170E Week 5 Discussion Notes

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Problem 1. Each day, a cat drinks $\mathrm{Unif}(0, 1)$ cups of milk. What is the expected number of days it will take for the cat to drink more than 1 cup of milk?

Solution. For $0 \leq x \leq 1$, let $f(x)$ be the expected number of days it will take for the cat to drink more than x cups of milk. We want to determine $f(1)$, and we already know $f(0) = 1$. Conditioning on how many cups $y \in \mathrm{Unif}(0, 1)$ it drinks the first day, we have

$$f(x) = 1 + \int_0^x f(x - y)\,\mathrm{d}y = 1 + \int_0^x f(u)\,\mathrm{d}u$$

by changing variables $u = x - y$. Taking the derivative with respect to x gives

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f$$

by the fundamental theorem of calculus. Thus $f(x) = Ae^x$ for some scalar A, and our boundary condition $f(0) = 1$ forces $f(x) = e^x$. We conclude that $f(1) = e$. $\square$

Alternate solution. Let X be the number of days it takes. Note that X is a random variable whose values are positive integers. More generally, for such a random variable X, we have

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{\infty} k\mathbb{P}(X = k) \\
&= \sum_{k=1}^{\infty} \sum_{\ell=1}^{k} \mathbb{P}(X = k) \\
&= \sum_{k=1}^{\infty} \sum_{\ell=k}^{\infty} \mathbb{P}(X = \ell) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(X \geq k).
\end{aligned}$$

Here, $\mathbb{P}(X \geq k)$ is the probability that $k-1$ independent samples from $\mathrm{Unif}(0,1)$ does not exceed 1, which is the volume of the region $x_1 + \cdots + x_{k-1} \leq 1$ in the unit cube $[0,1]^{k-1}$. We can now invoke the well-known fact that the area of the n-simplex in $\mathbb{R}^n$ is $1/n!$. To see this, inductively, it is

$$\int_0^1 \int_0^{x_1} \cdots \int_0^{x_1+\cdots+x_{n-1}} \mathrm{d}x_n \cdots \mathrm{d}x_2 \, \mathrm{d}x_1 = \int_0^1 x_1^{n-1} \frac{1}{(n-1)!} \, \mathrm{d}x_1 = \frac{1}{n!},$$

and of course our base case is that the area of the 1-simplex in $\mathbb{R}^1$ is 1. We conclude that

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) = \sum_{k=0}^{\infty} \frac{1}{k!} = e,$$

which agrees with our previous answer. $\square$

Problem 2. A continuous random variable is *memoryless* if

$$\mathbb{P}(X > t+s \mid X > s) = \mathbb{P}(X > t) \qquad \text{for any } t \geq 0 \text{ and } s \geq 0.$$

Show that exponential random variables are memoryless. Optionally, show that if a continuous random variable is memoryless, then it follows an exponential distribution.

Solution. Let us first show that exponential distributions are memoryless. Let $\lambda > 0$, and let $X \sim \mathrm{Exp}(\lambda)$ be an exponential distribution with rate λ. Recall that X has pdf

$$f_X(x) = \lambda e^{-\lambda x}.$$

Then

$$\mathbb{P}(X > t+s \mid X > s) = \frac{\mathbb{P}(X > t+s \text{ and } X > s)}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > s)}.$$

More generally

$$\mathbb{P}(X > x) = \int_x^{\infty} f_X(x) \, \mathrm{d}x = \int_x^{\infty} \lambda e^{-\lambda x} \, \mathrm{d}x = -e^{-\lambda x} \Big|_x^{\infty} = e^{-\lambda x}.$$

Thus

$$\frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t),$$

as required.

Now let X be a continuous random variable that is memoryless. We abbreviate $S(t) = \mathbb{P}(X > t)$ (named the survival function) so that our memoryless assumption reads

$$\frac{S(t+s)}{S(t)} = S(s).$$

Thus $S(kt) = S(t)^k$ for any nonnegative integer k, so also $S(t/k) = S(t)^{1/k}$. Therefore, for any rational number $q \in \mathbb{Q}$, we have $S(qt) = S(t)^q$. Taking $t = 1$, we get $S(q) = S(1)^q$, so since S is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$, this implies $S(t) = S(1)^t = e^{t \ln S(1)}$ for any real number $t \geq 0$. We conclude that $X \sim \mathrm{Exp}(-\ln S(1))$. $\square$

Problem 3. Suppose that an administrator takes $Y = 1 + \frac{1}{U}$ hours to respond an email, where $U \sim \mathrm{Unif}(0,1)$. What is their average response time? Suppose now that we re-send the email if they do not respond in $t > 1$ hours, which resets the time it takes for them to respond. Find (an approximate) t that minimizes the time it takes for the administrator to respond.

Solution. We have

$$\mathbb{E}(Y) = 1 + \mathbb{E}\left(\frac{1}{U}\right) = 1 + \int_0^1 \frac{1}{x}\,\mathrm{d}x = \infty,$$

so their average response time is infinite. Suppose now that we re-send after $t > 1$ hours, and let A denote the new average response time. Note that they respond within t hours if and only if $U > \frac{1}{t-1}$. We condition on their response time for our first email:

$$A = \int_0^{\frac{1}{t-1}} (t + A)\,\mathrm{d}x + \int_{\frac{1}{t-1}}^1 \left(1 + \frac{1}{x}\right)\,\mathrm{d}x = \frac{t+A}{t-1} + \frac{t-2}{t-1} - \ln\left(\frac{1}{t-1}\right),$$

so

$$A = \frac{(t-1)(2 + \ln(t-1))}{t-2}.$$

This is minimized at approximately $t \approx 5.505$ according to this Wolfram Alpha query. $\square$