ALGEBRAIC TOPOLOGY DONE RIGHT

COLIN NI

Abstract. This paper exposits parts of May [1] and the first chapter of Bott and Tu [2]. These two books take two different approaches to algebraic topology, both strikingly different from the one taken in Hatcher [3]. The title of this paper refers to the former one, in the author's impetuous opinion.

Contents

The category of smooth manifolds lacks desirable formal properties. To put it bluntly, the objects are nice, but the category is terrible. The smooth category does not have natural exponential objects. It is not closed under quotients, equalizers, or coequalizers. It is not closed under pullbacks. In particular, an intersection of smooth manifolds is certainly not in general a smooth manifold, and even when it is, its codimension is poorly behaved. Instead, the de Rham theory takes advantage of the nice tools and operations that come with smooth manifolds. Integration of forms, partitions of unity, good covers, and even elementary calculus play a foundational role in developing the theory, and more than that, they make the de Rham theory explicit and computable.

In opposition, the category of compactly generated spaces has all the properties one could wish for. It is complete and cocomplete. Its limits, sequential colimits, and pushouts are well behaved. It not only has exponential objects, but these objects satisfy adjoint relationships not just as sets but as spaces. Any reasonable construction is well behaved, so typical constructs from the categorical and homotopical viewpoint such as cofiber sequences, generalized cohomology theories, and Thom spaces often involve swaths of formal properties and terribly enormous objects. As a result, this modern viewpoint is abstract and difficult but powerful.

1. The categorical and homotopical viewpoint

Category theory is the language of algebraic topology. Even the core idea of algebraic topology, transforming topological problems into algebraic ones, is most accurately expressed through functoriality. Using category theory at the outset clarifies results and often even makes them easier to prove. Moreover, the language

Version: June 16, 2019

of category theory is so ingrained in the deeper theory, for instance of characteristic classes, K-theory, and cobordism, that avoiding the language would be harmful to further study. *Concise* upholds these virtues. Furthermore, *Concise* swears by cofiber and fiber sequences, which are derived homotopically, to thematically develop much of the theory it covers, and these sequences are similarly incorporated into the deeper theory.

The van Kampen theorem for fundamental groupoids, which specializes to the van Kampen theorem for fundamental groups, is a playful but useful example of the power of categorical language. Groupoids are the same thing as categories where every morphism is an isomorphism, and groups are such categories that contain only one object. The *fundamental groupoid* $\Pi(X)$ of a space X is constructed to have objects the points of x and morphisms $\mathrm{Mor}(x, y)$ the homotopy classes of paths $x \to y$. Thus a valid definition for $\pi_1(X, x_0)$ is the full subcategory x_0, whence $\pi_1(X, x_0)$ is a skeleton for $\Pi(X)$ and therefore is equivalent to it as categories. Note that this construction $\Pi(X)$ is functorial, so the fundamental groupoid is a functor

$$\Pi\colon \mathrm{Top} \to \mathrm{Groupoid}.$$

The van Kampen theorem for fundamental groupoids then states that given an open cover $\mathscr{O} = \{U\}$ consisting of path connected sets that is closed under finite intersection, we have

$$\Pi(X) \cong \mathrm{colim}_{U\in\mathscr{O}}\Pi(U),$$

where the colimit is taken over the diagram $\Pi|_{\mathscr{O}}\colon \mathscr{O} \to \mathrm{Groupoid}$ and where $\mathscr{O}$ is viewed as a category with morphisms the inclusions. The proof of van Kampen for groupoids is simpler because it does not involve a basepoint. There is no need for trips back to x_0 to make sense of a composition of loops in $\pi_1(X, x_0)$ because in $\Pi(X)$ everything that makes sense composes.

It is convenient in discussing the theory of cofibrations, fibrations, and their sequences to view homotopies $X \times I \to Y$ adjointly as $X \to Y^I$. More generally, we crave a homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$ so that for instance in the based setting we would have $[\Sigma X, Y] \cong [X, \Omega Y]$, where we recall the definitions $\Sigma X = S^1 \wedge X$ and $\Omega Y = Y^{S^1}$ and the fact $\pi_0(Y^X) = [X, Y]$. They obviously are in bijection as sets, but in Top when equipped with the usual compact open topologies, they are not in general homeomorphic under the natural bijection. In fact the core-compact spaces in Top are precisely the spaces that satisfy this duality and thus are known as the exponentiable spaces. These include the locally compact Hausdorff spaces, but the naive attempt of restricting attention to the full subcategory of such spaces does not work because the resulting category, among other issues, is neither complete nor cocomplete.

A preliminary before discussing (co)fibrations is therefore in order, namely to decide on a *convenient category of topological spaces* in which to work. Hilariously, this is a known technical term coined by Steenrod to refer to a cartesian closed, complete, and cocomplete category in which every CW complex is an object. The primary example of such a category is the category $\mathscr{U}$ of weak Hausdroff k-spaces, which *Concise* calls the compactly generated spaces. This category is where most modern algebraic topology takes place. Most familiar spaces are compactly generated, for instance metric spaces, topological manifolds, and CW complexes.

In $\mathscr{U}$ the desired homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$ holds, and furthermore the popular constructions such as limits, pushouts, and sequential colimits are still valid and still behave nicely with respect to core theory of algebraic topology. In other words $\mathscr{U}$ nicely retains the completeness and most of the cocompleteness of Top. The overarching mechanics are summarized in the following diagram of categories and adjoint functors:

$$\begin{tikzcd}
\mathscr{U} = \text{compactly generated spaces} \arrow[d, shift right, "\text{forget}"'] \\
w\mathscr{U} = \text{weak Hausdorff spaces} \arrow[u, shift right, "k\text{-ification}"'] \arrow[u, phantom, "\dashv"] \arrow[d, shift left, "\text{forget}"] \\
\text{Top} \arrow[u, shift left, "(\text{weak Hausdorff})\text{-ification}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

Recalling that left adjoints preserve colimits and that right adjoints preserve limits, the functors move these constructions into $\mathscr{U}$ without, under the lens of most constructions in algebraic topology, affecting the spaces. Taking a limit in $\mathscr{U}$ versus in Top differs at most by a k-ification, which is not detectable by maps from compact spaces such as spheres and simplexes. Even better, pushouts and sequential limits in $\mathscr{U}$ are just the point set limits, and these comprise most of the colimits taken in algebraic topology.

Detail. We cite the facts that point set limits of weak Hausdorff spaces are weak Hausdorff and that if a point set colimit of compactly generated spaces is weak Hausdorff, then it is also a k-space and hence compactly generated. The left adjoint (weak Hausdorff)-ification functor carries colimits over from Top, and the right adjoint k-ification functor carries limits over from $w\mathscr{U}$. Finally, $\mathscr{U}$ is cartesian closed and moreover admits homeomorphisms $Z^{X\times Y} \cong (Z^Y)^X$ for the following reasons. If Y is a k-space, then Z^Y is a k-space. Similarly, if X and Y are k-spaces, then $Z^{X\times Y} \cong (Z^Y)^X$ is a homeomorphism. Finally, if Z is weak Hausdorff, then Z^X is weak Hausdorff.

