Approximation Algorithms for Stochastic Optimization Problems in Operations Management

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Abstract

This article provides an introduction to approximation algorithms in stochastic optimization models arising in various application domains, including central areas of operations management, such as scheduling, facility location, vehicle routing problems, inventory and supply chain management and revenue management. Unfortunately, these models are very hard to solve to optimality both in theory and practice. We will survey recent development on approximation algorithms for these stochastic optimization models and their performance analysis techniques with worst-case performance guarantees.

Key words: approximation algorithms; stochastic optimization; operations management

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1 Introduction

The difficulty of sifting through large amount of data in order to make an informed choice is ubiquitous nowadays thanks to the advances in information technologies and high-speed networking. One of the promises of the information technology era is that many decisions can now be made rapidly by computers, such as deciding inventory levels, routing vehicles, planning facility locations, managing revenue and so on. Many of these applications can be modeled as discrete optimization problems. Unfortunately, most interesting discrete optimization problems and their stochastic variants are NP-hard. Thus, we cannot simultaneously have algorithms that (1) find optimal solution (2) in polynomial time (3) for any instance. In order to deal with such optimization problems, we need to relax at least one of three requirements above: relaxing the 'for any instance' requirement, the requirement of polynomial-time solvability, or the requirement of finding an optimal solution. The third relaxation is the most common approach, where we only need to find a 'good enough' solution, instead of the best one. Next, we formally define the notion of *approximation algorithms* for discrete optimization problems.

Definition 1. *An α-**approximation algorithm** for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of α of the value of an optimal solution.*

For an α-approximation algorithm, we always call α the *performance guarantee*, *approximation ratio* or *approximation factor* of the algorithm. And we will follow the convention that $\alpha > 1$ for minimization problems, while $\alpha < 1$ for maximization problems.

For a particular class of problems of interest (e.g., knapsack problem, Euclidean traveling salesman problem), we are able to obtain extremely good approximation algorithms; in fact, these problems have *polynomial-time approximation schemes.*

Definition 2. *A **polynomial-time approximation scheme (PTAS)** is a family of algorithms $\{P(\epsilon)\}$, where there is an algorithm for each $\epsilon > 0$, such that $P(\epsilon)$ is a $(1+\epsilon)$-approximation algorithm for minimization problems, or a $(1-\epsilon)$-approximation algorithm for maximization problems.*

However, there exists a large class of interesting (but not so easy) problems called MAX SNP (e.g., max satisfiability problem, max cut problem), which fails to have a PTAS, unless $\mathbf{P} = \mathbf{NP}$.

There is a vast body of literature in both computer science and operations research devoted to designing approximation algorithms for deterministic discrete optimization problems. For an excellent and detailed exposition on deterministic models, we refer interested readers to the following books, Ausiello [2], Vazirani [79] and Williamson and Shmoys [80]. This article mainly focuses on approximation algorithms for stochastic optimization models arising in operations management, which have gained much momentum recently. Before giving an overview or a survey on the recent development, we would like to first provide readers some basic ideas of discrete optimization models *under uncertainty* through the following example.

1.1 A stochastic vertex cover problem

We first use a stochastic version of the vertex cover problem to motivate our discussion of how approximation algorithms can be designed for stochastic optimization problems. We review the 2-approximation algorithm and its worst-case analysis by Ravi and Sinha [61].

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. Formally, a vertex-cover of an undirected graph $G = (V, E)$ (vertices and edges) is a subset S of V such that if an edge $(u, v) \in E$, then either $u \in S$ or $v \in S$ (or both). The set S is said to *cover* the edges of G. Since picking each vertex $i \in V$ incurs a cost c_i, a *minimum vertex cover* is a vertex cover of smallest possible cost. A simple deterministic LP rounding algorithm yields a 2-approximation. The best known approximation algorithm has performance ratio of $2 - \log\log|V|/(2\log|V|)$, due to Monien and Speckenmeyer [60]. A lower bound of 1.16 on the hardness of approximating the problem was shown by Håstad [36]. We refer readers to Williamson and Shmoys [80] and Vazirani [79] for the extensive discussion of the deterministic vertex cover problem.

Now we describe a two-stage stochastic version of the vertex cover problem. In the first stage, we are given a (undirected) graph $G = (V, E_0)$. In the stochastic version, the edges E that will be present in the second stage are uncertain a priori, and we only know its distributional information, i.e., there are m possible scenarios, each consisting of a set of *realized* edges E_k with probability of occurrence p_k. Note that E_k $(k = 1, \ldots, m)$ may or may not be subsets of the first-stage edge set E_0. The first-stage cost of vertex v is c_v^0, and its cost in scenario k at the second-stage is c_v^k. The edges in $E_k \cap E_0$ may be covered in either the first or second stage, while edges in $E_k \setminus E_0$ must be covered in the second stage. The objective is to identify a set of vertices to be selected in the first stage, so that the expected cost of extending this set to a vertex cover of the edges of the realized second-stage scenario is minimized.

Ravi and Sinha [61] provided a primal-dual algorithm which rounds the integer programming (IP) formulation of stochastic vertex cover. Variable x_v^k indicates whether or not vertex v is picked in scenario k (where $k = 0$ corresponds to the first stage). The IP formulation is given by

$$\begin{aligned}
\min \quad & \sum_v c_v^0 x_v^0 + \sum_{k=1}^m p_k c_v^k x_v^k, \qquad \text{(IP-SVC)} \\
\text{s.t.} \quad & x_u^0 + x_v^0 + x_u^k + x_v^k \geq 1, \qquad \forall (u, v) \in E_k \cap E_0, \forall k, \\
& x_u^k + x_v^k \geq 1, \qquad \forall (u, v) \in E_k \setminus E_0, \forall k, \\
& x \in \mathbb{Z}_0^+.
\end{aligned}$$

The LP relaxation of IP-SVC, called LP-SVC, replaces the constraint $x \in \mathbb{Z}_0^+$ by $x \geq 0$. Then we write the dual of LP-SVC below. The dual variable y_e^k packs edge e in E_k if $e \in E_k$, and it packs $e \in E_0$ if $e \in E_k \cap E_0$.

$$
\begin{aligned}
\max \quad & \sum_{k=1}^{m} \sum_{e \in E_0 \cup E_k} y_e^k, && \text{(DLP-SVC)} \\
\text{s.t.} \quad & \sum_{e \in E_k : v \in e} y_e^k \le p_k c_v^k, && \forall v, \forall k, \\
& \sum_{k=1}^{m} \sum_{e \in E_0 \cap E_k : v \in e} y_e^k \le c_v^0, && \forall v, \\
& y \ge 0.
\end{aligned}
$$

The algorithm π is a greedy dual-ascent type of primal-dual algorithm with two phases.

(a) In the first phase, we raise the dual variable y_e^k uniformly for all edges in $E_k \setminus E_0$, separately for each k. All vertices which become tight (the first dual constraint packed to $p_k c_v^k$) have x_v^k set to 1, and deleted along with adjacent edges. We proceed this way until all edges in $E_k \setminus E_0$ are covered and deleted.

(b) In the second phase, we raise the dual variable y_e^k uniformly for all *uncovered* edges in E_k. Note that these uncovered edges are contained in $E_k \cap E_0$. If a vertex is tight for x_v^0 (i.e., second dual constraint packed to c_v^0), then we pick it in the first stage solution by setting $x_v^0 = 1$, and if it is not tight for x_v^0 (the first dual constraint packed to $p_k c_v^k$), then we pick it in the second stage by setting $x_v^k = 1$ as a recourse decision.

