MATH:1860 Activity 10 – (Sections 11.7-11.11) **Apr. 17**

Name:________________________________

Instructions: Work with others or independently to complete the activity.

1. Find the radius of convergence R and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\,x^{n+1}}{2^{n+1}((n+1)^2+1)} \cdot \frac{2^n(n^2+1)}{n x^n}\right| = \frac{(n+1)(n^2+1)}{2n((n+1)^2+1)}|x|.$$ Thus, $$\lim_{n\to\infty} \frac{(n+1)(n^2+1)}{2n((n+1)^2+1)}|x| = \frac{1}{2}|x|$$

Thus the series converges if $\frac{1}{2}|x| < 1 \Rightarrow |x| < 2 \Rightarrow -2 < x < 2$ $R=2$

Now check bounds. when $x=-2$: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n^2+1}$ converges since $\frac{n}{n^2+1} \to 0$.

when $x=2$: $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. use LCT.

$$\lim_{n\to\infty} \frac{\left(\frac{n}{n^2+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0.$$ Thus, they both converge or diverge. Since $\frac{1}{n}$ diverges, this does too. $R=2$ $I=[-2,2)$

2. Find a power series representation of $f(x) = \arctan x$, starting from $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, $|x| < 1$. What is the radius of convergence R? (Fun fact: It can be proven that the power series representation for $\arctan x$ is also valid at the endpoint $x = 1$. In this case, we obtain $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, called the *Leibniz formula for* π.)

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n, \quad |-x^2| < 1$$

$$\int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}, \quad |x| < 1$$

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

if $x=0$, $C=0$

thus,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}, \quad |x| < 1, \quad R=1$$

3. Find the Maclaurin series of $f(x) = 2^x$ using the definition of a Maclaurin series.

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$f(x) = 2^x$ $f(0) = 1$

$f'(x) = \ln 2 \cdot 2^x$ $f'(0) = \ln 2$

$f''(x) = (\ln 2)^2 \cdot 2^x$ $f''(0) = (\ln 2)^2$

$f'''(x) = (\ln 2)^3 \cdot 2^x$ $f'''(0) = (\ln 2)^3$

So $a_n = \dfrac{(\ln 2)^n}{n!}$

Maclaurin Series: $$\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} \cdot x^n$$

4. Find a power series representation of $f(x) = \dfrac{x}{(1+4x)^2}$ and determine the radius of convergence R.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{d}{dx}\left(\frac{1}{1+4x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-4x)^n, \quad |-4x|<1\right)$$

$$\frac{-4}{(1+4x)^2} = \sum_{n=0}^{\infty} (-4)^n \cdot n x^{n-1}, \quad |x|<\frac{1}{4}$$

$$\frac{1}{(1+4x)^2} = \sum_{n=0}^{\infty} (-4)^{n-1} \cdot n x^{n-1}, \quad |x|<\frac{1}{4}$$

$$\frac{x}{(1+4x)^2} = \sum_{n=0}^{\infty} (-4)^{n-1} \cdot n x^{n}, \quad |x|<\frac{1}{4}$$

5. Let $i = \sqrt{-1}$ be the *imaginary unit* (treat i as a constant). Then $i^2 = -1$.

Recall $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ and $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x. Then

$$\begin{aligned}
\cos x + i \sin x &= \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \left(\sum_{n=0}^{\infty}(i^2)^n \frac{x^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty}(i^2)^n \frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty} i^{2n} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i^{2n+1} \frac{x^{2n+1}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty} i^n \frac{x^n}{n!},
\end{aligned}$$

where the last equality holds because the first sum consists of the even-indexed terms and the second sum consists of the odd-indexed terms of the resulting series. Now recall $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Use the above result to show $e^{ix} = \cos x + i \sin x$. (Fun fact: If $x = \pi$, then we obtain $e^{i\pi} + 1 = 0$, called *Euler's identity*, a famous equation because it relates five fundamental mathematical constants: $e, i, \pi, 1, 0$).

Replace x by ix for e^x

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \text{ for all } x$$

$$e^{ix} = \sum_{n=0}^{\infty} i^n \cdot \frac{x^n}{n!} \text{ for all } x$$

Thus $e^{ix} = \cos x + i \sin x$