The k-ification and (weak Hausdorff)-ification functors behave well in algebraic topology. Indeed, k-ifying a space simply considers more subsets to be closed, namely the subsets which the compact spaces consider closed, or to be precise those subsets A such that $g^{-1}(A)$ is closed for every map g from a compact space. Moreover, pushouts and sequential colimits of compactly generated spaces are weak Hausdorff. □

Let us finally discuss (co)fibrations. A map $i\colon A \to X$ is a *cofibration* if it satisfies the homotopy extension property (HEP):

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & Y^I \arrow[d, "-(0)"] \\
X \arrow[r] \arrow[ur, dotted, "\exists!"] & Y.
\end{tikzcd}$$

Pushouts preserve cofibrations, and any map $f\colon X \to Y$ factors via the mapping cylinder $Mf = Y \cup_f (X \times I)$ into a cofibration followed by a homotopy equivalence $X \to Mf \to Y$. In particular, whether i is a cofibration is determined solely by whether the mapping cylinder Mi fills out the diagram because it is initial among the objects that do. Therefore Mi is a retract of $X\times I$ if and only if i is a cofibration.

Dually, a surjective map $p\colon E \to B$ is a *fibration* if it satisfies the homotopy covering property (CHP):

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r] \arrow[ur, dotted, "\exists!"] & B. \end{tikzcd}$$

Many results dualize: pullbacks preserve fibrations, any map $f\colon X \to Y$ factors via the mapping path space $Nf = X \times_f Y^I$ into a homotopy equivalence followed by a fibration $X \to Nf \to Y$, and whether p is a fibration is determined solely by the mapping path space Np. The analogy ends with detecting fibrations, where now the condition is that p is a fibration if and only if its restriction $p^{-1}(U) \to U$ is a fibration for every U in a numerable open cover of B.

The fundamental sequences of based spaces are the cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma Cf \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \cdots$$

for a based map $f\colon X \to Y$ and the fiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Ff \xrightarrow{\pi} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega i} \Sigma Ff \xrightarrow{-\Omega\pi} \Omega^2 X \xrightarrow{\Omega^2 f} \cdots .$$

Here Cf is the *homotopy cofiber* $Y \cup_f CX$, and Ff is the *homotopy fiber* $X \times_f PY$. Respectively these yield the long exact sequences

$$\cdots \to [\Sigma^2 X, Z] \to [\Sigma Cf, Z] \to [\Sigma Y, Z] \to [\Sigma X, Z] \to [Cf, Z] \to [Y, Z] \to [X, Z]$$

and

$$\cdots \to [Z, \Omega^2 Y] \to [Z, \Omega Ff] \to [Z, \Omega X] \to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y]$$

consisting of abelian groups, groups, or pointed sets depending on location.

These fundamental sequences are core to the homotopical viewpoint, and they are entrenched in much of the core theory. For instance, they give the long exact sequence

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(X)$$

of homotopy groups for a pair (X, A) by setting $Z = S^0$ to obtain

$$\cdots \to [S^0, \Omega^n A] \to [S^0, \Omega^n X] \to [S^0, \Omega^{n-1} Fi] \to [S^0, \Omega^{n-1} A] \to \cdots \to [S^0, X],$$

where $i\colon A \to X$ is the inclusion. Indeed by adjunction

$$[S^0, \Omega^n -] = [\Sigma^n S^0, -] = \pi_n(-),$$

and by definition $\pi_n(X, A) = \pi_{n-1}(P(X, A))$ for $n \geq 1$ where $P(X, A)$ is the space of paths in X that begin at the basepoint and end in A. In a similar fashion, we obtain the long exact sequence

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E)$$

for a fibration. In proving that a reduced homology theory determines a homology theory, the fundamental sequences are used in constructing the connecting homomorphism via

$$E_q(X, A) = \tilde{E}_q(X_+/A_+) \xrightarrow{\partial_*} \tilde{E}_q(\Sigma A_+) \xrightarrow{\Sigma^{-1}} \tilde{E}_{q-1}(A_+) = E_{q-1}(A),$$

where Σ is the assumed natural isomorphism.

Perhaps most importantly, these fundamental sequences provide the means for constructing cellular homology from scratch, and in turn this construction is core to proving the uniqueness of this ordinary homology theory on CW complexes.

Detail. As usual, $C_*(X)$ is defined to be the free abelian group generated by the n-cells j. Given an n-cell j, set

$$\begin{aligned} d_n \colon C_n(X) &\longrightarrow C_{n-1}(X) \\ j &\longmapsto \textstyle\sum_i a_{ij} i, \end{aligned}$$

where for an $(n-1)$-cell i the coefficient a_{ij} is the degree of the composite

$$S^{n-1} \xrightarrow{\Phi_j} X^{n-1} \xrightarrow{\rho} X^{n-1}/X^{n-2} \xrightarrow{\pi_i} S^{n-1}.$$

We have defined the differentials d_n in terms of the cell structure, but to prove $d^2 = 0$ we now realize them in terms of the cofiber sequence. The compositions

$$\partial_n \colon X^n/X^{n-1} \xrightarrow{\text{h.e.}} C(X^{n-1} \hookrightarrow X^n) \xrightarrow{\pi} \Sigma X^{n-1} \xrightarrow{\Sigma\rho} \Sigma(X^{n-1}/X^{n-2})$$

induce d_n via

$$C_n(X) = \tilde{H}'_n\left(\frac{X^n}{X^{n-1}}\right) \xrightarrow{(\partial_n)_*} \tilde{H}'_n\left(\Sigma\frac{X^{n-1}}{X^{n-2}}\right) \xrightarrow[\cong]{\Sigma^{-1}} \tilde{H}'_{n-1}\left(\frac{X^{n-1}}{X^{n-2}}\right) = C_{n-1}(X),$$

where $\tilde{H}'_n$ denotes the homotopy groups but altered in degree zero by removing the basepoint component and in degree one by abelianizing. To now see that $d^2 = 0$, observe first that the following diagram is homotopy commutative:

$$\begin{tikzcd} X^n \cup CX^{n-1} \arrow[r, "\pi"] \arrow[d, "\text{h.e.}"] & \Sigma X^{n-1} \arrow[r, "\Sigma i"] \arrow[d, "\Sigma\rho"] & \Sigma(X^{n-1} \cup CX^{n-2}) \arrow[r, "\Sigma\pi"] \arrow[d, "\Sigma(\text{h.e.})"] & \Sigma^2 X^{n-2} \arrow[d, "\Sigma^2\rho"] \\ X^n/X^{n-1} \arrow[r, "\partial_n"] & \Sigma\dfrac{X^{n-1}}{X^{n-2}} \arrow[r, equal] & \Sigma\dfrac{X^{n-1}}{X^{n-2}} \arrow[r, "\Sigma\partial_{n-1}"] & \Sigma^2\dfrac{X^{n-2}}{X^{n-3}}. \end{tikzcd}$$

This shows $\Sigma\partial_{n-1} \circ \partial_n$ is nullhomotopic since $\Sigma\pi \circ \Sigma i$ is trivial, so passing to $\tilde{H}'_n$

$$d^2 = \Sigma^{-1} \circ (\partial_{n-1})_* \circ \Sigma^{-1} \circ (\partial_n)_* = \Sigma^{-1} \circ \Sigma^{-1} \circ (\Sigma\partial_{n-1})_* \circ (\partial_n)_*. \qquad \square$$

2. de Rham cohomology

Compared to singular cohomology or the study of generalized cohomology theories, de Rham cohomology has a distinct flavor. This is because the smooth category is poorly behaved formally but has nice objects. Recall the introduction of this paper which spells out some of its shortcomings and redeeming qualities.

