

Joint Pricing and Inventory Management with Strategic Customers

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We consider a model wherein the seller sells a product to customers over an infinite horizon. At each time, the seller decides a set of purchase options offered to customers and the inventory replenishment quantity. Each purchase option specifies a price and a product delivery time. Customers are infinitesimal and arrive to the system with a constant rate. Customer product valuations are heterogenous and follow a stationary distribution. A customer's arrival time and product valuation are his private information. Customers are forward looking, i.e., they strategize their purchasing times. A customer incurs delay disutility from postponing to place an order and waiting for the product delivery. A customer's delay disutility rate is perfectly and positively correlated with his valuation. The seller has zero replenishment lead time. The seller incurs fixed ordering cost and inventory holding cost. The seller seeks a joint pricing, delivery and inventory policy that maximizes her long-run average profit. Through a tractable upper bound constructed by solving a mechanism design problem, we derive an optimal joint pricing, delivery and inventory policy, which is a simple cyclic policy. We also extend our policy to a stochastic setting and establish its asymptotic optimality.

Key words: joint pricing and inventory control, strategic customers, optimal policy, mechanism design

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1. Introduction

We consider a model wherein the seller sells a divisible product to customers over an infinite horizon. At each time, the seller decides a set of purchase options offered to customers and the inventory replenishment quantity. Each purchase option specifies a price and a product delivery time. Customers are infinitesimal and arrive to the system with a constant rate. Customer product valuations are heterogenous and follow a stationary distribution. A customer's arrival time and product valuation are his private information. Customers are forward looking, i.e., they strategize their purchasing times. A customer incurs delay disutility from postponing to place an order and waiting for the product delivery. A customer's delay disutility rate is perfectly and positively

correlated with his valuation. The seller has zero replenishment lead time. The seller incurs fixed ordering cost and inventory holding cost. The seller seeks a joint pricing, delivery and inventory policy that maximizes her long-run average profit.

One of the primary motivating examples is the pricing, delivery and inventory decision problem faced by major furniture retailers, such as IKEA, Overstock, Wayfair, Art Van, in their day-to-day operations. When a customer purchases from a furniture retailer, the firm typically specifies both the price that the customer needs to pay and the time that furniture will be delivered. Delayed delivery always happens in the furniture industry. In particular, if a customer purchases from IKEA, even in the case of stockout, a customer is still allowed to place an order and the store will promise him a future delivery date. In the furniture industry, customers typically have heterogeneous valuations of the furniture products and different patience levels of waiting for product deliveries. Moreover, they are typically strategic with regard to their times of purchases, since these furniture retailers constantly offer a large number of promotions, discounts, and giveaways on a daily basis (as seen on their front pages). Many products, such as leather sofas, can be sold at a deep discount (as much as 50% off the regular price). A patient customer could wait for these promotions at the expense of experiencing a delay disutility from either his (deliberately) deferred purchase decision or the delayed product delivery. In the presence of strategic customers, a retailer faces a complex decision-making problem of how to jointly price, deliver, and replenish her products.

The example of this furniture retailing chain does not stand alone. For many other consumer products, such as soda, seasonings, food produce, hair care products, firms are routinely running promotions (Boehning (1996), Chen and Natarajan (2013), Verhaar (2015)). Customers, who observe these dynamic pricing strategies, may strategically time their purchases. For instance, RedMart, one of the biggest online retailer in Singapore, often runs promotions for products like Coke. Many customers strategically delay their purchases of these products by waiting for promotions. They are also tolerant of delayed deliveries. On the supply side, a firm has to repeatedly make pricing, delivery and inventory replenishment decisions to satisfy customer demand and maximize her profit, while taking customer forward-looking behaviors into consideration (Hammond (1994)).

1.1. Main Results and Contributions

Our main results and their implications are summarized as follows.

We derive the seller's optimal policy (§3.3). Under this policy, the seller makes the inventory replenishment with constant time interval and order-up-to level. The purchase options that the seller offers to customers are characterized as follows.

1. At a time that is close to the latest inventory replenishment time, the seller offers customers a single purchase option that guarantees the instantaneous product delivery.
2. At a time that is not too close to either the latest inventory replenishment time or the next inventory replenishment time, the seller offers customers two alternative purchase options: one that charges customers a higher price and delivers the product instantaneously and the other one that charges customers a lower price and delivers the product at the next inventory replenishment time.
3. At a time that is close to the next inventory replenishment time, the seller offers customers a single purchase option with the product delivery at the next inventory replenishment time.

We show that under the optimal policy, every customer either selects a purchase option offered at his arrival time or permanently leaves the system without purchasing anything at his arrival time, i.e., no customer delays selecting a purchase option offered at a time later than his arrival (Lemma 3). A customer's purchase strategy at his arrival time is characterized as follows.

1. If a single purchase option is offered at his arrival time (either with the instantaneous product delivery or the delayed product delivery), then he selects this option if his valuation is no less than a time-dependent threshold value. Otherwise, he permanently leaves the system without purchasing anything.
2. If both the instantaneous product delivery purchase option and the delayed product delivery purchase option are offered at his arrival time, then there exist two time-dependent threshold values. If the customer's valuation is no less than the higher threshold value, then he selects the purchase option with the instantaneous product delivery. If the customer's valuation is below the higher threshold value but above the lower threshold value, then he selects the purchase option with the delayed product delivery. Otherwise, he permanently leaves the system without purchasing anything.

We also extend our model to a stochastic setting wherein customers arrive to the system according to a Poisson process and their valuations are randomly drawn from a distribution function. We propose a heuristic policy motivated by the optimal policy in the deterministic setting and establish its asymptotic optimality.

Our main methodological contributions are summarized as follows.

First, we establish a benchmark profit that serves as an upper bound of the seller's optimal profit (the first inequality in Proposition 1). Our benchmark is defined as the seller's optimal profit in a joint mechanism design and inventory problem, which is a classical inventory management problem with an "orthogonal" static adverse selection problem added onto it (see the optimization

problem defined in (2)). This problem is subject to the (IC) constraints on customer valuation dimension. Therefore, this problem is one dimensional (as customer valuation and delay disutility are perfectly correlated) and the seller screens customer valuation through price and delivery time. In this problem, “orthogonal” means that customers are ex ante homogeneous, their valuations are independent of their arrival times and their arrival times are inconsequential and generate no information rent. Hence, at any point of time, the seller faces the same screening problem.

Second, we establish a closed-form upper bound of the optimal value of the joint mechanism design and inventory problem (the second inequality in Proposition 1 and Lemma 2). The complication of the analysis comes from the fact that the product delivery time is used as one screening instrument and hence increases the difficulty of managing the inventory. We cope with this complication in the following way. We begin with employing the Myersonian approach (Myerson (1981)) to internalize the information rent given up to the customers as their virtual valuations and virtual delay costs. This step allows us to convert the joint mechanism design and inventory problem into a centralized inventory problem. We then exploit the tradeoffs of the revenue (customer virtual valuations), inventory holding cost, backordering cost (customer virtual delay costs) and the fixed ordering cost in this centralized inventory problem to compute its optimal value.

Third, we use the structural properties of the established closed-form upper bound (the optimal value of the aforementioned centralized inventory problem) to propose a joint pricing, delivery and inventory policy. We show that under this policy, every customer either selects one purchase option offered at his arrival time or permanently leaves the system at his arrival time, i.e., no customer delays to select a purchase option offered later than his arrival (Lemma 3). We show that our proposed policy achieves the established closed-form upper bound. This result immediately implies the optimality of our proposed policy (Theorem 1).

1.2. Literature Review

Joint Pricing and Inventory Management. The benefits of joint pricing and inventory control have been long recognized in the research community, since the seminal work by Whitin (1955). We refer readers to Chen and Simchi-Levi (2012) for an overview of this field. Federgruen and Heching (1999) study a multi-period stochastic joint pricing and inventory control problem and prove the optimality of the base-stock list-price policy. Subsequently, Chen and Simchi-Levi (2004a,b) prove that the (s, S, p) and (s, S, A, p) policies are optimal with the fixed ordering cost. Li and Zheng (2006) and Chen et al. (2010) extend the optimality of base-stock list-price or (s, S, p) policies to incorporate random yield and concave ordering cost, respectively. Pang et al. (2012) identify various properties of the optimal policy with positive lead times. Feng et al. (2013) analyze a model in which

demand follows a generalized additive model. [Chen et al. \(2014\)](#) characterize the optimal policies for joint pricing and inventory control with perishable products. [Lu et al. \(2014\)](#) establish the optimality of base-stock list-price policies for models with incomplete demand information and non-concave revenue function. Besides backlogging models, there have been several studies devoted to the lost sales counterpart models (see [Chen et al. \(2006\)](#), [Huh and Janakiraman \(2008\)](#), [Song et al. \(2009\)](#)). In contrast to the above literature, this paper incorporates *strategic customer behaviors* and endogenizes the demand fulfillment rule, and our solution approach departs significantly from previous studies.

The following three papers are also relevant to our setting. [Hu et al. \(2016a\)](#) formulate a multi-period two-phase model on a firm's dynamic inventory and markdown decisions for perishable goods. Customers strategically decide whether they purchase in the clearance phase at the discounted price for future consumption or in the regular phase at the regular price for immediate consumption. The optimal policy is that the firm should either put all of the leftover inventory on discount (if it is higher than some threshold level) or dispose all of it. [Lu et al. \(2014\)](#) consider a joint pricing and inventory model with quantity-based price differentiation. At the beginning of each period, the firm decides how much to replenish and whether she sells at a unit selling price, or offers a quantity discount, or jointly uses both modes. They show that the optimal inventory decision follows a base-stock policy, and the optimal selling strategy depends on the optimal base-stock level. [Wu et al. \(2015\)](#) study a problem that the seller dynamically makes the joint pricing and inventory replenishment decisions over multiple periods. Each period consists of two stages. In the first stage, the seller replenishes inventory for selling within the period and charges customers at a regular price for the first stage. In the second stage, the seller offers a markdown price. In each period, customers arrive in the first stage. Customers decide whether to purchase in the first stage or the second stage. Customers decide when to buy by using the reference price that is formed based on historic markdown prices. This paper shows that the customer reference price exhibits a mean-reverting pattern under certain conditions.

The paper by [Chen and Chu \(2016\)](#) is closely related to our present paper. This paper also studies a joint pricing and inventory problem with forward-looking customers. However, one key distinction between this paper and our paper is that they restrict the purchase options to only depend on demand fulfillment time. By contrast, we allow the purchase options to also depend on customer purchasing time.

Mechanism Design in Operations Management. The methodology that we use to study our problem is mechanism design. One stream of literature study mechanism design problems in the revenue management context. [Gallien \(2006\)](#) studies an infinite horizon continuous time model

that assumes that the seller and all customers have the same time discount rate. He shows that the optimal mechanism can be implemented as a dynamic pricing policy. Board and Skrzypacz (2016) consider a finite horizon discrete time version of the same model. They show that the optimal mechanism is no longer a purely dynamic pricing mechanism but requires an end-of-season ‘clearing’ auction. Chen and Farias (2018) study a model that generalizes the models considered in the antecedent literature above by allowing for heterogeneity in customer delay disutility. Those authors propose a class of ‘robust’ dynamic pricing policies that are guaranteed to garner at least 29% of the seller’s optimal revenue. Chen et al. (2018) further show that a simple fixed price policy is asymptotically optimal.

Outside of the references discussed above, a nice variety of algorithmic work studies a special class of mechanisms in the presence of forward-looking customers: anonymous posted dynamic pricing mechanisms. Borgs et al. (2014) study a setting where a firm with time-varying capacity sets prices over time to maximize revenues in the face of forward-looking customers. The firm knows customers arrival times, deadlines and valuations. Authors contribute a surprising dynamic programming formulation. Besbes and Lobel (2015) characterize optimal ‘cyclic’ pricing policies for a special class of utility functions. Liu and Cooper (2015), Lobel (2016) study patient customer models by assuming that a customer immediately makes the purchase when the price drops down to a certain level. They also exhibit the optimality of ‘cyclic’ pricing policies. It is interesting to note that other researchers have motivated cyclic pricing policies by considering price reference effects; see Hu et al. (2016b) and Wang (2016).

Lobel and Xiao (2017), Zhang et al. (2010) study dynamic mechanism design problems in the inventory management framework. They study models wherein a manufacturer and a retailer repeatedly interact over multiple periods. The retailer’s demand forecast in each period is his private information. In each period, the retailer orders from the manufacturer. Lobel and Xiao (2017) show that the optimal long-term contract in the backlogged demand model is that the retailer selects a wholesale price in the first period and sticks to this price in future. Zhang et al. (2010) show that the optimal short-term contract in the lost sales model is the batch-order contract. Kakade et al. (2013) study a more general problem wherein the seller and customers repeatedly interact over time and each customer’s private information dynamically evolves over time. The distinction between the mechanism design problems studied in these papers and the mechanism design approach adopted in the present paper are as follows. In the aforementioned papers, each agent is long-lived and his private information dynamically evolves over time. Therefore, the incentive compatibility (IC) constraints on each agent is imposed at each point of time and the principal needs to trade off each agent’s intertemporal incentives. By contrast, in our paper, although agents

dynamically arrive to the system, each agent's private information does not evolve over time. Therefore, the (IC) constraints on each agent is time-independent.

The paper by [Golrezaei et al. \(2017\)](#) is closely related to our present paper. This paper also uses the mechanism design approach to study a joint pricing and inventory management problem with forward-looking customers. This paper also allows customer delay disutility to be perfectly and positively correlated with customer valuation. However, one key distinction between this paper and our paper is that they assume that all items are produced and stored prior to the selling season. By contrast, we allow inventory to be replenished during the season.

1.3. Structure of the Paper

The remainder of the paper is organized as follows. §2 presents our model in a deterministic setting: we introduce the notion of joint pricing, delivery and inventory policies, and model customer utilities. §3 presents the seller's optimal policy in the deterministic model. §4 extends the model presented in §2 to a stochastic setting. We present a policy that is near optimal in the stochastic setting. §5 concludes the paper and points out several future research directions.

In the interest of space and better readability, we relegate the proof of key Lemma 3 to the Appendix, and all other technical proofs to the Electronic Companion.

2. Model

We consider a model in which a seller sells a divisible product over an infinite selling horizon $[0, \infty)$. At each point of time $t \geq 0$, the seller makes inventory ordering decisions with quantity $q_t \geq 0$ and offers a set of purchase options Ω_t that a customer chooses from. Each purchase option $(p, s) \in \Omega_t$ consists of an anonymously posted price $p \geq 0$ and the seller's committed product delivery time $s \geq t$. Offering a set of purchase options with each purchase option that prescribes both the payment amount and the product delivery time is consistent with many industrial practices (e.g., RedMart, IKEA and Amazon) and many operations management literature (see, e.g., [Afèche and Pavlin \(2016\)](#), [Gurvich et al. \(2018\)](#), [Nazerzadeh and Randhawa \(2018\)](#)).

Customers are infinitesimal and arrive to the system with a constant rate λ . We denote by a generic symbol ϕ , the 'type' of an arriving customer which we understand to be the tuple

$$\phi \triangleq (t_\phi, v_\phi),$$

where $t_\phi \geq 0$ denotes the time that customer ϕ arrives to the system and starts to consider to purchase the product, $v_\phi \in [0, V)$ with $V \in \mathbb{R}_{++}$ denotes the product valuation for customer ϕ . Note

that we put subscript ϕ on customer ϕ 's all attributes to highlight their heterogeneities among different customers. Every customer ϕ 's type is his private information that cannot be observed by the seller or other customers.

Every customer either purchases one unit of the product or does not purchase anything. Every customer ϕ is forward looking who strategizes his time of purchase. At each customer ϕ 's arrival time t_ϕ , he decides to either purchase the product at a specified time (now or a specific time in future), or permanently leave the system now without purchasing anything. If customer ϕ decides to make a purchase, then he determines a time to purchase the product, $\tau_\phi \geq t_\phi$, and selects a purchase option (p_ϕ, s_ϕ) from the set of purchase options presented at time τ_ϕ , Ω_{τ_ϕ} . By choosing purchase option (p_ϕ, s_ϕ) , customer ϕ pays p_ϕ to the seller at time τ_ϕ and receives the product at time s_ϕ . As a convention, we set $\tau_\phi = t_\phi$ and $(p_\phi, s_\phi) = (0, \infty)$ if customer ϕ decides to permanently leave the system without purchasing anything at his arrival time t_ϕ .

Define the tuple

$$z_\phi \triangleq (\tau_\phi, p_\phi, s_\phi).$$

Customer ϕ garners utility

$$U(\phi, z_\phi) = \begin{cases} v_\phi - p_\phi - W(v_\phi)(s_\phi - t_\phi) & \text{if } s_\phi < \infty \\ 0 & \text{if } s_\phi = \infty \end{cases}. \quad (1)$$

Function $W(v)$ measures the per-unit-of-time *disutility* that a customer with valuation v incurs from having the willingness to possess the product but not yet having it on hand, hereafter called as the “*delay disutility rate function*”. We make the following assumptions on this function.

ASSUMPTION 1.

1. $W(v) > 0$ for $v \in [0, V)$.
2. $W(v)$ is continuous and differentiable in $v \in [0, V)$; denote $W(V) \triangleq \lim_{v \rightarrow V-} W(v)$ and $w(v) \triangleq \frac{dW(v)}{dv}$.
3. $W(v)$ is non-decreasing and (weakly) concave in $v \in [0, V)$.
4. $\frac{vw(v)}{W(v)}$ is non-decreasing in $v \in [0, V)$.

Let us interpret the conditions imposed by the assumptions on $W(\cdot)$. The first assumption simply formalizes our interpretation of $W(\cdot)$ as a *disutility*. The second assumption is made for analytical convenience and notational clarity. The third assumption captures the essence of the structure we impose on the disutility incurred due to a delay, and consists of two components. The first is that this disutility is non-decreasing in the customer's valuation so that high value customers incur a

larger cost of having delay of receiving the product. This assumption is natural and has widespread support in both theoretical and empirical literature; see, for instance, Afèche and Mendelson (2004), Afèche and Pavlin (2016), Doroudi et al. (2013), Gurvich et al. (2018), Katta and Sethuraman (2005), Kilcioglu and Maglaras (2015), Moon et al. (2017), Nazerzadeh and Randhawa (2018). The second part of the assumption can be interpreted as controlling the *rate* at which this disutility can grow with the customer's value. Our requirement of the (weak) concavity implies that this growth must be sub-linear. The fourth assumption has the interpretation that the disutility growth rate in valuation does not change drastically.

As an example, the following family of disutility functions fit the assumptions above:

$$W(v) = \theta_1 + \theta_2 v^{\theta_3}, \forall (\theta_1, \theta_2, \theta_3) \in \mathbb{R}_{++} \times \mathbb{R}_+ \times [0, 1].$$

This family of disutility functions includes many frequently used disutility functions. We give two examples below.

1. Constant delay disutility rate: $\theta_2 = 0$ (see, e.g., Besbes and Maglaras (2009), Chen and Frank (2001), Chen and Chu (2016)).
2. Affine delay disutility rate: $\theta_2 > 0$ and $\theta_3 = 1$ (see, e.g., Afèche and Mendelson (2004), Afèche and Pavlin (2016), Katta and Sethuraman (2005), Kilcioglu and Maglaras (2015), Nazerzadeh and Randhawa (2018)).

We assume that a customer's valuation v is independent of his arrival time t , which is a common assumption in the literature (see, e.g., Besbes and Lobel (2015), Board and Skrzypacz (2016), Chen and Chu (2016), Nazerzadeh and Randhawa (2018)). We denote by $F(v)$ the fraction of customers whose valuations are no more than v . We denote $f(v) \triangleq \frac{dF(v)}{dv}$ and $\bar{F}(v) \triangleq 1 - F(v)$. We make a standard assumption on the valuation distribution:

ASSUMPTION 2. *The hazard rate function $\frac{f(v)}{\bar{F}(v)}$ is non-decreasing in v . The virtual value function $v - \frac{\bar{F}(v)}{f(v)} \triangleq g_v$ admits a root v^* , i.e., $g_{v^*} = 0$.*

We make a convention that $\frac{f(V)}{\bar{F}(V)} = \infty$. The assumption on the hazard rate function is widely adopted in the operations management literature (see, e.g., Chen and Farias (2013), Golrezaei et al. (2017), Lobel and Xiao (2017), Özer and Wei (2006)). Many distribution functions, such as exponential, Gamma, Gumbel, satisfy this assumption. This assumption implies that the virtual value function g_v is strictly increasing in v . Therefore, v^* is the unique solution to the equation $g_v = 0$. All information above is common knowledge.

The seller is endowed with zero inventory at time zero. We assume that the delivery lead time is zero, i.e., inventory can be immediately replenished when the seller places an order. We denote

by $\mathcal{Q}^t \triangleq \{t' \in [0, t] : q_{t'} > 0\}$ the collection of times up to time t at which the seller places orders. The seller incurs a fixed ordering cost K while placing an order. Without loss of generality, we normalize the per-unit ordering cost to zero. The seller incurs a holding cost $h > 0$ for carrying a unit of the product for a unit time.

