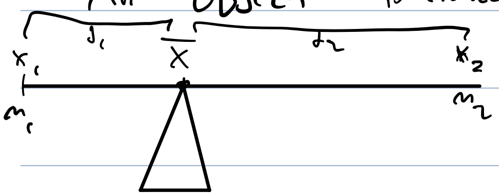
8.3 Applications to physics and engineering

1. Centers of mass a.k.a. centroids

An object balances horizontally at its centroid.



if rod balances, $m_1 d_1 = m_2 d_2$

So $m_1(\bar{x}-x_1) = m_2(x_2-\bar{x})$

$$\Rightarrow \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

$\leftarrow$ moments

$\leftarrow$ total mass

in general, a system of n objects, the centroid is given by $(\bar{x},\bar{y})$ $= \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$ where $m = \sum_{i=1}^{n} m_i$ $\leftarrow$ total mass

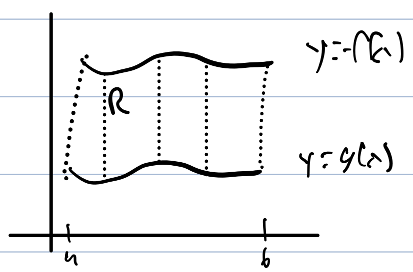
$M_y = \sum_{i=1}^{n} m_i x_i$, $M_x = \sum_{i=1}^{n} m_i y_i$ are moments

Let R be 2D object with uniform density ρ

where curves $y=f(x)$, $y=g(x)$, $f(x)\geq g(x)$ on $[a,b]$

We want to find centroid $\bar{x},\bar{y}$ of R

So we approximate R with rectangles and take limit



- Choose midpoint $\bar{x}_i$ from each subinterval
- centroids $\bar{x}_i, \frac{1}{2}\left(f(\bar{x}_i)+g(\bar{x}_i)\right)$

Mass of r_i is $\rho\Delta x\left(f(\bar{x}_i)+g(\bar{x}_i)\right)$

So we compute moment is $\bar{x}_i\,\rho\Delta x\left(f(\bar{x}_i)+g(\bar{x}_i)\right)$

$$\sum_{i=1}^{n} \bar{x}_i\,\rho\Delta x\left(f(\bar{x}_i)+g(\bar{x}_i)\right),$$ So the moment of R limit

$M_y = \lim_{n\to\infty} \bar{x}_i \, \rho \Delta x \left(f(\bar{x}_i) + g(\bar{x}_i)\right)$

x axis

$= \rho \int_a^b x \left(f(x) - g(x)\right) dx$

y axis moment

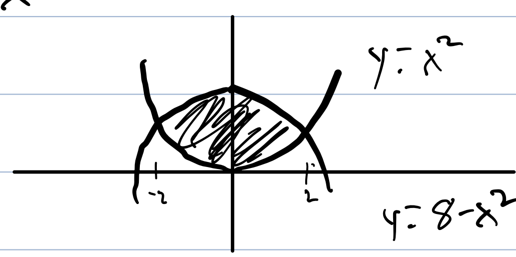
is $M_x = \int_a^b \frac{1}{2}\left(f(x)^2 - g(x)^2\right) dx$

Mass of R is $m = \rho A = \rho \int_a^b f(x) - g(x)\, dx$

Thus centroid of R is

$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{1}{A}\int_a^b x \cdot \left(f(x) - g(x)\right) dx, \ \frac{1}{A}\int_a^b \frac{1}{2}\left(f(x)^2 - g(x)^2\right) dx\right)$

Ex: find centroid of region bounded by curves $y = x^2$, $y = 8 - x^2$



$y = x^2$

$y = 8 - x^2$

$x^2 = 8 - x^2$

$2x^2 = 8$

$x = \pm 2$

$\bar{x} = \frac{1}{A}\int_{-2}^{2} x\left(f(x) - g(x)\right) dx$

$\bar{y} = \frac{1}{A}\int_{-2}^{2} \frac{1}{2}\left(f(x)^2 - g(x)^2\right) dx$

$A = \int_{-2}^{2} 8 - x^2 - x^2 \, dx$

$= \int 8 - 2x^2$

$= \int 8x - \frac{2x^3}{3} \Big|_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= \frac{64}{3}$

$\bar{x} = \frac{3}{64}\int_{-2}^{2} x\left(8 - 2x^2\right) dx$

$= \frac{3}{64}\int 8x - 2x^3 = 4x^2 - \frac{2x^4}{4}\Big|_{-2}^{2} = 0$

$\bar{y} = \frac{3}{64}\int_{-2}^{2} \frac{1}{2}\left((8 - x^2)^2 - x^4\right) = \frac{3}{128}\int_{-2}^{2} \left(64 - 16x^2 + x^4\right) - x^4$

$= 64x - \frac{16x^3}{3} + \cancel{\frac{x^5}{5}} - \cancel{\frac{x^5}{5}}$

$$= \frac{3}{128}\left(64x - \frac{16x^3}{3}\right)\Big|_{-2} = \frac{3}{128}\left(128 - \frac{128}{3}\right) - \left(-128 + \frac{128}{3}\right.$$

$$= \boxed{4}$$

Centroid is $(0, 4)$