

170E Week 6 Discussion Notes

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Exercise. (Book 3.1-4) If the mgf of X is

$$M_X(t) = \begin{cases} \frac{e^{5t} - e^{4t}}{t} & t \neq 0 \\ 1 & t = 0, \end{cases}$$

then find $\mathbb{E}(X)$, $\text{Var}(X)$, and $\mathbb{P}(4.2 < X \leq 4.7)$.

Solution. It is not so easy to take derivatives and evaluate at $t = 0$. Instead, we determine the pdf of X (assuming we do not recognize it somehow). Observe that

$$\frac{e^{5t} - e^{4t}}{t} = \left. \frac{e^{xt}}{t} \right|_4^5 = \int_4^5 e^{xt} dx,$$

so we deduce that $f_X(x) = 1\{4 \leq x \leq 5\}$ and $X \sim \text{Unif}(4, 5)$. Thus $\mathbb{E}(X) = 9/2$ and $\text{Var}(X) = 1/12$ and $\mathbb{P}(4.2 < X \leq 4.7) = 1/2$. \square

Exercise. (Book 3.3-10) If $X \sim \text{Norm}(\mu, \sigma^2)$, then show that $Y = aX + b \sim \text{Norm}(a\mu + b, a^2\sigma^2)$, where $a \neq 0$.

Solution. It suffices to show that the cdf of Y is the same as the cdf of $\text{Norm}(a\mu + b, a^2\sigma^2)$. To that effect, observe that the cdf of Y is

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(aX + b \leq y) \\ &= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) \\ &= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{\sigma}} e^{\frac{1}{2\sigma}(x-\mu)^2} dx. \end{aligned}$$

Changing variables to $u = ax + b$, we get

$$\begin{aligned} \int_{-\infty}^y \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{\sigma}} e^{\frac{1}{2\sigma^2}\left(\frac{u-b}{a}-\mu\right)^2} \frac{1}{a} du &= \int_{-\infty}^y \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{a^2\sigma^2}} e^{\frac{1}{2a^2\sigma^2}(u-(a\mu+b))^2} du \\ &= \int_{-\infty}^y f_{\text{Norm}(a\mu+b, a^2\sigma^2)}(u) du \\ &= F_{\text{Norm}(a\mu+b, a^2\sigma^2)}(y), \end{aligned}$$

as required. \square

Exercise. (Lecture notes 3.2) The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

for $\alpha > 0$. It generalizes factorials in the sense that $\Gamma(n) = (n-1)!$ for any $n \in \mathbb{N}$. The Gamma distribution $\Gamma(\alpha, \theta)$ has support \mathbb{R}_+ and pdf

$$f_{\Gamma(\alpha, \theta)}(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}.$$

show that the *Gamma distribution* is well-defined.

Solution. We need to show that the pdf integrates to 1. At once,

$$\begin{aligned} \int_0^\infty f_{\Gamma(\alpha, \theta)}(x) dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} dx \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty (\theta u)^{\alpha-1} e^{-u} \theta du \\ &\quad (\text{change variables to } u = x/\theta) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du \\ &= 1. \end{aligned} \quad \square$$

Exercise. (Lecture notes 3.3) The *chi-squared distribution* is defined as $\chi^2(r) = \Gamma(r/2, 2)$ for $r \in \mathbb{N}$. Find $\mathbb{E}(\chi^2(r))$, $\text{Var}(\chi^2(r))$, and the mgf of $\chi^2(r)$.

Solution. We may as well determine these things for the Gamma distribution and then plug in $\alpha = r/2$ and $\theta = 2$. Recall that the Gamma function satisfies the functional equation

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

Our strategy is to use the previous exercise:

$$\mathbb{E}(\Gamma(\alpha, \theta)) = \int_0^\infty x f_{\Gamma(\alpha, \theta)}(x) dx = \frac{\Gamma(\alpha + 1)\theta^{\alpha+1}}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty f_{\Gamma(\alpha+1, \theta)}(x) dx = \alpha\theta.$$

Similarly

$$\begin{aligned} \mathbb{E}(\Gamma(\alpha, \theta)^2) &= \int_0^\infty x^2 f_{\Gamma(\alpha, \theta)}(x) dx \\ &= \frac{\Gamma(\alpha + 2)\theta^{\alpha+2}}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty f_{\Gamma(\alpha+2, \theta)}(x) dx \\ &= \alpha(\alpha + 1)\theta^2, \end{aligned}$$

so

$$\text{Var}(\Gamma(\alpha, \theta)) = \mathbb{E}(\Gamma(\alpha, \theta)^2) - \mathbb{E}(\Gamma(\alpha, \theta))^2 = \alpha(\alpha + 1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2.$$

Finally, the mgf is

$$\begin{aligned}
M_{\Gamma(\alpha, \theta)}(t) &= \mathbb{E}(e^{t\Gamma(\alpha, \theta)}) \\
&= \int_0^\infty e^{tx} f_{\Gamma(\alpha, \theta)}(x) dx \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-(\frac{1}{\theta}-t)x} dx \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{u}{\frac{1}{\theta}-t}\right)^{\alpha-1} e^{-u} \frac{1}{\frac{1}{\theta}-t} du \\
&\quad (\text{change variables to } u = (\frac{1}{\theta}-t)x) \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{1}{\frac{1}{\theta}-t}\right)^\alpha \int_0^\infty u^{\alpha-1} e^{-u} du \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{1}{\frac{1}{\theta}-t}\right)^\alpha \int_0^\infty \Gamma(\alpha) f_{\Gamma(\alpha, 1)}(u) du \\
&= \frac{1}{\theta^\alpha} \left(\frac{1}{\frac{1}{\theta}-t}\right)^\alpha \\
&= \left(\frac{1}{1-\theta t}\right)^\alpha.
\end{aligned}$$

Thus

$$\mathbb{E}(\chi^2(r)) = r, \quad \text{Var}(\chi^2(r)) = 2r, \quad M_{\chi^2(r)}(t) = \left(\frac{1}{1-2t}\right)^{\frac{r}{2}}. \quad \square$$