Theorem 1 (Ravi and Sinha [61])**.** *The integer program IP-SVC can be bounded by the primal-dual algorithm described above within a factor of* 2 *in polynomial time.*

Proof. Let the cost of DLP-SVC generated by π be $\mathcal{C}^\pi$(DLP-SVC). Also, let the optimal costs of LP-SVC and IP-SVC be $\mathcal{C}^*$(LP-SVC) and $\mathcal{C}^*$(IP-SVC), respectively. By linear programming duality,

$$\mathcal{C}^\pi(\text{DLP-SVC}) \le \mathcal{C}^*(\text{LP-SVC}) \le \mathcal{C}^*(\text{IP-SVC}).$$

Now if π generates a feasible solution for IP-SVC and a cost $\mathcal{C}^\pi$(IP-SVC), then it suffices to show that

$$\mathcal{C}^\pi(\text{IP-SVC}) \le 2 \cdot \mathcal{C}^\pi(\text{DLP-SVC}) = 2 \sum_{k=1}^{m} \sum_{e \in E_0 \cup E_k} y_e^k.$$

The feasibility is rather obvious. Consider an edge $e = (u, v) \in E_k$ in scenario k . We must have picked one of its end-points in either the first phase or the second phase (or both) by the construction of algorithm π.

To complete the proof, we shall show that each time we set $x_v^k = 1$, we assign some dual variables to it such that (i) the sum of dual variables assigned to each such x_v^k variable has to equal $p_k c_v^k$ (where $p_0 = 1$), and (ii) each dual variable or each edge is assigned at most twice.

Consider a vertex v which was selected in scenario k in either the first or second phase. We assign all dual variables y_e^k such that v is incident to e. By construction of π, v is only chosen when the first constraint becomes tight, i.e.,

$$\sum_{e \in E_k : v \in e} y_e^k = p_k c_v^k, \qquad \forall k, \tag{1}$$

thereby guaranteeing (i). An edge $e \in E_k \setminus E_0$ is assigned to vertex v only if x_v^k is set to 1 for $k \ne 0$. Thus, (ii) for $e \in E_k \setminus E_0$ is ensured since each edge has at most 2 vertices.

Then we consider a vertex v for which x_v^0 is set to 1. By our construction of π, the second constraint has to be tight, i.e.,

$$\sum_{k=1}^{m} \sum_{e \in E_0 \cap E_k : v \in e} y_e^k = c_v^0,$$

and all edges in the sum are assigned to the variable x_v^0, ensuring (i). In addition, since these edges in the sum are not assigned to any other variable x_v^k for $k \neq 0$ and each such edge has only two vertices, (ii) is maintained. □

As shown in this example, instead of actually solving the dual LP (DLP-SVC), we can construct a feasible dual solution maintaining some desired properties. In this case, constructing the dual solution is much faster than solving the dual LP, and hence leads to a much faster algorithm. Other common techniques for designing approximation algorithms include various rounding techniques, greedy algorithms or randomized algorithms.

1.2 Literature review

Traditionally, approximation algorithm techniques have been applied primarily to deterministic combinatorial optimization problems, for instance, the set cover problem, the knapsack problem, the bin-packing problem, the traveling salesman problem, scheduling problems and so on. We refer interested readers to Ausiello [2], Vazirani [79] and Williamson and Shmoys [80] for more details on various deterministic models. Our literature review will mainly focus on the design of approximation algorithms for stochastic optimization models, with emphasis on the recent development in various operations management models.

Stochastic optimization is a vast field, beginning with the works of Dantzig [16] and Beale [6] in the 1950s, and seeing much activity to this day; we refer interested readers to the following books that survey the field, Birge and Louveaux [8], Kall and Wallace [41], Stougie and Van Der Vlerk [74], and Ruszczyński and Shapiro [63]. Stochastic optimization problems are often computationally quite difficult, and often more difficult than their deterministic counterparts, both from the viewpoint of complexity theory, as well as from a practical perspective. In many settings the computational difficulty stems from the fact that the distribution might assign a non-zero probability to an exponential number of scenarios, leading to considerable increase in the problem complexity, a phenomenon often called the *curse of dimensionality*.

The work on approximation algorithms for stochastic combinatorial problems goes back to the work on stochastic scheduling problem of Möhring et al. [57, 58] and the more recent work of Möhring et al. [59]. By its very nature, scheduling algorithms often have to account for uncertainty in the sizes and arrival times of future jobs. Next we will survey some recent development on approximation algorithms for these stochastic optimization models related to operations management. We attempt to divide the relevant work into three groups. The first group surveys the growing stream of approximation results for two-stage or multistage stochastic optimization models. These results are usually derived for general purposes, and can be potentially applied to specific problems in operations management. The second group discusses the recent advances in designing approximation algorithms for stochastic inventory systems, with a summary of many key results. The third group presents relevant work in other core operations management models, such as scheduling, facility location, vehicle routing and revenue management.

General stochastic optimization models. The first worst-case analysis of approximation algorithms for two-stage stochastic programming problems was the work on service provisioning in a telecommunication network by Dye et al. [20]. Ravi and Sinha [61] studied two-stage stochastic versions of several combinatorial optimization problem (e.g., shortest path, vertex cover, facility location, set cover, bin-packing problems), and provided nearly tight approximation algorithms for them when the number of scenarios is polynomial. Independently, Immorlica et al. [38] gave approximation algorithms for several covering and packing problems. Their model allowed for exponential scenarios (through an independent activation model) but the costs in the two stages have to be proportional.

Gupta et al. [31] were the first to consider the *black-box* model, and gave sampling-based approximation algorithms for various two-stage problems with a proportional cost structure. In the black-box model, the underlying distribution is not specified exactly but the algorithm has access to an oracle that can be used to sample from the underlying distribution, and the running time is measured in terms of the number of calls to this oracle. Gupta et al. [32] extended this boosted sampling framework for multistage stochastic optimization problems with recourse. Shmoys and Swamy [69, 70, 76] showed that one could derive a PTAS using sample average approximation and adapted ellipsoid methods for the exponentially-large LP relaxations, and then use a simple rounding approach to derive approximation algorithms for the original black-box model without any proportional cost assumption. Swamy and Shmoys [75] extended their results to multistage stochastic optimization problems, and showed that the LP solution for each stage can be rounded to an integer solution independent of other stages. Charikar [13] gave a general technique based on a sample average approximation that reduced the problem of obtaining a good approximation algorithm for the two-stage black-box model, to the problem of obtaining the analogous results in the polynomial scenario setting. Srinivasan [73] improved upon work of Swamy and Shmoys [75] by showing an approximability which does not depend multiplicatively on the number of stages.

Dhamdhere [19] introduced the robust version of two-stage combinatorial covering problems under uncertainty and gave approximation algorithms for them. There are also some recent work related to robust or risk-averse version under uncertainty (see, e.g., Gupta et al. [30] on covering problems using a guess and prune idea, Golovin [23] on two-stage min-cut and shortest path problems, So et al. [72] on problems with controllable risk aversion level).

Stochastic inventory systems. The concept of approximation algorithms has been applied to several deterministic problems in inventory management, see, e.g., Silver and Meal [71], Roundy [62], Levi et al. [48, 45, 50], Shen et al. [65], Cheung et al. [14]. We then focus on stochastic inventory systems.

The recent stream of research on designing approximation algorithms for the multiperiod stochastic inventory control problems was initiated by Levi et al. [46] who proposed a 2-approximation algorithm for the basic uncapacitated backlogged model. Subsequently, Levi et al. [51] proposed a 2-approximation algorithm for the capacitated backlogged model. Levi et al. [44] designed a 2-approximation algorithms for uncapacitated models with lost-sales. Levi and Shi [54] gave a 3-approximation algorithm for the uncapacitated backlogged model with setup costs. Subsequently, Shi et al. [67] gave a 4-approximation algorithm for the capacitated backlogged problem with setup costs.

