REPRESENTATIONS OF SL_2

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ABSTRACT. The Lie group SL_2 and its Lie algebra $\mathfrak{sl}_2$ are key examples in representation theory. This note exposits some Lie theory, the Borel-Weil-Bott theorem, and some of the work of Harish-Chandra, using SL_2 as a running example.

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1. ELEMENTARY METHODS

A canonical example in representation theory is

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a + d = 0 \right\}.$$

Its representation theory is simple and concrete. Each of its representation splits into a direct sum of irreducibles, and there is only one irreducible representation per dimension. For a nice illustration of the $(d+1)$-dimensional irreducible representation, take the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}_2$, which has the commutation relations

$$\begin{aligned} he - eh &= 2e, \\ hf - fh &= -2f, \\ ef - fe &= h. \end{aligned}$$

Define the representation on the space of degree d homogeneous polynomials by

$$e = Y\partial_X, \quad f = X\partial_Y, \quad h = \deg_X - \deg_Y .$$

Visually we have

$$\begin{tikzcd}
0 & X^d \arrow[l, bend left, "f=\times 0"] \arrow[loop above, "h=\times d"] \arrow[r, bend left, "e=\times d"] & X^{d-1}Y \arrow[l, bend left, "f=\times 1"] \arrow[loop above, "h=\times(d-2)"] \arrow[r, bend left, "e=\times(d-1)"] & \cdots & XY^{d-1} \arrow[l, bend left, "f=\times(d-1)"] \arrow[loop above, "h=\times(2-d)"] \arrow[r, bend left, "e=\times 1"] & Y^d \arrow[l, bend left, "f=\times d"] \arrow[loop above, "h=\times -d"] \arrow[r, bend left, "e=\times 0"] & 0
\end{tikzcd}$$

It is enlightening to work these representations out by hand. The methods used suggest a way to find irreducible representations of general Lie Algebras; for example the h operator in $\mathfrak{sl}_2$ motivates the Cartan subalgebra $\mathfrak{h}$ in a general Lie algebra, and the Casmir element plays a similar role in the unitary representation theory of $SL_2(\mathbb{R})$.

In the following 11 steps, we will deduce this complete picture of the representations of $\mathfrak{sl}_2$. First we show that e and f behave as raising and lowering operators on the eigenspaces of h. Then we explicitly compute the actions of e, f, h on these eigenspaces, thereby showing that any irreducible representation is of the form we described. Finally, using the Casmier operator we show that any representation splits into a direct sum of irreducibles.

Reference. The following 11 steps are taken from Exercise 2.15.1 in Etingoff's book on representation theory.

Let V be a representation of $\mathfrak{sl}_2$.

Step 1. Take eigenvalues of H and pick one with the biggest real part. Call it λ. Let $\overline{V}(\lambda)$ be the generalized eigenspace corresponding to λ. Show that $E|_{\overline{V}(\lambda)} = 0$.

Solution. Let $v \in \overline{V}(\lambda)$ so that $(h - \lambda I)^n v = 0$ for some n. We will show that $ev \in V(2+\lambda)$, which would imply $ev = 0$ by the choice of λ as the eigenvalue with the biggest real part.

Write $(h-(2+\lambda)I)e = e(h - \lambda I)$ using the commutativity relations. It follows that

$$(h-(2+\lambda)I)^n ev = e(h-\lambda I)^n v = 0,$$

as desired. □

Step 2. Let W be any representation of $\mathfrak{sl}(2)$ and let $w \in W$ be a nonzero vector such that $ew = 0$. For any $k > 0$ find a polynomial $P_k(x)$ of degree k such that $e^k f^k w = P_k(h)w$.

Solution. Let us always work on $\ker e$ so that any operator ending with e vanishes. Following the hint, we first compute ef^k. Repeatedly using the commutativity relation $ef = h + fe$ yields

$$\begin{aligned}
ef^k &= hf^{k-1} + fef^{k-1} \\
&= hf^{k-1} + fhf^{k-2} + f^2ef^{k-2} \\
&\vdots \\
&= hf^{k-1} + fhf^{k-2} + \cdots + F^{k-2}hf + f^{k-1}h,
\end{aligned}$$

where the last term $f^{k+1}e$ vanishes. The leading term hf^{k-1} merges into the second term fh^{k-1} with a cost of $-2f^{k-1}$ via the relation $hf = fh - 2f$. Now the new leading term $2fhf^{k-2}$ merges into the new second term f^2h^{k-2} with a cost of $-4f^{k-1}$ via the same relation. Continuing in this fashion shows

$$\begin{aligned} ef^k &= kf^kh - (2+4+\ldots+2(k-1))f^{k-1} \\ &= kf^{k-1}h - k(k-1)f^{k-1}. \end{aligned}$$

Hence by induction

$$\begin{aligned} e^kf^k &= e^{k-1}(kf^{k-1}h - k(k-1)f^{k-1}) \\ &= e^{k-1}f^{k-1}k(h-(k-1)) \\ &\vdots \\ &= k!h(h-1)(h-2)\cdots(h-(k-1)), \end{aligned}$$

which is indeed a degree k polynomial in h. □

Step 3. Let $v \in \overline{V}(\lambda)$ be a generalized eigenvector of h with eigenvalue λ. Show that there exists $N > 0$ such that $f^Nv = 0$.

Solution. Let n be such that $(h-\lambda I)^n v = 0$. Using a similar strategy as in step 1, we will show that $f^Nv \in \overline{V}(\lambda - 2N)$ for every N. Since our vector space is finite dimensional, this would mean $f^Nv = 0$.

Write $(h-(\lambda-2)I)f = f(h-\lambda I)$ using the commutativity relations. Like in step 1, it follows that

$$(h-(\lambda-2)I)^n f = f(h-\lambda I)^n,$$

so

$$(h-(\lambda-2N)I)^n f^N = f^N(h-\lambda I)^n.$$

Now apply these operators to v. □

Step 4. Show that h is diagonalizable on $\overline{V}(\lambda)$.

Solution. We will show that any vector $v \in \overline{V}(\lambda)$ is an eigenvector of H. Following the hint, by step 3 and taking the maximum over a basis, there exists N such that $f^N = 0$ on $\overline{V}(\lambda)$. By step 1 we have $Ev = 0$, so v satisfies the hypothesis of step 2 which gives $P_N(h)v = e^Nf^Nv = 0$. The solution to step 2 gives an explicit formula

$$P_N(h) = N!h(h-1)(h-2)\cdots(h-(N-1)),$$

and this demonstrates that $P_N(h)$ has distinct roots. In general with the same objects in play, if $(h-a_1)\cdots(h-a_n)v = 0$, then $\lambda = a_i$ for some a_i. Here, since $P_N(h)v = 0$ and $v \in \overline{V}(\lambda)$ is a generalized eigenvector, in fact $\lambda \in \{1, \ldots, N-1\}$. Now $(h-\lambda)v$ is still a generalized eigenvector and

$$\left(\prod_{k\neq\lambda}(h-k)\right)((h-\lambda)v) = N!P_N(h) = 0,$$

so by the general case it must be that $(h-\lambda)v = 0$. □

Step 5. Let N_v be the smallest N satisfying step 3. Show that $\lambda = N_v - 1$.

Solution. One direction is clear, and the other is by choice. □

Step 6. Show that for each $N > 0$, there exists a unique up to isomorphism irreducible representation of $\mathfrak{sl}(2)$ of dimension N. Compute the matrices e, f, h in this representation using a convenient basis.

Solution. Following the hint, letting V be such a representation that is irreducible, take λ as in step 1 and let $v \in V(\lambda)$ be an eigenvector of h using step 4. First let us argue that $v, fv, \ldots, f^\lambda v$ is a basis. It certainly has length N by step 5. By the solution to step 3 each $f^i v \in \overline{V}(\lambda - 2i)$ has generalized eigenvalue $\lambda - 2i$, so in particular they are generalized eigenvectors with distinct eigenvalues, hence are linearly independent by a linear algebra argument.

