Vote Today!

(Trumpet)

Exam 2!

Today from 6:30pm - 8pm
in LR2 Van Allen Hall (same as before)

1. An ant crawls up and down a stalk of grass so that its position at any time t is given by $y(t) = \sqrt{t}(t-2)$. Find the velocity and acceleration of the ant at $t = 1$. Is the ant speeding up or slowing down at $t = 1$?

velocity $= v(t) = y'(t) = \frac{1}{2}t^{-\frac{1}{2}}(t-2) + \sqrt{t}$

$v(1) = \frac{1}{2}$

acceleration $= a(t) = y''(t) = -\frac{1}{4}t^{-\frac{3}{2}}(t-2) + \frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{1}{2}}$

$a(1) = \frac{5}{4}$

velocity positive and acceleration positive $\Rightarrow$ speeding up

2. A particle moves horizontally through space so that its position at any time t is given by $x(t) = \frac{1}{2}t^4 + 3t^2 - 5t$. Find the velocity and acceleration of the particle at $t = 3$. Determine if the particle is speeding up or slowing down at $t = 3$.

$$v(t) = x'(t) = 2t^3 + 6t - 5$$

$$v(3) = 67$$

$$a(t) = x''(t) = 6t^2 + 6$$

$$a(3) = 60$$

velocity positive and acceleration positive $\Rightarrow$ speeding up

3. Use linear approximation to estimate the value of $f(x) = x^4$ at $x = 1.999$.

$f(x) = x^4 \qquad x = 1.999 \qquad a = 2$

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ &\approx 16 + 3(2)^3(-.001) \\ &\approx 16 + 24(-.001) \\ (1.999)^4 &\approx \boxed{15.976} \end{aligned}$$

4. Use linear approximation to estimate the value of $\frac{1}{4.002}$

$f(x) = \frac{1}{x}$ $\quad x = 4.002 \quad a = 4$

$$f(x) \approx f(a) + f'(a)(x-a)$$
$$\approx \frac{1}{4} - \frac{1}{16}(.002)$$
$$\approx \boxed{.249875}$$

5. An initial population of 20 rats live in the sewers beneath Paris. Their population doubles after 3 weeks. What will the population of rats be after 5 weeks?

Pe^{rt}

$20e^{r\cdot 3} = 40$

$e^{3r} = 2$

$\ln(e^{3r}) = \ln(2)$

$3r = \ln(2)$

$r = \frac{\ln(2)}{3} \longrightarrow \boxed{20e^{\left(\frac{\ln(2)}{3}\right)\cdot 5}}$

6. Unobtanium, a highly radioactive material found on an alien planet, has a half-life of 5 years. If you start with 10 grams of this material, how much will remain after 20 years?

Pe^{rt}

$$10e^{r(5)} = 5$$
$$e^{5r} = \frac{1}{2}$$
$$\ln(e^{5r}) = \ln\left(\frac{1}{2}\right)$$
$$5r = \ln\left(\frac{1}{2}\right)$$
$$r = \frac{\ln\left(\frac{1}{2}\right)}{5} \longrightarrow \boxed{10e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{5}\right)20}}$$