The seller and customers are playing a Stackelberg game specified as follows. At $t = 0$, the seller determines and commits to a policy $\pi \triangleq \{\Omega_t^\pi : t \geq 0\} \cup \mathcal{Q}^{\infty, \pi} \cup \{q_t^\pi : t \in \mathcal{Q}^{\infty, \pi}\}$ that consists of a deterministic process of the purchase option sets, $\{\Omega_t^\pi : t \geq 0\}$, all scheduled inventory replenishment times, $\mathcal{Q}^{\infty, \pi}$, and deterministic order quantities at replenishment times, $\{q_t^\pi : t \in \mathcal{Q}^{\infty, \pi}\}$. The deterministic process of the purchase option sets, $\{\Omega_t^\pi : t \geq 0\}$, is public information for all customers.

We assume that the seller has the power to commit to a policy. There is a plethora of wide-ranging justifications for this assumption in the literature; for example, [Chen and Chu \(2016\)](#), [Golrezaei et al. \(2017\)](#), [Lobel and Xiao \(2017\)](#) provide comprehensive justifications from a joint pricing and inventory management perspective, [Correa et al. \(2016\)](#), [Liu and Van Ryzin \(2008\)](#) from a revenue management perspective, and [Board and Skrzypacz \(2016\)](#) from an economics perspective.

Given the seller's purchase option policy $\{\Omega_t^\pi : t \geq 0\}$, in response, customers are forward looking and seek to maximize their derived utility, employing (symmetric) purchasing rules contingent on their types that constitute a symmetric Nash Equilibrium. In particular, customer ϕ follows actions $z_\phi^\pi = (\tau_\phi^\pi, p_\phi^\pi, s_\phi^\pi)$ that solve the following optimization problem:

$$\begin{aligned} & \max_{z_\phi} U(\phi, z_\phi) \\ & \text{subject to } \tau_\phi \geq t_\phi, (p_\phi, s_\phi) \in \Omega_{\tau_\phi}^\pi, \text{ or } \tau_\phi = t_\phi, (p_\phi, s_\phi) = (0, \infty), \end{aligned}$$

assuming that other customers use symmetric purchasing rules.

Policy π is feasible if the total quantity the seller orders from the supplier up to any time $t \geq 0$ is no less than the total quantity the seller commits to deliver to customers up to time t , i.e., the seller's (after ordering) on-hand inventory at each time t is non-negative:

$$I_t^\pi \triangleq \sum_{t' \in \mathcal{Q}^{t, \pi}} q_{t'} - \int_{\phi \in \mathcal{H}^t} \mathbf{1}\{s_\phi^\pi \leq t\} \geq 0, \quad \forall t \geq 0,$$

where $\mathcal{H}^t \triangleq \{\phi : t_\phi \leq t\}$ denotes the collection of customers who arrive up to time t . (Note that the integral $\int_{\phi \in \mathcal{H}^t} g(\phi)$ with any integrable function $g(\phi)$ is a concise expression of the integral $\int_{t'=0}^t \int_{v=0}^V g(t', v) f(v) dv dt'$.) We denote by Π the set of all feasible policies.

Under a joint pricing, delivery and inventory policy $\pi \in \Pi$ and customer corresponding purchasing rule $z^\pi = (\tau^\pi, p^\pi, s^\pi)$, the seller's long-run average profit is given by

$$J^{\pi, z^\pi} = \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\int_{\phi \in \mathcal{H}^T} p_\phi^\pi \mathbf{1}\{\tau_\phi^\pi \leq T\} - h \int_{t=0}^T I_t^\pi dt - K|\mathcal{Q}^{T, \pi}| \right).$$

We denote by

$$\pi^* \in \arg \max_{\pi \in \Pi} J^{\pi, z^\pi}$$

the seller's optimal policy, and

$$J^* \triangleq \max_{\pi \in \Pi} J^{\pi, z^\pi}$$

the seller's optimal long-run average profit.

Our goal in this paper is to derive the seller's optimal policy π^* , exhibit a corresponding customer purchasing rule z^{π^*} and compute the seller's optimal long-run average profit J^* . For the convenience of readers, we summarize the major notation in this paper in Table 1.

Customers
$\phi = (t_\phi, v_\phi)$: customer type that consists of customer arrival time and valuation
$z_\phi = (\tau_\phi, p_\phi, s_\phi)$: customer ϕ 's decisions on the purchasing time and the selected purchase option
(p_ϕ, s_ϕ) : customer ϕ 's payment and the time to receive the product
$g_v = v - \frac{\bar{F}(v)}{f(v)}$: virtual value function
v^* : root of the equation $g_v = 0$
$\theta_v = W(v) - \frac{\bar{F}(v)}{f(v)}w(v)$: virtual delay cost function
Seller
q_t : order quantity at time t
Ω_t : the set of purchase options offered at time t
Policies
π^* : optimal policy in the deterministic model
π_L : a cyclic policy in the deterministic model, with the cycle length L
$\tilde{\pi}_L$: a cyclic policy in the stochastic model, with the cycle length L
Profit Functions
J^{π, z^π} : long-run average profit under policy π in the deterministic model
J^* : optimal long-run average profit in the deterministic model
\bar{J} : optimal long-run average profit in the mechanism design problem in the deterministic model
J^L : (1) a computable profit function that is used to establish an upper bound profit
(2) this profit function also denotes the long-run average profit under policy π_L in the deterministic model
$L^* = \arg \max_{L > 0} J^L$
\tilde{J}^{π, z^π} : long-run average expected profit under policy π in the stochastic model
\tilde{J}^* : optimal long-run average expected profit in the stochastic model
\tilde{J}^L : long-run average expected profit under policy $\tilde{\pi}_L$ in the stochastic model

Table 1 Summary of Major Notation.

3. The Seller's Optimal Policy

Deriving the seller's optimal policy π^* and computing her optimal long-run average profit J^* require us to solve a Stackelberg game between the seller and all customers. To avoid the complexity of analyzing this game, we adopt the following approach. First, we establish an easy-to-compute upper bound of the seller's optimal profit J^* . Second, we use the structural properties of this upper bound to propose a feasible joint pricing, delivery and inventory policy. We show that the seller's long-run average profit under this policy achieves the upper bound. Therefore, this result immediately implies that our proposed policy is optimal.

3.1. Upper Bound of the Seller's Optimal Profit - Mechanism Design Approach

In this subsection, we use the mechanism design approach to establish an easy-to-compute upper bound of the seller's optimal profit J^* . We begin with presenting the formal definition of the joint mechanism design and inventory problem. Because the mechanism that we present here and the policy defined in §2 are closely related to each other, in this subsection, we keep on using some notation defined §2, such as z_ϕ . However, such notation that appears in this subsection shall only be interpreted in the mechanism design context, rather than the context in §2.

We restrict ourselves to direct mechanisms. A mechanism specifies the product allocation and payment transfer rule and the inventory replenishment rule that we encode as follows.

The product allocation and payment transfer rules are as follows. If the seller decides to sell the product to customer ϕ , then she determines the time that customer ϕ makes the payment, $\tau_\phi \geq t_\phi$, the payment amount, $p_\phi \geq 0$, and the committed product delivery time, $s_\phi \in [\tau_\phi, \infty)$. Otherwise, if the seller decides not to sell the product to customer ϕ , then she sets $\tau_\phi = t_\phi$, $p_\phi = 0$ and $s_\phi = \infty$. We define a tuple $z_\phi = (\tau_\phi, p_\phi, s_\phi)$.

The inventory replenishment rule is as follows. We denote by \mathcal{Q}^t the collection of times over the selling season $[0, \infty)$ at which the seller places orders from the supplier. At time $t = 0$, the seller determines all inventory replenishment times over the selling season $[0, \infty)$, \mathcal{Q}^∞ , and the order quantities at all replenishment times $\{q_t : \forall t \in \mathcal{Q}^\infty\}$. Therefore, $q_t = 0$ for all $t \notin \mathcal{Q}^\infty$.

We shall say that a mechanism $(z, \mathcal{Q}, q) = \{z_\phi : \forall \phi\} \cup \mathcal{Q}^\infty \cup \{q_t : \forall t \in \mathcal{Q}^\infty\}$ is feasible if the total quantity the seller orders from the supplier up to any time $t \geq 0$ is no less than the total quantity the seller commits to deliver to customers up to time t , i.e., the seller's on-hand inventory at each time t is non-negative:

$$I_t \triangleq \sum_{t' \in \mathcal{Q}^t} q_{t'} - \int_{\phi \in \mathcal{H}^t} \mathbf{1}\{s_\phi \leq t\} \geq 0, \forall t \geq 0.$$

We denote by \mathcal{M} the set of all feasible mechanisms.

Under mechanism $(z, \mathcal{Q}, q) \in \mathcal{M}$, the seller's long-run average profit is given by

$$\Pi(z, \mathcal{Q}, q) = \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\int_{\phi \in \mathcal{H}^T} p_\phi \mathbf{1}\{\tau_\phi \leq T\} - h \int_{t=0}^T I_t dt - K|\mathcal{Q}^T| \right).$$

Let us denote by $\phi_{v'} \triangleq (t_\phi, v')$ customer ϕ 's report to the seller when he distorts his valuation to v' . The utility garnered by customer ϕ when he reports his true type as $\phi_{v'}$ (v' is not necessarily the same as v_ϕ) is given by

$$U(\phi, z_{\phi_{v'}}) = \begin{cases} v_\phi - p_{\phi_{v'}} - W(v_\phi)(s_{\phi_{v'}} - t_\phi) & \text{if } s_{\phi_{v'}} < \infty \\ 0 & \text{if } s_{\phi_{v'}} = \infty \end{cases}.$$

Consider the following mechanism design problem:

$$\begin{aligned} \max_{(z, \mathcal{Q}, q) \in \mathcal{M}} \quad & \Pi(z, \mathcal{Q}, q) \\ \text{subject to} \quad & U(\phi, z_\phi) \geq U(\phi, z_{\phi_{v'}}), \forall \phi, v' \quad (\text{IC}) \\ & U(\phi, z_\phi) \geq 0, \forall \phi. \quad (\text{IR}) \end{aligned} \tag{2}$$

We denote by \bar{J} the optimal value obtained in the mechanism design problem above.

Now, we use \bar{J} to establish an easy-to-compute upper bound of the seller's optimal profit J^* . To present our result, we define

$$\theta_v \triangleq W(v) - \frac{\bar{F}(v)}{f(v)} w(v).$$

As an analogy to the name of the virtual value function g_v , we hereafter follow [Nazerzadeh and Randhawa \(2018\)](#) to call θ_v the “*virtual delay cost*”.

PROPOSITION 1. *We have*

$$J^* \leq \bar{J} \leq \max_{L > 0} J^L \triangleq J^{L*},$$

where

$$J^L \triangleq \frac{1}{L} \int_{\phi \in \mathcal{H}^L} (g_{v_\phi} - \min\{ht_\phi, \theta_{v_\phi}(L - t_\phi)\})^+ \mathbf{1}\{v_\phi \geq v^*\} - \frac{K}{L}. \tag{3}$$

The key idea to prove the first inequality is that the seller's every policy and the corresponding customer best response behaviors can be replicated by an associated mechanism design problem. The key idea to prove the second inequality is as follows. First, we employ the Myersonian approach ([Myerson \(1981\)](#)) to internalize the information rent given up to the customers as their virtual valuations and virtual delay costs. This step allows us to convert the joint mechanism design and

inventory problem into a centralized inventory problem. Second, we exploit the tradeoffs of the revenue (customer virtual valuations), inventory holding cost, backordering cost (customer virtual delay costs) and the fixed ordering cost in this centralized inventory problem to compute its optimal value.

The profit function J^L enjoys the following salient features. First, it is easy to compute. We only need to do a simple integral, rather than solving an optimization problem or a game. Second, J^L has a very nice managerial interpretation. Consider a setting wherein the seller makes decisions on each customer ϕ of whether and when to allocate one unit of the product to him. Therefore, J^L is interpreted as the seller's long-run average profit under the following cyclic policy with the constant cycle length L (without loss of generality, we characterize the seller's policy within the first cycle $[0, L)$):

1. The seller only considers to sell the product to customers whose valuations are at least v^* .
2. If the seller decides to sell the product to customer ϕ with $t_\phi \in [0, L)$, then she has two choices of either using an inventory replenished at time 0 to meet his demand and thus incurring an inventory holding cost ht_ϕ , or delaying to meet his demand at time L and thus incurring a delayed delivery cost $\theta_{v_\phi}(L - t_\phi)$. This cost can be interpreted as the seller's backlogging cost of satisfying customer ϕ 's demand at a delayed time. The seller chooses the one with less cost.
3. The seller sells the product to customer ϕ with $v_\phi \geq v^*$ if and only if the revenue the seller collects from him, the virtual value g_{v_ϕ} , dominates the lower cost of meeting his demand (i.e., the minimum of the inventory holding cost ht_ϕ and the delayed delivery cost $\theta_{v_\phi}(L - t_\phi)$).

3.2. Computing J^L

In this subsection, we get a close-form expression of J^L . Following from the definition of J^L given by (3), it is crucial to characterize the impact of a customer's type $\phi \in [0, L) \times [v^*, V)$ on the order relations of g_{v_ϕ} (hereafter called as “customer ϕ 's virtual value”), ht_ϕ (hereafter called as “customer ϕ 's inventory holding cost”) and $\theta_{v_\phi}(L - t_\phi)$ (hereafter called as “customer ϕ 's delayed delivery cost”).

To characterize the order relations of these three quantities, we introduce the following three time-dependent threshold valuations. For every $t \in [0, L]$, we define

$$\begin{aligned} v_t^h &\triangleq \sup \{v \in [v^*, V) : g_v - ht < 0\}, \\ v_t^\theta &\triangleq \sup \{v \in [v^*, V) : g_v - \theta_v(L - t) < 0\}, \\ \tilde{v}_t &\triangleq \begin{cases} \sup \{v \in [v^*, V) : \theta_v(L - t) < ht\} & \text{if } v_t^h \neq v_t^\theta \\ v_t^h & \text{if } v_t^h = v_t^\theta \end{cases}. \end{aligned}$$

The threshold valuation v_t^h is used to determine the order relation of a customer's virtual value and inventory holding cost. The threshold valuation v_t^θ is used to determine the order relation of a customer's virtual value and delayed delivery cost. The threshold valuation \tilde{v}_t is used to determine the order relation of a customer's inventory holding cost and delayed delivery cost. We note that the threshold valuations v_t^θ and \tilde{v}_t hinge on the cycle length L . For expositional clarity, we suppress their dependency on L .

First, we characterize the order relations of these three time-dependent threshold valuations, v_t^h , v_t^θ , \tilde{v}_t .

LEMMA 1. Consider any $L > 0$. We define the following threshold times:

$$\begin{aligned}\underline{t}_L &\triangleq \sup \{t \in [0, L] : v_t^h \leq v_t^\theta\}, \\ \bar{t}_L &\triangleq \max \left\{ \underline{t}_L, \frac{W(V)}{h + W(V)} L \right\}.\end{aligned}$$

We have the following results:

$$0 < \underline{t}_L \leq \bar{t}_L < L, \quad \begin{cases} \tilde{v}_t \leq v_t^h \leq v_t^\theta & \text{if } t \in [0, \underline{t}_L) \\ \tilde{v}_t = v_t^h = v_t^\theta & \text{if } t = \underline{t}_L \\ \tilde{v}_t \geq v_t^h > v_t^\theta & \text{if } t \in (\underline{t}_L, L] \end{cases}, \quad \tilde{v}_t = V, \forall t \in (\bar{t}_L, L].$$

Figure 1 gives an example that illustrates the order relations of v_t^h , v_t^θ and \tilde{v}_t .

Next, we use the order relations of the three time-dependent threshold valuations, v_t^h , v_t^θ , \tilde{v}_t , to compute J^L .

LEMMA 2. For any $L > 0$,

$$\begin{aligned}J^L &= \frac{\lambda}{L} \left(\int_{t=0}^{\underline{t}_L} (p_t^{L,h} - ht) \bar{F}(v_t^h) dt + \int_{t=\underline{t}_L}^{\bar{t}_L} ((p_t^{L,h} - ht) \bar{F}(\tilde{v}_t) + p_t^{L,\theta} (\bar{F}(v_t^\theta) - \bar{F}(\tilde{v}_t))) dt \right. \\ &\quad \left. + \int_{t=\bar{t}_L}^L p_t^{L,\theta} \bar{F}(v_t^\theta) dt \right) - \frac{K}{L},\end{aligned}\tag{4}$$

where

$$\begin{aligned}p_t^{L,\theta} &\triangleq v_t^\theta - W(v_t^\theta)(L - t), \forall t \in [\underline{t}_L, L), \\ p_t^{L,h} &\triangleq \begin{cases} v_t^h & \text{if } t \in [0, \underline{t}_L] \\ p_t^{L,\theta} + W(\tilde{v}_t)(L - t) & \text{if } t \in (\underline{t}_L, \bar{t}_L] \end{cases}.\end{aligned}$$

Figure 2 gives an example that illustrates the price dynamics $\{p_t^{L,h}\}$ and $\{p_t^{L,\theta}\}$.

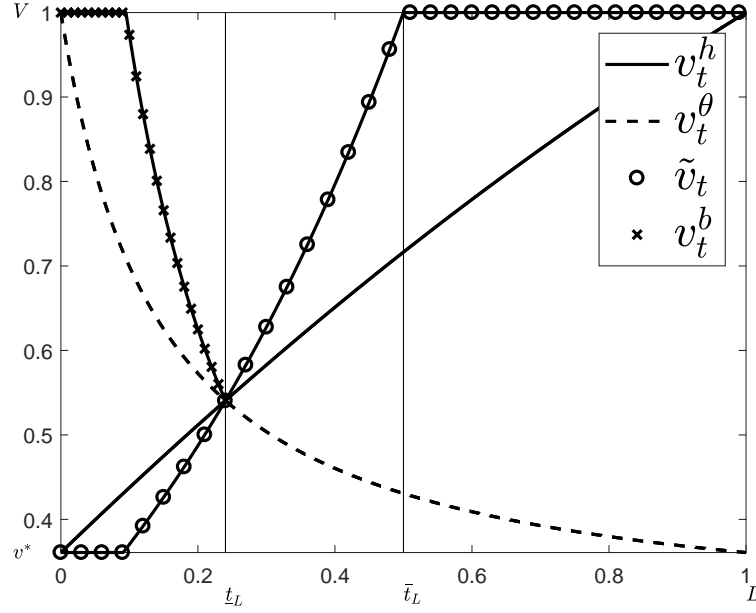


Figure 1 An example that illustrates the order relations of v_t^h , v_t^θ and \tilde{v}_t : $V = 1$, $F(v) = (1 - e^{-2v})/(1 - e^{-2V})$, $W(v) = 0.1 + 0.9v$, $h = 1$, $L = 1$. (Note: the curve for v_t^b will be introduced and explained in the next section.)

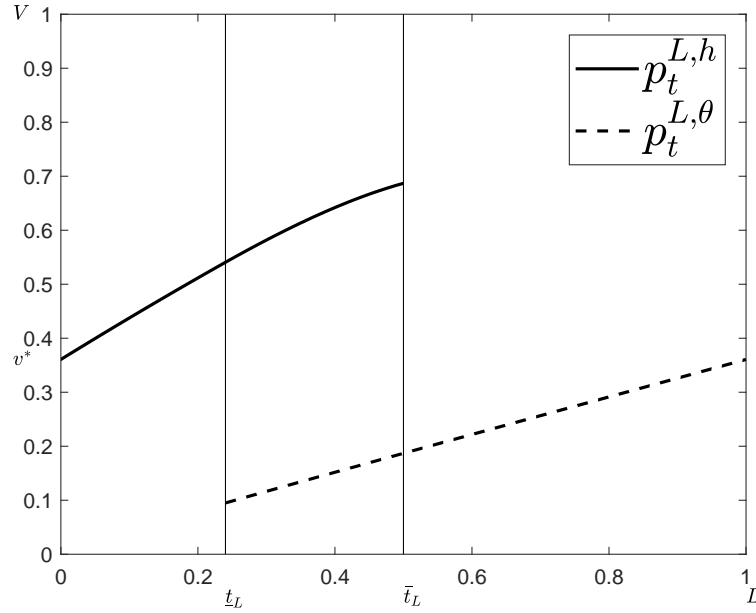


Figure 2 An example that illustrates the price dynamics under the instantaneous delivery option, $\{p_t^{L,h}\}$, and the price dynamics under the delayed delivery option, $\{p_t^{L,\theta}\}$: $V = 1$, $F(v) = (1 - e^{-2v})/(1 - e^{-2V})$, $W(v) = 0.1 + 0.9v$, $h = 1$, $L = 1$.

3.3. The Seller's Optimal Policy π^*

In this subsection, we derive the seller's optimal policy π^* . We proceed in the following three steps. First, we exploit the structural properties of the closed-form expression of J^L , (4), to propose a joint pricing, delivery and inventory policy π_L . Second, we show that under policy π_L , every customer either purchases the product at his time of arrival or permanently leaves the system without any purchase at his time of arrival, i.e., no customer makes a delayed purchase. Third, we show that the seller's profit under policy π_L achieves J^L . This result implies that the seller's profit under policy π_{L^*} achieves the upper bound of the seller's optimal profit J^{L^*} , i.e., policy π_{L^*} is optimal.