With respect to this basis $v, fv, \ldots, f^\lambda v$, certainly f has ones down its -1 diagonal and zeros everywhere else. To compute h, observe that $f^k v$ is an eigenvector of h with eigenvalue $\lambda - 2k$ because inductively $v \in V(\lambda)$ and

$$hf^k v = (fh - 2f)f^{k-1}v = (\lambda - 2k)f^k v.$$

Hence h has $\lambda, \lambda - 2, \ldots, -\lambda + 2, -\lambda$ down its diagonal and zeros everywhere else. Finally we can determine e by examining the relation $ef - fe = h$. The matrix ef is just e shifted down by one and fe is just e shifted left by one. For the difference $ef - fe$ to be h, we deduce that e must be zero everywhere except for the $+1$ diagonal. Here, starting from the top, we see that the numbers must be

$$\lambda, \quad 2\lambda - 2, \quad 3\lambda - 4, \quad \ldots, \quad N\lambda - 2(N-1).$$

Conversely, we can check by brute force that these matrices actually define and characterize an N-dimensional representation of $\mathfrak{sl}(2)$. Moreover, this representation is irreducible because any invariant subspace has an eigenvector v of h, but then $v, fv, \ldots, f^{N-1}$ is a basis for the space. $\square$

Step 7. Show that the operator $C = ef + fe + h^2/2$ (the so-called Casimir operator) commutes with e, f, h and equals $\frac{\lambda(\lambda+2)}{2}\mathrm{id}$ on V_λ.

Solution. Indeed

$$\begin{aligned}
[c, e] &= efe + fee + \frac{h^2 e}{2} - eef - efe - \frac{eh^2}{2} \\
&= e(ef - h) + (ef - h)e + \frac{h(eh + 2e)}{2} \\
&\qquad - eef - efe - \frac{(he - 2e)h}{2} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[c, f] &= eff + fef + \frac{h^2 f}{2} - fef - Ffe - \frac{fh^2}{2} \\
&= (h + fe)f + \frac{h(fh - 2f)}{2} \\
&\qquad - f(ef - h) - \frac{(fh + 2f)h}{2} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[c,h] &= efh + feh + \frac{h^3}{2} - hef - hfe - \frac{h^3}{2} \\
&= [ef + fe, h] \\
&= [ef + (ef - h), h] \\
&= 2(efh - hef) \\
&= 2(e(hf + 2h) - (2e + eh)f) \\
&= 0.
\end{aligned}$$

Moreover to show it equals $\frac{\lambda(\lambda+2)}{2}\mathrm{id}$ on V_λ, it suffices to show that C takes this value on each monomial, which it does:

$$\begin{aligned}
C(X^{\lambda-i}Y^i) &= \left(ef + fe + \frac{h^2}{2}\right)(X^{\lambda-i}Y^i) \\
&= \left(Y\partial_X X\partial_Y + X\partial_Y Y\partial_X + \frac{(\deg_X - \deg_Y)^2}{2}\right) X^{\lambda-i}Y^i \\
&= \left((\lambda - i + 1)i + (i+1)(\lambda - i) + \frac{(\lambda - 2i)^2}{2}\right) X^{\lambda-i}Y^i \\
&= \frac{\lambda(\lambda+2)}{2}(X^{\lambda-i}Y^i).
\end{aligned}$$

□

Now it is easy to prove the direct sum decomposition. Namely, assume the contrary, and let V be a reducible representation of the smallest dimension which is not a direct sum of smaller representations.

Step 8. Show that C has only one eigenvalue on V, namely $\frac{\lambda(\lambda+2)}{2}$ for some non-negative integer λ.

Solution. Since C is central and V is indecomposable, C acts a scalar, namely the scalar by which it acts on an irreducible sub-representation according to step 7. □

Step 9. Show that V has a subrepresentation $W = V_\lambda$ such that $V/W = nV_\lambda$ for some n.

Solution. Since V is reducible, it has a proper irreducible subrepresentation, so by the classification it must be V_λ where λ is as in step 8. Now $\dim V/W = \dim V - \lambda < \dim V$, so V/W is a direct sum of smaller representations by definition of V. Since C still has eigenvalue $\frac{\lambda(\lambda+2)}{2}$, each direct summand must be V_λ. Thus indeed $V/W = nV_\lambda$ for some n. □

Step 10. Deduce from step 9 that the eigenspace $V(\lambda)$ of h is $(n+1)$-dimensional. If $v_1, \ldots, v_{n+1}$ is its basis, show that $f^j v_i$ for $1 \le i \le n+1$ and $0 \le j \le \lambda$ are linearly independent and therefore form a basis of V.

Solution. At once

$$\dim V(\lambda) = \dim(V/W)(\lambda) + \dim W(\lambda) = n + 1$$

since $W(\lambda) = V_\lambda(\lambda)$ is spanned by X^d and $V/W = nV_\lambda$ by step 9.

Now note $f^j v_i$ has eigenvalue $\lambda - 2j$, so the powers of f must all be the same in any linear relation. It thus remains to show the list $f^j v_1, \dots, f_j v_{n+1}$ is linearly independent. Applying e^j to the list, we get by step 2 the list $P_k(\lambda)v_1, \dots, P_k(\lambda)v_{n+1}$. This list is linearly independent, so the original one was as well. □

Step 11. Define $W_i = \operatorname{span}(v_i, fv_i, \dots, f^\lambda v_i)$. Show that W_i are subrepresentations of W and derive a contradiction to the fact that V cannot be decomposed.

Solution. Since $v_i \in V(\lambda)$, the solution to step 6 shows that W_i is a subrepresentation. Moreover since the $f^j v_i$ form a basis of V, the W_i are direct summands of V, contradicting V being indecomposable. □

2. Lie theory and highest weights

Notation. Let k be an algebraically closed field of characteristic 0.

Just as the representations of $\mathfrak{sl}_2$ are indexed by their highest eigenvalues and thus by the positive integers, the representations of any semisimple Lie algebra $\mathfrak{g}$ over k are indexed by their highest weights and thus by the weight lattice of $\mathfrak{g}$. This is the theorem of the highest weight.

Let us review some basic Lie theory on the way to stating this theorem.

There are many important matrix Lie algebras which live in $\mathfrak{gl}_n(k)$ as Lie subalgebras, subspaces closed under the Lie bracket. For example

$$\mathfrak{so}_n(k) = \{x \in \mathfrak{gl}_n(k) \mid {}^t x = -x\}$$
$$\mathfrak{sp}_{2n}(k) = \{x \in \mathfrak{gl}_{2n}(k) \mid {}^t xJ = -Jx\}$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

are clearly subspaces, and it is easy to check they are closed under the Lie bracket:

$${}^t(xy - yx) = {}^t y {}^t x - {}^t x {}^t y = yx - xy = -(xy - yx)$$
$${}^t(xy - yx)J = {}^t y {}^t xJ - {}^t x {}^t yJ = -{}^t yJx + {}^t xJy = -J(xy - yx).$$

Example ($\mathfrak{sl}_n(k)$ is a Lie subalgebra of $\mathfrak{gl}_n(k)$)**.** Similarly

$$\mathfrak{sl}_n(k) = \{x \in \mathfrak{gl}_n(k) \mid \operatorname{Tr} x = 0\}$$

and

$$\operatorname{Tr}(xy - yx) = \operatorname{Tr}(xy) - \operatorname{Tr}(yx) = \operatorname{Tr}(xy) - \operatorname{Tr}(xy) = 0.$$

Just as finite representations of finite groups are completely reducible, so too are finite representations of semisimple Lie algebras; this is Weyl's theorem on complete reducibility. A semisimple Lie algebra is a direct sum of simple Lie algebras, those which are non-abelian with no nontrivial ideals. Cartan's criterion gives an equivalent condition for semisimplicity in characteristic 0, namely that its Killing form $B_{\mathfrak{g}}(x, y) = \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y))$ is nondegenerate.