Differential forms generalize integrands to manifolds. On $\mathbb{R}^n$, a differential form looks like an integrand $f\,dx_{i_1}\dots dx_{i_k}$ consisting of a smooth function f and differentials dx_i, where swapping consecutive differentials flips the sign and repeated differentials become zero. To be more precise, define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1, \dots, dx_n$ with the relations $dx_i^2 = 0$ and $dx_i dx_j = -dx_j dx_i$ for $i \neq j$, then set

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_R \Omega^*.$$

We interpret an element $f\,dx_{i_1}\dots dx_{i_k}$ to be integrable on a k-dimensional surface in $\mathbb{R}^n$ in the usual way from multivariable calculus, and we grade $\Omega^*(\mathbb{R}^n)$ in this fashion. The usual calculus operations gradient, curl, and divergence in $\mathbb{R}^3$ are maps

$$\Omega^0(\mathbb{R}^3) \xrightarrow{\nabla(-)} \Omega^1(\mathbb{R}^3) \xrightarrow{\nabla\times -} \Omega^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot(-)} \Omega^3(\mathbb{R}^3)$$

where for instance

$$\begin{aligned} d(f\,dx + g\,dy + h\,dz) = (\partial_y f - \partial_x g)dx\,dy \\ +(\partial_z f - \partial_x h)dx\,dz + (\partial_z g - \partial_y h)dy\,dz. \end{aligned}$$

They are generalized by the exterior derivative

$$d(f\,dx_I) = \sum_{i=1}^{n} \partial_{x_i} f dx_I\,dx_i,$$

and the identities $\nabla\times\nabla(-) = 0$ and $\nabla\cdot(\nabla\times(-)) = 0$ carry over to $d^2 = 0$. Hence we have the *de Rham chain complex*

$$0 \longrightarrow \Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \Omega^2(\mathbb{R}^n) \xrightarrow{d} \cdots$$

in which forms in the kernel of d are called *closed* and forms in the image of d are called *exact*. Its homology is called the *de Rham cohomology*. Since forms pull back along smooth maps $\mathbb{R}^m \to \mathbb{R}^n$, in fact Ω^* is a contravariant functor from Euclidean spaces and smooth maps to commutative differential graded algebras and their homomorphisms. The generalization to a manifold arises from functoriality by demanding compatibility with restrictions, so in other words Ω^* extends to a contravariant functor on smooth manifolds. For instance, on a manifold M with atlas $\phi_\alpha\colon U_\alpha \to \mathbb{R}^n$, a differential form is a collection of forms ω_U such that the restrictions

$$\begin{tikzcd} \Omega^*(U) \arrow[dr, "\mathrm{incl}_*"] & \\ & \Omega^*(U\cap V) \\ \Omega^*(V) \arrow[ur, "\mathrm{incl}_*"'] & \end{tikzcd}$$

of ω_U and ω_V agree in $\Omega^*(U\cap V)$. Similarly, we obtain the de Rham chain complex of a manifold and hence its de Rham cohomology.

Unlike singular cohomology, de Rham cohomology is computable from its definition and often yields explicit representatives of classes. For instance, the form

$$\frac{1}{r}\sum_{i=1}^{n+1}(-1)^{i-1}x_i\,dx_1\,\cdots\,\widehat{dx_i}\,\cdots\,dx_{n+1}$$

generates $H^n(S^n(r)) = \mathbb{R}$ where $S^n(r) \subset \mathbb{R}^{n+1}$ is the sphere of radius r. An excellent exercise to familiarize oneself with the definitions is the involved computation

$$H^*(\mathbb{R}^2 - 0) = \begin{cases} \mathbb{R} & \text{in dimensions 0 and 1} \\ 0 & \text{elsewhere.} \end{cases}$$

Detail. Denote $X = \mathbb{R}^2 - 0$. The de Rham complex is

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d_0} \Omega^1(X) \xrightarrow{d_1} \Omega^2(X) \longrightarrow 0 \longrightarrow \cdots,$$

where $\Omega^3(X) = 0$ by the pigeonhole principle.

To see that $H^0(X) = \mathbb{R}$, given $f \in \Omega^0(X)$, compute $d_0 f = \partial_x f dx + \partial_y f dy$ so that $d_0 f = 0$ if and only if $\partial_x f = \partial_y f = 0$ if and only if f is constant.

For $H^1(X) = \mathbb{R}$, first define

$$\begin{aligned} \lambda\colon \Omega^1(X) &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \textstyle\int_{\partial R} \omega, \end{aligned}$$

where R is any rectangle containing the origin and where by Green's theorem the choice of R does not affect the value of the integral. This map detects exact 1-forms in the sense that $\ker \lambda = \operatorname{im} d$. Indeed if ω is a gradient, then the fundamental theorem of path integrals implies $\lambda(\omega) = 0$. Conversely, let $\omega = f\,dx + g\,dy$ and assume $\lambda(\omega) = 0$. Define a smooth map $\alpha\colon f\colon \mathbb{R}^2 - 0 \to \mathbb{R}$ by fixing a point x_0 away from the origin and setting $\alpha(x) = \int_\gamma \omega$, where γ is a path from x_0 to x along the boundary of a rectangle; the condition $\lambda(\omega) = 0$ ensures α is well defined. Choosing γ to approach x horizontally gives $\partial_x \alpha = f$ and vertically gives $\partial_y \alpha = g$. Hence $d\alpha = \omega$ exhibits the exactness of ω. For a generator of $H^1(X) = \mathbb{R}$, observe that the form

$$d\theta = \frac{-xdy + ydx}{x^2 + y^2}$$

is closed by a straightforward computation $d(d\theta) = 0$, and it is exact (despite this standard notation) because $\lambda(d\theta) = 2\pi \neq 0$. Now the first cohomology is spanned by $d\theta$ since any closed form $\omega \in \Omega^1(X)$ may be written as a sum of an exact form and an element in the span of $d\theta$:

$$\omega = \left(\omega - \frac{\lambda(\omega)}{2\pi} d\theta\right) + \frac{\lambda(\omega)}{2\pi} d\theta.$$

Finally $H^2(X) = 0$ because every 2-form on X is exact. Indeed any $\omega = f\,dr\,d\theta \in \Omega^2 X$ is such that

$$d\int_1^r -f(\rho,\theta)\,d\rho\,d\theta = \partial_r \int_1^r -f(\rho,\theta)\,d\rho\,d\theta dr = f dr d\theta. \qquad \square$$

It is a major theorem, namely de Rham's, that de Rham cohomology agrees with singular cohomology on smooth manifolds. In particular, de Rham cohomology is indeed invariant of homotopy type, which is immediate by the Poincaré lemma which computes

