CHARACTERISTIC CLASSES AND OBSTRUCTION THEORY

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Abstract. A characteristic class assigns to a vector bundle a cohomology class of its base space in order to detect twists and non-triviality in the bundle. The Stiefel-Whitney classes in particular are obstructions to constructing linearly independent sections: if $w_i\xi \neq 0$ for an n-dimensional bundle ξ, then there cannot exist $n-i+1$ linearly independent sections of ξ. This paper proves this property using obstruction theory, the study of extending sections of fiber bundles over the skeleta of a CW complex.

Contents

This paper proves the following obstruction property for Stiefel-Whitney classes: if $w_i\xi \neq 0$ for an n-dimensional bundle ξ, then there cannot exist $n-i+1$ linearly independent sections of ξ. Along the way, this paper gives an overview of the surrounding theory. Sections 1 and 2 discuss vector bundles and the classifying properties of the universal bundle, assuming familiarity with point-set topology and manifolds. Section 3 describes the Stiefel-Whitney and Euler characteristic classes and their applications, further assuming knowledge of cohomology. Section 4 develops some obstruction theory using well-known results from homotopy theory. Finally, section 5 applies the results from these previous sections to prove the desired obstruction property.

1. Vector Bundles

Vector bundles are the central object of study in the theory of characteristic classes. Roughly speaking, a vector bundle is analogous to a covering space with real vector spaces as fibers instead of discrete spaces. Its total space is thus locally a cartesian product of its base space with Euclidean space, and if the entire total space is globally such a product, then the bundle is said to be trivial. What can make a vector bundle nontrivial are twists in the total space that prevent this product structure.

A geometric way to detect such twists is by looking for linearly independent sections of the total space. Intuitively, a section of a vector bundle is an embedding of the base space into the total space, and the linear structure in the total space lets us discuss, say, sections that vanish nowhere or sections that are linearly independent everywhere. We will show that global triviality of an n-dimensional vector bundle is equivalent to the existence of n linearly independent sections. Later in this paper, we will discuss the relationship between a weaker notion of triviality and admitting a fewer number of sections and how characteristic classes detect each.

Definition 1.1. An n-dimensional *vector bundle* $\xi = (\mathrm{E}\xi, \mathrm{B}\xi, \pi)$ consists of a surjection $\pi\colon \mathrm{E}\xi \to \mathrm{B}\xi$ of topological spaces whose fibers have the structure of real vector spaces and that satisfies the following *local triviality condition*: every point in $\mathrm{B}\xi$ has a neighborhood U along with a homeomorphism $h : \pi^{-1}U \to U \times \mathbb{R}^n$ that takes fibers $\pi^{-1}u$ to $\{u\} \times \mathbb{R}^n$ for $u \in U$ so that the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "h", "\sim"'] \arrow[d, "\pi"] & U \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

Intuitively, this *local triviality condition* says that $\mathrm{E}\xi$ is locally the product of $\mathrm{B}\xi$ with Euclidean space. This condition also forces the dimension of the fibers to be constant on connected components, so we will always assume that $\mathrm{B}\xi$ is connected. Typically, $\mathrm{B}\xi$ is referred to as the *base space,* $\mathrm{E}\xi$ the *total space,* π the *projection,* and h a *local trivialization.*

Definition 1.2. A *sub-bundle* of ξ is a bundle ζ that shares the same base space and whose fibers are vector subspaces of the fibers of ξ, so in particular $\mathrm{E}\zeta \subset \mathrm{E}\xi$.

Example 1.3 (Trivial bundle ϵ^n)**.** A trivial bundle can be trivialized over its entire base space. To construct an n-dimensional trivial bundle ϵ^n over a space X, let $X \times \mathbb{R}^n$ be the total space with projection onto the first coordinate.

Example 1.4 (Tautological line bundle γ_n on $\mathbb{R}\mathrm{P}^n$)**.** Recall that real projective space $\mathbb{R}\mathrm{P}^n$ may be defined as the quotient of S^n under the identification $q\colon S^n \to \mathbb{R}\mathrm{P}^n$ of antipodal points; we will write a point in $\mathbb{R}\mathrm{P}^n$ as $\pm x = \{-x, +x\}$. Let this be the base space of the bundle γ^n and let the total space be

$$\mathrm{E}\gamma_n = \{(\pm x, v) \mid v \in \operatorname{span} x\} \subset \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}.$$

Define the projection $\pi\colon \mathrm{E}\gamma_n \to \mathbb{R}\mathrm{P}^n$ by $\pi(\pm x, v) = \pm x$ so that any fiber is homeomorphic to a line. To construct a local trivialization around a point $\pm x_0 \in \mathbb{R}\mathrm{P}^n$, take the open hemisphere U centered around $x_0 \in S^n$. Then the map $U \times \mathbb{R}^1 \to \pi^{-1}qU$ defined by $(x, \lambda) \mapsto (\pm x, \lambda x)$ factors as

$$U \times \mathbb{R}^1 \xrightarrow{q\times\mathrm{id}} qU \times \mathbb{R}^1 \longrightarrow \pi^{-1}qU,$$

whence the second map is a local trivialization of the neighborhood $qU \ni \pm x_0$. Note that the $n = 1$ case is the open Möbius band.

Example 1.5 (Tangent and normal bundles of a manifold)**.** The tangent spaces of a manifold M form a vector bundle denoted by τM and referred to as the *tangent bundle* of M. The base space is the manifold itself, and the total space is the collection $\mathrm{T}M$ of tangent spaces. The projection takes points $(x, v) \in \mathrm{T}_xM$ to x. Similarly, the *normal bundle* of an embedded manifold M, denoted νM, has total space the orthogonal complements $(\mathrm{T}_xM)^\perp$ with the same base space and corresponding projection.

A homomorphism between vector bundles respects the structure of the topological and vector spaces involved by respectively requiring continuous and linear maps. Fibers are taken linearly to fibers, and to forbid jumping dimensions the definition further requires that the linear maps be isomorphisms. Explicitly, a map $\zeta \to \xi$ between bundles makes the diagram

$$\begin{array}{ccc} \mathrm{E}\zeta & \longrightarrow & \mathrm{E}\xi \\ \downarrow & & \downarrow \\ \mathrm{B}\zeta & \longrightarrow & \mathrm{B}\xi, \end{array}$$

commute, and the map $\mathrm{E}\zeta \to \mathrm{E}\xi$ restricted to $\pi^{-1}b$ is a linear isomorphism for all $b \in \mathrm{B}\zeta$.

Definition 1.6 (Bundle map)**.** A *bundle map* between vector bundles is a map of base spaces and a map of total spaces that takes fibers linearly and isomorphically to fibers. A *bundle isomorphism* is a bundle map with homeomorphisms of total and base spaces or, equivalently, a bundle map with an inverse bundle map.

Sections of a vector bundle reveal information about how trivial the bundle is. In general, a section of a map f is a right inverse of f or, in other words, a map g going the other way

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f"] & \cdot \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $fg = \mathrm{id}$. With vector bundles, we are interested in sections of the projection. Intuitively, a section of a vector bundle embeds a copy of the base space into the total space by picking for each point in the base space a point in its fiber. For instance, a section of a tangent bundle is a vector field. The linear structure in the total space allows us to introduce the further notions of non-vanishing and linearly independent sections.

Definition 1.7 (Section and linearly independent sections)**.** A *section* of a vector bundle ξ is a map $s \colon \mathrm{B}\xi \to \mathrm{E}\xi$ such that $\pi s = \mathrm{id}$. It is said to *vanish* at a point b if $s(b)$ is the zero vector in the vector space $\pi^{-1}b$. A collection of sections $s_1, \dots, s_n$ are *linearly independent* if $s_1b, \dots, s_nb$ are linearly independent as vectors in $\pi^{-1}b$ for every $b \in \mathrm{B}\xi$.

Every vector bundle ξ has a section called the *zero section* that vanishes everywhere since the map $\mathrm{B}\xi \to \mathrm{E}\xi$ taking $b \mapsto (b, 0)$ is well-defined. On the other hand, not all vector bundles have a nowhere vanishing section, and moreover trivial bundles are characterized by having a maximal number of linearly independent sections.

Theorem 1.8. *An n-dimensional vector bundle is trivial if and only if it admits n linearly independent sections.*

Proof. For a trivial bundle ξ, let $h\colon \mathrm{E}\xi \to \mathrm{B}\xi \times \mathbb{R}^n$ be a trivialization, and denote e_i to be the ith vector in the standard basis of $\mathbb{R}^n$. Then the sections $\mathrm{B}\xi \to \mathrm{E}\xi$ defined by $b \mapsto h^{-1}(b, e_i)$ are linearly independent since for every point b in the base space the trivialization on its fiber is a linear map which allows linear relations to pass through:

$$c_1 h^{-1}(b, e_1) + \ldots + c_n h^{-1}(b, e_n) = h^{-1}(b, c_1 e_1 + \ldots + c_n e_n).$$

Conversely, suppose an n-dimensional vector bundle ξ has n linearly independent sections $s_1, \ldots, s_n\colon \mathrm{B}\xi \to \mathrm{E}\xi$. To construct a bundle isomorphism $\epsilon^n \to \xi$ from the n-dimensional trivial bundle on $\mathrm{B}\xi$ to the bundle ξ, form a map of total spaces $\varphi\colon \mathrm{E}\xi = \mathrm{B}\xi \times \mathbb{R}^n \to \mathrm{E}\xi$ by

$$\varphi(x, v_1, \ldots, v_n) = v_1 s_1 x + \ldots + v_n s_n x.$$

This map φ takes fibers linearly to fibers, and it does so isomorphically since the n linearly independent vectors form a basis in each fiber. Moreover, because each section is continuous, so is φ. The following proposition, isolated for future use, then proves the theorem. □

Proposition 1.9 (Bundle isomorphism condition)**.** *If ζ and ξ are bundles over the same base space and $\varphi\colon \mathrm{E}\zeta \to \mathrm{E}\xi$ takes fibers isomorphically to fibers, then φ is a bundle isomorphism.*

Proof. Since ζ and ξ share the same base space, φ is a bijection of total spaces, and moreover it descends to a continuous map of base spaces since projections are continuous. It remains to prove the continuity of φ^{-1}, which would make φ a homeomorphism and $\zeta \to \xi$ a bundle isomorphism.

That φ^{-1} is continuous boils down to the fact that the inverse operation on real invertible matrices $\mathrm{GL}_n\mathbb{R}$ is continuous: if a matrix A is invertible, then

$$A^{-1} = \frac{1}{\det A} C^T,$$

where C is the cofactor matrix of A and C^T is known as the classical adjoint, and both maps $A \mapsto C^T$ and $A \mapsto \det A$ are continuous upon inspection of their explicit formulas. Examining φ^{-1} locally by working over a trivializing neighborhood U and composing with a trivialization h, the map φ is reduced to

$$U \times \mathbb{R}^n \xrightarrow{\varphi} \pi^{-1} U \xrightarrow{h} U \times \mathbb{R}^n,$$

which is the cartesian product of the identity map with a $\mathrm{GL}_n\mathbb{R}$-valued map varying continuously on U. □

The following two examples take advantage of the necessary condition for a vector bundle to be trivial.

Example 1.10 (The open Möbius band γ_1 is nontrivial)**.** The open Möbius band is just the tautological line bundle γ_1 on $\mathbb{R}\mathrm{P}^1$ from Example 1.4. To see that it is nontrivial, take any section $s\colon \mathbb{R}\mathrm{P}^1 \to \mathrm{E}\gamma_1$. Consider the composition

$$S^1 \to \mathbb{R}\mathrm{P}^1 \xrightarrow{s} \mathrm{E}\gamma_1,$$

where the first map is the quotient identifying $x \sim -x$. By continuity, the composition $S^1 \to \mathrm{E}\gamma_1$ takes the form $x \mapsto (\pm x, f(x)x)$, where f is some continuous real function on S^1 satisfying $f(-x) = -f(x)$. Fixing $x_0 \in S^1$ so that $qs(x_0) = (x_0, \lambda x_0)$ for some $\lambda \in \mathbb{R}^1$, compute $f(-x_0) = (\pm x, -\lambda x_0)$. The intermediate value theorem implies that f vanishes somewhere on the circle S^1 and hence somewhere on $\mathbb{RP}^1$, so γ_1 is not trivial since every trivial bundle has an everywhere nonzero section.

Example 1.11 (τS^{2k} is nontrivial)**.** A section of a tangent bundle S^n is a continuous choice of tangent vector at every point on the sphere, which is a vector field on S^n. By the hairy ball theorem, such a vector field must vanish somewhere if n is even, so the tangent bundle on S^{2k} cannot be trivial.

Of the many ways to create new bundles from old, a particularly important one is the pullback operation.

