Homework 11

11.9

8.) $f(x) = \frac{4}{2x+3} = \frac{\frac{4}{3}}{\frac{2x}{3}+1} = \frac{\frac{4}{3}}{1-\left(-\frac{2x}{3}\right)}$

Thus $\frac{1}{1-\left(-\frac{2x}{3}\right)} = \sum_{n=0}^{\infty} \left(\frac{-2x}{3}\right)^n, \quad \left|\frac{-2x}{3}\right| < 1$

$\frac{4}{3} \cdot \frac{1}{1-\left(-\frac{2x}{3}\right)} = \sum_{n=0}^{\infty} \left(\frac{4}{3}\right)\left(\frac{-2x}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n x^n}{3^n}\right) \cdot \frac{4}{3} = \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{2^n \cdot 2^1 \cdot 2^1}{3^{n+1}}\right) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^{n+2}}{3^{n+1}}\right) \cdot x^n$

$\left|\frac{2}{3}x\right| < 1 \Rightarrow \frac{2}{3} \cdot |x| \Rightarrow |x| < \frac{3}{2} \qquad I = \left(-\frac{3}{2}, \frac{3}{2}\right) \qquad R = \frac{3}{2}$

10.) $f(x) = \frac{x}{2x^2+1} = \frac{x}{1-(-2x^2)}$

$\frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty} (-2x^2)^n, \quad |-2x^2| < 1$

$x \cdot \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty} (x) \cdot (-2x^2)^n = \sum_{n=0}^{\infty} (-2)^n \cdot x^{2n+1}, \quad |-2x^2| < 1 \Rightarrow x^2 < \frac{1}{2} \Rightarrow \frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

$I = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

16.a) $\int \frac{1}{1-x} = \ln|1-x| + c$

Thus $\frac{-1}{1-x} = \sum_{n=0}^{\infty} -x^n, |x| < 1$

$\ln|1-x| + c = \sum_{n=0}^{\infty} \int -x^n \, dx, |x| < 1$

$\ln|1-x| = c + \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}, \quad |x| < 1 \qquad R = 1$

Solve for c

let $x = 0$, $\ln(1-1) = 0 + c \quad c = 0$

16.b) we know $\ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$.

So, $x\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \cdot x = \boxed{-\sum_{n=0}^{\infty} \frac{x^{n+2}}{n+1}}$

16.c) $\sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$, let $x=\frac{1}{2}$, $\ln(1-\frac{1}{2}) = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$

Shift indices. $\sum_{n=1}^{\infty} -\frac{x^n}{n}$. negate both sides: $-\ln(1-x) = -\sum_{n=1}^{\infty} -\frac{x^n}{n} \Rightarrow \boxed{\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}}$

18.) $f(x) = \left(\frac{x}{2-x}\right)^3$. $\frac{1}{2}\cdot\frac{1}{1-(\frac{x}{2})} = \sum_{n=0}^{\infty} \frac{1}{2}\cdot\left(\frac{x}{2}\right)^n, \left|\frac{x}{2}\right|<1$

$\frac{1}{2}\cdot\frac{1}{1-(\frac{x}{2})} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}, |x|<2 \Rightarrow \frac{d}{dx}\left(\frac{1}{2}\cdot\frac{1}{1-(\frac{x}{2})}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}\right)$

$= \frac{1}{(2-x)^2} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot n x^{n-1}$ $\quad \frac{d}{dx}\left(\frac{1}{(2-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}\cdot n x^{n-1}\right) =$

$\frac{2}{(2-x)^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot n(n-1)\cdot x^{n-2} \Rightarrow \frac{1}{2-x^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} \cdot n(n-1)\cdot x^{n-2}$

$x^3 \cdot \frac{1}{(2-x)^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} \cdot n(n-1)\cdot x^{n+1} \Rightarrow \left(\frac{x}{2-x}\right)^3 = \boxed{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} \cdot n(n-1)\cdot x^{n+1}}$

20.) $f(x) = \frac{x^2+x}{(1-x)^3}$. $\left(\frac{1}{1-x}\right)'' = \frac{2}{(1-x)^3}$

$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \Rightarrow \frac{x^2+x}{2}\cdot\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{x^2+x}{2}\cdot n(n-1)x^{n-2}$

$\frac{x^2+x}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2}\cdot\left((x^2+x)x^{n-2}\right)$

$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2}\cdot\left(x^n + x^{n-1}\right) \Rightarrow \sum_{n=2}^{\infty} \frac{n(n-1)}{2}x^n + \sum_{n=1}^{\infty} \frac{(n+1)n}{2}x^n$

$= \sum_{n=1}^{\infty} \frac{n(n-1)}{2}x^n + \frac{n(n+1)}{2}x^n \Rightarrow \sum_{n=1}^{\infty} \frac{nx^n}{2}\left((n-1)+(n+1)\right) = \boxed{\sum_{n=1}^{\infty} n^2x^n, \ |x|<1 \quad R=1}$

28.) $\frac{t}{1+t^3}$. $t\cdot\frac{1}{1-(-t^3)} = \sum_{n=0}^{\infty} (-t^3)^n \cdot t = \sum_{n=0}^{\infty} (-1)^n \cdot t^{3n+1}$

$\int \frac{t}{1+t^3} = \int \sum_{n=0}^{\infty} (-1)^n \cdot t^{3n+1}$

$= C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^{3n+2}}{3n+2}, \ |t|<1 \quad R=1$