First, we use the structural properties of the closed-form expression of J^L , (4), to propose the following policy $\pi_L = \{\Omega_t^{\pi_L} : t \geq 0\} \cup \mathcal{Q}^{\infty, \pi_L} \cup \{q_t^{\pi_L} : t \in \mathcal{Q}^{\infty, \pi_L}\}$. We define $r_L(t) \triangleq t - \lfloor \frac{t}{L} \rfloor \cdot L$. We have

$$\Omega_t^{\pi_L} = \begin{cases} \left\{ \left(p_{r_L(t)}^{L,h}, t \right) \right\} & \text{if } r_L(t) \in [0, \underline{t}_L] \\ \left\{ \left(p_{r_L(t)}^{L,h}, t \right), \left(p_{r_L(t)}^{L,\theta}, t - r_L(t) + L \right) \right\} & \text{if } r_L(t) \in (\underline{t}_L, \bar{t}_L] \\ \left\{ \left(p_{r_L(t)}^{L,\theta}, t - r_L(t) + L \right) \right\} & \text{if } r_L(t) \in (\bar{t}_L, L) \end{cases},$$

and

$$\mathcal{Q}^{\infty, \pi_L} = \{(n-1)L : n \in \mathbb{N}\},$$

and

$$q_t^{\pi_L} = \begin{cases} \lambda \left(\int_{t'=0}^{\underline{t}_L} \bar{F}(v_{t'}^h) dt' + \int_{t'=\underline{t}_L}^{\bar{t}_L} \bar{F}(\tilde{v}_{t'}) dt' \right) & \text{if } t = 0 \\ \lambda \left(\int_{t'=0}^{\underline{t}_L} \bar{F}(v_{t'}^h) dt' + \int_{t'=\underline{t}_L}^{\bar{t}_L} \bar{F}(v_{t'}^\theta) dt' + \int_{t'=\bar{t}_L}^L \bar{F}(v_{t'}^\theta) dt' \right) & \text{if } t = nL, \forall n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Under policy π_L , the seller repeats the same decisions on the sets of the purchase options offered to customers and the inventory replenishment quantities over cycles with the identical cycle length L . In each cycle (without loss of generality, we focus on the first cycle $[0, L)$), at a time that is very close to the latest replenishment time, $t \in [0, \underline{t}_L]$, because keeping too much inventory is too costly, the seller only offers customers the instantaneous delivery option $\Omega_t^{\pi_L} = \{(p_t^{L,h}, t)\}$ to expedite the inventory liquidation. At a time that is not too close to the latest or the next replenishment time, $t \in (\underline{t}_L, \bar{t}_L]$, the seller offers two purchase options to customers: one option with a higher price and an instantaneous delivery and the other option with a lower price and a delayed delivery, $\Omega_t^{\pi_L} = \{(p_t^{L,h}, t), (p_t^{L,\theta}, L)\}$. At a time that is close to the next replenishment time, $t \in (\bar{t}_L, L)$, because the seller may not have inventory on hand, the seller only offers customers one purchase option with the delayed delivery, $\Omega_t^{\pi_L} = \{(p_t^{L,\theta}, L)\}$.

Next, we characterize customer purchasing behaviors under policy π_L .

LEMMA 3. Under policy π_L , every customer ϕ 's following behavior is a dominant strategy:

$$\tau_\phi^{\pi_L} = t_\phi, (p_\phi^{\pi_L}, s_\phi^{\pi_L}) = \begin{cases} (p_{r_L(t_\phi)}^{L,h}, t_\phi) & \text{if } \phi \in \mathcal{R}_L^h \\ (p_{r_L(t_\phi)}^{L,\theta}, t - r_L(t_\phi) + L) & \text{if } \phi \in \mathcal{R}_L^\theta \\ (0, \infty) & \text{otherwise} \end{cases},$$

where

$$\begin{aligned} \mathcal{R}_L^h &\triangleq \{(t, v) : r_L(t) \in [0, \underline{t}_L], v \geq v_{r_L(t)}^h\} \cup \{(t, v) : r_L(t) \in (\underline{t}_L, \bar{t}_L], v \geq \tilde{v}_{r_L(t)}\}, \\ \mathcal{R}_L^\theta &\triangleq \{(t, v) : r_L(t) \in (\underline{t}_L, \bar{t}_L], v \in [v_{r_L(t)}^\theta, \tilde{v}_{r_L(t)}]\} \cup \{(t, v) : r_L(t) \in (\bar{t}_L, L), v \geq v_{r_L(t)}^\theta\}. \end{aligned}$$

Figure 1 illustrates regions \mathcal{R}_L^h and \mathcal{R}_L^θ in the first cycle $[0, L)$. \mathcal{R}_L^h consists of the region above the solid line for $t \in [0, \underline{t}_L]$ and the region above the line with circles for $t \in (\underline{t}_L, \bar{t}_L]$. \mathcal{R}_L^θ consists of the region between the dashed line and the line with circles for $t \in [\underline{t}_L, L)$.

Now, we discuss customer purchase behaviors under policy π_L . Due to the cyclic nature of policy π_L , without loss of generality, we focus on the first time interval $[0, L)$. First, every customer either purchases the product at his time of arrival or permanently leaves the system without any purchase at his time of arrival ($\tau_\phi = t_\phi$), i.e., no customer makes a delayed purchase. Second, a customer's decision on whether to purchase and which purchase option to select if he faces two alternative purchase options is fully characterized by the three threshold valuations $v_t^h, v_t^\theta, \tilde{v}_t$. To be specific, if customer ϕ arrives at a time at which only the instantaneous delivery option is available, $t_\phi \in [0, \underline{t}_L]$, then he selects this option if and only if his valuation reaches the threshold value, $v_{t_\phi}^h$. If customer ϕ arrives at a time at which both the instantaneous delivery option and the delayed delivery option are available, $t_\phi \in (\underline{t}_L, \bar{t}_L]$, then he selects the instantaneous delivery option if his valuation is sufficiently high, $v_\phi \geq \tilde{v}_{t_\phi}$, or he selects the delayed delivery option if his valuation is not too high or too low, $v_\phi \in [v_{t_\phi}^\theta, \tilde{v}_{t_\phi})$, or he does not purchase if his valuation is too low, $v_\phi < v_{t_\phi}^\theta$. If customer ϕ arrives at a time at which only the delayed purchase option is available, $t_\phi \in (\bar{t}_L, L)$, then he selects this option if and only if his valuation reaches the threshold value, $v_{t_\phi}^\theta$.

Following from the closed-form expression of J^L , (4), the definition of policy π_L and the corresponding customer purchasing behaviors characterized in Lemma 3, we immediately have the result that the seller's long-run average profit under policy π_L is equal to J^L :

$$J^{\pi_L, z^{\pi_L}} = J^L, \forall L > 0.$$

To ease our notation, we hereafter abuse our notation to use J^L to denote the seller's long-run average profit under policy π_L .

Recall from Proposition 1 that the seller's optimal profit is upper bounded by J^{L^*} and the property above that the policy π_{L^*} achieves this upper bound. We immediately have the result that policy π_{L^*} is optimal.

THEOREM 1. *We have*

$$\pi^* = \pi_{L^*} \quad \text{and} \quad J^* = J^{L^*}.$$

It is worth to note that the fixed ordering cost K plays an important role in determining the cycle length L^* . The following proposition characterizes the relationship between the cycle length L^* and the fixed ordering cost K .

PROPOSITION 2. *The optimal cycle length L^* is non-decreasing in K . In one extreme case that $K = 0$, we have $L^* = 0$. In another extreme case that $K \rightarrow \infty$, we have $L^* = \infty$.*

When it is more expensive to place an order (K increases), to avoid incurring too high ordering cost, the seller orders in a less frequent way (L^* increases). In one extreme case that it is costless to place an order ($K = 0$), the seller orders in an extremely high frequent way ($L^* = 0$) to avoid incurring any inventory holding cost or product delivery delay. In another extreme case that it is very expensive to place an order ($K \rightarrow \infty$), because costs that arise from holding inventory and incurring delayed delivery are much cheaper than the cost of placing an order, the seller has no incentive to order for the second time ($L^* = \infty$).

4. Joint Pricing and Inventory Management in a Stochastic Setting

The previous sections study a joint pricing and inventory management problem in a deterministic setting. In this section, we extend our model to a stochastic setting. We will present an easy-to-implement joint pricing, delivery and inventory policy that is proven to be near optimal in the stochastic setting.

4.1. Model

The stochastic joint pricing and inventory management problem that we study in this section is very similar to its deterministic counterpart problem defined in §2. Therefore, we only present their differences.

In the stochastic model, we assume that customers are indivisible and they arrive to the system according to a Poisson process with a constant arrival rate λ . Customer valuations are i.i.d. random variables with the c.d.f. $F(\cdot)$ and the p.d.f. $f(\cdot)$.

Because of the system's stochastic nature, we allow the seller's decisions on the set of purchase options and order quantities to be stochastic processes, which will be specified later. Therefore, unlike the deterministic model wherein every customer ϕ is clairvoyant and is able to decide whether and when to purchase at her arrival time t_ϕ , in the stochastic model, every customer makes these decisions in a progressive manner. To be precise, at each point of time, a customer who stays in the system needs to make one of the following three decisions: immediately purchase the product, permanently leave the system without purchasing anything, or continue to stay in the system and delay to make the purchasing decision at a future time. We use notation $\tau_\phi \geq t_\phi$ to denote the time that customer ϕ makes the purchasing decision. At time τ_ϕ , if customer ϕ decides to purchase, then she selects a purchase option (p_ϕ, s_ϕ) presented at that point of time. Otherwise, if customer ϕ decides to permanently leave the system without purchasing anything at time τ_ϕ , then we make a convention that he selects the no purchase option $(p_\phi, s_\phi) = (0, \infty)$ at time τ_ϕ .

Define the tuple

$$z_\phi \triangleq (\tau_\phi, p_\phi, s_\phi).$$

Customer ϕ garners utility

$$U(\phi, z_\phi) = \begin{cases} v_\phi - p_\phi - W(v_\phi)(s_\phi - t_\phi) & \text{if } s_\phi < \infty \\ -W(v_\phi)(\tau_\phi - t_\phi) & \text{if } s_\phi = \infty \end{cases}.$$

The seller and customers are playing a Stackelberg game specified as follows. At $t = 0$, the seller determines and commits to a joint pricing, delivery and inventory policy $\pi \triangleq \{\Omega_t^\pi : t \geq 0\} \cup \mathcal{Q}^{\infty, \pi} \cup \{q_t^\pi : t \in \mathcal{Q}^{\infty, \pi}\}$. Under policy π , the seller determines all replenishment times over the entire season, \mathcal{Q}^∞ , at time 0. Policy π needs to satisfy the causality property that the seller's purchase option decision at any time and her order quantity decision at any replenishment time are the functions of her information up to the time that she makes these decisions. Such information includes the customer-related information that pertains to historic customer purchase times and the selected purchase options, the seller's historic decisions on the sets of the purchase options presented to customers and her historic order quantity decisions. The seller's policy on the purchase option sets is public information for all customers.

Given the seller's purchase option policy, in response, customers are forward looking and seek to maximize their expected derived utility, employing (symmetric) purchasing rules contingent on their types that constitute a symmetric Markov Perfect Equilibrium. In particular, customer ϕ follows actions $z_\phi^\pi = (\tau_\phi^\pi, p_\phi^\pi, s_\phi^\pi)$ that solve the following optimization problem:

$$\begin{aligned} \max_{z_\phi} \quad & \mathbf{E} \left[U(\phi, z_\phi) \middle| \phi, \Omega_{t_\phi} \right] \\ \text{subject to} \quad & \tau_\phi \geq t_\phi, (p_\phi, s_\phi) \in \Omega_{\tau_\phi}^\pi \cup \{(0, \infty)\}, \end{aligned}$$

with the expectation above assuming that other customers use symmetric purchasing rules.

Policy π is feasible if the total quantity the seller orders from the supplier up to time t is no less than the total quantity the seller commits to deliver to customers up to time t , i.e., the seller's (after ordering) on-hand inventory at each time t is non-negative:

$$I_t^\pi \triangleq \sum_{t' \in \mathcal{Q}^{t,\pi}} q_{t'} - \sum_{\phi \in \mathcal{H}^t} \mathbf{1}\{s_\phi \leq t\} \geq 0 \quad \text{a.s.} \quad \forall t \geq 0.$$

We denote by $\tilde{\Pi}$ the set of all feasible policies.

Under policy $\pi \in \tilde{\Pi}$ and customer corresponding purchasing rule $z^\pi = (\tau^\pi, p^\pi, s^\pi)$, the seller's long-run average expected profit is given by

$$\tilde{J}^{\pi, z^\pi} = \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\mathbb{E} \left[\sum_{\phi \in \mathcal{H}^T} p_\phi^\pi \mathbf{1}\{\tau_\phi^\pi \leq T\} - h \int_{t=0}^T I_t^\pi dt \right] - K |\mathcal{Q}^{T,\pi}| \right).$$

The seller's objective is to find a policy that maximizes her long-run average expected profit:

$$\max_{\pi \in \tilde{\Pi}} \tilde{J}^{\pi, z^\pi} \triangleq \tilde{J}^*.$$

4.2. Asymptotic Optimal Policy in the Stochastic Setting

Deriving the seller's optimal policy in the stochastic setting requires us to solve a complicated Stackelberg game between the seller and stochastically arriving customers. To avoid the computational complexity, rather than deriving the seller's optimal policy, we seek an easy-to-implement heuristic policy and show that it is near optimal.

A good candidate could be policy π_L defined in §3.3 when we study the deterministic setting. As discussed in §3.3, this policy enjoys a simple cyclic structure and rich managerial insights. However, we cannot directly implement this policy in the stochastic setting. Under this policy, the instantaneous delivery options are offered at fixed times, $\cup_{n=1}^\infty [(n-1)L, (n-1)L + \bar{t}_L]$. However, in the stochastic setting, the presence of customer stochastic arrivals and random valuations may lead to the seller's inventory shortage at some of these times. Therefore, it is no longer feasible to offer an instantaneous delivery option at a time in $\cup_{n=1}^\infty [(n-1)L, (n-1)L + \bar{t}_L]$ at which the seller runs out of inventory.

Therefore, in the stochastic setting, rather than directly implementing π_L , we propose a modified policy, $\tilde{\pi}_L = \left\{ \Omega_t^{\tilde{\pi}_L} : t \geq 0 \right\} \cup \mathcal{Q}^{\infty, \tilde{\pi}_L} \cup \left\{ q_t^{\tilde{\pi}_L} : t \in \mathcal{Q}^{\infty, \tilde{\pi}_L} \right\}$, that both preserves the key structures of

π_L and avoids the aforementioned feasibility issue. The definition of $\tilde{\pi}_L$ is as follows:

$$\Omega_t^{\tilde{\pi}_L} = \begin{cases} \left\{ \left(p_{\max\{r_L(t), \underline{t}_L\}}^{L, \theta}, t - r_L(t) + L \right) \right\} & \forall t \text{ with } r_L(t) \in (0, \bar{t}_L] \text{ and } I_{t-} = 0 \\ \Omega_t^{\pi_L} & \text{otherwise} \end{cases},$$

and

$$\mathcal{Q}^{\infty, \tilde{\pi}_L} = \mathcal{Q}^{\infty, \pi_L},$$

and

$$q_t^{\tilde{\pi}_L} = \begin{cases} q_0^{\pi_L} & \text{if } t = 0 \\ \sum_{\phi \in \mathcal{H}^{nL}} a_{\phi}^{\pi_L} \mathbf{1} \{ \tau_{\phi}^{\pi_L} \in [(n-1)L, nL) \} & \text{if } t = nL, \forall n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Now, we highlight the distinctions between $\tilde{\pi}_L$ defined above and π_L defined in the deterministic setting.

1. In the purchase option decision part, $\tilde{\pi}_L$ is only distinguished from π_L at those points of times at which the seller runs out of inventory but instantaneous delivery options are offered under π_L . At all these times, because the seller has no inventory on hand, the policy $\tilde{\pi}_L$ stops the seller offering the instantaneous delivery options. Alternatively, if such a time t is close to the latest inventory replenishment time, $r_L(t) \in (0, \underline{t}_L]$, then the seller offers customers a delayed delivery option $(p_{\underline{t}_L}^{L, \theta}, t - r_L(t) + L)$ to replace the instantaneous delivery option offered under π_L . Otherwise, if such a time t is with $r_L(t) \in (\underline{t}_L, \bar{t}_L]$, the seller offers customers the delayed delivery option that is also offered under π_L , $(p_{t_L(t)}^{L, \theta}, t - r_L(t) + L)$.
2. In the inventory decision part, under $\tilde{\pi}_L$, the seller makes replenishment over cycles with a constant cycle length L , and the quantity that the seller orders at the beginning of each cycle, nL with $n \in \mathbb{N}$, is equal to the total number of sales in the previous cycle $[(n-1)L, nL)$. Hence, at the beginning of each cycle, the seller's on-hand inventory after being instantly replenished and clearing the backlogged demand reaches the same level. This is called (S, T) policy in inventory management literature (see, e.g., [Liu and Song \(2012\)](#)). However, because customer arrivals are stochastic, the seller may order different quantities at the beginning of different cycles. This is distinguished from π_L under which the seller orders the same amount at the beginning of each cycle.

Now, we present here our principle results for the family of policies $\{\tilde{\pi}_L : L > 0\}$ in the stochastic setting. First, we establish an equilibrium purchasing rule for customers when the seller follows $\tilde{\pi}_L$.

LEMMA 4. Consider the stochastic setting. Assume that the seller adopts policy $\tilde{\pi}_L$. Then every customer ϕ 's following behavior is a dominant strategy:

$$\tau_{\phi}^{\tilde{\pi}_L} = t_{\phi}, \quad \left(p_{\phi}^{\tilde{\pi}_L}, p_{\phi}^{\tilde{\pi}_L}\right) = \begin{cases} \left(p_{r_L(t_{\phi})}^{L,h}, t_{\phi}\right) & \text{if } \phi \in \mathcal{R}_L^h, \text{ and } I_{t_{\phi}-} > 0 \text{ if } r_L(t_{\phi}) > 0 \\ \left(p_{r_L(t_{\phi})}^{L,\theta}, t_{\phi} - r_L(t_{\phi}) + L\right) & \text{if } \phi \in \mathcal{R}_L^{\theta}, \text{ or } r_L(t_{\phi}) \in (\underline{t}_L, \bar{t}_L], v_{\phi} \geq \tilde{v}_{r_L(t_{\phi})}, I_{t_{\phi}-} = 0 \\ \left(p_{\underline{t}_L}^{L,\theta}, t_{\phi} - r_L(t_{\phi}) + L\right) & \text{if } r_L(t_{\phi}) \in (0, \underline{t}_L], v_{\phi} \geq v_{r_L(t_{\phi})}^b, I_{t_{\phi}-} = 0 \\ (0, \infty) & \text{otherwise} \end{cases},$$

where the threshold value v_t^b is defined as

$$v_t^b \triangleq \sup \{v \in [v^*, V) : v - W(v)(L - t) < p_{r_L(t)}^{L,\theta}\} \quad \forall t \in (0, \underline{t}_L].$$

Customer purchasing behaviors under $\tilde{\pi}_L$ in the stochastic setting are almost the same as their behaviors under π_L in the deterministic setting. The only distinction occurs at those points of times at which the seller runs out of inventory but instantaneous delivery options are offered under π_L . If such a time t is close to the latest replenishment time, $r_L(t) \in (0, \underline{t}_L]$, then a customer who arrives at this time selects the delayed delivery option offered at this time under policy $\tilde{\pi}_L$ if and only if his valuation is no less than the threshold valuation $v_{r_L(t)}^b$. Otherwise, if such a time t is with $r_L(t) \in (\underline{t}_L, \bar{t}_L]$, then a customer who arrives at this time selects the delayed delivery option offered at this time under policy $\tilde{\pi}_L$ if and only if his valuation is no less than the threshold valuation $v_{r_L(t)}^{\theta}$. Figure 1 gives an example that illustrates the threshold valuation function $v_{r_L(t)}^b$.

Now, we characterize the financial performance of policy $\tilde{\pi}_{L^*}$. To ease our notation, we hereafter use notation \tilde{J}^L to denote the seller's long-run average expected profit under policy $\tilde{\pi}_L$ in the stochastic setting.

THEOREM 2. In the stochastic strategic customer setting, we have

$$\frac{\tilde{J}^{L^*}}{\tilde{J}^*} \geq \frac{\tilde{J}^{L^*}}{J^{L^*}} \geq 1 - \frac{V + hL^*}{2J^{L^*}} \sqrt{\frac{\lambda}{L^*}}.$$

Consider a sequence of instances indexed by $i \in \mathbb{N}$. In the i -th instance, $\lambda^{(i)} = i\lambda$ and $K^{(i)} = iK$. We denote $L^{*,(i)} \triangleq \arg \max_{L > 0} J^{L,(i)}$. Therefore, $L^{*,(i)} = L^*$ and

$$\frac{\tilde{J}^{L^*}(i)}{\tilde{J}^*(i)} \geq \frac{\tilde{J}^{L^*}(i)}{J^{L^*}(i)} \geq 1 - O\left(\frac{1}{\sqrt{i}}\right).$$

Theorem 2 establishes a lower bound of the seller's long-run average expected profit under policy $\tilde{\pi}_{L^*}$ relative to her optimal profit. In the asymptotic regime wherein the customer arrival rate λ and the seller's fixed ordering cost K proportionally grow large, policy $\tilde{\pi}_{L^*}$ is optimal. Note that

we proportionally scale λ and K so that K stays relevant in the problem. In other words, if we were only to scale λ , the impact of having a fixed ordering cost K would be negligible.