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Detail. The Poincaré lemma is proved by showing that integration along fibers

$$\begin{aligned} \Omega^q(M \times \mathbb{R}) &\longrightarrow \Omega^{q-1}(M \times \mathbb{R}) \\ (\pi^*\phi) f(x,t) &\longmapsto 0 \\ (\pi^*\phi) f(x,y)\,dt &\longmapsto (\pi^*\phi) \textstyle\int_0^t f. \end{aligned}$$

form a chain homotopy from $\pi^* \circ s^*$ to the identity, where π^* and s^* are the projection and zero section of $M \times \mathbb{R} \to M$. Since $s^* \circ \pi^*$ is trivially the identity, this shows π^* and s^* are inverse isomorphisms. Now given a smooth homotopy $M \times I \to N$ between two maps $f, g\colon M \to N$, extend it in the obvious way to $M \times \mathbb{R} \to N$. The zero and one sections $s_0, s_1\colon M \to M \times \mathbb{R}$ agree on cohomology because they both invert π^*, so $f^* = s_1^* \circ F^* = s_0^* \circ F^* = g^*$. $\square$

The Mayer-Vietoris argument is core in the de Rham setting, and it is the theme in most proofs of results from singular cohomology. It works well in the category of smooth manifolds because it relies on partitions of unity and having a *good cover*, an open cover where all nonempty finite intersections are diffeomorphic to $\mathbb{R}^n$. A partition of unity is used in establishing the Mayer-Vietoris sequence, in particular to fill out surjectivity in the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(M) & \xrightarrow{(\mathrm{incl}_*,\mathrm{incl}_*)} & \Omega^*(U)\oplus\Omega^*(V) & \xrightarrow{\text{difference}} & \Omega^*(U\cap V) & \longrightarrow & 0 \\ & & & & (-\rho_V\omega,\rho_U\omega) & \longmapsto & \omega, & & \end{array}$$

where ρ is subordinate to the cover U, V. As a bonus, tracing the connecting homomorphism given by the snake lemma

$$\begin{tikzcd} & (-\rho_V\omega,\rho_U\omega) \arrow[r, mapsto] \arrow[d, mapsto] & \omega \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\ d^*\omega \arrow[r] & (-d(\rho_V\omega), d(\rho_U\omega)) \arrow[r, mapsto] & 0, & \end{tikzcd}$$

the coboundary map in the Mayer-Vietoris sequence is given explicitly by

$$d^*\omega = \begin{cases} -d(\rho_V\omega) & \text{on } U \\ d(\rho_U\omega) & \text{on } V. \end{cases}$$

Furthermore, every (compact) manifold has a (finite) good cover by endowing it with a Riemannian structure and then citing from differential geometry the existence around any point of, using the language of do Carmo, a totally normal neighborhood, which satisfies the requirements.

The Mayer-Vietoris argument reasons that if a property of $\mathbb{R}^n$ holds on $U\cup V$ whenever it holds for open sets U, V, and $U\cap V$, then it must hold on any manifold with a finite good cover. This works by induction because good sets are by definition diffeomorphic to $\mathbb{R}^n$ and because $U_n\cap(U_0\cup\ldots\cup U_{n-1})$ has a good cover consisting of the n sets $U_0\cap U_n,\ldots,U_{n-1}\cap U_n$.

To illustrate the power of the Mayer-Vietoris argument, let us use it to prove the following three theorems for a manifold M with a finite good cover, namely the finite dimensionality of de Rham cohomology, Poincaré duality, and the Thom isomorphism theorem. The same kind of argument is also used to prove the Künneth formula and the Leray-Hirsch theorem.

Theorem. The de Rham cohomology $H^*(M)$ is finite dimensional.

Proof. The Poincaré lemma establishes that $H^*(\mathbb{R}^n)$ is finite dimensional. Moreover, if $H^*(U)$, $H^*(V)$, and $H^*(U\cap V)$ are finite dimensional, then so is $H^*(U\cup V)$ since by the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^{q-1}(U\cap V) \xrightarrow{d^*} H^q(U\cup V) \xrightarrow{r} H^q(U)\oplus H^q(V) \longrightarrow \cdots$$

and by linear algebra $H^q(U\cup V) = \operatorname{im} d^* \oplus \operatorname{im} r$. □

Theorem. (Poincaré duality.) The pairing

$$\int(-\wedge-)\colon H^q(M)\otimes H_c^{n-q}(M)\to\mathbb{R}$$

is nondegenerate. In particular, $H^q(M)\cong H_c^{n-q}(M)^*$.

Proof. The pairing is nondegenerate on $H^*(\mathbb{R}^n)$ because it is nondegenerate on the bump form generator. The leg work goes into showing that

$$\begin{array}{ccccccccc}\cdots & \longrightarrow & H^q(U\cup V) & \longrightarrow & H^q(U)\oplus H^q(V) & \longrightarrow & H^q(U\cap V) & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle\int_{U\cup V}} & & \Big\downarrow{\scriptstyle\int_U+\int_V} & & \Big\downarrow{\scriptstyle\int_{U\cap V}} & & \\ \cdots & \longrightarrow & H_c^{n-q}(U\cup V)^* & \longrightarrow & H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^* & \longrightarrow & H_c^{n-q}(U\cap V)^* & \longrightarrow & \cdots\end{array}$$

commutes, which involves working with the coboundary operator for the Mayer-Vietoris sequence for compact support. By the five lemma, if the middle and right-hand pairings are nondegenerate, then so is the left-hand one. □

To formulate the Thom isomorphism, we require the notion of the compact vertical cohomology of an oriented n-dimensional vector bundle E over M, which arises from the forms $\omega\in\Omega^*_{cv}(E)$ that are compact along fibers or, more precisely, are such that $\pi^{-1}(K)\cap\operatorname{Supp}\omega$ is compact for every compact $K\subset M$. Given a function $f(x,t_1,\dots,t_n)$ on the bundle E with compact support for every x and a form ϕ on M, we may integrate along fibers as in the proof of the Poincaré lemma. Explicitly, over $\mathbb{R}^n$ we define the integration along fibers map by

$$\begin{array}{rcl}\Omega^*_{cv}(E) & \xrightarrow{\quad\pi_*\quad} & \Omega^{*-n}(M)\\ (\pi^*\phi)f(x,t_1,\dots,t_n)\,dt_1\dots dt_n & \longmapsto & \phi\int_{\mathbb{R}^n} f(x,t_1,\dots,t_n)dt_1\dots dt_n\\ (\pi^*\phi)f(x,t_1,\dots,t_n)\,dt_{i_1}\;\dots\;dt_{i_r} & \longmapsto & 0 \text{ where } r<n.\end{array}$$

Over a manifold, we can patch together forms that are defined on trivializations using the fact that on intersections the different fiber coordinates are related by an element of $\mathrm{GL}_n^+(\mathbb{R})$. Integration by forms commutes with the exterior derivative and satisfies the projection formula

$$\pi_*((\pi^*\tau)\cdot\omega)=\tau\cdot\pi_*\omega.$$

Theorem. (Thom isomorphism.) $H^*_{\mathrm{cv}}(E)\cong H^{*-n}(M)$ for an orientable n-dimensional vector bundle E over M.

