Explaining a Non-Linear Simple Pendulum Exact Solution to Novice Differential Equation Students

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In my introductory physics classes, I studied the linear simple pendulum equation:

$$\ddot{\theta} + \omega_0^2 \theta = 0 \tag{1}$$

, which approximately models the motion of a massless, rigid rod connecting a hinge and a point-like mass. When the rod displaces from a vertical position, gravity oscillates the mass. The equation neglects the force of friction. I was curious about the exact model, so as a project in my first differential equations (DE) course, I studied the non-linear simple pendulum equation:

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \tag{2}$$

and its exact solution in terms of an elliptic function and integral— specifically, the solution derived by João Plínio Juchem Neto (2010),

$$\theta(t) = 2\arcsin\left[\frac{\gamma_0}{\sqrt{k}}\,\mathrm{sn}\,\left(F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) + H\omega_0 t, \gamma_0^2 k\right)\right] \tag{3}$$

With an audience of my peers in this course, I simply explain the Equation 2′s historical development and derivation. I explain how and why Equation 2 and Equation 3 are applied. I derive Equation 3.

Historical Development of Equation 2 and Equation 3

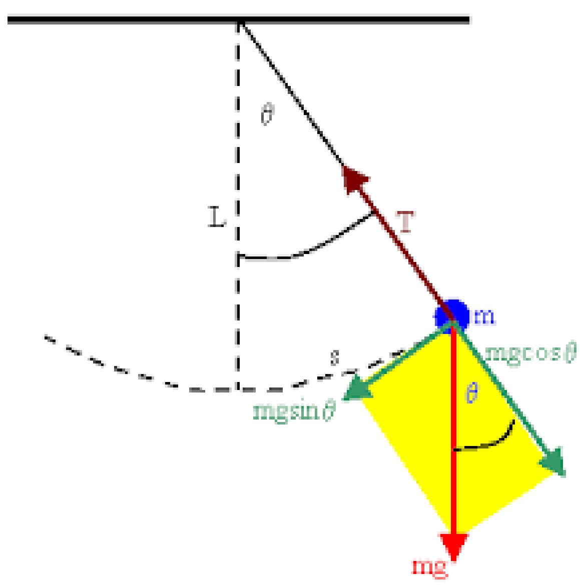
Christiaan Huygens and Sir Isaac Newton indirectly contribute to Equation 2. Adrien-Marie Legendre, Niels Henrik Abel, and Carl Gustav Jacob Jacobi develop the math used in Equation 3. In the 17th century, Christiaan Huygens invents the first dependable pendulum clock. In the first mathematical analysis of pendulums, he found that period grows with *angular amplitude*, or absolute maximum angular displacement. He suggests the pendulum's non-linear nature (*Huygens Invents the Pendulum Clock, Increasing Accuracy Sixty Fold*, 2023). He does not use Newton's laws of motion, which are released 13 years later. I derive Equation 2 with Newton's laws of motion in the next paragraph. The special functions used in Equation 3 are developed over the next several hundred years. In the 18th century, Legendre studies elliptic integrals and their application to mechanics. In the 19th century, Abel and Jacobi develop elliptic functions (Kovacic et al., 2016, p. 2). Many others worked on Equation 2 and its solution in terms of elliptic functions and integrals. Note that Juchem Neto (2010) developed a unique solution, but he did not develop Equation 2′s first solution in terms of elliptic functions and integrals.

The Derivation of Equation 2

To derive Equation 2, I refer to a force diagram, Newton's second law of motion, and the arc length equation.

Figure 1

Force Diagram of a Simple Pendulum



Note. θ, l, T, s, m, and g are the angle with the vertical, length of the rod connecting the mass to the pivot, force of tension, arc length created by θ, mass, and gravitational acceleration respectively. The forces acting on the mass are colored in bright and dark red. The force of gravity is colored bright red, and its components are colored in green. $mg\sin\theta$, $mg\cos\theta$, and

mg are the force of gravity along s, the force of gravity perpendicular to s, and the force of gravity, respectively. Neglect this diagram's capitalization of L; I use its lower-case counterpart, l, in this essay. From [Force diagram of simple pendulum]. University of Tennessee, Knoxville. http://labman.phys.utk.edu/phys221core/modules/m11/The%20pendulum.html.

