

Practice Final Exam

Math 170S (Winter 2025)
Instructor: Koffi Enakoutsa
TA: Colin Ni

Name: _____

Student ID: _____

Show all work. You may use one sheet (front and back) of notes. You may not use a calculator. Please ensure your phone is silenced and stowed away.

Duration: 3 hours.

The following is my own work, without the aid of any other person.

Signature: _____

Problem 1

Consider the following 20 samples X_1, \dots, X_{20} drawn from the random variable $X \sim \text{Norm}(1, 2)$:

0.60	1.28	2.53	2.04	0.13	2.06	2.15	0.77	3.69	1.74
-0.10	2.60	2.04	3.23	3.05	1.65	2.73	1.39	1.51	1.57.

The sum of the samples and the sum of the squares of the samples are respectively

$$\sum_{i=1}^{20} X_i = 36.66 \quad \text{and} \quad \sum_{i=1}^{20} X_i^2 = 86.20.$$

- (i) (10 points) Compute the sample mean \bar{X} and the sample variance s_X^2 .
- (ii) (10 points) Compute the third quartile $\pi_{0.75}$.
- (iii) (5 points, challenge) More generally, let X_1, \dots, X_n be samples drawn from a random variable X with variance $\text{Var}(X) = \sigma^2$. The sample variance s_X^2 is an estimator for σ^2 , but so is

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2.$$

In one or two sentences, explain the key difference between these two estimators and how it relates to Bessel's correction.

Problem 2

In this problem, we will explore the order statistics of the so-called power distribution. The *power distribution* $P(\alpha)$ with power $\alpha > 0$ is defined by the pdf

$$f_{P(\alpha)}(x) = \alpha x^{\alpha-1} \quad \text{for } 0 \leq x \leq 1.$$

Let P_1, \dots, P_n be samples from $P(\alpha)$.

- (i) (5 points) Check that $f_{P(\alpha)}$ is a valid pdf.
- (ii) (10 points) Determine the distribution of the n th order statistic $P_{(n)}$ in terms of a power distribution.
- (iii) (10 points) The *Kumaraswamy distribution* $K(a, b)$ with shape $a, b \geq 0$ is defined by the pdf

$$f_{K(a,b)}(x) = abx^{a-1}(1-x)^{b-1} \quad \text{for } 0 \leq x \leq 1.$$

(Note the similarity with $\text{Beta}(\alpha, \beta)$ which has pdf proportional to $x^{\alpha-1}(1-x)^{\beta-1}$.) Determine the distribution of the 1st order statistic $P_{(1)}$ in terms of a Kumaraswamy distribution.

Solution. For (i), we simply need to check that

$$\int_0^1 f_{P(\alpha)}(x) dx = \int_0^1 \alpha x^{\alpha-1} dx = x^\alpha \Big|_0^1 = 1.$$

For (ii), let us use the general formula

$$f_{X_{(r)}}(x) = r \binom{n}{r} F_X(x)^{r-1} (1 - F_X(x))^{n-r} f_X(x)$$

for the r th order statistic of a random variable X from week 1 discussion section. Since $F_{P(\alpha)}(x) = x^\alpha$, we have

$$f_{P_{(n)}}(x) = nx^{\alpha(n-1)} \cdot \alpha x^{\alpha-1} = n\alpha x^{n\alpha-1} = f_{P(n\alpha)}(x).$$

Thus the n th order statistic $P_{(n)} \sim P(n\alpha)$ follows the power distribution with power $n\alpha$.

For (iii), by the same general formula, we have

$$f_{P_{(1)}}(x) = \alpha nx^{\alpha-1} (1-x^\alpha)^{n-1} = f_{K(\alpha,n)}(x).$$

Thus the 1st order statistic $P_{(1)} \sim K(\alpha, n)$ follows the Kumaraswamy distribution with shape α, n . □

Problem 3

In this problem, we will find the MLE for the probability of success for a geometric random variable. Recall that a geometric random variable $\text{Geom}(p)$ with probability of success $0 \leq p \leq 1$ has pdf

$$f_{\text{Geom}(p)}(x) = (1-p)^{x-1}p \quad \text{for } x \geq 0.$$

- (i) (5 points) Describe precisely the likelihood function.
- (ii) (15 points) Find the maximum likelihood estimator \hat{p} of the parameter p .
- (iii) (5 points) Explain why the estimator \hat{p} you found makes sense intuitively in terms of what a geometric random variable models and what p represents.

