

170E Week 10 Discussion Notes

Colin Ni

December 11, 2025

Here are some review problems for the final exam. I've included most of the solutions.

Problem 1. Recall that the moment generating function $M_X(t)$ of a random variable X in general may not exist or may only be defined on a neighborhood of 0. In this problem, for simplicity we will assume that our random variables have mgf's that are defined everywhere.

- (i) Show that $M_{X+Y}(t) = M_X(t)M_Y(t)$ for independent random variables X and Y .
- (ii) Show that $M_{aX+b}(t) = e^{bt}M_X(at)$ for a random variable X and scalars $a, b \in \mathbb{R}$.
- (iii) Deduce that if $U \sim \text{Unif}(0, 1)$, then $5U + 3 \sim \text{Unif}(3, 8)$.
- (iv) Is true that if $U, V \sim \text{Unif}(0, 1)$ are independent, then $U + V \sim \text{Unif}(0, 2)$?
- (v) Prove the central limit theorem, using that Taylor's theorem gives $\exp(tX) = 1 + tX + \frac{1}{2}t^2X^2 + t^2h(t)$ where $\lim_{t \rightarrow 0} h(t) = 0$.

Solution. For (i), recall that since X and Y are independent, $f(X)$ and $g(Y)$ are independent for any functions f and g . Thus

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}(\exp(t(X+Y))) \\ &= \mathbb{E}(\exp(tX)\exp(tY)) \\ &= \mathbb{E}(\exp(tX))\mathbb{E}(\exp(tY)) \\ &= M_X(t)M_Y(t). \end{aligned}$$

For (ii), observe that

$$M_{aX+b}(t) = \mathbb{E}(\exp(t(aX+b))) = \exp(bt)\mathbb{E}(\exp(atX)) = \exp(bt)M_X(at).$$

For (iii), recall that

$$M_{\text{Unif}(a,b)}(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$

(for $t \neq 0$ and 1 for $t = 0$). Thus since

$$M_{5U+3}(t) = e^{3t} M_U(5t) = e^{3t} \cdot \frac{e^{5t-1}}{5t} = \frac{e^{8t} - e^{3t}}{5t} = M_{\text{Unif}(3,8)}(t),$$

we conclude that $5U + 3 \sim \text{Unit}(3, 8)$ because their mgf's agree.

For (iv), it is not true because

$$M_{U+V}(t) = M_U(t)M_V(t) = \frac{(e^t - 1)^2}{t^2} \neq \frac{e^{2t} - 1}{2t} = M_{\text{Unif}(0,2)}(t).$$

For (v), we want to show that if $X_1, \dots, X_n \sim X$ are i.i.d. random variables with mean μ and variance σ^2 , then

$$Z_n = \frac{X_1 + \dots + X_n - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to $\text{Norm}(0, 1)$ as $n \rightarrow \infty$. By standardizing, we can assume $\mu = 0$ and $\sigma^2 = 1$, so

$$M_X(t) = \mathbb{E}(\exp(tX)) = \mathbb{E}\left(1 + tX + \frac{1}{2}t^2X^2 + t^2h(t)\right) = 1 + \frac{1}{2}t^2 + t^2h(t)$$

using Taylor's theorem as described in the problem. By (i) and (ii), we have

$$M_{Z_n}(t) = M_X\left(\frac{t}{\sqrt{n}}\right)^n = \left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)^n.$$

Thus

$$\lim_{n \rightarrow \infty} M_{Z_n}(t) = \lim_{n \rightarrow \infty} \left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)^n = \exp\left(\frac{t^2}{2}\right) = M_{\text{Norm}(0,1)}(t),$$

where (if we want to be extra careful) we evaluated the limit by Taylor expanding $\log(1+x) = x + xg(x)$ where $g(x) \rightarrow 0$ as $x \rightarrow 0$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)^n &= \exp\left(\lim_{n \rightarrow \infty} n \log\left(1 + \frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)\right) \\ &= \exp\left(\lim_{n \rightarrow \infty} n \left(\frac{t^2}{2n} + \frac{t^2}{n}h\left(\frac{t}{\sqrt{n}}\right)\right)\right) \\ &= \exp\left(\frac{t^2}{2}\right). \quad \square \end{aligned}$$

Problem 2. Before doing this problem, please review Exercises 4.2(ii) and 4.3 in the lecture notes, which we did in discussion section in weeks 8 and 9. Let (X, Y) be bivariate normal with correlation ρ .

