Mathematics of the USSR-Izvestiya

DISCRETE LINEAR GROUPS GENERATED BY REFLECTIONS

To cite this article: È B Vinberg 1971 *Math. USSR Izv.* **5** 1083

Izv. Akad. Nauk SSSR
Ser. Mat. Tom 35 (1971), No. 5

Math. USSR Izvestija
Vol. 5 (1971), No. 5

DISCRETE LINEAR GROUPS GENERATED BY REFLECTIONS

UDC 519.45

È. B. VINBERG

Abstract. We investigate linear groups generated by reflections in the faces of a convex polyhedral cone and operating discretely on an open convex cone. These groups generalize the discrete groups of motions in simply connected spaces of constant curvature, generated by reflection. Like the latter, they turn out to be Coxeter groups.

Introduction

1. In his fundamental paper [1], Coxeter investigated and classified the discrete groups of motions in Euclidean space E^n generated by reflections. (By reflection we always mean a reflection in a hyperplane.)

It is natural to consider groups generated by reflections in simply connected spaces of constant curvature, i.e. Euclidean space E^n, the sphere S^n and Lobačevskiĭ space Λ^n. In these spaces the following general theorems hold, with proofs exactly like those in the Euclidean case.

Let P be a convex (not necessarily bounded) polyhedron in a simply connected space X^n of constant curvature; let P_i, $i = 1, \cdots, m$, be the $(n-1)$-dimensional faces of P. Suppose that the dihedral angles between adjacent faces P_i and P_j are π/n_{ij}, the n_{ij} being integers. Let R_i denote the reflection in the plane containing P_i. Then the R_i generate a discrete group Γ of motions of X^n of which P is a fundamental domain. Moreover, a set of defining relations of Γ consists of $R_i^2 = 1$, $(R_iR_j)^{n_{ij}} = 1$.

Every discrete group of motions of X^n generated by reflections may be obtained in this way.

The classification of discrete groups generated by reflections in Euclidean space and on the sphere is based on the fact that every convex polyhedron in E^n with dihedral angles not exceeding $\pi/2$ is a direct product of simplices, simplicial cones and a Euclidean space [1]. This is no longer true in Lobačevskiĭ space.

In Λ^n the most interesting case occurs when P is bounded, or at least has finite volume. E. M. Andreev [11] showed that in Λ^3 there exist bounded polyhedra with arbitrarily many faces whose dihedral angles are submultiples of π. Many further examples of polyhedra in Λ^n, bounded or at least of finite volume and having submultiples of π for their dihedral angles, are contained in [5], [6], [8], [9], [11], [12], and [14]. At the present moment, however, a complete classification of these polyhedra

---

AMS 1970 *subject classifications*. Primary 20H15, 50C15; Secondary 52F05, 52A25, 50C20.

still appears extremely difficult.

2. Discrete groups generated by reflections in a simply connected space of constant curvature may be considered as particular cases of discrete *linear* groups generated by reflections.

Definition 1. A linear transformation R of a vector space V is a *reflection* if $R^2 = 1$ and -1 is a simple eigenvalue of R.

A reflection R is completely determined by the subspace U of fixed points and an eigenvector h corresponding to the eigenvalue -1. If α is the linear functional on V vanishing on U and such that $\alpha(h) = 2$, then

$$Rv = v - \alpha(v)h. \tag{1}$$

Let K be a convex polyhedral cone in V defined by a system of linear inequalities

$$\alpha_i \geqslant 0, \qquad i = 1, \dots, m, \tag{2}$$

and suppose that none of these inequalities follows from the remaining ones. Let h_i $(i = 1, \cdots, m)$ be elements of V satisfying $\alpha_i(h_i) = 2$. Let R_i be the reflection defined by (1) for $\alpha = \alpha_i$, $h = h_i$.

Definition 2. The group $\Gamma \subset GL(V)$ generated by the R_i will be called a discrete linear group generated by reflections, or simply a *linear Coxeter group* if

$$\gamma K^0 \cap K^0 = \emptyset \quad \text{for every} \quad \gamma \in \Gamma \setminus \{1\}.\text{(1)} \tag{3}$$

K will be called a *fundamental chamber* of Γ.

(For $\delta \in GL(V)$, δK is also a fundamental chamber of Γ if δ belongs to the normalizer of Γ. It is very probable that all fundamental domains of Γ are obtained in this way.)

In § 3 we shall prove (Theorem 2) that linear Coxeter groups have the following properties:

1) $\bigcup_{\gamma \in \Gamma} \gamma K$ is a convex cone.

2) Γ operates discretely in the interior C of this cone.

3) $x \in K$ is contained in C if and only if Γ_x, the group generated by the R_i such that $\alpha_i(x) = 0$, is finite.

4) A system of defining relations for Γ is given by

$$R_i^2 = 1, \quad (R_i R_j)^{n_{ij}} = 1. \tag{4}$$

Definition 3. The *Cartan matrix* of Γ is the m by m matrix $A = (a_{ij})$, $a_{ij} = \alpha_i(h_j)$.

(We choose this name because the matrix constructed in this manner for the Weyl group of a semisimple Lie algebra $\mathfrak{g}$, with a suitable norming for the α_i, coincides with the Cartan matrix of $\mathfrak{g}$, in the usual meaning of this expression.)

(1) K^0 denotes the interior of K.

We shall prove in this paper (Theorem 1 and Propositions 6 and 17) that (3) holds if and only if the Cartan matrix of Γ satisfies the following conditions.

(C1) $a_{ij} \leq 0$ for $i \neq j$ and if $a_{ij} = 0$ then $a_{ji} = 0$.

(C2) $a_{ii} = 2$; $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2\pi/n_{ij}$, n_{ij} an integer.

These n_{ij} are exactly the exponents in (4); if $a_{ij}a_{ji} \geq 4$, then (4) contains no relation for these indices.

If the b_i generate V and the α_i generate V^*, then Γ is completely determined by its Cartan matrix. (In the general case, additional invariants are needed; see § 5.) Theorem 5 asserts the *existence* of a linear Coxeter group with a given Cartan matrix and set of additional invariants, subject to certain additional necessary conditions which, however, do not impose any restriction on the Cartan matrix except (C1) and (C2).

3. Every simply connected space X^n of constant curvature can be imbedded as a hypersurface in a vector space V^{n+1} such that the motions of X^n extend to linear transformations of V^{n+1}. By a suitable choice of coordinates, these hypersurfaces may be described by the following equations:

$$\begin{aligned} &x_0^2 + x_1^2 + \cdots + x_n^2 \quad \text{for } S^n, \\ &x_0 = 1 \quad \text{for } E^n, \\ &x_0^2 - x_1^2 - \cdots - x_n^2 = 1,\ x_0 > 0 \quad \text{for } \Lambda^n. \end{aligned}$$

Under this imbedding, the k-dimensional planes of X^n are exactly the intersections of the $(k+1)$-dimensional subspaces of V^{n+1} with X^n. In particular, every hyperplane H in X^n is the intersection of X^n and an n-dimensional subspace U of V^{n+1}, and the halfspaces in X^n bounded by H correspond in a natural manner to the halfspaces in V^{n+1} bounded by U.

Let P be a convex polyhedron in X^n; that is, the intersection of finitely many halfspaces of X^n. We define the polyhedral cone K in V^{n+1} corresponding to P as the intersection of the corresponding halfspaces of V^{n+1}. Let $\mathfrak{C} = \bigcup_{t>0} tX^n$. It is easy to see that

1) $P = K \cap X^n$;

2) P is bounded if and only if $K \backslash \{0\} \subset \mathfrak{C}$;

3) in the case $X^n = \Lambda^n$, P has finite volume if and only if $K \subset \overline{\mathfrak{C}}$.

Let Γ be a discrete group of motions in X^n generated by reflections in the faces of a convex polyhedron P, and suppose that P is a fundamental domain for Γ. By extending the action of Γ to V^{n+1}, we obtain a linear group generated by reflections (in the sense of Definition 1) in the faces of the convex polyhedral cone K corresponding to P. This group operates discretely on $\mathfrak{C}$. It will be an easy consequence of our Theorem 1 that K is a fundamental chamber of this group. This is, of course, quite obvious if $K^0 \subset \mathfrak{C}$ (this is true, e. g., if P is bounded).

Thus every discrete group generated by reflections in X^n operates on V^{n+1} as a linear Coxeter group. In § 6.4 we shall give a characterization of the linear Coxeter groups obtainable in this way.

Definition 4. A linear Coxeter group Γ is *orthogonal* if there exists a Γ-invariant scalar product in the subspace $[b]$ of V spanned by the b_i, such than $(b_i, b_i) > 0$ for all i.

All those linear Coxeter groups are orthogonal which are obtained from groups generated by reflections in spaces of constant curvature. The first examples of nonorthogonal linear Coxeter groups were constructed in [7].

In §6 it will be shown that a group is orthogonal if and only if, with a suitable norming of the α_i, its Cartan matrix is symmetric (Theorem 6).

4. Linear Coxeter groups may be considered as the images of abstract Coxeter groups under linear representations.

Definition 5. A group Γ with a distinguished system $\{r_1, \cdots, r_m\}$ of generators is called an (abstract) *Coxeter group* if it has the following system of defining relations: $r_i^2 = 1$ for all i, $(r_ir_j)^{n_{ij}} = 1$ for some i and j, with $n_{ij} = n_{ji} \geq 2$.

It is convenient to define $n_{ij} = \infty$ if r_i and r_j are not connected by a defining relation, and $n_{ii} = 1$. The n_{ij} are called the *exponents* of Γ.

Theorem 5 will show how to find all representations of a given Coxeter group as a discrete linear group generated by reflections.

One of these, called the *canonical representation*, was known previously. It was noted by Coxeter [2], and Tits [3] proved that its range is indeed a linear Coxeter group. Its Cartan matrix is the so-called *cosine matrix* of Γ:

$$\operatorname{Cos}\Gamma = \left(-2\cos\frac{\pi}{n_{ij}}\right), \tag{5}$$

where $\pi/n_{ij} = 0$ if $n_{ij} = \infty$. The range of the canonical representation is an orthogonal Coxeter group. Its fundamental chamber is a simplicial cone.

Besides the result quoted just now, Tits also proved in [3], for the range of the canonical representation, the assertions 1)–4) of subsection 2. (His results are also set forth in Bourbaki [4].) All these results of Tits are particular cases of our Theorems 1 and 2. It ought to be remarked that our proofs, in contrast to those of Tits, do not rely upon a preceding algebraic investigation of abstract Coxeter groups; indeed their algebraic properties are easily obtained later from our geometric considerations.

5. In §4 we study matrices satisfying (C 1) and systems of homogeneous linear inequalities related to them. While these results are interesting in themselves, they are needed here for the proof of Theorem 5 and serve as a starting point for studying the *combinatorial structure* of the fundamental chambers of linear Coxeter groups. This investigation is contained in §8.

Definition 6. Let K be a convex polyhedron in $\mathbf{R}^n$. The *complex* of K, denoted $\mathfrak{F}K$, is the set of its (closed) faces, partially ordered by inclusion.

Let $K_1, \cdots, K_m$ be the $(n-1)$-dimensional faces of K. For any face L, let

$$\sigma(L) = \{i : K_i \supset L\}. \tag{6}$$

Obviously $L_1 \supset L_2$ if and only if $\sigma(L_1) \subset \sigma(L_2)$. So the complex $\mathfrak{F}K$ is completely determined once it is known which subsets of $\mathbf{I}_m = \{1, \cdots, m\}$ belong to $\sigma(\mathfrak{F}K)$.

Definition 7. Let Γ be a group with a distinguished system of generators, $\{r_1, \cdots, r_m\}$. For every $S \subset \mathbf{I}_m$, let Γ_S be the subgroup generated by the r_i, $i \in S$. These Γ_S are called *standard subgroups* of Γ.

Now let K be a fundamental chamber of a linear Coxeter group Γ. There is a natural bijective correspondence between the faces $K_1, \cdots, K_m$ of codimension one of K and the reflections $R_1, \cdots, R_m$ generating Γ.

Theorem 7 below asserts among other things that $S \in \sigma(\mathfrak{F}K)$ if Γ_S is finite. For groups generated by reflections in Λ^n, this was proved by E. M. Andreev in [13].

Definition 8. A linear Coxeter group Γ will be called *perfect* if its fundamental chamber is strictly convex (i.e. does not contain any line) and

$$K \setminus \{0\} \subset C \tag{7}$$

(see subsection 2 for the definition of C).

Let PV be the projective space associated with the linear space V in which Γ operates; let PC be the image of C under the canonical mapping of $V \setminus \{0\}$ onto PV. (7) is equivalent to the assertion that PC/Γ is compact. Hence a linear Coxeter group obtained from a discrete group of motions in a space X^n of constant curvature (see subsection 3) is certainly perfect if it satisfies the following condition:

For $X^n = S^n$, the fundamental polyhedron does not contain any pair of opposite points.

For $X^n = E^n$ or Λ^n, the fundamental polyhedron is bounded.

Combining Theorem 7 with result 3) of subsection 2, we obtain for perfect groups that

$$S \in (\mathfrak{F}K) \text{ if and only if } \Gamma_S \text{ is finite or } S = \mathbf{I}_m. \tag{8}$$

So the combinatorial structure of the fundamental chamber of a perfect Coxeter group is completely determined by the abstract structure of the group; that is, by its exponents.

In §8, along with perfect linear Coxeter groups, we also study a class with similar properties, the *quasiperfect groups.* This class contains the linear Coxeter groups derived from discrete groups of motions in Λ^n whose fundamental polyhedron has finite volume.

6. The fundamental results of this paper were announced in [10].

§1. The universal spaces of abstract Coxeter groups

1. Let Γ be an abstract Coxeter group with generators $r_1, \cdots, r_m$, and let K be a topological space with a distinguished system $\{K_1, \cdots, K_m\}$ of closed subsets. For every $x \in K$, let Γ_x be the standard subgroup of Γ generated by the r_i such that $x \in K_i$.

We shall construct a topological space $\mathfrak{U} = \mathfrak{U}(\Gamma, K)$, called the *universal space*, and an action of Γ on $\mathfrak{U}$, the *universal action* of Γ.

