# Geometric structures and representations of surface groups

Colin Davalo

Ruprecht-Karls-Universität Heidelberg

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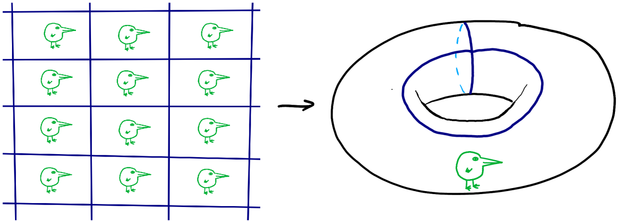
# Fundamental group of a torus



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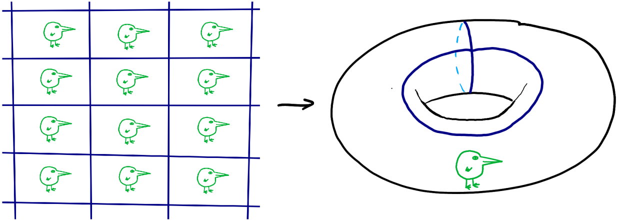

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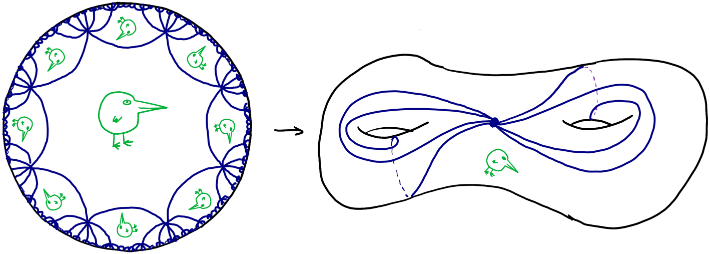
$$\rho : \mathbb{Z}^2 \to \mathrm{Isom}(\mathbb{E}^2),$$

$$T \simeq \mathbb{E}^2/\rho(\mathbb{Z}^2).$$

# Representations of surface groups



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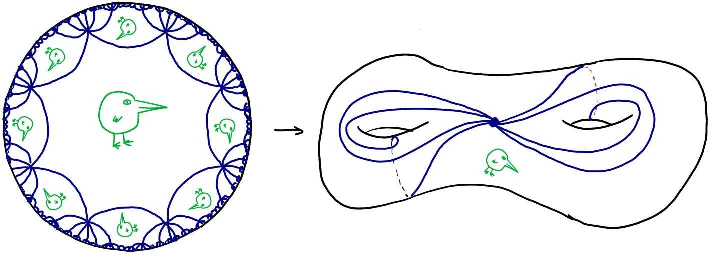
$$S_g \simeq \mathbb{H}^2/\Gamma_g.$$

# Representations of surface groups



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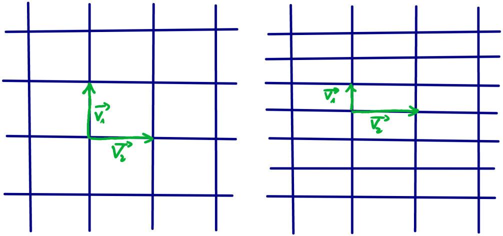
$$\rho : \Gamma_g \to \mathrm{Isom}(\mathbb{H}^2) \text{ Fuchsian, } S_g \simeq \mathbb{H}^2/\rho(\Gamma_g).$$

## Properties of Fuchsian representations

Representations $\rho : \mathbb{Z}^2 \to \mathrm{Isom}(\mathbb{E}^2)$ can degenerate to non-discrete representations !

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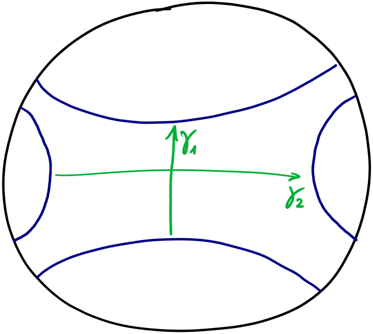
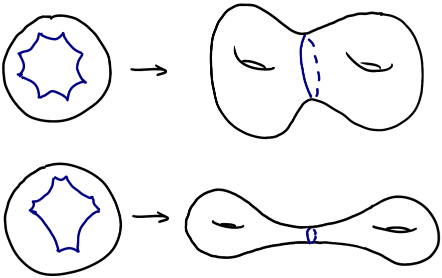


# Teichmüller space



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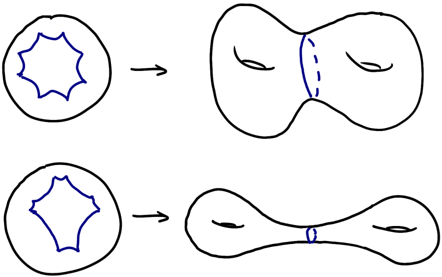
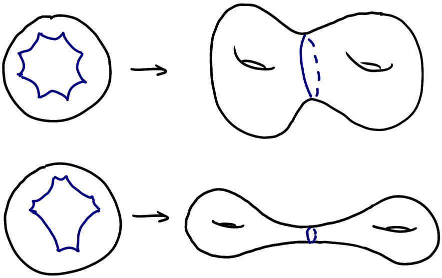


| $\mathcal{F}(S_g)$ | $\simeq$ | $\chi(S_g)$ | $\simeq$ | $\mathcal{T}(S_g)$ |
|---|---|---|---|---|
| *Fricke space* | | Character variety | | Teichmüller space |

Space of *hyperbolic structures* on $S_g$.