Section 11.10

14.) $f(x)=e^{-2x}$ $f(0)=1$

$f'(x)=-2e^{-2x}$ $f'(0)=-2$

$f''(x)=4e^{-2x}$ $f''(0)=4$

$f'''(x)=-8e^{-2x}$ $f'''(0)=-8$

$f^{4}(x)=16e^{-2x}$ $f^{4}(0)=16$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n=1-2x+\frac{4x^2}{2!}-\frac{8x^3}{3!}+\frac{16}{4!}x^4\ldots$$

Thus $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n\cdot x^n}{n!}$. $\displaystyle\lim_{n\to\infty}\left|\frac{2x}{n+1}\right|=0<1$

So $R=\infty$

24.) $f(x)=\frac{1}{x}$ $f(-3)=-\frac{1}{3}$ Centered at -3

$f'(x)=-x^{-2}$ $f'(-3)=-\frac{1}{3^2}$

$f''(x)=2x^{-3}$ $f''(-3)=\frac{-2!}{3^3}$

$f'''(x)=-6x^{-4}$ $f'''(-3)=-\frac{3!}{3^4}$

$f^{4}(x)=24x^{-5}$ $f^{4}(-3)=\frac{-4!}{3^5}$

$$\frac{f^{(n)}(0)}{n!}(x-a)^n=-\frac{1}{3}+\frac{-1}{3^2}(x+3)+\frac{\frac{2!}{3^3}}{!}(x+3)^2+\frac{\frac{-3!}{3^4}}{3!}(x+3)^3$$

$$=-\sum_{n=0}^{\infty}\frac{(x+3)^n}{3^{n+1}}$$

$\displaystyle\lim_{n\to\infty}\left|\frac{x+3}{3}\right|=\left|\frac{x+3}{3}\right|<1 \Rightarrow |x+3|<3$ $\quad -6<x<0$ $\boxed{R=3}$

28.) $f(x)=\cos x$ $f(\frac{\pi}{2})=0$ Centered at $\frac{\pi}{2}$

$f'(x)=-\sin x$ $f'(\frac{\pi}{2})=-1$

$f''(x)=\cos x$ $f''(\frac{\pi}{2})=0$

$f'''(x)=\sin x$ $f'''(\frac{\pi}{2})=1$

$f^{4}(x)=\cos x$ $f^{4}(\frac{\pi}{2})=0$

$$\frac{f^{(n)}(0)}{n!}(x-a)^n=0-\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5\ldots$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)!}\cdot\left(x-\frac{\pi}{2}\right)^{2n-1}$$

$R=\infty$ since $\lim\left|\frac{a_{n+1}}{a_n}\right|=0$

40.) from table 1: $\displaystyle\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot x^{2n+1}=(x)-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$

So, $\displaystyle\sin\left(\frac{\pi}{4}\cdot x\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\left(\frac{\pi x}{4}\right)^{2n+1}}{(2n+1)!}$ Ratio test shows $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0$ thus $R=\infty$

42.) from table 1, $\displaystyle e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\ldots$ $R=\infty$. So $\displaystyle e^{3x}-e^{2x}=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}-\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\sum_{n=0}^{\infty}\frac{(3x)^n-(2x)^n}{n!}$

$$=\sum_{n=0}^{\infty}\frac{3^n x^n-2^n x^n}{n!}=\sum_{n=0}^{\infty}\frac{(3^n-2^n)x^n}{n!}$$ and $R=\infty$

Section 11.11

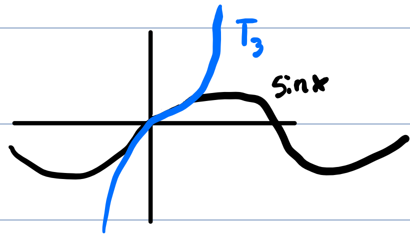
4.) $f(x)=\sin x \quad f(\frac{\pi}{6})=\frac{1}{2}$

$f'(x)=\cos x \quad f'(\frac{\pi}{6})=\frac{\sqrt{3}}{2}$

$f''(x)=-\sin x \quad f''(\frac{\pi}{6})=-\frac{1}{2}$

$f'''(x)=-\cos x \quad f'''(\frac{\pi}{6})=-\frac{\sqrt{3}}{2}$

$$\boxed{T_3(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3}$$



6.) $f(x)=e^{-x}\cdot\sin x$ from table 1: $\sin x=\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)!}\cdot x^{2n+1}=(x)-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$

from table 1, $e^x=\sum\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\dots$ Thus $e^{-x}=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{n!}=1-x+\frac{x^2}{n!}-\frac{x^3}{3!}$

So $e^{-x}\sin x=\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\right)\left(x-\frac{x^3}{3!}\right)=x-x^2+\frac{x^3}{2!}-\frac{x^4}{3!}-\frac{x^3}{3!}+\frac{x^4}{3!}-\frac{x^5}{2!3!}+\frac{x^6}{3!3!}$

$=x-x^2+\frac{x^3}{2}-\frac{x^3}{6}-\frac{x^5}{2!3!}+\frac{x^6}{3!3!}=x-x^2+\frac{x^3}{3}-\frac{x^5}{2!3!}+\frac{x^6}{3!3!}$. So $T_3(x)=x-x^2+\frac{x^3}{3}$