Consider the following equivalence relation on the topological product $\Gamma \times K$, Γ being considered discrete:

$$(\gamma, x) \sim (\delta, y) \Leftrightarrow x = y \text{ and } \gamma^{-1}\delta \in \Gamma_x. \tag{9}$$

The *universal space* $\mathfrak{U}$ is defined as the factor space of $\Gamma \times K$ modulo this equivalence. Let π denote the canonical projection of $\Gamma \times K$ onto $\mathfrak{U}$. We shall use the notation $\pi((\gamma, x)) = [\gamma, x]$.

The action of Γ on $\Gamma \times K$ is defined by $\delta(\gamma, x) = (\delta\gamma, x)$. It is easily seen that this action is compatible with the equivalence (9); hence it induces an action of Γ on $\mathfrak{U}$, namely $\delta[\gamma, x] = [\delta\gamma, x]$. This is the *universal action.*

The following proposition shows that the term "universal" is justified.

Proposition 1. *Let an action of* Γ *on a topological space* $\mathfrak{B}$ *be given, and let* ϕ: $K \to \mathfrak{B}$ *be a continuous map satisfying*

$$r_i\varphi(x) = \varphi(x) \quad \textit{for} \quad x \in K_i. \tag{10}$$

Then there exists a unique continuous map ψ: $\mathfrak{U} \to \mathfrak{B}$ *commuting with the action of* Γ *such that* $\psi([1, x]) = \phi(x)$ *for* $x \in K$.

Proof. The mapping $\tilde{\psi}$: $\Gamma \times K \to \mathfrak{B}$ defined by $\tilde{\psi}((\gamma, x)) = \gamma\phi(x)$ is continuous and commutes with the action of Γ; further, it identifies equivalent points, as is shown by (10). Hence there exists a continuous ψ: $\mathfrak{U} \to \mathfrak{B}$ making the diagram

$$\begin{tikzcd} \Gamma \times K \arrow[rr, "\tilde{\Psi}"] \arrow[dr, "\pi"'] & & \mathfrak{B} \\ & \mathfrak{U} \arrow[ur, "\psi"'] & \end{tikzcd}.$$

commutative. This ψ has the required properties. Obviously $\psi([\gamma, x]) = \gamma\phi(x)$.

Proposition 2. *The quotient space* $\mathfrak{U}/\Gamma$ *is homeomorphic to* K. *Let* τ *be the canonical mapping from* $\mathfrak{U}$ *to* $\mathfrak{U}/\Gamma$. *Then the mapping* σ: $K \to \mathfrak{U}/\Gamma$ *defined by*

$$\sigma(x) = \tau([1, x]) \tag{11}$$

is a homeomorphism.

Proof. The equivalence defined in $\Gamma \times K$ by the action of Γ is coarser than the equivalence (9). Hence there exists a natural homeomorphism between $\mathfrak{U}/\Gamma$ and $(\Gamma \times K)/\Gamma$; the latter space is obviously homeomorphic to K.

2. It is clear from the definition of $\mathfrak{U}$ that the stabilizer of $[1, x] \in \mathfrak{U}$ is just Γ_x. Let W be a neighborhood of x in K disjoint from all K_i which do not contain x. Then

$$O(x, W) = \{[\gamma, w] : \gamma \in \Gamma_x, \ w \in W\}$$

is a Γ_x-invariant neighborhood of $[1, x]$. Such a set will be called a *special neighborhood* of $[1, x]$.

Proposition 3. *If Γ_x is finite, the special neighborhoods form a neighborhood basis of $\mathfrak{U}$ at $[1, x]$.*

Proof. Let O be any neighborhood of $[1, x]$. Then $\bigcap_{\gamma\in\Gamma} O = O_1$ is also a neighborhood of $[1, x]$. Let W be a neighborhood of x in K such that

1) W is disjoint from all K_i which do not contain x, and

2) $[1, w] \in O_1$ for all $w \in W$.

Then $O(x, W)$ is a special neighborhood of $[1, x]$ contained in O.

Let $\mathfrak{U}^f$ denote the subspace of $\mathfrak{U}$ consisting of the points whose stabilizer is finite. Obviously $\mathfrak{U}^f$ is invariant under Γ. Every special neighborhood of any $[1, x] \in \mathfrak{U}^f$ is contained in $\mathfrak{U}^f$; so that these neighborhoods form a neighborhood basis for $\mathfrak{U}^f$ at $[1, x]$.

Let

$$K^f = \{x \in K : \Gamma_x \text{ is finite}\} = K \cap \mathfrak{U}^f.$$

Proposition 4. *If K is a Hausdorff space, then Γ operates discretely in $\mathfrak{U}^f$, and $\mathfrak{U}^f/\Gamma$ is homeomorphic to K^f, a homeomorphism being defined by* (11).

Proof. Since $\mathfrak{U}^f$ is open in $\mathfrak{U}$, it follows easily that the quotient topology on $\mathfrak{U}^f/\Gamma$ coincides with the topology of that space as a subspace of $\mathfrak{U}/\Gamma$. Hence our second assertion follows immediately from Proposition 2.

In order to prove the first assertion, note that the group Γ is said to operate discretely on a space X if the following conditions are satisfied.

1) X/Γ is a Hausdorff space.

2) Every $p \in X$ has a neighborhood U such that $\gamma U \cap U \neq \emptyset$ implies $\gamma p = p$.

3) Every point has a finite stabilizer.

In our case $\mathfrak{U}^f/\Gamma$ is homeomorphic to K^f, so 1) holds. 3) follows from the definition of $\mathfrak{U}^f$. To check 2), it is enough to consider points $[1, x]$; in this case, any special neighborhood will serve as U.

§ 2. Linear Coxeter groups with two generators

1. In this section we prove a proposition which shows the importance of studying linear Coxeter groups with two generators.

Let $\Gamma \subset GL(V)$ be a linear Coxeter group generated by reflections $R_1, \cdots, R_m$. Let S be any subset of I_m. Using the notation of subsection 2 of the Introduction, we define

$$K(S) = \{v \in V : \alpha_i(v) \geqslant 0 \quad \text{for} \quad i \in S\}. \tag{12}$$

$K(S)$ is obviously a convex polyhedral cone containing K.

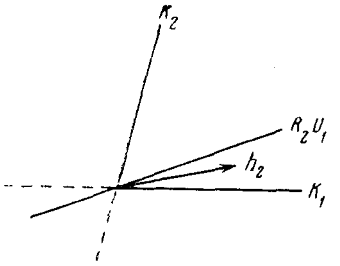

Figure 1

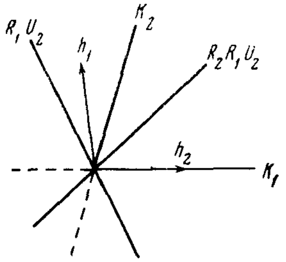

Figure 2

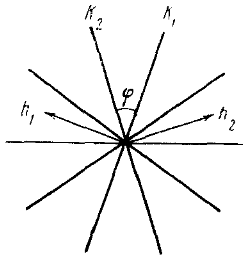

Figure 3

Proposition 5. *Let $S \in \sigma(\mathfrak{F}K)$ (see subsection 5 of the Introduction). Then Γ_S is a linear Coxeter group having $K(S)$ as a fundamental chamber.*

Proof. It is only necessary to check condition (3) for Γ_S and $K(S)$. Assume $v \in \gamma K(S)^0 \cap K(S)^0$ for some $\gamma \in \Gamma_S$. Let v_0 be an interior point of the face labeled S. All points lying on the segment vv_0 sufficiently close to v_0 are contained in $\gamma K^0 \cap K^0$. Hence $\gamma = 1$.

2. Now we shall find all linear Coxeter groups with two generators.

Let K be a dihedral cone bounded by hyperplanes U_1 and U_2. Let h_i $(i = 1, 2)$ be a vector not lying in U_i. We define reflections R_i and linear functionals α_i as in subsection 2 of the Introduction.

Proposition 6. *The dihedral cone K is a fundamental chamber of the group Γ generated by the reflections R_1 and R_2 if and only if the following conditions are satisfied:*

1) *$\alpha_1(h_2)$ and $\alpha_2(h_1)$ are either both negative or both zero.*

2) *$\alpha_1(h_2)\alpha_2(h_1) \geq 4$ or $= 4\cos^2\pi/k$, k an integer ≥ 2.*

Proof. The action of Γ on V/U, where $U = U_1 \cap U_2$, is generated by reflections in the sides of a planar angle. It is easily recognized that K is a fundamental chamber for Γ if and only if the image of K under projection into V/U is a fundamental chamber for the action of Γ on V/U. Hence we may assume $\dim V = 2$.

To show that 1) is necessary, suppose $\alpha_1(h_2) > 0$. Then (see Figure 1) the line R_2U_1 intersects K^0. Since $R_2R_1R_2^{-1}$ is a reflection in this line, we have $R_2R_1R_2^{-1}K^0 \cap K^0 \neq \emptyset$, i.e. (3) is violated. Suppose $\alpha_1(h_2) = 0$, $\alpha_2(h_1) < 0$. Then the line $R_2R_1U_2$ intersects K^0 (see Figure 2), and again (3) is violated, as a similar argument will show.

Now let 1) be satisfied, and assume $\alpha_1(h_2)\alpha_2(h_1) < 4$. By making use of the freedom inherent in the definition of the α_i, we may obtain $\alpha_1(h_2) = \alpha_2(h_1)$. Define a scalar multiplication in V by

$$(h_1, h_1) = (h_2, h_2) = 1, \quad (h_1, h_2) = \frac{1}{2}\alpha_1(h_2). \tag{13}$$

Since $|\alpha_1(h_2)| < 2$, this scalar product is positive definite. It is clear from (13) that $\alpha_i(v) = 2(h_i v)$ $(i = 1, 2)$. Hence U_i is orthogonal to h_i, and the R_i are orthogonal reflections in the sense of our scalar product (see Figure 3). The angle ϕ enclosed by K_1 and K_2 is given by $\alpha_1(h_2) = 2\cos\phi$, or equivalently, $\alpha_1(h_2)\alpha_2(h_1) = 4\cos^2\phi$. Obviously R_1R_2 is a rotation by an angle of 2ϕ, and (3) holds if and only if ϕ is a submultiple of π. This proves our assertion in the case $\alpha_1(h_2)\alpha_2(h_1) < 4$.

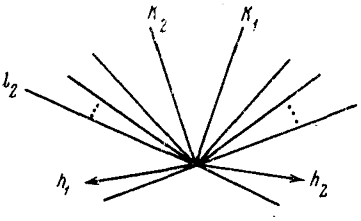

Figure 4

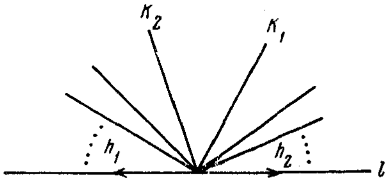

Figure 5

If $\alpha_1(h_2)\alpha_2(h_1) > 4$, we proceed as in the preceding case except for the fact that the scalar product is now indefinite, and R_1R_2 is a hyperbolic rotation. (3) is automatically satisfied in this case, and the images of K under Γ fill the angle between the isotropic lines l_1 and l_2 (see Figure 4).

Finally, if $\alpha_1(h_2)\alpha_2(h_1) = 4$, we may obtain $\alpha_1(h_2) = \alpha_2(h_1) = -2$. It follows that $h_1 + h_2 = 0$, and the line l through h_1 and h_2 is invariant under Γ. R_1R_2 is unipotent transformation. Condition (3) is satisfied, and the images of K under Γ fill a halfplane bounded by l (see Figure 5).

3. While proving Proposition 6, we have established several properties of linear Coxeter groups generated by two elements. We collect them in the following proposition.

Proposition 7. *Let Γ be a linear Coxeter group generated by R_1 and R_2.*

1) *If $\alpha_1(h_2)\alpha_2(h_1) = 4\cos^2\pi/k$, then the order of R_1R_2 is equal to k; in all other cases it is infinite.*

2) *$\bigcup_{\gamma\in\Gamma}\gamma K$ is a convex cone. Specifically if $\alpha_1(h_2)\alpha_2(h_1) < 4$ then $\bigcup_{\gamma\in\Gamma}\gamma K = V$; if $\alpha_1(h_2)\alpha_2(h_1) = 4$, then $\bigcup_{\gamma\in\Gamma}\gamma K$ is a halfspace; if $\alpha_1(h_2)\alpha_2(h_1) > 4$, then $\bigcup_{\gamma\in\Gamma}\gamma K$ is a dihedral cone.*

Note that a linear Coxeter group with two generators, like any group generated by two elements of order two, is an abstract Coxeter group.

§ 3. Linear Coxeter groups with an arbitrary number of generators

1. In this section we prove the following two theorems.

Theorem 1. *Let Γ be a linear group generated by reflections $R_1, \cdots, R_m$ in the faces of a convex polyhedral cone K. K is a fundamental chamber of Γ if and only if for every pair of adjacent faces K_i and K_j the dihedral cone $K(i, j)$ defined by* (12) *for* $S = \{i, j\}$ *is a fundamental chamber of the subgroup* Γ_{ij} *generated by* R_i *and* R_j.

A method for verifying this assumption is included in Proposition 6.

Theorem 2. *Let* Γ *be a discrete linear group generated by reflections* $R_1, \dots, R_m$ *in the faces of a convex polyhedral cone* K. *For any* $x \in K$ *let* Γ_x *denote the subgroup of* Γ *generated by reflections in those faces of* K *which contain* x. *Define* $K^f = \{x \in K: \Gamma_x$ *is finite*$\}$. *Then the following assertions are true.*

1) $\bigcup_{\gamma\in\Gamma}\gamma K$ *is a convex cone.*

2) Γ *acts discretely on the interior* C *of this cone.*

3) $C \cap K = K^f$.

4) *The canonical map from* K^f *to* C/Γ *is a homeomorphism.*

5) *For every* $x \in K$, Γ_x *is the stabilizer of* x *in* Γ.

6) *For every pair of adjacent faces* K_i, K_j *of* K, *let* n_{ij} *denote the order of* R_iR_j (n_{ij} *may be infinite*). *Then*

$$R_i^2 = 1, \quad (R_iR_j)^{n_{ij}} = 1$$

is a system of defining relations for Γ.

The last assertion implies that Γ is an abstract Coxeter group.

The conditions of Theorem 1 are necessary by Proposition 5. Their sufficiency will be proved along with Theorem 2.

2. Let Γ be a group of linear transformations in an n-dimensional linear space V generated by reflections in the faces of a convex polyhedral cone K. Suppose that Γ satisfies the assumptions of Theorem 1.