Recall 1.12. Given maps f and g as in the following diagram

$$\begin{tikzcd}
\cdot \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& f^*Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

the *pullback of Z along f* is an object f^*Z with maps into X and Z that make the square in the diagram commute and that satisfies the following universal property: any object with maps into X and Z making the diagram commute factors uniquely through f^*Z. For spaces, the pullback is

$$f^*Z = \{(x,z) \mid f(x) = g(z)\} \subset X \times Z.$$

Proposition 1.13 (Vector bundle operations)**.** *Let ξ be a vector bundle with projection π. The following constructions form new bundles.*

- *Restriction to a subset of the base space: Take any subset $X \subset \mathrm{B}\xi$ to be the base space and $\pi^{-1}X$ to be the total space, letting projection be the restriction $\pi \mid \pi^{-1}X$. The fibers under the new projection are the same as the fibers under the old map. This new bundle is denoted $\xi \mid X$.*
- *Pullback along a map into the base space: Given a map $f\colon X \to \mathrm{B}\xi$, the pullback bundle is denoted $f^*\xi$. Visually,*

$$\begin{tikzcd}
f^*\mathrm{E}\xi \arrow[r] \arrow[d] & \mathrm{E}\xi \arrow[d] \\
X \arrow[r, "f"] & \mathrm{B}\xi.
\end{tikzcd}$$

  The bundle has base space X and total space $f^*\mathrm{E}\xi \subset X \times \mathrm{E}\xi$, and the projection map projects to X. The fibers of the new bundle are mapped isomorphically along the top map.
- *Cartesian product of two bundles: If ζ is a bundle with projection ρ, the cartesian product $\zeta \times \xi$ has total space $\mathrm{E}\zeta \times \mathrm{E}\xi$, base space $\mathrm{B}\zeta \times \mathrm{B}\xi$, and projection $\pi \times \rho$. The new fibers are cartesian products of the old fibers.*
- *Whitney sum of two bundles: The motivation is to construct a new bundle whose fibers are direct sums. Let ζ be a bundle with projection ρ over the same base space B as ξ. We can explicitly define the Whitney sum by defining the total space of $\zeta \oplus \xi$ to be $\mathrm{E}(\zeta \oplus \xi) = \{(x,y) \mid \pi x = \rho y\} \subset \mathrm{E}\zeta \times \mathrm{E}\xi$*

and letting the projection take $(x, y) \mapsto \pi x = \rho y$ so that the fibers are indeed direct sums $\pi^{-1}b \oplus \rho^{-1}b$.

$$\begin{array}{ccc} \Delta^*(\mathrm{E}\zeta \times \mathrm{E}\xi) & \longrightarrow & \mathrm{E}\zeta \times \mathrm{E}\xi \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

Alternatively, the Whitney sum is the pullback of $\zeta \times \xi$ along the diagonal map $\Delta\colon B \to B \times B$.

Remark 1.14. The Whitney sum gives the set of vector bundles over a fixed space X an abelian group structure via the Grothendieck construction. This group, denoted $K(X)$, is the starting point for K-theory (see chapter two in [6]).

2. Grassmannians and Universal Bundles

In homotopy theory, the universal cover of a space is the cover that covers any cover. Likewise, with vector bundles the n-dimensional *universal bundle* is the bundle that pulls back to form any n-dimensional bundle. For this reason, the base space of the nth universal bundle, which we will see is the *Grassmannian manifold* $\mathrm{Gr}_n\mathbb{R}^\infty$, is called the *classifying space* for n-dimensional vector bundles.

The properties of the universal bundle give rise to strong statements about the classifying space $\mathrm{Gr}_n\mathbb{R}^\infty$. For instance, we can assert that the bundles up to isomorphism over a space X, denoted $\mathscr{E}_nX$, are in bijection with the maps $X \to \mathrm{Gr}_n\mathbb{R}^\infty$ up to homotopy. In the language of category theory, this means that the functor $\mathscr{E}_n$ is representable by the classifying space $\mathrm{Gr}_n\mathbb{R}^\infty$.

We will briefly describe the construction of the universal bundle and take for granted some properties that follow from it. Then, we will prove some technical statements about pullback bundles and how they behave with homotopic maps and the universal bundle. Finally, we will conclude the section by formally stating and proving the classifying properties described above.

Recall 2.1. The partition function p is the number of ways to partition an integer into a sum of positive integers. For instance, $p(4) = 5$ since $1+1+1+1 = 2+1+1 = 2+2 = 3+1 = 4$.

We further define p_n to be the number of ways to partition an integer into a sum of at most n positive integers. For instance, $p_2(4) = 3$.

Construction 2.2 (Grassmannian and Stiefel manifolds)**.** We will say that an *n-frame* in $\mathbb{R}^{n+k}$ is an ordered tuple of n linearly independent vectors in $\mathbb{R}^{n+k}$. The *Stiefel manifold* $V_n\mathbb{R}^{n+k}$ is the set of orthonormal n-frames in $\mathbb{R}^{n+k}$. Viewing each n-frame as living in $\mathbb{R}^{n(n+k)}$, the Stiefel manifold forms an open set in Euclidean space and is equipped with the subspace topology. The *Grassmannian manifold* $\mathrm{Gr}_n\mathbb{R}^{n+k}$ is the set of n-planes through the origin in $\mathbb{R}^{n+k}$. There is a canonical map $q\colon V_n\mathbb{R}^{n+k} \to \mathrm{Gr}_n\mathbb{R}^{n+k}$ taking an orthonormal n-frame to the n-plane that it spans, so equip the Grassmannian with the quotient topology under q using that the Stiefel manifold is open. Note that $\mathrm{Gr}_1\mathbb{R}^{n+1} \cong \mathbb{R}\mathrm{P}^n$ and that $\mathrm{Gr}_n\mathbb{R}^{n+k} \cong \mathrm{Gr}_k\mathbb{R}^{n+k}$ since the orthogonal complement of an n-plane in $\mathbb{R}^{n+k}$ determines a k-plane. The *nth infinite Grassmannian manifold* $\mathrm{Gr}_n\mathbb{R}^\infty$ is the set of n-planes in $\mathbb{R}^\infty$, equipped with the direct limit topology under the inclusions into the first coordinates

$$* = \mathrm{Gr}_n\mathbb{R}^n \hookrightarrow \mathrm{Gr}_n\mathbb{R}^{n+1} \hookrightarrow \dots,$$

where $\mathrm{Gr}_n\mathbb{R}^\infty = \bigcup_i \mathrm{Gr}_n\mathbb{R}^{n+i}$ as sets.

Theorem 2.3 (Properties of the Grassmannian)**.** *The Grassmannian* $\mathrm{Gr}_n\mathbb{R}^{n+k}$ *is a closed manifold. The infinite Grassmannian* $\mathrm{Gr}_n\mathbb{R}^\infty$ *has an infinite cell structure consisting of* $p_n(r)$ *cells in the* r*th dimension.*

Construction 2.4 (Universal bundle)**.** Define the *nth universal bundle* γ^n to have base space the nth Grassmannian $\mathrm{Gr}_n\mathbb{R}^\infty$ and total space the set of pairs $(p, v) \in \mathrm{Gr}_n\mathbb{R}^\infty \times \mathbb{R}^\infty$ such that the vector v is in the plane p, and let the projection take $(p, v) \mapsto p$.

Definition 2.5 (Bundle homotopy)**.** Two bundle maps $f, g\colon \zeta \to \xi$ are *bundle-homotopic* if there exists between them a *bundle-homotopy*, a map $\mathrm{E}\zeta \times I \to \mathrm{E}\xi$ that is f for $t = 0$, g for $t = 1$, and a bundle map for all t.

Recall 2.6. A paracompact second-countable Hausdorff space X admits a locally finite countable subcover U_i for any cover. Moreover, it admits a partition of unity subordinate to this subcover, which is a collection of maps $\varphi_i\colon X \to I$ such that their sum is the constant unity function and such that the support of φ_i is contained in U_i. Most familiar spaces, in particular CW complexes, satisfy these conditions.

Theorem 2.7 (Properties of the universal bundle)**.** *Every n-dimensional bundle ξ over a paracompact base space admits a bundle map, unique up to bundle homotopy, into to the nth universal bundle γ^n. Moreover, γ^n is the unique n-dimensional bundle up to isomorphism with this property since γ^n is the universal object in the category of bundles with homotopy classes of bundles as morphisms.*

Having described the universal bundle and its behavior, we set out to describe its classifying properties. The following three propositions describe the relationships among homotopies between maps, isomorphisms of bundles, and admitted maps into the universal bundle. These nice relationships will allow us to lay out strong categorical statements about the universal bundle.

Proposition 2.8 (Pullbacks are isomorphic)**.** *If $f\colon \zeta \to \xi$ is a bundle map, then $f^*\xi \cong \zeta$. In particular, any two pullbacks along the same base map are isomorphic.*

Proof. Take a bundle map $f\colon \zeta \to \xi$ and consider the pullback bundle $f^*\xi$.

$$\begin{tikzcd}
\mathrm{E}f^*\xi \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& \mathrm{E}\zeta \arrow[ul, dotted, "(\pi,f)"'] \arrow[r, "f"] \arrow[d, "\pi"'] & \mathrm{E}\xi \arrow[d, "\rho"] \\
& \mathrm{B}\zeta \arrow[r, "\overline{f}"'] & \mathrm{B}\xi
\end{tikzcd}$$

Using the notation in the above diagram, $\rho f e = \overline{f}\pi e$ for all $e \in \mathrm{E}\zeta$, so $(\pi e, fe) \in \mathrm{E}f^*\xi$ by definition of the pullback. Define a map $\mathrm{E}\zeta \to \mathrm{E}f^*\xi$ by $e \mapsto (\pi e, fe)$, which is continuous since π and f are each continuous. This map takes fibers isomorphically to fibers since a fiber $\pi^{-1}b$ over ζ is mapped to $(b, f\pi^{-1}b) = (b, \rho^{-1}\overline{f}b)$, which is precisely the fiber over $f^*\xi$. Hence, $f^*\xi \cong \zeta$ by the bundle isomorphism condition in Proposition 1.9. $\square$

Proposition 2.9 (Pullbacks along homotopic maps are isomorphic)**.** *Let ξ be a vector bundle and X a paracompact space. If $f, g\colon X \to \mathrm{B}\xi$ are homotopic maps, then $f^*\xi \cong g^*\xi$.*

Proof. Supposing f and g are homotopic maps $X \to \mathrm{B}\xi$, take a homotopy $F\colon X \times I \to \mathrm{B}\xi$ from f to g, along which ξ pulls back to give a bundle $F^*\xi$ on $X \times I$.

It is routine to check that restricting $F^*\xi$ gives isomorphisms $F^*\xi \mid (X \times 0) \cong f^*\xi$ and $F^*\xi \mid (X \times 1) \cong g^*\xi$. For the details, include $i\colon X \times \{0\} \hookrightarrow X \times I$ so that $Fi = f$ and so that ξ pulls back to give a bundle $f^*\xi$ over $X \times \{0\}$.

$$\begin{tikzcd}
\mathrm{E}f^*\xi \arrow[r, dotted, "\exists!"] \arrow[d] & \mathrm{E}F^*\xi \arrow[r] \arrow[d, "\pi"] & \mathrm{E}\xi \arrow[d] \\
X \times \{0\} \arrow[r, hook, "i"] & X \times I \arrow[r, "F"] & \mathrm{B}\xi
\end{tikzcd}$$

There is a unique map $\mathrm{E}f^*\xi \to \mathrm{E}F^*\xi$ making the diagram commute by the universal property of $F^*\xi$, so the map $\mathrm{E}f^*\xi \to E\xi$ is the composition along the top row. Then the left square induces a bundle map $f^*\xi \to F^*\xi \mid \pi^{-1}(X \times 0)$ which takes fibers isomorphically to fibers, from which the bundle isomorphism condition in Proposition 1.9 implies the result.

Thus, it suffices to show that for the bundle $\zeta = F^*\xi$ on $X \times I$, there is an isomorphism $\zeta \mid (X \times 0) \cong \zeta \mid (X \times 1)$. Following this direction, for a fixed point $x \in X$, pick trivializing neighborhoods for each point $(x, t) \in X \times I$. These neighborhoods cover the compact interval I, so all but a finite number may be thrown out. Projecting the remaining neighborhoods to X and then taking their intersection still leaves an open set of X. Repeating for all such $x \in X$ yields an open cover of X, out of which a countable number may be selected using the paracompactness of X. Thus, we have countable number of open sets U_i of X such that $U_i \times I$ is a trivializing neighborhood under the bundle ζ. Take a partition of unity $\varphi_i\colon X \to I$ (Recall 2.6), and let X_i be the graph of $\varphi_1 + \ldots + \varphi_i$, or explicitly

$$X_i = \{(x, \varphi_1 x + \ldots + \varphi_i x) \mid x \in X\} \subset X \times I.$$

Roughly speaking, the plan to construct a homeomorphism of the total spaces of $\zeta \mid (X \times 0)$ and $\zeta \mid (X \times 1)$ is as follows. One by one, project each X_i up to X_{i+1}. Since the difference is contained in a trivial neighborhood, it lifts to a homeomorphism of the total spaces of $\zeta \mid X_i$ and $\zeta \mid X_{i+1}$. Then compose the homeomorphisms to get a homeomorphism $\zeta \mid (X \times 0) = \zeta \mid X_0 \to \zeta \mid (X \times 1)$.