# Teichmüller space



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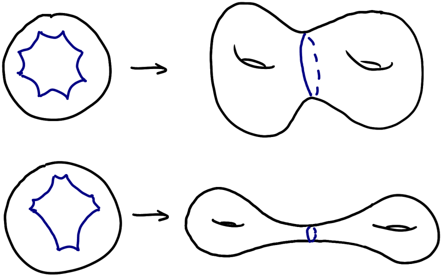
Complex structures on $S_g$.

# Teichmüller space



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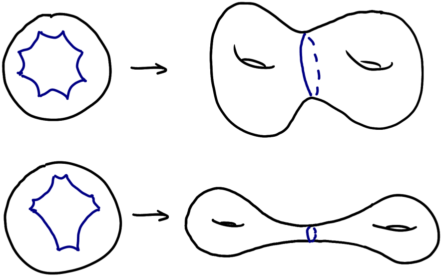
Connected component of the space of *representations*

$$\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{R}).$$

# Teichmüller space



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The point of view of character varieties can be generalized !

# Higher rank Teichmüller spaces

Let $G$ be a *Lie group* (group of matrices).
One can construct some $\rho : \Gamma_g \to G$ using $\rho_0 : \Gamma_g \to \mathrm{SL}(2, \mathbb{R})$ a Fuchisan representation, together with a representation:

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A connected component of $\mathrm{Hom}(\Gamma_g, G)$ with only discrete and faithful representations is a *higher rank Teichmüller component*.

Examples of such components are *maximal components* and *Hitchin components*.

# Maximal representations in $\mathrm{Sp}(4,\mathbb{R})$

Take $\iota : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(4,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{R})$ :

$$\iota(M) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

One can obtain the *unit tangent bundle* over $S_g$ by "folding" a *domain of discontinuity* inside the projective space ($\mathbb{RP}^3$) using $\iota \circ \rho_0$.

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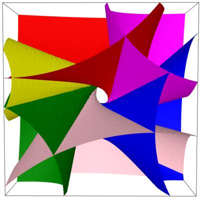

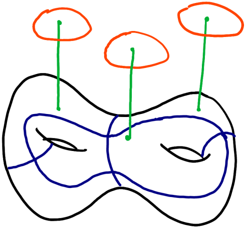

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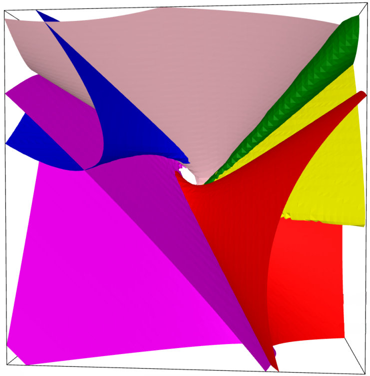
# Maximal representations in $\mathrm{Sp}(4, \mathbb{R})$

Any deformation of $\rho_0$ into $\mathrm{Sp}(4, \mathbb{R})$ remains discrete and faithful.



# Geometric structures and representations

For *Anosov* representations $\rho : \Gamma_g \to G$, Guichard-Wienhard and Kapovich-Leeb-Porti constructed *domains of discontinuity* $\Omega$ in some *flag manifolds* $\mathcal{F}$ that can be "folded" by $\rho$ into compact manifolds $M = \Omega/\rho(\Gamma)$.

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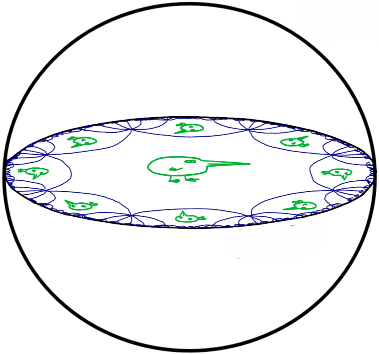
**Question**

- What is $M$ ?
- How to characterize the structures obtained ?

In many cases, M is a *fiber bundle* over $S_g$.