State Newton's second law of motion along the mass' trajectory:

$$f = ma$$

where f and a are force along the trajectory and acceleration along the trajectory, respectively. Evaluate f and a according to Figure 1.

$$-mg\sin\theta = m\ddot{s}$$

where $\ddot{s}$ denotes the second derivative of s with respect to time. State the equation for a circle's arc length:

$$s = l\theta$$

Define the natural angular frequency:

$$\omega_0 := \sqrt{\frac{g}{l}}$$

where $:=$ denotes "is defined as." Combine the previous 3 equations to form Equation 2:

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0$$

Justification for Equation 3

After deriving Equation 2, I focus on its solution. Studying a solution in terms of special functions is advantageous, particularly for Equation 3, because math, science, and engineering use elliptic functions and integrals.

To understand the behavior of Equation 2, I seek its analytical solution. Unfortunately, the analytical solution is not an elementary function, or an algebraic combination of logarithmic, trigonometric, and exponential functions (Blinder, 2013, p. 230). Alternatively, I can write the solution with special functions, or functions studied frequently in math and science. Variable coefficient, second order ordinary DEs are commonly solved this way (Blinder, 2013, p. 230). Equation 3 uses special functions called the incomplete elliptic integral of the first kind and the Jacobi elliptic sine function.

Justification for DE Solutions in Terms of Special Functions

Understanding a DE solution in terms of special functions bolsters understanding of similar DEs, the DE solution, and the special functions. Firstly, elliptic functions solve several DE types (Kovacic et al., 2016, pp. 7-11). As a common, but advanced DE practice problem, deriving Equation 3 likely increases understanding of other DEs solved by elliptic functions. Secondly, understanding special functions' properties leads to understanding the solution's behaviors and limits and vice versa. As an example for novice DE students, i cite Equation 1′s solution:

$$\theta(t) = A\cos(\omega_0 t + \Phi) \tag{4}$$

where A and Φ are constants. Understanding a cosine property leads to Equation 4; Since cosine's maximum absolute value is one, the pendulum's angular amplitude is A radians. Conversely, understanding Equation 4 leads to understanding cosine functions; I visualize cosine as a linear pendulum when solving other cosine problems. Although this example uses an elementary instead of a special function, the logic is analogous for special functions.

Benefit of Understanding the Special Functions in Equation 3

Equation 3′s special functions apply to geometry and astronomy. The incomplete elliptic integral of the first kind can calculate the arc length of ellipse-like curves such as an ellipse or a *lemniscate*— which looks like an infinity sign, ∞ (Villanueva, n.d., pp. 5-6). Additionally, the integral can calculate Mercury's *perihelion*, or its closest location to the sun (Villanueva, n.d., pp. 8-11). The Jacobi sine function is analogous to the trigonometric sine function, but for an ellipse (Kovacic et al., 2016, pp. 3-4).

Benefit of Understanding Equation 2

In science and engineering, Equation 2 models pendulum-like motion when friction and angular displacement are small and large, respectively. In other cases, alternative equations are used. Firstly, friction is negligible or else an damping term is added to the equation. For example, my DE course studied damped oscillator equations of the form:

$$\ddot{\theta} + B\dot{\theta} + C\theta = 0$$

where B and C are constants. Secondly angular displacement is large or else one uses Equation 1 and Equation 4. In exchange for small inaccuracies, these equations are more conceptually simple and computationally efficient than their non-linear counterparts. See the Appendix for further analysis of these two equations.

Derivation of Equation 3

After deciding to study Equation 3, I derive it as written by Juchem Neto (2010). I abridged the derivation due to its long length. In short, I eliminate $\ddot{\theta}$. I use substitution and calculus to manipulate the expression into elliptic integrals of the first kind. I substitute the integrals with special functions and isolate theta as a function of time.

