170E Week 7 Discussion Notes

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Problem 1. A player is dealt 5 cards from a shuffled standard 52-card deck. Let D be the number of diamonds and F the number of face cards (J, Q, K) in the player's hand.

(a) Determine the support of (D, F).

(b) Determine the pdf $f_{(D,F)}$ of (D, F).

(c) Describe the marginal distributions of D and of F.

(d) Let L be the event that the player was dealt a flush (five of the same suit). Describe the conditional distribution of (D, F) given L.

Solution. For (a), we need to describe the support and pdf of (D, F). Let us abbreviate $[n] = \{0, 1, \ldots, n\}$. For the support, the tricky part is that there are only three diamond face cards. If $F \in [3]$, then any value $D \in [5]$ is possible. However, if $F = 4$, then only the values $D \in [4]$ are possible, and if $F = 5$, then only the values $D \in [3]$ are possible. Thus our support is

$$\operatorname{supp}(D, F) = ([5] \times [3]) \cup ([4] \times \{4\}) \cup ([3] \times \{5\}).$$

For (b), to determine the pdf, let $(d, f) \in \operatorname{supp}(D, F)$. The key ingredient is to introduce the auxiliary random variable I which is the number of diamond face cards in the player's hand. We then condition on I:

$$\begin{aligned} f_{(D,F)}(d, f) &= \mathbb{P}(D = d, F = f) \\ &= \sum_{i=0}^{3} \mathbb{P}(D = d, F = f \mid I = i)\mathbb{P}(I = i) \\ &= \sum_{i=0}^{3} \mathbb{P}(D = d, F = f, I = i) \\ &= \frac{1}{\binom{52}{5}} \sum_{i=0}^{3} \binom{10}{d-i}\binom{9}{f-i}\binom{3}{i}\binom{30}{5-d-f+i}, \end{aligned}$$

where we adopt the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$. In the last line, we are counting the number of 5-card hands that have $D = d, F = f, I = i$.

This is equivalent to picking i of the 3 diamond face cards, then picking $d-i$ of the 10 non-face diamond cards, then picking $f-i$ of the 9 non-diamond face cards, then picking $5-d-f+i$ of the non-diamond non-face cards.

For (c), the marginal distribution of D has support $[5]$ and has pdf

$$f_D(d) = \frac{\binom{13}{d}\binom{39}{5-d}}{\binom{52}{5}},$$

and the marginal distribution of F has support $[5]$ and has pdf

$$f_F(f) = \frac{\binom{12}{f}\binom{40}{5-f}}{\binom{52}{5}}.$$

These follow so-called hypergeometric distributions.

For (d), the support is

$$\operatorname{supp}(D, F \mid L) = (\{0\} \times [3]) \cup (\{5\} \times [3]).$$

To compute the pdf, we condition on the suit of the flush:

$$\begin{aligned}
f_{(D,F)\mid L}(d,f) &= \mathbb{P}(D=d, F=f \mid L) \\
&= \mathbb{P}(D=d, F=f \mid \text{diamond flush})\mathbb{P}(\text{diamond flush} \mid L) \\
&\quad + \mathbb{P}(D=d, F=f \mid \text{non-diamond flush})\mathbb{P}(\text{non-diamond flush} \mid L) \\
&= \begin{cases} \dfrac{3}{4} \cdot \mathbb{P}(F=f \mid \text{non-diamond flush}) & d=0 \\[2ex] \dfrac{1}{4} \cdot \mathbb{P}(F=f \mid \text{diamond flush}) & d=5, \end{cases} \\
&= \begin{cases} \dfrac{3}{4} \cdot \dfrac{\binom{3}{f}\binom{10}{5-f}}{\binom{13}{5}} & d=0 \\[2ex] \dfrac{1}{4} \cdot \dfrac{\binom{3}{f}\binom{10}{5-f}}{\binom{13}{5}} & d=5, \end{cases}
\end{aligned}$$

where in the last line we are computing the probability of having f face cards given that all five cards are the same suit. □

Problem 2. Let P and Q be continuous random variables supported on $\mathbb{R}$ (*i.e.* their pdfs are positive everywhere). The *Kullback-Leibler divergence* of P from Q is defined to be

$$D_{\mathrm{KL}}(P \parallel Q) = \int_{-\infty}^{\infty} f_P(x) \log\left(\frac{f_P(x)}{f_Q(x)}\right) \mathrm{d}x.$$

It measures how different the distribution of P is from Q.