# Geometric structures and representations

$G$ semi-simple Lie group of non-compact type
$\iota : \mathrm{SL}(2, \mathbb{R}) \to G$, $\rho = \iota \circ \rho_0$



$\mathbb{X}$ symmetric space
$\mathcal{H} \subset \mathbb{X}$ totally geodesic

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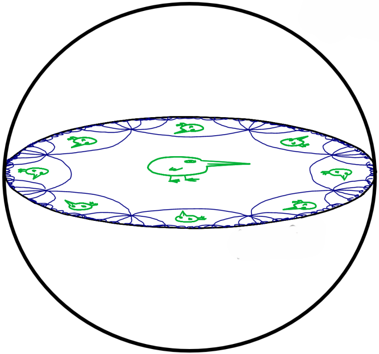

$\mathbb{X}$ symmetric space
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Fix some $\omega \in \mathfrak{a}^*$
$\mathcal{F}_\omega \subset \mathbb{X}$ flag manifold.

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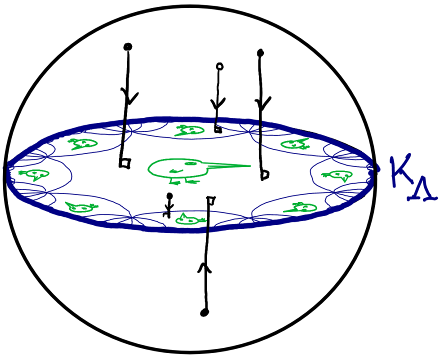


$\Omega = \mathcal{F}_\omega \backslash K_\Lambda$

*domain of discontinuity*

# Fibration of domain of discontinuity

$\mathcal{H}$ *totally geodesic* copy of $\mathbb{H}^2$ preserved by $\iota$ and $\rho$.

**Theorem (D.)**

Suppose that $\mathcal{H}$ is *$\omega$-regular*. The nearest point projection to $\mathcal{H}$ extends to a smooth fibration of a domain $\Omega \subset \mathcal{F}_\omega$.

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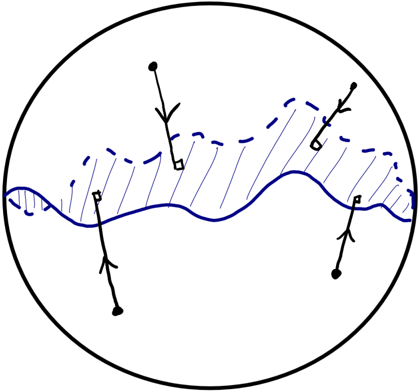
The domain $\Omega$ is a domain of discontinuity constructed by (Kapovich, Leeb, and Porti, 2017), $M = \Omega/\rho(\Gamma_g)$ fibers over $S_g$.

# Nearly geodesic immersions

The extension of the nearest point projection on $\mathcal{F}_\omega$ still works for *$\omega$-nearly geodesic immersions*.



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**Theorem (D.)**

If $\rho : \Gamma_g \to G$ admits an equivariant $\omega$-nearly geodesic immersion $u : \widetilde{S_g} \to \mathbb{X}$, then $\rho$ is *$\omega$-undistorted*, and hence $\Theta$-Anosov for some $\Theta$.

# Maximal representations

Take $\iota : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$.

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For $\rho_0$ Fuchsian, every deformation of $\iota \circ \rho_0$ is *maximal*.

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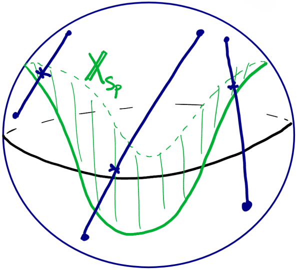
**Theorem (D.)**

If $n = 2$, the deformations of $\iota \circ \rho_0$ are never $\{1\}$-Anosov.

# Fitting immersions

Let $\mathcal{X}$ be the *projective model* for the symmetric space of $\mathrm{SL}(2n, \mathbb{R})$, and $\mathbb{X}_{\mathrm{Sp}}$ be the symmetric space of $\mathrm{Sp}(2n, \mathbb{R})$.



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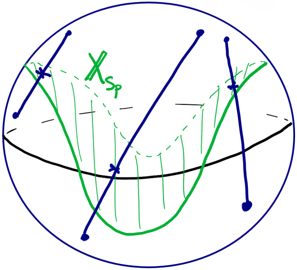


## Definition

A *fitting map* is a map into the space of codimension 2 projective subspaces, locally defining a fibration of $\overline{\mathcal{X}}$.

# Characterization of maximal representations

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These define a fibration of a domain of discontinuity in projective space by *pencils of quadrics*.

Thank you for your attention !

# References I

Burger, Marc et al. (2005). "Maximal representations of surface groups: symplectic Anosov structures". In: *Pure Appl. Math. Q.* 1.3, Special Issue: In memory of Armand Borel. Part 2, pp. 543–590. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2005.v1.n3.a5. URL: https://doi.org/10.4310/PAMQ.2005.v1.n3.a5.

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