4.3. Numerical Analysis of $\tilde{\pi}_{L^*}$

In this subsection, we numerically study the performance of $\tilde{\pi}_{L^*}$. We show that this policy is near optimal in a wide range of regimes. Throughout the numerical analysis, we assume that customer valuations are uniformly distributed between 0 and 1, the inventory holding cost is $h = 0.1$, the delay disutility function takes the following form:

$$W(v) = \theta_1 + \theta_2 v^{\theta_3}, \forall (\theta_1, \theta_2, \theta_3) \in \mathbb{R}_{++} \times \mathbb{R}_+ \times [0, 1].$$

We study the following three types of delay disutility functions: (1) constant delay disutility function ($\theta_2 = 0$), (2) linear delay disutility function ($\theta_2 > 0$ and $\theta_3 = 1$), (3) strictly concave delay disutility function ($\theta_2 > 0$ and $\theta_3 \in (0, 1)$). For each type of delay disutility function, we conduct two sets of experiments. In the first set of experiments, we vary the values of customer arrival rate λ and the seller's fixed ordering cost K by enumerating over $(\lambda, K) \in \{10, 15, 20, 25\} \times \{0.5, 1\}$. In the second set of experiments, we set $\lambda = 10$ and $K = 0.5$ as their base values and proportionally scale these two parameters by a factor that takes values from 1 to 4.5, with an increment of 0.5. For each experiment, we compute the optimal cycle length L^* and the relative performance lower bound \tilde{J}^{L^*}/J^{L^*} . All numerical results are reported in Tables 2–4.

We make the following observations. First, our proposed heuristic policy π_{L^*} performs consistently well, even when the traffic intensity is relatively low. The relative profit loss is within 5% for most instances. Second, when the customer arrival rate λ increases and the seller's fixed ordering cost K remains the same, both the optimal cycle length L^* and the relative profit loss decrease. The intuition is as follows. Since the fixed ordering cost per customer becomes lower with more customers, the seller orders more frequently to avoid carrying too much inventory that leads to an excessively high inventory holding cost. Also, when customer arrival rate increases, the customer arrival process is relatively less volatile, i.e., the stochastic system is closer to its deterministic counterpart system. Therefore, the relative profit loss of our proposed policy diminishes. Third, when the seller's fixed ordering cost K increases and the customer arrival rate λ remains the same, both the optimal cycle length L^* and the relative profit loss increases. The intuition is as follows. When placing an order is more costly, the seller orders less frequently to avoid incurring an excessively high fixed ordering cost. Also, because the volatility of the customer arrival process remains the same and the increasing order cycle length makes the supply less elastic, the relative

	λ	K	L^*	\tilde{J}^{L^*}/J^{L^*}	λ	K	L^*	\tilde{J}^{L^*}/J^{L^*}
$\theta_1=0.2$	10	0.5	2.1	88%	10	0.5	2.1	88%
	15	0.5	1.7	92%	15	0.75	2.1	92%
	20	0.5	1.4	96%	20	1	2.1	94%
	25	0.5	1.1	97%	25	1.25	2.1	95%
	10	1	2.7	88%	30	1.5	2.1	96%
	15	1	2.3	92%	35	1.75	2.1	96%
	20	1	2.1	94%	40	2	2.1	96%
	25	1	1.8	95%	45	2.25	2.1	97%
$\theta_1=0.3$	10	0.5	2	89%	10	0.5	2	89%
	15	0.5	1.6	92%	15	0.75	2	93%
	20	0.5	1.3	96%	20	1	2	94%
	25	0.5	1	98%	25	1.25	2	95%
	10	1	2.5	88%	30	1.5	2	96%
	15	1	2.2	92%	35	1.75	2	96%
	20	1	2	94%	40	2	2	97%
	25	1	1.7	96%	45	2.25	2	97%
$\theta_1=0.4$	10	0.5	1.9	91%	10	0.5	1.9	91%
	15	0.5	1.6	93%	15	0.75	1.9	93%
	20	0.5	1.3	96%	20	1	1.9	94%
	25	0.5	1	98%	25	1.25	1.9	95%
	10	1	2.4	89%	30	1.5	1.9	96%
	15	1	2.1	93%	35	1.75	1.9	97%
	20	1	1.9	95%	40	2	1.9	97%
	25	1	1.7	97%	45	2.25	1.9	98%

Table 2 Performance of π_{L^*} under constant delay disutility $W(v) = \theta_1$ where $\theta_1 \in \{0.2, 0.3, 0.4\}$.

profit loss of our proposed policy increases. Fourth, when the customer's delay disutility increases (by varying parameters $\theta_1, \theta_2, \theta_3$ in the delay disutility functions) and the customer arrival rate λ remains the same, both the optimal cycle length L^* and the relative profit loss decrease. The intuition is as follows. When delaying is more costly, the seller orders more frequently to ensure the customers do not suffer from unnecessarily long waiting times. Also, decreasing order cycle length makes the supply more elastic, the relative profit loss of our proposed policy decreases. Fifth, when the customer arrival rate λ and the seller's fixed ordering cost K proportionally grow large, the optimal cycle length L^* remains the same, and the relative profit loss decreases. The intuition is as follows. When these two parameters proportionally grow large, the fixed ordering cost per customer remains the same. Therefore, the optimal cycle length does not change. Also, when these two parameters proportionally grow large, the system is relatively less volatile, i.e., the stochastic system is closer to its deterministic counterpart system. Therefore, the relative profit loss of our proposed policy diminishes, echoing our theoretical asymptotic analysis.

	λ	K	L^*	\tilde{J}^{L^*}/J^{L^*}	λ	K	L^*	\tilde{J}^{L^*}/J^{L^*}
$\theta_2=0.2$	10	0.5	2	88%	10	0.5	2	88%
	15	0.5	1.5	93%	15	0.75	2	93%
	20	0.5	1.2	95%	20	1	2	94%
	25	0.5	1.1	96%	25	1.25	2	95%
	10	1	2.5	87%	30	1.5	2	95%
	15	1	2.1	91%	35	1.75	2	96%
	20	1	2	94%	40	2	2	96%
	25	1	1.9	96%	45	2.25	2	97%
$\theta_2=0.3$	10	0.5	1.8	88%	10	0.5	1.8	88%
	15	0.5	1.4	94%	15	0.75	1.8	93%
	20	0.5	1.1	95%	20	1	1.8	94%
	25	0.5	1	96%	25	1.25	1.8	95%
	10	1	2.3	87%	30	1.5	1.8	96%
	15	1	1.9	91%	35	1.75	1.8	96%
	20	1	1.8	94%	40	2	1.8	96%
	25	1	1.7	96%	45	2.25	1.8	97%
$\theta_2=0.4$	10	0.5	1.7	89%	10	0.5	1.7	89%
	15	0.5	1.3	94%	15	0.75	1.7	93%
	20	0.5	1	95%	20	1	1.7	94%
	25	0.5	0.9	96%	25	1.25	1.7	95%
	10	1	2.2	87%	30	1.5	1.7	96%
	15	1	1.8	92%	35	1.75	1.7	96%
	20	1	1.7	94%	40	2	1.7	97%
	25	1	1.6	96%	45	2.25	1.7	97%

Table 3 Performance of π_{L^*} under linear delay disutility $W(v) = 0.1 + \theta_2 v$ where $\theta_2 \in \{0.2, 0.3, 0.4\}$.

5. Conclusion

This paper studies a joint pricing, delivery and inventory problem in the presence of forward-looking customers. We allow customers to be heterogeneous in their arrival times and product valuations. We allow a customer's delay disutility rate to be perfectly and positively correlated with his product valuation. In the deterministic setting, we use the mechanism design approach to determine the seller's optimal policy. In the stochastic setting, we propose a policy that is proven to be near optimal in a wide range of regimes.

We remark that our results can be extended in the following two ways. First, our results hold in a flexible payment scheme wherein a customer is allowed to pay a cheaper price if it comes out after he places an order and before he receives the product. The intuition is as follows. We note that under policy π_L , the price associated with the delayed delivery purchase option is non-decreasing over time in each cycle (Lemma 7 Part 1) and the price associated with the delayed delivery purchase option is no less than the price associated with the instantaneous delivery purchase option offered at the same time (Lemma 2). Therefore, under policy π_L , if a customer chooses a purchase option with the delayed delivery, then he will never see a cheaper price posted before he receives the product. Second, our results hold in a model that takes into account every customer's

	λ	K	L^*	\tilde{J}^{L^*}/J^{L^*}	λ	K	L^*	\tilde{J}^{L^*}/J^{L^*}
$\theta_3=0.9$	10	0.5	1.9	88%	10	0.5	1.9	88%
	15	0.5	1.4	93%	15	0.75	1.9	91%
	20	0.5	1.3	95%	20	1	1.9	93%
	25	0.5	1.1	96%	25	1.25	1.9	94%
	10	1	2.5	87%	30	1.5	1.9	94%
	15	1	2.4	90%	35	1.75	1.9	95%
	20	1	1.9	93%	40	2	1.9	95%
	25	1	1.8	95%	45	2.25	1.9	96%
$\theta_3=0.5$	10	0.5	1.9	88%	10	0.5	1.9	88%
	15	0.5	1.4	93%	15	0.75	1.9	91%
	20	0.5	1.3	95%	20	1	1.9	93%
	25	0.5	1.1	96%	25	1.25	1.9	94%
	10	1	2.5	87%	30	1.5	1.9	94%
	15	1	2.3	90%	35	1.75	1.9	95%
	20	1	1.9	93%	40	2	1.9	95%
	25	1	1.8	95%	45	2.25	1.9	96%
$\theta_3=0.1$	10	0.5	1.9	88%	10	0.5	1.9	88%
	15	0.5	1.4	93%	15	0.75	1.9	91%
	20	0.5	1.3	95%	20	1	1.9	93%
	25	0.5	1.1	96%	25	1.25	1.9	94%
	10	1	2.4	87%	30	1.5	1.9	94%
	15	1	2.2	91%	35	1.75	1.9	95%
	20	1	1.9	93%	40	2	1.9	95%
	25	1	1.8	95%	45	2.25	1.9	96%

Table 4 Performance of π_{L^*} under concave delay disutility $W(v) = 0.1 + 0.3v^{\theta_3}$ where $\theta_3 \in \{0.9, 0.5, 0.1\}$.

heterogeneous price monitoring cost incurred before the customer places an order and leaves the system, in addition to the waiting disutility studied in the previous sections. The intuition is as follows. In the model without the price monitoring cost, under policy π_L , no customer has incentive to select any purchase option offered at a time later than his arrival. Therefore, by taking into account the price monitoring cost, a customer is even worse off by doing so.

To close this paper, we point out a direction that deserves further exploration. This paper gives an asymptotic analysis of the stochastic model, showing that the results and insights from the deterministic model continue to hold. It would be conceivably more challenging to theoretically analyze the stochastic problem in a non-asymptotic regime. Perhaps other types of inventory replenishment policies, such as some variants of (r, Q) policies (i.e., a fixed quantity Q is ordered when the inventory level drops to the reorder point r), could be proven effective.

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APPENDIX: Proof of Lemma 3

Before proving Lemma 3, we need to establish the following preliminary results Lemmas 5, 6 and 7. We only state their results and relegate their detailed proofs to the Electronic Companion.

LEMMA 5.

1. For any $v \in [v^*, V)$, $\theta_v \in (0, W(V)]$.
2. θ_v is non-decreasing in $v \in [0, V)$.
3. For any $K \geq 0$, if $g_v - \theta_v K \geq 0$ for some $v \in [v^*, V)$, then for all $v' \in (v, V)$, $g_{v'} - \theta_{v'} K > 0$.
4. For any $K > 0$ and any $v \in (0, V)$, if $v - W(v)K \geq 0$, then
 - (a) If $v' \in (v, V]$, then $(v' - W(v')K) - (v - W(v)K) \in (0, v' - v]$.
 - (b) If $v' \in [0, v)$, then $(v' - W(v')K) - (v - W(v)K) < 0$.

LEMMA 6. Consider any $L > 0$.

1. v_t^h is non-decreasing and continuous in $t \in [0, L]$, with $v_0^h = v^*$ and $v_t^h > v^*$ for all $t \in (0, L]$.
2. v_t^θ is non-increasing and continuous in $t \in [0, L]$, with $v_L^\theta = v^*$ and $v_t^\theta > v^*$ for all $t \in [0, L)$.

LEMMA 7. Function $v_t^\theta - W(v_t^\theta)(L - t)$ has the following properties.

1. $v_t^\theta - W(v_t^\theta)(L - t)$ is non-decreasing in $t \in [0, L]$.
2. $v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$.

With these preliminary results, we proceed to prove Lemma 3.

Proof of Lemma 3.

Due to the cyclic nature of policy π_L , without loss of generality, we only need to characterize the equilibrium behaviors of customers who arrive during the first cycle $[0, L)$.

In this proof, we introduce notation $\phi_{t'} \triangleq (t', v_\phi)$. We extend the definition of $p_t^{L,h}$ to the support $t \in (\underline{t}_L, L)$ as $p_t^{L,h} \triangleq p_t^{L,\theta} + W(\tilde{v}_t)(L - t)$.

We make the proof by taking the following steps.

Step 1: We show that for any customer $\phi \in (\bar{t}_L, L) \times [0, V)$ and any z_ϕ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi L})$.

In this scenario, at any time $t \in [t_\phi, L)$, the seller only offers the delayed delivery option $\Omega_t^{\pi L} = \{(p_t^{L,\theta}, L)\}$. Therefore,

$$\begin{aligned} U(\phi, z_\phi) &= \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \\ &\leq \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+. \end{aligned}$$

The inequality follows from Lemma 7 Part 1 that $p_t^{L,\theta} = v_t^\theta - W(v_t^\theta)(L - t)$ is non-decreasing in t .

Now, we show that the RHS is equal to $U(\phi, z_\phi^{\pi L})$. We have

$$\begin{aligned} v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) &= (v_\phi - W(v_\phi)(L - t_\phi)) - \left(v_{t_\phi}^\theta - W(v_{t_\phi}^\theta)(L - t_\phi) \right) \\ &\begin{cases} \geq 0 & \text{if } v_\phi \geq v_{t_\phi}^\theta, \\ \leq 0 & \text{if } v_\phi < v_{t_\phi}^\theta, \end{cases} \end{aligned}$$

where the equality follows from the definition of $p_t^{L,\theta}$, the inequality follows from the property that Lemma 7 Parts 1 and 2 and condition $t_\phi > \underline{t}_L$ imply $v_{t_\phi}^\theta - W(v_{t_\phi}^\theta)(L - t_\phi) \geq 0$, condition $L - t_\phi > 0$, and Lemma 5 Part 4.

Hence,

$$\begin{aligned} \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ &= \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\{v_\phi \geq v_{t_\phi}^\theta\} \\ &= U(\phi, z_\phi^{\pi L}). \end{aligned}$$

Step 2: We show that for any customer $\phi \in (\underline{t}_L, \bar{t}_L] \times [0, V)$ and any z_ϕ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi L})$.

We have

$$\begin{aligned} U(\phi, z_\phi) &\leq \max \left\{ \left(v_\phi - p_{\tau_\phi}^{L,h} - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\ &= \max \left\{ \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - \tau_\phi) - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - \tau_\phi) - W(v_\phi)(\tau_\phi - t_\phi) \right)^+ & \text{if } v_\phi \geq \tilde{v}_{\tau_\phi} \\ \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ & \text{if } v_\phi < \tilde{v}_{\tau_\phi} \end{cases} \\
&\leq \begin{cases} \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - t_\phi) \right)^+ & \text{if } v_\phi \geq \tilde{v}_{\tau_\phi} \\ \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ & \text{if } v_\phi < \tilde{v}_{\tau_\phi} \end{cases} \\
&= \left(v_\phi - p_{t_\phi}^{L,\theta} - W(\tilde{v}_{t_\phi})(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi \geq \tilde{v}_{t_\phi}\} + \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi < \tilde{v}_{t_\phi}\} \\
&= \left(v_\phi - p_{t_\phi}^{L,\theta} - W(\tilde{v}_{t_\phi})(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi \geq \tilde{v}_{t_\phi}\} + \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi \in [v_{t_\phi}^\theta, \tilde{v}_{t_\phi}]\} \\
&= \left(v_\phi - p_{t_\phi}^{L,h} \right)^+ \mathbf{1}\{v_\phi \geq \tilde{v}_{t_\phi}\} + \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi \in [v_{t_\phi}^\theta, \tilde{v}_{t_\phi}]\} \\
&= U(\phi, z_\phi^{\pi_L}).
\end{aligned}$$

The first equality follows from the definition of $p_t^{L,h}$. The second equality, the second inequality and the third equality follow from Assumption 2 Part 1 that $W(\cdot) > 0$, Part 3 that $W(v)$ is non-decreasing in v , and the property that $t_\phi \leq \tau_\phi \leq L$. The fourth equality follows from the property that Lemma 7 Parts 1 and 2 and condition $t_\phi > \underline{t}_L$ imply $p_{t_\phi}^{L,\theta} = v_{t_\phi}^\theta - W(v_{t_\phi}^\theta)(L - t_\phi) \geq 0$, the property that Lemma 1 that $\bar{t}_L < L$ and condition $t_\phi \leq \bar{t}_L$ imply $L - t_\phi > 0$, and Lemma 5 Part 4. The fifth equality follows from the definition of $p_t^{L,h}$.

Step 3: We show that for any customer $\phi \in [0, \underline{t}_L] \times [0, V)$ and any z_ϕ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi_L})$.

We prove this result by taking the following steps.

Step 3.1: Consider the case that $\tau_\phi \in [t_\phi, \underline{t}_L]$.

We have

$$\begin{aligned}
U(\phi, z_\phi) &\leq \max \left\{ \left(v_\phi - p_{\tau_\phi}^{L,h} - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - v_{\tau_\phi}^h - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left((v_\phi - W(v_\phi)(L - t_\phi)) - (v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L)) \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left((v_\phi - W(v_\phi)(L - \underline{t}_L)) - (v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L)) \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left(v_\phi - v_{\underline{t}_L}^\theta \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left(v_\phi - v_{\underline{t}_L}^h \right)^+ \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left(v_\phi - v_{t_\phi}^h \right)^+ \\
&= U \left(\phi, z_{\phi}^{\pi_L} \right).
\end{aligned}$$

The first equality follows from the definition of $p_t^{L,h}$. The second inequality follows from Lemma 6 Part 1 that v_t^h is non-decreasing in t , property $\tau_\phi \geq t_\phi$, and Assumption 1 Part 1 that $W(v) \geq 0$. The second equality follows from the definition of $p_t^{L,\theta}$. The third inequality follows from Assumption 1 Part 1 that $W(v) \geq 0$ and condition $t_\phi \leq \underline{t}_L$. The fourth inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^\theta - W \left(v_{\underline{t}_L}^\theta \right) (L - \underline{t}_L) \geq 0$, Lemma 1 that $L - \underline{t}_L > 0$, and Lemma 5 Part 4. The third equality follows from Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$. The fourth equality follows from Lemma 6 Part 1 that v_t^h is non-decreasing in t and property $t_\phi \leq \underline{t}_L$.

Step 3.2: Consider the case that $\tau_\phi \in (\underline{t}_L, L)$ and $t_\phi = \underline{t}_L$.

For any $t' \in (\underline{t}_L, \tau_\phi]$, we have

$$\begin{aligned}
U(\phi, z_\phi) &= U(\phi_{t'}, z_\phi) - W(v_\phi)(t' - \underline{t}_L) \\
&\leq U\left(\phi_{t'}, z_{\phi_{t'}}^{\pi_L}\right) - W(v_\phi)(t' - \underline{t}_L) \\
&\leq U\left(\phi_{t'}, z_{\phi_{t'}}^{\pi_L}\right) \\
&\leq \max \left\{ (v_\phi - p_{t'}^{L,h})^+, (v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t'))^+ \right\} \\
&= \max \left\{ (v_\phi - p_{t'}^{L,\theta} - W(\tilde{v}_{t'})(L - t'))^+, (v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t'))^+ \right\} \\
&\leq \max \left\{ (v_\phi - p_{t'}^{L,\theta} - W(v_{t'}^h)(L - t'))^+, (v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t'))^+ \right\}.
\end{aligned}$$

The first inequality follows from results that we prove in Steps 1-2 above. The second inequality follows from Assumption 1 Part 1 that $W(v) > 0$. The second equality follows from the definition of $p_t^{L,h}$. The fourth inequality follows from Lemma 1 that $\tilde{v}_t \geq v_t^h$ if $t > \underline{t}_L$ and Assumption 1 Part 3 that $W(v)$ is non-decreasing in v .