Proof. Over $\mathbb{R}^n$ this reduces to the corresponding Poincaré lemma for compact vertical supports. Again we invoke the five lemma, so the leg work goes into checking commutativity of the diagram

$$\begin{array}{ccccccc}H^*_{cv}(E|_{U\cup V}) & \longrightarrow & H^*_{cv}(E|_U)\oplus H^*_{cv}(E|_V) & \longrightarrow & H^*_{cv}(E|_{U\cap V}) & \xrightarrow{d^*} & H^{*+1}_{cv}(E|_{U\cup V})\\ \Big\downarrow{\scriptstyle\pi_*} & & \Big\downarrow{\scriptstyle\pi_*} & & \Big\downarrow{\scriptstyle\pi_*} & & \Big\downarrow{\scriptstyle\pi_*}\\ H^{*-n}(U\cup V) & \longrightarrow & H^{*-n}(U)\oplus H^{*-n}(V) & \longrightarrow & H^{*-n}(U\cap V) & \xrightarrow{d^*} & H^{*-n+1}(U\cup V).\end{array}$$

The first two squares are straightforward, and the right-hand square involves using the explicit formula for the coboundary operator and using the projection formula:

$$\pi_*d^*\omega=\pi_*d((\pi^*\rho_U)\omega)=\pi_*((\pi^*d\rho_U)\cdot\omega)=(d\rho_U)\cdot\pi_*\omega=d(\rho_U\pi_*\omega)=d^*\pi_*\omega.\qquad\square$$

Certain things such as the Poincaré dual of a submanifold and the Thom class of a vector bundle can be realized concretely in the de Rham setting. If M is an oriented n-manifold and S is a closed oriented k-submanifold, then the *Poincaré dual* of S is the cohomology class $\eta_S\in H^{n-k}(M)$ is constructed as follows. Given a closed k-form ω with compact support in M, the restriction $i^*\omega$ has compact support in S, so $\int_S i^*\omega$ is defined. This linear functional on $\Omega_c^k(M)$ descends by Stoke's theorem to a linear functional on $H_c^k(M)$, hence by Poincaré duality defines an element $\eta_S \in H^{n-k}(M)$. It is characterized as being the unique form satisfying

$$\int_S i^*\omega = \int_M \omega \cdot \eta_S.$$

For example, the Poincaré dual of the ray $\{(x,0) \mid x > 0\}$ in $\mathbb{R}^2 - 0$ is $d\theta/2\pi \in H^1(\mathbb{R}^2 - 0)$.

Detail. Let $f dr + g d\theta \in H_c^1 M$ so that $\partial_\theta f = \partial_r g$ and f, g are compactly supported. Observe

$$\partial_\theta \int_0^\infty f(r,\theta)\, dr = \int_0^\infty \partial_\theta f(r,\theta)\, dr = \int_0^\infty \partial_r g(r,\theta)\, dr = 0$$

since g has compact support. Since r is a valid coordinate on S we have

$$\int_S (f dr + g d\theta) = \int_0^\infty f(r,0)\, dr = \frac{1}{2\pi}\int_M f\, dr\, d\theta = \int_M (f dr + g d\theta) \wedge \frac{d\theta}{2\pi}$$

where the second equality holds by the observation. □

Along the same lines, the Thom class $\Phi \in H_{cv}^n(E)$ of an n-dimensional oriented vector bundle E is the form in $H_{\mathrm{cv}}^n(E)$ constructed as the image of $1 \in H^0(M)$ under the Thom isomorphism $\mathscr{T} \colon H^*(M) \cong H_{cv}^{*+n}(E)$. As in singular cohomology, it is the unique such class which restricts to the generator $H_{\mathrm{c}}^n(F)$ for each fiber F and realizes the Thom isomorhpism via

$$\mathscr{T}(-) = \pi^*(-) \cdot \Phi.$$

In the de Rham setting it is furthermore the inverse to integration on fibers since

$$\pi_* \mathscr{T}\omega = \pi_*(\pi^*\omega \cdot \Phi) = \omega \cdot \pi_*\Phi = \omega$$

by the projection formula.

Particularly concrete, the Poincaré dual of S is the Thom class of its normal bundle $\nu(S)$. Recall that $\nu(S)$ may be defined by the exact sequence

$$0 \longrightarrow \tau(S) \longrightarrow \tau(M)|_S \longrightarrow \nu(S) \longrightarrow 0$$

of vector bundles, oriented to satisfy $\nu(S) \oplus \tau(S) = \tau(M)|_S$. Further, recall that by the tubular neighborhood theorem $\nu(S)$ is diffeomorphic to a tubular neighborhood T of S. Hence the Thom isomorphism fits into the composition

$$H^*(S) \xrightarrow{\mathscr{T}} H_{\mathrm{cv}}^{*+n-k}(T) \xrightarrow{j_* = \text{ extend by zero}} H^{*+n-k}(M).$$

In symbols, then, the claim is that $\eta_S = j_*\Phi$, and checking this amounts to checking that $j_*\Phi$ satisfies the characterization of the Poincaré dual η_S.

Detail. Let $\omega \in H_{\mathrm{c}}^k(M)$, let $\pi \colon S \to T$ be the projection, and let $i \colon T \to S$ be the zero section. Note that π^* and i_* are inverses on cohomology because π is a deformation retraction, so ω and $\pi^* i^* \omega$ differ by an exact form. Hence

$$\int_M \omega \cdot j_*\Phi = \int_T \omega \cdot \Phi = \int_T \pi^* i^* \omega \cdot \Phi = \int_S i^*\omega \cdot \pi_*\Phi = \int_S i^*\omega,$$

respectively since $j_*\Phi$ has support in T, Stoke's theorem, the projection formula, and the characterization of the Thom class. □

3. The modern approach to homology and cohomology

The construction of singular (co)homology via singlar (co)chain complexes is at odds with the modern approach to (co)homology. While it is intuitive and concrete, it does not shine brightly enough to expose the surrounding cohesive study of generalized (co)homology theories which pervades modern algebraic topology. Singular (co)homology is only one of many instances of a generalized (co)homology theory, namely the one defined by the Eilenberg-MacLane (Ω-)spectrum. Importantly, this abstract view has the benefit of easing the study of, for instance, cohomology operations and characteristic classes, and the way of thinking extends into areas such as K-theory and cobordism.

The following axiomization of generalized cohomology theories encompasses the one by Eilenberg and Steenrod for ordinary cohomology. A *generalized cohomology theory* is a sequence of functors $E^q(X,A;\pi)\colon (h\mathscr{U}^2)^{\mathrm{op}} \to \mathrm{Ab}$ and natural transformations $\delta^q\colon E^q(A) \to E^{q+1}(X,A;\pi)$ that satisfy the usual exactness, excision, and additivity axioms. It is well known that singular cohomology satisfies the additional dimension axiom, which demands that $E^q(S^0;\pi)$ is π in degree zero and zero in other degrees, and that such cohomology theories agree on CW complexes. A *reduced generalized cohomology theory*, which reduced singular cohomology is certainly an example of, is a sequence of functors $\tilde{E}^q\colon (h\mathscr{T})^{\mathrm{op}} \to \mathrm{Ab}$ satisfying the following axioms:

- Exactness: If $i\colon A \to X$ is a cofibration, then the sequence $\tilde{E}^q(X/A) \to \tilde{E}^q(X) \to \tilde{E}^q(A)$ is exact.
- Suspension: There are natural isomorphisms $\Sigma\colon \tilde{E}^q(\Sigma X) \cong \tilde{E}^{q+1}(X)$
- Additivity: If $X = \bigvee X_i$, then the inclusions $X_i \to X$ induce an isomorphism $\tilde{E}^*(X) \to \prod_i \tilde{E}^*(X_i)$
- Weak equivalence: If $f\colon X \to Y$ is a weak equivalence, then $f_*\colon \tilde{E}^*(Y) \to \tilde{E}^*(X)$ is an isomorphism of abelian groups.

