182 Week 9 Discussion Notes

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This week for discussion section we will discuss two homework problems from a previous iteration of Math 182.

Notation. For today, let $G = (V, E)$ be a connected, weighted, undirected graph. We abbreviate “minimum spanning tree” as MST.

Problem 1. Prove or disprove:

(a) If T is a spanning tree of G such that every edge in T is in some MST of G, then T is a MST of G.

(b) If T is a MST of G, then T is a MST of the graph G' formed by squaring the edge weights of G.

(c) If G has exactly $|V|$ edges and has a unique heaviest edge e, then no MST of G contains e.

Hint: One takeaway from this problem is that MST's are slippery.

Solution. All three are statements are false! Here is a counterexample for each:

□

Recall. Prim's algorithm constructs a MST of G by growing a connected tree, at each stage extending the tree by one vertex along a minimal weight edge. For example:

On the other hand, Kruskal's algorithm constructs a MST of G by growing a forest, at each stage adding to the forest an edge of minimal weight that does not create a cycle. For example:

Problem 2. Here is another algorithm for constructing a MST of G. We start with G, and at each stage we remove a maximal weight edge that is in a cycle. For example:

Show that the resulting tree is a MST.

Solution. We begin by describing an operation on spanning trees. Let T be a spanning tree of G, and let e be an edge not in T. Adding e to T creates a subgraph $T + e$ with a unique cycle C containing e, and removing any edge e' in C results in a spanning tree $T + e - e'$. Picture:

Let us show that $T+e-e'$ is a spanning tree. Recall that a spanning tree is a subgraph that is a tree with $|V|-1$ edges. Since $T+e-e'$ has $|V|-1$ edges, it suffices to show it is a tree, *i.e.* has no cycles. Let D be a cycle in $T+e$; we will show that D contains e'. Since T has no cycles, D contains the edge $e = (v, w)$. The rest of the cycle $D - e$ is a path from v to w in T. But $C - e$ is also a path from v to w in T, so since paths between two given vertices in a tree are unique, we have $D - e = C - e$. Thus $D = C$, which contains e'.

Now let us show that the resulting tree T of the algorithm is a MST. Let T' be a spanning tree of G. We will show that $\mathrm{wgt}(T) \leq \mathrm{wgt}(T')$, and we proceed by induction on the number n of edges in T that are not in T'. Of course if $n = 0$, then $T = T'$, so we are done. Otherwise, let e be an edge in T that is not in T'. We perform our operation: we add e to T', creating a subgraph $T' + e$ with a unique cycle C, and we remove a maximal weight edge e' in C, resulting in a spanning tree $T' + e - e'$. Since T is the result of removing maximal weight edges, we have $\mathrm{wgt}(e) \leq \mathrm{wgt}(e')$. Thus

$$\begin{aligned}
\mathrm{wgt}(T) &\leq \mathrm{wgt}(T' + e - e') && \text{(by induction)} \\
&= \mathrm{wgt}(T') + \mathrm{wgt}(e) - \mathrm{wgt}(e') \\
&\leq \mathrm{wgt}(T'),
\end{aligned}$$

as required. $\square$