Let $\overline{\Gamma}$ be the *abstract* Coxeter group with generators $r_1, \dots, r_m$ and exponents n_{ij} defined as follows, for $i \neq j$:

1) if K_i and K_j are adjacent faces, n_{ij} is the (possibly infinite) order of R_iR_j;

2) if K_i and K_j are not adjacent, $n_{ij} = \infty$.

Let η be the homomorphism of $\overline{\Gamma}$ onto Γ defined by $\eta(r_i) = R_i$; η induces an action of $\overline{\Gamma}$ on V, viz. $\gamma v = \eta(\gamma)v$ for $\gamma \in \overline{\Gamma}$.

It is clear from the definition of n_{ij} that the standard subgroup $\overline{\Gamma}_{ij}$ of $\overline{\Gamma}$ generated by any two generators r_i and r_j is mapped by η isomorphically onto the subgroup Γ_{ij} of Γ generated by R_i and R_j.

Consider the universal space $\mathfrak{U} = \mathfrak{U}(\overline{\Gamma}, K)$, where the $(n-1)$-dimensional faces of K are the K_i and where $\psi: \mathfrak{U} \to V$ is the mapping described in Proposition 1, for $\mathfrak{V} = V$ and ϕ equal to the identity embedding of K in V.

Our immediate goal will consist in proving

Proposition 8. ψ *is injective, and* $\psi(\mathfrak{U}) = \bigcup_{\gamma\in\Gamma}\gamma K$ *is a convex cone.*

The following assertions are consequences of Proposition 8:

1) K is a fundamental chamber for Γ, i.e. Theorem 1;

2) η is an isomorphism, i.e. 6) of Theorem 2;

3) part 5) of Theorem 2;

4) part 1) of Theorem 2;

For the proof of Proposition 8 we introduce the concept of a *segment* in $\mathfrak{U}$; this is a continuous curve $s(t)$, $0 \leq t \leq 1$, whose image is a segment in V with the usual parametrization. Further, a *chamber* in $\mathfrak{U}$ will be any set $K(\gamma) = \{[\gamma, x] : x \in K\}$. Every chamber is canonically homeomorphic to K. The intersection of a segment and a chamber is a segment.

Lemma 1. *Every segment in $\mathfrak{U}$ is contained in the union of a finite number of chambers.*

Proof. Since segments are compact, we need only show that every point of a segment has a neighborhood contained in the union of a finite number of chambers. Without loss of generality we may assume that the point is of the form $[1, x]$. Let W be a star-shaped neighborhood of x in K disjoint from every K_i to which x does not belong. Then the special neighborhood $O(x, W)$ of $[1, x]$ in $\mathfrak{U}$ is also star-shaped. The intersection of any segment containing $[1, x]$ with this neighborhood is contained in the union of at most two chambers.

In subsections 3 and 4 we shall prove

Proposition 9. *Any two points of $\mathfrak{U}$ may be connected by a segment.*

This entails Proposition 8. Indeed let $u_1, u_2 \in \mathfrak{U}$ and $\psi(u_1) = \psi(u_2)$. Let u_1u_2 be the connecting segment. Its image is a single point. Since ψ is one-to-one on every chamber, the intersection of u_1u_2 with any chamber cannot consist of more than one point. By Lemma 1 this cannot occur unless u_1u_2 consists of one point only; that is, $u_1 = u_2$. The second assertion of Proposition 8 follows from Proposition 9 in an obvious way.

3. Let L be the union of all those $(n-2)$-dimensional faces $K_{ij} = K_i \cap K_j$ of K such that $n_{ij} = \infty$, and of all (closed) $(n-3)$-dimensional faces of K. Define

$$K' = K/L \quad \text{and} \quad \mathfrak{U}' = \{[\gamma, x] \in \mathfrak{U} : x \in K'\}.$$

For every $x \in K'$, $\eta : \overline{\Gamma} \to \Gamma$ induces an isomorphism from $\overline{\Gamma}_x$ to Γ_x. $\mathfrak{U}'$ is obviously open in $\mathfrak{U}$. Moreover if $x \in K'$, then every special neighborhood of $[1, x]$ is contained in $\mathfrak{U}'$.

Lemma 2. *The restriction of ψ to $\mathfrak{U}'$ is a local homeomorphism.*

Proof. It is enough to show that every $u \in \mathfrak{U}'$ has a neighborhood which is mapped bijectively on a neighborhood of $\psi(u)$ in V. Since ψ commutes with the action of $\overline{\Gamma}$ there is no loss in generality in considering only the case $u = [1, x]$, $\psi(u) = x \in K'$. Let $O(x, W)$ be any special neighborhood of $[1, x]$. Its image is $\bigcup_{\gamma \in \Gamma_x} \gamma W$. Three cases are possible:

1) $x \in K^0$.

2) x is interior to some $(n-1)$-dimensional face.

3) x lies on a $(n-2)$-dimensional face.

In all three cases $\bigcup_{\gamma\in\Gamma_x}\gamma W$ is obviously a neighborhood of x in V, and for γ_1, $\gamma_2\in\Gamma_x$ and $W_1, W_2\in W$ we have $\gamma_1 w_1=\gamma_2 w_2$ only if $w_1=w_2$ and $\gamma_1^{-1}\gamma_2\in\Gamma_{w_1}$; that is, ψ is one-to-one on $O(x, W)$.

It follows from this lemma that if ψ is one-to-one on a compact set $\mathfrak{K}\subset\mathfrak{U}'$, then ψ is still one-to-one on some neighborhood of $\mathfrak{K}$. Specifically, we obtain

Lemma 3. *Every segment $S\subset\mathfrak{U}'$ has a neighborhood which is mapped homeomorphically to an open set in V.*

Let $\mathfrak{S}$ be the space of segments in $\mathfrak{U}$, topologized by pointwise convergence. Consider the continuous mapping $\epsilon : S\to\mathfrak{U}\times\mathfrak{U}$ defined by

$$\varepsilon(s)=(s(0), s(1)),\quad s\in\mathfrak{S}. \tag{14}$$

Let $\mathfrak{S}'$ be the subspace of $\mathfrak{S}$ consisting of the segments contained in $\mathfrak{U}'$.

Lemma 4. *The restriction of ϵ to $\mathfrak{S}'$ is one-to-one and open.*

Proof. Let $S\in\mathfrak{S}'$; let O be a neighborhood of s in $\mathfrak{U}'$ which is mapped homeomorphically to an open convex set $\psi(O)\subset V$. Let $\mathfrak{S}_0$ be the set of segments contained in O. Clearly $\mathfrak{S}_0=\epsilon^{-1}(O\times O)$ and ϵ is one-to-one on $\mathfrak{S}_0$. Moreover, convergence of segments in $\mathfrak{S}_0$ is implied by convergence of their endpoints. Hence $\mathfrak{S}_0$ is mapped homeomorphically onto $O\times O$.

4. We proceed now to the proof of Proposition 9.

For every $\gamma\in\overline{\Gamma}$, denote by $\mathfrak{S}(\gamma)$ (respectively, by $\mathfrak{S}'(\gamma)$) the set of those segments $s\subset\mathfrak{U}$ ($s\subset\mathfrak{U}'$) for which

$$s(0)\in K(1),\quad s(1)\in K(\gamma). \tag{15}$$

It follows from Lemma 4 that $\epsilon(\mathfrak{S}'(\gamma))$ is open in $K(1)\times K(\gamma)$ and homeomorphic to $\mathfrak{S}'(\gamma)$.

Assume $\mathfrak{S}'(\gamma)\neq\emptyset$, and let $\mathfrak{S}'_{\mathrm{conn}}(\gamma)$ be a connected component of $\mathfrak{S}'(\gamma)$.

Lemma 5. *There exists a finite set $\Delta(\gamma)\subset\Gamma$ such that $\bigcup_{\delta\in\Delta(\gamma)}K(\delta)$ contains all segments $s\in\mathfrak{S}'_{\mathrm{conn}}(\gamma)$.*

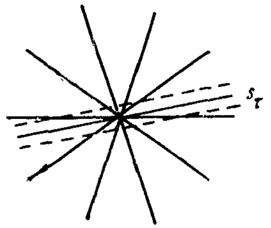

Figure 6

Proof. Let $\mathfrak{S}''(\gamma)$ consist of those $s\in\mathfrak{S}'_{\mathrm{conn}}(\gamma)$ which satisfy the following conditions.

1) $s(0) \in K^0(1)$ and $s(1) \in K^0(\gamma)$.

2) The intersections of s with the $(n-1)$-dimensional faces of all chambers consist of one point at most.

3) s does not intersect any $(n-2)$-dimensional face of any chamber.

It is easy to see that $\mathfrak{S}''(\gamma)$ is open and dense in $\mathfrak{S}''_{\mathrm{conn}}(\gamma)$. $\mathfrak{S}''(\gamma)$ need not be connected, but any two segments s_0, $s_1 \in \mathfrak{S}''(\gamma)$ may be connected in $\mathfrak{S}'_{\mathrm{conn}}(\gamma)$ by a continuous curve s_τ, $0 \le \tau \le 1$, such that every s_τ satisfies 1) and 2). Define $M_\tau = \{\delta \in \overline{\Gamma} : s_\tau \cap K(\delta) \neq \emptyset\}$. As τ runs from 0 to 1, M_τ cannot change anywhere except for values of τ such that s_τ intersects an $(n-2)$-dimensional face of some chamber. But in this situation the number of chambers meeting s remains unchanged, as a look at Figure 6 will make clear. So all $s \in \mathfrak{S}''(\gamma)$ intersect the same number of chambers, say $l(\gamma)$.

Let $\Delta(\gamma)$ consist of those $\delta \in \overline{\Gamma}$ such that there exists a sequence $\delta_1, \delta_2, \cdots, \delta_l \in \overline{\Gamma}$ satisfying the following conditions:

1) $\delta_1 = 1$ and $\delta_l = \delta$.

2) $l \le l(\gamma)$.

3) $K(\delta_i)$ and $K(\delta_{i-1})$ are adjacent chambers for $i = 2, \cdots, l$.

$\Delta(\gamma)$ is clearly finite, and every segment in $\mathfrak{S}''(\gamma)$ is contained in $\bigcup_{\delta \in \Delta(\gamma)} K(\delta)$; so also every segment in $\mathfrak{S}'_{\mathrm{conn}}(\gamma)$ lies in $\bigcup_{\delta\in\Delta(\gamma)} K(\delta)$.

Corollary. $\overline{\epsilon(\mathfrak{S}''(\gamma))} \subset \overline{\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma))} \subset \epsilon(\mathfrak{S}(\gamma))$.

Proof. Let $s_1, s_2, \cdots$ be a sequence of segments in $\mathfrak{S}'_{\mathrm{conn}}(\gamma)$ whose endpoints converge. Since all segments are contained in the compact set $\bigcup_{\delta\in\Delta(\gamma)} K(\delta)$, a convergent subsequence may be extracted. Call its limit s. Then $\epsilon(s) = \lim \epsilon(s_k)$.

Lemma 6. *Let u be an interior point of an $(n-2)$-dimensional face $K_{ij}(\delta)$ of $K(\delta)$, corresponding to the face K_{ij} of K. If $n_{ij} = \infty$, any segment s containing u as an interior point has further points in common with $K_{ij}(\delta)$ besides u.*

Proof. Assume $\delta = 1$. Let $s \subset \mathfrak{U}$ be a segment such that $s \cap K_{ij}(1) = \{u\}$. Let $\tilde{s}$ be the part of s lying in some special neighborhood O of u. It is easy to see that $\psi(\tilde{s}) \cap K_{ij} = \{\psi(u)\}$. It is clear from the structure of $\psi(O)$ (see Figures 4 and 5) that $\psi(u)$ cannot be an interior point of $\psi(\tilde{s})$; so u is not an interior point of s.

Now let $A(\gamma)$ be the subset of $K'(1) \times K'(\gamma)$ consisting of those (u_1, u_2) such that the segment in V connecting $\psi(u_1)$ and $\psi(u_2)$ either intersects an $(n-3)$-dimensional face of some cone δK, $\delta \in \Delta(\gamma)$, or has two points in common with an $(n-2)$-dimensional face of some δK. It is easy to see that $A(\gamma)$ is the union of finitely many manifolds of dimensions $\le 2n-2$. Hence

$$B(\gamma) = (K'(1) \times K'(\gamma)) \setminus A(\gamma) \tag{16}$$

is connected and dense in $K(1) \times K(\gamma)$.

Lemma 7. $\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma)) \supset B(\gamma)$.

Proof. Let $s \in \mathfrak{S}'_{\mathrm{conn}}(\gamma)$ be such that $\epsilon(s) \in B(\gamma)$. By Lemma 5, $s \in \bigcup_{\delta \in \Delta(\gamma)} K(\delta)$, and it follows from the definition of $B(\gamma)$ and from Lemma 6 that $s \subset \mathfrak{U}'$; so $s \in \mathfrak{S}'(\gamma)$. Moreover, $s \in \mathfrak{S}'_{\mathrm{conn}}(\gamma)$, since the components of $\mathfrak{S}'(\gamma)$ are closed in $\mathfrak{S}'(\gamma)$. So

$$\overline{\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma))} \cap B(\gamma) \subset \epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma)).$$

Now, applying the Corollary following Lemma 5, we find that $\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma)) \cap B(\gamma)$ is closed in $B(\gamma)$. On the other hand, $\epsilon(\mathfrak{S}'(\gamma))$ is open in $K(1) \times K(\gamma)$, so its component $\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma))$ is open as well. Hence $\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma)) \cap B(\gamma)$ is open relative to $B(\gamma)$. Since $B(\gamma)$ is connected, this implies that $\epsilon(\mathfrak{S}'_{\mathrm{conn}}(\gamma)) \cap B(\gamma) = B(\gamma)$, and Lemma 7 is proved.

Since $B(\gamma)$ is dense in $K(1) \times K(\gamma)$, Lemma 7 and the Corollary following Lemma 5 imply

$$\varepsilon(\mathfrak{S}(\gamma)) = K(1) \times K(\gamma), \tag{17}$$

which means that any two points $u_1 \in K(1)$ and $u_2 \in K(\gamma)$ may be connected by a segment.

So we have the following result: if $\mathfrak{S}'(\gamma) \neq \emptyset$, then $\epsilon(\mathfrak{S}'(\gamma)) \supset B(\gamma)$ and $\epsilon(\mathfrak{S}'(\gamma)) = K(1) \times K(\gamma)$.