For reference, I restate Equation 2:

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0$$

The first major step is to eliminate $\ddot{\theta}$. The following three minor steps resemble the solution method for Bernoulli DEs, or DEs of the form:

$$y' + P(x)y = f(x)y^n$$

, except, here, the *integrating factor* is $\dot{\theta}$ and u-subsittuion is omitted. First, multiply $\frac{d\theta}{dt}$ on both sides to get

$$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\omega_0^2 \sin\theta = 0$$

Use the calculus product rule in reverse, or rewrite the equation as a derivative:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2\cos\theta\right] = 0$$

Integrate from 0 to a positive time, t.

$$\int_0^t \frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2\cos\theta\right]\mathrm{d}t = 0 \tag{5}$$

The next major step is to manipulate the equation into an elliptic integral by variable substitution. Define variables and constants:

$$y(t) := \sin\left(\frac{\theta(t)}{2}\right), \qquad k := \sin^2\left(\frac{\theta_0}{2}\right), \qquad \tau(t) := \omega_0 t$$

$$z(t) := \frac{y}{\sqrt{k}}, \qquad \gamma_0^2 := 1 + \frac{\phi_0^2}{4\omega_0^2 k}, \qquad \theta(0) := \theta_0, \qquad \dot{\theta}(0) := \phi_0$$

Substitute them into Equation 5, yielding

$$\left(\frac{dz}{d\tau}\right)^2 = (1-z^2)(1-kz^2) + \frac{\phi_0^2}{4\omega_0^2 k}(1-kz^2) = (1-kz^2)(\gamma_0^2 - z^2)$$

Separate the variables, or isolate $\mathrm{d}\tau$:

$$\pm d\tau = \frac{\mathrm{d}z}{\sqrt{(1-kz^2)(\gamma_0^2-z^2)}}$$

Integrate both sides with respect to time, and rearrange into

$$\pm\tau = \frac{1}{\sqrt{k}}\left[\int_0^z \frac{dz}{\sqrt{\left(\frac{1}{k}-z^2\right)(\gamma_0^2-z^2)}} - \int_0^1 \frac{dz}{\sqrt{\left(\frac{1}{k}-z^2\right)(\gamma_0^2-z^2)}}\right] \tag{6}$$

, which resembles elliptic integrals of the first kind. Evaluate with special functions.

$$\pm\tau = \mathrm{sn}^{-1}\left(\frac{z}{\gamma_0}, \gamma_0^2 k\right) - F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) \tag{7}$$

where $\mathrm{sn}^{-1}(*,*)$ and $F(*,*)$ are the inverse Jacobi sine function and the incomplete integral of the first kind function, respectively. Both functions each input two values and output one value. Note that Equation 6 and Equation 7 both contain incomplete integrals of the first kind noted as an integral and function, respectively. The explanation for these special functions lie beyond the scope of this paper. Undo the substitutions and isolate θ to get Equation 3:

$$\theta(t) = 2\arcsin\left[\frac{\gamma_0}{\sqrt{k}}\,\mathrm{sn}\,\left(F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) + H\omega_0 t, \gamma_0^2 k\right)\right]$$

and

$$H := \begin{cases} -1, & \phi_0 \leq 0, \\ 1, & \phi_0 > 0. \end{cases}$$

For reference, I print my previous definitions:

$$\gamma_0^2 := 1 + \frac{\phi_0^2}{4\omega_0^2 k}, \qquad k := \sin^2\left(\frac{\theta_0}{2}\right), \qquad \omega_0^2 := \sqrt{\frac{g}{l}}$$

Summary and Future Directions

Over several hundred years, individuals developed Equation 3′s math and Equation 2. Equation 2 is derived using force equations. Equation 2 effectively models pendulum-like motion when friction and angular displacement are small and large, respectively. Equation 3 uses special functions for insight into Equation 3, the special functions, and similar DEs' solutions. To derive Equation 3, use calculus and algebra to manipulate Equation 2 into a form resembling special functions. Apply special funcitons and isolate θ. Through this project, I introduced myself to complicated DEs and special functions, which are common in my major, physics. This project's knowledge aids me in my future studies. During my research, I was curious about the following topics, and I would explore them if given sufficient time: Equation 2′s phase space, or its graph with axes θ and $\dot{\theta}$ and visual representations of elliptic functions and integrals. Interestingly, the linear and non-linear simple pendulum solutions are described with the x-axis projection on circles and ellipses, respectively (Kovacic et al., 2016, p. 3)

