

# 170E Week 6 Discussion Notes

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**Exercise.** (Book 3.1-4) If the mgf of  $X$  is

$$M_X(t) = \begin{cases} \frac{e^{5t} - e^{4t}}{t} & t \neq 0 \\ 1 & t = 0, \end{cases}$$

then find  $\mathbb{E}(X)$ ,  $\text{Var}(X)$ , and  $\mathbb{P}(4.2 < X \leq 4.7)$ .

*Solution.* It is not so easy to take derivatives and evaluate at  $t = 0$ . Instead, we determine the pdf of  $X$  (assuming we do not recognize it somehow). Observe that

$$\frac{e^{5t} - e^{4t}}{t} = \frac{e^{xt}}{t} \Big|_4^5 = \int_4^5 e^{xt} dx,$$

so we deduce that  $f_X(x) = 1\{4 \leq x \leq 5\}$  and  $X \sim \text{Unif}(4, 5)$ . Thus  $\mathbb{E}(X) = 9/2$  and  $\text{Var}(X) = 1/12$  and  $\mathbb{P}(4.2 < X \leq 4.7) = 1/2$ .  $\square$

**Exercise.** (Book 3.3-10) If  $X \sim \text{Norm}(\mu, \sigma^2)$ , then show that  $Y = aX + b \sim \text{Norm}(a\mu + b, a^2\sigma^2)$ , where  $a \neq 0$ .

*Solution.* Following the hint given in the book, it suffices to show that the cdf of  $Y$  is the same as the cdf of  $\text{Norm}(a\mu + b, a^2\sigma^2)$ . To that effect, observe that the cdf of  $Y$  is

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(aX + b \leq y) \\ &= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) \\ &= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{\sigma}} e^{\frac{1}{2\sigma^2}(x-\mu)^2} dx. \end{aligned}$$

Changing variables to  $u = ax + b$ , we get

$$\begin{aligned} \int_{-\infty}^y \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{\sigma}} e^{\frac{1}{2\sigma^2}\left(\frac{u-b}{a}-\mu\right)^2} \frac{1}{a} du &= \int_{-\infty}^y \frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{a^2\sigma}} e^{\frac{1}{2a^2\sigma^2}(u-(a\mu+b))^2} du \\ &= \int_{-\infty}^y f_{\text{Norm}(a\mu+b, a^2\sigma^2)}(u) du \\ &= F_{\text{Norm}(a\mu+b, a^2\sigma^2)}(y), \end{aligned}$$

as required.  $\square$

*Alternate solution.* Here is another way to do this problem. Observe that in general, the mgf of a linear transformation  $aX + b$  of a random variable  $X$  is

$$M_{aX+b}(t) = \mathbb{E}(e^{t(aX+b)}) = e^{bt} \mathbb{E}(e^{atX}) = e^{bt} M_X(at).$$

For  $X \sim \text{Norm}(\mu, \sigma^2)$ , we have  $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$ , so

$$\begin{aligned} M_{aX+b}(t) &= e^{bt} M_X(at) \\ &= e^{bt} e^{\mu(at) + \sigma^2(at)^2/2} \\ &= e^{(a\mu+b)t + (a^2\sigma^2)t^2/2} \\ &= M_{\text{Norm}(a\mu+b, a^2\sigma^2)}(t). \end{aligned}$$

It follows that  $aX + b \sim \text{Norm}(a\mu + b, a^2\sigma^2)$  because their mgf's agree.  $\square$

**Exercise.** (Lecture notes 3.2) The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

for  $\alpha > 0$ . It generalizes factorials in the sense that  $\Gamma(n) = (n-1)!$  for any  $n \in \mathbb{N}$ , and it satisfies the functional equation

$$\alpha \Gamma(\alpha) = \Gamma(\alpha + 1).$$

The Gamma distribution  $\Gamma(\alpha, \theta)$  has support  $\mathbb{R}_+$  and pdf

$$f_{\Gamma(\alpha, \theta)}(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}.$$

Show that the *Gamma distribution* is well-defined.

*Solution.* We need to show that the pdf integrates to 1. At once,

$$\begin{aligned} \int_0^\infty f_{\Gamma(\alpha, \theta)}(x) dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} dx \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty (\theta u)^{\alpha-1} e^{-u} \theta du \\ &\quad \text{(change variables to } u = x/\theta) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du \\ &= 1. \end{aligned} \quad \square$$

**Exercise.** (Lecture notes 3.3) The *chi-squared distribution* is defined as  $\chi^2(r) = \Gamma(r/2, 2)$  for  $r \in \mathbb{N}$ . Find  $\mathbb{E}(\chi^2(r))$ ,  $\text{Var}(\chi^2(r))$ , and the mgf of  $\chi^2(r)$ .

*Solution.* We may as well determine these things for the Gamma distribution and then plug in  $\alpha = r/2$  and  $\theta = 2$ . Recall that the Gamma function satisfies the functional equation

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

Our strategy is to use the previous exercise:

$$\mathbb{E}(\Gamma(\alpha, \theta)) = \int_0^\infty x f_{\Gamma(\alpha, \theta)}(x) dx = \frac{\Gamma(\alpha + 1)\theta^{\alpha+1}}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty f_{\Gamma(\alpha+1, \theta)}(x) dx = \alpha\theta.$$

Similarly

$$\begin{aligned} \mathbb{E}(\Gamma(\alpha, \theta)^2) &= \int_0^\infty x^2 f_{\Gamma(\alpha, \theta)}(x) dx \\ &= \frac{\Gamma(\alpha + 2)\theta^{\alpha+2}}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty f_{\Gamma(\alpha+2, \theta)}(x) dx \\ &= \alpha(\alpha + 1)\theta^2, \end{aligned}$$

so

$$\text{Var}(\Gamma(\alpha, \theta)) = \mathbb{E}(\Gamma(\alpha, \theta)^2) - \mathbb{E}(\Gamma(\alpha, \theta))^2 = \alpha(\alpha + 1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2.$$

Finally, the mgf is

$$\begin{aligned} M_{\Gamma(\alpha, \theta)}(t) &= \mathbb{E}(e^{t\Gamma(\alpha, \theta)}) \\ &= \int_0^\infty e^{tx} f_{\Gamma(\alpha, \theta)}(x) dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-(\frac{1}{\theta}-t)x} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{u}{\frac{1}{\theta}-t}\right)^{\alpha-1} e^{-u} \frac{1}{\frac{1}{\theta}-t} du \\ &\quad \text{(change variables to } u = (\frac{1}{\theta}-t)x) \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{1}{\frac{1}{\theta}-t}\right)^\alpha \int_0^\infty u^{\alpha-1} e^{-u} du \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{1}{\frac{1}{\theta}-t}\right)^\alpha \int_0^\infty \Gamma(\alpha) f_{\Gamma(\alpha, 1)}(u) du \\ &= \frac{1}{\theta^\alpha} \left(\frac{1}{\frac{1}{\theta}-t}\right)^\alpha \\ &= \left(\frac{1}{1-\theta t}\right)^\alpha. \end{aligned}$$

Thus

$$\mathbb{E}(\chi^2(r)) = r, \quad \text{Var}(\chi^2(r)) = 2r, \quad M_{\chi^2(r)}(t) = \left(\frac{1}{1-2t}\right)^{\frac{r}{2}}. \quad \square$$