*Probability in the Engineering and Informational Sciences* (2022), **36**:4 941–949
doi:10.1017/S0269964821000164

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**RESEARCH ARTICLE**

# Depths in hooking networks

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**Keywords:** Distance in graph, Limit law, Network, Preferential attachment, Random graph, Small world

**Abstract**
A hooking network is built by stringing together components randomly chosen from a set of building blocks (graphs with hooks). The vertices are endowed with "affinities" which dictate the attachment mechanism. We study the distance from the master hook to a node in the network chosen according to its affinity after many steps of growth. Such a distance is commonly called the depth of the chosen node. We present an exact average result and a rather general central limit theorem for the depth. The affinity model covers a wide range of attachment mechanisms, such as uniform attachment and preferential attachment, among others. Naturally, the limiting normal distribution is parametrized by the structure of the building blocks and their probabilities. We also take the point of view of a visitor uninformed about the affinity mechanism by which the network is built. To explore the network, such a visitor chooses the nodes uniformly at random. We show that the distance distribution under such a uniform choice is similar to the one under random choice according to affinities.

## 1. Introduction

Several types of networks grow by repeatedly attaching components chosen from a given set of building blocks. For instance, a social network may start out from a clique of friends, and later additional cliques adjoin themselves to the network. Many species of random trees (such as random recursive trees and random plane-oriented recursive trees) grow this way, where the blocks are simply paths of length 1. Recursive trees built from blocks that are themselves trees are investigated in [5].

In the context of networks, a *building block* is defined to be a connected graph $G = (V, E)$ with a set of *vertices* $V$ (also called *nodes*) and a set of edges $E$ containing at least one edge; a particular vertex in $V$ is distinguished as the *hook*, the node designated for connecting to larger graphs. A set of building blocks is used as components strung together into a bigger network in discrete time steps. A network grown from *one* building block is investigated in [10], where the initial building block is called a *seed*. Because all the components in a collection of size one are the same, the author of [10] calls such a network a self-similar hooking network. The focus in [10] is on local and global degree profiles.

As we consider here a set of possibly more than one building block, we think that it is more appropriate to call such networks simply hooking networks. Desmarais and Holmgren [3] considers the global degree profile for a hooking network grown from a set of building blocks, as a generalization of [10]. In the present manuscript, we consider depths in hooking networks, extending similar results on random trees in [4,5,8,9].

## 2. The set-up of hooking networks

We have a finite set of *random* building blocks (each equipped with a hook) from which we grow the hooking network. By random, we mean that there is a probability distribution on the set of blocks.

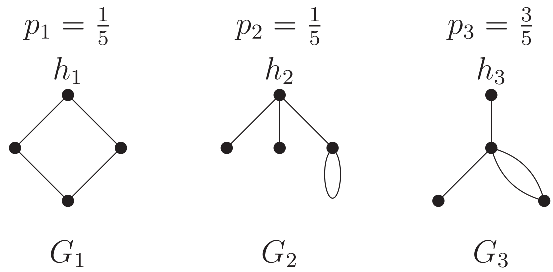


***Figure 1.*** *A collection of random building blocks. The probability of a block appears above the block.*

More precisely, we build a sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ of hooking networks from a collection of building blocks $C = \{G_1, G_2, \ldots, G_m\}$, where each $G_i$ has a labeled vertex $h_i$ called the *hook* and a selection probability $p_i \in [0, 1]$ such that $\sum_{i=1}^{m} p_i = 1$. Note that we allow the graphs in the set of building blocks to have loops and multiple edges.