Example ($\mathfrak{sl}_2(k)$ is semisimple)**.** Let us use Cartan's criterion to show that $\mathfrak{g} = \mathfrak{sl}_2(k)$ is semisimple [cf Notation]. Using the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the relations

$$[e,f]=h,\quad [h,f]=-2f,\quad [h,e]=2e,$$

we can compute for instance

$$\operatorname{Tr}(\operatorname{ad}(h)\operatorname{ad}(h)) = \operatorname{Tr}\begin{pmatrix}4 & * & * \\ * & 4 & * \\ * & * & 0\end{pmatrix} = 8$$

by writing

$$\begin{gathered}\operatorname{ad}(h)\operatorname{ad}(h)e = [h,[h,e]] = [h,2e] = 4e\\ \operatorname{ad}(h)\operatorname{ad}(h)f = [h,[h,f]] = [h,-2f] = 4f\\ \operatorname{ad}(h)\operatorname{ad}(h)h = 0.\end{gathered}$$

Doing this for the other pairs of basis elements and using symmetry, in summary the matrix of $B_{\mathfrak{g}}$ is

$$\begin{pmatrix} \operatorname{Tr}\begin{pmatrix}0 & * & * \\ * & 0 & * \\ * & * & 0\end{pmatrix} & \operatorname{Tr}\begin{pmatrix}2 & * & * \\ * & 0 & * \\ * & * & 2\end{pmatrix} & \operatorname{Tr}\begin{pmatrix}0 & * & * \\ * & 0 & * \\ * & * & 0\end{pmatrix} \\ \operatorname{Tr}\begin{pmatrix}2 & * & * \\ * & 0 & * \\ * & * & 2\end{pmatrix} & \operatorname{Tr}\begin{pmatrix}0 & * & * \\ * & 0 & * \\ * & * & 0\end{pmatrix} & \operatorname{Tr}\begin{pmatrix}0 & * & * \\ * & 0 & * \\ * & * & 0\end{pmatrix} \\ \operatorname{Tr}\begin{pmatrix}0 & * & * \\ * & 0 & * \\ * & * & 0\end{pmatrix} & \operatorname{Tr}\begin{pmatrix}0 & * & * \\ * & 0 & * \\ * & * & 0\end{pmatrix} & \operatorname{Tr}\begin{pmatrix}4 & * & * \\ * & 4 & * \\ * & * & 0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8\end{pmatrix}$$

which is nondegenerate since it has determinant -128.

A subalgebra $\mathfrak{h}$ is Cartan if it is commutative, is such that $\operatorname{ad}(h)$ is semisimple for all $h \in \mathfrak{h}$, and is maximal among those subalgebras satisfying these two conditions. Recall that a linear transformation is semisimple if every T-invariant subspace has a complementary T-invariant subspace.

Example (diagonal matrices form a Cartan subalgebra in $\mathfrak{sl}_n(k)$)**.** Let $\mathfrak{h}$ denote the diagonal matrices in $\mathfrak{sl}_n(k)$. It is commutative because diagonal matrices commute, and $\operatorname{ad}(h)$ is semisimple because any diagonal matrix such as h is semisimple. It is maximal because the centralizer

$$\mathfrak{z}_{\mathfrak{sl}_n(k)}(\mathfrak{h}) = \{x \in \mathfrak{sl}_n(k) \mid [\mathfrak{h}, x] = 0\}$$

of $\mathfrak{h}$ is precisely $\mathfrak{h}$, so any larger subalgebra would not be commutative. To compute the centralizer, note any element x that centralizes $\mathfrak{h}$ cannot have a nonzero element x_{ij} off the diagonal because then

$$(e_{ii}x - xe_{ii})_{ij} = (i\text{th row of } x - i\text{th column of } x)_{ij} = x_{ij} - 0 \neq 0,$$

which would mean $[e_{ii}, x] \neq 0$.

Fixing a Cartan subalgebra $\mathfrak{h}$, the adjoints $\operatorname{ad}(h)$ are simultaneously diagonalizable, hence for semisimple $\mathfrak{g}$ we have a decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$$

where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \operatorname{ad}(h)x = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Now

$$\Delta = \{\lambda \in \mathfrak{h}^* - 0 \mid \mathfrak{g}_\lambda \neq 0\}$$

forms the root system of $\mathfrak{g}$.

Example (root system and decomposition of $\mathfrak{sl}_n(k)$)**.** Suppose that for some $x \in \mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$ we have $\operatorname{ad}(h)x = \lambda(h)x$ for all $h \in \mathfrak{h}$. Then

$$((h_{ii} - h_{jj})x_{ij})_{ij} = hx - xh = \operatorname{ad}(h)x = \lambda(h)x = \left(\sum_k \lambda(e_{kk})h_{kk}\right)x,$$

and picking a spot off the diagonal where $x_{ij} \neq 0$, it follows that

$$h_{ii} - h_{jj} = \sum_k \lambda(e_{kk})h_{kk}.$$

Since this must hold for all h, it follows that λ must send e_{ii}, e_{jj} to $1, -1$ respectively and all others to zero. Thus the root system is

$$\Delta = \{\lambda_i - \lambda_j \mid i \neq j\}$$

where $\lambda_i = e_{ii}^*$. To compute $\mathfrak{g}_{\lambda_i - \lambda_j}$, observe that when $h = e_{kk}$, the matrix $((h_{ii} - h_{jj})x_{ij})_{ij}$ is the difference of the kth row and the kth column of x, and $\sum_k (\lambda_i - \lambda_j)(e_{kk})h_{kk}$ is 1 if $k = i$ and -1 if $k = j$ and 0 otherwise. Now e_{ij} satisfies these $h = e_{kk}$ cases, hence $x \in \mathfrak{g}_{\lambda_i - \lambda_j}$, and it follows that $\mathfrak{g}_{\lambda_i - \lambda_j}$ is just the span of e_{ij} because the dimension of such a summand is always 1.

The subgroup generated by reflections through the hyperplanes orthogonal to the roots form the Weyl group.

Example (Weyl group of $\mathfrak{sl}_n(k)$ is S_n)**.** The reflection through $\lambda_i - \lambda_j$ swaps the i and j coordinates since

$$\begin{aligned} s_{\lambda_i - \lambda_j}(v) &= v - 2\frac{\langle v, \lambda_i - \lambda_j\rangle}{\|\lambda_i - \lambda_j\|^2}(\lambda_i - \lambda_j) \\ &= v - \langle v, \lambda_i - \lambda_j\rangle(\lambda_i - \lambda_j) \\ &= v - (v_i - v_j)\lambda_i + (v_i - v_j)\lambda_j. \end{aligned}$$

Transpositions generate the symmetric group, so the Weyl group is S_n.

In any root system Δ there is a positive root system Δ^+ such that $\Delta = \Delta^+ \sqcup -(\Delta^+)$ and if $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta^+$. The simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ are those which cannot be expressed as a sum of two positive roots.