More recently, Truong [78] provided a 2-approximation algorithm for the stochastic inventory problem via a look-ahead (myopic) optimization approach. Tao and Zhou [77] designed a 2-approximation algorithm for stochastic inventory systems with remanufacturing. Chao et al. [11] proposed an approximation algorithm with a worst-case guarantee between 2 and 3 for perishable inventory systems. Subsequently, Chao et al. [10, 12] studied perishable inventory systems with capacity and with setup cost, respectively. There are also recent works on designing approximation algorithms for multi-echelon inventory systems (e.g., Chu and Shen [15] for one-warehouse-multi-retailer problems, Levi et al. [52] for serial inventory systems, Levi and Shi [53] for joint replenishment problems). For the distribution-free or the black-box model, Levi et al. [49] also proposed an approximation scheme based on a sample average approximation approach for multiperiod stochastic inventory problems.

Other core stochastic operations management models. Our list below is by no means exhaustive.

(a) *Scheduling.* Dean [17] in his doctoral thesis gave approximation algorithms for a broad class of stochastic scheduling problems. He also conducted an exhaustive survey of this topic prior to his thesis. More recently, Shmoys and Sozio [68] designed approximation algorithms based on the approach of Charikar [13] for two-stage stochastic scheduling problems, extending the results of Bar-Noy [5].

(b) *Vehicle routing.* Gupta et al. [29] gave randomized approximation algorithms with factor $1 + \alpha$ for split-delivery vehicle routing problem with stochastic demands, and $2 + \alpha$ for unsplit-delivery counterpart, where α is the best approximation guarantee for the traveling salesman problem. They also showed that the cyclic heuristic for split-delivery achieves a constant approximation ratio, thereby confirming the conjecture in Bertsimas [7]. More recently, Gørtz [24] formulated the stochastic vehicle routing problem via a two-stage stochastic optimization with recourse, and gave approximation results.

(c) *Facility location.* For the two-stage recourse model, Shmoys and Swamy [69] gave a $(3.225 + \epsilon)$-approximation algorithm, whereas Gupta et al. [31] gave a 8.45-approximation algorithm in the black-box model with proportional cost, and Srinivasan [73] presented a 3.25-approximation algorithm for facility location in the polynomial scenario setting. So et al. [72] gave an 8-approximation algorithm for the risk-adjusted two-stage stochastic facility location problem. Shen et al. [66] studied a reliable facility location problem wherein some facilities are subject to failure from time to time and proposed a 4-approximation algorithm.

(d) *Stochastic network design.* Gupta et al. [33, 34] gave LP rounding approximation algorithms for stochastic Steiner tree and stochastic network design problems. Krishnaswamy et al. [43] gave approximation algorithms for node-capacitated network design to minimize the energy consumption.

(e) *Resource allocation and revenue management.* Dean et al. [18] presented approximation algorithms for stochastic knapsack problems. Chan and Farias [9] gave a 2-approximation algorithm for multiperiod stochastic depletion problems. Geunes et al. [22] proposed a 1.582-approximation algorithm via LP rounding scheme for supply chain planning and logistics problems with market choice under demand uncertainties. Goyal et al. [25] devised a PTAS for the assortment planning problem under dynamic substitution and stochastic demand. Levi and Radovanoic [47] gave a 2-approximation algorithm for revenue management models with reusable resources. Subsequently, Levi and Shi [55] then extended the same results allowing for advanced reservations.

Many if not most of the core problems in operations management fall into the category of multistage stochastic optimization models. Particularly, one has to make multiple, typically dependent decisions over time to optimize a certain objective function under uncertainty on how the system will evolve over the future time horizon. Unfortunately, it is often computationally intractable to find the exact optimal solutions for these fundamental and important models. Approximation algorithms seem to be the natural remedy for overcoming this prohibitive computing resource requirement.

Indeed, most of the heuristics and algorithms that have been proposed for operations management models were evaluated merely through computational experiments on randomly generated instances. This does not necessarily provide strong indications that the proposed heuristics are good in general, beyond the instances that were actually tested. In contrast, approximation algorithms have the advantage that they provide a priori and posteriori guarantees on the quality of the solution produced by the algorithm. Moreover, the performance analysis provides insights on how to design algorithms that have good empirical performance, which is in most cases significantly better than the worst-case performance guarantees.

1.3 Organization of the paper

The remainder of the paper is organized as follows. In Section 2, we discuss recent results on designing approximation algorithms for stochastic inventory systems. In Section 3, we present recent results on designing approximation algorithms for revenue management problems. Section 4 concludes our article and points out plausible avenues for future research.

2 Approximation Algorithms on Stochastic Inventory Models

In this section, we focus our attention on designing approximation algorithms for stochastic inventory models that allow for generally correlated demand structures capturing demand seasonality and forecast updates. Table 1 shows the current available results on various stochastic inventory systems.

Stochastic Inventory Control Systems	Approximation Ratio	References
Backlogged	2	[46], [78]
Lost-sales	2	[44]
Backlogged, Capacity	2	[51]
Backlogged, Setup Cost	3	[54]
Backlogged, Setup Cost, Capacity	4	[67]
Backlogged, Perishable	2 to 3	[11]
Backlogged, Perishable, Capacity	2 to 3	[10]
Backlogged, Perishable, Setup Cost	3 to 4	[12]
Backlogged, Remanufacturing	2	[77]
Backlogged, Serial System	2	[52]
Service-level, One-warehouse-multi-retailer	1.26	[15]

Table 1: Summary of current results on stochastic inventory control systems

When decision makers want to incorporate any forecast update mechanisms, the future demands can be represented as a function of the current information set consisting of past realized demand information and some other possible exogenous information that is available to them. This unavoidably introduces correlations between the future demands which make the dynamic programming formulation computationally intractable, since the state space grows exponentially fast. This is usually referred to as *the curse of dimensionality*. In this article, we present a single-echelon problem called the *stochastic lot-sizing problem*. The term lot-sizing captures the setup cost, or interchangeably the fixed ordering cost whenever a strictly positive order is placed. We demonstrate how randomized approximation algorithms can be designed for this problem.

2.1 Model

We consider a finite planning horizon of T periods indexed $t = 1, \dots, T$. The demands over these periods are random variables, denoted by $D_1, \dots, D_T$, and the goal is to coordinate a sequence of orders over the planning horizon to satisfy these demands with minimum expected cost. As a general convention, from now on we will refer to a random variable and its realization using capital and lower-case letters.

In each period $t = 1, \dots, T$, four types of costs are incurred, a per-unit ordering cost c_t for ordering any number of units at the beginning of period t, a per-unit holding cost h_t for holding excess inventory from period t to $t+1$, a per-unit backlogging penalty b_t that is incurred for each unsatisfied unit of demand at the end of period t, and a *fixed ordering* cost K that is incurred in each period with strictly positive ordering quantity. (It should be noted that our analysis remains valid for nonstationary K_t satisfying $\alpha K_{t+1} \leq K_t$, which is commonly assumed in the literature. Without loss of generality, we can assume that the discount factor $\alpha = 1$.) Unsatisfied units of demand are usually called *backorders*. Each unit of unsatisfied demand incurs a per-unit backlogging penalty cost b_t in each period t until it is satisfied. In addition, we consider a model with a lead time of L periods between the time an order is placed and the time at which it actually arrives. We assume that the lead time is a known integer L. We assume without loss of generality that the discount factor is equal to 1, and that $c_t = 0$ and h_t, $b_t \geq 0$, for each t.

