182 HW 6 Problem 2 Solution

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Recall Problem 2 from HW 6.

Problem 2. Given a directed acyclic graph $G = (V, E)$, determine whether G has a Hamiltonian path.

Our algorithm is as follows: take a topological sort $v_1, \ldots, v_n$ of the vertices V and then check whether $(v_i, v_{i+1}) \in E$ for all $j = 1, \ldots, n-1$. Proving correctness of this algorithm means proving the following statement:

Statement. There exists a Hamiltonian path in G if and only if for any topological sort $v_1, \ldots, v_n$ of our graph G we have $(v_i, v_{i+1}) \in E$ for all $j = 1, \ldots, n-1$.

Direct proof

Here is a direct proof of the Statement. Note that it is somewhat intricate and still missing some details.

Direct proof of statement. Suppose there exists a Hamiltonian path in G, and let us denote it by $v_1 \to \cdots \to v_n$. Let $[v_{i_1}, \ldots, v_{i_n}]$ be a topological sort of G.

We claim that this topological sort is unique in the sense that

$$[v_{i_1}, \ldots, v_{i_n}] = [v_1, \ldots, v_n].$$

It would then follow that $(v_{i_j}, v_{i_{j+1}}) = (v_j, v_{j+1}) \in E$ for all $j = 1, \ldots, n-1$, as required. To prove the claim, we proceed by induction. The case $n = 1$ is trivial. Consider the full subgraph G' of G containing the vertices $v_1, \cdots, v_{n-1}$, and let S denote the length $n-1$ subsequence of $[v_{i_1}, \ldots, v_{i_n}]$ that does not contain v_n. Then $v_1 \to \cdots \to v_{n-1}$ is a Hamiltonian path in G', and S is a topological sort of G'. Thus by induction $S = [v_1, \ldots, v_{n-1}]$. Since $(v_{n-1}, v_n) \in E$, the vertex v_n must come after S in the sequence $[v_{i_1}, \ldots, v_{i_n}]$, so in fact

$$[v_{i_1}, \ldots, v_{i_n}] = S + [v_n].$$

Note that the reverse direction is trivial because any directed acyclic graph has a topological sort. $\square$

Truth-revealing proof

Alternatively, here is in my opinion the best proof of the Statement. We start with the following general fact which uncovers the general relationship between paths in a DAG and path-subsequences of any topological sort.

Proposition. Let $G = (V, E)$ be a directed acyclic graph, and let $[v_1, \ldots, v_n]$ be a topological sort of G. Then there exists a length-preserving bijection

$$\{\text{paths in } G\} \cong \left\{ \begin{array}{c} \text{subsequences } [v_{i_1}, \ldots, v_{i_k}] \text{ of} \\ [v_1, \ldots, v_n] \text{ such that} \\ (v_{i_j}, v_{i_{j+1}}) \in E \text{ for all} \\ j = 1, \ldots, k-1 \end{array} \right\} \tag{1}$$

Proof. We will construct mutually inverse, length-preserving functions

$$\{\text{paths in } G\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{array}{c} \text{subsequences } [v_{i_1}, \ldots, v_{i_k}] \text{ of} \\ [v_1, \ldots, v_n] \text{ such that} \\ (v_{i_j}, v_{i_{j+1}}) \in E \text{ for all} \\ j = 1, \ldots, k-1 \end{array} \right\}.$$

To construct f, let $w_1 \to \cdots \to w_k$ be a path in G. Since $V = \{v_1, \ldots, v_n\}$, we can express our path uniquely as $v_{i_1} \to \ldots \to v_{i_k}$ for some $i_1, \ldots, i_k$. Consider the sequence $[v_{i_1}, \ldots, v_{i_k}]$. Certainly $(v_{i_j}, v_{i_{j+1}}) \in E$ for all $j = 1, \ldots, k-1$, and this sequence is a subsequence of $[v_1, \ldots, v_n]$ because $(v_{i_j}, v_{i_{j+1}}) \in E$ implies $i_j < i_{j+1}$, by virtue of $[v_1, \ldots, v_n]$ being a topological sort of G. We may thus define

$$f(w_1 \to \cdots \to w_k) = [v_{i_1}, \ldots, v_{i_k}].$$

To construct g, let $[v_{i_1}, \ldots, v_{i_k}]$ be a subsequence of $[v_1, \ldots, v_n]$ such that $(v_{i_j}, v_{i_{j+1}}) \in E$ for all $j = 1, \ldots, k-1$. Then of course $v_{i_1} \to \cdots \to v_{i_k}$ is a path in G, and we set

$$g([v_{i_1}, \ldots, v_{i_k}]) = v_{i_1} \to \cdots \to v_{i_k}.$$

It is clear that they are mutually inverse (*i.e.* fg and gf are both the identity functions) because they do not change the vertices in a path or subsequence or their order, and they visibly preserve lengths. $\square$

The statement now slides out of the general fact.

Proof of statement. Suppose there exists a Hamiltonian path in G, and let $v_1, \ldots, v_n$ be a topological sort of our graph G. Observe that a Hamiltonian path G is the same thing as a length n path in G and that a subsequence of $v_1, \ldots, v_n$ of length n must be the entire sequence $v_1 \ldots, v_n$. Thus, restricting our length-preserving bijection (1) to length n paths and subsequences gives

$$\left\{ \begin{array}{c} \text{Hamiltonian} \\ \text{paths in } G \end{array} \right\} \cong \begin{cases} \{[v_1, \ldots, v_n]\} & \text{if } (v_i, v_{i+1}) \in E \text{ for all } j = 1, \ldots, n-1 \\ \{\} & \text{otherwise.} \end{cases}$$

Since there exists a Hamiltonian path in G, the LHS is nonempty, so the RHS is nonempty. Thus $(v_i, v_{i+1}) \in E$ for all $j = 1, \ldots, n-1$.

Note that the reverse direction is trivial because any directed acyclic graph has a topological sort. $\square$