We denote the degree of vertex $v$ in a graph by $\deg(v)$. Note that a self-loop on $v$ contributes 2 to the degree of $v$. To cover a wide variety of growth models, we endow the network with two real parameters: $\chi$ and $\rho$. The *affinity* or "attractive power" of a vertex $v$ in the hooking network $\mathcal{G}_{n-1}$ is the value

$$\chi \deg(v) + \rho,$$

where $\deg(v)$ is the degree of $v$ in $\mathcal{G}_{n-1}$. The sum of these quantities over all vertices in $\mathcal{G}_{n-1}$, denoted by $\tau_{n-1}$, is the *total affinity* of $\mathcal{G}_{n-1}$. The network $\mathcal{G}_n$ is grown from $\mathcal{G}_{n-1}$ by "randomly" choosing a *latch* among the vertices of $\mathcal{G}_{n-1}$, where the probability of choosing $v$ as a latch is

$$\frac{\chi \deg(v) + \rho}{\tau_{n-1}}.$$

So, when $\chi = 0$, the choice is made uniformly at random among all the vertices of $\mathcal{G}_{n-1}$, and when $\rho = 0$, the choice is made proportionally to the degree of $v$ in $\mathcal{G}_{n-1}$. Once a latch $v$ is identified, we choose a block $G_i \in C$ with probability $p_i$ independently from history and attach a copy isomorphic to it to $\mathcal{G}_{n-1}$ by merging the hook $h_i$ to the latch $v$. The initial graph $\mathcal{G}_0$ is isomorphic to $G_i$, where $G_i$ is chosen from $C$ with probability $p_i$.

Let $B_n$ denote the $n$th block attached to the hooking network. The vertex corresponding to the hook of the initial block, $B_0$, used to launch the network with $\mathcal{G}_0$, is called the *master hook* of the network and is denoted by $H$. Notice that by construction, the blocks $B_0, B_1, \ldots, B_n$ are all independently and identically distributed.

## 3. Illustrative example

Let $G_1, G_2, G_3$ be the blocks in Figure 1, with probabilities $p_1 = 1/5$, $p_2 = 1/5$, and $p_3 = 3/5$. The hook of $G_i$, for $i = 1, 2, 3$, is labeled with $h_i$. Figure 2 shows the step-by-step evolution of one possible realization of the hooking networks $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ grown from $C = \{G_1, G_2, G_3\}$. The hooking network $\mathcal{G}_3$ is shown with its blocks $B_0, B_1, B_2$, and $B_3$ identified.

Under uniform attachment ($\chi = 0$), the probability of the sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ is

$$\frac{1}{5} \times \left(\frac{1}{4} \times \frac{1}{5}\right) \times \left(\frac{1}{7} \times \frac{1}{5}\right) \times \left(\frac{1}{10} \times \frac{3}{5}\right) = \frac{3}{175{,}000},$$

and under preferential attachment ($\rho = 0$), this probability is given by

$$\frac{1}{5} \times \left(\frac{1}{4} \times \frac{1}{5}\right) \times \left(\frac{1}{16} \times \frac{1}{5}\right) \times \left(\frac{5}{24} \times \frac{3}{5}\right) = \frac{1}{64{,}000}.$$

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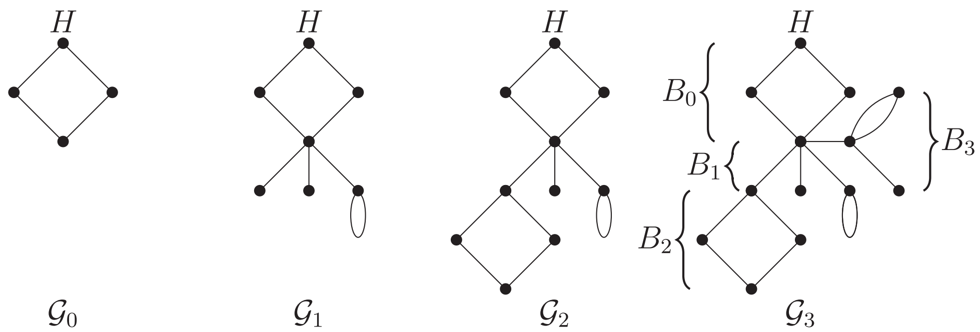