Example (Positive and simple roots of $\mathfrak{sl}_n(k)$)**.** For $\Delta = \{\lambda_i - \lambda_j \mid i \neq j\}$ a positive root system is $\Delta^+ = \{\lambda_i - \lambda_j \mid i < j\}$: indeed $(\lambda_i - \lambda_j) + (\lambda_{i'} - \lambda_{j'}) \in \Delta^+$ if and only if $j = i'$, whence $\lambda_i - \lambda_{j'} \in \Delta^+$. The simple roots Π are then

$$\alpha_1 = \lambda_1 - \lambda_2, \quad \ldots, \quad \alpha_{n-1} = \lambda_{n-1} - \lambda_n$$

From the simple roots we form a Cartan matrix $(\langle \alpha_i, \alpha_j\rangle)$, which has the properties that $a_{ii} = 2$, that $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. This gives a Dynkin diagram, with vertices given by $\alpha_1, \ldots, \alpha_n$, and a type of edge from α_i to α_j depending on the following:

$$\begin{cases} \text{no edge} & a_{ij} = a_{ji} = 0 \\ \text{single edge} & a_{ij} = a_{ji} = -1 \\ \text{double directed edge} & a_{ij} = -1, a_{ji} = -2 \\ \text{triple directed edge} & a_{ij} = -1, a_{ji} = -3. \end{cases}$$

Example (Dynkin diagram of $\mathfrak{sl}_n(k)$ is A_{n-1})**.** The $(n-1) \times (n-1)$ Cartan matrix thus has 2's down the diagonal and -1's down the ± 1 diagonals, with zeros everywhere else. This gives the Dynkin diagram A_{n-1}.

In fact the Dynkin diagram of an irreducible root system is a member of one of the infinite families $A_\ell, B_\ell, C_\ell, D_\ell$, or one of E_6, E_7, E_8, F_4, G_2. Thus this classifies the simple Lie algebras over the algebraically closed field k of characteristic 0.

For the simple roots $\alpha_1, \ldots, \alpha_\ell$ of a Lie algebra $\mathfrak{g}$, the fundamental weights π_i are those satisfying $\langle \alpha_j^*, \pi_i \rangle = \delta_{ij}$. Set

$$Q = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \quad \text{and} \quad P = \bigoplus_{i=1}^{\ell} \mathbb{Z}\pi_i$$

and denote by Q^+ and P^+ the positive summands. Both Q and P are lattices in $\mathfrak{g}$ and are respectively called the root and weight lattices. To get a notion of highest weight, define a partial ordering on P by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Example (Roots and weights in $\mathfrak{sl}_2(k)$)**.** Here $\alpha = \lambda_1 - \lambda_2$ spans $\mathfrak{h}^*$, and we have $\alpha(h) = 2$. Thus setting $\rho = \frac{\alpha}{2}$ (more generally known as the Weyl vector), we have

$$\Delta = \{\pm\alpha\}, \quad \Delta^+ = \Pi = \{\alpha\}, \quad W = \{\pm 1\}, \quad P = \mathbb{Z}\rho.$$

Note that the partial ordering on P is just the ordering of $\mathbb{Z}$.

Given a finite-dimensional representation $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(V)$, there are the weights λ of the representation, which are those such that

$$V_\lambda = \{v \in V \mid \sigma(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$$

is nonzero, and the decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda.$$

Theorem (Theorem of the highest weight)**.** *The irreducible representations of $\mathfrak{g}$ are parametrized by its weights P^+. Namely, any irreducible representation has a highest weight in P^+, and conversely for any weight $\lambda \in P^+$ there exists a unique irreducible representation $L^+(\lambda)$ with highest weight λ.*

Example (the weights of $\mathfrak{sl}_2(k)$ parametrize its representations)**.** Let σ_n denote the n-dimensional irredudible representation of $\mathfrak{sl}_2(k)$ which we worked out before. The h-eigenvalues $n-2i$ with $0 \leq i \leq n$ correspond to the weights $(n-2i)\rho$: indeed for the generator $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}$ and $a\rho \in \mathfrak{h}^*$, the condition $\sigma_n(h)X^{n-i}Y^i = a\rho(h)X^{n-i}Y^i$ translates to $(n-2i)X^{n-i}Y^i = aX^{n-i}Y^i$. Thus the theorem asserts that the σ_n exhaust all irreducible representations of $\mathfrak{sl}_2$ which is what we deduced by hand.

3. Borel-Weil-Bott

The Borel-Weil-Bott theorem realizes the highest weight representations as the sheaf cohomology groups of line bundles on the flag variety.

The flag manifold X of a semisimple algebraic group G is the set of Borel subgroups of G and that $X = G/B$, where B is a fixed Borel subgroup. Here we invoke a theorem that B is the stabilizer of itself under the action of conjugation.

Example (flag manifold for $SL_n(k)$ is the flags in k^n)**.** Indeed $G = SL_n(k)$ acts on the set of flags in k^n via

$$g(V_1 \subset \cdots \subset V_n) = g(V_1) \subset \cdots \subset g(V_n),$$

and the upper triangular matrices B in G form a Borel subgroup. The stabilizer of the standard flag

$$ke_1 \subset ke_1 \oplus ke_2 \subset \cdots \subset ke_1 \oplus ke_2 \oplus \cdots \oplus ke_n$$

is just B, so indeed G/B is the flag manifold.

Example (flag manifold for $SL_2(k)$ is $\mathbb{P}^1$)**.** Continuing above, when $n = 2$ a flag is determined by a line through the origin, hence gives an element of $\mathbb{P}^1$. The action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \langle (x, y) \rangle = (xa + yb, cx + yd)$$

corresponds to the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

where we think of $z = x/y$. given by Mobius transformations on $\mathbb{C}$.

$$\begin{array}{rcl} B \setminus (G \times U) & \longleftarrow\!\shortmid & U \\ V & \longmapsto & V_B \end{array}$$

$$\begin{tikzcd}
\begin{matrix}G\text{-equivariant}\\ \text{locally free}\\ \mathcal{O}_X\text{-modules of}\\ \text{finite rank}\end{matrix} \arrow[r, shift left] & \begin{matrix}G\text{-equivariant}\\ \text{vector bundles on}\\ X = G/B\end{matrix} \arrow[l, shift left, "\mathcal{O}_X"] \arrow[r, shift left, "\text{pg } 255"] & \begin{matrix}\text{finite-dimensional}\\ B\text{-modules}\end{matrix} \arrow[l, shift left] \\
 & \begin{matrix}G\text{-equivariant line}\\ \text{bundles}\end{matrix} \arrow[u, hook] \arrow[r, leftrightarrow] & \begin{matrix}\text{one-dimensional}\\ B\text{-modules}\end{matrix} \arrow[u, hook] \\
 & & \begin{matrix}\text{characters of}\\ H = B/N\end{matrix} \arrow[u, leftrightarrow, "\text{pg } 256"'] \arrow[ul, "\Lambda"] \arrow[uull, bend left, "\mathcal{L}"]
\end{tikzcd}$$

There is a bijection between G-equivariant vector bundles on X and finite-dimensional B-modules, as follows. In the forward direction V_B is a finite-dimensional B-module. Conversely, given a finite-dimensional B-module U, consider the locally free B-action on the trivial vector bundle $G \times U$ on G given by $b(g, u) = (gb^{-1}, bu)$. The quotient space $V = B \setminus (G \times U)$ obtained by this action is an algebraic vector bundle on $X = G/B$, with trivializations given by the fact that the morphism

$N^- \to G/B$ given by $n \mapsto gnB$ is an open embedding, and the flag variety X is covered by the open subsets $gN^-B/B \subset X$ which are isomorphic to $N^- \cong k^{|\Delta^+|}$:

$$X = \bigcup_{g \in G} gN^-B/B.$$

Moreover the action $g'(g,u) = (g'g,u)$ descends to an action on V that satisfies $V_B = U$. In particular, G-equivariant line bundles correspond to one dimensional B-modules, so since actions by the unipotent radical $N = R_u(B)$ are trivial, they correspond to characters $\lambda \in L = \mathrm{Hom}(B,k^*)$ of $H = B/N$. Set $\mathcal{L}(\lambda) = \mathcal{O}_X(\Lambda(\lambda))$.