Therefore,

$$\begin{aligned}
U(\phi, z_\phi) &\leq \limsup_{t' \rightarrow \underline{t}_L^+} \max \left\{ (v_\phi - p_{t'}^{L,\theta} - W(v_{t'}^h)(L - t'))^+, (v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t'))^+ \right\} \\
&= \max \left\{ (v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_{\underline{t}_L}^h)(L - \underline{t}_L))^+, (v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - \underline{t}_L))^+ \right\} \\
&= \max \left\{ (v_\phi - v_{\underline{t}_L}^h)^+, \left((v_\phi - W(v_\phi)(L - \underline{t}_L)) - (v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L)) \right)^+ \right\} \\
&\leq \max \left\{ (v_\phi - v_{\underline{t}_L}^h)^+, (v_\phi - v_{\underline{t}_L}^\theta)^+ \right\} \\
&= \max \left\{ (v_\phi - v_{\underline{t}_L}^h)^+, (v_\phi - v_{\underline{t}_L}^h)^+ \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left(v_\phi - v_{\underline{t}_L}^h \right)^+ \\
&= U \left(\phi, z_\phi^{\pi_L} \right).
\end{aligned}$$

The first equality follows from Lemma 6 Part 1 that v_t^h is continuous in t , Part 2 that v_t^θ is continuous in t , and Assumption 1 Part 2 that $W(v)$ is continuous in v . The second equality follows from the definition of $p_{\underline{t}_L}^{L,\theta}$ and Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$. The second inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^\theta - W \left(v_{\underline{t}_L}^\theta \right) (L - \underline{t}_L) \geq 0$, Lemma 1 that $L - \underline{t}_L > 0$, and Lemma 5 Part 4(a). The third equality follows from Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$.

Step 3.3: Consider the case that $\tau_\phi \in (\underline{t}_L, L)$ and $t_\phi \in [0, \underline{t}_L)$.

We have

$$\begin{aligned}
U(\phi, z_\phi) &= U(\phi_{\underline{t}_L}, z_\phi) - W(v_\phi)(\underline{t}_L - t_\phi) \\
&\leq U(\phi_{\underline{t}_L}, z_{\phi_{\underline{t}_L}}^{\pi_L}) - W(v_\phi)(\underline{t}_L - t_\phi) = U(\phi, z_\phi^{\pi_L}) \leq U(\phi, z_\phi^{\pi_L}).
\end{aligned}$$

The first inequality follows from the result that we prove in Step 3.2 above. The second inequality follows from the result that we prove in Step 3.1 above.

Therefore, all results in Steps 3.1-3.3 jointly imply $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi_L})$.

Step 4: We show that for any customer $\phi \in [0, L) \times [0, V)$, for any z_ϕ with $\tau_\phi = L$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi_L})$. We have

$$\lim_{t \rightarrow L^-} p_t^{L,\theta} = \lim_{t \rightarrow L^-} v_t^\theta - W(v_t^\theta)(L - t) = v^* = v_0^h = p_0^{L,h},$$

where the first equality follows from the definition of $p_t^{L,\theta}$, the second equality follows from Lemma 6 Part 2 that v_t^θ is continuous in t and $v_L^\theta = v^*$, the third equality follows from Lemma 6 Part 1, the fourth equality follows from the definition of $p_0^{L,h}$.

Therefore, this result, the definition of π_L that $\Omega_L^{\pi_L} = \{(p_0^{L,h}, L)\}$, and all results that we prove in Steps 1-3 jointly imply $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi_L})$.

Step 5: We show that for any customer $\phi \in [0, L) \times [0, V)$, for any z_ϕ with $\tau_\phi = nL + \tau$ for $n \in \mathbb{N}$ and $\tau \in (0, L]$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\pi_L})$.

First, for any $n \in \mathbb{N}$, we have

$$U(\phi, z_\phi) = U(\phi_{nL}, z_\phi) + W(v_\phi)nL \leq U(\phi_{nL}, z_{\phi_{nL}}^{\pi_L}) + W(v_\phi)nL = U(\phi, z_{\phi_{nL}}^{\pi_L}),$$

where the inequality follows from all results that we prove in Steps 1-4 above.

Second, for any $n \in \mathbb{N}$, we have

$$U\left(\phi, z_{\phi(n+1)L}^{\pi_L}\right) = U\left(\phi_{nL}, z_{\phi(n+1)L}^{\pi_L}\right) + W(v_\phi)nL \leq U\left(\phi_{nL}, z_{\phi_{nL}}^{\pi_L}\right) + W(v_\phi)nL = U\left(\phi, z_{\phi_{nL}}^{\pi_L}\right),$$

where the inequality follows from the result that we prove in Step 4 above.

Therefore, for any z_ϕ with $\tau_\phi = nL + \tau$ for $n \in \mathbb{N}$ and $\tau \in (0, L]$, we have

$$U(\phi, z_\phi) \leq U\left(\phi, z_{\phi_{nL}}^{\pi_L}\right) \leq U\left(\phi, z_{\phi_L}^{\pi_L}\right) \leq U\left(\phi, z_\phi^{\pi_L}\right),$$

where the first inequality follows from the first result that we prove in this step, the second inequality follows from the second result that we prove in this step, the third inequality follows from the result that we prove in Step 4 above.

All results that we prove in Steps 1-5 complete the proof of this lemma.

Q.E.D.

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Electronic Companion to

“Joint Pricing and Inventory Management with Strategic Customers”

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EC.1. Proofs of Lemmas 5, 6 and 7

Proof of Lemma 5.

Part 1.

For any $v \in [v^*, V)$, we have

$$\theta_v = W(v) - \frac{\bar{F}(v)}{f(v)}w(v) = W(v) - (v - g_v)w(v) = W(v) \left(1 - \frac{vw(v)}{W(v)}\right) + g_v w(v) > 0.$$

The inequality holds due to the following reasons. First, Assumption 1 Part 1 assumes $W(v) > 0$ for all $v \in [0, V)$. Second, Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$. Third, the properties that g_v is strictly increasing in v and $g_{v^*} = 0$ imply $g_v \geq 0$ for $v \in [v^*, V)$. Fourth, Assumption 1 Part 3 that $W(v)$ is non-decreasing in v implies $w(v) \geq 0$.

For any $v \in [v^*, V)$, we have

$$\theta_v = W(v) - \frac{\bar{F}(v)}{f(v)}w(v) \leq W(v) \leq W(V).$$

The first and the second inequalities follow from Assumption 1 Part 3 that $W(v)$ is non-decreasing in v .

Part 2.

For any $v, v' \in [0, V)$ with $v' > v$, we have

$$\theta_{v'} = W(v') - \frac{\bar{F}(v')}{f(v')}w(v') \geq W(v) - \frac{\bar{F}(v)}{f(v)}w(v) = \theta_v.$$

The inequality holds due to the following reasons. First, Assumption 1 Part 3 that $W(v)$ is non-decreasing and concave in v and condition $v' > v$ imply $W(v') \geq W(v)$ and $0 \leq w(v') \leq w(v)$. Second, Assumption 2 and condition $v' > v$ imply $\frac{\bar{F}(v')}{f(v')} \leq \frac{\bar{F}(v)}{f(v)}$.

Part 3.

Consider the first case that $K = 0$.

We have $g_v - \theta_v K = g_v$. Note that g_v is strictly increasing in v and $g_{v^*} = 0$. Hence, for any $v^* \leq v < v' < V$, $g_{v'} > g_v \geq 0$.

Consider the second case that $K > 0$.

We notice that $g_{v^*} - \theta_{v^*}K = -\theta_{v^*}K < 0$, where the inequality follows from Part 1 in this lemma that $\theta_{v^*} > 0$. Hence, to prove the statement in this part, we only need to consider $v, v' \in (v^*, V)$ with $v < v'$.

Consider any $v, v' \in (v^*, V)$ with $v < v'$. Suppose $g_v - \theta_v K \geq 0$. We notice that

$$\frac{1}{v}(g_v - \theta_v K) = \left(1 - \frac{\bar{F}(v)}{vf(v)}\right)(1 - w(v)K) - \frac{W(v)}{v} \left(1 - \frac{vw(v)}{W(v)}\right)K.$$

First, we note that Assumption 1 Part 1 assumes that $W(v) > 0$ for $v \in [0, V]$, Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$, and $K > 0$. Hence, the second term on the RHS is strictly positive. Second, we note that the properties that g_v is strictly increasing in v , $g_{v^*} = 0$, and condition $v > v^*$ jointly imply $1 - \frac{\bar{F}(v)}{vf(v)} = \frac{g_v}{v} > \frac{g_{v^*}}{v^*} = 0$. Therefore, $1 - w(v)K > 0$.

For v' , we have

$$\begin{aligned} \frac{1}{v'}(g_{v'} - \theta_{v'}K) &= \left(1 - \frac{\bar{F}(v')}{v'f(v')}\right)(1 - w(v')K) - \frac{W(v')}{v'} \left(1 - \frac{v'w(v')}{W(v')}\right)K \\ &> \left(1 - \frac{\bar{F}(v)}{vf(v)}\right)(1 - w(v)K) - \frac{W(v)}{v} \left(1 - \frac{vw(v)}{W(v)}\right)K \\ &= \frac{1}{v}(g_v - \theta_v K) \\ &\geq 0. \end{aligned}$$

The first inequality holds due to following reasons. First, Assumption 2, condition $v' > v > v^*$, and the property that g_v is strictly increasing in v with $g_{v^*} = 0$ imply $1 - \frac{\bar{F}(v')}{v'f(v')} > 1 - \frac{\bar{F}(v)}{vf(v)} = \frac{g_v}{v} > \frac{g_{v^*}}{v^*} \geq 0$. Second, Assumption 1 Part 3 that $W(v)$ is concave in v and condition $v' > v$ imply $1 - w(v')K \geq 1 - w(v)K > 0$. Third, Assumption 1 Part 1 that $W(v) > 0$ for $v \in [0, V]$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$ jointly imply $\frac{d}{dv} \frac{W(v)}{v} = -\frac{W(v)}{v^2} \left(1 - \frac{vw(v)}{W(v)}\right) < 0$. Hence, condition $v' > v$ implies $\frac{W(v')}{v'} < \frac{W(v)}{v}$. Fourth, the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$, Assumption 1 Part 4 that $\frac{vw(v)}{W(v)}$ is non-decreasing in v and condition $v' > v$ jointly imply $0 < 1 - \frac{v'w(v')}{W(v')} \leq 1 - \frac{vw(v)}{W(v)}$.

To summarize two cases above, for any $v, v' \in [v^*, V]$ with $v < v'$, if $g_v - \theta_v K \geq 0$, then $g_{v'} - \theta_{v'} K > 0$.

Part 4.

Consider any $K > 0$ and any $v \in (0, V)$. Suppose $v - W(v)K \geq 0$.

Consider the first case that $v' \in (v, V]$.

We have

$$(v' - W(v')K) - (v - W(v)K) = \frac{v'}{v} \left(v - v \frac{W(v')}{v'} K \right) - (v - W(v)K) > \left(\frac{v'}{v} - 1 \right) (v - W(v)K) \geq 0.$$

The first inequality holds due the following reasons. First, Assumption 1 Part 1 that $W(v) > 0$ for $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$ jointly imply $\frac{d}{dv} \frac{W(v)}{v} = -\frac{W(v)}{v^2} \left(1 - \frac{vw(v)}{W(v)} \right) < 0$. Hence, condition $v' > v$ implies $\frac{W(v')}{v'} < \frac{W(v)}{v}$. Second, we have $K > 0$. The second inequality follows from condition $v' \geq v$ and property $v - W(v)K \geq 0$.

In addition, we have

$$(v' - W(v')K) - (v - W(v)K) = v' - v - (W(v') - W(v))K \leq v' - v,$$

where the inequality follows from Assumption 1 Part 3 that $W(v)$ is non-decreasing in v , condition $v' > v$, and condition $K > 0$.

Consider the second case that $v' \in [0, v)$.

If $v' > 0$, then we have

$$(v' - W(v')K) - (v - W(v)K) = \frac{v'}{v} \left(v - v \frac{W(v')}{v'} K \right) - (v - W(v)K) < \left(\frac{v'}{v} - 1 \right) (v - W(v)K) \leq 0.$$

The first inequality holds due the following reasons. First, Assumption 1 Part 1 that $W(v) > 0$ for $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$ jointly imply $\frac{d}{dv} \frac{W(v)}{v} = -\frac{W(v)}{v^2} \left(1 - \frac{vw(v)}{W(v)} \right) < 0$. Hence, condition $v' < v$ implies $\frac{W(v')}{v'} > \frac{W(v)}{v}$. Second, we have $K > 0$. The second inequality follows from condition $v' < v$ and property $v - W(v)K \geq 0$.

If $v' = 0$, then we have

$$(v' - W(v')K) - (v - W(v)K) = -W(0)K - (v - W(v)K) \leq -W(0)K < 0,$$

where the first inequality follows from the assumption that $v - W(v)K \geq 0$, the second inequality follows from Assumption 1 Part 1 that $W(0) > 0$ and the condition $K > 0$.

The analysis of three cases above completes the proof of this part.

Q.E.D.

Proof of Lemma 6.

Part 1.

Because g_v is strictly increasing and continuous in v , we have $v_t^h = g^{-1}(\min\{ht, g_V\})$, which is non-decreasing and continuous in $t \in [0, L]$, with $v_0^h = g^{-1}(0) = v^*$ and $v_t^h > g^{-1}(0) = v^*$ for all $t \in (0, L]$.

Part 2.

Consider any $t, t' \in [0, L]$ with $t > t'$. For any $v \in [v^*, V)$, if $g_v - \theta_v(L - t) < 0$, then $g_v - \theta_v(L - t') = g_v - \theta_v(L - t) - \theta_v(t - t') < 0$, where the inequality follows from Lemma 5 Part 1 that $\theta_v > 0$ for $v \in [v^*, V)$ and condition $t > t'$. Therefore, v_t^θ is non-increasing in $t \in [0, L]$.

Because g_v is strictly increasing in v and $g_{v^*} = 0$, we have $v_L^\theta = \sup\{v \in [v^*, V) : g_v < 0\} = v^*$.

Suppose there exists $t \in [0, L)$, such that $v_t^\theta = v^*$. Hence, the definition of v_t^θ and the property that $g_v - \theta_v(L - t)$ is continuous in v imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) \geq 0$. We note that the condition $v_t^\theta = v^*$ implies $g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) = g_{v^*} - \theta_{v^*}(L - t) = -\theta_{v^*}(L - t)$. Thus, $\theta_{v^*} \leq 0$. This contradicts with Lemma 5 Part 1 that $\theta_v > 0$ for $v \in [v^*, V)$. Therefore, we have $v_t^\theta > v^*$ for all $t \in [0, L)$.

Next, we prove that v_t^θ is continuous in $t \in [0, L]$.

First, we prove that v_t^θ is right-continuous in $t \in [0, L]$.

Consider any $t \in [0, L)$. Because $t < L$, $v_t^\theta > v^*$. Consider any $\epsilon \in (0, v_t^\theta - v^*]$. The definition of v_t^θ and the property that $g_v - \theta_v(L - t)$ is continuous in v imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) \leq 0$. Hence, Lemma 5 Part 3 implies that for any $v \in [v^*, v_t^\theta)$, $g_v - \theta_v(L - t) < 0$. Define $\Delta \triangleq -\sup_{v \in [v^*, v_t^\theta - \epsilon]} (g_v - \theta_v(L - t)) > 0$. Define $\delta \triangleq \frac{\Delta}{2W(V)}$. Note that Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and the condition $V < \infty$ imply $W(V) \in (0, \infty)$. Hence, $\delta \in (0, \infty)$.

Therefore, for any $t' \in (t, \min\{t + \delta, L\}]$ and any $v \in [v^*, v_t^\theta - \epsilon]$,

$$g_v - \theta_v(L - t') = g_v - \theta_v(L - t) + \theta_v(t' - t) \leq -\Delta + W(V)\delta = -\frac{\Delta}{2} < 0,$$

where the inequality follows from Lemma 5 Part 1 that $\theta_v \leq W(V)$. Therefore, v_t^θ is right-continuous in t .

Second, we prove that v_t^θ is left-continuous in $t \in [0, L]$.

Note that v_t^θ is non-increasing in t and $v_t^\theta \leq V$. Hence, if there exists $t \in [0, L]$, such that $v_t^\theta = V$. Then for all $t' \in [0, t)$, $v_{t'}^\theta = V$. This implies that v_t^θ is left-continuous in t .

Now, we consider $t \in (0, L]$ with $v_t^\theta < V$. Consider any $\epsilon \in (0, V - v_t^\theta)$. The definition of v_t^θ and the property that $g_v - \theta_v(L - t)$ is continuous in v imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) \geq 0$. Hence, Lemma 5 Part 3 implies that for any $v \in (v_t^\theta, V)$, $g_v - \theta_v(L - t) > 0$. Define $\Delta \triangleq \inf_{v \in [v_t^\theta + \epsilon, V)} (g_v - \theta_v(L - t)) > 0$. Define $\delta \triangleq \frac{\Delta}{2W(V)}$. Note that Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and the condition $V < \infty$ imply $W(V) \in (0, \infty)$. Hence, $\delta \in (0, \infty)$.

Therefore, for any $t' \in \left[(t - \delta)^+, t\right)$ and any $v \in [v_t^\theta + \epsilon, V)$,

$$g_v - \theta_v(L - t') = g_v - \theta_v(L - t) - \theta_v(t - t') \geq \Delta - W(V)\delta = \frac{\Delta}{2} > 0,$$

where the inequality follows from Lemma 5 Part 1 that $\theta_v \leq W(V)$. Therefore, v_t^θ is left-continuous in t .

Because v_t^θ is both right-continuous and left-continuous in t , it is continuous in t .

Q.E.D.

Proof of Lemma 7.

Part 1.

Define $s \triangleq \sup \{t \in [0, L] : v_t^\theta = V\}$. Following from Lemma 6 Part 2, we have that $v_t^\theta \in (v^*, V)$ for all $t \in (s, L)$, and $v_t^\theta = V$ for all $t \in [0, s)$. Following from the definition of v_t^θ and the property that $g_v - \theta_v(L - t)$ is continuous in v , we have that for all $t \in (s, L)$, v_t^θ satisfies condition $g_{v_t^\theta} = \theta_{v_t^\theta}(L - t)$.

Therefore, for any $t \in (s, L)$, we have

$$v_t^\theta - W(v_t^\theta)(L - t) = v_t^\theta - W(v_t^\theta) \frac{g_{v_t^\theta}}{\theta_{v_t^\theta}} = v_t^\theta - \frac{v_t^\theta - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)}}{1 - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)} \frac{w(v_t^\theta)}{W(v_t^\theta)}} = \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)} \frac{1 - \frac{v_t^\theta w(v_t^\theta)}{W(v_t^\theta)}}{1 - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)} \frac{w(v_t^\theta)}{W(v_t^\theta)}}.$$

The RHS term is non-decreasing in $t \in (s, L)$ due to following reasons. First, we have Assumption 2 that $\frac{f(v)}{\bar{F}(v)}$ is non-decreasing in v . Second, we have Assumption 1 Part 4 that $\frac{vw(v)}{W(v)}$ is non-decreasing in v and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$. Third, following from Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and Part 3 that $W(v)$ is concave in v , we have $\frac{d}{dv} \frac{w(v)}{W(v)} = \frac{W(v)w'(v) - w(v)^2}{W(v)^2} \leq 0$. Fourth, Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and Lemma 5 Part 1 that $\theta_v > 0$ for all $v \geq v^*$ imply that $1 - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)} \frac{w(v_t^\theta)}{W(v_t^\theta)} = \frac{\theta_{v_t^\theta}}{W(v_t^\theta)} > 0$ for $v \geq v^*$. Fifth, we have Lemma 6 Part 2 that v_t^θ is non-increasing in t . Therefore, all properties above jointly imply that $v_t^\theta - W(v_t^\theta)(L - t)$ is non-decreasing in $t \in (s, L)$.

For $t \in [0, s)$, because $v_t^\theta = V$ and Assumption 1 Part 1 assumes that $W(v) \geq 0$ for all $v \in [0, V)$, we have that $v_t^\theta - W(v_t^\theta)(L - t) = V - W(V)(L - t)$ is non-decreasing in $t \in [0, s)$.

We also note from Lemma 6 Part 2 that v_t^θ is continuous at $t = s$ and L . Therefore, all results above imply that $v_t^\theta - W(v_t^\theta)(L - t)$ is non-decreasing in $t \in [0, L]$.

Part 2.

Following from Lemma 6 Part 1 that v_t^h is non-decreasing in t and Part 2 that v_t^θ is non-increasing in t and the definition of \underline{t}_L , we have that $v_t^\theta < V$ for all $t \in (\underline{t}_L, L]$. In addition, we have Lemma 6

Part 2 that $v_t^\theta > v^*$ for $t < L$ and Lemma 1 that $\underline{t}_L < L$. Hence, $v_t^\theta \in (v^*, V)$ for all $t \in (\underline{t}_L, L)$. The definition of v_t^θ and the property that $g_v - \theta_v(L - t)$ is continuous in v imply that v_t^θ satisfies the condition $g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) = 0$.

We notice that

$$g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) = \frac{1}{v_t^\theta} g_{v_t^\theta} (v_t^\theta - W(v_t^\theta)(L - t)) - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)} \frac{W(v_t^\theta)}{v_t^\theta} \left(1 - \frac{v_t^\theta w(v_t^\theta)}{W(v_t^\theta)}\right) (L - t).$$

We note that the properties that g_v is strictly increasing in v and $g_{v^*} = 0$ imply $g_{v_t^\theta} > 0$. In addition, we have Assumption 1 Part 1 that $W(v) \geq 0$ for all $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$. Therefore, for any $t \in (\underline{t}_L, L)$, $v_t^\theta - W(v_t^\theta)(L - t) \geq 0$. Following from Lemma 6 Part 2 that v_t^θ is continuous in t , we have $v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$.

Q.E.D.

EC.2. Proof of Proposition 1

We provide the proof of the main result in §3.1, Proposition 1. We begin with proving the first inequality in Proposition 1.