More precisely, the projection $p_i\colon X_i \to X_{i+1}$ taking $(x, t) \mapsto (x, t + \varphi_{i+1} x)$ is continuous since φ_{i+1} is continuous, and it is nontrivial only within the trivialized neighborhood U_{i+1}. Thus, denoting the projection of ζ to be π, define a homeomorphism $\mathrm{E}(\zeta \mid X_i) \to \mathrm{E}(\zeta \mid X_{i+1})$ over trivialized neighborhoods to take

$$(x, v) \mapsto \begin{cases} (p_i x, v) & x \in \pi^{-1}(U_i \times I) \\ (x, v) & \text{otherwise.} \end{cases}$$

The composition of all such homeomorphisms is defined since, due to the partition of unity, each x has a neighborhood where only a finite number of $p_i x$ are nonzero. Hence, this composition exhibits a homeomorphism $\zeta \mid (X \times 0) \cong \zeta \mid (X \times 1)$. $\square$

Proposition 2.10. *Isomorphic bundles admit bundle-homotopic maps into the universal bundle, which in turn induce homotopic base maps.*

Proof. Suppose ζ is isomorphic to ξ with a homeomorphism $h\colon \mathrm{E}\zeta \to \mathrm{E}\xi$ of total spaces. Take maps $f\colon \mathrm{E}\zeta \to \mathrm{E}\gamma^n$ and $g\colon \mathrm{E}\xi \to \mathrm{E}\gamma^n$ into the total space of the universal bundle. Then f and gh induce bundle maps $\zeta \to \gamma^n$ into the universal bundle which are thus bundle homotopic, say through a bundle homotopy $F\colon \mathrm{E}\zeta \times [0,1] \to \mathrm{E}\gamma^n$. For a quick argument that their base maps are homotopic, note that F restricts to the base space through the zero section. More precisely, $(x,t) \mapsto \pi(F(s(x),t))$, where $s\colon X \to \mathrm{E}\zeta$ is the zero section, exhibits a homotopy. □

The language of category theory neatly states the classifying properties of the universal bundle. Let $\mathscr{E}_n X$ denote the set of isomorphism classes of n-dimensional bundles over X. Then $\mathscr{E}_n$ may be viewed as a contravariant functor Top → Sets from the category of topologtical spaces to the category of sets taking objects $X \mapsto \mathscr{E}_n X$ and morphisms $f\colon X \to Y$ to $f^*\colon \mathscr{E}_n Y \to \mathscr{E}_n X$, where f^* is the pullback operation. It is functorial since $(gf)^*\xi = f^*g^*\xi$ for a bundle ξ: to see this, apply the universal property of the pullback three times in order from right to left as shown in the following diagram.

$$\begin{tikzcd}
(gf)^*\xi \arrow[dr, dotted, bend right, "\exists!"'] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& f^*g^*\xi \arrow[r] \arrow[d] \arrow[ul, dotted, bend right, "\exists!"'] & g^*\xi \arrow[r] \arrow[d] \arrow[ull, dotted, bend right, "\exists!"'] & \mathrm{E}\xi \arrow[d] \\
& \cdot \arrow[r, "f"] & \cdot \arrow[r, "g"] & \mathrm{B}\xi
\end{tikzcd}$$

Now let $[X, \mathrm{Gr}_n\mathbb{R}^\infty]$ denote the set of homotopy classes of maps from X into the Grassmannian. Then $[-, \mathrm{Gr}_n\mathbb{R}^\infty]$ is a contravariant functor Top → Sets taking objects $X \mapsto [X, \mathrm{Gr}_n\mathbb{R}^\infty]$ and morphisms $f\colon X \to Y$ to their induced maps $f^*\colon [Y, \mathrm{Gr}_n\mathbb{R}^\infty] \to [X, \mathrm{Gr}_n\mathbb{R}^\infty]$ defined by $f^*[\phi] = [\phi f]$. It is functorial since

$$(fg)^*[\phi] = [\phi f g] = g^*[\phi f] = (g^* f^*)[\phi].$$

The following two theorems use the properties of the universal bundles to establish a strong relationship between the functors $\mathscr{E}_n$ and $[-, \mathrm{Gr}_n\mathbb{R}^\infty]$.

Theorem 2.11. *The mapping $[f] \mapsto f^*\gamma^n$ is a bijection $[X, \mathrm{Gr}_n\mathbb{R}^\infty] \to \mathscr{E}_n X$ for paracompact X.*

Proof. The fact from Proposition 2.9 that pullbacks depend only on homotopy class guarantees that the map is well-defined. To construct an inverse, take each n-dimensional bundle ξ over X to the base map of the bundle map $\xi \to \gamma^n$, unique up to homotopy, that it admits into the Grassmannian. This map is well-defined up to the isomorphism class of ξ since Proposition 2.10 asserts that isomorphic bundles induce homotopic base maps.

It is indeed an inverse. Starting on the right, if ξ is a bundle over X and $f\colon \mathrm{B}\xi \to \mathrm{Gr}_n\mathbb{R}^\infty$ is the base map of the unique bundle map up to homotopy admitted into the universal bundle, then $f^*\gamma^n \cong \xi$ since pullbacks are isomorphic, which was Proposition 2.8. The composition starting on the left is the identity since for a pullback $f^*\gamma^n$ of a map $f\colon X \to \mathrm{Gr}_n\mathbb{R}^\infty$, the induced base and total maps commute with the projections, making the base map the unique map up to homotopy admitted by the pullback bundle into the Grassmannian. □

Theorem 2.12 ($\mathscr{E}_n$ is representable by $\mathrm{Gr}_n\mathbb{R}^\infty$)**.** *The functors $\mathscr{E}_n$ and $[-, \mathrm{Gr}_n\mathbb{R}^\infty]$ are naturally isomorphic.*

Proof. Fix a map $f\colon X \to Y$ so that the functors induce maps $f^*\colon [Y, \mathrm{Gr}_n\mathbb{R}^\infty] \to [X, \mathrm{Gr}_n\mathbb{R}^\infty]$ and $f^*\colon \mathscr{E}_n Y \to \mathscr{E}_n X$. Use the bijection established in Theorem 2.11 to construct maps

$$\begin{tikzcd} {[Y, \mathrm{Gr}_n\mathbb{R}^\infty]} \arrow[r, "f^*"] \arrow[d] & {[X, \mathrm{Gr}_n\mathbb{R}^\infty]} \arrow[d] \\ \mathscr{E}_n Y \arrow[r, "f^*"'] & \mathscr{E}_n X \end{tikzcd}$$

between the images of X and the images of Y. The diagram commutes since pushing a map $[\phi] \in [Y, \mathrm{Gr}_n\mathbb{R}^\infty]$ right then down $[\phi] \mapsto f^*[\phi] = [\phi f] \mapsto (\phi f)^*\gamma^n = f^*\phi^*\gamma^n$ agrees with pushing down then right $[\phi] \mapsto \phi^*\gamma^n \mapsto f^*\phi^*\gamma^n$. Hence, the bijection, an isomorphism in the category of sets, is a natural isomorphism between the functors. $\square$

3. Characteristic Classes

Characteristic classes assign to a vector bundle a cohomology class of its base space. The assigned class can immediately give information such as whether the bundle is nontrivial, whether there exists a certain number of linearly independent sections on the bundle, or whether a manifold is orientable when the bundle is a tangent bundle. With more work, the theory of characteristic classes classify manifolds up to cobordism, describe the structure of important cohomology rings, and even generalize the Euler characteristic. There are four main kinds of characteristic classes: Stiefel-Whitney, Chern, Euler, and Pontryagin. This paper will focus on the first and third.

Definition 3.1. A *characteristic class* is a natural transformation from the functor $\mathscr{E}_n$ to a cohomology functor $H^*(-; G)$.

In other words, a characteristic class c assigns to each vector bundle ξ a cohomology class $c\xi \in H^*(\mathrm{B}\xi; G)$, where G is an abelian group, so that $f^*c\xi = cf^*\xi$ for any map $f\colon X \to \mathrm{B}\xi$, where $f^*\xi$ is the pullback bundle. Visually, the following square commutes:

$$\begin{tikzcd} \mathscr{E}_n \mathrm{B}\xi \arrow[r, "f^*"] \arrow[d, "c"'] & \mathscr{E}_n X \arrow[d, "c"] \\ H^i(\mathrm{B}\xi; G) \arrow[r, "f^*"] & H^i(X; G). \end{tikzcd}$$

Proposition 3.2. *Characteristic classes are in bijection with $H^*(\mathrm{Gr}_n\mathbb{R}^\infty; G)$.*

Proof. Yoneda's lemma proves this at once since $\mathscr{E}_n$ is representable by $\mathrm{Gr}_n\mathbb{R}^\infty$ by Theorem 2.12. For an explicit bijection, note that any characteristic class c is determined by its assignment $c\gamma^n$ on the universal bundle since if ξ is a vector bundle and $f\colon \mathrm{B}\xi \to \mathrm{Gr}_n\mathbb{R}^\infty$ is the unique base map up to homotopy it admits into the Grassmannian, then $c\xi = cf^*\gamma^n = f^*c\gamma^n$, which is well-defined by Propositions 2.9 and 2.10. Thus, any $\varphi \in H^*(\mathrm{Gr}_n\mathbb{R}^\infty; G)$ determines a characteristic class c by demanding that $c\gamma^n = \varphi$, and any characteristic class c determines a cohomology class $c\gamma^n \in H^*(\mathrm{Gr}_n\mathbb{R}^\infty; G)$. These two associations are inverses and hence exhibit a bijection. $\square$

3.1. **Stiefel-Whitney Classes.** The Stiefel-Whitney classes $w_0, w_1, w_2, \ldots$ are a sequence of characteristic classes beginning with 1 and vanishing after the dimension of a bundle. The classes satisfy several axioms, and they are moreover characterized by these axioms in the sense that the Stiefel-Whitney classes are the unique characteristic class satisfying them. This is proved in page 86 of [1], and the construction of the classes is given in chapter 8 of [1]. Incidentally by this uniqueness property, sections 4 and 5 of this paper provide an obstruction-theoretic construction of the Stiefel-Whitney classes.

Definition 3.3. The *Stiefel-Whitney classes* are a sequence $w_0, w_1, w_2, \ldots$ of characteristic classes with coefficients in $\mathbb{Z}_2$ that satisfy the following three axioms: for an n-dimensional bundle ξ,

(i) Dimensionality: $w_0\xi = 1$, and $w_i\xi = 0$ when $i > n$.
(ii) Whitney Product Theorem: $w(\zeta \oplus \xi) = w\zeta \smile w\xi$, where $w = \sum_i w_i$ is called the *total Stiefel-Whitney class* and ζ is another bundle over the same base space as ξ.
(iii) Nontriviality: The first class $w_1\gamma^1$ of the universal line bundle on $\mathbb{R}\mathrm{P}^\infty$ is nonzero, which makes it a generator of $H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$.

Stiefel-Whitney classes have many applications. The crux of many of these applications is whether a certain Stiefel-Whitney class is nontrivial since, intuitively, a bundle having nontrivial ith Stiefel-Whitney class means that the bundle is nontrivial in the dimension i. The below list briefly describes some of these applications, and the string of lemmas and theorems following it provide some details.

- Each Stiefel-Whitney class is a *primary obstruction* to constructing linearly independent sections on a bundle. More precisely, if the ith Stiefel-Whitney class $w_i\xi$ of an n-dimensional bundle ξ over a CW complex is nonzero, then there cannot exist $n-i+1$ linearly independent sections of ξ. Sections 4 and 5 are dedicated to proving this property using the theory of obstructions.
- Only 2^n-dimensional division algebras can exist. Indeed, the real numbers, complex numbers, quaternions, and octonions have respective dimensions one, two, four, and eight, and it is further known that no other division algebras can exist. To prove this, we begin with the computation $w\tau\mathbb{R}\mathrm{P}^n = (1 + x)^{n+1}$ (Lemma 3.4) of the total Stiefel-Whitney class of the tangent bundle on $\mathbb{R}\mathrm{P}^n$. A result by Stiefel (page 48 of [1]) says that an n-dimensional division algebra can exist only when $\tau\mathbb{R}\mathrm{P}^{n-1}$ is trivial or, in other words, when $\mathbb{R}\mathrm{P}^{n-1}$ is *parallelizable*. A quick examination of $w\tau\mathbb{R}\mathrm{P}^{n-1}$ reveals that $\mathbb{R}\mathrm{P}^{n-1}$ is parallelizable only when n is a power of two (Lemma 3.5).
- The Whitney embedding theorem, which states that any smooth compact n-dimensional manifold may be embedded in $(2n-1)$-dimensional Euclidean space, is a sharp bound. Indeed, $\mathbb{R}\mathrm{P}^{2^r}$ cannot be embedded in fewer than $2 \cdot 2^r - 1$ dimensions (Theorem 3.7). Proving this starts with noting that total Stiefel-Whitney classes have inverses in the cohomology ring, putting to use the superficial axiom $w_0 = 1$. Then since the tangent and normal bundles of an embedded manifold form the trivial bundle via Whitney sum, their total Stiefel-Whitney classes are inverses (Lemma 3.6). The punchline is that for any manifold embedded in Euclidean space, the dimension of the

highest nonvanishing class of the normal bundle is a requirement on how large the dimension of the Euclidean space must be.