Now Proposition 9 is easily proved. Indeed, it is obvious that $\mathfrak{S}'(1) \neq \emptyset$. We shall show that $\mathfrak{S}'(\gamma) \neq \emptyset$ implies $\mathfrak{S}'(\gamma r_i) \neq \emptyset$. We have $K(\gamma) \cap K(\gamma r_i) = K_i(\gamma)$, the $(n-1)$-dimensional face of $K(\gamma)$ corresponding to K_i. Further, $(K(1) \times K(\gamma)) \cap B(\gamma) \neq \emptyset$ since $\dim\,(K(1) \times K_i(\gamma)) = 2n - 1$. Hence $(K(1) \times K(\gamma)) \cap \epsilon(\mathfrak{S}') \neq \emptyset$, so $\mathfrak{S}'(\gamma r_i) \neq \emptyset$. Now every $\gamma \in \Gamma$ may be written as a product of generators r_i: so $\mathfrak{S}'(\gamma) \neq \emptyset$ for every γ. Hence $\epsilon(\mathfrak{S}) \supset K(1) \times \mathfrak{U}$: since $\epsilon(\mathfrak{S})$ is invariant under the action of Γ, $\epsilon(\mathfrak{S}) = \mathfrak{U} \times \mathfrak{U}$.

5. It remains to prove parts 2)–4) of Theorem 2. To this end we prove

Proposition 10. *The restriction of ψ to $\mathfrak{U}^f$ (see § 1.2) is a homeomorphism onto the interior C of $\bigcup_{\gamma \in \Gamma} \gamma K$.*

Note that from Proposition 8 and Lemma 2 it is already known that ψ maps $\mathfrak{U}'$ homeomorphically to an open subset C' of C.

By a *mirror* of Γ in V, we shall mean any subspace γU_i $(\gamma \in \Gamma)$; that is, the subspace of fixed points for the reflection $\gamma R_i \gamma^{-1}$. (It is easy to see that all reflections in Γ are of this kind.)

Lemma 8. *Let $x \in \bigcup_{\gamma \in \Gamma} \gamma K$. The stabilizer of x is finite if and only if the number of mirrors of Γ containing x is finite.*

Proof. We may assume $x \in K$. A mirror U, corresponding to the reflection R, contains x if and only if $Rx = x$. By 5) of Theorem 2, this is equivalent to $R \in \Gamma_x$. Hence, if Γ_x is finite, so is the number of mirrors containing x. Conversely, let the latter number be finite. Since $x \in \gamma K$ for $\gamma \in \Gamma_x$, all these γK are decomposed by the mirrors containing x. Hence there are only a finite number of these chambers, i.e. Γ_x is finite.

Lemma 9. *Any two points $x, y \in \bigcup_{\gamma\in\Gamma}\gamma K$ are separated by only a finite number of mirrors of Γ.*

Proof. Let xy denote the segment from x to y. A mirror U separates x and y if and only if it intersects xy without containing either x or y. If $x, y \in C'$, then xy intersects only finitely many mirrors. Now let x and y be arbitrary in $\bigcup_{\gamma\in\Gamma}\gamma K$, and select star-shaped neighborhoods O_x and O_y of their inverse images in $\mathfrak{U}$ such that $\psi(O_x)$ ($\psi(O_y)$) is disjoint from all mirrors not containing x (resp. y). Since $\mathfrak{U}'$ is dense in $\mathfrak{U}$, there exist points $x' \in \psi(O_x) \cap C'$ and $y' \in \psi(O_y) \cap C'$. A mirror separating x and y will also separate x' and y'; hence the lemma is true.

It follows from Lemmas 8 and 9 that $\psi(\mathfrak{U}^f)$ is convex. Indeed, let $x, y \in \psi(\mathfrak{U}^f)$. By Lemma 8, only finitely many mirrors of Γ contain x and y. Let z be an interior point of the segment xy. By Lemma 9 only finitely many mirrors contain z but not x and y. Hence z lies on a finite number of mirrors only; so, again by Lemma 8, the stabilizer of z is finite, i.e. $z \in \psi(\mathfrak{U}^f)$.

Lemma 10. $\bigcup_{\gamma\in\Gamma}\gamma K = V$ *if Γ is finite.*

Proof. If Γ is finite, there exists in V a positive definite scalar product invariant under Γ; so Coxeter's results [1] apply.

By Lemma 10, ψ maps every special $\mathfrak{U}$-neighborhood of $x \in K^f$ onto an open set in V. Since ψ commutes with the action of Γ, this means that ψ is open on $\mathfrak{U}^f$.

So $\psi(\mathfrak{U}^f)$ is open and convex. Moreover, $\psi(\mathfrak{U}^f)$ is dense in $\bigcup_{\gamma\in\Gamma}\gamma K$; hence it is identical to the interior of the latter cone; that is, $\psi(\mathfrak{U}^f) = C$. The restriction of ψ to $\mathfrak{U}^f$ is continuous, one-to-one and open; this means that it is a homeomorphism. The proof of Proposition 10 is complete.

Part 3) of Theorem 2 follows directly from Proposition 10; just as immediately, parts 2) and 4) are derived from Proposition 4. So Theorems 1 and 2 are completely proved.

§4. Some systems of linear inequalities

1. The present section will be based on certain well-known results on systems of linear inequalities. They are all variants of the basic theorem which says that if the homogeneous linear inequality $\alpha \geq 0$ is a consequence of the system of homogeneous linear inequalities $\alpha_i \geq 0$, $i = 1, \cdots, m$, then $\alpha = \sum_{i=1}^{m} c_i\alpha_i$, $c_i \geq 0$. We mention two further theorems of this sort.

The system of linear inequalities $\alpha_i > 0$, $i = 1, \cdots, m$, has a solution if and only if the α_i do not obey any nontrivial linear relation with nonnegative coefficients.

For the second theorem, we introduce the following notation: if $X = (X_1, \cdots, X_m) \in \mathbf{R}^m$, we write $X > 0$ if $X_i > 0$ for every i, and $X \geq 0$ if $X_i \geq 0$ for every i.

Now let A be any matrix. If there does not exist a column vector $Y \geq 0$, $Y \neq 0$, such that $A'Y \geq 0$, then there exists $X > 0$ such that $AX < 0$ (A' denotes the transpose of A).

2. Let A be a square matrix. We shall say that A is the *direct sum* of $A_1, \cdots, A_k$, denoted $A = A_1 \oplus \cdots \oplus A_k$, if some permutation applied simultaneously to the rows and columns transforms A into

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & A_k \end{pmatrix}. \tag{18}$$

A matrix will be called *indecomposable* if it cannot be represented as the direct sum of two matrices. Clearly every matrix decomposes in a unique way into a direct sum of indecomposable matrices. The latter will be called (indecomposable) *components* of A.

3. In this section we consider $m \times m$ matrices satisfying

(C1) $a_{ij} \le 0$ for $i \neq j$; $a_{ij} = 0$ implies $a_{ji} = 0$.

By $X, Y, \cdots$ we denote column vectors.

Lemma 11. *Let A be an indecomposable matrix satisfying* (C1). *Then*

$$X \geqslant 0, \quad X \neq 0, \quad AX \geqslant 0 \Rightarrow X > 0.$$

Proof. Suppose $X_i = 0$ for $i \le k$ and $X_i \neq 0$ for $i > k$. Since $AX \ge 0$, it follows from (C1) that $a_{ij} = a_{ji} = 0$ for $i \le k$, $j > k$, contradicting the hypothesis that A is indecomposable.

Theorem 3. *Suppose that A is indecomposable and satisfies* (C1). *Then exactly one of the following three assertions holds, simultaneously for A and its transpose*:

(P) *A is nonsingular, and $AX \ge 0$ implies $X > 0$ or $X \ge 0$.*

(Z) *A has rank $m-1$; there exists $X > 0$ such that $AX = 0$; $AX \ge 0$ implies $AX = 0$.*

(N) *There exists $X > 0$ such that $AX < 0$; $X \ge 0$ and $AX \ge 0$ imply $X = 0$.*

According as (P), (Z) or (N) holds, we shall refer to A as a matrix of *positive*, *zero* or *negative type*.

Proof. On replacing X by $-X$, we find in cases (P) and (Z) that

$$\nexists X \geqslant 0 : AX \leqslant 0, \quad AX \neq 0. \tag{19}$$

Hence (P) and (Z) are incompatible with (N). (P) and (Z) are mutually exclusive by the difference in rank.

Now assume

$$\exists X \geqslant 0, \quad X \neq 0 : AX \geqslant 0. \tag{20}$$

We shall show that either (P) or (Z) holds. Define

$$K_A = \{X : AX \geqslant 0\}. \tag{21}$$

By Lemma 11, $K_A \cap \{X : X \geq 0\} \subset \{X : X > 0\} \cup \{0\}$. This is possible in two cases only: either

$$K_A \subset \{X : X > 0\} \cup \{0\}, \tag{22}$$

or

$$K_A \text{ is a line.} \tag{23}$$

In the first case, (P) holds. Indeed, the second part of (P) is identical with (22), and A is a nonsingular in this case since K_A is strictly convex.

If (23) holds, we have obviously $K_A = \{X : AX = 0\}$, so that (Z) is true.

From what we have proved, it follows that (20) implies (19); but in view of subsection 1, (19) in its turn implies that (20) holds for A'. So if (20) holds, both A and A' satisfy the same one of conditions (P) and (Z).

If (20) does not hold for A and A', by subsection 1 they both satisfy (N). This completes the proof of the theorem.

Remark. By Frobenius' theorem, any matrix A satisfying the assumptions of Theorem 3 has real eigenvalues, and if a is the smallest real eigenvalue, there exists a corresponding eigenvector whose components are positive. It is not hard to see that $a > 0$, $a = 0$ or $a < 0$ according as A is of positive, zero or negative type.

Let $A = (a_{ij})$ be an $m \times m$ matrix and S a subset of $\mathbf{I}_m = \{1, 2, \cdots, m\}$. Let A_S be the submatrix of A consisting of the a_{ij} such that $i, j \in S$. We shall call these submatrices *principal submatrices* of A.

Proposition 11. *If A is of positive or zero type, every principal proper submatrix A_S of A is the direct sum of matrices of positive type.*

Proof. Let $X > 0$ and $AX \geq 0$. Let X_S be the vector consisting of the X_i such that $i \in S$. Obviously $A_S X_S \geq 0$, and $A_S X_S = 0$ only if $a_{ij} = 0$ for all $i \in S$, $j \notin S$, which cannot occur since A is indecomposable. Hence A_S cannot have any components of negative or zero type.

Proposition 12. *A symmetric indecomposable matrix satisfying* (C1) *is of positive* (*resp. zero*) *type if and only if it is positive definite* (*resp., positive semidefinite and singular*).

Proof. Let A be a symmetric matrix of positive or zero type; let $X > 0$ and $AX \geq 0$. Then $(A + \lambda E)\, X > 0$ for every positive λ, so $A + \lambda E$ is of positive type, hence nonsingular. It follows that A does not possess negative eigenvalues, QED.

Now let A be symmetric and of negative type; choose $X > 0$ such that $AX < 0$. Then $X'AX < 0$; so A is not positive semidefinite.

4. For any matrix A we shall denote by $L_r(A)$ the space of linear relations among its rows, and by $L_r^+(A)$ the convex cone in $L_r(A)$ consisting of the relations with nonnegative coefficients.

If A satisfies (C1), let A^+ (resp. A^0, A^-) denote the direct sum of its components of positive (zero, negative) type. We have $A = A^+ \oplus A^0 \oplus A^-$ and

$$L_r^+(A) = L_r^+(A^0) \tag{24}$$

under the canonical identification of $L_r(A^0)$ with a subspace of $L_r(A)$.

Let $\alpha_1, \cdots, \alpha_m$ be nonvanishing linear functionals on V, and let $b_1, \cdots, b_m$ be vectors in V. Set $a_{ij} = \alpha_i(b_j)$ and $A = (a_{ij})$. Every linear relation among the α_i implies the same relation among the rows of A. Let L_α be the space of linear relations among the α_i. We have

$$L_\alpha \subset L_r(A). \tag{25}$$

Let K denote the convex cone defined by the system of inequalities

$$\alpha_i \geqslant 0, \qquad i = 1, \ldots, m. \tag{26}$$

Proposition 13. *Suppose that A satisfies* (C1). *K has nonempty interior if and only if*

$$L_\alpha \cap L_r^+(A^0) = \{0\}. \tag{27}$$

If all a_{ii} are positive, then among the inequalities (26) *no one is redundant, i.e. K is an m-sided cone.*

Proof. K has interior points if and only if the system of strict inequalities $\alpha_i > 0$, $i = 1, \cdots, m$, is solvable. Hence the first assertion follows from the theorem mentioned in subsection 1, on considering (24). For the second assertion, it suffices to remark that $a_{ii} > 0$ implies $-b_i \notin K$ even though $\alpha_j(b_i) \leq 0$ for $j \neq i$.

Let

$$Z(S) = \{i : a_{ij} = 0 \text{ for all } j \in S\} \tag{28}$$

for every subset S of $\mathbf{I}_m$. Let S^+ (resp. S^0, S^-) denote the subset T of S such that $A_T = A_S^+$ (A_S^0, A_S^-).

Theorem 4. *Let $\alpha_1, \cdots, \alpha_m \in V^*$, $\alpha_i \neq 0$, $b_1, \cdots, b_m \in V$; suppose that the matrix $A = (a_{ij})$, $a_{ij} = \alpha_i(b_j)$, satisfies* (C1), *and that the cone K defined by* (26) *has nonempty interior. Let S be a subset of $\mathbf{I}_m$ satisfying one of the following assumptions:*

1) $S^0 \cup S^- \in \sigma(\mathfrak{F}K)$ *(this holds in particular if $S = S^+$).*
2) $S = S^0$ *and* $Z(S)^0 = \emptyset$.

Then $S \in \sigma(\mathfrak{F}K)$.

(On $\mathfrak{F}K$ and σ, cf. subsection 5 of the Introduction.)

Proof. Note first that $S \in \sigma(\mathfrak{F}K)$ if and only if

$$\alpha_i = 0 \text{ for } i \in S, \quad \alpha_i > 0 \text{ for } i \notin S \tag{29}$$

is solvable. This is the case if and only if no linear relation $\Sigma X_i \alpha_i = 0$ holds such that the X_i for $i \notin S$ are nonnegative and not all of them are zero.