7. Given the equation $x^3y^2 - \cos(\pi y) = 3$, find the equation of the tangent line at the point $(1, 2)$.

$$3x^2y^2 + x^3 2y\frac{dy}{dx} + \pi\sin(\pi y)\frac{dy}{dx} = 0$$

$$x^3 2y\frac{dy}{dx} + \pi\sin(\pi y)\frac{dy}{dx} = -3x^2y^2$$

$$\frac{dy}{dx}\left(x^3 2y + \pi\sin(\pi y)\right) = -3x^2y^2$$

$$\frac{dy}{dx} = \frac{-3x^2y^2}{x^3 2y + \pi\sin(\pi y)}$$

$$\frac{-3(1)^2(2)^2}{(1)^3 2(2) + \pi\sin(2\pi)} = \frac{-12}{4+0} = -3$$

$$\rightarrow \boxed{y - 2 = -3(x-1)}$$

8. Given the equation $e^{2x}y^2 - xy^2 = 1$, find the equation of the tangent line at the point $(0, 1)$.

$$2e^{2x}y^2 + e^{2x}2y\frac{dy}{dx} - y^2 - x2y\frac{dy}{dx} = 0$$

$$e^{2x}2y\frac{dy}{dx} - x2y\frac{dy}{dx} = y^2 - 2e^{2x}y^2$$

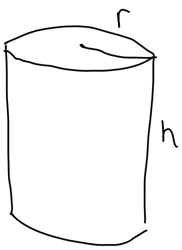
$$\frac{dy}{dx}\left(e^{2x}2y - x2y\right) = y^2 - 2e^{2x}y^2$$

$$\frac{dy}{dx} = \frac{y^2 - 2e^{2x}y^2}{e^{2x}2y - x2y}$$

$$\frac{(1)^2 - 2e^{2(0)}(1)^2}{e^{2(0)}2(1) - (0)2(1)} = -\frac{1}{2}$$

$$\rightarrow \boxed{y - 1 = -\frac{1}{2}(x - 0)}$$

9. A closed cylindrical canister is being fabricated to contain 200 cubic meters of soda. What is the radius of the base of the cylinder that will minimize the surface area?



① $V = 200 = \pi r^2 h \longrightarrow h = \dfrac{200}{\pi r^2}$

Plug in

② $SA = 2\pi r^2 + 2\pi r h$

minimize

$$SA = 2\pi r^2 + \frac{400}{r}$$

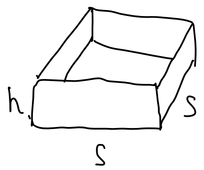
$$0 = 4\pi r - \frac{400}{r^2}$$

$$\frac{400}{r^2} = 4\pi r$$

$$\frac{100}{\pi} = r^3$$

$$\boxed{r = \sqrt[3]{\frac{100}{\pi}}}$$

10. An open top rectangular box with a square base is to be made. The volume of the box is 108 cubic units. Find the height of the box that will minimize the surface area.



$$V = 108 = s^2 h \rightarrow h = \frac{108}{s^2}$$

$$SA = s^2 + 4sh$$

$$= s^2 + \frac{436}{s}$$

minimize

$$0 = 2s - \frac{436}{s^2}$$

$$\frac{436}{s^2} = 2s$$

$$216 = s$$

$$\boxed{s = 6 \rightarrow h = \frac{108}{36} = 3}$$

11. Evaluate the limit using L'Hospital's rule:

$$\lim_{x\to 0} x^3 \ln(x^2) \longrightarrow 0 \cdot (-\infty)$$

$$\lim_{x\to 0} x^3 \ln(x^2)$$

$$= \lim_{x\to 0} \frac{\ln(x^2)}{x^{-3}}$$

L'H only works on fractions, so use this trick to make a fraction!

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{x^2}\cdot 2x}{-3x^{-3}}$$

$$= \lim_{x\to 0} \frac{\frac{2}{x^2}\cdot x^3}{-3}$$

$$= \lim_{x\to 0} \frac{2x}{-3}$$

$$\boxed{= 0}$$

12. Evaluate the limit using L'Hospital's rule:

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{x^2} \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{2x} \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{4e^{2x} - 2}{2} \boxed{= 1}$$

13. Use L'Hospital's rule to find:

$$\lim_{x\to 1^+} x^{1/(1-x)} \longrightarrow 1^{\infty}$$

$$\lim_{x\to 1^+} \ln\left(x^{1/(1-x)}\right) = \ln y$$

$$\lim_{x\to 1^+} \frac{1}{1-x} \cdot \ln(x) = \ln y$$

$$\lim_{x\to 1^+} \frac{\ln(x)}{1-x} = \ln y$$

$$\overset{L'H}{=} \lim_{x\to 1^+} \frac{1/x}{-1} = \lim_{x\to 1^+} \frac{-1}{x} = \ln y$$