- Stiefel-Whitney numbers classify manifolds up to cobordism. For a closed, smooth, n-dimensional manifold M, there exists a unique *fundamental class* $[M] \in H_n(M;\mathbb{Z}_2)$ (see page 236 of [4]) that characterizes its orientation. On the other hand, there are degree n monomials in the Stiefel-Whitney classes $w_1M, \dots, w_nM$ of the tangent bundle of M; in fact, there are $p(n)$ of them, where p is the partition function, each taking the form $(w_1M)^{e_1} \smile \dots \smile (w_nM)^{e_n}$, where $e_1 + 2e_2 + \dots + ne_n = n$. Evaluating every monomial on the fundamental class yields a collection of $\mathbb{Z}_2$-numbers called the *Stiefel-Whitney numbers.* It is easy to show that these numbers all vanish if M is a boundary (Lemma 3.8). But a magical theorem due to René Thom (Theorem 3.9) says that the converse is true as well: if Stiefel-Whitney numbers of a manifold M vanish, then M is the boundary of a smooth compact manifold. It follows that two smooth closed manifolds are *cobordant* if and only if all of their Stiefel-Whitney numbers are equal (Corollary 3.10). The sophisticated theory surrounding Thom's theorem is far beyond the scope of this paper (see chapters 23 and 25 in [7]).
- The cohomology ring of the Grassmannian is a polynomial ring generated by the Stiefel-Whitney classes of the universal bundle (Theorem 3.11): $H^*(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1\gamma^n, \dots, w_n\gamma^n]$. Broadly speaking, it is useful to know what this ring looks like because it is where the Stiefel-Whitney classes of the universal bundle live. In particular, this computation will be crucial later when we construct a different sequence of characteristic classes $\mathfrak{p}_i$ meant to agree with the Stiefel-Whitney classes w_i. To show they agree, it will suffice by naturality to show that $\mathfrak{p}_i\gamma^n = w_i\gamma^n \in H^*(\mathrm{Gr}_n\mathbb{R}^\infty;\mathbb{Z}_2)$, so we will need to use the explicit polynomial structure of the ring.

Lemma 3.4. *The total Stiefel-Whitney class of the tangent bundle on* $\mathbb{R}\mathrm{P}^n$ *is*

$$w\tau\mathbb{R}\mathrm{P}^n = (1+x)^{n+1} \in \frac{\mathbb{Z}_2[x]}{(x^{n+1})} \cong H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2).$$

Sketch of proof. Most of the work in the proof (page 45 in [1]) goes towards establishing the isomorphism of bundles

$$\tau\mathbb{R}\mathrm{P}^n \oplus \epsilon^1 \cong \bigoplus^{n+1} \gamma_n,$$

where ϵ^1 is the trivial line bundle on $\mathbb{R}\mathrm{P}^n$ and γ_n is the tautological line bundle on $\mathbb{R}\mathrm{P}^n$. Then taking total Stiefel-Whitney classes on each side yields

$$w\tau\mathbb{R}\mathrm{P}^n = w\tau\mathbb{R}\mathrm{P}^n \smile w\epsilon^1 = w(\mathbb{R}\mathrm{P}^n \oplus \epsilon^1) = w\bigoplus^{n+1}\gamma_1^n = (1+x)^{n+1}. \qquad \square$$

Lemma 3.5. *The class* $w\tau\mathbb{R}\mathrm{P}^n$ *is trivial if and only if* $n+1$ *is a power of two.*

Proof. If $n+1 = 2^r$ for some r, then

$$w\tau\mathbb{R}\mathrm{P}^n = (1+x)^{n+1} = (1+x)^{2^r} = 1^{2^r} + x^{2^r} = 1,$$

where the third equality uses the identity $(1+x)^2 = 1 + x^2 \bmod 2$ and the final equality uses the relation $x^{2^r} = 0$ in the polynomial ring structure of $H^*(\mathbb{R}\mathrm{P}^{2^r};\mathbb{Z}_2)$

described in the previous lemma. Conversely, if $n+1$ is not a power of two, then $n+1 = 2^r m$ for some odd m, so

$$w\tau\mathbb{R}\mathrm{P}^n = (1+x)^{2^r m} = \left(1+x^{2^r}\right)^m \neq 1$$

since there is a nonzero term mx^{2^r}. $\square$

Lemma 3.6 (Whitney duality theorem)**.** *For M a manifold, $w\nu M = (w\tau M)^{-1}$.*

Proof. If M has dimension n, then $\tau M \oplus \nu M = \epsilon^n$ is the trivial bundle on M, so

$$w\tau M \smile w\nu M = w(\tau M \oplus \nu M) = w\epsilon^n = 1. \qquad \square$$

Theorem 3.7. *The manifold $\mathbb{R}\mathrm{P}^{2^r}$ cannot be embedded in fewer than $2 \cdot 2^r - 1$ dimensions.*

Proof. If an n-dimensional manifold M is embedded in $\mathbb{R}^{n+k}$, then the tangent bundle τM and normal bundle νM have respective dimensions n and k, so $w_i \nu M = 0$ for $i > k$. Computing the total Stiefel-Whitney class for $\mathbb{R}\mathrm{P}^{2^r}$, write

$$w\tau\mathbb{R}\mathrm{P}^{2^r} = (1+x)^{2^r+1} = 1 + x + x^{2^r}$$

and

$$w\nu\mathbb{R}\mathrm{P}^{2^r} = \left(w\tau\mathbb{R}\mathrm{P}^{2^r}\right)^{-1} = \left(1+x+x^{2r}\right)^{-1} = 1 + x + x^2 + \ldots + x^{2^r-1}.$$

Hence, if $\mathbb{R}\mathrm{P}^{2^r}$ is embedded in $\mathbb{R}^{2^r+k}$, then k must be at least $2^r - 1$. $\square$

Lemma 3.8. *All of the Stiefel-Whitney numbers of the boundary of a smooth compact manifold vanish.*

Proof. Let M be an n-dimensional manifold and let B be its boundary. Putting an inner product on τM, the orthogonal complement of $\tau M \mid B$ in τM is a trivial line bundle ϵ^1 since the unique outward normal vector field on M is a non-vanishing section. Thus, the tangent bundle on M splits as $\tau M = (\tau M \mid B) \oplus \epsilon^1$, so $w_i \tau B = w_i(\tau M \mid B) = i^* w_i \tau M$, where i^* is the restriction on cohomology induced by the inclusion $B \hookrightarrow M$. Now let $[M] \in H_{n+1}(M, B)$ denote the fundamental class of the pair (M, B), and note that the connecting homomorphism $\partial\colon H_{n+1}(M,B) \to H_n B$ takes $[M] \mapsto [B]$ (see appendix, section A, of [1]). Hence, the Stiefel-Whitney numbers

$$\begin{aligned}(w_1\tau B)^{e_1} \smile \ldots \smile (w_n\tau B)^{e_n}[B] &= i^*[(w_1\tau M)^{e_1} \smile \ldots \smile (w_n\tau M)^{e_n}]\partial[M] \\ &= \partial^* i^*[(w_1\tau M)^{e_1} \smile \ldots \smile (w_n\tau M)^{e_n}][M] \\ &= 0\end{aligned}$$

vanish, where the final equality follows by exactness of the sequence

$$H^n M \xrightarrow{i^*} H^n B \xrightarrow{\partial^*} H^{n+1}(M, B). \qquad \square$$

Theorem 3.9 (Thom's theorem)**.** *All of the Stiefel-Whitney numbers of a smooth closed manifold M vanish if and only if M is the boundary of a smooth compact manifold.*

Corollary 3.10 (Classification up to cobordism)**.** *Two smooth closed manifolds are cobordant if and only if all of their Stiefel-Whitney numbers agree.*

Proof. Let M and N be two smooth closed n-dimensional manifolds so that the fundamental class of their disjoint union is $[M \sqcup N] = [M] + [N]$, and take a degree n monomial $m = w_1(M \sqcup N)^{e_1} \smile \ldots \smile w_{n+1}(M \sqcup N)^{e_n}$. Then M and N are cobordant if and only if $M \sqcup N$ is the boundary of some smooth compact manifold if and only if $m[M] + m[N] = m[M \sqcup N] = 0 \bmod 2$ by Thom's theorem. □

Theorem 3.11. *The cohomology ring of the Grassmannian has structure*

$$H^*(\mathrm{Gr}_n\mathbb{R}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1\gamma^n, \ldots, w_n\gamma^n].$$

Proof. We will use $\mathbb{Z}_2$-coefficients throughout this proof. Recall that γ^1 is the tautological line bundle over $\mathbb{R}\mathrm{P}^\infty$, whose first Stiefel-Whitney class generates $\mathbb{Z}_2[w\gamma^1] \cong H^*\mathbb{R}\mathrm{P}^\infty$ by the nontriviality axiom. The n-fold cartesian product $\prod^n \gamma^1$ has base space $\prod^n \mathbb{R}\mathrm{P}^\infty$, whose cohomology is given by Kunneth's formula as

$$H^* \prod^n \mathbb{R}\mathrm{P}^\infty \cong \mathbb{Z}_2[\alpha_1, \ldots, \alpha_n].$$

Pulling γ^1 along the projection $\pi_i \colon \prod^n \mathbb{R}\mathrm{P}^\infty \to \mathbb{R}\mathrm{P}^\infty$ onto the ith coordinate gives a bundle $\pi_i^*\gamma^1$ over $\prod^n \mathbb{R}\mathrm{P}^\infty$, and $\prod^n \gamma^1 \cong \bigoplus^n \pi_i^*\gamma^1$. Taking Stiefel-Whitney classes

$$w \prod^n \gamma^1 = w \bigoplus^n \pi_i^*\gamma^1 = \overset{n}{\smile} w\pi_i^*\gamma^1 = (1 + \alpha_1) \smile \ldots \smile (1 + \alpha_n) \in \mathbb{Z}_2[\alpha_1, \ldots, \alpha_n]$$

shows that, for instance, $w_1 \prod^n \gamma^1 = \alpha_1 + \ldots + \alpha_n$ and $w_n \prod^n \gamma^1 = \alpha_1 \smile \ldots \smile \alpha_n$. In general, $w_i \prod^n \gamma^1 = \sigma_i$ is the ith symmetric polynomial on the variables $\alpha_1, \ldots, \alpha_n$.

Now consider the bundle map up to homotopy

$$\begin{array}{ccc} \mathrm{E}\prod^n \gamma^1 & \xrightarrow{f} & \mathrm{E}\gamma^n \\ \downarrow & & \downarrow \\ \prod^n \mathbb{R}\mathrm{P}^\infty & \longrightarrow & \mathrm{Gr}_n\mathbb{R}^\infty \end{array}$$

that $\prod^n \gamma^1$ admits into the Grassmannian. Pulling back the universal bundle gives a bundle $f^*\gamma^n$ over $\prod^n \mathbb{R}\mathrm{P}^\infty$. By naturality the induced map takes the ith class of the universal bundle to $f^*w_i\gamma^n = w_i f^*\gamma^n = w_i \prod^n \gamma^1 = \sigma_i$ the ith symmetric polynomial. So f induces a map

$$f^* \colon \langle w_1\gamma^n, \ldots, w_n\gamma^n \rangle \to H^*\mathrm{Gr}_n\mathbb{R}^\infty \to H^* \prod^n \mathbb{R}\mathrm{P}^\infty \cong \mathbb{Z}_2[\alpha_1, \ldots, \alpha_n]$$

taking $w_i\gamma^n \mapsto \sigma_i$. There are no relations among the $w_i\gamma^n$ since there are none among the symmetric polynomials: if p is a nonzero polynomial in n variables, then

$$f^*p(w_1\gamma^n, \ldots, w_n\gamma^n) = p(f^*w_1\gamma^n, \ldots, f^*w_n\gamma^n) = p(\sigma_1, \ldots, \sigma_n)$$

is also nonzero by a classical result in algebra (see page 192 in [8]). Hence, f^* is an injective ring homomorphism, which establishes the inclusion

$$\langle w_1\gamma^n, \ldots, w_n\gamma^n \rangle \hookrightarrow H^*\mathrm{Gr}_n\mathbb{R}^\infty.$$

Conversely, to show that $H^*\mathrm{Gr}_n\mathbb{R}^\infty$ is precisely this polynomial ring, it suffices to show that $|\mathbb{Z}_2[w_1\gamma^n, \ldots, w_n\gamma^n]_i| \geq |H^i\mathrm{Gr}_n\mathbb{R}^\infty|$, where $\mathbb{Z}_2[w_1\gamma^n, \ldots, w_n\gamma^n]_i$ denotes the degree i monomials, which is the goal of the following counting argument. Recall that there is a cell structure on $\mathrm{Gr}_n\mathbb{R}^\infty$ that has $p_n(i)$ cells in the ith dimension, where $p_n(i)$ is the number of ways to partition i into a sum of at most n positive integers. Letting C_i denote the free abelian group generated by these i-cells, on the level of the cellular cochain complex

$$\cdots \longrightarrow \operatorname{Hom}(C_{i-1},\mathbb{Z}_2) \xrightarrow{\partial_i^*} \operatorname{Hom}(C_i,\mathbb{Z}_2) \xrightarrow{\partial_{i+1}^*} \operatorname{Hom}(C_{i+1},\mathbb{Z}_2) \longrightarrow \cdots$$

the ith cohomology group is defined to be $H^i\mathrm{Gr}\mathbb{R}^\infty = \ker\partial_{i+1}^*/\operatorname{im}\partial_i^*$. Thus,

$$|H^i\mathrm{Gr}\mathbb{R}^\infty| = \left|\frac{\ker\partial_{i+1}^*}{\operatorname{im}\partial_i^*}\right| \leq |\ker\partial_{i+1}^*| \leq |\operatorname{Hom}(C_i,\mathbb{Z}_2)| = |C_i| = p_n(i).$$

Now take a degree i monomial $(w_1\gamma^n)^{e_1}\ldots(w_n\gamma^n)^{e_n} \in \mathbb{Z}_2[w_1\gamma^n,\ldots,w_n\gamma^n]$ such that $e_1+2e_2+\ldots+ne_n = i$. The association of this monomial to the partition

$$e_n + (e_n+e_{n-1}) + \ldots + (e_n+e_{n-1}+\ldots+e_1) = i$$

of i into at most n positive integers is bijective, so $p_n(i)$ is the number of degree i monomials in the polynomial ring. Hence, patching the inequalities together yields

$$|H^i\mathrm{Gr}_n\mathbb{R}^\infty| \leq p_n(i) = |\mathbb{Z}_2[w_1\gamma^n,\ldots,w_n\gamma^n]_i|.$$

This concludes the proof. □

3.2. **The Euler Class.** The *Euler class* is a refinement of the Stiefel-Whitney classes for oriented bundles, using $\mathbb{Z}$ coefficients rather than $\mathbb{Z}_2$ coefficients. It is named after Leonhard Euler since it generalizes the Euler characteristic. Unlike the Stiefel-Whitney classes which are sequence of characteristic classes assigning a cohomology class at every dimension, the Euler class is a single characteristic class assigning only one class in the dimension of a bundle. We will construct this class and, along the way, introduce machinery from algebraic topology which we will use later in the paper. The Euler class has the following several important properties, the first three of which follow without much work from the construction. The naturality property in particular implies that the Euler class is indeed a characteristic class.

