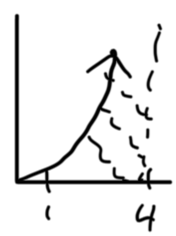
7.8: Improper integrals

Definite integrals can be extended where
1. the interval is ∞
2. f has an infinite discontinuously $[a,b]$

if not,

Ex: $\int_{-\infty}^{2} f(x)dx$ $\int_{-1}^{\infty} f(x)dx$ $\int_{-\infty}^{\infty} f(x)dx$, $\int_{1}^{4} f(x)dx$ infinite discontinuous at $x=4$

Def: if $\int_a^t f(x)$ exists for all $t>0$ then we define $\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$



$$\int_1^4 f(x)dx = \lim_{t\to 4} \int_1^t f(x)dx$$

When evaluating a limit, it may be intermeniate then we can use L'hopital rule

$\infty + \infty = \infty$ $\infty \cdot \infty = \infty$

$-\infty - \infty = -\infty$

$\infty + \infty = \infty$

$\infty \pm c = \infty$

$\infty \cdot -c = -\infty \quad \infty \cdot c = \infty$

Ex: Find $\int_1^\infty \frac{1}{x^2}\,dx$ and $\int_1^\infty \frac{1}{x}\,dx$

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = \left(\frac{-1}{\infty} + \frac{1}{1}\right)$$

$\frac{1}{\infty} = \lim_{x \to \infty} \frac{1}{x}$

$\frac{1}{\infty} = 0$

$= 0 + 1$ convergent

$$\int_1^\infty \frac{1}{x^2}\,dx = 1 \quad \text{convergent}$$

$$\int_1^\infty \frac{1}{x}\,dx = \ln|x|\Big|_1^\infty = \ln(\infty) - \ln(1)$$

$= \infty - 0$

$\ln\infty = \lim_{x \to \infty} \ln x = \infty$

$= \infty$

$$\int_1^\infty \frac{1}{x}\,dx = \infty \quad \text{divergent}$$

Ex: for what values of p is the integral $\int^\infty \frac{1}{x^p}$ to be convergent

$\int_1 \quad x^p$

$$= \frac{x^{-p+1}}{-p+1}\Bigg|_1^{\infty} = \frac{1}{1-p}\left(\infty^{1-p} - 1\right)$$

if $p<1$ $\quad 1-p>0$

$\infty^{1-p} = \infty$

if $p>1$

then $1-p<0$

So $\infty^{1-p} = 0$

$$= \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & p<1 \end{cases}$$

if $p=1$

$=\infty$

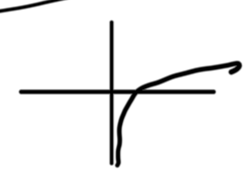
$\int_1^{\infty} \frac{1}{x^p}$ is convergent if $p>1$

and divergent $p \le 1$

Ex: $$\int_4^5 \frac{1}{\sqrt{x-4}}\,dx \qquad u = x-4$$

$du = dx$

$$\int_0^1 \frac{1}{\sqrt{u}}\,du = 2u^{1/2}\Big|_0^1 = 2$$



Ex: $\int_0^1 \ln x\,dx$ $\qquad u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$$\int_0^1 \ln x = x\ln x - \int_0^1 x \cdot \frac{1}{x}\,dx$$

$$= (1\cdot\ln 1 - 0\cdot\ln 0) - \int_0^1 dx$$
$$= (0 - 0\ln 0) - (1-0)$$
$$= -1 - 0\ln 0$$

$0\ln 0 = 0\cdot -\infty$
indeterminite

$0\ln 0 = \lim_{x\to 0^+} x\ln x$

$$0\ln 0 = \lim_{x\to 0^+} x\ln x \overset{LH}{=} \frac{\ln x}{1/x} \overset{\frac{-\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{1/x}{\frac{-1}{x^2}} = \frac{1}{-1/x} = \lim_{x\to 0^+} -x = 0$$

$$\boxed{-1 - 0 = 0}$$

$u = x^2+1 \qquad du = 2x\,dx$

$\frac{du}{2} = x\,dx$

Ex: $\int_{-\infty}^{0} \frac{x}{(x^2+1)^3}\,dx$

New bounds
$(-\infty)^2+1 = \infty$
$0^2+1 = 1$

$$= \int \frac{1}{u^3}\frac{du}{2} = \frac{1}{2}\int \frac{1}{u^3}\,du$$

$$= \int_{\infty}^{1} u^{-3}\,du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) = \frac{-1}{4}\left(u^{-2}\right) = \frac{-1}{4}\left(1 - \infty^{-2}\right)$$

$$= \frac{1}{-4}(1-0) = \boxed{\frac{-1}{4}}$$