(a) Show that $D_{\mathrm{KL}}(P \parallel Q) \geq 0$. (*Hint*: $\log(x) \leq x - 1$ for all $x > 0$.)

(b) Show that if P follows the same distribution as Q, then $D_{\mathrm{KL}}(P \parallel Q) = 0$. (The converse is also true.)

(c) Suppose $P \sim \mathrm{Norm}(\mu_P, \sigma_P^2)$ and $Q \sim \mathrm{Norm}(\mu_Q, \sigma_Q^2)$. Compute $D_{\mathrm{KL}}(P \parallel Q)$.

(d) Deduce that $D_{\mathrm{KL}}(P \parallel Q)$ is not a symmetric function in P and Q.

Solution. For (a),

$$\begin{aligned}
-D_{\mathrm{KL}}(P \parallel Q) &= \int_{-\infty}^{\infty} f_P(x) \log\left(\frac{f_Q(x)}{f_P(x)}\right) \mathrm{d}x \\
&\leq \int_{-\infty}^{\infty} f_P(x) \left(\frac{f_Q(x)}{f_P(x)} - 1\right) \mathrm{d}x \\
&\leq \int_{-\infty}^{\infty} (f_Q(x) - f_P(x))\, \mathrm{d}x \\
&= 0.
\end{aligned}$$

Thus $D_{\mathrm{KL}}(P \parallel Q) \geq 0$.

For (b), if P and Q follow the same distribution, then

$$\begin{aligned}
D_{\mathrm{KL}}(P \parallel Q) &= \int_{-\infty}^{\infty} f_P(x) \log\left(\frac{f_P(x)}{f_Q(x)}\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f_P(x) \log(1)\, \mathrm{d}x \\
&= 0.
\end{aligned}$$

For (c), we compute

$$\begin{aligned}
D_{\mathrm{KL}}(P \parallel Q) &= \int_{-\infty}^{\infty} f_P(x) \log\left(\frac{f_P(x)}{f_Q(x)}\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f_P(x) \log\left(\frac{\sigma_Q}{\sigma_P} \exp\left(\frac{(x-\mu_Q)^2}{2\sigma_Q^2} - \frac{(x-\mu_P)^2}{2\sigma_P^2}\right)\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f_P(x) \left(\log\frac{\sigma_Q}{\sigma_P} + \frac{(x-\mu_Q)^2}{2\sigma_Q^2} - \frac{(x-\mu_P)^2}{2\sigma_P^2}\right) \mathrm{d}x \\
&= \log\frac{\sigma_Q}{\sigma_P} + \frac{1}{2\sigma_Q^2}\int_{-\infty}^{\infty} f_P(x)\,(x-\mu_Q)^2\, \mathrm{d}x - \frac{1}{2\sigma_P^2}\int_{-\infty}^{\infty} f_P(x)\,(x-\mu_P)^2\, \mathrm{d}x \\
&= \log\frac{\sigma_Q}{\sigma_P} + \frac{1}{2\sigma_Q^2}\mathbb{E}_P\big[(X-\mu_Q)^2\big] - \frac{1}{2\sigma_P^2}\mathbb{E}_P\big[(X-\mu_P)^2\big] \\
&= \log\frac{\sigma_Q}{\sigma_P} + \frac{\sigma_P^2 + (\mu_P - \mu_Q)^2}{2\sigma_Q^2} - \frac{1}{2}.
\end{aligned}$$

For (d), we observe that our answer to part (c) is not symmetric in P and Q. Indeed, taking $\mu_P = \mu_Q$ and $\sigma_P^2 = 1/\sigma_Q^2 = \sigma^2$ gives

$$-\log(\sigma^2) + \frac{\sigma^4}{2} - \frac{1}{2},$$

whereas taking $\mu_P = \mu_Q$ and $\sigma_Q^2 = 1/\sigma_P^2 = \sigma^2$ gives

$$\log(\sigma^2) + \frac{1}{2\sigma^4} - \frac{1}{2}.$$

These are not equal because log is not a rational function. $\square$