LEMMA EC.1. *We have*

$$J^* \leq \bar{J}.$$

Proof of Lemma EC.1.

Proving this lemma requires us to compare the policy defined in §2 and the mechanism design problem defined in §3. Note that some notation that appears in both contexts have different definitions, such as z_ϕ . Therefore, to distinguish the differences of the same notation that appears in both contexts, throughout this proof, we add ‘ \wedge ’ on the notation that appears in the mechanism design context.

We use the seller’s any feasible policy $\pi \in \Pi$ and customer corresponding purchasing rules $z^\pi = (\tau^\pi, p^\pi, s^\pi)$ to construct the following mechanism $(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi)$:

$$\hat{Q}^{\infty, \pi} = Q^{\infty, \pi}, \quad \hat{z}_\phi^\pi = z_\phi^\pi \forall \phi, \quad \hat{q}_t^\pi = q_t^\pi \forall t \geq 0.$$

Because mechanism $(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi)$ replicates the seller’s decisions and customer purchase decisions under π , we have $(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi) \in \mathcal{M}$ and $\Pi(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi) = J^{\pi, z^\pi}$.

Now, we show $(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi)$ satisfies the (IC) and (IR) constraints in (2).

For any ϕ and $\phi_{v'}$, we have

$$U(\phi, \hat{z}_\phi^\pi) = U(\phi, z_\phi^\pi) \geq U(\phi, z_{\phi_{v'}}^\pi) = U(\phi, \hat{z}_{\phi_{v'}}^\pi),$$

where the first and the second equalities follow from property $\hat{z}^\pi = z^\pi$, the inequality follows from the property that z_ϕ^π is the optimal solution of customer ϕ 's optimization problem. Therefore, mechanism $(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi)$ satisfies the (IC) constraints in (2).

For any ϕ , define $z_\phi^0 \triangleq (t_\phi, p_\phi^0, s_\phi^0)$. We have

$$U(\phi, \hat{z}_\phi^\pi) = U(\phi, z_\phi^\pi) \geq U(\phi, z_\phi^0) = 0,$$

where the first equality follows from property $\hat{z}^\pi = z^\pi$, the first inequality follows from the property that z_ϕ^π is the optimal solution of customer ϕ 's optimization problem. Therefore, mechanism $(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi)$ satisfies the (IR) constraints in (2).

Therefore, for any $\pi \in \Pi$, we have $J^{\pi, z^\pi} = \Pi(\hat{z}^\pi, \hat{Q}^{\infty, \pi}, \hat{q}^\pi) \leq \bar{J}$. **Q.E.D.**

Next, we prove a series of lemmas that will jointly imply the second inequality in Proposition 1. To lighten the notation in presenting and proving Lemmas EC.2-EC.5, we introduce the following notation

$$a_\phi \triangleq \mathbf{1}\{s_\phi < \infty\}.$$

We begin with establishing the following two lemmas.

LEMMA EC.2. *If (IC) and (IR) hold, then for any ϕ ,*

$$p_\phi = (v_\phi - W(v_\phi)(s_\phi - t_\phi)) a_\phi - \int_{v'=0}^{v_\phi} (1 - w(v')(s_{\phi_{v'}} - t_{\phi_{v'}})) a_{\phi_{v'}} dv'.$$

Proof of Lemma EC.2.

First, we show that (IR) implies that

$$U(\phi_0, z_{\phi_0}) = 0. \tag{EC.1}$$

To see this notice that by definition and Assumption 1 Part 1 that $W(0) > 0$,

$$U(\phi_0, z_{\phi_0}) = (0 - p_{\phi_0} - W(0)(s_{\phi_0} - t_{\phi_0})) a_{\phi_0} \leq 0.$$

But since (IR) requires $U(\phi_0, z_{\phi_0}) \geq 0$, we must have (EC.1).

Now, define $u(\phi, z) \triangleq \frac{\partial}{\partial v_\phi} U(\phi, z)$. Applying the envelope theorem, we have:

$$\begin{aligned} U(\phi, z_\phi) &= \int_{v'=0}^{v_\phi} u(\phi_{v'}, z_{\phi_{v'}}) dv' + U(\phi_0, z_{\phi_0}) \\ &= \int_{v'=0}^{v_\phi} (1 - w(v')(s_{\phi_{v'}} - t_{\phi_{v'}})) a_{\phi_{v'}} dv' + U(\phi_0, z_{\phi_0}) \\ &= \int_{v'=0}^{v_\phi} (1 - w(v')(s_{\phi_{v'}} - t_{\phi_{v'}})) a_{\phi_{v'}} dv'. \end{aligned} \quad (\text{EC.2})$$

The first equality follows from Fubini's theorem and the envelope theorem (specifically, Theorem 2 of [Milgrom and Segal \(2002\)](#)). The second equality follows from the definition of $u(\cdot)$, and the final equality follows from [\(EC.1\)](#). Consequently,

$$\begin{aligned} p_\phi &= (v_\phi - W(v_\phi)(s_\phi - t_\phi)) a_\phi - U(\phi, z_\phi) \\ &= (v_\phi - W(v_\phi)(s_\phi - t_\phi)) a_\phi - \int_{v'=0}^{v_\phi} (1 - w(v')(s_{\phi_{v'}} - t_{\phi_{v'}})) a_{\phi_{v'}} dv'. \end{aligned}$$

The first equality follows from the definition of $U(\cdot)$. The second equality follows from our application of the envelope theorem above. **Q.E.D.**

The next lemma establishes a second implication of the constraints (IC) and (IR).

LEMMA EC.3. *If (IC) and (IR) hold, then for any ϕ with $v_\phi > 0$, we have:*

$$(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi \geq 0.$$

Proof of Lemma EC.3.

Consider any ϕ and any $v, v' \in [0, V]$. (IC) implies

$$\begin{aligned} U(\phi_v, z_{\phi_v}) &\geq U(\phi_v, z_{\phi_{v'}}), \\ U(\phi_{v'}, z_{\phi_{v'}}) &\geq U(\phi_{v'}, z_{\phi_v}). \end{aligned}$$

Adding these two inequalities, and writing them explicitly (using the definition of $U(\cdot)$), yields:

$$\begin{aligned} (v - v')(a_{\phi_v} - a_{\phi_{v'}}) &\geq (W(v) - W(v'))(s_{\phi_v} - t_{\phi_v}) a_{\phi_v} - (W(v) - W(v'))(s_{\phi_{v'}} - t_{\phi_{v'}}) a_{\phi_{v'}} \\ &\geq w(v)(v - v')(s_{\phi_v} - t_{\phi_v}) a_{\phi_v} - w(v')(v - v')(s_{\phi_{v'}} - t_{\phi_{v'}}) a_{\phi_{v'}} \\ &= (v - v')(w(v)(s_{\phi_v} - t_{\phi_v}) a_{\phi_v} - w(v')(s_{\phi_{v'}} - t_{\phi_{v'}}) a_{\phi_{v'}}), \end{aligned}$$

where the second inequality follows from Assumption 1 Part 3 that $W(v)$ is concave in v . Thus,

$$(v - v')((1 - w(v)(s_{\phi_v} - t_{\phi_v})) a_{\phi_v} - (1 - w(v')(s_{\phi_{v'}} - t_{\phi_{v'}})) a_{\phi_{v'}}) \geq 0.$$

Therefore, $(1 - w(v)(s_{\phi_v} - t_\phi))a_{\phi_v}$ is non-decreasing in v . But (EC.2) and (IR) imply that for any $v_\phi \geq 0$,

$$U(\phi, z_\phi) = \int_{v'=0}^{v_\phi} (1 - w(v')(s_{\phi_{v'}} - t_\phi))a_{\phi_{v'}}dv' \geq 0,$$

which with the fact that $(1 - w(v)(s_{\phi_v} - t_\phi))a_{\phi_v}$ is non-decreasing in v immediately lets us conclude that

$$(1 - w(v)(s_{\phi_v} - t_\phi))a_{\phi_v} \geq 0$$

for all $v > 0$.

Q.E.D.

Now, we use Lemmas EC.2 and EC.3 to establish an upper bound of the optimal mechanism design problem (2). This upper bound only requires us to solve a pure dynamic optimization problem without (IC) constraints.

LEMMA EC.4. *If (IC) and (IR) hold, then*

$$\Pi(z, \mathcal{Q}, q) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\int_{\phi \in \mathcal{H}^T} (g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi - h \int_{t=0}^T I_t dt - K|\mathcal{Q}^T| \right).$$

Proof of Lemma EC.4.

For any $T > 0$, we have

$$\begin{aligned} & \int_{\phi \in \mathcal{H}^T} p_\phi \mathbf{1}\{\tau_\phi \leq T\} \\ & \leq \int_{\phi \in \mathcal{H}^T} p_\phi \\ & = \int_{\phi \in \mathcal{H}^T} \left((v_\phi - W(v_\phi)(s_\phi - t_\phi))a_\phi - \int_{v'=0}^{v_\phi} a_{\phi_{v'}}dv' + \int_{v'=0}^{v_\phi} w(v')(s_{\phi_{v'}} - t_{\phi_{v'}})a_{\phi_{v'}}dv' \right) \quad (\text{EC.3}) \end{aligned}$$

where the inequality follows from the properties that $p_\phi \geq 0$ and $\tau_\phi \geq t_\phi$, the second equality follows from Lemma EC.2.

For the third term in (EC.3), we have

$$\begin{aligned} \int_{\phi \in \mathcal{H}^T} \int_{v'=0}^{v_\phi} a_{\phi_{v'}}dv' &= \int_{t=0}^T \lambda \int_{v=0}^V f(v) \int_{v'=0}^v a_{(t,v')}dv' dv dt \\ &= \int_{t=0}^T \lambda \int_{v'=0}^V a_{(t,v')} \int_{v=v'}^V f(v) dv dv' dt \\ &= \int_{t=0}^T \lambda \int_{v'=0}^V \frac{\bar{F}(v')}{f(v')} a_{(t,v')} f(v') dv' dt \\ &= \int_{\phi \in \mathcal{H}^T} \frac{\bar{F}(v_\phi)}{f(v_\phi)} a_\phi. \end{aligned}$$

Here the first equality follows from the fact that v_ϕ is independent of t_ϕ , the second equality follows from an exchange in the order of integration, and the fourth equality again employs the fact that v_ϕ is independent of t_ϕ .

Following the similar analysis, for the fourth term in (EC.3), we have

$$\int_{\phi \in \mathcal{H}^T} \int_{v'=0}^{v_\phi} w(v') (s_{\phi_{v'}} - t_{\phi_{v'}}) a_{\phi_{v'}} dv' = \int_{\phi \in \mathcal{H}^T} \frac{\bar{F}(v_\phi)}{f(v_\phi)} w(v_\phi) (s_\phi - t_\phi) a_\phi.$$

Therefore,

$$\Pi(z, \mathcal{Q}, q) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\int_{\phi \in \mathcal{H}^T} (g_{v_\phi} - \theta_{v_\phi} (s_\phi - t_\phi)) a_\phi - h \int_{t=0}^T I_t dt - K |\mathcal{Q}^T| \right).$$

Q.E.D.

We denote by \hat{J} the optimal value of the following optimization problem:

$$\begin{aligned} \max_{(z, \mathcal{Q}, q) \in \mathcal{M}} \quad & \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\int_{\phi \in \mathcal{H}^T} (g_{v_\phi} - \theta_{v_\phi} (s_\phi - t_\phi)) a_\phi - h \int_{t=0}^T I_t dt - K |\mathcal{Q}^T| \right) \\ \text{subject to} \quad & (1 - w(v_\phi) (s_\phi - t_\phi)) a_\phi \geq 0, \forall \phi \text{ with } v_\phi > 0. \end{aligned} \quad (\text{EC.4})$$

Therefore, Lemmas EC.3 and EC.4 immediately imply $\bar{J} \leq \hat{J}$.

In the next step, we establish an upper bound of \hat{J} . We introduce the following notation. We denote by $L_n(\mathcal{Q}^\infty)$ the n -th inventory replenishment time in the replenishment schedule \mathcal{Q}^∞ . We make a convention that $L_0(\mathcal{Q}^\infty) = 0$. We denote by $\mathcal{H}_n \triangleq \{\phi : t_\phi \in [L_{n-1}(\mathcal{Q}^\infty), L_n(\mathcal{Q}^\infty))\}$ the collection of customers who arrive between the $(n-1)$ -th and the n -th inventory replenishment times. To ease notation, in the rest of this paper, unless we cause confusion, we suppress the argument \mathcal{Q}^∞ in $L_n(\mathcal{Q}^\infty)$.

LEMMA EC.5. *We have*

$$\hat{J} \leq \max_{\mathcal{Q}^\infty} J^{\mathcal{Q}^\infty},$$

where

$$J^{\mathcal{Q}^\infty} \triangleq \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} (g_{v_\phi} - \min \{h(t_\phi - L_{n-1}), \theta_{v_\phi}(L_n - t_\phi)\})^+ \mathbf{1}\{v_\phi \geq v^*\} - KN \right).$$

Proof of Lemma EC.5.

We complete the proof by taking the following steps.

First, under the demand fulfillment commitment, for any $N \in \mathbb{N}$, we have

$$\int_{t=L_{N-1}}^{L_N} I_t dt \geq \int_{\phi \in \mathcal{H}_n} (s_\phi - L_{n-1}) \mathbf{1}\{s_\phi < L_n\}.$$

Second, we show that for any $v_\phi \in (0, v^*)$, under the constraint that $(1 - w(v_\phi)(s_\phi - t_\phi))a_\phi \geq 0$, we have

$$(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi \leq 0.$$

Suppose this statement is not true for some ϕ with $v_\phi \in (0, v^*)$. Hence, we have

$$\begin{aligned} (g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi &= g_{v_\phi}(1 - w(v_\phi)(s_\phi - t_\phi))a_\phi - W(v_\phi) \left(1 - \frac{v_\phi w(v_\phi)}{W(v_\phi)}\right) (s_\phi - t_\phi) a_\phi \\ &> 0. \end{aligned}$$

Note that we have the following properties. First, following from Assumption 2, we have that g_{v_ϕ} is strictly increasing in v_ϕ . Hence, for $v_\phi < v^*$, $g_{v_\phi} < 0$. Second, Assumption 1 Part 1 that $W(v) > 0$ for $v \geq 0$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{v_\phi w(v_\phi)}{W(v_\phi)} < 1$ jointly imply $W(v_\phi) \left(1 - \frac{v_\phi w(v_\phi)}{W(v_\phi)}\right) > 0$. Hence, the two properties above imply $(1 - w(v_\phi)(s_\phi - t_\phi))a_\phi < 0$. This contradicts with the property $(1 - w(v_\phi)(s_\phi - t_\phi))a_\phi \geq 0$. Therefore, for any $v_\phi \in (0, v^*)$, under the constraint that $(1 - w(v_\phi)(s_\phi - t_\phi))a_\phi \geq 0$, we have

$$(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi \leq 0.$$

Third, following from the results that we derive from the previous two steps, under the constraint that $(1 - w(v_\phi)(s_\phi - t_\phi))a_\phi \geq 0$ for any $v_\phi > 0$, we have

$$\begin{aligned} J^{\mathcal{Q}^\infty} &= \liminf_{T \rightarrow \infty} \frac{1}{T} \left(\int_{\phi \in \mathcal{H}^T} (g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi - h \int_{t=0}^T I_t dt - K|\mathcal{Q}^T| \right) \\ &= \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} (g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi - h \int_{t=L_{N-1}}^{L_N} I_t dt - KN \right) \\ &\leq \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} (g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi - h(s_\phi - L_{n-1}) \mathbf{1}\{s_\phi < L_n\} - KN \right) \\ &\leq \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} \left((g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi \right. \right. \\ &\quad \left. \left. - h(s_\phi - L_{n-1}) \mathbf{1}\{s_\phi < L_n\} \right) \mathbf{1}\{v_\phi \geq v^*\} - KN \right) \\ &= \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} \left((g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi))a_\phi \right. \right. \end{aligned}$$

$$\begin{aligned}
& -h(s_\phi - L_{n-1}) \mathbf{1}\{s_\phi < L_n\} \Big) \mathbf{1}\{v_\phi \geq v^*\} - KN \Big) \\
& \leq \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} \left((g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)) a_\phi \right. \right. \\
& \quad \left. \left. - h(s_\phi - L_{n-1}) \mathbf{1}\{s_\phi < L_n\} \right) \mathbf{1}\{v_\phi \geq v^*\} - KN \right) \\
& \leq \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N \int_{\phi \in \mathcal{H}_n} (g_{v_\phi} - \min\{h(t_\phi - L_{n-1}), \theta_{v_\phi}(L_n - t_\phi)\})^+ \mathbf{1}\{v_\phi \geq v^*\} - KN \right).
\end{aligned}$$

The first inequality follows from the result that we derive in the first step above. The second inequality follows from the result that we derive in the second step above and property $h > 0$. The third equality follows from the law of total expectation. The third and the fourth inequalities follow from Lemma 5 Part 1 that $\theta_{v_\phi} > 0$ for $v_\phi \geq v^*$ and the property $a_\phi \in \{0, 1\}$.

Q.E.D.

We notice that $J^{\mathcal{Q}^\infty}$ is the function of the inventory replenishment time schedule, \mathcal{Q}^∞ . Next, we explore an inventory replenishment time schedule that maximizes $J^{\mathcal{Q}^\infty}$.

LEMMA EC.6. *For any \mathcal{Q}^∞ , we have*

$$J^{\mathcal{Q}^\infty} \leq \max_{L > 0} J^L = J^{L^*}.$$

Proof of Lemma EC.6.

For any \mathcal{Q}^∞ , we have

$$J^{\mathcal{Q}^\infty} = \liminf_{N \rightarrow \infty} \frac{1}{L_N} \left(\sum_{n=1}^N (L_n - L_{n-1}) J^{L_n - L_{n-1}} \right) \leq \max_{L > 0} J^L = J^{L^*}.$$

Q.E.D.

EC.3. Proofs of Lemmas 1 and 2 and Proposition 2

Proof of Lemma 1.

Following from Lemma 6 Parts 1 and 2, we have that $v_t^h - v_t^\theta$ is non-decreasing and continuous in $t \in [0, L]$, with $v_0^h - v_0^\theta = v^* - v_0^\theta < 0$ and $v_L^h - v_L^\theta = v_L^h - v^* > 0$. Therefore, the definition of \underline{t}_L implies

$$\underline{t}_L \in (0, L), \quad \begin{cases} v_t^h \leq v_t^\theta & \text{if } t \in [0, \underline{t}_L) \\ v_t^h = v_t^\theta & \text{if } t = \underline{t}_L \\ v_t^h > v_t^\theta & \text{if } t \in (\underline{t}_L, L] \end{cases}.$$

Following from Assumption 1 Part 1 that $W(v) > 0$ for $v \geq 0$ and the property $V < \infty$, we have $W(V) \in (0, \infty)$. The definition of \bar{t}_L and the properties that $h, W(V) \in (0, \infty)$ imply $\bar{t}_L \in [\underline{t}_L, L)$.

Next, we characterize the properties of \tilde{v}_t .

First, we show that $v_t^h < v_t^\theta$ implies $\tilde{v}_t \leq v_t^h$.

Following from the definition of \tilde{v}_t , condition $v_t^h < v_t^\theta$ implies $\tilde{v}_t = \sup \{v \in [v^*, V) : \theta_v(L - t) < ht\}$. Consider any t with $v_t^h < v_t^\theta$. Consider any $v \in (v_t^h, v_t^\theta)$. The definition of v_t^h and the property that g_v is strictly increasing in v imply $g_v - ht > 0$. The definition of v_t^θ , property $v_t^\theta > v_t^h \geq v^*$, and the property that $g_v - \theta_v(L - t)$ is continuous in v imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L - t) \leq 0$. Hence, condition $v < v_t^\theta$ and Lemma 5 Part 3 imply $g_v - \theta_v(L - t) < 0$. Therefore, for any $v \in (v_t^h, v_t^\theta)$, $\theta_v(L - t) > ht$. Following from Lemma 5 Part 2 that θ_v is non-decreasing in v , we have that $\theta_v(L - t) > ht$ for all $v \in (v_t^h, V)$. Therefore, $\tilde{v}_t \leq v_t^h$.

Second, we show that $v_t^h > v_t^\theta$ implies $\tilde{v}_t \geq v_t^h$.

Following from the definition of \tilde{v}_t , condition $v_t^h > v_t^\theta$ implies $\tilde{v}_t = \sup \{v \in [v^*, V) : \theta_v(L - t) < ht\}$. Consider any t with $v_t^h > v_t^\theta$. Consider any $v \in (v_t^\theta, v_t^h)$. The definition of v_t^h and the property that g_v is strictly increasing in v imply $g_v - ht < 0$. The definition of v_t^θ implies $g_v - \theta_v(L - t) \geq 0$. Hence, $\theta_v(L - t) < ht$ for all $v \in (v_t^\theta, v_t^h)$. Therefore, $\tilde{v}_t \geq v_t^h$.

Third, we show that for any $t \in (\bar{t}_L, L]$, $\tilde{v}_t = V$.

Consider any $t \in (\bar{t}_L, L]$. Because $\bar{t}_L \geq \underline{t}_L$, we note from the result that we prove earlier in this lemma that $v_t^h > v_t^\theta$ for $t \in (\bar{t}_L, L]$. Following from the definition of \tilde{v}_t , condition $v_t^h < v_t^\theta$ implies $\tilde{v}_t = \sup \{v \in [v^*, V) : \theta_v(L - t) < ht\}$.