Proposition 3.12 (Euler Class properties)**.** *The Euler class e assigns to an n-dimensional vector bundle ξ a class $e\xi \in H^n(\mathrm{B}\xi;\mathbb{Z})$ in the nth cohomology group of its base space. The Euler class has the following four properties:*

(*i*) *Naturality: $ef^*\xi = f^*e\xi$ for a map f into $\mathrm{B}\xi$.*
(*ii*) *Agrees with Stiefel-Whitney: The coefficient homomorphism $H^n(\mathrm{B}\xi;\mathbb{Z}) \to H^n(\mathrm{B}\xi;\mathbb{Z}_2)$ takes $e\xi \mapsto w_n\xi$.*
(*iii*) *$e\xi = 0$ if ξ has a nonvanishing section.*
(*iv*) *Generalizes the Euler characteristic: $(e\tau M)[M] = \chi M$ for a smooth compact oriented manifold M, where $\chi M = \sum_i(-1)^i \operatorname{rank} H^iM$ denotes the Euler characteristic of M.*

To begin the construction of the Euler class, we need to first identify some more structure in general vector bundles by constructing two associated *fiber bundles.* A fiber bundle is a generalization of vector bundles and hence of covering spaces, where the fibers are allowed to be any topological space.

Definition 3.13. A *fiber bundle* (E,B,π,F) consists of a surjection $\pi\colon E\to B$ of topological spaces whose fibers are homeomorphic to a space F and that satisfies the usual local triviality condition. Fiber bundles are usually written as $F\to E\to B$ and referred to by its total space E.

The simple case is when an n-dimensional vector bundle ξ has an inner product, which is a map $\mathrm{E}\xi \oplus \mathrm{E}\xi \to \mathbb{R}$ that restricts to a vector space inner product on every fiber. Such an inner product always exists when the base space $\mathrm{B}\xi$ is paracompact (see page 171 of [5]). With such an inner product, there is an obvious associated disk bundle $D^n \to \mathrm{E}\xi \to \mathrm{B}\xi$ and sphere bundle $S^{n-1} \to \mathrm{E}\xi \to \mathrm{B}\xi$, which are respectively denoted $D\xi$ and $S\xi$. A fiber in the associated disk bundle is the set of vectors in the total space with norm less than or equal to one, and a fiber in the sphere bundle is the set of vectors with norm precisely one. More generally, the following construction demonstrates that this works for any vector bundle, regardless of whether it admits an inner product.

Construction 3.14 (Sphere and disk bundle)**.** The sphere bundle $S\xi$ of a vector bundle is obtained by identifying each vector in the total space with its positive scalar multiples within its fiber, ignoring the zero vector. The disk bundle $D\xi$ is obtained from the sphere bundle by taking the mapping cylinder of the projection $S\xi \to \mathrm{B}\xi$, where we recall that the mapping cylinder of a map $f\colon X \to Y$ is the quotient

$$\frac{(X \times I) \sqcup Y}{(x, 0) \sim f(x)}.$$

Intuitively, this construction of the disk bundle works because if the bundle has dimension n, then for a fiber S^{n-1} of $S\xi$ over a point $b \in \mathrm{B}\xi$, the mapping cylinder identifies the bottom boundary $S^{n-1} \times \{0\}$ of $S^{n-1} \times I$ to the point b, which results in a disk.

The Thom isomorphism from algebraic topology gives an isomorphism between the cohomology of the base space of a bundle and the cohomology of the disk total space relative to its sphere total space. It begins with the *Thom class* of the bundle. For an n-dimensional vector bundle ξ and any commutative ring R, the Thom class is the unique cohomology class $c \in H^n(DE, SE; R)$ whose restriction to each fiber is a generator of $H^n(D^n, S^{n-1}; R) \cong \mathbb{Z}$. All real vector bundles have a Thom class with mod 2 coefficients, and further all real orientable vector bundles have a Thom class with integer coefficients (see page 442 in [4]).

Lemma 3.15 (Thom isomorphism)**.** *Cupping by the Thom class is an isomorphism*

$$H^i(B; R) \to H^{i+n}(DE, SE; R).$$

Explicitly, the map is defined by $b \mapsto \pi^* b \smile c$, *where* $c \in H^n(DE, SE; R)$ *is the Thom class and* π *is the projection of the bundle.*

The Gysin sequence uses the Thom isomorphism to establish for any sphere bundle $S^{n-1} \to E \to B$ a long exact sequence

$$\cdots \longrightarrow H^{i-n}B \xrightarrow{\smile e} H^i B \xrightarrow{p^*} H^i SE \xrightarrow{\Phi^{-1}\partial^*} H^{i-n+1}B \longrightarrow \cdots.$$

To explain the maps, consider it in the context of the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(DE, SE) \arrow[r, "j^*"] & H^i DE \arrow[r, "i^*"] & H^i SE \arrow[r, "\partial^*"] & H^{i+1}(DE, SE) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{i-n}B \arrow[u, "\Phi"] \arrow[r, "\smile e"] & H^i B \arrow[u, "\pi^*"] \arrow[r, "\pi^*"] & H^i SE \arrow[u, equal] \arrow[r, "\Phi^{-1}\partial^*"] & H^{i-n+1}B \arrow[u, "\Phi"] \arrow[r] & \cdots .
\end{tikzcd}$$

The sequence along the top is the long exact sequence for the pair (DE, SE), and the vertical map π^* is an isomorphism since B and DE are homotopy equivalent. The maps Φ are Thom isomorphisms, and the *Euler class* of the bundle is defined by $e = (\pi^*)^{-1}j^*c \in H^nB$, where $c \in H^n(DE, SE; R)$ denotes the Thom class. The middle and right squares are clearly commutative, and the left square is commutative since

$$j^*\Phi b = j^*(\pi^*b \smile c) = \pi^*b \smile j^*c = \pi^*b \smile \pi^*e = \pi^*(b \smile e).$$

From this computation, note that the Euler class may alternatively be defined as $e = \Phi^{-1}(c \smile c)$ since

$$\Phi e = \pi^*e \smile c = j^*c \smile c = c \smile c.$$

Definition 3.16. The *Euler class* of a vector bundle is the restriction of the Thom class to the zero section of the bundle. More precisely, if $c \in H^n(DE, SE; R)$ is the Thom class of a bundle ξ, then the Euler class is defined by $e\xi = (\pi^*)^{-1}j^*c$ or alternatively $e\xi = \Phi^{-1}(c \smile c)$.

The Euler class of the tangent bundle τS^n of a sphere will play a role later in applying obstruction theory to the Stiefel-Whitney classes. In light of the hairy ball theorem, it is not surprising that eS^n depends on the parity of n. If n is odd, then there is a non-vanishing section of τS^n, so $eS^n = 0$ since the Euler class detects such non-vanishing sections by Proposition 3.12. If n is even, then the situation is more complicated.

Proposition 3.17 (Euler class of S^n)**.** *The Euler class $eS^n \in H^n(S^n; \mathbb{Z})$ of the tangent bundle of a sphere vanishes if n is odd and is twice a generator of $H^n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ if n is even.*

Proof. The odd case is explained above, so suppose n is even. Let $F \to E \to B$ denote the tangent bundle of a sphere S^n.

The Euler class $eS^n \in H^n(S^n; \mathbb{Z})$ is defined to be $eS^n = \Phi^{-1}(c \smile c)$, where $c \in H^n(E, E - B)$ is the Thom class of τS^n and $\Phi\colon H^nS^n \to H^{2n}(E, E - B)$ is the Thom isomorphism. Thus, showing that eS^n is twice a generator of H^nS^n is equivalent to showing that $c \smile c$ is twice a generator of $H^{2n}(E, E - B)$ since Φ is an isomorphism. To do this, we will first rewrite $H^*(E, E-B)$ into a ring that has generators that are easy to work with and then directly compute the Thom class.

Denote the diagonal of $S^n \times S^n$ and the antipodal pairs respectively by

$$\Delta = \{(x, x) \in S^n \times S^n\} \quad \text{and} \quad A = \{(x, -x) \in S^n \times S^n\}.$$

Note that of course $B \cong S^n$, but moreover $E \cong S^n \times S^n - \Delta$ and $E \cong S^n \times S^n - A$ since for a fixed point $x \in S^n$, the elements $(x, v) \in E$ may vary along $S^n - \{x\} \cong \mathbb{R}^n$. It follows that $E - B \cong S^n \times S^n - \Delta - A$. The following arguments establish the isomorphisms

$$\begin{aligned} H^*(E, E - B) &\cong H^*(S^n \times S^n, S^n \times S^n - \Delta) \\ &\cong H^*(S^n \times S^n, A) \\ &\cong H^*(S^n \times S^n, \Delta) \\ &\subset H^*(S^n \times S^n). \end{aligned}$$

Excision gives the first isomorphism by viewing the subspaces as

$$S^n \times S^n - \Delta \hookrightarrow (S^n \times S^n - A) \cup (S^n \times S^n - \Delta) = S^n \times S^n$$

and

$$E - B \cong (S^n \times S^n - \Delta) \cap (S^n \times S^n - A) \hookrightarrow S^n \times S^n - A \cong E.$$

The second isomorphism holds since A is a deformation retract of $S^n \times S^n - \Delta$ by taking any two distinct points x and y so that $(x, y) \in S^n \times S^n - \Delta$ and sliding y along the great circle to $-x$, resulting in an antipodal pair in A. For the final isomorphism, note that the homeomorphism $S^n \times S^n$ defined by $(x, y) \mapsto (x, -y)$ takes A to Δ and hence induces an isomorphism $H^*(S^n \times S^n, \Delta) \to H^*(S^n \times S^n, A)$.

Consider the short exact sequence

$$0 \to H^n(S^n \times S^n, \Delta) \to H^n(S^n \times S^n) \to H^n\Delta \to 0$$

taken from the long exact sequence of the pair $(S^n \times S^n, \Delta)$, where the left end is zero since $\Delta \cong S^n$ gives $H^{n-1}\Delta \cong H^{n-1}S^n = 0$ and where the right end is zero by the surjectivity of the final map. Writing

$$H^*S^n \otimes_{\mathbb{Z}} H^*S^n \cong \frac{\mathbb{Z}[x]}{(x^2)} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[y]}{(y^2)} \cong \frac{\mathbb{Z}[x, y]}{(x^2, y^2)}$$

shows that $H^n(S^n \times S^n) \cong \mathbb{Z}^2$ has two generators α and β, which may be chosen to be the pullbacks of a generator of $H^n\Delta \cong \mathbb{Z}$ under the two projections $S^n \times S^n \to \Delta$. The kernel is then generated by $\alpha - \beta$, so by exactness $H^n(S^n \times S^n, \Delta)$ is also generated by $\alpha - \beta$. By uniqueness, $\alpha - \beta$ is the Thom class since its restriction to any fiber is a generator. To compute its square, begin with

$$(\alpha - \beta) \smile (\alpha - \beta) = -(\alpha \smile \beta) - (\beta \smile \alpha),$$

then using that n is even, the identity

$$\alpha \smile \beta = (-1)^{n^2} \beta \smile \alpha = \beta \smile \alpha$$

shows that the square of the Thom class is $-2\alpha \smile \beta$. Considering dimensions, $\alpha \smile \beta$ generates $H^{2n}(S^n \times S^n, \Delta) \cong H^{2n}(E, E - B)$, so the square of the Thom class is twice a generator, as desired. $\square$

4. Obstruction Theory

This section takes a break from characteristic classes to develop some basic obstruction theory. The results here will be used in the next section to prove the following obstruction property for Stiefel-Whitney classes: if the ith Stiefel-Whitney class $w_i\xi$ of an n-dimensional bundle ξ over a CW complex is nonzero, then there cannot exist $n - i + 1$ linearly independent sections of ξ. The introduction to the next section gives the big picture of how this theory will be applied, so we encourage the reader to look ahead.