Suppose $\Sigma X_i \alpha_i = 0$ and $X_i \geq 0$ for $i \notin S$. Let X be the row vector with components X_i. Then $XA = 0$.

For any $M \subset \mathbf{I}_m$, let X_M be the part of X formed by the X_i such that $i \in M$. Obviously $X_M A_M \geq 0$ if $X_i \geq 0$ for all $i \notin M$.

In particular, $X_S A_S \geq 0$; hence $H_{S+} \geq 0$. If $S^0 \cup S^- \in \sigma(\mathfrak{F}K)$, we must have $X_i = 0$ for all $i \notin S^0 \cup S^-$, so a fortiori for all $i \notin S$.

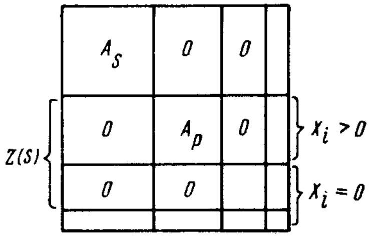

Figure 7

Assume that 2) holds. Then if $X_S A_S \geq 0$, we have $X_S A_S = 0$. Define $P = \{i \notin S: X_i > 0\}$. By combining the relations $XA = 0$ and $X_S A_S = 0$, we find $P \subset Z(S)$ and $X_P A_P = 0$. Since $X_P > 0$, we have $A_P = A_P^0$. Moreover, from $\Sigma_i X_i a_{ij} = 0$ for $j \in Z(S) \setminus P$ we find $a_{ij} = 0$ for $i \in P$, $j \in Z(S) \setminus P$ (see Figure 7). So $P \subset Z(S)^0$. Under our assumption, this implies $P = \emptyset$, i.e. $X_i = 0$ for $i \notin S$, QED.

Remark. By applying Theorem 4 to a one-element set S, we obtain once more the second part of Proposition 13.

5. Suppose that the inequalities $\alpha_i \geq 0$, $i = 1, \cdots, m$, determine an m-sided cone. Let K_i denote the face $\alpha_i = 0$.

Proposition 14. *Let $j \in \mathbf{I}_m$, let S consist of those s such that K_s is adjacent to K_j, and let $i \notin S \cup \{j\}$. Then there exist numbers $c > 0$ and $c_s \geq 0$ ($s \in S$) such that $\alpha_i = -c\alpha_j + \Sigma_{s \in S} c_s \alpha_s$.*

Proof. The system

$$\alpha_j = 0, \quad \alpha_s \geqslant 0 \text{ for } s \in S \tag{31}$$

is obviously sufficient to describe K_j. So $\alpha_i \geq 0$ follows from (31). Hence $\alpha_i = c\alpha_j + \Sigma_{s \in S} c_s \alpha_s$, $c_s \geq 0$ for $s \in S$ (see subsection 1). But c has to be positive, for otherwise the system $\alpha_s \geq 0$, $s \in S \cup \{j\}$, would imply $\alpha_i \geq 0$, in contradiction to our hypothesis.

§5. The Cartan matrix and characteristic of a linear Coxeter group

1. Let V be a linear space, and let V^* be its dual. For systems $b_1, \cdots, b_m \in V$,

$\alpha_1, \cdots, \alpha_m \in V^*$ we define the following invariants:

1) the *Cartan matrix* $A = (a_{ij})$, $a_{ij} = \alpha_i(h_j)$;

2) the space L_h of linear relations among the h_i, considered as a subspace of $\mathbf{R}^m$;

3) the space L_α of linear relations among the α_i;

4) the defect $d = \dim \operatorname{Ann}[\alpha]/[h] \cap \operatorname{Ann}[\alpha]$, where $[h]$ (resp. $[\alpha]$) denotes the linear envelope of the h_i (the α_i).(2)

The set $\{A, L_h, L_\alpha, d\}$ will be called the *characteristic* of the system $\{h_i, \alpha_i\}$. Obviously L_h (resp. L_α) is a subspace of $L_{\mathbf{c}}(A)$ ($L_{\mathbf{r}}(A)$), the space of linear relations among the columns (rows) of A. Let d_h denote the codimension of L_h in $L_{\mathbf{c}}(A)$, and d_α the codimension of L_α in $L_{\mathbf{r}}(A)$.

Systems $\{h_i \in V, \alpha_i \in V^*, i = 1, \cdots, m\}$ and $\{\tilde{h}_i \in \tilde{V}, \tilde{\alpha}_i \in \tilde{V}^*, i = 1, \cdots, m\}$ will be called *isomorphic* if there exists an isomorphism from V to $\tilde{V}$ transforming h_i into $\tilde{h}_i$ and α_i into $\tilde{\alpha}_i$.

Proposition 15. *Let A be any m by m matrix, let L_h and L_α be subspaces of $L_{\mathbf{c}}(A)$ and $L_{\mathbf{r}}(A)$, respectively, and let k be a nonnegative integer. Then there exists a system $\{h_i \in V, \alpha_i \in V^*, i = 1, \cdots, m\}$ unique up to isomorphism, having $\{A, L_h, L_\alpha, d\}$ as its characteristic. Also,*

$$\dim V = \operatorname{rank} A + d_h + d_\alpha + d. \tag{32}$$

Proof. Let $\{h_i \in V, \alpha_i \in V^*\}$ have characteristic $\{A, L_h, L_\alpha, d\}$. Denote the rank of A by r. Suppose for simplicity that the principal r by r submatrix in the upper left corner of A has nonzero determinant. We define

$$[h]_0 = [h] \cap \operatorname{Ann}[\alpha], \quad [\alpha]_0 = [\alpha] \cap \operatorname{Ann}[h].$$

Clearly

$$[h] = [h_1, \ldots, h_r] \oplus [h]_0, \quad [\alpha] = [\alpha_1, \ldots, \alpha_r] \oplus [\alpha]_0. \tag{33}$$

Let V_0 and V_1 be the subspace of V such that

$$\operatorname{Ann}[\alpha] = [h]_0 \oplus V_0, \tag{34}$$

$$\operatorname{Ann}[\alpha_1, \ldots, \alpha_r] = \operatorname{Ann}[\alpha] \oplus V_1.$$

Then

$$V = [h_1, \ldots, h_r] \oplus [h]_0 \oplus V_1 \oplus V_0, \tag{35}$$

$$V^* = [\alpha_1, \ldots, \alpha_r] \oplus [\alpha]_0 \oplus W_1 \oplus W_0, \tag{36}$$

(2)For $M \subset V^*$, Ann M denotes the set $\{v \in V : \alpha(v) = 0 \text{ for all } \alpha \in M\}$.

where

$$W_1 = \mathrm{Ann}([h_1, \ldots, h_r] \oplus V_1 \oplus V_0),$$

$$W_0 = \mathrm{Ann}([h] \oplus V_1).$$

In the decompositions (35) and (36) there is duality between the following pairs of spaces: $[h_1, \cdots, h_r]$ and $[\alpha_1, \cdots, \alpha_r]$, $[h]_0$ and W_1, V_1 and $[\alpha]_0$, and V_0 and W_0; the remaining pairs are orthogonal by definition.

$[h]_0$ identifies canonically with $L_c(A)/L_h$; hence dim $[h]_0 = d_h$. Similarly $[\alpha]_0$ identifies with $L_r(A)/L_\alpha$, and dim $[\alpha]_0 = d_\alpha$. It is clear from (34) that dim $V_0 = d$. From these relations we obtain (32).

We write the columns of A as linear combinations of the first r columns:

$$a_{si} = \sum_{k=1}^{r} p_{ik} a_{sk}.$$

By the definition of $[h]_0$,

$$h_i \equiv \sum_{k=1}^{r} p_{ik} h_k \pmod{[h]_0}.$$

Thus, if A is known, the components of the vectors h_i in (35) may be computed. Similarly, the components of the α_i in (36) are determined.

It is clear from the preceding considerations that the system $\{h_i, \alpha_i\}$ is uniquely determined by its characteristic, and a method to compute it from its characteristic has been obtained. It consists in the following operations:

1) For V, take

$$V = \mathbf{R}^r \oplus (L_c(A)/L_h) \oplus (L_r(A)/L_\alpha)^* \oplus \mathbf{R}^d;$$

2) For h_i, $1 \le i \le r$, take the unit vectors of $\mathbf{R}^r$; for $i > r$, define h_i by

$$h_i = \sum_{k=1}^{r} p_{ik} h_k + \pi((-p_{i1}, \ldots, -p_{ir}, 0, \ldots, 1, \ldots, 0)),$$

where π is the canonical mapping from $\mathbf{R}^m$ to $\mathbf{R}^m/L_h$.

3) By 1),

$$V^* = \mathbf{R}^r \oplus (L_r(A)/L_\alpha) \oplus (L_c(A)/L_h)^* \oplus \mathbf{R}^d.$$

Now for $i = 1, \cdots, r$ define $\alpha_i = (a_{i1}, \cdots, a_{ir}) \in \mathbf{R}^r$, while for $i = r+1, \cdots, m$ the α_i are defined similarly as the corresponding h_i.

2. Let Γ be the subgroup of $GL(V)$ generated by the reflections

$$R_i : v \to v - \alpha_i(v) h_i \quad (i = 1, \ldots, m), \tag{37}$$

$\alpha_i(b_i) = 2$. The *characteristic* (and in particular, the *Cartan matrix*) of Γ is defined as the characteristic (resp. the Cartan matrix) of $\{b_i, \alpha_i\}$.

The b_i and α_i are defined up to transformations $b_i \to c_i^{-1} b_i$ and $\alpha_i \to c_i \alpha_i$. Hence the Cartan matrix of Γ is defined up to transformation by a diagonal matrix. If Γ is a discrete linear group generated by reflections and K is a fixed fundamental chamber of Γ, we adopt the convention of choosing the α_i so that K is contained in every halfspace $\alpha_i \geq 0$. This determines the Cartan matrix up to transformation by diagonal matrices with positive diagonal elements.

Matrices A and B will be called *equivalent* if $A = DBD^{-1}$ for a diagonal matrix D having positive diagonal elements. For distinct values of $i_1, \cdots, i_k$ the expresions

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}, \tag{38}$$

will be called *cyclic products* of $A = (a_{ij})$.

Proposition 16. *Matrices A and B satisfying* (C1) *are equivalent if and only if their cyclic products are identical.*

Proof. The fact that equivalent matrices have the same cyclic product is obvious. Conversely suppose that A and B have identical cyclic products. Then $a_{ij} = 0$ implies $a_{ij}a_{ji} = b_{ij}b_{ji} = 0$; hence $b_{ij} = 0$ by (C1). Similarly, $b_{ij} = 0$ implies $a_{ij} = 0$. If a_{ij} and b_{ij} are different from zero, define $c_{ij} = a_{ij}b_{ij}^{-1}$. We have $c_{ij} > 0$ and $c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_k i_1} = 1$ whenever all factors on the left are defined. Hence there exist $d_i > 0$ such that $c_{ij} = d_i d_j^{-1}$. It follows that $a_{ij} = d_i b_{ij} d_j^{-1}$ for all i and j, QED.

3. The characteristics of linear Coxeter groups have certain special properties.

Proposition 17. *The Cartan matrix of a linear Coxeter group always satisfies conditions* (C1) *and* (C2), *stated in subsection* 2 *of the Introduction.*

In particular, the theory of § 4 and Proposition 16 apply to the Cartan matrix of a linear Coxeter group.

Proof. Propositions 5 and 6 show that (C1) and (C2) hold for $\{i, j\}$ if K_i and K_j are adjacent faces of the fundamental chamber. If K_i and K_j are not adjacent, Proposition 14 shows that $\alpha_i = -c\alpha_j + \Sigma_{s\in S} c_s \alpha_s$ where $c > 0$, $c_s \geq 0$, and S consists of those s such that K_s is adjacent to K_j. In particular, $\alpha_i(b_j) = -2c + \Sigma_{s\in S} c_s \alpha_s(b_j) < 0$. So (C1) holds for all i and j. In order to prove (C2), we need only consider the case $a_{ij}a_{ji} < 4$. In this case $\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ is clearly of positive type. Theorem 4 applied to $S = \{i, j\}$ shows that K_i and K_j are adjacent, and, by the first part of the proof, (C2) holds in the case under consideration.

4. Linear groups $\Gamma \subset GL(V)$ and $\widetilde{\Gamma} \subset GL(\widetilde{V})$ generated by reflections $R_1, \cdots, R_m$ and $\widetilde{R}_1, \cdots, \widetilde{R}_m$ will be called *isomorphic* if there exists an isomorphism from V to $\widetilde{V}$ transforming R_i into $\widetilde{R}_i$. It is obvious that isomorphic groups have identical characteristics.

Theorem 5. *Let A be an m by m matrix satisfying* (C1) *and* (C2). *Let L_b be a subspace of $L_c(A)$, let L_α be a subspace of $L_r(A)$, and assume*

$$L_\alpha \cap L_r^+ (A^0) = \{0\}, \tag{39}$$

Let d be a nonnegative integer. Then there exists a linear Coxeter group with characteristic $\{A, L_b, L_\alpha, d\}$; moreover, this group is unique up to isomorphism.

Proof. Uniqueness follows from Proposition 15, since a linear Coxeter group is uniquely determined by the b_i and α_i.

To prove its existence, we recall that by Proposition 15, $\{A, L_b, L_\alpha, d\}$ is the characteristic of some system $\{b_i \in V, \alpha_i \in V^*, i = 1, \cdots, m\}$. Let Γ be the group generated by the reflections (37), and let K be the cone $\alpha_i \geq 0$, $i = 1, \cdots, m$. Proposition 13 shows that K is an m-sided cone; by Theorem 1 and Proposition 6, K is a fundamental chamber of Γ.

Corollary 1. *Let A be an m by m matrix satisfying* (C1) *and* (C2), *and suppose A has no component of zero type. Then there exists a linear Coxeter group with Cartan matrix A operating in a space V whose dimension is equal to the rank of A. This group is unique up to isomorphism.*

Proof. Formula (32), which describes the dimension of V, shows that a group with the required properties necessarily has characteristic $\{A, L_c(A), L_r(A), 0\}$. Since A has no components of zero type, (39) holds trivially; so there exists a group with this characterstic.

Corollary 2. *Every abstract Coxeter group is isomorphic to some linear Coxeter group.*

Proof. Let Γ be an abstract Coxeter group with exponents n_{ij}. For the linear Coxeter group we may take, for example, a group with characteristic $\{\text{Cos}\,\Gamma, L_c(\text{Cos}\,\Gamma), 0, 0\}$ (see subsection 4 of the Introduction).

