Section 11.7

10.) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ take dominant terms $= \frac{n}{n^3} = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n-1}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(n-1)n^2}{n^3+1} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3+1} \overset{LH}{=} \lim_{n\to\infty} \frac{3n^2-2n}{3n^2} \overset{LH}{=} \lim_{n\to\infty} \frac{6n-2}{6n} \overset{LH}{=} \frac{6}{6} = 1 > 0$$

Converges by LCT

12.) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2-1}{n^2+1}$ $\quad \displaystyle\lim_{n\to\infty} \frac{n^2-1}{n^2-1} \overset{LH}{=} \frac{2}{2} = 1 \neq 0$ AST [illegible]

Divergence.

14.) $$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}} = \left(\frac{n^2}{(1+n)^3}\right)^n = \lim_{n\to\infty} \frac{n^2}{(1+n)^3} = \lim_{n\to\infty} \frac{\frac{n^2}{n^3}}{\frac{(1+n)^3}{n^3}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\left(\frac{n+1}{n}\right)^3} = \frac{0}{(1+0)^3} = 0 < 1$$

Converges by Root Test

16. $$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n^4}{4^n} \quad \lim_{n\to\infty} \frac{\frac{(n+1)^4}{4^{(n+1)}}}{\frac{n^4}{4^n}} = \lim_{n\to\infty} \frac{(n+1)^4}{n^4} \cdot \frac{1}{4} = \frac{1}{4} \lim_{n\to\infty} \frac{n^4}{n^4} = \frac{1}{4} \cdot 1 = \frac{1}{4} < 1$$

Converges by Ratio Test

18.) let $f(x) = \frac{x^2}{e^{x^3}}$ for $x \geq 1$. it is decreasing

$u = x^3 \quad du = 3x^2$

$$\int_1^{\infty} \frac{x^2}{e^{x^3}} dx = \frac{1}{3}\int_1^{\infty} e^{-u} du = \frac{1}{3}\left(-e^{-u}\right)\Big|_1^{\infty} = \frac{1}{3e}.$$

Converges

20.) $$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}} = \lim_{k\to\infty} \frac{a_k}{b_k} = \frac{\frac{1}{k\sqrt{k^2+1}}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k\sqrt{k^2+1}} = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}} = \frac{1}{\sqrt{1+\frac{1}{k^2}}} \to 1$$

Converges by LCT

22.) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n} \leq \frac{1}{2^n}$ which converges

So by Comparison test

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ Converges

24.) $a_n = \sqrt{n^4+1} \quad b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^3+n} \cdot n = \lim_{n\to\infty} \frac{n^3}{n^3} = 1$

diverges by LCT

Section 11.8

12.) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2} = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{(n+1)^2}}{\frac{(-1)^n x^n}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| \cdot \frac{n^2}{(n+1)^2}$

$= \lim_{n\to\infty} |x| \cdot \left(\frac{n+1}{n+1} - \frac{1}{n+1} \right)^2 = |x| \cdot (1-0)^2 = |x|$, thus $|x|<1$

So $I = [-1,1]$ $\quad R = \frac{1-(-1)}{2} = 1$

14.) $\sqrt[n]{|a_n|} = \sqrt[n]{|n^n x^n|} = \sqrt[n]{n^n \cdot |x|^n} = n \cdot |x| \to \infty$. Diverges when $x \neq 0$

thus $R=0$ $I=[0,0]$

16.) $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^4 \cdot 4^{n+1}}}{\frac{x^n}{n^4 \cdot 4^n}} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^4 4^{n+1}} \cdot \frac{n^4 4^n}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{4} \cdot \left(\frac{n}{n+1} \right)^4 = \lim_{n\to\infty} \frac{|x|}{4} \cdot \left| \frac{\frac{n}{n}}{\frac{1}{n}+\frac{n}{n}} \right|^4$

$= \frac{|x|}{4} \cdot \left| \frac{1}{0+1} \right|^4 = \frac{|x|}{4}$ $\quad -4<x<4$

when $x=-4$ $\sum \frac{(-1)^n}{n^4}$ converges

when $x=4$ $\sum \frac{1}{n^4}$ converges

$I = [-4,4]$ $\quad R=4$

18.) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} \cdot x^n = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{(n+1) \cdot 5^{n+1}}}{\frac{(-1)^{n-1}}{n5^n}} \right| = \lim_{n\to\infty} \frac{|x|}{5} \cdot \frac{n}{n+1} = \lim_{n\to\infty} \frac{|x|}{5} \cdot \frac{1}{\frac{1}{n}+1} = \frac{|x|}{5}$ $\quad -5<x<5$

When $x=-5$. $\sum \frac{-1}{n}$ diverges. when $x=5$ $\sum \frac{(-1)^{n-1}}{n}$ converges

$R=5$ $\quad I=(-5,5]$

20.) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!} = \lim_{n\to\infty} \frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}} = \lim_{n\to\infty} \frac{x^{2n+2}}{x^{2n}} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} x^2 \cdot \frac{n!}{n! \cdot (n+1)} = x^2 \cdot \lim_{n\to\infty} \frac{n!}{n! \cdot (n+1)} = x^2 \cdot 0 = 0 < 1$

$I = (-\infty, \infty)$

22. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{(2n-1)2^n} = \lim_{n\to\infty} \sqrt[n]{\frac{(-1)^n (x-1)^n}{(2n-1)2^n}} = \lim_{n\to\infty} \frac{|x-1|}{2(2n-1)^{1/n}} = \frac{|x-1|}{2}$, $-1 < x < 3$
$R=2$

let $x = -1$ $\sum \frac{1}{2n}$ diverges

$x = 3$ converges $R = 2$ $I = (-1, 3]$

24.) $\sum_{n=1}^{\infty} \frac{\sqrt{n}(x+6)^n}{8^n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}(x+6)^{n+1}}{8^{n+1}}}{\frac{\sqrt{n}(x+6)^n}{8^n}} = \lim_{n\to\infty} \frac{|x+6|}{8} \sqrt{\frac{n+1}{n}}$

$= \lim_{n\to\infty} \frac{|x+6|}{8} \sqrt{1+\frac{1}{n}} = \frac{|x+6|}{8}\sqrt{1+0} = \frac{|x+6|}{8}$ $-14 < x < 2$

$I = (-14, 2)$ $R = 8$