Solution. For (i), the likelihood of a sample X_1, \dots, X_n given a value of p is

$$\begin{aligned} L(X_1, \dots, X_n \mid p) &= f_{\text{Geom}(p)}(X_1) \cdots f_{\text{Geom}(p)}(X_n) \\ &= (1-p)^{X_1-1}p \cdots (1-p)^{X_n-1}p \\ &= (1-p)^{n\bar{X}-n}p^n. \end{aligned}$$

For (ii), maximizing the likelihood is equivalent to maximizing the log-likelihood

$$\ln L(X_1, \dots, X_n \mid p) = (n\bar{X} - n) \ln(1-p) + n \ln(p).$$

The critical points of the log-likelihood occur where

$$0 = \frac{d \ln L}{dp}(X_1, \dots, X_n \mid p) = -\frac{n\bar{X} - n}{1-p} + \frac{n}{p}.$$

Solving for p shows the log-likelihood has a unique critical point $\hat{p} = 1/\bar{X}$. To see that this is a maximum, let us compute the second derivative:

$$\frac{d^2 \ln L}{dp^2}(X_1, \dots, X_n \mid p) = -\frac{n\bar{X} - n}{(1-p)^2} - \frac{n}{p^2}.$$

In order to plug in $\hat{p} = 1/\bar{X}$, we must handle the case where $1/\bar{X} = 1$ separately. In this case the log-likelihood is simply $n \ln(p)$, so since $0 \leq p \leq 1$, the value $p = 1 = 1/\bar{X}$ is the unique maximizing value. Otherwise $1/\bar{X} \neq 1$, in which case

$$\frac{d^2 \ln L}{dp^2}\left(X_1, \dots, X_n \mid \frac{1}{\bar{X}}\right) = -\frac{n\bar{X} - n}{(1 - \frac{1}{\bar{X}})^2} - \frac{n}{\frac{1}{\bar{X}}^2} < 0.$$

Therefore, the MLE \hat{p} of the parameter p for the geometric distribution $\text{Geom}(p)$ is given by

$$\hat{p} = \frac{1}{\bar{X}}.$$

For (iii), this makes sense intuitively because \bar{X} is the average number of trials needed to get a success, so $\frac{1}{\bar{X}}$ estimates the probability of success. \square

Problem 4

In this problem, we will find the method of moments estimators for the parameters of a Gamma distribution. Recall that a Gamma distribution $\text{Gamma}(\alpha, \theta)$ has parameters $\alpha > 0$ and $\theta > 0$ and is defined by the pdf

$$f_{\Gamma(\alpha, \theta)}(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} \quad \text{for } x \geq 0.$$

Recall further that it has mean $\alpha\theta$ and variance $\alpha\theta^2$.

- (i) (10 points) Let $X \sim \text{Gamma}(\alpha, \theta)$. Find the method of moments estimators $\tilde{\alpha}$ and $\tilde{\theta}$ for the parameters α and θ respectively.
- (ii) (5 points) Explain why

$$V = \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{n-1}{n} s_X^2$$

is a biased estimator of variance (for any random variable X). Write $\tilde{\alpha}$ and $\tilde{\theta}$ in terms of V and \bar{X} .

- (ii) (10 points) Method of moments estimators can sometimes produce nonsensical estimates. Elaborate on this point, and provide a concrete example using the estimators $\tilde{\alpha}$ and $\tilde{\theta}$.

Solution. For (i), we are given that the theoretical first moment is $\mathbb{E}(X) = \alpha\theta$, and we deduce that the theoretical second moment is

$$\mathbb{E}(X^2) = \text{Var}(X) + \mathbb{E}(X)^2 = \alpha\theta^2 + \alpha^2\theta^2 = \alpha\theta^2(1 + \alpha).$$

Suppose our sample is X_1, \dots, X_n . Our estimators for the first and second moments are respectively

$$\hat{M}_1 = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad \hat{M}_2 = \frac{1}{n} \sum_{i=1}^n X_i^2.$$

Thus, the method of moments estimators $\tilde{\alpha}$ and $\tilde{\theta}$ are the solution to the system of equations

$$\tilde{\alpha}\tilde{\theta} = \hat{M}_1 \quad \text{and} \quad \tilde{\alpha}\tilde{\theta}^2(1 + \tilde{\alpha}) = \hat{M}_2.$$

Some algebra shows that

$$\tilde{\alpha} = \frac{\hat{M}_1}{\hat{M}_2 - \hat{M}_1^2} \quad \text{and} \quad \tilde{\theta} = \frac{\hat{M}_2 - \hat{M}_1^2}{\hat{M}_1}.$$

For (ii), simply write

$$\tilde{\alpha} = \frac{\bar{X}}{V} \quad \text{and} \quad \tilde{\theta} = \frac{V}{\bar{X}}.$$

