this. Use this to determine the probability of getting at least one even number when we roll two 6-faced dice. (Similar to Exercise 1.1-9 from the book.)

Solution. Let A be the event that the first die gives an even number, and let B be the event that the second die gives an even number. The probability of getting at least one even number is $\mathbb{P}(A \cup B)$. Since $\mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$ and $\mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}$ and $\mathbb{P}(A \cap B) = \frac{9}{36} = \frac{1}{4}$, we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{3}{4}. \qquad \square$$

Exercise. Compute the same thing by computing its complement.

Solution. Using the same notation, we have

$$\mathbb{P}(A \cup B) = 1 - \mathbb{P}((A \cup B)') = 1 - \mathbb{P}(A' \cap B') = \frac{3}{4}$$

by De Morgan's laws. $\square$

Exercise. Show that

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ -\mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) \\ +\mathbb{P}(A \cap B \cap C).\end{aligned}$$

(Exercise 1.1-10 from the book.) Use this to compute the probability of getting at least one even number when we roll three 6-faced dice.

Solution. Writing $A \cup B \cup C = (A \cup B) \cup C$, we have

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}((A \cup B) \cup C) \\ &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C).\end{aligned}$$

Since $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, we have

$$\begin{aligned}\mathbb{P}((A \cup B) \cap C) &= \mathbb{P}((A \cap C) \cup (B \cap C)) \\ &= \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}((A \cap C) \cap (B \cap C)) \\ &= \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C).\end{aligned}$$

Plugging this in immediately gives the result. To use this to compute the probability of getting at one even number when we roll three 6-faced dice, let A, B, C be the event that we get an even number on the first, second, and third throws, respectively. Then

$$\mathbb{P}(A \cup B \cup C) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

This agrees with what we get by computing the complement:

$$\mathbb{P}(A \cup B \cup C) = 1 - \mathbb{P}(A' \cap B' \cap C') = \frac{7}{8}. \qquad \square$$

Probability on (uncountably) infinite sample spaces

So far we have only been discussing finite sample spaces. How can we define a probability function on $\mathbb{R}$? Assigning probabilities to individual numbers does not work because a sum of uncountably many positive numbers is infinite. Instead, we assign probabilities to intervals of numbers using probability density functions.

Example. The uniform distribution on $[0, 1]$ has probability density function $f(x) = 1\{0 \le x \le 1\}$. The probability $\mathbb{P}((a, b))$ that we select a number in an interval (a, b) is

$$\int_a^b f(x)\,\mathrm{d}x = \text{the length of the interval } (a, b) \text{ inside } [0, 1].$$

Exercise. What is $\mathbb{P}((\frac{1}{3}, \frac{2}{3}))$? What about $\mathbb{P}((\frac{1}{3}, 100))$? What about $\mathbb{P}([\frac{1}{3}, \frac{2}{3}])$? What about $\mathbb{P}(\mathbb{Q})$ (recall that $\mathbb{Q}$ is the set of rational numbers)? (Similar to Exercise 1.1-12 from the book.)

Solution. We have $\mathbb{P}((\frac{1}{3}, \frac{2}{3})) = \frac{1}{3}$ and $\mathbb{P}((\frac{1}{3}, 100)) = \frac{2}{3}$. The length of $[\frac{1}{3}, \frac{2}{3}]$ is the same as the length of $(\frac{1}{3}, \frac{2}{3})$, so $\mathbb{P}([\frac{1}{3}, \frac{2}{3}]) = \frac{1}{3}$. This suggests that $\mathbb{P}(\{\frac{1}{3}\}) = 0$, and of course this holds true for any element: $\mathbb{P}(\{x\}) = 0$ for any $x \in \mathbb{R}$. By countable additivity (part of the definition of a probability function), we have $\mathbb{P}(\mathbb{Q}) = 0$.

Another way to see this last fact is that we can cover $\mathbb{Q}$ with open intervals with arbitrarily small length. Indeed, let $\varepsilon > 0$, and enumerate $q_1, q_2, \ldots$ the elements of $\mathbb{Q}$. Cover q_i with an open interval of length $\frac{\varepsilon}{2^i}$. Then all elements of $\mathbb{Q}$ are covered with total length ε. $\square$

Exercise. Take a stick of length 1. If we break it at a uniformly random spot, what is the probability that the longer segment is at least two times longer than the shorter segment? (Exercise 1.1-13 from the book.)

Solution. Our break point x is uniformly chosen in $[0, 1]$. The x for which the condition holds is $[0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$, which has probability $\frac{2}{3}$. $\square$

Stars and bars / Balls and urns

The following puzzle and the technique used to solve it are super standard, but it is not in our textbook, as far as I can tell.

Puzzle. Suppose we have n indistinguishable balls and k distinguishable urns. How many ways can we put the n balls into the k urns?

Example. For $n = 5$ and $k = 3$, we can enumerate them by hand: $(5, 0, 0)$, $(4, 1, 0)$, $(4, 0, 1)$, $(3, 2, 0)$, $(3, 1, 1)$, $(3, 0, 2)$, $(2, 3, 0)$, $(2, 2, 1)$, $(2, 1, 2)$, $(2, 0, 3)$, $(1, 4, 0)$, $(1, 3, 1)$, $(1, 2, 2)$, $(1, 1, 3)$, $(1, 0, 4)$, $(0, 5, 0)$, $(0, 4, 1)$, $(0, 3, 2)$, $(0, 2, 3)$, $(0, 1, 4)$, $(0, 0, 5)$. There are $21 = \binom{7}{2} = \binom{5+3-1}{3-1}$ ways.

Solution. There are $\binom{n+k-1}{k-1}$ ways. The point is that there is a bijection with the $\binom{n+k-1}{k-1}$ strings of length $n + k - 1$ consisting of n stars $*$ and $k - 1$ bars $|$. Indeed, the $k - 1$ bars divide the string into k regions, which we think of the urns, and the stars in each region correspond to the balls in each urn. For example $(1, 2, 2)$ corresponds to $*|**|**$. $\square$

Exercise. Suppose we uniformly choose a way to put 100 balls into 35 urns. What is the probability that the first and last urns have exactly 1 ball?

Solution. There are $\binom{134}{34}$ ways to put 100 balls into 35 urns. To count the number of ways where the first and last urns have exactly 1 ball, observe that this is the number of ways of putting 98 balls into 33 urns, of which there are $\binom{130}{32}$ ways. So our answer is

$$\frac{\binom{130}{32}}{\binom{134}{34}} = \frac{34 \cdot 33 \cdot 100 \cdot 99}{134 \cdot 133 \cdot 132 \cdot 131} \approx 3.60\%. \qquad \square$$

Exercise. Now suppose instead that we put 100 balls into 35 urns by iteratively choosing a random urn and placing a ball into that urn. Does the above probability change? If so, what is the new probability?

Solution. If we instead place the balls one by one, then we do not get a uniformly random configuration. For example, it is much more likely to get the configuration $3, \ldots, 3, 1, 0$ than the configuration $100, 0, \ldots, 0$. Thus, intuitively the probability should change. We can compute the new probability as follows. When we place the 100 balls into the urns, we track which urn each ball went into. There are 35^{100} possibilities, all equally likely, and we are interested in

those where exactly 1 is the first urn and exactly 1 is the last urn. There are $100 \cdot 99 \cdot 33^{98}$ of these, so our answer is

$$\frac{100 \cdot 99 \cdot 33^{98}}{35^{100}} \approx 2.53\%. \qquad \square$$