Consider any $v \in [v^*, V)$, we have

$$\begin{aligned} \theta_v(L - t) - ht &\leq \theta_V(L - t) - ht = W(V)(L - t) - ht \\ &= W(V)L - (h + W(V))t < W(V)L - (h + W(V))\bar{t}_L \leq 0, \end{aligned}$$

where the first inequality follows from Lemma 5 Part 2 that θ_v is non-decreasing in v , the second inequality follows from Assumption 1 Part 1 that $W(v) > 0$, property $h > 0$, and condition $t > \bar{t}_L$, the third inequality follows from the definition of \bar{t}_L . Therefore, for $t \in (\bar{t}_L, L]$, we have $\tilde{v}_t = V$.

Q.E.D.

The following lemma will be used to prove Lemma 2.

LEMMA EC.7. *For any customer $(t, v) \in [0, L) \times [v^*, V)$, we have*

$$(g_v - \min\{ht, \theta_v(L-t)\})^+ = \begin{cases} g_v - ht & \text{if } t \in [0, \underline{t}_L], v \in [v_t^h, V), \text{ or } t \in (\underline{t}_L, \bar{t}_L], v \in [\tilde{v}_t, V) \\ g_v - \theta_v(L-t) & \text{if } t \in (\underline{t}_L, \bar{t}_L], v \in [v_t^\theta, \tilde{v}_t), \text{ or } t \in (\bar{t}_L, L), v \in [v_t^\theta, V) \\ 0 & \text{otherwise} \end{cases}.$$

Proof of Lemma EC.7.

First, we analyze the order relation of g_v and ht .

Consider the first case that $v \in [v_t^h, V)$. Following from the definition of v_t^h and the property that g_v is continuous in v , we have $g_v \geq ht$.

Consider the second case that $v \in [v^*, v_t^h)$. Following from the definition of v_t^h and the property that g_v is strictly increasing in v , we have $g_v < ht$.

Second, we analyze the order relation of g_v and $\theta_v(L-t)$.

Consider the first case that $v \in [v_t^\theta, V)$. Following from the definition of v_t^θ and the property that g_v and θ_v are continuous in v , we have $g_v \geq \theta_v(L-t)$.

Consider the second case that $v \in [v^*, v_t^\theta)$. Following from the definition of v_t^θ and Lemma 5 Part 3, we have $g_v < \theta_v(L-t)$.

Third, we analyze the order relation of $\theta_v(L-t)$ and ht .

We analyze two scenarios that $v_t^h \neq v_t^\theta$ and $v_t^h = v_t^\theta$, respectively. Consider the first scenario that $v_t^h \neq v_t^\theta$. Following from the definition of \tilde{v}_t , we have $\tilde{v}_t = \sup\{v \in [v^*, V) : \theta_v(L-t) < ht\}$. In this scenario, we consider two cases that $v \in [\tilde{v}_t, V)$ and $v \in [v^*, \tilde{v}_t)$, respectively.

Consider the first case that $v \in [\tilde{v}_t, V)$. Following from the definition of \tilde{v}_t and the property that θ_v is continuous in v , we have $\theta_v(L-t) \geq ht$.

Consider the second case that $v \in [v^*, \tilde{v}_t)$. Following from the definition of \tilde{v}_t and Lemma 5 Part 2 that θ_v is non-decreasing in v , we have $\theta_v(L-t) < ht$.

Consider the second scenario that $v_t^h = v_t^\theta \triangleq v^{**}$. If $v^{**} = V$, then the analysis in this proof above implies that for any $v \in [0, V)$, $(g_v - \min\{ht, \theta_v(L-t)\})^+ = 0$. Therefore, we only need to analyze the case that $v^{**} < V$. Following from the definition of \tilde{v}_t , we have $\tilde{v}_t = v^{**}$. Following from Lemma 6 Parts 1 and 2, we have $v^{**} > v^*$. Therefore, the definition of v_t^h , the property that g_v is continuous in v , and property $v^{**} \in (v^*, V)$ imply $g_{v^{**}} - ht = 0$. The definition of v_t^θ , the property that g_v

and θ_v are continuous in v , and property $v^{**} \in (v^*, V)$ imply $g_{v^{**}} - \theta_{v^{**}}(L - t) = 0$. Therefore, $\theta_{v^{**}}(L - t) = ht$. Therefore, following from Lemma 5 Part 2 that θ_v is non-decreasing in v , we have

$$\begin{cases} \theta_v(L - t) \geq ht & \text{if } v \in [\tilde{v}_t, V) \\ \theta_v(L - t) \leq ht & \text{if } v \in [v^*, \tilde{v}_t) \end{cases}.$$

Therefore, all results above complete the proof.

Q.E.D.

Proof of Lemma 2.

Following from the definition of J^L , (3), we have

$$\begin{aligned} J^L &= \frac{\lambda}{L} \int_{t=0}^L \int_{v=v^*}^V (g_v - \min\{ht, \theta_v(L - t)\})^+ f(v) dv dt - \frac{K}{L} \\ &= \frac{\lambda}{L} \left(\int_{t=0}^{\bar{t}_L} \int_{v=v_t^h}^V (g_v - ht) f(v) dv dt \right. \\ &\quad \left. + \int_{t=\bar{t}_L}^{\bar{t}_L} \left(\int_{v=\tilde{v}_t}^V (g_v - ht) f(v) dv + \int_{v=v_t^\theta}^{\tilde{v}_t} (g_v - \theta_v(L - t)) f(v) dv \right) dt \right. \\ &\quad \left. + \int_{t=\bar{t}_L}^L \int_{v=v_t^\theta}^V (g_v - \theta_v(L - t)) f(v) dv dt \right) - \frac{K}{L} \\ &= \frac{\lambda}{L} \left(\int_{t=0}^{\bar{t}_L} (v_t^h - ht) \bar{F}(v_t^h) dt + \int_{t=\bar{t}_L}^{\bar{t}_L} \left((\tilde{v}_t - ht) \bar{F}(\tilde{v}_t) \right. \right. \\ &\quad \left. \left. + (v_t^\theta - W(v_t^\theta)(L - t)) \bar{F}(v_t^\theta) - (\tilde{v}_t - W(\tilde{v}_t)(L - t)) \bar{F}(\tilde{v}_t) \right) dt \right. \\ &\quad \left. + \int_{t=\bar{t}_L}^L (v_t^\theta - W(v_t^\theta)(L - t)) \bar{F}(v_t^\theta) dt \right) - \frac{K}{L} \\ &= \frac{\lambda}{L} \left(\int_{t=0}^{\bar{t}_L} (p_t^{L,h} - ht) \bar{F}(v_t^h) dt + \int_{t=\bar{t}_L}^{\bar{t}_L} \left((v_t^\theta - W(v_t^\theta)(L - t) + W(\tilde{v}_t)(L - t) - ht) \bar{F}(\tilde{v}_t) \right. \right. \\ &\quad \left. \left. + (v_t^\theta - W(v_t^\theta)(L - t)) (\bar{F}(v_t^\theta) - \bar{F}(\tilde{v}_t)) \right) dt \right. \\ &\quad \left. + \int_{t=\bar{t}_L}^L (v_t^\theta - W(v_t^\theta)(L - t)) \bar{F}(v_t^\theta) dt \right) - \frac{K}{L} \\ &= \frac{\lambda}{L} \left(\int_{t=0}^{\bar{t}_L} (p_t^{L,h} - ht) \bar{F}(v_t^h) dt + \int_{t=\bar{t}_L}^{\bar{t}_L} ((p_t^{L,h} - ht) \bar{F}(\tilde{v}_t) + p_t^{L,\theta} (\bar{F}(v_t^\theta) - \bar{F}(\tilde{v}_t))) dt \right. \\ &\quad \left. + \int_{t=\bar{t}_L}^L p_t^{L,\theta} \bar{F}(v_t^\theta) dt \right) - \frac{K}{L}. \end{aligned}$$

The second equality follows from Lemma EC.7.

Q.E.D.

Proof of Proposition 2.

Following from the definition of J^L , (3), we have

$$J^L = \frac{\lambda}{L} \int_{t=0}^L \int_{v=v^*}^V (g_v - \min\{ht, \theta_v(L-t)\})^+ f(v) dv dt - \frac{K}{L}.$$

Therefore,

$$\frac{\partial^2 J^L}{\partial L \partial K} = \frac{1}{L^2} \geq 0.$$

Therefore, following from Topkis's theorem (see [Topkis \(2011\)](#)), we have that L^* is non-decreasing in K .

Consider one extreme case that $K = 0$.

For any $L > 0$, we have

$$\begin{aligned} J^L &= \frac{\lambda}{L} \int_{t=0}^L \int_{v=v^*}^V (g_v - \min\{ht, \theta_v(L-t)\})^+ f(v) dv dt \\ &\leq \frac{\lambda}{L} \int_{t=0}^L \int_{v=v^*}^V g_v f(v) dv dt \\ &= \lambda \int_{v=v^*}^V g_v f(v) dv \\ &= J^0. \end{aligned}$$

The inequality holds due to following reasons. First, the properties that g_v is strictly increasing in v and $g_{v^*} = 0$ imply $g_v > 0$ for any $v \in [v^*, V)$. Second, we have $h > 0$. Third, we have Lemma 5 Part 1 that $\theta_v > 0$ for $v \geq v^*$.

Consider the other extreme case that $K \rightarrow \infty$.

Define $T \triangleq \max\left\{\frac{V}{h}, \frac{V}{\theta_{v^*}}\right\}$. Hence, if $L \geq 2T$, then for any $t \in [T, L-T]$ and any $v \geq v^*$, we have

$$(g_v - \min\{ht, \theta_v(L-t)\})^+ \leq (V - \min\{hT, \theta_v T\})^+ \leq (V - \min\{hT, \theta_{v^*} T\})^+ = 0,$$

where the first inequality follows from property $g_v = v - \frac{\bar{F}(v)}{f(v)} \leq v \leq V$, condition $t \in [T, L-T]$, property $h > 0$ and Lemma 5 Part 1 that $\theta_v > 0$ for $v \geq v^*$, the second inequality follows from Lemma 5 Part 2 that θ_v is non-decreasing in v and condition $v \geq v^*$.

For any $L > 0$, define

$$G^L \triangleq \lambda \int_{t=0}^L \int_{v=v^*}^V (g_v - \min\{ht, \theta_v(L-t)\})^+ f(v) dv dt.$$

Hence, for $L > 2T$, we have $G^L = G^{2T}$.

Consider any $K > G^{2T}$.

For $L \leq 2T$, we have $J^L = \frac{1}{L} (G^L - K) \leq \frac{1}{L} (G^{2T} - K) < 0$, where the first inequality follows from the property that G^L is non-decreasing in L .

For $L > 2T$, we have $J^L = \frac{1}{L} (G^L - K) = \frac{1}{L} (G^{2T} - K) < 0$, where the second equality follows from the property that $G^L = G^{2T}$ for $L > 2T$.

In addition, we note that $\lim_{L \rightarrow \infty} J^L = \lim_{L \rightarrow \infty} \frac{1}{L} (G^{2T} - K) = 0$.

Therefore, for $K > G^{2T}$, $L^* = \infty$.

Q.E.D.

EC.4. Proofs of Lemma 4 and Theorem 2

Proof of Lemma 4.

Due to the cyclic nature of policy $\tilde{\pi}_L$, without loss of generality, we only need to characterize the equilibrium behaviors of customers who arrive during the first cycle $[0, L)$.

In this proof, we introduce notation $\phi_{t'} \triangleq (t', v_\phi)$. We extend the definition of $p_t^{L,h}$ to the support $t \in (\underline{t}_L, L)$ as $p_t^{L,h} \triangleq p_t^{L,\theta} + W(\tilde{v}_t)(L - t)$.

We make the proof by taking the following steps.

Step 1: We show that for any customer $\phi \in (0, L) \times [0, V)$, if at time t_ϕ , $\Omega_{t_\phi}^{\tilde{\pi}_L} = \left\{ \left(p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta}, L \right) \right\}$, then for any z_ϕ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\tilde{\pi}_L})$.

In this scenario, at any time $t \in [t_\phi, L)$, the seller only offers the delayed delivery option $\Omega_t^{\tilde{\pi}_L} = \left\{ \left(p_{\max\{t, \underline{t}_L\}}^{L,\theta}, L \right) \right\}$. This entails that either $t_\phi \in (0, \bar{t}_L]$ and $I_{t_\phi-} = 0$ or $t_\phi \in (\bar{t}_L, L)$. Therefore,

$$\begin{aligned} U(\phi, z_\phi) &= \left(v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \\ &\leq \left(v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+. \end{aligned}$$

The inequality follows from Lemma 7 Part 1 that $p_t^{L,\theta} = v_t^\theta - W(v_t^\theta)(L - t)$ is non-decreasing in t .

Now, we show that the RHS is equal to $U(\phi, z_\phi^{\tilde{\pi}_L})$. We prove this result by taking the following steps.

Step 1.1: Consider the case that $t_\phi \in (\underline{t}_L, L)$.

We have

$$v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta} - W(v_\phi)(L - t_\phi) = v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi)$$

$$\begin{aligned}
&= (v_\phi - W(v_\phi)(L - t_\phi)) - (v_{t_\phi}^\theta - W(v_{t_\phi}^\theta)(L - t_\phi)) \\
&\quad \begin{cases} \geq 0 & \text{if } v_\phi \geq v_{t_\phi}^\theta, \\ \leq 0 & \text{if } v_\phi < v_{t_\phi}^\theta, \end{cases}
\end{aligned}$$

where the first equality follows from condition $t_\phi > \underline{t}_L$, the second equality follows from the definition of $p_{t_\phi}^{L,\theta}$, the inequality follows from the property that Lemma 7 Parts 1 and 2 and condition $t_\phi > \underline{t}_L$ imply $v_{t_\phi}^\theta - W(v_{t_\phi}^\theta)(L - t_\phi) \geq 0$, condition $L - t_\phi > 0$, and Lemma 5 Part 4.

Hence,

$$\begin{aligned}
(v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta} - W(v_\phi)(L - t_\phi))^+ &= (v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi)) \mathbf{1}\{v_\phi \geq v_{t_\phi}^\theta\} \\
&= U(\phi, z_\phi^{\tilde{\pi}L}).
\end{aligned}$$

Step 1.2: Consider the case that $t_\phi \in (0, \underline{t}_L]$ and $v_{t_\phi}^b \in (v^*, V)$.

Following from the definition of v_t^b , condition $v_{t_\phi}^b \in (v^*, V)$, and the property that $v - W(v)K$ is continuous in v , we have $v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi) = p_{\underline{t}_L}^{L,\theta}$. Following from Lemma 7 Part 2 that $p_{\underline{t}_L}^{L,\theta} = v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$, we have $v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi) = p_{\underline{t}_L}^{L,\theta} \geq 0$.

Therefore,

$$\begin{aligned}
v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta} - W(v_\phi)(L - t_\phi) &= v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \\
&= (v_\phi - W(v_\phi)(L - t_\phi)) - (v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi)) \\
&\quad \begin{cases} \geq 0 & \text{if } v_\phi \geq v_{t_\phi}^b, \\ \leq 0 & \text{if } v_\phi < v_{t_\phi}^b, \end{cases}
\end{aligned}$$

where the first equality follows from condition $t_\phi \leq \underline{t}_L$, the inequality follows from property $v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi) \geq 0$, the property that Lemma 1 that $\underline{t}_L < L$ and condition $t_\phi \leq \underline{t}_L$ jointly imply $L - t_\phi > 0$, and Lemma 5 Part 4.

Hence,

$$\begin{aligned}
(v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L,\theta} - W(v_\phi)(L - t_\phi))^+ &= (v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi)) \mathbf{1}\{v_\phi \geq v_{t_\phi}^b\} \\
&= U(\phi, z_\phi^{\tilde{\pi}L}).
\end{aligned}$$

Step 1.3: Consider the case that $t_\phi \in (0, \underline{t}_L]$ and $v_{t_\phi}^b = V$.

Following from the definition of v_t^b , condition $v_{t_\phi}^b = V$, and the property that $v - W(v)K$ is continuous in v , we have $v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi) \leq p_{\underline{t}_L}^{L,\theta}$.

Suppose there exists ϕ with $t_\phi \in (0, \underline{t}_L]$, such that $v_\phi - W(v_\phi)(L - t_\phi) > p_{\underline{t}_L}^{L, \theta}$. Following from Lemma 7 Part 2 that $p_{\underline{t}_L}^{L, \theta} = v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$, we have $v_\phi - W(v_\phi)(L - t_\phi) > 0$. Therefore,

$$\begin{aligned} v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi) &= (V - W(V)(L - t_\phi)) \\ &> (v_\phi - W(v_\phi)(L - t_\phi)) \\ &> p_{\underline{t}_L}^{L, \theta}, \end{aligned}$$

where the first equality follows from condition $v_{t_\phi}^b = V$, the first inequality follows from property $v_\phi - W(v_\phi)(L - t_\phi) > 0$, the property that Lemma 1 that $\underline{t}_L < L$ and condition $t_\phi \leq \underline{t}_L$ jointly imply $L - t_\phi > 0$, property $v_\phi < V$, and Lemma 5 Part 4. This result contradicts with property $v_{t_\phi}^b - W(v_{t_\phi}^b)(L - t_\phi) \leq p_{\underline{t}_L}^{L, \theta}$.

Therefore,

$$\begin{aligned} \left(v_\phi - p_{\max\{t_\phi, \underline{t}_L\}}^{L, \theta} - W(v_\phi)(L - t_\phi) \right)^+ &= \left(v_\phi - p_{\underline{t}_L}^{L, \theta} - W(v_\phi)(L - t_\phi) \right)^+ \\ &= 0 \\ &= \left(v_\phi - p_{\underline{t}_L}^{L, \theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\{v_\phi \geq V\} \\ &= \left(v_\phi - p_{\underline{t}_L}^{L, \theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\{v_\phi \geq v_{t_\phi}^b\} \\ &= U(\phi, z_\phi^{\bar{\pi}^L}), \end{aligned}$$

where the first equality follows from condition $t_\phi \leq \underline{t}_L$, the second equality follows from the property that we prove above that $v_\phi - W(v_\phi)(L - t_\phi) \leq p_{\underline{t}_L}^{L, \theta}$ if $t_\phi \in (0, \underline{t}_L]$ and $v_{t_\phi}^b = V$, the third equality follows from property $v_\phi < V$, the fourth equality follows from condition $v_{t_\phi}^b = V$.

Step 1.4: We show that if $t_\phi \in (0, \underline{t}_L]$, then we must have $v_{t_\phi}^b > v^*$.

For any $t \in (0, \underline{t}_L]$, we have

$$v^* - W(v^*)(L - t) \leq v^* - W(v^*)(L - \underline{t}_L) < v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) = p_{\underline{t}_L}^{L, \theta},$$

where the first inequality follows from condition $t \leq \underline{t}_L$ and Assumption 1 Part 1 that $W(v) > 0$, the second inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$, Lemma 1 that $\underline{t}_L < L$, Lemma 6 Part 2 that $v_t^\theta > v^*$ for $t \in [0, L)$, property $v^* > 0$, and Lemma 5 Part 4(b).

Therefore, this property, the property that $v - W(v)(L - t)$ is continuous in v , and the definition of v_t^b jointly imply $v_{t_\phi}^b > v^*$.

Therefore, all results in Steps 1.1-1.4 jointly imply $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\bar{\pi}^L})$.

Step 2: We show that for any customer $\phi \in (\underline{t}_L, \bar{t}_L] \times [0, V)$, if at time t_ϕ , $\Omega_{t_\phi}^{\tilde{\pi}_L} = \left\{ \left(p_{t_\phi}^{L,h}, t_\phi \right), \left(p_{t_\phi}^{L,\theta}, L \right) \right\}$, then for any z_ϕ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\tilde{\pi}_L})$.

In this scenario, because the seller is able to offer a purchase option with the instantaneous product delivery, $\left(p_{t_\phi}^{L,h}, t_\phi \right)$, following from the definition of $\tilde{\pi}_L$, we must have $I_{t_\phi-} > 0$.