These axioms seem weaker, for instance because they lack long exactness and excision axioms. However, a generalized cohomology theory in fact determines and is determined by a reduced generalized cohomology theory. Moreover, by CW approximation, for which there is an analogy for based spaces, such theories are determined on their corresponding category of CW complexes. The following diagram summarizes the results:

$$\begin{tikzcd}
\begin{array}{c}\text{generalized cohomology}\\ \text{theory on } h\mathscr{U}^2\end{array} \arrow[r, shift left, "\tilde{E}^*(X)=E^*(X,*)"] \arrow[d, shift right, "E^*(X,A)=E^*(\Gamma X,\Gamma A)"'] & \begin{array}{c}\text{generalized reduced}\\ \text{cohomology theory on } h\mathscr{T}\end{array} \arrow[l, shift left, "E^*(X,A)=\tilde{E}^*(C(i_+))"] \arrow[d, shift right, "\tilde{E}^*(X)=\tilde{E}^*(\Gamma X)"'] \\
\begin{array}{c}\text{generalized cohomology}\\ \text{theory on } h\mathrm{CW}^2\end{array} \arrow[u, shift right, "\text{include}"'] & \begin{array}{c}\text{generalized reduced}\\ \text{cohomology theory on } h\mathrm{CW}_*,\end{array} \arrow[u, shift right, "\text{include}"']
\end{tikzcd}$$

where Γ denotes CW approximation and $C(i_+)$ is the cofiber of the based inclusion.

Before discussing how Ω-spectra define cohomology theories, recall the Eilenberg-MacLane Ω-spectrum. Brown's representability theorem implies the existence of Eilenberg-MacLane spaces $K(\pi,n)$ by application to reduced singular cohomology:

$$\pi_q(K(\pi,n)) = [S^q, K(\pi,n)] = \tilde{H}^n(S^q;\pi) = \begin{cases} \pi & \text{if } n = q \\ 0 & \text{otherwise.} \end{cases}$$

In fact the spaces $K(\pi, n)$ form a spectrum as well as an Ω-spectrum as follows. In general any based space has a fibration $\Omega(X, x_0) \to P(X, x_0) \to X$ where $P(X, x_0)$ is contractible by shrinking paths to x_0. Hence by the long exact sequence and uniqueness of representing objects, there is a homotopy equivalence $K(\pi, n) \to \Omega K(\pi, n+1)$ which is certainly a weak homotopy equivalence. Furthermore, by adjointness this yields a map $\Sigma K(\pi, n) \to K(\pi, n+1)$.

An Ω-spectrum T_n defines a reduced cohomology theory on $h\mathrm{CW}_*$ via

$$\tilde{E}^q(X) = \begin{cases} [X, T_q] & \text{if } q \geq 0 \\ [X, \Omega^{|q|} T_0] & \text{if } q < 0. \end{cases}$$

and similarly a spectrum T_n such that T_n is a $(n-1)$-connected CW complex defines a reduced homology theory via

$$\tilde{E}_q(X) = \mathrm{colim}_n\, \pi_{q+n}(X \wedge T_n).$$

Detail. Here the colimit is taken over the maps

$$\begin{aligned} \pi_{q+n}(X \wedge T_n) &\xrightarrow{\Sigma} \pi_{q+n+1}(\Sigma(X \wedge T_n)) \\ &\cong \pi_{q+n+1}(X \wedge \Sigma T_n) \\ &\xrightarrow{(\mathrm{id}\wedge\sigma)_*} \pi_{q+n+1}(X \wedge T_{n+1}). \end{aligned}$$

□

It is in this fashion that reduced singular (co)homology arises from the Eilenberg-MacLane Ω-spectrum. Many other important objects of study in algebraic topology arise in this fashion, for instance topological K-theory arises from the spectrum $BU \times \mathbb{Z}$, where $\mathbb{Z}$ is given the discrete topology and BU is the colimit of the classifying spaces of the unitary groups. Brown's representability theorem asserts that this is true in general: every generalized cohomology theory is represented by an Ω-spectrum.

Before discussing the theory of cohomology operations, let us describe some of its applications. The Steenrod cohomology operations, which we will introduce shortly, construct the Stiefel-Whitney characteristic classes via

$$w_i(\xi) = \Phi^{-1}\mathrm{Sq}^i\Phi(1) = \Phi^{-1}\mathrm{Sq}^i\mu,$$

where μ is a Thom class for ξ and Φ is the Thom isomorphism. In 1962, J. Frank Adams used the Steenrod algebra, which the Steenrod squares generate for $p = 2$, to solve the vector fields on spheres problem. The problem asks how many linearly independent smooth vector fields there are on the sphere in $\mathbb{R}^N$; for instance the answer is zero for N odd by the generalized hairy ball theorem. The answer that Adams found is as follows. For any positive N, express $N = A \cdot 2^B$ with A odd, then write $B = c + 4d$ with $0 \leq c < 4$. Setting $\rho(N) = 2^c + 8d$, the answer is $\rho(N) - 1$. The first few values of $\rho(N)$ are

$$1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 9, 1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 10, \ldots.$$

These numbers $\rho(N)$ are the Radon-Hurwitz numbers, and they appeared before Adams' paper from the Hurwitz problem in quadratic forms and in matrix to count the maximum dimension of a linear subspace of $\mathbb{R}^{n\times n}$ for which each non-zero matrix is a similarity transformation, that is, a product of an orthogonal matrix and a scalar matrix. Earlier in his celebrated 1960 paper, Adams again used the Steenrod algebra, this time to solve the Hopf invariant one problem. Together, Adams and Atiyah later found a much simpler proof using cohomology operations but now in topological K-theory, a generalized cohomology theory. The Hopf invariant, whatever it is, is assigned to homotopy classes of maps $S^{2n-1} \to S^n$, hence elements of $\pi_{2n-1}(S^n)$. Adams showed that the only maps with Hopf invariant one occur in $n = 1, 2, 4$ and 8. In these dimensions, there are indeed maps with Hopf invariant one which correspond to the multiplication in the real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, and indeed there are close connections to parallelizability of spheres and the existence of H-space structures on spheres. Steenrod squares arise here from the fact that $\pi_{2n-1}(S^n)$ contains no element with Hopf invariant one if and only if for any attachment of an $(m+n)$-cell to S^m, the following Steenrod square is zero:

$$\mathrm{Sq}^n\colon H^m(K;\mathbb{Z}_2) \to H^{m+n}(K;\mathbb{Z}_2).$$

Having seen the applications, we will we will describe the beginning theory of cohomology operations and Steenrod squares. A *cohomology operation* of degree n between cohomology theories $\tilde{E}^*$ and $\tilde{F}^*$ is a natural transformation $\tilde{E}^q \to \tilde{F}^{q+n}$, and a *stable cohomology operation* is one that commutes with the suspension homomorphisms $\Sigma\colon \tilde{E}^q(X) \to \tilde{E}^{q+1}(\Sigma X)$. Recall the Yoneda lemma, which states that if k is a contravariant set valued functor and $[-, Z]$ is representable, then there is a natural bijection $\mathrm{Nat}([-, Z], k) \cong k(Z)$. If the former theory arises from an Ω-spectra, then by the Yoneda lemma the cohomology operations $\tilde{E}^q \to \tilde{F}^{q+n}$ are in natural bijection with $\tilde{F}^{q+n}$ of the representing object for $\tilde{E}^q$. In particular, for singular cohomology there is the bijection

$$\{\text{cohomology operations } \tilde{H}^q(-;\pi) \to \tilde{H}^{q+n}(-,\rho)\} \cong \tilde{H}^{q+n}(K(\pi,q);\rho).$$

Thus the determination of cohomology operations on singular cohomology reduces to computing the cohomology of the Eilenberg-MacLane spaces.