Obstruction theory is based on the problem of finding a section of a fiber bundle over a CW complex X. This is an inductive problem on the skeleta of X: there always exists a section of the zero skeleton by choosing an arbitrary element in the fiber for each point in X^0, so the question becomes whether a section of the i-skeleton may be extended to a section of the $(i+1)$-skeleton. We fix the following notations and assumptions.

Notation 4.1. Let X be a CW complex and $F \to E \to X$ be a fiber bundle ξ on X. To avoid technical difficulties with basepoints, assume that the fiber F is path-connected and that the action of $\pi_1 F$ on $\pi_i F$ is trivial for all i. Further, to avoid difficulties with twisted coefficients (see section 30 of [3]), assume that the action of $\pi_1 X$ on $\pi_n F$ is also trivial.

Suppose there is a section $s\colon X^i \to E$ defined on the i-skeleton of X. Since X is assumed to have a cell structure, all cohomology in the remainder of the section will be cellular. Denote $C_iX = H_i(X^i, X^{i-1})$ to be the ith cellular chain group, which has basis in bijection with the i-cells of X.

An algebraic object $\mathfrak{o}_s$ called the *obstruction cocycle*, which we will construct, answers this problem of extendability. As its name suggests, it represents a class in the cohomology ring of X, where the coefficients are the homotopy group π_iF. The following fundamental theorem in obstruction theory describes the properties of this object, and the remainder of this section is dedicated to proving it.

Theorem 4.2. *There exists a cellular cochain $\mathfrak{o}_s \colon C_{i+1}X \to \pi_iF$ that vanishes if and only if s may be extended to X^{i+1}. Moreover, $\mathfrak{o}_s$ is a cocycle, and its cohomology class $[\mathfrak{o}_s] \in H^{i+1}(X; \pi_iF)$ vanishes if and only if the restriction $s \mid X^{i-1}$ extends to X^{i+1}. If $\pi_kF = 0$ for $k < i$, then $\mathfrak{o}_s$ is independent of the choice of section, so $\mathfrak{o}_i\xi \colon C_{i+1}X \to \pi_iF$ is well defined.*

On the level of a single cell, intuitively the obstruction cocycle measures how twisted the section is along the boundary. This nontriviality is measured by the ith homotopy group of the fiber, using that the boundary of a cell is an i-sphere before attaching.

Roughly speaking, the construction uses the section to lift the boundary of the map $(D^{i+1}, S^i) \to (X^{i+1}, X^i)$ into the total space and then nullhomotopes the lifted map to be contained within a fiber. This yields a map $S^i \to F$, which represents a class in π_iF. The technical crux of being able to nullhomotope the lifted map relies on fiber bundles having the homotopy lifting property with respect to CW complexes, which describes when a homotopy may be lifted to agree with a starting map.

$$\begin{tikzcd} & E \arrow[d, two heads] \\ W \arrow[ur, "\tilde{g}_0"'] \arrow[ur, dotted, bend left, "\tilde{g}_t"] \arrow[r, "g_t"'] & X \end{tikzcd}$$

With our notation, this means that for a homotopy $g_t \colon W \times I \to X$ and any map $\tilde{g}_0 \colon W \to E$, there exists a homotopy $\tilde{g}_t \colon W \times I \to E$ that agrees with $\tilde{g}_0$ at $t = 0$ and makes the above diagram commute, which justifies the notation $\tilde{g}_t$.

Construction 4.3 (Obstruction cocycle)**.** This construction defines $\mathfrak{o}_s$ on maps $\Phi \colon (D^{i+1}, S^i) \to (X^{i+1}, X^i)$. In particular, viewing $\mathfrak{o}_s$ as taking the attaching map of a cell, which is a map taking the form of Φ, it becomes a cochain $\mathfrak{o}_s \colon C_{i+1}X \to \pi_iF$ after extending linearly. We will write $\mathfrak{o}_s$ either way, abusing notation.

$$\begin{tikzcd} & & F \arrow[d, hook] \\ & & E \arrow[d, bend right, "\pi"'] \\ S^i \arrow[r, hook] \arrow[urr, "\tilde{g}_0 = s\varphi", "\tilde{g}_t"'] \arrow[uurr, dotted, "\mathfrak{o}_s\Phi"] & (D^{i+1}, S^i) \arrow[r, "g_t", "\Phi"'] & (X^{i+1}, X^i) \arrow[u, bend right, "s"'] \end{tikzcd}$$

Let $\varphi = \Phi \mid S^i$ denote the restriction to the boundary. There is a nullhomotopy of φ to a constant map $S^i \to \{*\}$, where $*$ is a point in X^{i+1}, defined by $g_t = \Phi \mid$

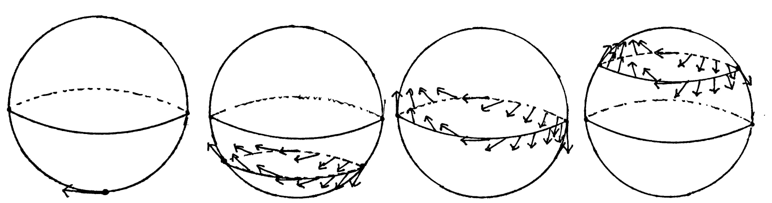
$(1-t)S^i$ that intuitively shrinks the original sphere through smaller and smaller concentric spheres. Moreover, φ lifts through the section s into the total space, giving a map $\tilde{g}_0 = s\varphi\colon S^i \to E$. By the homotopy lifting property, the nullhomotopy g_t lifts to a homotopy $\tilde{g}_t\colon S^i \to E$ from $\tilde{g}_0$ to a map $\tilde{g}_1\colon S^i \to \pi^{-1}\{*\} = F$. Define $\mathfrak{o}_s\Phi = \tilde{g}_1$, which is an element of $\pi_i F$. The bundle is trivial over the interior of each cell since the interior is contractible, so the construction is well-defined up to the choice of lift $\tilde{g}_t$ as well as the homotopy class of Φ (see section 29.6 of [3]).

Proposition 4.4 ($\mathfrak{o}_s$ vanishes iff the section extends)**.** *For an attaching map $\Phi\colon (D^{n+1}, S^n) \to (X^{n+1}, X^n)$ of a cell that restricts to s on its boundary, $\mathfrak{o}_s\Phi = 0$ if and only if s extends across the interior of the cell.*

Proof. The crux of the argument is that a map $S^n \to F$ is nullhomotopic if and only if it extends to a map $D^{n+1} \to F$. To see this, let $H\colon S^n \times I \to F$ be a nullhomotopy of a map $S^n \to F$ to a constant map $S^n \to \{*\}$. Recall that $CS^n = (S^n \times I)/(S^n \times \{0\})$ is the cone on S^n and that $CS^n \cong D^{n+1}$ since CS^n is the bottom half of the suspension $SS^n \cong S^{n+1}$ of the n-sphere. Using this, define $\overline{H}\colon CS^n \to F$ by $\overline{H}(x,t) = H(x, 1-t)$, which is well defined since $\overline{H}(\,\cdot\,, 0) = H(\,\cdot\,, 1)$ is the constant map. Similarly and conversely, an extension $D^{n+1} \to F$ of a map $S^n \to F$ is a map $H\colon CS^n \to F$, which defines a nullhomotopy of the map $S^n \to F$ via $(x,t) \mapsto H(x, 1-t)$.

By construction, $\mathfrak{o}_s$ vanishing on an attaching map $\Phi\colon (D^{n+1}, S^n) \to (X^{n+1}, X^n)$ means that, using the notation in the previous construction, the result $\tilde{g}_1\colon S^n \to F$ of the lifted homotopy $\tilde{g}_t$ is nullhomotopic. Since the interior of a cell is homeomorphic to a disk, the interior is contractible and so the bundle over it is trivial. Hence, running both the homotopy $\tilde{g}_t$ and the nullhomotopy yields a nullhomotopy of $\tilde{g}_0\colon S^n \to F$, which is an extension $D^{n+1} \to F$ of $s\varphi\colon S^n \to F$. Conversely, an extension of s over D^{n+1} gives a nullhomotopy of $\tilde{g}_0$. □

Example 4.5 (Extending vector field on S^2)**.** Consider the sphere bundle V_1S^2 with fiber $S^1 \cong V_1\mathbb{R}^2$ associated to the tangent bundle on S^2. Explicitly, $\mathrm{E}V_1S^2$ is the set of pairs (x, v) where $x \in S^2$ and v is a unit vector tangent to x. Put a cell structure on S^2 consisting of one 0-cell and one 2-cell with attaching map Φ, and pick a section on the 0-cell, which corresponds to choosing a unit tangent vector v at, say, the south pole s. This is also section on the 1-skeleton since there are no 1-cells. Then $s\varphi\colon (D^2, S^1) \to \{(s, v)\} \in \mathrm{E}V_1S^2$ is the constant map. The nullhomotopy of the boundary S^1 of D^2 to the center of the disk composed with the attaching map Φ is, up to homotopy, a homotopy of the constant loops at the south pole to the north pole.



Lifting this homotopy to agree with $s\varphi$ at $t = 0$ results in twice a generator of $\pi_1 S^1$ at $t = 1$. Hence, the section on the 1-skeleton cannot be extended to the 2-skeleton, which in particular proves the hairy ball theorem on S^2.

Example 4.6 (Extending vector field on $\mathbb{RP}^n$)**.** Consider the sphere bundle $V_1\mathbb{RP}^n$ associated to the tangent bundle on $\mathbb{RP}^n$. A pair $(\pm x, v) \in \mathrm{E}V_1\mathbb{R}^{n+1}$, where v is a unit vector tangent to $+x$, may be visualized by imagining a copy of v sitting at each of $+x$ and $-x$ pointing in the same direction. To define a section on $\mathbb{RP}^{n-1}$, imagined as the boundary of the n-cell of $\mathbb{RP}^n$, choose the tangent vector that points towards the north pole. A nullhomotopy of a loop starting at the equator to the north pole lifts to a homotopy in the total space that results in a generator of $\pi_{n-1}S^{n-1}$. Thus, this section cannot be extended to all of $\mathbb{RP}^n$.

The Hurewicz homomorphism $h\colon \pi_k(X, A, x_0) \to H_k(X, A)$ in homotopy theory establishes a strong link between homotopy and homology groups. The Hurewicz homomorphism is natural in the sense that a map $f\colon (X, x_0) \to (Y, y_0)$ induces a commutative diagram

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{f_*} & \pi_n(Y, y_0) \\ \downarrow h & & \downarrow h \\ H_nX & \xrightarrow{f_*} & H_nY. \end{array}$$

Moreover, the absolute and relative Hurewicz homomorphisms together form a chain map

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \longrightarrow & \pi_{n-1}(A, x_0) & \longrightarrow & \cdots \\ & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \\ \cdots & \longrightarrow & H_nA & \longrightarrow & H_nX & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}A & \longrightarrow & \cdots \end{array}$$

between the long exact sequences of homotopy and homology groups. The homomorphism is defined by $h[f] = f_*\alpha$, where $[f] \in \pi_k(X, A, x_0)$ is viewed as a map $f\colon (D^k, S^{k-1}, s_0) \to (X, A, x_0)$ which induces the map $f_*\colon H_k(D^k, S^{k-1}) \to H_k(X, A)$ on homology and where α is a fixed generator of $H_n(D^n, S^{n-1})$. The following theorem collects some facts about the Hurewicz theorem that are used in the next proposition.

Recall 4.7. A space X is said to be *n-connected* if $\pi_iX = 0$ for $i \leq n$. Relatively, a pair (X, A) is said to be *n-connected* if $\pi_i(X, A) = 0$ for $i \leq n$ and every path-component of X contains a point of A.

Lemma 4.8 (Hurewicz Theorem)**.** *When X is $(n-1)$-connected with $n > 1$, the Hurewicz homomorphism $h\colon \pi_k(X, x_0) \to H_kX$ is abelianization for $k = 1$ and an isomorphism for $k \leq n$. Relatively, if X and A are connected and (X, A) is $(n-1)$-connected, then $H_k(X, A) = 0$ for $k < n$ and the Hurewicz homomorphism gives an isomorphism $\pi_n(X, A)/\pi_1A \to H_n(X, A)$, where $\pi_n(X, A)/\pi_1A$ denotes the quotient by the action of π_1A.*

Proposition 4.9. *The cochain $\mathfrak{o}_s$ is a cellular cocycle.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
\pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, two heads, "h"] \arrow[d, "\partial"] & C_{n+2}X \arrow[d, "\partial"] \arrow[dd, bend left=60, "d"] \\
\pi_{n+1}X^{n+1} \arrow[r, "h"] \arrow[d, "j_*"] & H_{n+1}X^{n+1} \arrow[d, "q_*"] \\
\pi_{n+1}(X^{n+1}, X^n) \arrow[r, "h"] \arrow[d, "\mathfrak{o}_s"'] & C_{n+1}X \arrow[dl, "\mathfrak{o}_s"] \\
\pi_n F &
\end{tikzcd}$$

The diagonal map $\mathfrak{o}_s$ on the bottom is an abuse of notation, explained in the construction of the obstruction cocycle. Recalling that the cellular chain groups are defined as $C_nX = H_n(X^n, X^{n-1})$, the horizontal maps are Hurewicz homomorphisms. The top-most one is surjective since for any $(n+2)$-cell e^{n+2} corresponding to a basis element of $C_{n+2}X$, the Hurewicz homomorphism takes the class of its attaching map to $h[\Phi] = \Phi_*\alpha = e^{n+2}$, where $\Phi_* \colon H_{n+2}(D^{n+2}, S^{n+1}) \to C_{n+2}X$ is the induced map and α is a fixed generator of $H_{n+2}(D^{n+2}, S^{n+1}) \cong \mathbb{Z}$. The maps ∂ and q_* are from the long exact sequences of the pairs (X^{n+2}, X^{n+1}) and (X^{n+1}, X^n), and the hemisphere on the right formed by these maps defines the boundary map d in cellular homology.