Example ($\mathcal{L}(n\rho) = \mathcal{O}(-n)$)**.** Let us see that the $SL_2(k)$-equivariant vector bundle $\mathcal{L}(n\rho)$ is just the twisted sheaf $\mathcal{O}_{\mathbb{P}^1}(-n)$. It is constructed as $(SL_2(k) \times \mathbb{C}_{n\rho})/B$, where $b(g,u) = (gb, b^{-1}u)$. Here $\mathbb{C}_{n\rho}$ is the one-dimensional representation corresponding to the character $e^{n\rho} = (e^\rho)^n$ which behaves as

$$(e^\rho)^n \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} = a^n$$

of H. By above, the pair

$$\left[\begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix}, u \right] \in (SL_2(k) \times \mathbb{C}_{n\rho})/B$$

has no other element in its equivalence class. Since $x_{1/0}$ and t each vary over $\mathbb{C}$, this shows that the vector bundle $\mathcal{L}(n\rho)$ is trivial on U_0, and similarly on U_1. On their intersection where $x_{1/0} \neq 0$, we have in particular the following identification by the B-action:

$$\begin{aligned} \left(\begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix}, u \right) &\sim \left(\begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix} \begin{pmatrix} x_{1/0}^{-1} & -1 \\ 0 & x_{1/0} \end{pmatrix}, \begin{pmatrix} x_{1/0} & 1 \\ 0 & x_{1/0}^{-1} \end{pmatrix} u \right) \\ &\sim \left(\begin{pmatrix} x_{1/0}^{-1} & -1 \\ 1 & 0 \end{pmatrix}, x_{1/0}^n u \right). \end{aligned}$$

On $\mathbb{P}^1$ we have the typical transition from U_0 to U_1, and on the bundle we have multiplication by $x_{1/0}^n$. Thus this sheaf is $\mathcal{O}_{\mathbb{P}^1}(-n)$.

Example ($SL_2(k)$-action on $\mathcal{L}(n\rho)$)**.** We can compute the $SL_2(k)$-action $g(g_1,u) = (gg_1,u)$ on the vector bundle as

$$\begin{pmatrix} a & b. \\ c & d \end{pmatrix} (z,u) = \left(\frac{az+b}{cz+d}, (cz+d)^n u \right).$$

Indeed over U_0 we have

$$\begin{aligned} \begin{pmatrix} a & b. \\ c & d \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix}, u \right) &= \left(\begin{pmatrix} a + bx_{1/0} & b \\ c + dx_{1/0} & d \end{pmatrix}, u \right) \\ &\sim \left(\begin{pmatrix} a + bx_{1/0} & b \\ c + dx_{1/0} & d \end{pmatrix} \begin{pmatrix} \frac{1}{a+bx_{1/0}} & -b \\ 0 & a + bx_{1/0} \end{pmatrix}, \begin{pmatrix} a + bx_{1/0} & b \\ 0 & \frac{1}{a+bx_{1/0}} \end{pmatrix} u \right) \\ &= \left(\begin{pmatrix} 1 & 0 \\ \frac{c+dx_{1/0}}{a+bx_{1/0}} & 1 \end{pmatrix}, (a + bx_{1/0})^n u \right), \end{aligned}$$

and similarly over U_1 we have

$$\begin{pmatrix} a & b. \\ c & d \end{pmatrix}\left(\begin{pmatrix} x_{0/1} & -1 \\ 1 & 0 \end{pmatrix}, u\right) = \left(\begin{pmatrix} ax_{0/1}+b & -a \\ cx_{0/1}+d & -c \end{pmatrix}, u\right)$$
$$\sim \left(\begin{pmatrix} ax_{0/1}+b & -a \\ cx_{0/1}+d & -c \end{pmatrix}\begin{pmatrix} \frac{1}{cx_{0/1}+d} & c \\ 0 & cx_{0/1}+d \end{pmatrix}, \begin{pmatrix} cx_{0/1}+d & -c \\ 0 & \frac{1}{cx_{0/1}+d} \end{pmatrix} u\right)$$
$$= \left(\begin{pmatrix} \frac{ax_{0/1}+b}{cx_{0/1}+d} & -1 \\ 1 & 0 \end{pmatrix}, (cx_{0/1}+d)^n u\right).$$

There is an equivalence of categories between algebraic vector bundles on X and locally free $\mathcal{O}_X$-modules. Indeed we get $\mathcal{V} = \mathcal{O}_X(V)$ by taking the sheaf of algebraic sections on X, and in the reverse direction we take the spectrum of the symmetric algebra of $V^\bullet$. Moreover a G-action on a vector bundle is G-equivariant if $g(V_x) = V_{gx}$ and $g\colon V_x \to V_{gx}$ is a linear isomorphism. The vector space $\Gamma(X, \mathcal{V})$ has a G-action via $(gs)(x) = g(s(g^{-1}x))$, and by categorizing this we obtain G-actions on each cohomology group $H^i(X, \mathcal{V})$ (see page 254-255). Thus from vector bundles we obtain representations.

Example ($SL_2(k)$-action on $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$)**.** Fix the following notations. Let $[x_0 : x_1]$ be the coordinates on $\mathbb{P}^1$. Let U_0 be the open set where x_0 does not vanish, and let $x_{1/0}$ be the coordinate on U_0. Similarly define U_1 and $x_{0/1}$. Pick

$$\begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_{0/1} & 1 \\ -1 & 0 \end{pmatrix}$$

to be a representative for an arbitrary element of U_0 and U_1 respectively. Note the only element of B that fixes these matrices is the identity: for U_0 we have

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix} = \begin{pmatrix} a+bx_{1/0} & b \\ a^{-1}x_{1/0} & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_{1/0} & 1 \end{pmatrix}$$

if and only if $b = 0$ and $a = 1$, and similarly for U_1.

Let us compute the G-action $(gs)(x) = g(s(g^{-1}x))$. The global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$ are homogeneous degree n polynomials in the variables x_0, x_1. Restricting to U_0 and dividing out by x_0, we get the sections $\bigoplus_{k=0}^n \mathbb{C}x_{1/0}^k$. If $x \in \mathbb{P}^1$ and $s = x_{1/0}^k$, then we compute

$$\begin{aligned} gs(g^{-1}x) &= gs\left(\frac{dx-b}{-cx+a}\right) \\ &= g\left(\frac{dx-b}{-cx+a}, \left(\frac{dx-b}{-cx+a}\right)^k\right) \\ &= \left(x, \left(c\left(\frac{dx-b}{-cx+a}\right)+d\right)^{-n}\left(\frac{dx-b}{-cx+a}\right)^k\right) \\ &= \left(x, \left(\frac{1}{-cx+a}\right)^{-n}\left(\frac{dx-b}{-cx+a}\right)^k\right) \\ &= (x, (-cx+a)^{n-k}(dx-b)^k). \end{aligned}$$

There is a similar computation on U_1. Together, we can write globally that the G-action is described by

$$g \cdot x_0^k x_1^{n-k} = (-cx_1 + ax_0)^{n-k}(dx_1 - bx_0)^k.$$

Example ($\mathfrak{sl}_2(k)$-action on $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$**.** Before getting the action of the Lie algebra $\mathfrak{sl}_2$, let us orient ourselves. This G-action by g is a linear operator on the $(n+1)$-dimensional vector space $V = \langle x_0^k x_1^{n-k} \rangle_k$, in other words we have a representation $G \to GL(V)$. The associated representation of the Lie algebra is the derivation of this map at the tangent plane $dG_1 = \mathfrak{g}$ at the identity. To be precise, then, we should take a chart around the identity, think of $GL(V)$ as embedded in $\mathbb{R}^{(n+1)^2}$, compute the Jacobian of the resulting smooth map $\mathbb{R}^3 \to \mathbb{R}^{(n+1)^2}$ and then evaluate at the identity. The resulting linear map is represented by a matrix in $\mathbb{R}^{3\times(n+1)^2}$, and to get the action of e, f, h on the space $\mathbb{R}^{(n+1)^2}$ we multiply the matrix by the points representing e, f, h in the chart. We can then read off the actions, which should agree with the matrices we obtained by hand.