At the beginning of each period s, we observe what is called an *information set* denoted by f_s. The information set f_s contains all of the information that is available at the beginning of time period s. More specifically, the information set f_s consists of the realized demands $d_1, \dots, d_{s-1}$ over the interval $[1, s)$, and possibly some exogenous information. The information set f_s in period s is one specific realization in the set of all possible realizations of the random vector $F_s = (D_1, \dots, D_{s-1})$. The set of all possible realizations is denoted by $\mathcal{F}_s$. The observed information set f_s induces a given conditional joint distribution of the future demands $(D_s, \dots, D_T)$. For ease of notation, D_t will always denote the random demand in period t according to the conditional joint distribution in some period $s \leq t$, where it will be clear from the context to which period s it refers. The index t will be used to denote a general time period, and s will always refer to the current period. The only assumption on the demands is that for each $s = 1, \ldots, T$, and each $f_s \in \mathcal{F}_s$, the conditional expectation $\mathbb{E}[D_t \mid f_s]$ is well defined and finite for each period $t \geq s$. In particular, we allow for non-stationary and correlation between the demands in different periods.

The traditional approach to study these models has been dynamic programming. Using the dynamic programming approach it can be shown that *state-dependent* (s, S) *policies are optimal* (see, e.g., Zipkin [82]). However, the computational complexity of the resulting dynamic programs is very sensitive to the dimension of the sets $\mathcal{F}_s$. In particular, in many practical scenarios these sets are of high dimension, which leads to dynamic programming formulations that are computationally intractable. In fact, it has been shown in Halman et al. [35] that this model is NP-hard, even for the relatively simple special case of independent discrete, finite support demands. In fact, for this special case it is possible to construct a polynomial approximation scheme (PTAS); that is, the problem can be approximated to an arbitrary degree of accuracy with a running time that depends on the degree of accuracy.

In addition, Guan and Miller [28, 27], Huang and Küçükyavuz [37], and Jiang and Guan [40] proposed exact and polynomial-time algorithms for the stochastic lot-sizing problem if the stochastic programming scenario tree is polynomially representable. These models allow for stochastic and correlated demands. However, the scenario tree in our model is exponentially large. Besides the exact (dynamic programming) approaches to stochastic lot-sizing problems, Guan et al. [26] and Zhang et al. [81] also proposed branch-and-cut methods to this class of problems.

2.2 Randomized Cost-Balancing Policies

In this section, we shall describe the *randomized cost-balancing policy* following Levi and Shi [54]. This policy is based on two major ideas: 1) *Marginal cost accounting scheme.* The standard dynamic programming approach directly assigns to the decision of how many units to order in each period only the expected holding and backlogging costs incurred in that period, although this decision might effect the costs in future periods. Instead, marginal cost accounting scheme assigns to the decision in each period *all* the expected costs that, once this decision is made, become unaffected by any decision made in future periods. These cost may still depend on future demands. 2) *Randomizing cost-balancing.* The idea of cost balancing was used in the past to construct heuristics with constant performance guarantees for deterministic inventory problems (e.g., Silver and Meal [71]). The key observation in the above model is that any policy in any period incurs potential expected costs due to *over ordering* (namely, expected holding costs of carrying excess inventory) and *under ordering* (namely, expected backlogging costs incurred when demand is not met on time). To address the nonlinearity induced by the fixed costs, a randomized decision rule is employed to balance the expected fixed ordering costs, holding costs and backlogging costs, in each period. In particular, the order quantity in each period is decided based on a carefully designed randomized rule that chooses among various possible order quantities with carefully chosen probabilities.

2.2.1 Marginal cost accounting scheme

We first introduce a *marginal holding cost accounting approach.* Without loss of generality, assume that the ordered supply units are consumed on *first-ordered, first-consumed basis.* The key observation under this assumption is that once an order is placed in some period, then the expected holding cost that the units just ordered will incur over the rest of the planning horizon is a function only of the realized demands over the rest of the horizon, not of any future orders. Hence, within each period, we can associate the overall expected holding cost that is incurred by the units ordered in this period over the entire horizon. We note that similar ideas of holding cost accounting were used previously in the context of models with continuous time, infinite horizon, and stationary (Poisson distributed) demand (see, e.g., the work of Axsäter and Lundell [4] and Axsäter [3]). More specifically, let x_s be the *inventory position* at the beginning of period s that captures the total sum of the physical *on-hand inventory* and the *outstanding orders* (placed in past periods, but still on the way) minus the pending backlogged demand. Say now that q_s units were ordered in period s, and consider a future period $t \geq s+L$. Then the holding cost incurred by the q_s units ordered in period s at the end of period t is $h_t(q_s - (D_{[s,t]} - x_s)^+)^+$, where $x^+ = \max(x, 0)$ and $D_{[s,t]} = \sum_{j=s}^{t} D_j$ is the cumulative demand over the interval $[s, t]$. Observe that if $D_{[s,t]} \leq x_s$, then none of the q_s units has been yet consumed. When $D_{[s,t]}$ exceeds x_s, the q_s units are used to satisfy the demand until all of them are consumed. It follows that the total holding cost incurred by the q_s units ordered in period s over the entire horizon is equal to

$$H_s = H_s(Q_s) \triangleq \sum_{t=s+L}^{T} h_t(Q_s - (D_{[s,t]} - X_s)^+)^+. \tag{2}$$

Because X_s and Q_s are realized at the beginning of period s (whereas, x_s and q_s are the realizations of X_s and Q_s, respectively), then, as seen from the beginning of period s, this quantity depends only on future demands and not on any future decision.

In addition, in an uncapacitated model the decision of how many units to order in each period affects the expected backlogging cost in only a single future period, namely, a lead time ahead. Now let Π_s be backlogging cost incurred in period $s+L$, for each $s = 1-L, \ldots, T-L$. In particular, it is straightforward to verify that

$$\Pi_s \triangleq b_{s+L}(D_{[s,s+L]} - (X_s + Q_s))^+, \tag{3}$$

where $D_j \triangleq 0$ with probability 1 for each $j \leq 0$. (Observe that the supply units captured by $X_s + Q_s$ will become available by time period $s+L$, and that no order placed after period s will arrive by time period $s+L$.)

Now let $\mathcal{C}(P)$ be the cost of a feasible policy P and use the superscript P to relate the respective quantities to that policy. Clearly,

$$\mathcal{C}(P) \triangleq \sum_{t=1-L}^{0} \Pi_t^P + H_{(-\infty,0]} + \sum_{t=1}^{T-L} (K \cdot \mathbb{1}(Q_t^P > 0) + H_t^P + \Pi_t^P), \tag{4}$$

where $H_{(-\infty,0]}$ denotes the total holding cost incurred by units ordered before period 1 (given as an input). We note that the first two expressions $\sum_{t=1-L}^{0} \Pi_t^P$ and $H_{(-\infty,0]}$ are the same for any feasible policy and each realization of the demand, and therefore we will omit them. Because they are nonnegative, this will not affect our approximation results. Also observe that, without loss of generality, it can be assumed that $Q_t^P = H_t^P = 0$ for any policy P and each period $t = T-L+1, \ldots, T$, because nothing that is ordered in these periods can be used within the given planning horizon. We now can write

$$\mathcal{C}(P) \triangleq \sum_{t=1}^{T-L} (K \cdot \mathbb{1}(Q_t^P > 0) + H_t^P + \Pi_t^P). \tag{5}$$

The cost accounting scheme in (5) above is marginal; i.e., in each period we account for all the expected costs that become unaffected by any future decision.