For (iii), for this example there are many ways these estimators can give nonsensical answers. For example, α and θ are always positive real numbers, but the estimator $\tilde{\alpha}$ can be zero when $\bar{X} = 0$ (this can happen because the support is $x \geq 0$) and the estimator $\tilde{\theta}$ can be zero when $V = 0$. Even worse, when $\bar{X} = 0$, the estimator $\tilde{\theta}$ is undefined, and when $V = 0$, the estimator $\tilde{\alpha}$ is undefined. \square

Problem 5

In this problem, we will work out an example of linear regression and discuss what happens when we introduce regularization. Consider the following dataset of x -values x_1, \dots, x_8 and y -values y_1, \dots, y_8 :

$$\begin{array}{cccccccc} x: & -2 & 4 & -1 & -2 & 0 & 3 & 1 & 1 \\ y: & 4 & -7 & 3 & 5 & 2 & -4 & -1 & -2. \end{array}$$

- (i) (5 points) Plot this dataset.
- (ii) (15 points) Find the exact line of best fit using linear regression. (*Hint:* Your answer should contain the number 32/17.)
- (iii) (5 points, challenge) Recall that for the linear model $y(x) = \alpha + x\beta$, linear regression finds the parameters α and β that minimizes the mean squared error. Consider the regularized mean squared error function

$$\frac{1}{n} \sum_{i=1}^n (y(x_i) - y_i)^2 + \frac{\lambda}{n} (\alpha^2 + \beta^2)$$

where $\lambda > 0$ is some hyperparameter. This penalizes α and β being large, resulting in a simpler model. Find α and β that minimize this regularized error function. (*Hint:* Write the function as $\frac{1}{n}(\mathbf{X}\gamma - \mathbf{y})^T(\mathbf{X}\gamma - \mathbf{y}) + \frac{\lambda}{n}\gamma^T\gamma$ which has derivative $\frac{2}{n}(\mathbf{X}^T\mathbf{X}\gamma - \mathbf{X}^T\mathbf{y}) + \frac{2\lambda}{n}\gamma$.)

Solution. For (i), see this Wolfram Alpha query.

For (ii), see week 5 discussion section. We have

$$\mathbf{X} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \\ 1 & -1 \\ 1 & -2 \\ 1 & 0 \\ 1 & 3 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 4 \\ -7 \\ 3 \\ 5 \\ 2 \\ -4 \\ -1 \\ -2 \end{pmatrix}.$$

So

$$\mathbf{X}^T\mathbf{X} = \begin{pmatrix} 8 & 4 \\ 4 & 36 \end{pmatrix} \quad \text{and} \quad \mathbf{X}^T\mathbf{y} = \begin{pmatrix} 0 \\ -64 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \frac{1}{272} \begin{pmatrix} 36 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ -64 \end{pmatrix} = \frac{1}{272} \begin{pmatrix} 256 \\ -512 \end{pmatrix}.$$

Therefore

$$y(x) = \frac{16}{17} - \frac{32}{17}x$$

is the line of best fit.

For (iii), setting the derivative equal to zero and solving for γ gives

$$\hat{\gamma} = (\mathbf{X}^T\mathbf{X} + \lambda I)^{-1}\mathbf{X}^T\mathbf{y}.$$

Unraveling gives $\hat{\alpha}$ and $\hat{\beta}$. Note that we have the theoretical benefit of $\mathbf{X}^T\mathbf{X} + \lambda I$ being guaranteed to be invertible, because $\lambda > 0$ makes this positive definite. \square

Problem 6

In this problem, we will give an example of a sufficient statistic and an example of a non-sufficient statistic. Let $X \sim \text{Pois}(\lambda)$ be an Poisson random variable with unknown rate $\lambda > 0$. Recall that the pdf is given by

$$f_{\text{Pois}(\lambda)}(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for } k \geq 0 \text{ an integer}$$

and that the MLE of the parameter λ is the sample mean: $\hat{\lambda} = \bar{X}$.

(i) (20 points) Show that the MLE $\hat{\lambda}$ is a sufficient statistic for λ .

(ii) (5 points, challenge) Show that the sample minimum is not a sufficient statistic for λ .

