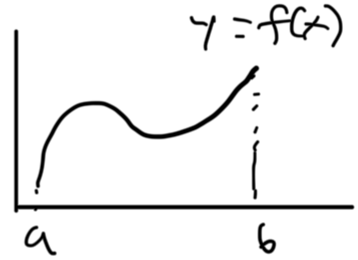
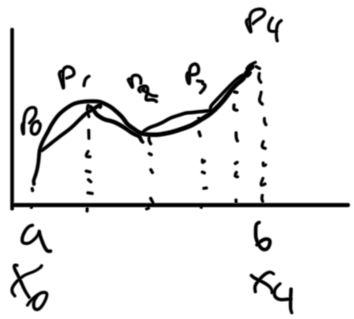
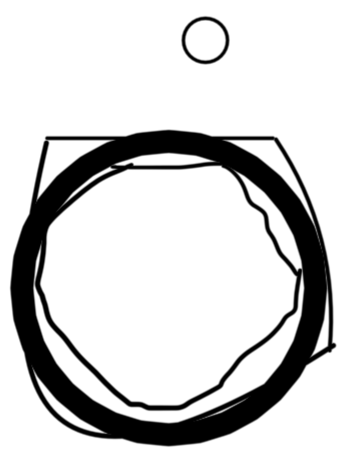
8.1 Arc length



fit piece of string to curve a → b, straighten it out and measure it.
This is arc length of curve.

Approxminate the curve with a polygonal path
a path of connected line segments,
then take the limit as the number of segments of the path increases,





$C = 2\pi r = \pi d$

$\frac{C}{d} = \pi$

let c denote the curve defined by f on [a,b]

Partion it into sub-intervals of = width Δx_i

$[x_0, x_1] \ldots\ldots [x_{n-1}, x_n]$

$x_0 = A$, $x_n = B$

Let P be (x,y)

Let $|P_{i-1}P_i|$ be length of line segment $P_{i-1} \to P_i$

$$\sum_{i=1}^{n} |P_{i-1}P_i| \text{ is length of path}$$

arc length L of C is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

to find Δy_i^2 is find slope of the line $P_{i-1}P_i$
apply MVT of f on $[x_{i-1}, x_i]$

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Thus $\Delta y_i = f'(x_i^*)\Delta x_i$
and $|P_{i-1}P_i| = \sqrt{\Delta x_i^2 + (f'(x_i^*)\Delta x_i)^2}$

$$= |\Delta x_i| \sqrt{1 + f(x_i^*)^2}$$

$$= \Delta x_i \cdot \sqrt{1 + f(x_i^*)^2}$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x_i \cdot \sqrt{1 + f'(x_i^*)^2} = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Arclength formula

$$y = f(x) \quad a \le x \le b$$

$$\boxed{L = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_a^b \sqrt{1 + \frac{dy}{dx}^2}\, dx}$$

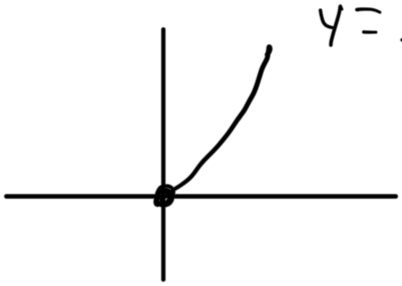
if the curve is given by $x = g(y)$ on $[c,d]$ then the formula is

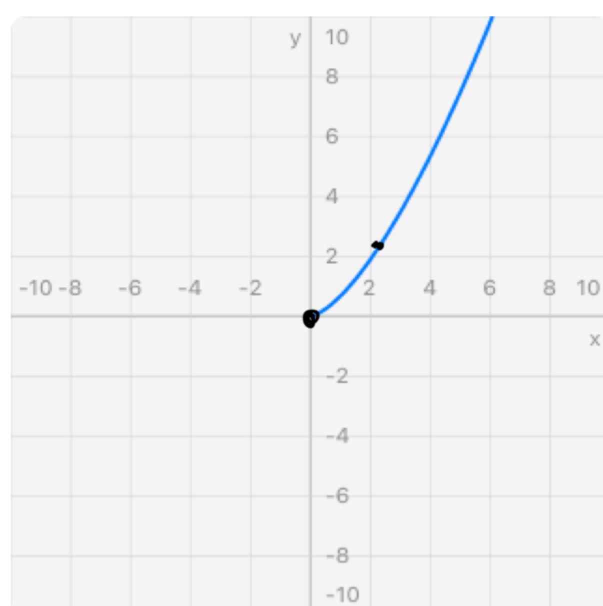
$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy = \int_c^d \sqrt{1 + \frac{dx}{dy}^2}\, dy$$

Example: Find arclength

$y = \frac{2}{3}x^{3/2}$ between $(0,0)$, $(2, \frac{4}{3}\sqrt{2})$

$y = \frac{2}{3}x^{3/2}$





$f'(x) = x^{1/2}$

$$L = \int_0 \sqrt{1 + (x^{1/2})^2}\, dx = \int_0^2 \sqrt{1+x}\, dx$$

$u = 1+x$
$du = dx$
$u = 1+0 = 1$
$= 1+2 = 3$

$$= \int_1^3 u^{1/2}\, du = \frac{2}{3} u^{3/2} \Big|_1^3$$

$$= \left(\left(\frac{2}{3}(1+3)^{3/2} - \frac{2}{3}(2(1+1))^{3/2} \right) = \frac{2}{3}\left(3\sqrt{3} - 1\right) \right.$$

?

Ex arclength $x = y^2 + y$ $(0,0)$, $(12,3)$

$x' = 2y+1$

$$L = \int_0^3 \sqrt{1 + (2y+1)^2}\, dy$$

$x = y^2 + y$
$u = 2y+1$
$du = 2\,dy$ $\quad \frac{du}{2} = dy$

$$= \frac{1}{2}\int_1^7 \sqrt{1+u^2}\, du$$

$2(3)+1 = 7$
$2(0)+1 = 1$

$$= \frac{1}{2}\int_1^7 \sqrt{\tan^2\theta + 1} = \int \sec\theta\, du \quad u = \tan\theta \qquad du = \sec^2\theta\, d\theta$$

$\theta = \tan^{-1} 7$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\tan^{-1} 7} \sec^3\theta\, d\theta = \frac{1}{2}\left(\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right)\Bigg|_{\frac{\pi}{4}}^{\tan^{-1} 7} =$$

$$\boxed{= \frac{1}{4}\left(34\sqrt{2} + \ln\left|\frac{5\sqrt{2}+7}{\sqrt{2}+1}\right|\right.}$$

The arclength function

The arclength function

$$S(x)=\int_a^x \sqrt{1+f'(t)^2}\,dt \quad \text{for } x \text{ in } [a,b]$$

$$S'(x)=\sqrt{1+f'(x)^2}$$