References

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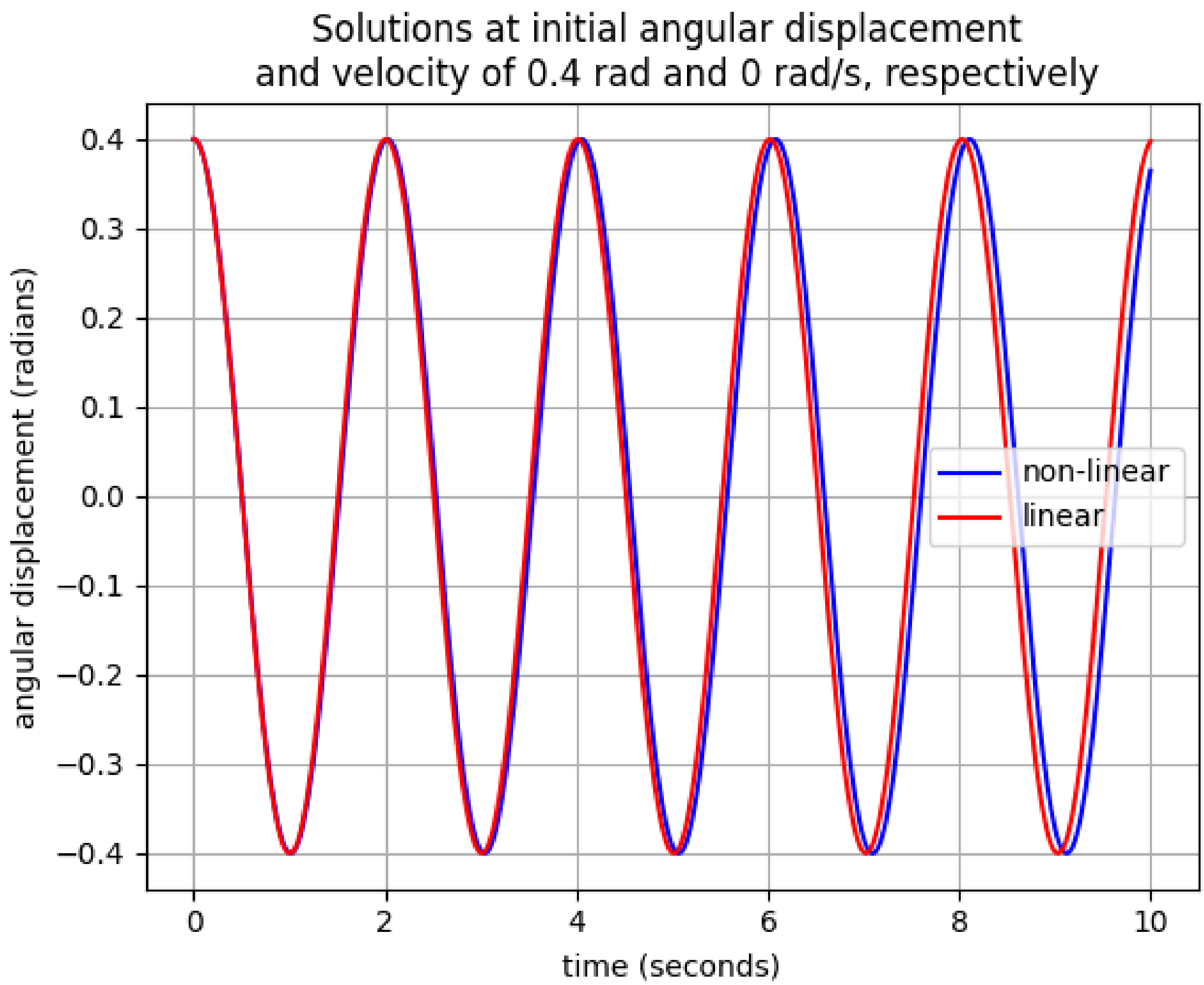
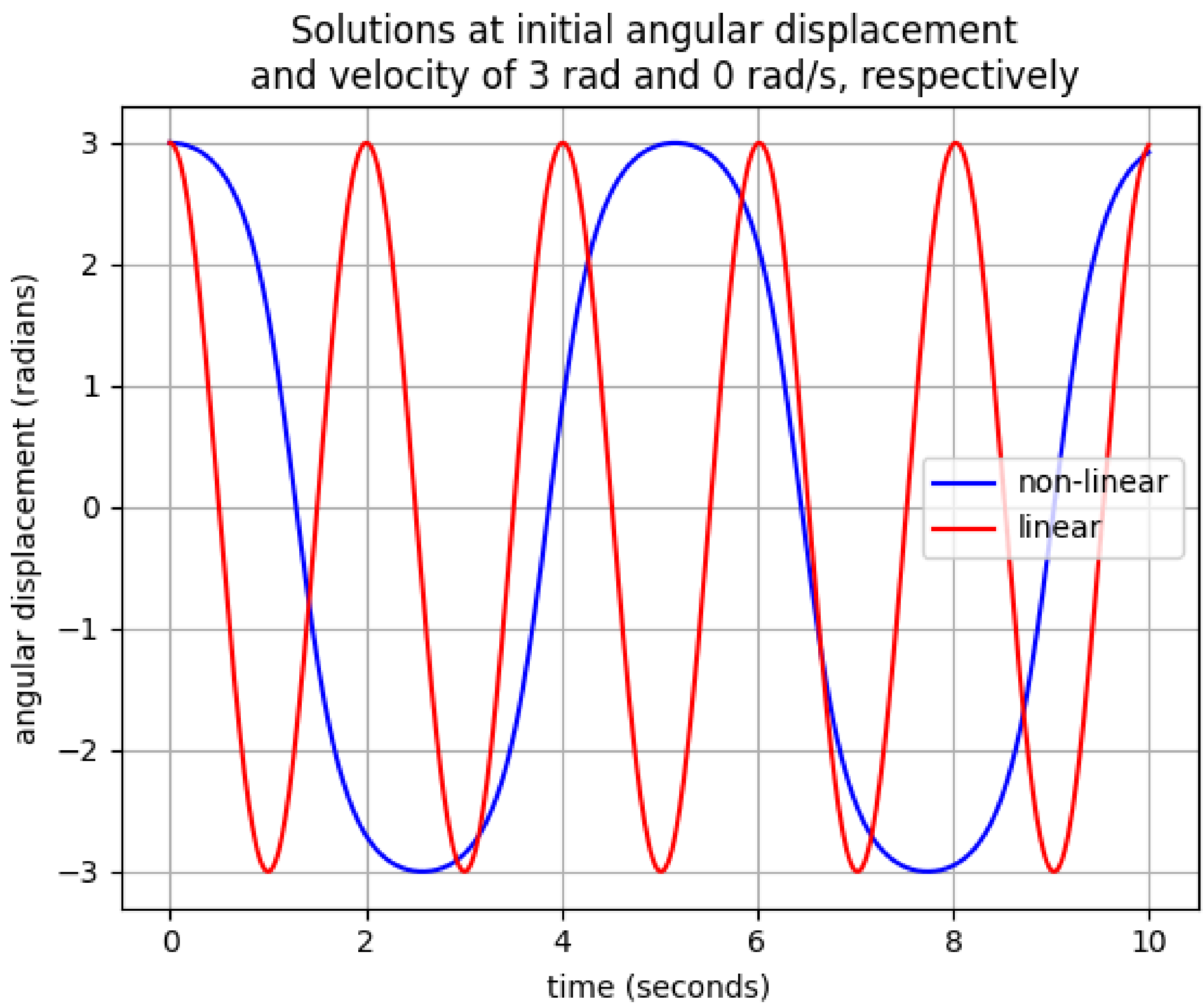
I wanted to connect this paper's research topic, Equation 2, with what I already knew in the past, Equation 1. I compared the graphs of Equation 2 and Equation 1.