Solution. For (i), the joint pdf of a sample X_1, \dots, X_n and a rate $\lambda > 0$ is

$$f_{\text{Pois}(\lambda)}(X_1) \cdots f_{\text{Pois}(\lambda)}(X_n) = \frac{\lambda^{X_1} e^{-\lambda}}{X_1!} \cdots \frac{\lambda^{X_n} e^{-\lambda}}{X_n!} = \frac{\lambda^{X_1 + \cdots + X_n} e^{-(X_1 + \cdots + X_n)}}{X_1! \cdots X_n!}.$$

Since $X_1 + \cdots + X_n = n\bar{X} = n\hat{\lambda}$, we can factor this joint pdf as

$$\lambda^{n\hat{\lambda}} \cdot \frac{e^{-(X_1 + \cdots + X_n)}}{X_1! \cdots X_n!},$$

where the factor on the LHS only depends on the MLE $\hat{\lambda}$ and the parameter λ , and the RHS only depends on the sample X_1, \dots, X_n . By our definition of sufficient statistics, this shows that $\hat{\lambda}$ is a sufficient statistic for λ .

For (ii), the intuition is that the sample minimum does not capture the same information as the sample mean. For example, the samples 0, 5 and 0, 6 have the same sample minimums but different means. More precisely, suppose for the sake of contradiction that for every n , the joint pdf can be factored as

$$\frac{\lambda^{X_1 + \cdots + X_n} e^{-(X_1 + \cdots + X_n)}}{X_1! \cdots X_n!} = \phi(X_{(1)}, \lambda) h(X_1, \dots, X_n)$$

for some functions ϕ and h . Let us absorb the $X_1! \cdots X_n!$ into the function h by defining $H(X_1, \dots, X_n) = h(X_1, \dots, X_n) X_1! \cdots X_n!$ so that

$$\lambda^{X_1 + \cdots + X_n} e^{-(X_1 + \cdots + X_n)} = \phi(X_{(1)}, \lambda) H(X_1, \dots, X_n).$$

Consider the case $n = 2$ and even more specifically when $X_1 = 0$:

$$\lambda^{X_2} e^{-X_2} = \phi(0, \lambda) H(0, X_2).$$

For $\lambda = 1$, we get $H(0, X_2) \propto e^{-X_2}$, and for $\lambda = 2$, we get $H(0, X_2) \propto 2^{X_2} e^{-X_2}$. This is a contradiction because e^{-X_2} and $2^{X_2} e^{-X_2}$ are not proportional. \square

Problem 7

In this problem, we will walk you through how to determine the Bayes estimate of the mean of a normal distribution with a normal prior, and then we will discuss how to interpret the Bayes estimate.

Let X_1, \dots, X_n be a sample from the normal random variable $\text{Norm}(\mu, \sigma^2)$ where σ^2 is known, and assume that the prior distribution of μ is the normal distribution $\text{Norm}(\mu_0, \sigma_0^2)$.

- (i) (5 points) Recall that the pdf of $\text{Norm}(\mu, \sigma^2)$ is

$$f_{\text{Norm}(\mu, \sigma^2)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

Find a, b, c such that the joint distribution of \bar{X} and μ is proportional to $\exp\left(\frac{a\mu^2 - b\mu}{c}\right)$.

- (ii) (5 points) Observe that

$$\exp\left(\frac{a\mu^2 - b\mu}{c}\right) = \exp\left(\frac{(\mu - \frac{b}{2a})^2 - \frac{b^2}{4a^2}}{\frac{c}{a}}\right) \propto \exp\left(\frac{(\mu - \frac{b}{2a})^2}{\frac{c}{a}}\right).$$

Find the normal distribution whose pdf this is proportional to.

- (iii) (5 points) Deduce that the posterior distribution of μ is

$$\text{Norm}\left(\frac{\bar{X}\sigma_0^2 + \mu_0\sigma^2/n}{\sigma_0^2 + \sigma^2/n}, \frac{\sigma^2\sigma_0^2/n}{\sigma_0^2 + \sigma^2/n}\right).$$

- (iv) (5 points) Let us take our Bayes estimate to be the mean of the posterior. Use this Bayes estimate to illustrate the idea that Bayes estimators are influenced by both the data and the prior.
- (v) (5 points) Explain what it means for n to be large, and explain what happens to the Bayes estimate in this case. Similarly, explain what it means for σ_0^2 to be small, and explain what happens to the Bayes estimate in this case.

Problem 8

In this problem, we will use a simple hypothesis test about a proportion to discuss the core intuition of hypothesis tests. Suppose USC claims that only 25% of its students are spoiled, and suppose a group of UCLA students wants to test whether this proportion is in fact higher.

- (i) (5 points) Write down the null hypothesis and the alternative hypotheses.
- (ii) (5 points) Say whether your alternative hypothesis is one-sided or two-sided, and explain why the other option is not appropriate.
- (iii) (5 points) Determine the exact critical region for the confidence coefficient $\alpha = 0.05$.
- (iv) (5 points) Explain why you can approximate this critical region using a normal distribution, and determine the approximated critical region. You should refer to the table of normal and t -distribution values on the last page of this exam.
- (v) (5 points) The group of UCLA students sample $n = 100$ USC students and determine that 37 of them are spoiled. Using this data, perform the hypothesis test using the approximated critical region, and report the p -value.