Alternatively, we can wave our hands a little and make the following analogous global computations:

$$\frac{d}{da}(g \cdot x_0^k x_1^{n-k})|_1 = (n-k)(-cx_1 + ax_0)^{n-k-1} x_0 (dx_1 - bx_0)^k|_1 = (n-k) x_0^{n-k} x_1^k$$
$$\frac{d}{db}(g \cdot x_0^k x_1^{n-k})|_1 = (-cx_1 + ax_0)^{n-k} k (dx_1 - bx_0)^{k-1}(-x_0)|_1 = -k x_0^{n-k+1} x_1^{k-1}$$
$$\frac{d}{dc}(g \cdot x_0^k x_1^{n-k})|_1 = (n-k)(-cx_1 + ax_0)^{n-k-1}(-x_1)(dx_1 - bx_0)^k|_1 = -(n-k) x_0^{n-k-1} x_1^{k+1}$$
$$\frac{d}{dd}(g \cdot x_0^k x_1^{n-k})|_1 = (-cx_1 + ax_0)^{n-k} k (dx_1 - bx_0)^{k-1}|_1 = k x_0^{n-k} x_1^k.$$

This shows that on the subspace $x_0^k x_1^{n-k}$ we have

$$e = \left.\frac{d}{db}\right|_1 = -x_0 \partial_1, \quad f = \left.\frac{d}{dc}\right|_1 = -x_1 \partial_0, \quad h = \left.\frac{d}{da}\right|_1 - \left.\frac{d}{dd}\right|_1 = n - 2k,$$

which agrees up to sign with what we obtained by hand.

To state the Borel-Weil-Bott theorem, first we define the singular and regular weights

$$P_{\text{sing}} = \{\lambda \in P \mid \exists \alpha \in \Delta \text{ such that } \langle \lambda - \rho, \alpha^\vee \rangle = 0\}$$
$$P_{\text{reg}} = P - P_{\text{sing}}$$

and the shifted action

$$w \star \lambda = w(\lambda - \rho) + \rho$$

for $w \in W$ and $\lambda \in P$. One can show that $W \star P_{\text{sing}} = P_{\text{sing}}$ and that $-P^+$ is a fundamental domain for this shifted action.

Example ($-P^+$ is a fundamental domain for $\star$ in $SL_2(k)$)**.** Indeed $-P^+ = -\mathbb{Z}^{\geq 0}\rho$ is a fundamental domain for the shifted W-action: for $-1 \in W$ we have

$$-1 \star n\rho = -1(n\rho - \rho) + \rho = (2-n)\rho,$$

so this flips the weight lattice about ρ.

Theorem (Borel-Weil-Bott)**.** *Let $\lambda \in L \subset P$. If $\lambda \in P_{\text{sing}}$, then $H^i(X, \mathcal{L}(\lambda)) = 0$ for $i \geq 0$. If $\lambda \in P_{\text{reg}}$ and $w \in W$ is such that $w \star \lambda \in -P^+$, then*

$$H^i(X, \mathcal{L}(\lambda)) = \begin{cases} L^-(w \star \lambda) & i = \ell(w) \\ 0 & i \neq \ell(w). \end{cases}$$

Example (Borel-Weil-Bott for $SL_2(k)$)**.** Recall that $\mathcal{L}(n\rho) = \mathcal{O}(-n)$. In the case $n \leq 0$ we have

$$\begin{aligned} L^-(1 \star n\rho) &= L^-(n\rho) \\ &= \text{homogeneous degree } n \text{ polynomials} \\ &= H^0(\mathbb{P}^1, \mathcal{O}(-n)), \\ &= H^0(\mathbb{P}^1, \mathcal{L}(n\rho)), \end{aligned}$$

which matches the theorem since we may take $w = 1$ which has length 0. In the case $n \geq 2$, recalling that Serre duality gives

$$H^0(\mathbb{P}^k, \mathcal{O}(i)) \cong H^k(\mathbb{P}^k, \mathcal{O}(-k-1-i))^\vee,$$

we have

$$\begin{aligned} L^-(-1 \star n\rho) &= L^-((2-n)\rho) \\ &= \text{homogeneous degree } n \text{ polynomials} \\ &= H^0(\mathbb{P}^1, \mathcal{O}(n-2)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}(-n)) \\ &= H^1(\mathbb{P}^1, \mathcal{L}(n\rho)) \end{aligned}$$

which also matches the theorem since we may take $w = -1$ which has length 1.

4. Harish-Chandra

For motivation, let us get our hands dirty and work through a representation of $SL_2(\mathbb{R})$: let $\mathscr{H} = L^2(\mathbb{R}, dx/(x^2+1))$ and $(\pi(g)f)(x) = f(g^{-1}x)$. On the circle group we compute

$$(\pi(k(\theta))f)(0) = f\left(\frac{\cos\theta \cdot 0 - \sin\theta}{-\sin\theta \cdot 0 + \cos\theta}\right) = f(-\tan\theta).$$

In general, representations $\rho\colon K \to U(\mathscr{H})$ of the circle group are quite structured; the images of the projections

$$P_n(v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\rho(k(\theta))v\, d\theta$$

split $\mathscr{H} = \bigoplus_n \mathscr{H}_n$ into closed orthogonal subspaces, and the behavior of ρ on these subspaces is predictable: $\rho(k(\theta))v = e^{in\theta}v$ in the subspace $\mathscr{H}_n$. Thus in the present case, if $f \in \mathscr{H}_n$, then

$$f(-\tan\theta) = (\pi(k(\theta))f)(0) = (e^{in\theta}f)(0) = e^{in\theta}f(0).$$

Taking $\theta = \pi$, we find $f(0) = (-1)^n f(0)$, whence n must be even, say $= 2m$. Therefore

$$f(x) = f(0)e^{-2im\tan^{-1}x}.$$

The functions $f_{2m}(x) = e^{-2im\tan^{-1}x}$ scaled by $1/\sqrt{m}$ are easily checked to form an orthonormal basis of $\mathscr{H}$.

A compatible representation on $\mathfrak{g} = \mathfrak{sl}_2$ had better agree with π on $\mathfrak{k}$; that is, its restriction to $\mathfrak{k}$ had better be $d\pi|_{\mathfrak{k}}$ and so satisfy

$$\pi(x)f = \lim_{t\to 0}\frac{1}{t}(\pi(\exp(tx))f - f)$$

for $x \in \mathfrak{k}$. In fact this defines a representation on all of $\mathfrak{g}$. For $h \in \mathfrak{g}_0$ we have

$$\exp(th) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

so

$$\begin{aligned}
(\pi(h)f)(x) &= \lim_{t\to 0} \frac{1}{t}((\pi(\exp(th)))f(x) - f(x)) \\
&= \lim_{t\to 0} \frac{1}{t}\left(f\left(\frac{e^{-t}x - 0}{-0\cdot x + e^t}\right) - f(x)\right) \\
&= \lim_{t\to 0} \frac{f(e^{-2t}x) - f(x)}{t} \\
&= -2x\frac{df}{dx}(x). && \text{(L'Hopital)}
\end{aligned}$$

Thus

$$(\pi(h)f_{2m})(x) = -2xf_{2m}'(x) = -\frac{4imx}{x^2+1}e^{-2im\tan^{-1}x}.$$

Writing $\theta = \tan^{-1} x$, we have

$$\frac{x}{x^2+1} = \frac{\tan\theta}{\sec^2\theta} = \sin\theta\cos\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = \frac{1}{4i}(e^{2i\theta} - e^{-2i\theta}).$$

Hence

$$\pi(h)f_{2m} = -mf_{2m+2} + mf_{2m-2}.$$

Similarly we can compute

$$\begin{gathered}
(\pi(e)f) = -\frac{df}{dx}(x) \\
\pi(e)f_{2m} = \frac{im}{2}f_{2m+2} + imf_{2m} + \frac{im}{2}f_{2m-2} \\
(\pi(f)f)(x) = x^2\frac{df}{dx} \\
\pi(f)f_{2m} = \frac{im}{2}f_{2m+2} - imf_{2m} + \frac{im}{2}f_{2m-2}.
\end{gathered}$$

Taking a step back, note that $(\pi, \mathscr{H})$ is certainly not irreducible because there obviously is a one-dimensional invariant subspace $\mathscr{H}_0$ of constant functions. In fact we see that $\mathscr{H}_K$ has exactly three proper subspaces corresponding to the negative even numbers, the positive ones, and zero, but the representations on the quotient $\mathscr{H}^{\pm}/\mathscr{H}_0$ are both irreducible.