The linear solution is much simpler than the non-linear equation. However, as Huygens realized, when θ is large, the linear model is inaccurate. To illustrate this, I used Python version 3.11.13, Matplotlib version 3.10.0, SciPy version 1.15.3, and NumPy version 2.0.2 to generate 2 graphs. My computer code is adapted from an example in Scipy's Odeint package Documentation, which is used to numerically solve ordinary DEs (Community, n.d.). The code, which was developed for this study, can be found in the back of the Appendix.



Note. Angular displacement, θ in radians is on the y-axis. Time, t in seconds is on the x-axis. The non-linear equation and the linear equation are graphed in blue and red, respectively. When θ_0 and φ_0 are 0.4 radians and 0 radians per second, respectively, the linear equation nearly overlaps with the non-linear equation. The linear equation is a good approximation for the non-linear equation. This graph was generated using the code on the Appendix's last page.



Note. The axis, color meanings, and code source are the same as Figure 1. When θ_0 and φ_0 are 3 radians and 0 radians per second, respectively, the linear equation deviates significantly from the non-linear equation. The linear equation is an inaccurate approximation of the non-linear equation, which has sharper turns than a sine curve.

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

def nl_vs_l(theta0, phi0, l, title):
    """Untitled0
    Function to compare the graphs of a simple linear and simple non-linear
pendulum. The x-axis is time in seconds. The y-axis is angular displacement
in radians.

    Parameters
    ----------
    theta0 : float
        initial angular displacement in rad

    phi0 : float
        inital angular velocity in rad/s

    l : float
        length of the rod that connects the mass to the pivot in meters

    title: string
        title of the graph

    Returns
    -------
    none
        This function doesn't return a value. It creates a graph.
    """

    # define gravitational acceleration in m/s^2
    g = 9.8

    # define he natural angular frequency
    omega_0 = (g / l)**0.5

    def diff_eq(y, t,linear):
```

```
        """
        Function to define variables relating to the pendulum differential
equation. We need this for the scipy ODE solver

        Parameters
        ----------
        y : array
            angular displacement and angular velocity in rad and rad/s,
respectively

        t : array
            time in seconds

        linear : boolean
            decide if you want to linearly or non-linearly model the DE

        Returns
        -------
        dydt: array
            the derivative of y. Contains angular velocity and angular
acceleration in rad/s and rad/s^2, respectively
        """

        #create a vector y. Theta and phi are angular displacement and
angular velocity, respectively.
        theta, phi = y

        # pick which equation to use
        if linear == True:
            # if linear is true, then pick the linear equation
            ang_accel =  -(omega_0**2) * theta

        else:
            # if linear is false, then pick this non-linear equation
            ang_accel = -(omega_0**2) * np.sin(theta)

        # define the derivative of y
```

```
        dydt = [phi, ang_accel]

        return dydt

    # define initial angular displacement and angular velocity as theta0 and
phi0, respectively
    y0 = [theta0, phi0]

    # generate 120 evenly spaced samples between t=0 and t=10
    t = np.linspace(0, 10, 500)

    # numerically solve the non-linear DE.
    nl_sol = odeint(diff_eq, y0, t, args=(False,))

    # numerically solve the linear DE
    l_sol = odeint(diff_eq, y0, t, args=(True,))

    # plot it
    plt.plot(t, nl_sol[:, 0], 'b', label=r'non-linear')
    plt.plot(t, l_sol[:, 0], 'r', label=r'linear')
    plt.ylabel('angular displacement (radians)')
    plt.xlabel('time (seconds)')
    plt.grid()
    plt.legend(loc='right')
    plt.title(title)
    plt.show()

# call the function twice to compare the graphs when intital angular
displacement is changed.
nl_vs_l(theta0=0.4, phi0=0, l=1, title = "Solutions at initial angular
displacement \n and velocity of 0.4 rad and 0 rad/s, respectively")
nl_vs_l(theta0=3, phi0=0, l=1, title = "Solutions at initial angular
displacement \n and velocity of 3 rad and 0 rad/s, respectively")
```