Corollary 3. *Every standard subgroup of a linear Coxeter group is a linear Coxeter group (cf. Proposition 5).*

Proof. The Cartan matrix of a standard subgroup Γ_S of Γ is the principal submatrix A_S of the Cartan matrix A of Γ. Our assertion follows from the fact that the validity of (C1) and (C2) is inherited by A_S from A. The polyhedral cone $K(S)$ defined by (12) is a fundamental chamber for Γ_S.

For linear groups $\Gamma_i \subset GL(V_i)$ $(i = 1, \cdots, m)$ we define the *direct product* $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$ operating in $V = V_1 \oplus \cdots \oplus V_k$ by the formula

$$(\gamma_1, \ldots, \gamma_k)\ (v_1, \ldots, v_k) = (\gamma_1 v_1, \ldots, \gamma_k v_k).$$

It is easy to see that the *direct product of linear Coxeter groups is a linear Coxeter group* having for its fundamental chamber the direct product of the fundamental chambers of the factors; for its Cartan matrix, the direct sum of the Cartan matrices of the factors; for its L_b and L_α, the direct sums of the corresponding spaces for the Γ_i; and for its defect d, the sum of the defects of the Γ_i. Now we obtain from Theorem 5

Corollary 4. *Let Γ be a linear Coxeter group with characteristic $\{A, L_b, L_\alpha, d\}$. Suppose $A = A_1 \oplus \cdots \oplus A_k$, $A_i = A_{S_i}$, $d = d_1 + \cdots + d_k$. Denote by $L_b^{(i)}$ ($L_\alpha^{(i)}$) the space of linear relations valid among the b_s (resp. α_s) such that $s \in S_i$. If $L_b = L_b^{(1)} \oplus \cdots \oplus L_b^{(k)}$ and $L_\alpha = L_\alpha^{(1)} \oplus \cdots \oplus L_\alpha^{(k)}$, then Γ is a direct product of linear Coxeter groups Γ_i having characteristic $\{A_i, L_b^{(i)}, L_\alpha^{(i)}, d_i\}$. This holds in particular if $d_b = d_\alpha = 0$; that is, if $L_b = L_c(A)$ and $L_\alpha = L_r(A)$.*

We wish to mention another obvious fact, namely

Proposition 18. *Suppose Γ is a linear Coxeter group with characteristic $\{A, L_b, L_\alpha, d\}$. The following properties are equivalent:*

1) *The fundamental chamber of Γ is strictly convex.*
2) $[\alpha] = V^*$.
3) $d_b = d = 0$.

A group satisfying these conditions will be called *reduced*. $\{A, L_\alpha\}$ will be called the *characteristic of the reduced group*. If Γ is reduced, then

$$\dim V = \operatorname{rank} A + d_\alpha, \quad \dim [h] = \operatorname{rank} A. \tag{40}$$

If $\Gamma \subset GL(V)$ is an arbitrary linear Coxeter group with characteristic $\{A, L_b, L_\alpha, d\}$, we obtain a reduced linear Coxeter group Γ^r with characteristic $\{A, L_\alpha\}$ by considering the action of Γ on $V^r = V/\operatorname{Ann}[\alpha]$. The inverse image of the fundamental chamber of Γ^r under the canonical projection $V \to V^r$ is a fundamental chamber of Γ.

5. We wish to determine the invariant subspaces for linear groups generated by reflections.

We shall say that a set $S \subset \mathbf{I}_m$ *reduces* A if $a_{ij} = 0$ for $i \notin S$, $j \in S$.

Let $\Gamma \subset GL(V)$ be a linear group generated by the reflections (37), and let A be its Cartan matrix. For any $S \subset \mathbf{I}_m$ let $[b]_S$ ($[\alpha]_S$) be the linear space generated by the b_i (the α_i) such that $i \in S$. S reduces A if and only if

$$[h]_S \subset \operatorname{Ann}[\alpha]_{\mathbf{I}_m \setminus S}. \tag{41}$$

Proposition 19. *Let S be a set reducing A. Then every space lying between $[b]_S$ and $\operatorname{Ann}[\alpha]_{\mathbf{I}_m \setminus S}$ is invariant under Γ. Conversely, every Γ-invariant subspace of V lies between $[b]_S$ and $\operatorname{Ann}[\alpha]_{\mathbf{I}_m \setminus S}$ for some set S reducing A.*

Proof. The first assertion is an immediate consequence of (37). Now let W be an invariant subspace of V. Define $S = \{i : b_i \in W\}$. Suppose that $\alpha_i(v) \neq 0$ for some $v \in W$; then it is clear from (37) that $b_i \in W$, i.e. $i \in S$. So $W \subset \operatorname{Ann}[\alpha]_{\mathbf{I}_m \setminus S}$.

Moreover, $W \supset [h]_S$ by the definition of S. It only remains to be shown that S reduces A; but this follows from (41).

Corollary. *The group* Γ *is irreducible if and only if* A *is irreducible and* $d_h = d_\alpha = d = 0$.

(A matrix is called *irreducible* if $\emptyset$ and I_m are the only reducing sets of indices. If (C1) is assumed, this concept is equivalent to indecomposability.)

§ 6. Orthogonal Coxeter groups

1. Now we turn to the question of orthogonality for linear Coxeter groups (see subsection 3 of the Introduction). We need a preliminary result.

Lemma 12. *Let* $\Gamma \subset GL(V)$ *be a linear Coxeter group; let* $(\ ,\ )$ *be a scalar product in* $[h]$ *such that* $(h_i, h_i) = 2$ *for every* i. *The scalar product is* Γ*-invariant if and only if*

$$(h_i, h_j) = a_{ij}, \tag{42}$$

i.e. the Gramian of $\{h_1, \cdots, h_m\}$ *is the Cartan matrix of* Γ.

Proof. If $(h_i, h_i) = 2$, then the reflection R_i defined by (37) is orthogonal in the subspace $[h]$ if and only if $\alpha_i(v) = (h_i, v)$ for every $v \in [h]$. On replacing v by h_j we obtain (42).

Theorem 6. *A linear Coxeter group is orthogonal if and only if its Cartan matrix* A *is equivalent to a symmetric matrix. For every scalar product fulfilling the conditions of Definition* 4, *the* α_i *may be normed in such a way that* A *coincides with the Gramian of* $\{h_1, \cdots, h_m\}$.

Proof Suppose that in V there exists an invariant scalar product such that $(h_i, h_i) > 0$ for every i. Norm the α_i so that $(h_i, h_i) = 2$. Then, according to Lemma 12, A is the Gramian of $\{h_1, \cdots, h_m\}$, hence symmetric.

Conversely, assume that A is symmetric. Then $L_r(A) = L_c(A)$. Since $L_h \subset L_c(A)$, there exists in $[h]$ a scalar product satisfying (42). By Lemma 12 this product is Γ-invariant.

2. The next assertion is an easy consequence of Proposition 16.

Proposition 20. *A matrix* A *satisfying* (C1) *is equivalent to a symmetric matrix if and only if*

$$a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} \ldots a_{i_1 i_k} \tag{43}$$

for all values of $i_1, \cdots, i_k$.

We call a matrix satisfying (C1) *acyclic* if all its cyclic products of length greater than two vanish. (43) holds for every acyclic matrix.

Lemma 13. *Let A be an indecomposable matrix satisfying* (C1) *and* (C2). *If A is of positive type, it is acyclic. If A is of zero type, it is either acyclic or equivalent to*

$$\begin{pmatrix} 2 & -1 & 0 \ldots & 0 & -1 \\ -1 & 2 & -1 \ldots & 0 & 0 \\ 0 & -1 & 2 \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots & 2 & -1 \\ -1 & 0 & 0 \ldots & -1 & 2 \end{pmatrix}. \tag{44}$$

Proof. If A contains a cycle of length > 2, then it has a principal submatrix of the form

$$B = \begin{pmatrix} 2 & -b_1 & 0 \ldots & 0 & -b_k' \\ -b_1' & 2 & -b_2 \ldots & 0 & 0 \\ 0 & -b_2' & 2 \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots & 2 & -b_{k-1} \\ -b_k & 0 & 0 \ldots & -b_{k-1}' & 2 \end{pmatrix}, \quad b_i, b_j' > 0, \quad k > 2.$$

By Proposition 11, B is of positive or zero type; the latter case occurs if and only if A is of zero type and $B = A$.

Let Y be a vector such that $Y > 0$ and $BY \geq 0$. After replacing B by some equivalent matrix, we may assume that Y has all its coordinates equal to unity. In this case the sum of the coordinates of BY is equal to the sum of the elements of B. Hence

$$2k - \sum_{i=1}^{k} (b_i + b_i') \geqslant 0. \tag{45}$$

Note that, because of (C2), $b_i b_i' \geq 1$. Hence equality obtains in (45), and, moreover, $b_i = b_i' = 1$ for every i, i.e. B has the form (44). So B is of zero type, and hence coincides with A.

Corollary. *If the Cartan matrix of a linear Coxeter group has no component of negative type, the group is orthogonal.*

Proposition 21. *Let Γ be a linear Coxeter group with Cartan matrix A. Assume one of the following conditions is satisfied*:

1) *A has no component of negative type.*

2) Cos Γ (*see* (5)) *possesses neither components of negative type nor components*

of zero type having the form (44) *or* $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Then A *and* Cos Γ *are equivalent.*

Proof. By Proposition 16, the equivalence of A and Cos Γ will follow as soon as we have proved that they have the same cyclic products. For products of length > 2 this follows from Lemma 13 under either assumption, since A and Cos Γ have their nonzero elements in the same places. Furthermore, if $n_{ij} < \infty$, then because of (C2), the cyclic product $a_{ij}a_{ji}$ of A is identical with the corresponding cyclic product of Cos Γ. So only the cyclic products $a_{ij}a_{ji}$ for $n_{ij} = \infty$ remain to be considered.

If 1) holds, then we may infer from Proposition 12 that A is equivalent to a nonnegative semidefinite symmetric matrix; so $a_{ij}a_{ji} \leq 4$. This means that for $n_{ij} = \infty$ the cyclic product $a_{ij}a_{ji}$ is equal to 4, and so to the corresponding cyclic product of Cos Γ. If 2) holds, then, again by Proposition 12, Cos Γ is nonnegative semidefinite and it follows from Proposition 11 that all its principal submatrices of order 2 are positive definite (for Cos Γ does not have any components of zero type and order two); so $n_{ij} = \infty$ is impossible.

3. Now we consider the situation described in subsection 3 of the Introduction: X^n is a space of constant curvature, and Γ is a linear Coxeter group derived from a discrete group generated by reflections in X^n. We shall call Γ an *elliptic, parabolic* or *hyperbolic Coxeter group* according as $X^n = S^n$, E^n or Λ^n, provided the following additional condition is satisfied: if $X^n = S^n$, the fundamental polyhedron of Γ in S^n does not contain any pair of diametrically opposite points; if $X^n = E^n$, the fundamental polyhedron of Γ in E^n is bounded; if $X^n = \Lambda^n$, neither any proper plane of Λ^n nor any point at infinity is Γ-invariant.

It follows from this condition that the fundamental chamber of Γ is strictly convex. This is obvious in the first two cases; in the third case we derive it from the following lemma.

Lemma 14. *Every hyperbolic Coxeter group is irreducible (as a linear group).*

Proof. Let $\Gamma \subset GL(V)$ be a hyperbolic Coxeter group. Then in V there exists a Γ-invariant nondegenerate scalar product whose index of negativity is unity; here Λ^n is embedded in V as a connected component of the hyperboloid $(v, v) = -1$. Let $W \subset V$ be an invariant subspace different from 0 and V. If W is hyperbolic, $W \cap \Lambda^n$ is a Γ-invariant plane in Λ^n. If W is elliptic, the orthogonal complement of W is hyperbolic, and we are thrown back to the preceding case. Finally, if W is parabolic, the intersection of W with the isotropic cone defines a Γ-invariant point at infinity in Λ^n.

4. In the following Propositions 22–24 we assume that V is a linear space of dimension $n + 1$, $\Gamma \subset GL(V)$ is a *reduced* linear Coxeter group with m generators, and A is the Cartan matrix of Γ.

Proposition 22. *The following properties of* Γ *are equivalent*:

1) Γ *is an elliptic Coxeter group.*

2) *There exists in* V *a* Γ*-invariant positive definite scalar product.*

3) $A = A^+$.

4) Γ *is finite.*

Proof. 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 4) is obvious.

2) $\Rightarrow$ 3). We norm the h_i so that $(h_i, h_i) = 2$. The fact that R_i is orthogonal means that

$$\alpha_i(v) = (h_i, v) \text{ for all } v \in V. \tag{46}$$

So A is the Gramian of $\{h_1, \cdots, h_m\}$, hence nonnegative semidefinite. Proposition 12 shows that $A = A^+ \oplus A^0$. Now components of zero type cannot exist. Assume the contrary: then there exists a nontrivial linear relation with nonnegative coefficients involving the h_i. By (46) the same relation also holds for the α_i, a contradiction.

3) $\Rightarrow$ 2). Since A is nondegenerate, $d_a = 0$; it follows (40) that $m = n + 1$. So $\{h_1, \cdots, h_m\}$ is a basis of V. Proposition 21 shows that A is equivalent to a symmetric matrix. By Theorem 6 there exists in V a Γ-invariant scalar such that, with a suitable norming of the h_i, A is the Gramian. By Proposition 12 this scalar product is positive definite.

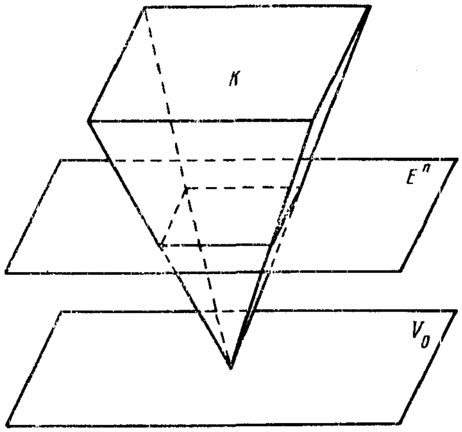

Figure 8

Proposition 23. *The following properties of* Γ *are equivalent.*

1) Γ *is a parabolic Coxeter group.*

2) *There exists a* Γ*-invariant n-dimensional subspace* V_0 *of* V *such that* $V_0 \cap K = \{0\}$, *and a* Γ*-invariant positive definite scalar product on* V_0.