$$e^{-1} = \cancel{e}^{\cancel{\ln} y}$$

$$\boxed{e^{-1} = y}$$

14. Use L'Hospital's rule to find:

$$\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x$$

$$\lim_{x\to 0^+} \ln\left(1+\frac{1}{x}\right)^x = \ln y$$

$$\lim_{x\to 0^+} x \ln\left(1+\frac{1}{x}\right) = \ln y$$

$$\lim_{x\to 0^+} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = \ln y$$

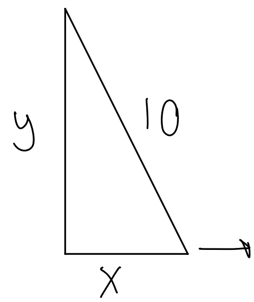
$$\lim_{x\to 0^+} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-x^{-2}} = \ln y$$

$$\lim_{x\to 0^+} \frac{\frac{1}{\frac{x+1}{x}} \left(-\frac{1}{x^2}\right)}{-x^{-2}} = \ln y$$

$$\lim_{x\to 0^+} \frac{\frac{x}{x+1} \left(-\frac{1}{x^2}\right) \cdot x^2}{-1} = \ln y$$

$$\lim_{x\to 0^+} \frac{x}{x+1} = \frac{0}{1} \boxed{= 0}$$

15. A 10-foot ladder is leaning against a wall. If the bottom of the ladder is being pulled away from the wall at a rate of 5 feet per second, how fast is the top of the ladder sliding down the wall when the bottom is 3 feet from the wall?



$\frac{dx}{dt} = 5$

find $\frac{dy}{dt}$ when $x = 3$

$x^2 + y^2 = 100$

$3^2 + y^2 = 100$

$y^2 = 91$

$y = \sqrt{91}$

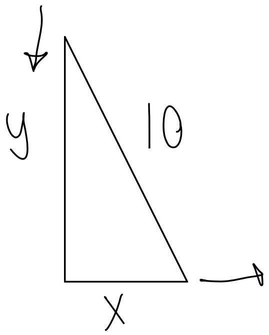
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(3)(5) + 2(\sqrt{91})\frac{dy}{dt} = 0$

$$\boxed{\frac{dy}{dt} = -\frac{15}{\sqrt{91}}}$$

16. A 10-foot ladder is leaning against a wall. If the bottom of the ladder is being pulled away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall after 3 seconds?

find $\frac{dy}{dt}$ when $t = 3$



$\frac{dx}{dt} = 2$

$x^2 + y^2 = 100$

$6^2 + y^2 = 100$

$y^2 = 64$

$y = 8$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

3 seconds $\cdot$ 2 ft/sec

$2(6)(2) + 2(8)\frac{dy}{dt} = 0$

$$\boxed{\frac{dy}{dt} = -\frac{3}{2}}$$

17. For function $f(x) = x^3 - 3x^2 + 4$, identify and label all critical points and inflection points. Mark all intervals in which the function is increasing and decreasing. Mark all intervals in which the function is concave up (CONVEX) or concave down (CONCAVE).