2.2.2 Randomized cost-balancing policy

To describe the policy, we modify the definition of the information set f_t to also include the randomized decisions of the randomized balancing policy up to period $t-1$. Thus, given the information set f_t, the inventory position at the beginning of period t is known. However, the order quantity in period t is still *unknown* because the policy randomizes among various order quantities. We denote the randomized cost-balancing policy by RB. The decision in each period, whether to order and how much to order, is based on the following quantities.

- Compute the *balancing quantity* $\hat{q}_t$ which balances the expected marginal holding cost incurred by the units ordered against the expected backlogging cost in period $t + L$. That is, $\hat{q}_t$ uniquely solves

$$\mathbb{E}\left[H_t^{RB}(\hat{q}_t) \mid f_t\right] = \mathbb{E}\left[\Pi_t^{RB}(\hat{q}_t) \mid f_t\right] \triangleq \theta_t. \tag{6}$$

- Compute the *holding-cost-K quantity* $\tilde{q}_t$ that solves $\mathbb{E}[H_t^{RB}(\tilde{q}_t) \mid f_t] = K$, i.e., $\tilde{q}_t$ is the order quantity that brings the expected marginal holding cost to K.
- Compute $\mathbb{E}[\Pi_t^{RB}(\tilde{q}_t) \mid f_t]$, i.e., the expected backlogging cost if one orders $\tilde{q}_t$ units in period t.
- Compute $\mathbb{E}[\Pi_t^{RB}(0) \mid f_t]$, i.e., the expected backlogging cost resulting from not ordering in period t.

Based on the above quantities computed, the following randomized rule is used in each period t. Let P_t denote our ordering probability which is a priori random. With the observed information set f_t, the ordering probability $p_t = P_t \mid f_t$ in period t is defined differently in the two cases below.

Case (I)

If the balancing cost exceeds K, i.e., $\theta_t \geq K$, the *RB* policy orders the balancing quantity $q_t^{RB} = \hat{q}_t$ with probability $p_t = 1$. The intuition is that when $\theta_t \geq K$, the fixed ordering cost K is less dominant compared to marginal holding and backlogging costs. Moreover, if the *RB* policy does not place an order, the conditional expected backlogging cost is potentially large. Thus, it is worthwhile to order the balancing quantity $q_t^{RB} = \hat{q}_t$ with probability $p_t = 1$.

Case (II)

If the balancing cost is less than K, i.e., $\theta_t < K$, the *RB* policy orders the holding-cost-K quantity (i.e., $q_t^{RB} = \tilde{q}_t$) with probability p_t and nothing with probability $1 - p_t$. That is,

$$q_t^{RB} = \begin{cases} \tilde{q}_t, & \text{with probability } p_t \\ 0, & \text{with probability } 1 - p_t \end{cases}. \tag{7}$$

The probability p_t is computed by solving the following equation

$$p_t K = p_t \cdot \mathbb{E}[\Pi_t^{RB}(\tilde{q}_t) \mid f_t] + (1 - p_t) \cdot \mathbb{E}[\Pi_t^{RB}(0) \mid f_t]. \tag{8}$$

The underlying reason behind the choice of this particular randomization in (8) is that the policy perfectly balances the three types of costs, namely, the marginal holding cost, the marginal backlogging cost and the fixed ordering cost associated with the period t. In particular, since we order the holding-cost-K quantity with probability p_t and nothing with probability $1 - p_t$, the conditional expected marginal holding cost in this case is

$$\mathbb{E}[H_t^{RB}(q_t^{RB}) \mid f_t] = p_t \mathbb{E}[H_t^{RB}(\tilde{q}_t) \mid f_t] + (1 - p_t)\mathbb{E}[H_t^{RB}(0) \mid f_t] = p_t K. \tag{9}$$

By the construction of p_t in (8), the conditional expected backlogging cost is

$$\mathbb{E}[\Pi_t^{RB}(q_t^{RB}) \mid f_t] = p_t \mathbb{E}[\Pi_t^{RB}(\tilde{q}_t) \mid f_t] + (1 - p_t)\mathbb{E}[\Pi_t^{RB}(0) \mid f_t] = p_t K. \tag{10}$$

Since p_t is the ordering probability in Case (II), the expected fixed ordering cost is $p_t K$. It can be shown that (8) has the following solution,

$$0 \leq p_t = \frac{\mathbb{E}[\Pi_t^{RB}(0) \mid f_t]}{K - \mathbb{E}[\Pi_t^{RB}(\tilde{q}_t) \mid f_t] + \mathbb{E}[\Pi_t^{RB}(0) \mid f_t]} < 1. \tag{11}$$

The inequalities in (11) follows from the fact that $\theta_t < K$ and $\tilde{q}_t > \hat{q}_t$, which implies that $\mathbb{E}[\Pi_t^{RB}(\tilde{q}_t) \mid f_t] < \mathbb{E}[\Pi_t^{RB}(\hat{q}_t) \mid f_t] = \theta_t < K$.

2.2.3 Worst-case performance analysis

To obtain a 3-approximation algorithm, one wishes to show that on expectation the cost of an optimal policy can 'pay' for at least one-third of the expected cost of the randomized cost-balancing policy. The periods are decomposed into subsets in which we will define explicitly. For certain well-behaved subsets, we want to show that the holding and backlogging costs incurred by an optimal policy can 'pay' for one-third of the cost incurred by the RB policy. The difficulty arises in analyzing the remaining subset of *problematic periods*, for which it is not a priori clear how to 'pay' for their cost. These problematic periods are further partitioned into intervals defined by each pair of two consecutive orders placed by the optimal policy. It can be shown that the total expected cost incurred by the RB policy in problematic periods within each interval, does not exceed $3K$. This implies that the fixed ordering cost incurred by an optimal policy can 'pay' on expectation one-third of the cost incurred by the randomized cost-balancing policy in problematic periods. Let Z_t^{RB} be a random variable defined as

$$Z_t^{RB} \triangleq \mathbb{E}[H_t^{RB}(Q_t^{RB}) \mid F_t] = \mathbb{E}[\Pi_t^{RB}(Q_t^{RB}) \mid F_t]. \tag{12}$$

Note that Z_t^{RB} is a random variable that is realized with the information set in period t. Observe that by the construction of the RB policy, the random variable Z_t^{RB} is well-defined since the expected marginal holding costs and the expected marginal backlogging costs are always balanced. That is, the conditional expected marginal holding cost is always equal to the conditional expected backlogging cost. In addition, the expected fixed ordering cost in period t is also Z_t^{RB} by the construction of the algorithm, and therefore we have the following lemma.