3) $A = A^0$ *and* rank $A = n$.

4) *As an abstract Coxeter group,* Γ *is isomorphic to a parabolic group; its Coxeter diagram has* $m - n$ *connected components, and every component of type* A_l *(in Coxeter's notation) satisfies*

$$|a_{i_0 i_1} a_{i_1 i_2} \ldots a_{i_l i_0}| = \begin{cases} 1 & \text{for } l > 1, \\ 4 & \text{for } l = 1, \end{cases} \tag{47}$$

where $i_0, i_1, \cdots, i_l$ are the labels of the nodes of the Coxeter diagram belonging to the component in question, in standard order (along a circumference).

Proof. 1) $\Leftrightarrow$ 2) is obvious (see Figure 8).

2) $\Rightarrow$ 3). The assumption $V_0 \cap K = \{0\}$ implies $V_0 \not\subset \operatorname{Ann} \alpha_i$, so $h_i \in V_0$ for all i (see Proposition 19). We norm the h_i in such a way that $(h_i, h_i) = 2$. Now the orthogonality of R_i can be expressed thus:

$$\alpha_i(v) = (h_i, v) \quad \text{for} \quad v \in V_0. \tag{48}$$

Hence A is the Gramian of $\{h_1, \cdots, h_m\}$. By Proposition 12, $A = A^+ \oplus A^0$. Now it follows from $V_0 \cap K = \{0\}$ that the restrictions of the α_i to V_0 satisfy a linear equation with positive coefficients (see § 4.1). This same linear relation is also satisfied by the rows of A. So $A = A^0$. Finally, from (48), $n + 1 = \dim [\alpha] \leq \dim [h] + 1$. So $\operatorname{rank} A = \dim [h] = n$.

3) $\Rightarrow$ 2). Define $V_0 = [h]$. We have $\dim V_0 = \operatorname{rank} A = n$. Suppose $v = \Sigma X_i h_i \in K$. Let X be the column vector with coordinates X_i. Then $AX \geq 0$, whence $AX = 0$. So $v = 0$; that is, $V_0 \cap K = \{0\}$. Further, A is equivalent to a symmetric matrix, by Proposition 21.

By Theorem 6 there exists in V_0 a Γ-invariant scalar product making A the Gramian of $\{h_1, \cdots, h_m\}$, if the vectors are normed suitably. By Proposition 12, A is positive definite, and since $\operatorname{rank} A = n$, the scalar product in V_0 is also positive definite.

1) and 3) $\Rightarrow$ 4). There is an obvious correspondence between the connected components of the Coxeter diagram of Γ and the indecomposable components of its Cartan matrix. Since the rank of an indecomposable matrix of zero type is one less than its order and rank $A = n$, it follows that A has $m - n$ components. Now (47) follows from Proposition 21.

4) $\Rightarrow$ 3). Let $\overline{\Gamma}$ be a parabolic Coxeter group having the same exponents as Γ. We have already shown that the Cartan matrix of $\overline{\Gamma}$ has components of zero type only; so by Proposition 21 it is equivalent to $\operatorname{Cos} \overline{\Gamma} = \operatorname{Cos} \Gamma$. The components of $\operatorname{Cos} \Gamma$ of the form (44) or $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ correspond to the components of the Coxeter diagram of Γ, of type $\tilde{A}_l$. Hence by Proposition 21 it follows from (47) that A and $\operatorname{Cos} \Gamma$ are equivalent. So $A = A^0$.

Proposition 24. *The following properties of Γ are equivalent:*

1) *Γ is a hyperbolic Coxeter group.*

2) *Γ operates irreducibly on V, and there exists a Γ-invariant nondegenerate scalar product with index of negativity one such that $(h_i, h_i) > 0$ for every i.*

3) *A is indecomposable, of negative type, and equivalent to a symmetric matrix of rank* $n + 1$ *and index of negativity* 1.

Proof. From Lemma 14, 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 3). Let the b_i be so normed that $(b_i, b_i) = 2$. By Lemma 12 this implies that A is the Gramian of $\{b_1, \cdots, b_m\}$. The Corollary after Lemma 19 shows that A is indecomposable and $[b] = V$, since Γ is irreducible. So A is symmetric of rank $n + 1$ and index of negativity 1. By Proposition 12, A is of negative type.

3) $\Rightarrow$ 1). Since dim $[b] = \text{rank } A$, we have $[b] = V$. By Theorem 6 there exists in V a Γ-invariant scalar product such that, with a suitable norming of $b_1, \cdots, b_m$, A is their Gramian. It follows from the assumptions about A that this scalar product is nondegenerate and has index of negativity 1. By the Corollary to Lemma 19, A is indecomposable. By Proposition 12, A is of negative type.

Let X be a column vector satisfying $X < 0$ and $AX > 0$, and let $v_0 = \Sigma X_i b_i$. v_0 is an interior point of the fundamental chamber K of Γ, and $(v_0, v_0) = X'AX < 0$. For all $v \in K$ we have $(v_0, v) \leq 0$, since $(b_i, v) = \alpha_i(v)$. This means that K lies on one side of the n-dimensional elliptic subspace $W = \{v \in V : (v_0, v) = 0\}$. Let Λ^n denote the connected component of $(v, v) = -1$ lying on the same side of W, and let $P = K \cap \Lambda^n$.

Now Γ contains the group generated by the reflections in the faces of P. To show that the two groups are equal, we have to verify that every face K_i of K intersects Λ^n, or, equivalently, that it intersects the convex cone $\mathfrak{C} = \bigcup_{t>0} t\Lambda^n = \{v \in V: (v, v) < 0$ and $(v_0, v) < 0\}$. Let us define vectors $v_i = v_0 + R_i v_0 = 2v_0 - \alpha_i(v_0)b_i$. We have $v_i \in \mathfrak{C}$ since $R_i\mathfrak{C} \subset \mathfrak{C}$. On the other hand, $\alpha_i(v_i) = 0$ and $\alpha_j(v_i) > 0$ for $j \neq i$; hence $v_i \in K_i$. So $K_i \cap \mathfrak{C} \neq \emptyset$, QED.

Remark. It is easy to show that in 2) of Proposition 24 the condition $(b_i, b_i) > 0$ may be omitted.

§7. The cone C

1. For an arbitrary linear Coxeter group $\Gamma \subset GL(V)$ we denote by V_0 the maximal subspace contained in the closure of $\bigcup_{\gamma \in \Gamma} \gamma K$, or, equivalently, in the closure of $C = \bigcup_{\gamma \in \Gamma} \gamma K^f$ (see Theorem 2).

We have $V_0 = C = V$ if and only if Γ is finite. For parabolic groups V_0 has codimension 1, so C is a halfspace. More precisely, in this case $V_0 = [b]$.

Lemma 15. *Let Γ be a reduced linear Coxeter group. If its Cartan matrix A is indecomposable and of negative type, then C is strictly convex, i.e. $V_0 = 0$.*

Proof. Assume $V_0 \neq 0$. We have $[b] \subset V_0$, since V_0 is an invariant subspace (cf. Proposition 19). Let $AX > 0$ for a column vector X. Then $v = \Sigma X_i b_i$ lies in the interior of the fundamental chamber, and also in V_0. So $V_0 = V$, and Γ is finite, contradicting the fact that A is of negative type (Proposition 22).

2. Let $\Gamma \subset GL(V)$ be a linear Coxeter group with Cartan matrix A, and let $\mathbf{I}_m = I^+ \cup I^0 \cup I^-$ be the decomposition of $\mathbf{I}_m$ corresponding to the representation of A as a direct sum $A = A^+ \oplus A^0 \oplus A^+$. For every $S \subset \mathbf{I}_m$, $[b]_S$ denotes the linear envelope of the b_i, $i \in S$.

Proposition 25. $V_0 = [b]_{I^+ \cup I^0} + \operatorname{Ann}[\alpha]$.

Proof. $\operatorname{Ann}[\alpha] \subset V_0$, since $\operatorname{Ann}[\alpha] \subset K$. As in §5.4, let Γ^r be the reduced group operating in $V^r = V/\operatorname{Ann}[\alpha]$. Let π be the canonical mapping from V to V^r, and let V_0^r be the subspace of V^r playing the role of V^0 for Γ^r. Obviously $V_0 = \pi^{-1}(V_0^r)$. Hence we need only prove the assertion for reduced groups.

Let Γ be a reduced linear Coxeter group of characteristic $\{A, L_\alpha\}$. Let $\widetilde{\Gamma}$ denote the reduced linear Coxeter group with characteristic $\{A, 0\}$. We shall denote all objects related to $\widetilde{\Gamma}$ by the symbols denoting the corresponding objects related to Γ with a tilde on top. Obviously $\widetilde{K}$ is a simplicial cone. V may be imbedded in $\widetilde{V}$ in such a way that $b_i = \widetilde{b}_i \in V$ and $\alpha_i = \widetilde{\alpha}_i/V$. Under this imbedding $K = \widetilde{K} \cap V$, and Γ is obtained by restricting the action of $\widetilde{\Gamma}$ to V. Hence $K^f = \widetilde{K}^f \cap V$, $C = \widetilde{C} \cap V$ and $V_0 = \widetilde{V}_0 \cap V$. Suppose that Proposition 25 holds for $\widetilde{\Gamma}$. Then $\widetilde{V}_0 = [b]_{I^+ \cup I^0} \subset V$, so $V_0 = \widetilde{V}_0 = [b]_{I^+ \cup I^0}$. So it is enough to prove our assertion for groups whose fundamental chamber is simplicial.

Let $\Gamma \subset GL(V)$ be a linear Coxeter group whose fundamental chamber is simplicial, and let $A = A_1 \oplus \cdots \oplus A_k$ be its Cartan matrix. According to Corollary 4 of Theorem 5, $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$, Γ_i being a linear Coxeter group with simplicial fundamental chamber and with Cartan matrix A_i. In this situation, $V_0 = V_{10} \oplus \cdots \oplus V_{k0}$, V_{i0} being related to Γ_i as V_0 is to Γ. So we need prove our assertion only for groups having a simplicial fundamental chamber and an indecomposable Cartan matrix.

Let Γ be such a group. If A is of positive type, then, according to Proposition 22, Γ is an elliptic Coxeter group, and $V_0 = [b] = V$. If A is of zero type, Proposition 23 shows that Γ is parabolic and $V_0 = [b]$. Finally, if A is of negative type, Lemma 15 shows that $V_0 = 0$. So Proposition 25 holds in all cases.

Remark. It is not hard to infer from Proposition 25 that if Γ is a reduced Coxeter group and C is a halfspace, then Γ is either parabolic or the direct product of a parabolic and an elliptic group.

§8. The combinatorial structure of the fundamental chamber

1. The combinatorial structure of a convex polyhedron $K \subset \mathbf{R}^n$ is described by its *complex* $\mathfrak{F}K$ (see Definition 6).

Let $K_1, \cdots, K_m$ be the $(n-1)$-dimensional faces of K. Let

$$K_S = \bigcap_{i \in S} K_i \tag{49}$$

for every $S \subset \mathbf{I}_m$. Obviously $S \in \sigma(\mathfrak{F}K)$ (see subsection 5 of the Introduction) if and only if $K_S \neq 0$ and $K_{S \cup \{i\}} \neq K_S$ for every $i \notin S$.

Let K be a strictly convex polyhedral cone in an $(n+1)$-dimensional linear space V. By PV we denote the projective space associated with V, and by PK the convex polyhedron in PV obtained from K under the canonical mapping from $V \setminus \{0\}$ to PV. On selecting a suitable hyperplane at infinity, PK becomes a bounded polyhedron in $\mathbf{R}^n$. Obviously

$$\sigma(\mathfrak{F}PK) \cup \{\mathbf{I}_m\} = \sigma(\mathfrak{F}K). \tag{50}$$

2. **Theorem 7.** *Let Γ be a linear Coxeter group generated by reflections $R_1, \dots, R_m$ in the faces of a convex polyhedral cone K lying in an $(n+1)$-dimensional linear space V. Let S be a subset of $\mathbf{I}_m$ satisfying one of the following conditions:*

1) *Γ_S is finite.*

2) *Γ_S is abstractly isomorphic to a parabolic Coxeter group; its number of generators minus the number of components of its Coxeter diagram is $n-1$; (47) holds for every component of type A_l; Γ is not parabolic; and $S \neq \mathbf{I}_m$.*

Then $S \in \sigma(\mathfrak{F}K)$, and in the first case $\dim K_S = n+1-s$, s being the number of elements of S, while in the second case $\dim K_S = 1$.

Proof. Let A_S be the principal submatrix of A corresponding to the set S. If 1) holds, it follows from Proposition 22 that $A_S = A_S^+$, and Theorem 4 shows that $S \in \sigma(\mathfrak{F}K)$. Moreover $\dim K_S = n+1-\dim[\alpha]_S = n+1-s$, since A_S is nondegenerate.

Now suppose that 2) holds. Exactly as in the proof of 4) $\Rightarrow$ 3) in Proposition 23, we find $A_S = A_S^0$. Hence, in particular, rank $A_S = n-1$. Since $L_r(A_S)$ cannot be wholly contained in L_α, we have $\dim[\alpha]_S \geq n$.

In order to apply Theorem 4 in our case, we have to verify that $Z(S)^0 = \emptyset$. Assume the contrary, and denote $Z(S)^0$ by T. Then $Z(S \cup T)^0 = \emptyset$, and $S \cup T \in \sigma(\mathfrak{F}K)$ by Theorem 4. Now rank $A_{S \cup T} \geq$ rank $A_S + 1 = n$; so $\dim[\alpha]_{S \cup T} = n+1$ and $K_{S \cup T} = \{0\}$. This means $S \cup T = \mathbf{I}_m$, and Γ is parabolic, contradicting the hypothesis.

So $S \in \sigma(\mathfrak{F}K)$ and $\dim K_S = n+1-\operatorname{rank}[\alpha]_S = 0$ or 1. But if $\dim K_S = 0$ then $S = \mathbf{I}_m$. The proof of the theorem is complete.

3. According to Theorem 2, a reduced linear Coxeter group Γ is *perfect* (Definition 8) if and only if $K^f \supset K \setminus \{0\}$.

