170S Week 3 Discussion Notes

Colin Ni

January 20, 2025

Warm-up

Let $Y_1, \dots, Y_{13}$ be i.i.d. random variables, and denote by $Y_{(1)} \le \cdots \le Y_{(13)}$ their order statistics. Suppose $\pi_{0.8} = 420$. Without overthinking it, what are $\mathbb{P}(Y_i \le 420)$ and $\mathbb{P}(Y_i > 420)$? Determine $\mathbb{P}(Y_{(3)} \le 420 \le Y_{(7)})$.

Solution. The assumption $\pi_{0.8} = 420$ says, without overthinking, that 80 percent of samples are ≤ 420, so $\mathbb{P}(Y_i \le 420) = 0.8$ and $\mathbb{P}(Y_i > 420) = 0.8$. Now

$$\begin{aligned}\mathbb{P}(Y_{(3)} \le 420 \le Y_{(7)}) &= \mathbb{P}(\text{exactly 3, 4, 5, 6, or 7 of the } Y_i \text{ are } \le 420)\\ &= \sum_{r=3}^{7} \binom{13}{r} 0.8^r 0.2^{13-r}.\end{aligned}$$

□

Computing some order statistics

Summary. Here is a table of the stuff we will compute, where order statistics are taken with respect to n independent samples.

distribution X	pdf $f_X(x)$	mean $\mathbb{E}(X)$	variance $\mathrm{Var}(X)$
$X_{(r)}$	$r\binom{n}{r} F_X(x)^{r-1}(1-F_X(x))^{n-r} f_X(x)$	$\int x f_{X_{(r)}}(x)\,\mathrm{d}x$	$\mathbb{E}(X_{(r)}^2) - \mathbb{E}(X_{(r)})^2$
$U \sim \mathrm{Unif}(0,1)$	1	$\frac{1}{2}$	$\frac{1}{12}$
$U_{(r)} \overset{\mathrm{d}}{=} e^{-E_{(r)}}$	$r\binom{n}{r} x^{r-1}(1-x)^{n-r}$	$\frac{r}{n+1}$	$\frac{r(n-r+1)}{(n+1)^2(n+1)}$
$E \sim \mathrm{Exp}(\lambda)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$E_{(r)}$	$r\binom{n}{r}(1-\lambda e^{-\lambda x})^{r-1}(\lambda e^{-\lambda x})^{n-r+1}$	$\frac{1}{\lambda}\sum_{i=1}^{r}\frac{1}{n-i+1}$	dwai

Moreover, we have the connections

$$\begin{aligned} E_{(1)} &\sim \mathrm{Exp}(n\lambda) \\ E_{(r+1)} - E_{(r)} &\sim \mathrm{Exp}((n-r)\lambda) \\ e^{-E_{(r)}} \overset{\mathrm{d}}{=} U_{(r)} &\sim \mathrm{Beta}(r, n-r+1). \end{aligned}$$

Remark. The information about U and E is of course imported from 170A/170E, and the information about $X_{(r)}$ and the pdf and mean of $U_{(r)}$ and the connection $U_{(r)} \sim \mathrm{Beta}(r, n-r+1)$ is from our week 1 discussion section. It remains to determine the variance of $U_{(r)}$, the pdf and mean of $E_{(r)}$, the information about $e^{-E_{(r)}}$, and the rest of the connections will be found along the way.

Variance of $U_{(r)}$. Observe

$$\begin{aligned} \mathbb{E}(U_{(r)}^2) &= \int_0^1 x^2 f_{U_{(r)}}(x)\,\mathrm{d}x \\ &= \int_0^1 r\binom{n}{r} x^{r+1}(1-x)^{n-r}\,\mathrm{d}x \\ &= \frac{r\binom{n}{r}}{r'\binom{n'}{r'}} \int_0^1 r'\binom{n'}{r'} x^{r'}(1-x)^{n'-r'}\,\mathrm{d}x \\ &\text{(set } n' = n+1 \text{ and } r' = r+1 \text{ and massage into } \mathbb{E}(U_{(r')}) \text{ with } n' \text{ samples)} \\ &= \frac{r\binom{n}{r}}{r'\binom{n'}{r'}} \mathbb{E}(U_{(r')}) \\ &= \frac{\frac{n!}{(r-1)!(n-r)!}}{\frac{(n+1)!}{r!(n-r)!}} \frac{r+1}{n+2} \qquad \text{(using } \mathbb{E}(U_{(r)}) = \tfrac{r}{n+1}) \\ &= \frac{r(r+1)}{(n+1)(n+2)} \qquad \text{(patiently cancel).} \end{aligned}$$