Lemma 1 (Levi and Shi [54]). *Let $\mathscr{C}(RB)$ be the total cost incurred by the RB policy. Then we have,*

$$\mathbb{E}[\mathscr{C}(RB)] \leq 3 \cdot \sum_{t=1}^{T-L} \mathbb{E}[Z_t^{RB}]. \tag{13}$$

To complete the worst-case analysis, we would like to show that the expected cost of an optimal policy denoted by OPT is at least $\sum_{t=1}^{T-L} \mathbb{E}[Z_t^{RB}]$. This will be done by amortizing the cost of OPT against the cost of the RB policy. In particular, we shall show that on expectation OPT pays for a large fraction of the cost of the RB policy. In the subsequent analysis, we will use a random partition of periods $t = \{1, 2, \ldots T - L\}$ to the following sets: The set $\mathscr{T}_{1H} \triangleq \{t : \Theta_t \geq K \text{ and } Y_t^{OPT} > Y_t^{RB}\}$ consists of periods in which the balancing cost Θ_t exceeds K and the optimal policy had higher inventory position than that of the RB policy after ordering (recall that if $\Theta_t \geq K$ then the RB policy orders the balancing quantity with probability 1 and the value Y_t^{RB} is known deterministically (i.e., realized) with F_t). The set $\mathscr{T}_{1\Pi} \triangleq \{t : \Theta_t \geq K \text{ and } Y_t^{OPT} \leq Y_t^{RB}\}$ consists of periods in which the balancing cost exceeds K and the inventory position of the optimal policy does not exceed that of the RB policy after ordering (see the comment above regarding $\mathscr{T}_{1H}$).The set $\mathscr{T}_{2H} \triangleq \left\{t : \Theta_t < K \text{ and } Y_t^{OPT} \geq X_t^{RB} + \tilde{Q}_t^{RB}\right\}$ consists of periods in which the balancing cost is less than K and, in such periods, the inventory position of the RB policy after ordering would be either X_t^{RB} if no order was placed, or $X_t^{RB} + \tilde{Q}_t^{RB}$ if the *holding-cost-K quantity* is ordered, depending on the randomized decision of the RB policy. However, the inventory position of OPT after ordering exceeds even $X_t^{RB} + \tilde{Q}_t^{RB}$. (Note again that the quantity $\tilde{Q}_t^{RB}$ is known deterministically (i.e., realized) with F_t.) Analogous to $\mathscr{T}_{2H}$, the set $\mathscr{T}_{2\Pi} \triangleq \{t : \Theta_t < K \text{ and } X_t^{RB} \geq Y_t^{OPT}\}$ consists of periods in which the inventory position of OPT after ordering is below X_t^{RB}. The set $\mathscr{T}_{2M} \triangleq \left\{t : \Theta_t < K \text{ and } X_t^{RB} < Y_t^{OPT} < X_t^{RB} + \tilde{Q}_t^{RB}\right\}$ consists of periods in which the balancing cost is less than K and the inventory position of OPT after ordering is within $(X_t^{RB}, X_t^{RB} + \tilde{Q}_t^{RB})$. Thus, whether the RB policy or OPT has more inventory depends on whether the RB policy placed an order. Note that the sets $(\mathscr{T}_{1H} - \mathscr{T}_{2M})$ are disjoint and the union makes a complete set. Conditioning on f_t, it is already known which part of the partition period t belongs.

Next we will show that the total holding cost incurred by OPT is higher than the marginal holding cost incurred by the RB policy in periods that belong to $\mathscr{T}_{1H} \cup \mathscr{T}_{2H}$, and that the total backlogging cost incurred by OPT is higher than the backlogging cost incurred by the RB policy associated with periods within $\mathscr{T}_{1\Pi} \cup \mathscr{T}_{2\Pi}$.

Lemma 2 (Levi and Shi [54])**.** *The overall holding cost and backlogging cost incurred by OPT are denoted by H^{OPT} and Π^{OPT}, respectively. Then we have, with probability 1,*

$$H^{OPT} \geq \sum_t H_t^{RB} \cdot \mathbb{1}(t \in \mathscr{T}_{1H} \cup \mathscr{T}_{2H}),\ \Pi^{OPT} \geq \sum_t \Pi_t^{RB} \cdot \mathbb{1}(t \in \mathscr{T}_{1\Pi} \cup \mathscr{T}_{2\Pi}). \tag{14}$$

The idea of Lemma 2 is as follows. In each period $t \in \mathscr{T}_{1H} \cup \mathscr{T}_{2H}$, the inequality $Y_t^{RB} < Y_t^{OPT}$ holds and implies that the Q_t^{RB} units ordered by the RB policy in period t have been ordered by OPT either in period t or even earlier. Thus, the holding cost they incur under OPT are higher than those incurred under the RB policy. On the other hand, in each period $t \in \mathscr{T}_{1\Pi} \cup \mathscr{T}_{2\Pi}$, the inequality $Y_t^{RB} \geq Y_t^{OPT}$ holds and implies that the backlogging incurred by OPT at the end of period $t+L$ will be higher than that of the RB in that period. We are still left with the problematic set $\mathscr{T}_{2M}$. Note that in this particular set, whether the RB policy or OPT has more inventory depends on whether the RB policy placed an order. Fortunately, Lemma 3 shows that the fixed ordering costs incurred by OPT can cover the randomized balancing costs in $\mathscr{T}_{2M}$.

Lemma 3 (Levi and Shi [54])**.** *The expected randomized cost in set $\mathscr{T}_{2M}$ is less than the total expected fixed ordering cost incurred by OPT, i.e.,*

$$\mathbb{E}\left[\sum_t Z_t^{RB} \cdot \mathbb{1}(t \in \mathscr{T}_{2M})\right] \leq \mathbb{E}\left[\sum_{t=1}^{T-L} K \cdot \mathbb{1}(Q_t^{OPT} > 0)\right]. \tag{15}$$

As an immediate consequence of Lemmas 2 and 3, we obtain the following lemma and theorem.

Lemma 4 (Levi and Shi [54])**.** *Let $\mathscr{C}(OPT)$ be the total cost incurred by the cost-balancing policy RB. Then we have,*

$$\mathbb{E}[\mathscr{C}(OPT)] \geq \sum_{t=1}^{T-L} \mathbb{E}[Z_t^{RB}]. \tag{16}$$

Theorem 2 (Levi and Shi [54])**.** *For each instance of the stochastic lot-sizing problem, the expected cost of the randomized cost-balancing policy RB is at most three times the expected cost of an optimal policy OPT, i.e.,*

$$\mathbb{E}[\mathscr{C}(RB)] \leq 3 \cdot \mathbb{E}[\mathscr{C}(OPT)]. \tag{17}$$

2.3 Some remarks and future directions

The worst-case analysis suggests that the theoretical worst-case performance bound is 3. However, extensive computational experiments (Levi and Shi [54]) show that the randomized cost-balancing policy performs significantly better than the worst-case guarantee, in most cases within 5% of optimum.

It has been shown (see, e.g., Levi and Shi [54]) that one can consider parametric versions of these policies and use known lower and upper bounds on the optimal ordering levels to devise more sophisticated policies with better empirical (typical) performance, some of which still admit worst-case analysis. The parameterization of these policies is motivated by the fact that, in the context of worst-case analysis, one wishes to choose policies that protect against all possible instances, whereas per a given (known) instance it is possible to choose the parameters of the policy optimally with respect to that instance. This naturally leads to improved empirical performance.

Besides the summary of current results shown in Table 1, designing approximation algorithms to stochastic assemble-to-order systems, stochastic one-warehouse multi-retailer problems, stochastic joint inventory and pricing models, stochastic perishable inventory with depletion decisions remains an open challenge, and would definitely require novel ideas and techniques. Another important future research direction is to study the performance of cost-balancing policies under various assumptions on the underlying demand distributions. As much as it is powerful to establish general worst-case analysis, it is equally important to refine this analysis to various parametric regimes of the underlying demand distributions and other key parameters of the problem. We call this *parametric worst-case analysis*.

3 Approximation Algorithms on Revenue Management Models

In this section, we consider a class of revenue management problems that arise in systems with *reusable resources* and *advanced reservations*. This work is motivated by both traditional and emerging application domains, such as hotel room management, car rental management and workforce management. For instance, in hotel industries, customers make requests to book a room in the future for a specified number of days. This is called advanced reservation. Rooms are allocated to customers based on their requests, and after one customer used a room it becomes available to serve other customers. One of the major issues in these systems is how to manage capacitated pool of reusable resources over time in a dynamic environment with many uncertainties.