$3x^2 - 6x = 0$

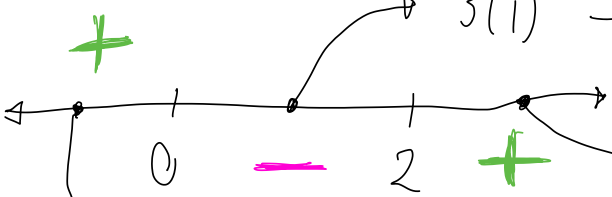
$3x(x-2) = 0$

critical values: $x = 0, 2$

$0^3 - 3(0)^2 + 4 = 4 \longrightarrow$ $(0,4)$ max

$2^3 - 3(2)^2 + 4 = 0 \longrightarrow$ $(2,0)$ min

$3(1)^2 - 6(1) = -3$



$3(3)^2 - 6(3) = 9$

$3(-1)^2 - 6(-1) = 9$

interval of increase: $(-\infty, 0) \cup (2, \infty)$

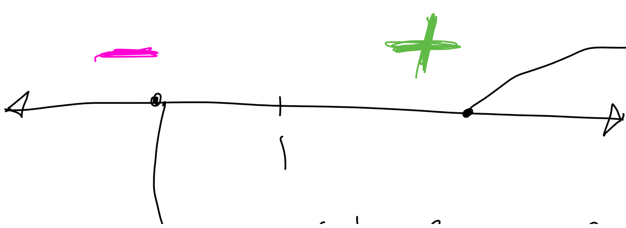
interval of decrease: $(0, 2)$

second deriv: $6x - 6 = 0$

$6x = 6$

$x = 1$

$1^3 - 3(1)^2 + 4 = 2 \longrightarrow$ $\boxed{\text{inflection point: } (1,2)}$



$6(2) - 6 = 6$

$\hookrightarrow 6(0)-6 = -6$

interval of concave up: $(1, \infty)$

interval of concave down: $(-\infty, 1)$

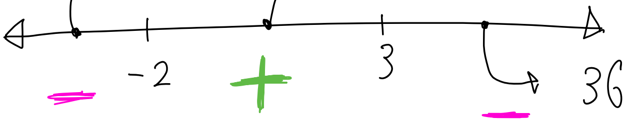
18. For function $f(x) = 36x + 3x^2 - 2x^3$, identify and label all critical points and inflection points. Mark all intervals in which the function is increasing and decreasing. Mark all intervals in which the function is concave up (CONVEX) or concave down (CONCAVE).

$$36 + 6x - 6x^2 = 0$$
$$6 + x - x^2 = 0$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2)$$
$$x = 3, -2 \quad \text{critical values}$$

$$36(3) + 3(3)^2 - 2(3)^3 = 189$$
$$36(-2) + 3(-2)^2 - 2(-2)^3 = -44$$

$\rightarrow$ $\boxed{\begin{array}{ll} (3, 189) & \text{max} \\ (-2, -44) & \text{min} \end{array}}$

$$36 + 6(-3) - 6(-3)^2 = -36$$
$$36 + 6(0) - 6(0)^2 = 36$$



$$36 + 6(4) - 6(4)^2 = -36$$

interval of increase: $(-2, 3)$
interval of decrease: $(-\infty, -2) \cup (3, \infty)$

$$6 - 12x = 0$$
$$-12x = -6$$
$$x = \frac{1}{2}$$

$$36\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 - 2\left(\tfrac{1}{2}\right)^3 = \frac{37}{2}$$

$\rightarrow$ inflection point: $\left(\frac{1}{2}, \frac{37}{2}\right)$

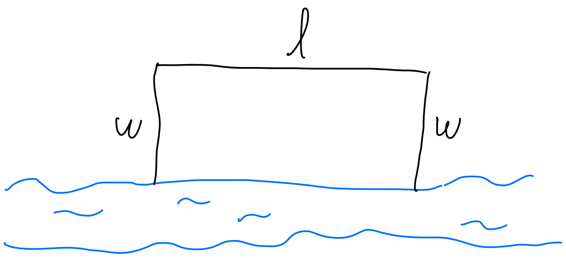
$6 - 12(0) = 6$ $\qquad$ $6 - 12(1) = -6$

$+$ $\qquad$ $\frac{1}{2}$ $\qquad$ $-$

interval of concave up: $(-\infty, \frac{1}{2})$

interval of concave down: $(\frac{1}{2}, \infty)$

19. A farmer wants to fence off a rectangular field along a river. If the farmer has 200 yards of fencing and needs no fencing along the river, what dimensions will maximize the area of the field?



maximize area

1 $l + 2w = 200$

$l = 200 - 2w$

2 $A = lw$

$A = (200 - 2w)w$

$A = 200w - 2w^2$

$200 - 4w = 0$

$-4w = -200$

$\boxed{w = 50} \rightarrow l = 200 - 2(50)$

$\boxed{l = 100}$