Thus

$$\mathrm{Var}(U_{(r)}) = \mathbb{E}(U_{(r)}^2) - \mathbb{E}(U_{(r)})^2 = \frac{r(r+1)}{(n+1)(n+2)} - \left(\frac{r}{n+1}\right)^2 = \frac{r(n-r+1)}{(n+1)^2(n+2)}.$$

Pdf of $E_{(r)}$. Recall $F_E(x) = 1 - \lambda e^{-\lambda x}$, so

$$\begin{aligned} f_{E_{(r)}}(x) &= r\binom{n}{r}(1-\lambda e^{-\lambda x})^{r-1}(\lambda e^{-\lambda x})^{n-r}\lambda e^{-\lambda x} \\ &\qquad \text{(using the formula for } f_{X_{(r)}}(x)) \\ &= r\binom{n}{r}(1-\lambda e^{-\lambda x})^{r-1}(\lambda e^{-\lambda x})^{n-r+1}. \end{aligned}$$

Mean of $E_{(r)}$. The key point is that $E_{(1)} \sim \mathrm{Exp}(n\lambda)$. This is because plugging $r = 1$ into

$$f_{E_{(r)}}(x) = r\binom{n}{r}(1-\lambda e^{-\lambda x})^{r-1}(\lambda e^{-\lambda x})^{n-r+1}$$

gives

$$f_{\mathrm{Exp}(n\lambda)}(x) = n\lambda e^{-n\lambda x}.$$

Now let us give a rough argument that $E_{(r+1)} - E_{(r)} \sim \mathrm{Exp}((n-r)\lambda)$. Consider the probability

$$\mathbb{P}(E_{(r+1)} - E_{(r)} > x \mid E_{(r)} = y) = \mathbb{P}(E_{(r+1)} > x + y \mid E_{(r)} = y).$$

The is the probability that out of $n-r$ independent $\mathrm{Exp}(\lambda)$ samples each larger than y, the smallest is larger than $x+y$. Since Exp is memoryless, this is the probability that out of $n-r$ independent $\mathrm{Exp}(\lambda)$ samples, the smallest is larger than x, which follows that of a first order statistic. In other words this probability is

$$\mathbb{P}(\mathrm{Exp}((n-r)\lambda) > x).$$

Since this is independent of y, conditioning on y gives the result.

This shows

$$E_{(r)} \sim \sum_{i=1}^{r} \mathrm{Exp}((n-i+1)\lambda),$$

so the mean is

$$\mathbb{E}(E_{(r)}) = \sum_{i=1}^{r} \mathbb{E}(\mathrm{Exp}((n-i+1)\lambda)) = \frac{1}{\lambda}\sum_{i=1}^{r}\frac{1}{n-i+1}.$$

Pdf of $e^{-E_{(r)}}$. Recall that for a monotonic function $g\colon \mathbb{R} \to \mathbb{R}$ and a random variable X, the pdf of $g(X)$ is

$$f_{g(X)}(y) = f_X(g^{-1}(y)) \left| \frac{\mathrm{d}(g^{-1})}{\mathrm{d}y}(y) \right|,$$

where g^{-1} denotes the inverse of g. Our $e^{-E_{(r)}}$ is of this form, with $g(x) = e^{-x}$ and $X = E_{(r)}$, so

$$\begin{aligned} f_{e^{-E_{(r)}}}(y) &= f_{E_{(r)}}(-\ln(y)) \left| -\frac{1}{y} \right| \\ &= r\binom{n}{r}(1-\lambda e^{\lambda \ln(y)})^{r-1}(\lambda e^{\lambda \ln(y)})^{n-r+1}\frac{1}{y} \\ &= r\binom{n}{r}(1-\lambda y^{\lambda})^{r-1}\lambda^{n-r+1}y^{\lambda(n-r+1)-1}. \end{aligned}$$

In particular, for $\lambda = 1$ this gives

$$r\binom{n}{r}(1-y)^{r-1}y^{n-r},$$

which shows that $e^{-E_{(r)}} \stackrel{d}{=} U_{(r)}$.

Remark. In general $F_X(X) \stackrel{d}{=} \mathrm{Unif}(0,1)$, so in particular $F_{E_{(r)}}(E_{(r)}) \stackrel{d}{=} \mathrm{Unif}(0,1)$. One reason the above result is cool is because it says $f_{\mathrm{Exp}(1)}(E_{(r)}) \stackrel{d}{=} \mathrm{Unif}(0,1)$.