3.1 Model

We consider revenue management problems of a single pool of reusable resources used to serve multiple classes of customers through advanced reservations. There is a single pool of resources of integer capacity $C < \infty$ that is used to satisfy the demands of M different classes of customers. The customers of each class $k = 1, \ldots, M$, arrive according to an independent Poisson process with respective rate λ_k. Each class-k customer requests to reserve one unit of the capacity for a specified *service time interval* in the future.

Let D_k be the reservation distribution of a class-k customer, and S_k be the respective service distribution with mean μ_k. We assume that they are non-negative, discrete and bounded. In particular, upon an arrival of a class-k customer at some random time t, the customer requests to reserve the service time interval $[t+d, t+d+s]$, where d is distributed according to D_k and s is distributed according to S_k. Note that D_k and S_k are independent of the arrival process and between customers; however, per customer, D_k and S_k can be correlated. (We assume that both D_k and S_k are finite discrete distributions.) During the time a customer is served (i.e., $[t+d, t+d+s]$), the requested unit cannot be used by any other customer; after the service is over, the unit becomes available again to serve other customers. If the resource is reserved, the customer pays a class-specific rate of r_k dollars per unit of service time. The resource can be reserved for an arriving customer only if upon arrival there is at least one unit of capacity that is available (i.e., not reserved) throughout the entire requested interval $[t+d, t+d+s]$. Specifically, a customer's request can be satisfied if the maximum number of already reserved resources throughout the requested service interval is smaller than the capacity C. However, customers can be rejected even if there is available capacity. Rejecting a customer now possibly enables serving more profitable customers in the future. Customers whose requests are not reserved upon arrival are *lost* and leave the system. The goal is to find a feasible admission policy that maximizes the expected long-run average revenue. Specifically, if $\mathcal{R}_\pi(T)$ denotes the revenue achieved by policy π over the interval $[0, T]$, then the expected long-run average revenue of π is defined as $\mathcal{R}(\pi) \triangleq \liminf_{T\to\infty}(\mathbb{E}[\mathcal{R}_\pi(T)]/T)$, where the expectation is taken with respect to the probability measure induced by π.

Like many stochastic optimization models, one can formulate this problem using dynamic programming approach. However, even in special cases (e.g., no advanced reservations allowed and with exponentially distributed service times), the resulting dynamic programs are computationally intractable due to the *curse of dimensionality*.

3.2 An LP-based approach

In this section, we describe a simple linear program (LP) that provides an upper bound on the achievable expected long-run average revenue. The LP conceptually resembles the one used by Levi and Radovanovic [47], Key [42] and Iyengar and Sigman [39] who study models without advanced reservations. It is also similar in spirit to the one used by Adelman [1] in the queueing networks framework with unit resource requirements again without advanced reservations. We shall show how to use the optimal solution of the LP to construct a simple admission control policy that is called *class selection policy* (CSP).

At any point of time t, the state of the system is specified by the entire booking profile consisting of the class, reservation and service information of each customer in the booking system as well as the customers currently served. Without loss of generality, we restrict attention to *state-dependent policies*. Note that each state-dependent policy induces a Markov process over the state-space, and one can show that the induced Markov process has a unique stationary distribution which is ergodic. The detailed technical proof can be found in Levi and Shi [55] following the arguments in Sevastyanov [64] and Lu and Radovanovic [56]. The key idea is to find a *fixed* auxiliary probability distribution that can be scaled (by positive constants) to lower and upper bound the transitional probability in the underlying Markov chain. This auxiliary probability distribution can be readily found if the state space is compact (which is true in our model).

Since any state-dependent policy induces a Markov process on the state-space of the system that is ergodic, for a given state-dependent policy π, there exists a long-run stationary probability α^{π}_{ijk} for accepting a class-k customer who wishes to start service in i units of time for j units of time, which is equal to the long-run proportion of accepted customers of this type while running the policy π. In other words, any state-dependent policy π is associated with the stationary probabilities α^{π}_{ijk} for all possible reservation time i, service time j and class k. Let $\lambda_{ijk} \triangleq \lambda_k \mathbb{P}(D_k = i, S_k = j)$ be the arrival rate of class-k customers with reservation time i and service time j. Therefore the mean arrival rate of accepted class-k customers with reservation time i and service time j is $\alpha^{\pi}_{ijk}\lambda_{ijk}$. By Little's Law and PASTA (see, e.g., Gallager [21]), the expected number of class-k customers with reservation time i and service time j being served in the system under π is $\alpha^{\pi}_{ijk}\lambda_{ijk}j$. It follows that under π the expected long-run average number of resource units being used to serve customers can be expressed as $\sum_{k=1}^{M}\sum_{i,j}\alpha^{\pi}_{ijk}\lambda_{ijk}j$. This gives rise to the following *knapsack* LP:

$$\max_{\alpha_{ijk}} \quad \sum_{k=1}^{M}\sum_{i,j} r_k \alpha^{\pi}_{ijk}\lambda_{ijk}j, \tag{18}$$

$$\text{s.t.} \quad \sum_{k=1}^{M}\sum_{i,j} \alpha^{\pi}_{ijk}\lambda_{ijk}j \le C, \tag{19}$$

$$0 \le \alpha^{\pi}_{ijk} \le 1, \qquad \forall i,j,k. \tag{20}$$

Note that for each feasible state-dependent policy π, the vector $\alpha^{\pi} = \{\alpha^{\pi}_{ijk}\}$ is a feasible solution for the LP with objective value equal to the expected long-run average revenue of policy π. In fact, the LP enforces the capacity constraint (19) of the system only in expectation, whereas in the original problem this constraint has to hold, for each sample path. It follows that the LP relaxes the original problem and provides an upper bound on the best obtained expected long-run average revenue. The LP can be solved optimally by applying the following greedy rule. Without loss of generality, assume that classes are re-numbered such that $r_1 \ge r_2 \ge \ldots \ge r_M$. Then, for each $k = 1, \ldots, M$, we sequentially set $\alpha_{ijk} = 1$ for all i and j as long as constraint (19) is satisfied. If there exists a class $M' \le M$ such that

$$C = (1-\gamma)\sum_{k=1}^{M'-1}\sum_{i,j}\lambda_{ijk}j + \gamma\sum_{k=1}^{M'}\sum_{i,j}\lambda_{ijk}j,$$

for some $\gamma \in (0,1)$, we set $\alpha_{ijM'} = \gamma$ for all i and j. Note that for each class k, the values of α_{ijk} are all equal regardless of i and j. We abuse the notation and drop the subscripts i and j of α_{ijk}. Then the optimal solution reduces to: for $k = 1, \ldots, M' - 1$, $\alpha_k = 1$; $\alpha_{M'} = \gamma$; and for $k = M' + 1, \ldots, M$, we have $\alpha_k = 0$.

Next, we shall use the optimal solution of the knapsack LP to construct a very simple admission policy. Let $\alpha^* = \{\alpha_k^*\}$ be the optimal solution of the knapsack LP. We propose a simple policy that is called *class selection policy* (CSP). Consider an arrival of a class-k customer ($k = 1, \ldots, M$). For each $k = 1, \ldots, M' - 1$, *accept* the customer upon arrival (regardless of the reservation time and the service time) as long as there is sufficient unreserved capacity throughout the requested service interval. If $k = M'$, *accept* the customer with probability γ (regardless of the reservation time and the service time) and as long as there is sufficient unreserved capacity throughout the requested service interval. For each $k = M' + 1, \ldots, M$, *reject*.