- (i) Set $Z_X = (X - \mu_X)/\sigma_X$ and $Z_Y = (Y - \mu_Y)/\sigma_Y$. Show that (Z_X, Z_Y) is bivariate normal with correlation ρ and with Z_X and Z_Y unit normal.

- (ii) Set $U_X = (Z_X - \rho Z_Y)/\sqrt{1 - \rho^2}$ and $U_Y = Y$. Show that (U_X, U_Y) is unit bivariate normal. Deduce that U_X and U_Y are independent.
- (iii) Suppose from now on that $\mu_X = -2$, $\mu_Y = -1$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and $\rho_{X,Y} = 1/2$. Find the conditional distribution of X given that $Y = y$ and the conditional distribution of Y given that $X = x$.
- (iv) Draw a picture of the joint pdf of (X, Y) (e.g. draw contour lines) and use it to corroborate the conditional expectations found in part (iii).
- (v) Find the conditional distribution of $2X - Y$ given that $Y = 3X - 3$.

Solution. For (i) and (ii), see the discussion notes from weeks 8 and 9.

For (iii), we have

$$(X \mid Y = y) \sim \text{Norm}\left(-2 + \frac{1}{4}(y + 1), \frac{3}{4}\right)$$

and

$$(Y \mid X = x) \sim \text{Norm}(-1 + 2(x + 2), 3).$$

For (v), note that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{Norm}\left(\begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\right),$$

so

$$\begin{aligned} \begin{pmatrix} 2X - Y \\ Y - 3X \end{pmatrix} &= \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &\sim \text{Norm}\left(\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}\right) \\ &= \text{Norm}\left(\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 & -5 \\ -5 & 7 \end{pmatrix}\right). \end{aligned}$$

Thus since the new correlation is $-5/2\sqrt{7}$, we have

$$(2X - Y \mid Y - 3X = -3) \sim \text{Norm}\left(\frac{19}{7}, \frac{3}{7}\right). \quad \square$$

Problem 3. Let $X_1, \dots, X_n \sim X$ be i.i.d. random variables. Consider the r th smallest value $X_{(r)}$ of X_1, \dots, X_n , so in particular $X_{(1)} \leq \dots \leq X_{(r)}$. This is called the r th order statistic of X_1, \dots, X_n .

- (i) Determine the pdf of the minimum $X_{(1)}$.
- (ii) Determine the pdf of the maximum $X_{(n)}$.
- (iii) In fact

$$f_{X_{(r)}}(x) = r \binom{n}{r} F_X(x)^{n-r} (1 - F_X(x))^{r-1} f_X(x)$$

(this is tricky to show). Use this to show that if $X \sim \text{Unif}(0, 1)$, then $X_{(r)} \sim \text{Beta}(r, n - r + 1)$.

Solution. For (i), the key is to look at the cdf, which is

$$\begin{aligned}
 F_{X_{(1)}}(x) &= \mathbb{P}(X_{(1)} \leq x) \\
 &= 1 - \mathbb{P}(X_{(1)} > x) \\
 &= 1 - \mathbb{P}(X_i > x \text{ for all } i = 1, \dots, n) \\
 &= 1 - \mathbb{P}(X > x)^n && \text{(independence)} \\
 &= 1 - (1 - F_X(x))^n,
 \end{aligned}$$

so the pdf is

$$f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x) = n(1 - F_X(x))^{n-1} f_X(x).$$

For (ii), the cdf is

$$\begin{aligned}
 F_{X_{(n)}}(x) &= \mathbb{P}(X_{(n)} \leq x) \\
 &= \mathbb{P}(X_i \leq x \text{ for all } i = 1, \dots, n) \\
 &= \mathbb{P}(X \leq x)^n && \text{(independence)} \\
 &= F_X(x)^n,
 \end{aligned}$$

so the pdf is

$$f_{X_{(n)}}(x) = \frac{d}{dx} F_{X_{(n)}}(x) = nF_X(x)^{n-1} f_X(x).$$

For (iii), simply compare their pdf's, recalling that $\Gamma(n) = (n-1)!$ for positive integers n . \square

Problem 4. Review the following problems from previous discussion sections:

- (i) The marbles problem from week 2
- (ii) The convex polygon problem from week 3
- (iii) The Russian roulette problem from week 3
- (iv) The coupon collectors problem from week 4
- (v) The slow administrator problem from week 5