Problem 9

In this problem, we will construct a confidence interval for the difference between two means assuming unknown but proportional variances. Assume $X \sim \text{Norm}(\mu_X, \sigma^2)$ and $Y \sim \text{Norm}(\mu_Y, \sigma^2)$ are normal, where the means are unknown and where the variances are equal but otherwise unknown. Suppose we collect a sample of size n_X from X with sample mean \bar{X} and sample variance s_X^2 , and suppose we collect a sample of size n_Y from Y with sample mean \bar{Y} and sample variance s_Y^2 .

(i) (5 points) Show that

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\sigma^2/n_X + \sigma^2/n_Y}} \sim \text{Norm}(0, 1).$$

(ii) (5 points) Recall that the chi-squared distribution $\chi^2(n)$ with n degrees of freedom can be defined as the distribution of a sum of squares of n independent unit normals. Show that

$$\frac{(n_X - 1)s_X^2}{\sigma^2} + \frac{(n_Y - 1)s_Y^2}{\sigma^2} \sim \chi^2(n_X + n_Y - 2).$$

Recall that the t -distribution $t(n)$ with n degrees of freedom can be defined as $Z/\sqrt{V/(n-1)}$, where $Z \sim \text{Norm}(0, 1)$ and $V \sim \chi^2(n-1)$ are independent. Let us denote the random variables described in parts (i) and (ii) by Z and V respectively. You may use the (kind of deep) fact that Z and V are independent.

(iii) (15 points) Show that

$$\left((\bar{X} - \bar{Y}) - t_0 s_P \sqrt{\frac{1}{n_X} + \frac{1}{n_Y}}, (\bar{X} - \bar{Y}) + t_0 s_P \sqrt{\frac{1}{n_X} + \frac{1}{n_Y}} \right)$$

where

$$s_P = \sqrt{\frac{(n_X - 1)s_X^2 + (n_Y - 1)s_Y^2}{n_X + n_Y - 2}}$$

and

$$t_0 = t_{\alpha/2}(n_X + n_Y - 2)$$

is a confidence interval for $\mu_X - \mu_Y$ with confidence coefficient $1 - \alpha$.

Problem 10

In this problem, we will test the hypothesis that the air quality in San Francisco is different from the air quality in Los Angeles. Let X and Y denote the concentration of suspended particles in the air in San Francisco and Los Angeles, respectively. As in Problem 9, assume $X \sim \text{Norm}(\mu_X, \sigma^2)$ and $Y \sim \text{Norm}(\mu_Y, \sigma^2)$ are normal, where the means are unknown and where the variances are equal but otherwise unknown. Suppose we collect data with the following statistics:

$$\begin{array}{lll} n_X = 5 & \bar{X} = 79 & s_X = 14 \\ n_Y = 10 & \bar{Y} = 101 & s_Y = 23. \end{array}$$

- (i) (20 points) Using the confidence interval described in Problem 8 part (iii), perform a hypothesis test on whether $\mu_X \neq \mu_Y$ at confidence level $\alpha = 0.05$, and report the p -value of your test. You should refer to the table of normal and t -distribution values on the last page of this exam.
- (ii) (5 points) For which values of n_X would the result of the test be different?

Table of normal and t -distribution values

The following table gives some common values of the cdf of some t -distributions. Since $t(\infty) = \text{Norm}(0, 1)$ this also contains common values of the cdf of the unit normal distribution. Recall that $t_\alpha(r)$ is defined as $t_\alpha(r) = F_{t(r)}(1 - \alpha)$, where $t(r)$ denotes the t -distribution with r degrees of freedom.

r	$t_{0.10}(r)$	$t_{0.05}(r)$	$t_{0.025}(r)$	$t_{0.01}(r)$	$t_{0.005}(r)$	$t_{0.001}(r)$
1	3.078	6.314	12.706	31.821	63.657	318.31
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
35	1.306	1.690	2.030	2.438	2.724	3.340
40	1.303	1.684	2.021	2.423	2.704	3.307
45	1.301	1.679	2.014	2.412	2.690	3.281
50	1.299	1.676	2.009	2.403	2.678	3.261
55	1.297	1.673	2.004	2.396	2.668	3.245
60	1.296	1.671	2.000	2.390	2.660	3.232
∞	1.282	1.645	1.960	2.326	2.576	3.090