Let us collect the following key features of the above example:

- $(\pi, \mathscr{H})$ is a representation of both $\mathfrak{g}$ and K
- the span of $\pi(K)v$ is finite
- $\pi(x)f = \lim_{t\to 0}\frac{1}{t}(\pi(\exp(th))f - f)$ for all $x \in \mathfrak{k}$.

In general, such a representation $(\pi, \mathscr{H})$ satisfying these key features is called a $(\mathfrak{g}, K)$-module, and in general the formula

$$\pi(x)v = \lim_{t\to 0}\frac{1}{t}(\pi(\exp(th))v - v) \in H_K$$

defines a representation of $\mathfrak{g}_0$ on the subspace $\mathscr{H}_K$ of K-finite vectors. This $(\mathfrak{g}, K)$-module is called the Harish-Chandra module $(\pi, \mathscr{H}_K)$. Moreover this construction induces a lattice isomorphism

$$\begin{array}{ccc} \text{closed invariant subspaces of } H & \longleftrightarrow & K\text{- and } \mathfrak{g}_0\text{-invariant subspaces } of H_K \\ \mathscr{S} & \longmapsto & \mathscr{S} \cap H_K \\ \overline{\mathscr{S}_K} & \longleftarrow & \mathscr{S}_K \end{array}$$

Harish-Chandra introduced $(\mathfrak{g}, K)$-modules to reduce the analytic study of irreducible unitary representations of a real reductive Lie group G to the algebraic study of irreducible $(\mathfrak{g}, K)$-modules. In the manner above, one associates to an admissible group representation $(\pi, \mathscr{H})$ the Harish-Chandra $(\mathfrak{g}, \mathscr{H})$-module $(\pi, \mathscr{H}_K)$, and it turns out that every irreducible $(\mathfrak{g}, K)$-module arises as a Harish-Chandra module in this way. Moreover the unitary irreducible representations are precisely the irreducible $(\mathfrak{h}, K)$-modules. In fact for $SL_2(\mathbb{R})$ the irreducible admissible representations can be found by decomposing the principal series representations into irreducible components and determining the isomorphisms. This is true in general for reductive Lie groups and is known as Casselman's subrepresentation theorem.

Our goal then is to determine structure of the Harish-Chandra modules of $SL(2, \mathbb{R})$. Set

$$H = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$X = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$
$$Y = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

so that we have $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$.

One computes immediately that on any $(\mathfrak{g}, K)$-module, X lowers, Y raises, and H scales the subspaces $\mathscr{H}_n$. For example $\pi(H)v = nv$ is by definition of a $(\mathfrak{g}, K)$-module, and then

$$\pi(H)\pi(X)v = \pi(X)\pi(H)v + 2\pi(X)v = (n + 2)\pi(X)v$$

shows that $\pi(X)v \in \mathscr{H}_{n+2}$ (similarly for $\pi(Y)v \in \mathscr{H}_{n-2}$).

Similarly, the Casmir operator

$$\Omega = (H - 1)^2 + 4XY = (H + 1)^2 + 4YX$$

scales on any irreducible admissible $(\mathfrak{g}, K)$-module, and we let λ denote a square root of this scalar. Note then that

$$\pi(XY)v = \frac{1}{4}(\lambda^2 - (n - 1)^2)v, \quad \pi(YX)v = \frac{1}{4}(\lambda^2 - (n + 1)^2)v.$$

The strategy for classifying the Harish-Chandra modules is to construct, for any irreducible admissible $(\mathfrak{g}, K)$-module (π, V), a collection of vectors $w_j \in V_j$ that satisfy the following things. Let n be such that $w_n \in V_n$ is nonzero.

- the $\mathfrak{g}$-action

$$\pi(H)w_j = jw_j$$
$$\pi(X)w_j = \frac{1}{2}(\lambda + (j + 1))w_{j+2}$$

$$\pi(Y)w_j = \frac{1}{2}(\lambda - (j-1))w_{j-2}$$

- $w_{n+2(m+1)} = 0$ if and only if $w_{n+2m} = 0$ or $\lambda = \pm(n+2m+1)$ when $m \geq 0$
- $w_{n-2(m+1)} = 0$ if and only if $w_{n-2m} = 0$ or $\lambda = \pm(n-2m-1)$ when $m \geq 0$

By irreducibility the nonzero w_j would form a basis of V, so such (π, V) would be classified up to the scalar $\pi(\Omega)$ and any n such that $V_n \neq 0$. Furthermore, such w_j determine the K-types of V by using the second and third conditions to spread out as far as possible from n. For example if λ is not an integer of the same parity as $\mu+1$, then taking $n = \mu$, the w_j are never 0 because

$$\lambda \neq \pm\mu \pm 2m \pm 1,$$

so the K-types are $\mu + 2m$ with $m \in \mathbb{Z}$. Similarly if λ is a nonzero integer of the same parity as $\mu+1$, then taking $n = 0$ in the definition of w_n, we see that the spread stops at $\pm 2r$ where $n = 2r+1$, so the K-types $|\lambda|-1, |\lambda|-3, \ldots, -|\lambda|+1$. If $\lambda = 0$ and $\mu = \pm 1$, then the K-types are $\mu + 2(\operatorname{sign}\mu)m$ as $m = 0, 1, 2, \ldots$.

To construct these w_j, choose $w_n \in \mathscr{H}_n$ nonzero, and inductively define

$$w_{n+2(m+1)} = \frac{2}{\lambda + (n+2m+1)}\pi(X)w_{n+2m}$$

$$w_{n-2(m+1)} = \frac{2}{\lambda - (n-2m-1)}\pi(Y)w_{n-2m},$$

where we interpret any number divided by zero to be zero. Let us show that $w_{n-2(m+1)} = 0$ if and only if $w_{n-2m} = 0$ or $\lambda = \pm(n-2m-1)$; the argument for the other statement is by symmetry of the raising and lowering.

If $\pi(X)v = 0$, then the elements $w'_{n-2m} = \pi(Y)^m v$ span V; in particular $\mathscr{H}_{n+2} = 0$. To see this, let W denote the span of w'_{n-2m}. It suffices to show that W is invariant under $\mathfrak{g}$ by irreducibility. Now $\pi(H)$ leaves W invariant since

$$\pi(H)w'_{n-2m} = (n-2m)w'_{n-2m},$$

and trivially $\pi(Y)$ does as well. For $\pi(X)$, consider $\pi(X)w'_{n-2m}$. If $m = 0$, then obviously $\pi(X)w'_n = \pi(X)v = 0 \in W$. If $m > 0$, then

$$\pi(X)w'_{n-2m} = \pi(XY)w'_{n-2(m-1)} \in \mathscr{H}_{n-2(m-1)} \subset W.$$

Thus if $\pi(XY)v = 0$, then $\pi(Y)v = 0$. Indeed suppose $v_1 = \pi(Y)v \neq 0$ so that $\pi(X)v_1 = \pi(XY)v = 0$. Since $v_1 \neq 0$, we apply the above paragraph to get that $\pi(Y)^m v_1$ spans V and that $\mathscr{H}_{(n-2)+2} = \mathscr{H}_n = 0$, contradicting $v \neq 0$.