The CSP has a very simple structure. It always admits customers from the classes for which the corresponding value α_k^* in the optimal LP solution equals to one as long as capacity permits. It never admits customers from classes for which the corresponding value α_k^* equals to zero, and it flips a coin for the possibly one class with fractional value $\alpha_{M'}^* = \gamma$. The CSP is conceptually very intuitive in that it splits the classes into profitable and nonprofitable that should be ignored. In fact, we can assume, without loss of generality, that there is no fractional variable in the optimal solution α^*, i.e., for each $k = 1, \ldots, M'$, $\alpha_k^* = 1$. (If $\alpha_{M'}^* = \gamma$ is fractional, we think of class M' as having an arrival rate $\lambda'_{M'} = \gamma\lambda_{M'}$ and then eliminate the fractional variable from α^*.)

3.3 Worst-case performance analysis

In this section, we discuss performance analysis of the CSP under models with advanced reservations. The CSP induces a well-structured stochastic process called loss networks with advanced reservations (i.e., a $M/G/C/C$ loss system with advanced reservations). Each class $k = 1, \ldots, M$ induces a Poisson arrival stream with respective rate $\alpha_k^*\lambda_k$, $1 \leq k \leq M$. Thus, for each class k with $\alpha_k^* = 1$, the arrival process is identical to the original process, and each class k with $\alpha_k^* = 0$ can be ignored. For each class $k = 1, \ldots, M'$, let S_k and D_k be the service and reservation distributions of class-k customers, respectively. We want to characterize the long-run *blocking probability* of class-k customers with reservation time i and service time j under the CSP, i.e., the stationary probability that a class-k customer with reservation time i and service time j arrives at a random time to the system and is rejected by the CSP because there is no available capacity at some point within the requested service interval. For each $k = 1, \ldots, M'$, let Q_{ijk} be the stationary probability of blocking a class-k customers with reservation time i and service time j under the CSP. Since the corresponding stochastic process is ergodic, Q_{ijk} is well-defined. Thus, the expected long-run average revenue of the CSP can be expressed as $\sum_{k=1}^{M'} \sum_{i,j} r_k\lambda_{ijk}j(1 - Q_{ijk})$. However, $\sum_{k=1}^{M'} \sum_{i,j} r_k\lambda_{ijk}j$ is the optimal value of the LP, which is an upper bound on the best achievable expected long-run average revenue, denoted by $\mathcal{R}(OPT)$. Thus, a key aspect of the performance analysis of the CSP is to obtain an upper bound on the probabilities Q_{ijk}'s. More specifically, if $1 - Q_{ijk} \geq \xi$, for each i, j, and k, it follows that

$$\mathcal{R}(CSP) = \sum_{k=1}^{M'} \sum_{i,j} r_k\lambda_{ijk}j(1 - Q_{ijk}) \geq \sum_{k=1}^{M'} \sum_{i,j} r_k\lambda_{ijk}j\xi \geq \xi\mathcal{R}(OPT). \tag{21}$$

We want to upper bound probabilities Q_{ijk}'s and analyze their asymptotic behavior under the Halfin-Whitt regime. The *traffic intensity* $\rho \triangleq \sum_{k=1}^{M} \sum_{i,j} \lambda_{ijk}j = \sum_{k=1}^{M} \lambda_k\mu_k$. Under the Halfin-Whitt regime, the capacity C and the arrival rates λ_k as well as the traffic intensity ρ are scaled together to infinity while keeping the service and the reservation distributions fixed (i.e., $C = \rho + \beta\sqrt{\rho} + o(\sqrt{\rho}) \to \infty$, for some positive spare capacity parameter β).

Theorem 3 (Levi and Shi [55]). *Consider the revenue management model with a single pool of capacitated reusable resources and advanced reservations under the CSP. Let $\Phi(\cdot)$ be the cumulative density function of a standard Normal. Then:*

(a) For each k and j, the blocking probability Q_{ijk} has the following asymptotic upper bound,

$$\lim_{\rho \to \infty} Q_{0jk} \leq \Phi(-\beta); \qquad \lim_{\rho \to \infty} Q_{ijk} = 0, \qquad \forall i \geq 1,$$

where $\beta > 0$ is the spare capacity parameter in the Halfin-Whitt regime.

(b) The CSP is guaranteed to obtain at least 0.5 of the optimal expected long-run average revenue in the Halfin-Whitt heavy-traffic limit.

The upper bound on the blocking probability is obtained by considering a counterpart system with infinite capacity, where all customers are admitted into a $M/G/\infty$ system with advanced reservations. The detailed stochastic analysis of this loss queueing system with advanced reservations falls out of scope of this article and we refer interested readers to Levi and Shi [55].

3.4 Some remarks and future directions

There are also several plausible extensions into pricing models. An interesting direction is to study both the static and dynamic pricing model of reusable resources with advanced reservations. The static pricing model allows the arrival rates being affected by prices. Specifically, consider a two-stage decision. At the first stage, we set the respective prices $r_1, \ldots, r_M$ for each class. This determines the respective arrival rates $\lambda_1(r_1), \ldots, \lambda_M(r_M)$. (The rate of class-$i$ customers is affected only by price r_i.) Then, given the arrival rates, we wish to find the optimal admission policy that maximizes the expected long-run revenue rate. We may assume that $\lambda_i(r_i)$ is nonnegative, differentiable, and decreasing in r_i for each $1 \leq i \leq M$. One might construct an upper bound on the achievable expected long-run revenue rate through a nonlinear program, and then use it to construct a similar policy with the same performance guarantees. In the dynamic pricing model, consider a single-class time-homogenous Poisson arrival process with rate λ. Each customer's reservation and service-time are drawn from D and S, respectively. The system offers a price from a fixed price menu $[r_1, \ldots, r_n]$ to an arriving customer with d and s, depending on the current state. The state is characterized by the booking profile, d and s. A reservation price distribution R has to be specified, i.e., the customer only accepts the offer if the price offered falls below the reservation price. One might construct a new linear program to obtain provably near-optimal randomized policies.

4 Conclusion and Future Directions

Mathematical programming techniques have been used extensively to obtain relaxations and provably near-optimal approximation algorithms for deterministic combinatorial optimization problems (see, e.g., Ausiello [2], Vazirani [79] and Williamson and Shmoys [80]). As mentioned in our literature review, more recent work has extended these approaches and/or developed novel techniques for two-stage and multistage stochastic optimization problems. These powerful techniques were developed for a wide range of combinatorial optimization problems, and could be potentially applied to many core operation management models. We would like to point out some plausible research avenues for future research. *(a) Data-driven models.* Many of the underlying stochastic applications in operations management often involve correlated data (e.g., the demands in stochastic inventory control problems are often correlated over periods due to economic and/or seasonal factors). The decision makers may not know the demand distributions exactly and can only rely on collected (unbiased) sampling data. The sampling-based framework developed by Gupta et al. [32] and Swamy and Shmoys [75] could be applied to many distribution-free operations management models. These black-box models are desirable in that they allow one to specify distributions with exponentially many scenarios and correlation in a compact way that makes it reasonable to talk about polynomial time algorithms. *(b) Models with risk.* The multistage stochastic model measures the expected cost associated with each stage, but often in applications one is also interested in the risk associated with each stage decisions, where risk is some measure of the variability in the random cost incurred in later stages. Gupta et al. [33] considered the use of budgets that bound the cost of each scenario, as a means of guarding against risk. It would be interesting to explore stochastic operations management models that incorporate risk. One may also consider designing approximation algorithms for the robust variants of these models. *(c) Models with endogenous uncertainty.* Another interesting and important research avenue, which brings us closer to Markov decision processes, is to investigate problems where the uncertainty is affected by the decisions taken. For instance, in stochastic scheduling problems, the scheduling decisions interact with the evolution of the random job sizes especially in a preemptive environment. Another salient example is the joint inventory and pricing problem where the pricing decisions influence the demands.

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