Each of the two rectangles commutes since the Hurewicz homomorphisms form a chain map between the homotopy and homology groups. To see that the triangle commutes, first note that because the construction of $\mathfrak{o}_s$ is well-defined up to homotopy of the attaching map of a cell, it is in particular well-defined up to translations of the basepoint. But translations of the basepoint are precisely the action of $\pi_1 X^n$ on $\pi_{n+1}(X^{n+1}, X^n)$, viewing based attaching maps as maps $(D^n, S^{n-1}, s_0) \to (X^n, X^{n-1}, x_0)$. By the Hurewicz theorem, modding out by this action induces an isomorphism

$$\overline{h} \colon \pi_{n+1}(X^{n+1}, X^n)/\pi_1 X^n \to H_{n+1}(X^{n+1}, X^n) = C_{n+1}X.$$

Hence, $\mathfrak{o}_s$ decomposes as

$$\pi_{n+1}(X^{n+1}, X^n) \xrightarrow{q} \pi_{n+1}(X^{n+1}, X^n)/\pi_1 X^n \xrightarrow{\overline{h}} C_{n+1}X \xrightarrow{\mathfrak{o}_s} \pi_n F,$$

which shows that the triangle commutes since $\overline{h}q = h$.

Explicitly, the goal is to show that $d^*\mathfrak{o}_s = \mathfrak{o}_s d = 0$, but since the top-most Hurewicz homomorphism is surjective, it suffices by commutativity to show that the composition $\mathfrak{o}_s j_* \partial$ down the left side of the diagram vanishes. Let

$$g \colon (D^{n+2}, S^{n+1}, s_0) \to (X^{n+2}, X^{n+1}, x_0)$$

represent an element of $\pi_{n+2}(X^{n+2}, X^{n+1})$. Viewing g as a homotopy, lift it through the section s using the homotopy lifting property to a map $\tilde{g}$ such that $\tilde{g}s_0 = sx_0$. By the definitions of the maps in the long exact sequence of homotopy groups, $\partial g \colon (S^{n+1}, s_0) \to (X^{n+1}, x_0)$ is the restriction to the sphere and

$$j_*\partial g \colon (S^{n+1}, s_0, s_0) \to (X^{n+1}, X^n, x_0)$$

is the composition with the induced inclusion. Viewing the maps $j_*\partial g$ and $\tilde{g} \mid S^{n+1}$ respectively as maps $(D^{n+1}, \partial D^{n+1}) \to (X^{n+1}, x_0)$ and $(D^{n+1}, \partial D^{n+1}) \to (E, sx_0)$, the latter is a lift of the former defined on D^{n+1} that extends the section s defined on ∂D^{n+1}. Hence, $\mathfrak{o}_s$ vanishes on $j_*\partial g$ by Proposition 4.4, so $\mathfrak{o}_s j_* \partial = 0$. □

The product of two CW complexes X and Y with respective cells e^n_α and e^m_β and characteristic maps Φ_α and Ψ_β consists of cells $e^n_\alpha \times e^m_\beta$ and attaching maps $\Phi_\alpha \times \Psi_\beta$. From cellular homology, the boundary of a product of cells is $d(e^n_\alpha \times e^m_\beta) = de^n_\alpha \times e^m_\beta + (-1)^n e^n_\alpha \times de^m_\beta$, which is in the $(n+m-1)$-skeleton. In particular, we put the cell structure on I consisting of 0-cells denoted $\overline{0}$ and $\overline{1}$ and a 1-cell denoted $\overline{I}$ so that a product $e^i_\alpha \times I$ has boundary $d(e^i_\alpha \times I) = de^i_\alpha \times \overline{I} \pm e^i_\alpha \times (\overline{1} - \overline{0})$.

Construction 4.10 (Difference cochain)**.** Let $\hat{X} = X \times I$ so that $\hat{X}^i = (X^i \times dI) \cup (X^{i-1} \times I)$. Then a section $s\colon \hat{X}^i \to E$ may be seen a pair of sections $s_0\colon X^i \times \{0\} \to E$ and $s_1\colon X^i \times \{1\} \to E$ along with a homotopy $s_t\colon X^{i-1} \to Y$ of their restrictions $s_0 \mid X^{i-1}$ and $s_1 \mid X^{i-1}$. Consequently, the obstruction cocycle $\mathfrak{o}_s\colon C_{i+1}\hat{X} \to \pi_i F$ measures the trouble in extending both the homotopy and the sections to $\hat{X}^{i+1}$. The *difference cochain* $\mathfrak{d}_s\colon C_i X \to \pi_i F$ is defined by $\mathfrak{d}_s e^i_\alpha = \mathfrak{o}_s(e^i_\alpha \times \overline{I})$, ignoring cells of the form $e^{i+1}_\alpha \times \{0\}$ or $e^{i+1}_\alpha \times \{1\}$ since on such cells the difference cochain and the obstruction cocycle agree: $\mathfrak{d}_s(e^{i+1}_\alpha \times \{0\}) = \mathfrak{o}_{s_0} e^{i+1}_\alpha$ and $\mathfrak{d}_s(e^{i+1}_\alpha \times \{1\}) = \mathfrak{o}_{s_1} e^{i+1}_\alpha$. Hence, the difference cochain measures the obstruction in constructing a homotopy between two sections s_0 and s_1 on the i-skeleton of X.

Proposition 4.11 (Sections agreeing on X^{i-1} form $[\mathfrak{o}_s]$)**.** *Set $s_0 = s$. If a section s_1 on X^i agrees with s_0 on X^{i-1}, then $[\mathfrak{o}_{s_0}] = [\mathfrak{o}_{s_1}]$. Conversely, every element cohomologous to $\mathfrak{o}_{s_0}$ is the obstruction cocycle of such a section s_1.*

Proof. Let s_1 be a section of X^i that agrees with $s_0 = s$ on X^{i-1} so that together with the identity homotopy on X^{i-1} they form a section $r\colon \hat{X}^i \to E$. The following computation shows that $\mathfrak{o}_{s_1} - \mathfrak{o}_{s_0} = \pm d^*\mathfrak{d}_r$, so $[\mathfrak{o}_{s_0}] = [\mathfrak{o}_{s_1}]$. Pick an $(i+1)$-cell e^{i+1}_α with boundary $de^{i+1}_\alpha = \sum_\beta n_{\alpha\beta} e^i_\beta$. Then $\mathfrak{d}_r e^i_\beta = \mathfrak{o}_r(e^i_\beta \times I)$ by definition of the difference cochain, and $\mathfrak{d}_r(e^{i+1}_\alpha \times \{0\}) = \mathfrak{o}_{s_0} e^{n+1}_\alpha$ and $\mathfrak{d}_r(e^{i+1}_\alpha \times \{1\}) = \mathfrak{o}_{s_1} e^{i+1}_\alpha$ as noted in its construction. Remembering that $\mathfrak{o}_r$ is a cocycle, the string of equalities

$$\begin{aligned} d^*\mathfrak{d}_r e^{i+1}_\alpha &\pm (\mathfrak{o}_{s_1} - \mathfrak{o}_{s_0}) e^{i+1}_\alpha \\ &= \sum_\beta n_{\alpha\beta} \mathfrak{o}_r(e^i_\beta \times \overline{I}) \pm \mathfrak{d}_r(e^{i+1}_\alpha \times \overline{1} - e^{i+1}_\alpha \times \overline{0}) \\ &= d^*\mathfrak{o}_r(e^{i+1}_\alpha \times I) \\ &= 0, \end{aligned}$$

follows immediately, so $\mathfrak{o}_{s_1} - \mathfrak{o}_{s_0} = \pm d^*\mathfrak{d}_r$.

Conversely, pick $\varphi\colon C_{i+1}X \to \pi_i F$ so that $\mathfrak{o}_{s_0} + d^*\varphi$ is an arbitrary element cohomologous to $\mathfrak{o}_{s_0}$. For a cell e^{i+1}_α with attaching map $\Phi\colon (D^{i+1}, S^i) \to (X^{i+1}, X^i)$, the section s_0 is defined on its boundary $\Phi \mid S^i$. The plan is to construct a map $s_1\colon (D^{i+1}, S^i) \to E$ that agrees with $s_0\Phi$ on the boundary such that together they define a section $S^{i+1} = D^{i+1} \cup D^{i+1} \to E$ in the same homotopy class as φe^{i+1}_α. Viewing each D^{i+1} as sitting at each end of $D^{i+1} \times I$ with the identity homotopy between then on $S^i \times I$, this becomes a section $r\colon (D^{i+1} \times I)^{i+1} \to E$ on the $(i+1)$-skeleton of $D^{i+1} \times I$, so the preceding paragraph will give the identity $\mathfrak{o}_{s_1} e^{i+1}_\alpha = \mathfrak{o}_{s_0} e^{i+1}_\alpha \pm d^*\varphi e^{i+1}_\alpha$.

Pick any map $f\colon S^i \to F$ representing φe^{i+1}_α. Let S^i_- and S^i_+ respectively denote the lower and upper hemisphere of S^i so that we may assume $s_0\Phi$ is defined on

S^i_-. Since S^i_- is contractible, there exist nullhomotopies of $s_0\Phi$ and $f \mid S^i_-$ and hence a homotopy $H\colon S^i_- \times I \to E$ from $f \mid S^i_-$ to $s_0\Phi$. The map f extends $f \mid S^i_-$ to all of S^i, so by the homotopy lifting property of the pair (S^i, S^i_-), there exists a lift $\tilde{H}\colon S^i \times I \to E$ of the homotopy H that agrees with f at $t = 0$. Set s_1 to be the restriction $H(\,\cdot\,, 1) \mid S^i_+$. Then s_1 is an extension of $s_0\Phi$ such that together they define a map $S^i \to F$ in the same homotopy class as f, with $\tilde{H}$ providing the homotopy. □

Corollary 4.12. *The class $[\mathfrak{o}_s]$ vanishes if and only if $s \mid X^{i-1}$ extends to X^{i+1}.*

Proof. To keep the notation consistent, set $s_0 = s$. Suppose $[\mathfrak{o}_{s_0}]$ vanishes so that $\mathfrak{o}_{s_0}$ is cohomologous to the zero cocycle. Then by the previous proposition, the zero cocycle arises as an obstruction cocycle $\mathfrak{o}_{s_1}$, where s_1 is a section on the i-skeleton that agrees with s_0 on the $(i-1)$-skeleton. Since $\mathfrak{o}_{s_1}$ vanishes on all cells, Proposition 4.4 allows s_1 to be extended to the $(i+1)$-skeleton. Then spelling it out, s_1 is an extension of $s_0 \mid X^{i-1}$ to the $(i+1)$-skeleton of X.

Conversely, if $s_0 \mid X^{i-1}$ extends to a section s_1 on X^{i+1}, then since s_0 and $s_1 \mid X^i$ agree on X^{i-1}, by this proposition the obstruction cocycles $\mathfrak{o}_{s_0}$ and $\mathfrak{o}_{s_1 \mid X^i}$ differ by a coboundary, so $[\mathfrak{o}_{s_0}] = [\mathfrak{o}_{s_1 \mid X^i}] = 0$. □

Corollary 4.13. *If $\pi_k F = 0$ for $k < i$, then $\mathfrak{o}_s \in H^{i+1}(X; \pi_i F)$ is independent of the section s. Hence, we denote it $\mathfrak{o}_i\xi$.*

Proof. Suppose $\pi_k F = 0$ for $k < i$, and pick a section s of the zero skeleton. Since $H^{k+1}(X; \pi_k F) = 0$ for $k < i$, the section s extends to the i-skeleton by inductively applying Proposition 4.4, using that its obstruction cocycle is automatically zero at each step. Moreover, the extensions at each step up to the $(i-1)$-skeleton are all homotopic since, by construction, the obstruction to constructing a homotopy between sections on the k-skeleton is inherent in the difference cochain in $H^k(X; \pi_k F)$, which is zero for $k < i$. Thus, there exists a unique section up to homotopy on the i-skeleton. In particular, the extension from the i-skeleton to the $(i+1)$-skeleton is unique up to homotopy rel the $(i-1)$-skeleton. □

5. Stiefel-Whitney Classes as Obstructions

Stiefel-Whitney classes were discovered by Eduard Stiefel and Hassler Whitney as obstructions to constructing linearly independent sections on a vector bundle. In the language of Stiefel-Whitney classes, they discovered the following property.