The Steenrod squares arise in this fashion as the generators of the mod 2 singular cohomology operations. The Steenrod operations are stable cohomology operations

$$\mathrm{Sq}^n\colon H^q(X;\mathbb{Z}_2) \to H^{q+n}(X;\mathbb{Z}_2)$$

that are characterized by the axioms

- Sq^0 is the identity
- $\mathrm{Sq}^n = \begin{cases} x^2 & \text{if } n = \deg x \\ 0 & \text{if } n > \deg x. \end{cases}$
- $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y)$, known as the Cartan formula.

Some of this can be made quite explicit, for instance the first Steenrod square can be realized as the Bockstein operation for the exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Detail. A homomorphism $f\colon \pi \to \rho$ of abelian groups induces cohomology operations $f^*\colon H^q(-;\pi) \to H^q(-;\rho)$, as follows. Observe that by universal coefficients and the Hurewicz theorem

$$[K(\pi,q), K(\rho,q)] = \tilde{H}^n(K(\pi,q);\rho) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(K(\pi,q)),\rho) = \mathrm{Hom}_{\mathbb{Z}}(\pi,\rho).$$

Hence we obtain a map $K(\pi,q) \to K(\rho,q)$, which gives by representability and the Yoneda lemma a map $H^q(-;\pi) \to H^q(-;\rho)$.

Furthermore, a short exact sequence

$$0 \longrightarrow \pi \xrightarrow{f} \rho \xrightarrow{g} \sigma \longrightarrow 0$$

of abelian groups induces a long exact sequence

$$\cdots \longrightarrow H^q(-;\pi) \xrightarrow{f^*} H^q(-;\rho) \xrightarrow{g^*} H^q(-;\sigma) \xrightarrow{\beta} H^{q+1}(-;\pi) \longrightarrow \cdots .$$

because singular chain groups are free abelian and in particular flat. The connecting homomorphisms $\beta\colon H^q(-;\sigma) \to H^{q+1}(-;\pi)$ are called *Bockstein operations*. For

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

in fact the Bockstein operation is $\mathrm{Sq}^1\colon H^q(-;\mathbb{Z}_2) \to H^{q+1}(-;\mathbb{Z}_2)$. Indeed, take a generator $\omega \in H^1(\mathbb{R}\mathrm{P}^2)$ so that automatically $\mathrm{Sq}^1(\omega) = \omega^2$ by the second axiom. In general $\beta\colon H^n(K(\mathbb{Z}_2;1);\mathbb{Z}_2) \to H^{n+1}(K(\mathbb{Z}_2,1);\mathbb{Z}_2)$ is an isomorphism for n odd and zero for n even since the triangle commutes in

$$\begin{tikzcd}
H^n(K(\mathbb{Z}_2,1);\mathbb{Z}) \arrow[r] & H^n(K(\mathbb{Z}_2,1);\mathbb{Z}_2) \arrow[r] \arrow[dr, "\beta"'] & H^{n+1}(K(\mathbb{Z}_2,1);\mathbb{Z}) \arrow[r] \arrow[d, "\text{mod } 2"] & H^n(K(\mathbb{Z}_2,1);\mathbb{Z}) \\
& & H^{n+1}(K(\mathbb{Z}_2,1);\mathbb{Z}_2), &
\end{tikzcd}$$

where the top sequence is induced by $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. □

The precise result states that $H^*(K(\mathbb{Z}_2,q);\mathbb{Z}_2)$ is a polynomial algebra generated by the *Serre-Cartan basis* of so-called admissible sequences of Steenrod operations applied to the fundamental class in of $K(\mathbb{Z}_2,q)$, where explicitly an *admissible* sequence is of the form $\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_j}$ with $i_1,\dots,i_j$ all positive, $i_r \geq 2i_{r+1}$ for $1 \leq r < j$, and $i_1 < i_2 + \cdots + i_j + q$.

The *Ádem relations*

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{0\leq k\leq[i/2]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

for $0 < i < 2j$ allow one to break down an arbitrary composition of Steenrod squares into a sum of Serre-Cartan basis elements, and this gives one way of defining the Steenrod algebra, namely as the free $\mathbb{F}_2$-algebra generated by Sq^i modulo the Ádem relations. This algebra is in fact generated by the elements Sq^{2^k}.

Detail. If n is not a power of two, then set $j = 2^k < n$, where k is chosen to be maximal, and $i = n - j$. Since $0 < i < j < 2j$, there is an Ádem relation for $\mathrm{Sq}^i\mathrm{Sq}^j$ where the $k = 0$ term in the sum is $\binom{j-1}{i}\mathrm{Sq}^n$. Write

$$j - 1 = 2^0 + \cdots + 2^{k-1} \quad \text{and} \quad i = i_0 2^0 + \cdots + i_{k-1}2^{k-1}$$

in base two. The identity

$$\binom{j_0 2^0 + \cdots j_n 2^n}{i_0 2^0 + \cdots i_n 2^n} = \binom{j_0}{i_0} \cdots \binom{j_n}{i_n} \pmod 2$$

always holds; in fact the same statement mod p is known as Lucas's theorem. In the present case $j_0,\dots,j_n$ are all 1, so in fact $\binom{j-1}{i} = 1$. Now Sq^n is written in terms of $\mathrm{Sq}^i\mathrm{Sq}^j$ and $\mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$ where $0 < k \leq [i/2]$, hence by induction in terms of elements of the form Sq^{2^k}. □

4. Cobordism

Whereas it is provably impossible to classify manifolds up to diffeomorphism, it is certainly possible to classify manifolds up to a weaker equivalence. In 1953, Thom classified all smooth closed manifolds up to cobordism in a strikingly elegant fashion. Recall that two manifolds of the same dimension are *cobordant* if their disjoint union is the boundary of a manifold one dimension higher, for instance $S^1 \sqcup S^1$ and S^1 are cobordant via the pair of pants. With disjoint union as addition, cartesian product as multiplication, and every element being 2-torsion, the manifolds up to cobordism form a graded $\mathbb{Z}_2$-algebra $\mathscr{N}_*$. Thom computed this to be

$$\mathscr{N}_* \cong \mathbb{Z}_2[u_i \mid i > 1 \text{ and } i \neq 2^r - 1].$$

Further recall that for a characteristic class c and given the fundamental class $z \in H_n(M; R)$ of an R-oriented manifold M, the tangential characteristic number of M is $\langle c(\tau(M)), z\rangle$ and similarly the normal number is $\langle c(\nu(M)), z\rangle$. As a result of Thom's work, we have the amazing fact that two R-oriented such n-manifolds are cobordant if and only if their tangential (or equivalently normal) Stiefel-Whitney numbers are equal. Philosophically this result is magical because it reduces the geometric question of deciding cobordism to checking a small set of numbers.

