## 182 Week 9 Discussion Notes

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This week for discussion section we will discuss two homework problems from a previous iteration of Math 182.

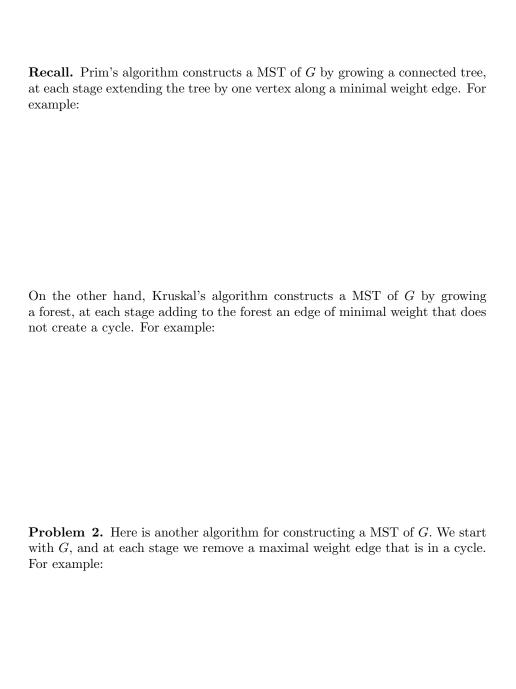
**Notation.** For today, let G = (V, E) be a connected, weighted, undirected graph. We abbreviate "minimum spanning tree" as MST.

## **Problem 1.** Prove or disprove:

- (a) If T is a spanning tree of G such that every edge in T is in some MST of G, then T is a MST of G.
- (b) If T is a MST of G, then T is a MST of the graph G' formed by squaring the edge weights of G.
- (c) If G has exactly |V| edges and has a unique heaviest edge e, then no MST of G contains e.

Hint: One takeaway from this problem is that MST's are slippery.

Solution. All three are statements are false! Here is a counterexample for each:



Show that the resulting tree is a MST.

Solution. We begin by describing an operation on spanning trees. Let T be a spanning tree of G, and let e be an edge not in T. Adding e to T creates a subgraph T + e with a unique cycle C containing e, and removing any edge e' in C results in a spanning tree T + e - e'. Picture:

Let us show that T+e-e' is a spanning tree. Recall that a spanning tree is a subgraph that is a tree with |V|-1 edges. Since T+e-e' has |V|-1 edges, it suffices to show it is a tree, *i.e.* has no cycles. Let D be a cycle in T+e; we will show that D contains e'. Since T has no cycles, D contains the edge e=(v,w). The rest of the cycle D-e is a path from v to w in T. But C-e is also a path from v to w in T, so since paths between two given vertices in a tree are unique, we have D-e=C-e. Thus D=C, which contains e'.

Now let us show that the resulting tree T of the algorithm is a MST. Let T' be a spanning tree of G. We will show that  $\operatorname{wgt}(T) \leq \operatorname{wgt}(T')$ , and we proceed by induction on the number n of edges in T that are not in T'. Of course if n=0, then T=T', so we are done. Otherwise, let e be an edge in T that is not in T'. We perform our operation: we add e to T', creating a subgraph T'+e with a unique cycle C, and we remove a maximal weight edge e' in C, resulting in a spanning tree T'+e-e'. Since T is the result of removing maximal weight edges, we have  $\operatorname{wgt}(e) \leq \operatorname{wgt}(e')$ . Thus

$$\operatorname{wgt}(T) \leq \operatorname{wgt}(T' + e - e')$$
 (by induction)  
=  $\operatorname{wgt}(T') + \operatorname{wgt}(e) - \operatorname{wgt}(e')$   
 $\leq \operatorname{wgt}(T'),$ 

as required.  $\Box$