Theorem 5.1 (Obstruction to linearly independent sections)**.** *If the ith Stiefel-Whitney class $w_i\xi$ of an n-dimensional bundle ξ over a CW complex is nonzero, then there cannot exist $n - i + 1$ linearly independent sections of ξ.*

For instance, $w_n\xi$ being nonzero says that every section on the n-dimensional bundle ξ must vanish somewhere. Further, if ξ is the tangent bundle of a manifold, then $w_1\xi$ being nonzero says that the manifold is not orientable.

To prove this obstruction property, consider the problem of finding k linearly independent sections of the n-dimensional vector bundle ξ. This is equivalent to finding a single section of the fiber bundle $V_k\mathbb{R}^n \to V_k\mathrm{E}\xi \to \mathrm{B}\xi$ since the fibers are tuples of k orthonormal vectors and since the Gram-Schmidt process translates between orthonormal vectors and linearly independent vectors. This reduces the problem of finding multiple sections of a vector bundle to a more palpable problem of finding a single section of a fiber bundle.

In the previous section, we developed some obstruction theory that detected when finding this single section is possible. The theory applies nicely here since Proposition 5.2 will show that $\pi_{n-k}V_k\mathbb{R}^n$ is the first non-vanishing homotopy group of $V_k\mathbb{R}^n$. Thus, the main theorem of the previous section yields an object

$$\mathfrak{o}_{n-k+1}V_k\xi \in H^{n-k+1}(X;\pi_{n-k}V_k\mathbb{R}^n)$$

that satisfies the following property: $\mathfrak{o}_{n-k+1}V_k\xi$ vanishes if and only if there exists a section of $V_k\xi$ on the $(n-k+1)$-skeleton of X. We say that $\mathfrak{p}_{n-k+1}\xi = \mathfrak{o}_{n-k+1}V_k\xi$ is a *primary obstruction* to finding a section of $V_k\xi$ since, of course, there cannot exist a section on X if there cannot exist one on X^{n-k+1}.

The primary obstruction begins to resemble a Stiefel-Whitney class. Proposition 5.2 will show that $\pi_{n-k}V_k\mathbb{R}^n$ is either $\mathbb{Z}$ or $\mathbb{Z}_2$, so we may opt to always take mod 2 coefficients, which avoids technical complications with bundles of coefficients (see section 30 of [3]). Further, setting $i = n-k+1$, the primary obstruction becomes an object $\mathfrak{p}_i\xi \in H^i(X;\mathbb{Z}_2)$ that, by the previous section, satisfies the obstruction property described above. Moreover, it is natural because $\mathfrak{o}_{f^*s} = \mathfrak{o}_s f_*$ (see page 102 of [2] for details). The closing argument in Theorem 5.4 demonstrates that the primary obstruction $\mathfrak{p}_i$ agrees with the ith Stiefel-Whitney class w_i, hence proving that Stiefel-Whitney classes have the obstruction property.

Proposition 5.2. *The Stiefel manifold $V_k\mathbb{R}^n$ is $(n-k-1)$-connected, and its first non-vanishing homotopy group is*

$$\pi_{n-k}V_k\mathbb{R}^n \cong \begin{cases} \mathbb{Z} & \text{if } n-k \text{ is even or if } k=1 \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Proof. If $k=1$, then the Stiefel manifold $V_1\mathbb{R}^n$ is the $(n-1)$-sphere, which indeed is $(n-2)$-connected and has homotopy group $\pi_{n-1}S^{n-1} \cong \mathbb{Z}$. Assume $k>1$.

There is a fiber bundle $V_{k-1}\mathbb{R}^{n-1} \to V_k\mathbb{R}^n \to S^{n-1}$ obtained by restricting a k-frame to its last vector. When $i < n-2$, both groups $\pi_{i+1}S^{n-1}$ and $\pi_i S^{n-1}$ vanish, so $\pi_i V_k\mathbb{R}^n \cong \pi_i V_{k-1}\mathbb{R}^{n-1}$ by the long exact sequence of homotopy groups. Thus, remembering $k>1$ we have the isomorphisms

$$\pi_i V_k\mathbb{R}^n \cong \pi_i V_{k-1}\mathbb{R}^{n-1} \cong \ldots \cong \pi_i V_1\mathbb{R}^{n-k+1} \cong 0$$

when $i < n-k$, where the last isomorphism holds since $V_1\mathbb{R}^{n-k+1}$ is the $(n-k)$-sphere. Hence, $V_k\mathbb{R}^n$ is $(n-k-1)$-connected.

When $i = n-k$, the same isomorphisms hold only up to the penultimate spot:

$$\pi_{n-k}V_k\mathbb{R}^n \cong \pi_{n-k}V_{k-1}\mathbb{R}^{n-1} \cong \ldots \cong \pi_{n-k}V_2\mathbb{R}^{n-k+2}.$$

To study the space $V_2\mathbb{R}^{n-k+2}$, we will study the unit tangent bundle $S^{n-k} \to V_2\mathbb{R}^{n-k+2} \to S^{n-k+1}$. We use the Gysin sequence starting at the $i = n-k$ spot:

$$\cdots \longrightarrow 0 \longrightarrow H^{n-k}V_2\mathbb{R}^{n-k+2} \longrightarrow H^0 S^{n-k+1} \xrightarrow{\smile e} H^{n-k+1}S^{n-k+1}$$
$$\xrightarrow{p^*} H^{n-k+1}V_2\mathbb{R}^{n-k+2} \longrightarrow 0 \longrightarrow \cdots .$$

Suppose $n-k$ is even. There exists a nowhere vanishing section of the tangent bundle of the odd-dimensional sphere S^{n-k+1}, so $e=0$ by Proposition 3.12, which makes the $\smile e$ map the zero map. By exactness $H^{n-k}V_2\mathbb{R}^{n-k+2} \cong H^0 S^{n-k+1} \cong \mathbb{Z}$.

Now the universal coefficient theorem together with the Hurewicz theorem implies that $\pi_{n-k}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z}$. For more detail, consider the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}(H_{n-k-1}V_2\mathbb{R}^{n-k+2}, \mathbb{Z}_n) \\ &\to H^{n-k}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z} \\ &\to \mathrm{Hom}(H_{n-k}V_2\mathbb{R}^{n-k+2}, \mathbb{Z}_n) \\ &\to 0 \end{aligned}$$

given by the universal coefficient theorem. Since $V_2\mathbb{R}^{n-k+2}$ is $(n-k-1)$-connected, the Hurewicz homomorphism gives an isomorphism

$$H_{n-k-1}V_2\mathbb{R}^{n-k+2} \cong \pi_{n-k-1}V_2\mathbb{R}^{n-k+2} = 0,$$

so the Ext functor vanishes. Moreover, applying the theorem for all n implies that

$$H^{n-k}V_2\mathbb{R}^{n-k+2} \cong H_{n-k}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z}$$

since $\mathrm{Hom}(-, \mathbb{Z}_n)$ picks out the free and n-torsion subgroups. Hence, by Hurewicz $\pi_{n-k}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z}$ since the action of $\pi_1 V_2\mathbb{R}^{n-k+2}$ is trivial on $\pi_{n-k}V_2\mathbb{R}^{n-k+2}$.

Now suppose $n-k$ is odd. The computation of the Euler class eS^{n-k+1} in Proposition 3.17 shows that it is twice a generator of $H^{n-k+1}S^{n-k+1} \cong \mathbb{Z}$. Thus, the map $\smile e$ is multiplication by 2, say, which by exactness implies that $H^{n-k}V_2\mathbb{R}^{n-k+2} = 0$ and $H^{n-k+1}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z}_2$. Using a similar argument as before, the universal coefficient theorem forces $H_{n-k}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z}_2$ and again the Hurewicz homomorphism gives $\pi_{n-k}V_2\mathbb{R}^{n-k+2} \cong H_{n-k}V_2\mathbb{R}^{n-k+2}$. Hence, $\pi_{n-k}V_2\mathbb{R}^{n-k+2} \cong \mathbb{Z}_2$. □

Corollary 5.3. *An n-dimensional vector bundle always has k linearly independent sections on its $(n-k)$-skeleton.*

Theorem 5.4. *The primary obstruction agrees with the Stiefel-Whitney classes: $\mathfrak{p}_i = w_i$ for all i.*

Proof. It suffices by naturality of $\mathfrak{p}_i$ and w_i to show that $\mathfrak{p}_i\gamma^n = w_i\gamma^n$, where γ^n is the n-dimensional universal bundle, since then $\mathfrak{p}_i\xi = f^*\mathfrak{p}_i\gamma^n = f^*w_i\gamma^n = w_i\xi$ for any bundle ξ and the map f it admits into the Grassmannian. According to Theorem 3.11, the cohomology ring of the Grassmannian $H^*(\mathrm{Gr}_n\mathbb{R}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1\gamma^n, \dots, w_n\gamma^n]$ is generated by the Stiefel-Whitney classes of the universal bundle, so in particular $\mathfrak{p}_i\gamma^n \in H^i(\mathrm{Gr}_n\mathbb{R}^\infty; \mathbb{Z}_2)$ is a polynomial in these classes. Considering dimensions, it is in fact of the form

$$\mathfrak{p}_i\gamma^n = p(w_1\gamma^n, \dots, w_{i-1}\gamma^n) + \lambda w_i\gamma^n,$$

where p is a polynomial in $i-1$ variables with $\mathbb{Z}_2$ coefficients and where $\lambda \in \mathbb{Z}_2$.

The polynomial p is the zero polynomial. Set $\xi_{i-1} = \gamma^{i-1} \oplus \epsilon^{n-i+1}$ and let $f\colon \xi_{i-1} \to \gamma^n$ be the bundle map it admits into the universal bundle. It suffices to justify the equalities

$$\begin{aligned} p(w_1\gamma^{i-1}, \dots, w_{i-1}\gamma^{i-1}) &= p(w_1\xi_{i-1}, \dots, w_{i-1}\xi_{i-1}) + \lambda w_i\xi_{i-1} \\ &= p(f^*w_1\gamma^n, \dots, f^*w_{i-1}\gamma^n) + \lambda f^*w_i\gamma^n \\ &= f^*\mathfrak{p}_i\gamma^n \\ &= \mathfrak{p}_i\xi_{i-1} \\ &= 0 \end{aligned}$$

since, again by Theorem 3.11, the classes

$$w_1\gamma^{i-1}, \dots, w_{i-1}\gamma^{i-1} \in \mathbb{Z}_2[w_1\gamma^{i-1}, \dots, w_{i-1}\gamma^{i-1}] \cong H^*(\mathrm{Gr}_{i-1}\mathbb{R}^\infty; \mathbb{Z}_2)$$

do not satisfy any nontrivial polynomial relations, implying $p = 0$. The Whitney product theorem shows that

$$w\xi_{i-1} = w(\gamma^{i-1} \oplus \epsilon^{n-i+1}) = w\gamma^{i-1} \smile w\epsilon^{n-i+1} = w\gamma^{i-1}$$

and moreover γ^{i-1} having dimension $i-1$ implies that $w_i\xi_{i-1} = 0$, so together they show the first equality. The naturality of the Stiefel-Whitney classes $w\xi_{i-1} = f^*w\gamma^n$ gives the second equality. The third and fourth equalities hold immediately from f^* being a ring homomorphism, the identity above involving $\mathfrak{p}_i\gamma^n$, and naturality of $\mathfrak{p}_i$. Finally, $\mathfrak{p}_i\xi_{i-1}$ vanishes since the trivial part of $\xi_{i-1} = \gamma^{i-1} \oplus \epsilon^{n-i+1}$ admits $n-i+1$ linearly independent sections.

To show that $\lambda = 1$, it suffices to construct a bundle ξ with $\mathfrak{p}_i\xi$ nonzero since then naturality implies that $f^*\mathfrak{p}_i\gamma^n = \mathfrak{p}_i\xi$ is nonzero and hence that $\mathfrak{p}_i\gamma^n$ is nonzero as well since f^* is a ring homomorphism. For the case $i = n$, Example 4.6 gives such a construction of a bundle $V_1\mathbb{R}^n \to \mathbb{R}\mathrm{P}^n \cong \mathrm{Gr}_1\mathbb{R}^n$ with fiber $V_1\mathbb{R}^n$ which has nonzero ith obstruction. For the remaining cases $i < n$, consider the direct sum ξ of the bundle $E_1\mathbb{R}^n \to \mathrm{Gr}_1\mathbb{R}^n$ with the trivial bundle of dimension $n-i$ over the same base space, which has fibers $V_{n-i+1}\mathbb{R}^n$. The inclusion $V_1\mathbb{R}^n \hookrightarrow V_{n-i+1}\mathbb{R}^n$ induces by the long exact sequence a surjection $\pi_{i-1}V_1\mathbb{R}^n \twoheadrightarrow \pi_{i-1}V_{n-i+1}\mathbb{R}^n$ under which the obstruction $\mathfrak{p}_i\xi$ is the image of the old obstruction. Hence, $\mathfrak{p}_i\xi$ is nonzero since the surjection is an isomorphism mod 2. □

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