Elliptic and parabolic Coxeter groups are obviously perfect. But in spaces of all dimensions there also exist perfect nonorthogonal linear Coxeter groups. An example is provided by the reduced linear Coxeter group with Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \ldots 0 & -a \\ -1 & 2 & -1 \ldots 0 & 0 \\ 0 & -1 & 2 \ldots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \ldots 2 & -1 \\ -a^{-1} & 0 & 0 \ldots -1 & 2 \end{pmatrix} \tag{51}$$

for arbitrary $a > 1$. As an abstract Coxeter group, this group is of type $\tilde{A}_l$ for every a. For $a = 1$ it is parabolic.

Let $\Gamma \subset GL(V)$ be a reduced linear Coxeter group. Consider, for arbitrary $S \subset \mathbf{I}_m$, the reduced linear Coxeter group Γ_S^r operating in $V/\mathrm{Ann}[\alpha]_S$ (cf. § 5.4). If Γ is perfect, then Γ_S^r is elliptic for all $S \in \sigma(\mathfrak{F}PK)$. Let us call Γ *quasiperfect* if Γ_S^r is an elliptic or parabolic Coxeter group for every $S \in \sigma(\mathfrak{F}PK)$.

Obviously

$$(\Gamma_S^r)_T^r = \Gamma_T^r \text{ for } S \supset T. \tag{52}$$

Hence we obtain

Lemma 16. *Let Γ be a reduced linear Coxeter group, and assume that for every vertex Q of PK, $\Gamma_{\sigma(Q)}^r$ is elliptic or parabolic. Then Γ is quasiperfect and $\Gamma_{\sigma(Q)}^r$ is elliptic for every face Q of PK which is not a vertex.*

Note that for every vertex Q of PK there is a natural way of identifying $\Gamma_{\sigma(Q)}^r$ with the action of Γ in the tangent space to PV at Q.

For quasiperfect Γ, $\sigma(\mathfrak{F}PK)$ consists of those S which satisfy one of the conditions in Theorem 7.

4. Let $\mathbf{Z}_2$ be the cyclic group of order 2 operating in $\mathbf{R}^1$ by multiplication with ± 1.

Lemma 17. *The direct product $\Gamma = \Gamma_1 \times \Gamma_2$ of two (nontrivial) linear Coxeter groups is quasiperfect exactly in two cases:*

1) *Both factors are elliptic.*

2) *One factor is parabolic, and the other is $\mathbf{Z}_2$.*

Proof. Suppose that Γ_1 is not elliptic. The fundamental chamber K of Γ is the direct product of the fundamental chambers K_1 and K_2 of Γ_1 and Γ_2. Let L_2 be any edge of K_2. Then $L = \{0\} \times L_2$ is an edge of K, and $\Gamma_{\sigma(L)}^r = \Gamma_1 \times (\Gamma_2)_{\sigma(L_2)}^r$. The group $\Gamma_{\sigma(L)}^r$ is necessarily elliptic or parabolic. The first case is impossible; the second case is possible only if Γ_1 is parabolic and $(\Gamma_2)_{\sigma(L_2)}^r$ is trivial, i.e. $\sigma(L_2) = \emptyset$. It remains to note that if $\sigma(L_2) = \emptyset$, then $\Gamma_2 = \mathbf{Z}_2$.

Lemma 18. *Let A be an m by m matrix satisfying* (C1), *and let A_S be a principal submatrix of A. If $A_S = A_S^+$ and* rank A = rank A_S, *then $A = A^+ \oplus A^0$.*

Proof. It is obviously enough to consider the case in which A is indecomposable. Let $a_1, \cdots, a_m$ be its columns. Let $i \notin S$. By assumption

$$a_i = \sum_{s \in S} X_s a_s. \tag{53}$$

Let X be the column with elements X_S, $s \in S$. By (53) we have $A_S X \leq 0$; hence $X \leq 0$. So $L_c^+(A) \neq \{0\}$, and A is of zero type.

Proposition 26. *Let $\Gamma \subset GL(V)$ be a quasiperfect linear Coxeter group, and let A be its Cartan matrix. Then exactly one of the following statements is true:*

1) Γ *is elliptic.*
2) Γ *is parabolic.*
3) Γ *is the direct product of a parabolic group and* $\mathbf{Z}_2$.
4) A *is indecomposable and of negative type, and* rank A = dim V.

Proof. First assume rank A = dim V. Then $d_\alpha = 0$ and $L_\alpha = K_r(A)$. If A is decomposable, then, according to Corollary 4 of Theorem 5, Γ is a direct product of two linear Coxeter groups; so by Lemma 17 either 1) or 3) holds. If A is indecomposable, it is either of positive or of negative type; accordingly 1) or 4) holds.

Now suppose rank $A <$ dim $V = n + 1$. Select a vertex Q of PK, and take $S = \sigma(Q)$. By assumption Γ_S^r is elliptic or parabolic.

If Γ_S^r is elliptic, then $A_S = A_S^+$ and rank $A_S = n$ = rank A. By Lemma 18, $A = A^+ \oplus A^0$. Corollary 4 of Theorem 5 implies that if $A^+ \neq \emptyset$, Γ is a direct product of two linear Coxeter groups, and Lemma 17 applies again. If $A = A^0$, then Γ is parabolic by Lemma 23.

If Γ_S^r is parabolic, we have $A_S = A_S^0$ and rank $A_S = n - 1$. We shall show that in this case

$$A = A_S \oplus A_T, \text{ where } T = \mathbf{I}_m \setminus S. \tag{54}$$

Let dim$[b]_S = n - 1$. Then all linear relations satisfied by the columns of A_S hold also for those columns of A which participate in A_S; so the latter satisfy a linear relation with positive coefficients. Hence $a_{ij} = 0$ for $i \notin S$, $j \in S$. Now let dim$[b]_S = n$. Since dim $[b]$ = rank $A \leq n$, we have $[b] = [b]_S$, and all columns of A are linearly expressible by the columns participating in A_S. Now on considering those sections of the columns formed by the coordinates whose indices belong to S, we find $a_{ij} = 0$ for $i \in S$, $j \notin S$, by the definition of a matrix of zero type.

From rank $A \leq n$ and (54) it follows that rank $A_T = 1$. It is easy to see that a matrix of rank 1 satisfying (C1) and (C2) is necessarily equivalent either to (2) or to $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. If $A_T = (2)$, assertion 3) of the Proposition holds; if $A_T = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, 2) holds.

5. We wish to determine the quasiperfect orthogonal Coxeter groups.

Proposition 27. *A quasiperfect orthogonal Coxeter group is either elliptic, parabolic or hyperbolic, or the direct product of a parabolic Coxeter group and* $\mathbf{Z}_2$.

Proof. Let Γ be a quasiperfect orthogonal Coxeter group. Then one of the four assertions of Proposition 26 holds for Γ. In the first three cases there is nothing to prove. So let us consider the fourth case. We shall show that Γ is hyperbolic. By Proposition 24 this is equivalent to saying that the symmetric matrix equivalent to A has index of negativity one.

We shall assume that A is itself symmetric. Then A is the Gramian of $\{h_1, \cdots, h_m\}$ relative to a Γ-invariant scalar product in V. Since rank $A = \dim V$, this scalar product is nondegenerate, and A, being of negative type, cannot be positive definite.

Select a vertex Q of PK; let $S = \sigma(Q)$. Then Γ_S^r is elliptic or parabolic. In either case A_S is nonnegative semidefinite. Hence the scalar product is nonnegative semidefinite on $[h]_S$. By orthogonality, putting $\dim V = n + 1$, we have $\dim [h]_S = \dim [\alpha]_S = n$. Hence the index of negativity of the scalar product, considered on the whole space V, cannot be larger than 1.

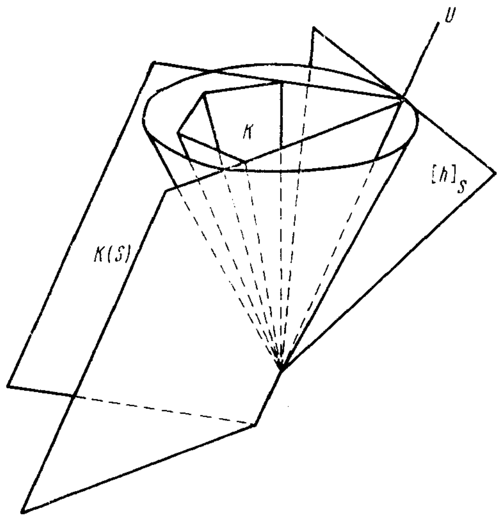

Figure 9

Proposition 28. *A hyperbolic Coxeter group* Γ *obtained by extending a discrete group of motions in* Λ^n *is perfect (quasiperfect) if and only if* Λ^n/Γ *is compact (resp. of finite volume).*

Proof. The volume of Λ^n/Γ is finite if and only if the fundamental chamber K of Γ is contained in the closure of $\mathfrak{C} = \bigcup_{t>0} t\Lambda^n$; and Λ^n/Γ is compact if and only if $K \setminus \{0\} \subset \mathfrak{C}$. Now $\mathfrak{C}$ is a connected component of $\{v \in V : (v, v) < 0\}$. In proving the part 3) $\Rightarrow$ 1) of Proposition 24 we found that K lies in some halfspace bounded by an elliptic hyperplane. So if $x \in K$ and $(x, x) < 0$ ($(x, x) \leq 0$), then $x \in \mathfrak{C}$ (resp. $x \in \overline{\mathfrak{C}}$).

Let L denote an edge of K; set $\sigma(L) = S$. Then $[b]_S$ is the orthogonal complement of L. If Γ^r_S is elliptic, then A_S is positive definite and $[b]_S$ is an elliptic subspace. So in this case $L \backslash \{0\} \subset \mathfrak{C}$. It can be proved similarly that $L \subset \overline{\mathfrak{C}}$ if Γ^r_S is parabolic. Since $\mathfrak{C}$ is convex, it follows that $K \backslash \{0\} \subset \mathfrak{C}$ ($K \subset \overline{\mathfrak{C}}$) if Γ is perfect (quasiperfect).

Conversely, assume $L \backslash \{0\} \subset \mathfrak{C}$. Let U be the one-dimensional subspace containing L. Γ^r_S acts on V/U, and there is a natural identification of V/U and $[b]_S$; the latter is a Γ_S-invariant subspace. This subspace is also elliptic; hence there exists a Γ^r_S-invariant positive definite scalar product, i.e. Γ^r_S is elliptic. Therefore Γ is perfect if $K \backslash \{0\} \subset \mathfrak{C}$.

Assume now $K \subset \overline{\mathfrak{C}}$, and let L be an edge of K lying on the boundary of $\mathfrak{C}$. In this case $[b]_S$ is a parabolic subspace, and the (invariant) scalar product in $[b]_S$ has U as its kernel of boundedness. Γ^r_S acts on $\widetilde{V} = V/U$, with fundamental chamber $\widetilde{K} = K(S)/U$ (see (9)). $\widetilde{V}_0 = [b]_S/U$ is an $(n-1)$-dimensional subspace of $\widetilde{V}$ invariant under Γ^r_S, and the scalar product on $[b]_S$ induces a positive definite Γ^r_S-invariant scalar product in $\widetilde{V}_0$. Since $K \subset \overline{\mathfrak{C}}$, we have $K(S) \cap [b]_S = U$ and so $\widetilde{K} \cap \widetilde{V}_0 = \{0\}$ (see Figure 9). By Proposition 23, Γ^r_S is a parabolic Coxeter group. In view of our earlier results in the case $L \backslash \{0\} \subset \mathfrak{C}$, we see that Γ is quasiperfect. The proof of the proposition is complete.

Examples of perfect (quasiperfect) hyperbolic Coxeter groups are known to me only for $n \leq 5$ ($n \leq 17$), n being the dimension of the Lobačevskiĭ space.

Received 16/JUNE/70

BIBLIOGRAPHY

[1] H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. 35 (1934), 588–621.

[2] ———, *The complete enumeration of finite groups of the form* $R_i^2 = (R_iR_j)^{k_{ij}} = 1$, J. London Math. Soc. 10 (1935), 21–25.

[3] J. Tits, *Groupes et géométries de Coxeter*, Inst. Haute Études Sci., Paris, 1961 (mimeographed notes).

[4] N. Bourbaki, *Éléments de mathématique*. Fasc. XXXIV. *Groupes et algèbres de Lie*, Actualités Sci. Indust., no. 1337, Hermann, Paris, 1968. MR 39 #1590.

[5] V. S. Makarov, *A class of decompositions of the Lobačevskiĭ space*, Dokl. Akad. Nauk SSSR 161 (1965), 277–278 = Soviet Math. Dokl. 6 (1965), 400–401. MR 30 #4840.

[6] ———, *A certain class of discrete Lobačevskiĭ space groups with an infinite fundamental region of finite measure*, Dokl. Akad. Nauk SSSR 167 (1966), 30–33 = Soviet Math. Dokl. 7 (1966), 328–331. MR 34 #244.

[7] È. B. Vinberg and V. G. Kac, *Quasi-homogeneous cones*, Mat. Zametki 1 (1967), 347–354. (Russian) MR 34 #8280.

[8] È. B. Vinberg, *Discrete groups generated by reflections in Lobačeveskiĭ spaces*, Mat. Sb. 72 (114) (1967), 471–488 = Math. USSR Sb. 1 (1967), 429–444. MR 34 #7667.

[9] ———, *Some examples of crystallographic groups in Lobačevskiĭ spaces*, Mat. Sb. **78** (120) (1969), 633–639 = Math. USSR Sb. 7 (1969), 617–622. MR 39 #7497.

[10] ———, *Geometric representations of Coxeter groups*, Uspehi Mat. Nauk 25 (1970), no. 2 (152), 267–268. (Russian) MR 41 #6985.

[11] E. M. Andreev, *On convex polyhedra in Lobačeveskiĭ spaces*, Mat. Sb. **81** (123) (1970), 445–478 = Math. USSR Sb. **10** (1970), 413–446. MR 41 #4367.

[12] ———, *On convex polyhedra of finite volume in Lobačevskiĭ space*, Mat. Sb. 83 (125) (1970), 256–260 = Math. USSR Sb. 12 (1970), 255–259. MR 42 #8388.

[13] ———, *On the intersection of plane faces of polyhedra with acute angles*, Mat. Zametki 8 (1970), 521–527. (Russian)

[14] V. S. Makarov, *The Fedorov groups of four-dimensional and five-dimensional Lobačevskiĭ space*, Studies in General Algebra, no. 1, Kišinev. Gos. Univ., Kishinev, 1968, pp. 120–129. (Russian) MR 41 #4368.

Translated by:
P. Flor