Therefore $w_{n-2(m+1)} = 0$ if and only if $w_{n-2m} = 0$ or $\lambda = \pm(n-2m-1)$. To see this, note we may assume $w_{n-2m} \neq 0$ and $\lambda \neq n-2m-1$ since otherwise $w_{n-2(m+1)} = 0$ by definition. Then $w_{n-2(m+1)}$ is a nonzero multiple of $\pi(Y)w_{n-2m}$, so we are reduced to showing that $\pi(Y)w_{n-2m} = 0$ if and only if $\lambda = -(n-2m-1)$. This is true because $\pi(Y)w_{n-2m} = 0$ if and only if $\pi(XY)w_{n-2m} = 0$ (by above) if and only if $\frac{1}{4}(\lambda^2 - (n-2m-1)^2)w_{n-2m} = 0$ if and only if $\lambda = \pm(n-2m-1)$ or $w_{n-2m} = 0$.

5. Unitary representations

The unitary representations of $SL_2(\mathbb{R})$ are best realized analytically, so let us describe some.

The discrete series $\mathscr{D}_n^\pm$ for $n \geq 2$ are defined on the Hilbert space

$$\left\{ f \text{ analytic on } \mathbb{H} \;\middle|\; \|f\|^2 = \iint_{\mathbb{H}} |f(z)|^2 y^n \frac{dx\,dy}{y^2} < \infty \right\}$$

as

$$\mathscr{D}_n^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (-bz+d)^{-n} f\left(\frac{ax-c}{-bx+d}\right)$$

and similarly for $\mathscr{D}_n^+$ but with complex conjugates. The principal series $\mathscr{P}^{\pm,iv}$ for $v \in \mathbb{R}$ are defined on the Hilbert space $L^2(\mathbb{R})$ as

$$\mathscr{P}^{\pm,iv} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = |-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right) \cdot (\operatorname{sign}(-bx+d) \text{ if } -).$$

The complementary series $\mathscr{C}^u$ for $0 < u < 1$ are defined on the Hilbert space

$$\left\{ f \colon \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)\overline{f(y)}\,dx\,dy}{|x-y|^{1-u}} < \infty \right\}$$

as

$$\mathscr{C}^u \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = |-bx+d|^{-1-u} f\left(\frac{ax-c}{-bx+d}\right).$$

There are also the limits $\mathscr{D}_1^\pm$ of the discrete series which are much like the discrete series but with norm

$$\|f\|^2 = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\,dx.$$

Bargmann classified the unitary irreducible representations of $SL_2(\mathbb{R})$ as follows.

Theorem 5.1. *The only unitary irreducible representations of $SL_2(\mathbb{R})$ are*

- *the discrete series $\mathscr{D}_n^\pm$ for $n \geq 2$*
- *the principal series $\mathscr{P}^{\pm,iv}$ for $v \neq 0$ and $\mathscr{P}^{+,0}$*
- *the complementary series $\mathscr{C}^u$ for $0 < u < 1$*
- *the limits $\mathscr{D}_1^\pm$ of the discrete series*
- *the trivial representation.*

The only unitary equivalences among these are $\mathscr{P}^{\pm,iv} \cong \mathscr{P}^{\pm,-iv}$.

It is easy to check that these are all unitary, so let us see why some of these representations are irreducible.

For irreducibility of the discrete series, the strategy is to work over the unit disk $\mathbb{D}$. Here

$$\mathscr{D}_n^+ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} z^N = e^{i(n+2N)i\theta} z^N,$$

and the K-finite elements are linear combinations of z^N. But

$$\frac{d}{dt} \mathscr{D}_n^+ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \Bigg|_{t=0} z^N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z^N = (n+N) z^{N+1} - N z^{N-1},$$

so any invariant subspace containing z^N must contain the entire space.

Now for irreducibility of the principal series, we use the following form of Schur's lemma: A unitary representation of a topological group G on a Hilbert space is irreducible if and only if the only bounded linear operators commuting with all G-actions are the scalars. Thus suppose B commutes with all $\mathscr{P}^{+,iv}(g)$. In particular B commutes with all translations

$$\mathscr{P}^{+,iv}\begin{pmatrix}1 & 0\\ z_0 & 1\end{pmatrix} f(z) = f(z-z_0),$$

so by Fourier theory $\widehat{Bf}(\zeta) = m(\zeta)\hat{f}(\zeta)$ for some bounded measurable function m. Similarly

$$\mathscr{P}^{+,iv}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} f(z) = |a|^{2+iv}\left(\frac{a}{|a|}\right)^k f(a^2z),$$

so pushing symbols gives (abbreviating $\mathscr{P}$ and setting $c =$ above constant)

$$B(f(a^2\cdot)) = B\left(\frac{1}{c}\mathscr{P}f\right) = \frac{1}{c}B(\mathscr{P}f) = \frac{1}{c}\mathscr{P}(Bf) = (Bf)(a^2\cdot).$$

Now multiplying by $e^{2\pi iz\cdot\zeta}$ and integrating, the left-hand side becomes

$$\begin{aligned} B\widehat{(f(a^2\cdot))}(\zeta) &= m(\zeta)\widehat{f(a^2\cdot)}(\zeta)\\ &= m(\zeta)\int_{\mathbb{C}} e^{2\pi iz\cdot\zeta} f(a^2z)\,dz\\ &= |a|^{-4}m(\zeta)\int_{\mathbb{C}} e^{2\pi iz\cdot a^{-2}\zeta} f(az)\,dz\\ &= |a|^{-4}m(\zeta)\widehat{f}(a^{-2}\zeta), \end{aligned}$$

and the right-hand side becomes

$$\begin{aligned} \int_{\mathbb{C}} e^{2\pi iz\cdot\zeta}(Bf)(a^2z)\,dz &= |a|^{-4}\int_{\mathbb{C}} e^{-2\pi z\cdot a^{-2}\zeta}(Bf)(z)\,dz\\ &= |a|^{-4}\widehat{Bf}(a^{-2}\zeta)\\ &= |a|^{-4}m(a^{-2}\zeta)\widehat{f}(a^{-2}\zeta). \end{aligned}$$

Therefore $m(a^{-2}\zeta) = m(\zeta)$ for almost all ζ and all a, but by Fubini's theorem it also holds for some ζ_0 for almost all a. Since $a^{-2}\zeta_0$ covers the right-half line as a varies, m must be a linear combination of 1 and sign. We will now show that the subspace U of functions whose Fourier transforms vanish on this right-half line is not stable under $\mathscr{P}^{\pm,iv}$ except in the case $\mathscr{P}^{-,0}$. Such functions are called "causal," and by Paley and Weiner (1936) $\widehat{f}$ extends to a holomorphic function on $\mathbb{H}$ which vanishes at infinity. Note $(x+i)^{-1}$ is in U because it is the boundary of the function $F(z) = (z+i)^{-1}$. Now suppose

$$\mathscr{P}^{\pm,iv}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(x+i)^{-1} = |x|^{-1-iv}i^{-1}(x+i)^{-1}\cdot(\mathrm{sign}(|x|^{-1-iv}) \text{ if } -)$$

is in U. Then $|x|^{iv}\cdot(\mathrm{sign}(x) \text{ if } +)$ is the nontangential boundary value of an analytic function F in $\mathbb{H}$. But now $F(z) - e^{-iv\mathrm{Log}\,z}$ has boundary value 0 on the right-half line but does not vanish on the left-half (because we have excluded $\mathscr{P}^{-,0}$). This contradicts the following theorem of Privalov: Let f be a single-valued analytic function in a domain D of the complex plane bounded by a rectifiable Jordan curve Γ. If on some set $E \subset \Gamma$ of positive Lebesgue measure f has non-tangential boundary values zero, then $f = 0$ in D.

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