There are known explicit generators of $\mathscr{N}_*$. For even dimensions $u_{2j} = [\mathbb{R}\mathrm{P}^{2j}]$ works, but this does not work in odd dimensions because odd real projective spaces are null-bordant. Seeking another option, Dold constructed his *Dold manifolds* two years after Thom's paper via

$$H_{n,m} = \{[x_0 : \ldots : x_n], [y_0 : \ldots : y_m] \mid x_0 y_0 + \ldots + x_n y_n = 0\} \quad \text{for } n < m.$$

Now if $i \neq 2^r - 1$ is odd, then we may write $i = 2^p(2q+1) - 1$ for some $p, q \geq 1$, so $u_i = H_{2^{p+1}q, 2^p}$ is a valid generator.

Thom established his results in his celebrated 1953 paper *Quelques propriétés globales des variétés différentiables*; in this paper, Thom invented his lastingly important Thom spaces. Given an n-dimensional vector bundle ξ, one point compactify each fiber to form an S^n bundle, then identify the points at infinity in each fiber to form the *Thom space* $T(\xi)$ associated to ξ. Equivalently, $T(\xi)$ is the cofiber of the inclusion $S(\xi) \hookrightarrow D(\xi)$ of the sphere bundle into the disk bundle, with metric induced by pulling back the universal bundle over $BO(n)$. Crucially, this construction is functorial, so we have Thom space functors $T\colon \mathscr{E}_n \to \mathscr{T}$.

Thom spaces of normal bundles are especially important since they allow the Pontryagin-Thom construction, which given the normal bundle ν of a smooth closed n-dimensional manifold M yields a map $S^{n+q} \to T(\nu)$.

Detail. Embed $\iota_0\colon M \hookrightarrow \mathbb{R}^{n+q}$ the manifold into Euclidean space using the Whitney embedding theorem, then using the Tubular Neighborhood theorem extend ι_0 to a diffeomorphism $\iota\colon D_\epsilon(\nu) \to T$ of an ϵ-neighborhood of M, viewed as the zero section of ν, with a tubular neighborhood $T \hookleftarrow \iota(M)$ of the embedding of M. Letting $S^{n+q} = \mathbb{R}^{n+q} \cup \{\infty\}$ be the one point compactification of the ambient space, a trivial point-set argument establishes continuity of the map $t\colon S^{n+q} \to T(\nu)$ which restricts to ι on $D_\epsilon(\nu)$ and sends everything else to ∞. □

Thom first showed that $\mathscr{N}_* \cong \pi_*(TO)$ as graded $\mathbb{Z}_2$-algebras. Here $TO(q)$ is the Thom space of the universal bundle γ_q, and the spectrum TO has structure maps

$\Sigma TO(q) \cong Th(\gamma_q \oplus \epsilon) \to TO(q+1)$ induced by the bundle map $\gamma_q \oplus \epsilon \to \gamma_{q+1}$. The homotopy groups of a spectrum T are defined to be

$$\pi_n(T) = \operatorname{colim} \pi_{n+q}(T_q)$$

where the Freudenthal suspension theorem motivates the maps

$$\pi_{n+q}(T_q) \xrightarrow{(-)\wedge \mathrm{id}} \pi_{n+q+1}(T_q \wedge S^1) = \pi_{n+q+1}(\Sigma T_q) \xrightarrow{(\mathrm{struct}_q)_*} \pi_{n+q+1}(T_{q+1}).$$

Detail. Furthermore, TO has a *commutative ring spectrum structure*, which in general for a spectrum T is given by a unit map $\eta\colon S^0 \to T^0$ and a multiplication map $\phi_{m,n}\colon T^m \wedge T^n \to T^{m+n}$ satisfying the obvious relations (see definition on page 218). In particular, the multiplication map for TO is given by

$$T(\gamma_m) \vee T(\gamma_n) \cong T(\gamma_m \times \gamma_n) \longrightarrow T(\gamma_{m+n}),$$

where we cite the facts that the Thom space functor preserves finite products and that there is a map $BO(m) \times BO(n) \to BO(m+n)$ along which γ_{m+n} pulls back to $\gamma_m \times \gamma_n$. This endows $\pi_*(TO)$ with a commutative graded ring structure by taking the smash and then composing

$$S^{m+p+n+q} \cong S^{m+p} \wedge S^{n+q} \longrightarrow TO(m) \wedge TO(n) \xrightarrow{\phi_{m,n}} TO(m+n). \qquad \square$$

It is enlightening to see why $\mathscr{N}_* \cong \pi_*(TO)$ holds just as graded abelian groups. For a map in the forward direction, given M with dimension n as before, the Pontryagin-Thom construction yields a map $S^{n+q} \to T(E(\nu))$. The normal bundle ν determines a bundle map $\nu \to \gamma_q$, so applying the Thom functor and composing yields

$$S^{n+q} \to T(E(\nu)) \to T(E(\gamma_q)),$$

a representative of $\pi_n(TO)$. It is believable that this association is well defined, but it is easily shown to be a group homomorphism. Indeed we may embed disjoint M and N into separate hemispheres so that by pinching the equator $T(\nu_{M\sqcup N}) = T(\nu_M) \wedge T(\nu_N)$, this carries over to the addition in π_*, which is given by the pinch map. For a map in the reverse direction, pick a continuous map $S^{n+q} \to TO(q) = T(E(\gamma_q^\infty))$. By compactness this factors through $T(E(\gamma_q^r))$ for sufficiently large r, and using the Whitney approximation theorem we get a homotopic smooth map $S^{n+q} \to T(E(\gamma_q^r))$. Applying the Thom transversality theorem, we may homotope this map to be transverse to the zero section of γ_q^r and also to avoid ∞ so that now the preimage of $\mathrm{Gr}_q(\mathbb{R}^r)$ is a smooth manifold embedded in $\mathbb{R}^{n+q}$. Recalling that $E(\gamma_q^r)$ consists of the pairs

$$(q\text{-plane in } \mathbb{R}^r, \text{vector in the } q\text{-plane}),$$

it follows that the preimage has dimension n since $\mathrm{Gr}_q(\mathbb{R}^r)$ has codimension q. To show well definedness, given two homotopic maps $S^{n+q} \to T(E(\gamma_q))$ we may smoothly approximate a homotopy between them and homotope it to be transverse to the zero section so that its preimage has as boundary a disjoint union of the two associated manifolds.

Acknowledgments

This expository paper was written for a reading course at UCSB organized in the Spring quarter of 2019. I thank Stephen Bigelow for supervising the reading course and Qingjing Chen for mentoring.

References

[1] May, J. Peter. 1999. *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics.
[2] Bott, Raoul; Tu, Loring W. 1982. *Differential Forms in Algebraic Topology.* Springer. Graduate Texts in Mathematics.
[3] Hatcher, Allen. 2002. *Algebraic Topology.* Cambridge University Press.