We have

$$\begin{aligned}
U(\phi, z_\phi) &\leq \max \left\{ \left(v_\phi - p_{\tau_\phi}^{L,h} - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - \tau_\phi) - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \begin{cases} \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - \tau_\phi) - W(v_\phi)(\tau_\phi - t_\phi) \right)^+ & \text{if } v_\phi \geq \tilde{v}_{\tau_\phi} \\ \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ & \text{if } v_\phi < \tilde{v}_{\tau_\phi} \end{cases} \\
&\leq \begin{cases} \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - t_\phi) \right)^+ & \text{if } v_\phi \geq \tilde{v}_{\tau_\phi} \\ \left(v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ & \text{if } v_\phi < \tilde{v}_{\tau_\phi} \end{cases} \\
&= \left(v_\phi - p_{t_\phi}^{L,\theta} - W(\tilde{v}_{t_\phi})(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi \geq \tilde{v}_{t_\phi}\} + \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \mathbf{1}\{v_\phi < \tilde{v}_{t_\phi}\} \\
&= \left(v_\phi - p_{t_\phi}^{L,\theta} - W(\tilde{v}_{t_\phi})(L - t_\phi) \right) \mathbf{1}\{v_\phi \geq \tilde{v}_{t_\phi}\} + \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\{v_\phi \in [v_{t_\phi}^\theta, \tilde{v}_{t_\phi})\} \\
&= \left(v_\phi - p_{t_\phi}^{L,h} \right) \mathbf{1}\{v_\phi \geq \tilde{v}_{t_\phi}\} + \left(v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\{v_\phi \in [v_{t_\phi}^\theta, \tilde{v}_{t_\phi})\} \\
&= U(\phi, z_\phi^{\tilde{\pi}_L}).
\end{aligned}$$

The first equality follows from the definition of $p_t^{L,h}$. The second equality, the second inequality and the third equality follow from Assumption 2 Part 1 that $W(\cdot) > 0$, Part 3 that $W(v)$ is non-decreasing in v , and the property that $t_\phi \leq \tau_\phi \leq L$. The fourth equality follows from the property that Lemma 7 Parts 1 and 2 and condition $t_\phi > \underline{t}_L$ imply $p_{t_\phi}^{L,\theta} = v_{t_\phi}^\theta - W(v_{t_\phi}^\theta)(L - t_\phi) \geq 0$, the property that Lemma 1 that $\bar{t}_L < L$ and condition $t_\phi \leq \bar{t}_L$ imply $L - t_\phi > 0$, and Lemma 5 Part 4. The fifth equality follows from the definition of $p_t^{L,h}$.

Step 3: We show that for any customer $\phi \in [0, \underline{t}_L] \times [0, V)$, if at time t_ϕ , $\Omega_{t_\phi}^{\tilde{\pi}_L} = \left\{ \left(p_{t_\phi}^{L,h}, t_\phi \right) \right\}$, then for any z_ϕ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\tilde{\pi}_L})$.

In this scenario, because the seller is able to offer a purchase option with the instantaneous product delivery, $\left(p_{t_\phi}^{L,h}, t_\phi \right)$, following from the definition of $\tilde{\pi}_L$, we must have $I_{t_\phi-} > 0$.

We prove this result by taking the following steps.

Step 3.1: Consider the case that $\tau_\phi \in [t_\phi, \underline{t}_L]$.

We have

$$\begin{aligned}
U(\phi, z_\phi) &\leq \max \left\{ \left(v_\phi - p_{\tau_\phi}^{L,h} - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - v_{t_\phi}^h - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left((v_\phi - W(v_\phi)(L - t_\phi)) - (v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L)) \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left((v_\phi - W(v_\phi)(L - \underline{t}_L)) - (v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L)) \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left(v_\phi - v_{\underline{t}_L}^\theta \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - v_{t_\phi}^h \right)^+, \left(v_\phi - v_{\underline{t}_L}^h \right)^+ \right\} \\
&= \left(v_\phi - v_{t_\phi}^h \right)^+ \\
&= U(\phi, z_\phi^{\tilde{\pi}_L}).
\end{aligned}$$

The first equality follows from the definition of $p_t^{L,h}$. The second inequality follows from Lemma 6 Part 1 that v_t^h is non-decreasing in t , property $\tau_\phi \geq t_\phi$, and Assumption 1 Part 1 that $W(v) > 0$. The second equality follows from the definition of $p_t^{L,\theta}$. The third inequality follows from Assumption 1 Part 1 that $W(v) > 0$ and condition $t_\phi \leq \underline{t}_L$. The fourth inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$, Lemma 1 that $L - \underline{t}_L > 0$, and Lemma 5 Part 4. The third equality follows from Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$. The fourth equality follows from Lemma 6 Part 1 that v_t^h is non-decreasing in t and property $t_\phi \leq \underline{t}_L$.

Step 3.2: Consider the case that $\tau_\phi \in (\underline{t}_L, L)$ and $t_\phi = \underline{t}_L$.

For any $t' \in (\underline{t}_L, \tau_\phi]$, we have

$$\begin{aligned}
U(\phi, z_\phi) &= U(\phi_{t'}, z_\phi) - W(v_\phi)(t' - \underline{t}_L) \\
&\leq U(\phi_{t'}, z_{\phi_{t'}}^{\tilde{\pi}_L}) - W(v_\phi)(t' - \underline{t}_L) \\
&\leq U(\phi_{t'}, z_{\phi_{t'}}^{\tilde{\pi}_L}) \\
&\leq \max \left\{ \left(v_\phi - p_{t'}^{L,h} \right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t') \right)^+ \right\} \\
&= \max \left\{ \left(v_\phi - p_{t'}^{L,\theta} - W(\tilde{v}_{t'})(L - t') \right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t') \right)^+ \right\} \\
&\leq \max \left\{ \left(v_\phi - p_{t'}^{L,\theta} - W(v_{t'}^h)(L - t') \right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t') \right)^+ \right\}.
\end{aligned}$$

The first inequality follows from results that we prove in Steps 1-2 above. The second inequality follows from Assumption 1 Part 1 that $W(v) > 0$. The second equality follows from the definition of $p_t^{L,h}$. The fourth inequality follows from Lemma 1 that $\tilde{v}_t \geq v_t^h$ if $t > \underline{t}_L$ and Assumption 1 Part 3 that $W(v)$ is non-decreasing in v .

Therefore,

$$\begin{aligned}
U(\phi, z_\phi) &\leq \limsup_{t' \rightarrow \underline{t}_L^+} \max \left\{ (v_\phi - p_{t'}^{L,\theta} - W(v_{t'}^h)(L - t'))^+, (v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t'))^+ \right\} \\
&= \max \left\{ (v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_{\underline{t}_L}^h)(L - \underline{t}_L))^+, (v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - \underline{t}_L))^+ \right\} \\
&= \max \left\{ (v_\phi - v_{\underline{t}_L}^h)^+, \left((v_\phi - W(v_\phi)(L - \underline{t}_L)) - (v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L)) \right)^+ \right\} \\
&\leq \max \left\{ (v_\phi - v_{\underline{t}_L}^h)^+, (v_\phi - v_{\underline{t}_L}^\theta)^+ \right\} \\
&= \max \left\{ (v_\phi - v_{\underline{t}_L}^h)^+, (v_\phi - v_{\underline{t}_L}^h)^+ \right\} \\
&= (v_\phi - v_{\underline{t}_L}^h)^+ \\
&= U(\phi, z_{\phi}^{\tilde{\pi}^L}).
\end{aligned}$$

The first equality follows from Lemma 6 Part 1 that v_t^h is continuous in t , Part 2 that v_t^θ is continuous in t , and Assumption 1 Part 2 that $W(v)$ is continuous in v . The second equality follows from the definition of $p_{\underline{t}_L}^{L,\theta}$ and Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$. The second inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^\theta - W(v_{\underline{t}_L}^\theta)(L - \underline{t}_L) \geq 0$, Lemma 1 that $L - \underline{t}_L > 0$, and Lemma 5 Part 4(a). The third equality follows from Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$.

Step 3.3: Consider the case that $\tau_\phi \in (\underline{t}_L, L)$ and $t_\phi \in [0, \underline{t}_L)$.

We have

$$\begin{aligned}
U(\phi, z_\phi) &= U(\phi_{\underline{t}_L}, z_\phi) - W(v_\phi)(\underline{t}_L - t_\phi) \\
&\leq U(\phi_{\underline{t}_L}, z_{\phi_{\underline{t}_L}}^{\tilde{\pi}^L}) - W(v_\phi)(\underline{t}_L - t_\phi) = U(\phi, z_{\phi_{\underline{t}_L}}^{\tilde{\pi}^L}) \leq U(\phi, z_{\phi}^{\tilde{\pi}^L}).
\end{aligned}$$

The first inequality follows from the result that we prove in Step 3.2 above. The second inequality follows from the result that we prove in Step 3.1 above.

Therefore, all results in Steps 3.1-3.3 jointly imply $U(\phi, z_\phi) \leq U(\phi, z_{\phi}^{\tilde{\pi}^L})$.

Step 4: We show that for any customer $\phi \in [0, L) \times [0, V)$, for any z_ϕ with $\tau_\phi = L$, $U(\phi, z_\phi) \leq U(\phi, z_{\phi}^{\tilde{\pi}^L})$.

We have

$$\lim_{t \rightarrow L^-} p_t^{L,\theta} = \lim_{t \rightarrow L^-} v_t^\theta - W(v_t^\theta)(L-t) = v^* = v_0^h = p_0^{L,h},$$

where the first equality follows from the definition of $p_t^{L,\theta}$, the second equality follows from Lemma 6 Part 2 that v_t^θ is continuous in t and $v_L^\theta = v^*$, the third equality follows from Lemma 6 Part 1, the fourth equality follows from the definition of $p_0^{L,h}$.

Therefore, this result, the definition of $\tilde{\pi}_L$ that $\Omega_L^{\tilde{\pi}_L} = \{(p_0^{L,h}, L)\}$, and all results that we prove in Steps 1-3 jointly imply $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\tilde{\pi}_L})$.

Step 5: We show that for any customer $\phi \in [0, L) \times [0, V)$, for any z_ϕ with $\tau_\phi = nL + \tau$ for $n \in \mathbb{N}$ and $\tau \in (0, L]$, $U(\phi, z_\phi) \leq U(\phi, z_\phi^{\tilde{\pi}_L})$.

First, for any $n \in \mathbb{N}$, we have

$$U(\phi, z_\phi) = U(\phi_{nL}, z_\phi) + W(v_\phi)nL \leq U(\phi_{nL}, z_{\phi_{nL}}^{\tilde{\pi}_L}) + W(v_\phi)nL = U(\phi, z_{\phi_{nL}}^{\tilde{\pi}_L}),$$

where the inequality follows from all results that we prove in Steps 1-4 above.

Second, for any $n \in \mathbb{N}$, we have

$$U(\phi, z_{\phi_{(n+1)L}}^{\tilde{\pi}_L}) = U(\phi_{nL}, z_{\phi_{(n+1)L}}^{\tilde{\pi}_L}) + W(v_\phi)nL \leq U(\phi_{nL}, z_{\phi_{nL}}^{\tilde{\pi}_L}) + W(v_\phi)nL = U(\phi, z_{\phi_{nL}}^{\tilde{\pi}_L}),$$

where the inequality follows from the result that we prove in Step 4 above.

Therefore, for any z_ϕ with $\tau_\phi = nL + \tau$ for $n \in \mathbb{N}$ and $\tau \in (0, L]$, we have

$$U(\phi, z_\phi) \leq U(\phi, z_{\phi_{nL}}^{\tilde{\pi}_L}) \leq U(\phi, z_{\phi_L}^{\tilde{\pi}_L}) \leq U(\phi, z_\phi^{\tilde{\pi}_L}),$$

where the first inequality follows from the first result that we prove in this step, the second inequality follows from the second result that we prove in this step, the third inequality follows from the result that we prove in Step 4 above.

All results that we prove in Steps 1-5 complete the proof of this lemma.

Q.E.D.

Proof of Theorem 2.

We use the mechanism design methodology to prove the first inequality of this theorem, $\tilde{J}^* \leq J^{L*}$. The mechanism design problem used in this proof is very similar to the mechanism design problem (2) studied in the deterministic setting, except that we replace every customer's dominant

(IC) and (IR) constraints by his Bayesian (IC) and (IR) constraints that are only conditional on this customer's type and all public information at time 0. The proof is analogous to the proof of Theorem 1. Therefore, we omit the proof.

Now, we prove the second inequality of this theorem.

Consider policy $\tilde{\pi}_L$ with any $L > 0$. Due to the cyclic nature of policy $\tilde{\pi}_L$, the seller's long-run average expected profit under policy $\tilde{\pi}_L$ is the same as her average expected profit over $[0, L)$. Therefore, without loss of generality, we only need to compute the seller's average expected profit over the first cycle $[0, L)$.

For each $t \in [0, \underline{t}_L]$, we denote by

$$N_t^h \triangleq \begin{cases} \sum_{\phi} \mathbf{1} \{t_{\phi} \in [0, t], v_{\phi} \geq v_t^h\} & \text{if } t \in [0, \underline{t}_L] \\ \sum_{\phi} \mathbf{1} \{t_{\phi} \in [0, \underline{t}_L], v_{\phi} \geq v_t^h\} + \mathbf{1} \{t_{\phi} \in (\underline{t}_L, t], v_{\phi} \geq \tilde{v}_t\} & \text{if } t \in (\underline{t}_L, \bar{t}_L] \end{cases}$$

the total number of customers who arrive no later than time t and choose the instantaneous delivery option, assuming that the seller has inventory on hand up to time t . Thus, N_t^h is a Poisson random variable with parameter

$$\lambda_t^h = \begin{cases} \lambda \int_{t'=0}^t \bar{F}(v_{t'}^h) dt' & \text{if } t \in [0, \underline{t}_L] \\ \lambda \int_{t'=0}^{\underline{t}_L} \bar{F}(v_{t'}^h) dt' + \lambda \int_{t'=\underline{t}_L}^t \bar{F}(\tilde{v}_{t'}) dt' & \text{if } t \in (\underline{t}_L, \bar{t}_L] \end{cases}.$$

For each $t \in (\underline{t}_L, L)$, we denote by

$$N_t^{\theta} \triangleq \begin{cases} \mathbf{1} \{t_{\phi} \in (\underline{t}_L, t], v_{\phi} \in [v_t^{\theta}, \tilde{v}_t)\} & \text{if } t \in (\underline{t}_L, \bar{t}_L] \\ \mathbf{1} \{t_{\phi} \in (\underline{t}_L, \bar{t}_L], v_{\phi} \in [v_t^{\theta}, \tilde{v}_t)\} + \mathbf{1} \{t_{\phi} \in (\bar{t}_L, t], v_{\phi} \geq v_t^{\theta}\} & \text{if } t \in (\bar{t}_L, L) \end{cases}$$

the total number of customers who arrive after time \underline{t}_L and no later than time t and choose the delayed delivery option, assuming that the seller has inventory on hand up to time t . Thus, N_t^{θ} is a Poisson random variable with parameter

$$\lambda_t^{\theta} = \begin{cases} \lambda \int_{t'=\underline{t}_L}^t (\bar{F}(v_{t'}^{\theta}) - \bar{F}(\tilde{v}_{t'})) dt' & \text{if } t \in (\underline{t}_L, \bar{t}_L] \\ \lambda \int_{t'=\underline{t}_L}^{\bar{t}_L} (\bar{F}(v_{t'}^{\theta}) - \bar{F}(\tilde{v}_{t'})) dt' + \lambda \int_{t'=\bar{t}_L}^t \bar{F}(v_{t'}^{\theta}) dt' & \text{if } t \in (\bar{t}_L, L) \end{cases}.$$

Therefore, under policy $\tilde{\pi}_L$, given customer behaviors characterized by Lemma 4, we have

$$\begin{aligned} \tilde{J}^L &\geq \frac{1}{L} \mathbb{E} \left[\int_{t=0}^{\bar{t}_L} p_t^{L,h} dN_t^h - V \left(N_{\bar{t}_L}^h - q_0^{\tilde{\pi}_L} \right)^+ + \int_{t=\underline{t}_L}^L p_t^{L,\theta} dN_t^{\theta} - h \int_{t=0}^{\bar{t}_L} \left(q_0^{\tilde{\pi}_L} - N_t^h \right)^+ dt \right. \\ &\quad \left. - h(L - \bar{t}_L) \left(q_0^{\tilde{\pi}_L} - N_{\bar{t}_L}^h \right)^+ \right] - \frac{K}{L} \\ &= \frac{1}{L} \mathbb{E} \left[\int_{t=0}^{\bar{t}_L} p_t^{L,h} d\lambda_t^h - V \left(N_{\bar{t}_L}^h - q_0^{\tilde{\pi}_L} \right)^+ + \int_{t=\underline{t}_L}^L p_t^{L,\theta} d\lambda_t^{\theta} - h \int_{t=0}^{\bar{t}_L} \left(q_0^{\tilde{\pi}_L} - N_t^h \right)^+ dt \right] \end{aligned}$$

$$\begin{aligned}
& -h(L - \bar{t}_L) \left(q_0^{\tilde{\pi}L} - N_{\bar{t}_L}^h \right)^+ \Big] - \frac{K}{L} \\
& = \frac{1}{L} \mathbb{E} \left[\int_{t=0}^{\bar{t}_L} p_t^{L,h} d\lambda_t^h - V(N_{\bar{t}_L}^h - \lambda_{\bar{t}_L}^h)^+ + \int_{t=\bar{t}_L}^L p_t^{L,\theta} d\lambda_t^\theta - h \int_{t=0}^{\bar{t}_L} (\lambda_{\bar{t}_L}^h - N_t^h)^+ dt \right. \\
& \quad \left. - h(L - \bar{t}_L) (\lambda_{\bar{t}_L}^h - N_{\bar{t}_L}^h)^+ \right] - \frac{K}{L}.
\end{aligned}$$

The first inequality follows from the property that the RHS of this inequality does not take into account the revenue from those customers who intend to select the instantaneous delivery purchase option but eventually select the delayed delivery purchase option due to inventory unavailability. The first equality follows from [Brémaud \(1981\)](#) Theorem II. The second equality follows from property $q_0^{\tilde{\pi}L} = \lambda_{\bar{t}_L}^h$.

We establish upper bounds of the second, fourth, and fifth terms on the RHS. For the second term, we have

$$\mathbb{E} \left[(N_{\bar{t}_L}^h - \lambda_{\bar{t}_L}^h)^+ \right] \leq \frac{\sqrt{\lambda_{\bar{t}_L}^h}}{2} \leq \frac{\sqrt{\lambda L}}{2},$$

where the first inequality follows from [Gallego and Van Ryzin \(1994\)](#) Equation (18).

For the fourth and the fifth terms, for any $t \in [0, \bar{t}_L]$, we have

$$\begin{aligned}
& \mathbb{E} \left[(\lambda_{\bar{t}_L}^h - N_t^h)^+ \right] = \mathbb{E} \left[(N_t^h - \lambda_{\bar{t}_L}^h)^+ \right] + \mathbb{E} [\lambda_{\bar{t}_L}^h - N_t^h] = \mathbb{E} \left[(N_t^h - \lambda_{\bar{t}_L}^h)^+ \right] + \lambda_{\bar{t}_L}^h - \lambda_t^h \\
& \leq \frac{1}{2} \left(\sqrt{\lambda_t^h + (\lambda_{\bar{t}_L}^h - \lambda_t^h)^2} - (\lambda_{\bar{t}_L}^h - \lambda_t^h) \right) + \lambda_{\bar{t}_L}^h - \lambda_t^h \leq \frac{\sqrt{\lambda_{\bar{t}_L}^h}}{2} + \lambda_{\bar{t}_L}^h - \lambda_t^h \leq \frac{\sqrt{\lambda L}}{2} + \lambda_{\bar{t}_L}^h - \lambda_t^h,
\end{aligned}$$

where the first inequality follows from [Gallego and Van Ryzin \(1994\)](#) Equation (18), the second inequality follows from the property that for $K \geq 0$ and $\sigma \geq 0$, $\frac{\sqrt{\sigma^2 + K^2} - K}{2} = \frac{\sigma^2}{2(\sqrt{\sigma^2 + K^2} + K)} \leq \frac{\sigma^2}{2\sigma} = \frac{\sigma}{2}$.

Therefore,

$$\begin{aligned}
\tilde{J}^L & \geq \frac{1}{L} \left(\int_{t=0}^{\bar{t}_L} p_t^{L,h} d\lambda_t^h + \int_{t=\bar{t}_L}^L p_t^{L,\theta} d\lambda_t^\theta - h \int_{t=0}^{\bar{t}_L} (\lambda_{\bar{t}_L}^h - \lambda_t^h) dt \right) - \frac{K}{L} - \frac{V}{2} \sqrt{\frac{\lambda}{L}} - \frac{hL}{2} \sqrt{\frac{\lambda}{L}} \\
& = \frac{1}{L} \left(\int_{t=0}^{\bar{t}_L} p_t^{L,h} d\lambda_t^h + \int_{t=\bar{t}_L}^L p_t^{L,\theta} d\lambda_t^\theta - h \int_{t=0}^{\bar{t}_L} t d\lambda_t^h \right) - \frac{K}{L} - \frac{V}{2} \sqrt{\frac{\lambda}{L}} - \frac{hL}{2} \sqrt{\frac{\lambda}{L}} \\
& = \frac{1}{L} \left(\int_{t=0}^{\bar{t}_L} (p_t^{L,h} - ht) d\lambda_t^h + \int_{t=\bar{t}_L}^L p_t^{L,\theta} d\lambda_t^\theta \right) - \frac{K}{L} - \frac{V + hL}{2} \sqrt{\frac{\lambda}{L}} \\
& = J^L - \frac{V + hL}{2} \sqrt{\frac{\lambda}{L}},
\end{aligned}$$

where the first equality follows from the formula for integration by parts.

Therefore,

$$\frac{\tilde{J}^{L^*}}{J^{L^*}} \geq 1 - \frac{V + hL^*}{2J^{L^*}} \sqrt{\frac{\lambda}{L^*}}.$$

Now, we consider the sequence of instances defined in this theorem. Because $\lambda^{(i)} = i\lambda$ and $K^{(i)} = iK$, for any $L > 0$, $J^{L,(i)} = iJ^L$. Hence, $L^{*,(i)} = L^*$. Therefore,

$$\frac{\tilde{J}^{L^*}(i)}{J^{L^*}(i)} \geq 1 - O\left(\frac{1}{\sqrt{i}}\right).$$

Q.E.D.