***Figure 2.*** *A sequence of hooking network.*

## 4. Scope

Define $D_n$ to be the shortest distance from the master hook $H$ to the latch $v$ chosen in $\mathcal{G}_{n-1}$ to construct $\mathcal{G}_n$; that is, the number of edges along the shortest path from $H$ to $v$. We call this distance the *depth* of the vertex $v$. Notice that $D_n$ is distributed as the depth of a vertex $u$ in $\mathcal{G}_{n-1}$ chosen at random proportionally to $\chi \deg(u) + \rho$, since the choice of a latch is made according to its affinity. For certain choices of $C$ and affinity parameters $\chi$ and $\rho$, we find the exact average of $D_n$ and prove a normal limit law. This result holds in the case of uniform attachment when all the blocks have the same number of vertices, and in the case of preferential attachment, when all the blocks have the same number of edges. The theorem holds in more generality, which is explained below.

We also consider an uninformed view of "random" depth, $\widetilde{D}_n$, according to which the network explorer is not aware of the underlying affinity. The explorer then resorts to a uniform choice of a random node. The result for $\widetilde{D}_n$ is similar to that for $D_n$.

Let us introduce some notation:

- Denote the cardinality of a set $\mathcal{E}$ by $|\mathcal{E}|$.
- For a graph $G$, we use $V(G)$ for the set of its vertices and $E(G)$ for the set its of edges.
- Let $v_i = |V(G_i)|$.
- Let $e_i = |E(G_i)|$.
- Let $s_i = 2\chi e_i + (v_i - 1)\rho$.
- Let $\mathbb{I}_{\mathcal{A}}$ be the indicator of event $\mathcal{A}$.

The quantity $s_i$ is the *affinity* of a copy $B_n$ of the block $G_i$ when it is hooked in the network. Note that after hooking, $h_i$ is absorbed in a node of a previous network $\mathcal{G}_{n-1}$, so only $v_i - 1$ vertices of $B_n$ participate in future hooking, while all the edges are intact and participate fully in all the future stages of hooking. The collections of building blocks we consider for the presentation of Gaussian laws have blocks with the same affinity, which we call $s$. This requirement is attained in several cases, for example, when $\chi = 0$ and $v_1 = v_2 = \cdots = v_m$ or when $\rho = 0$ and $e_1 = e_2 = \cdots = e_m$.

We define a few random variables that are used in our proofs.

**Definition 4.1.** *For a block $G_i$, let $d_i(v)$ be the distance from a vertex $v$ in $G_i$ to the hook $h_i$. For a block $G_i$, define the random variable $\delta_i$ such that*

$$\delta_i = \begin{cases} d_i(v) & \text{with probability } (\chi \deg(v) + \rho)/s_i \text{ if } v \neq h_i, \\ 0 & \text{with probability } \chi \deg(h_i)/s_i. \end{cases}$$

*For $n \geq 1$, define the random variable*

$$\Delta_n = \sum_{i=1}^{m} \mathbb{I}_{\{B_n = G_i\}} \delta_i.$$

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The random variable $\delta_i$ measures the depth of a vertex $v$ within a block in the network that is a copy of $G_i$, where $v$ is chosen according to the affinity $v$ contributes to the block after hooking (against the total affinity $s_i$ of the block). Since the hook $h_i$ is absorbed after hooking, it only contributes $\chi \deg(h_i)$ to the affinity of the block. The random variable $\Delta_n$ is the depth within the $n$th block $B_n$ added to the network of a vertex $v$ chosen according to the affinity $v$ contributes to the block $B_n$ after hooking. Note that $\Delta_n$ has an "average distribution" over the $\delta_i$'s. In other words, $\Delta_n$ is a mixture of the distributions of $\delta_i$, where the mixing is taken according to the probabilities $p_i$. Each $\Delta_n$ is identically and independently distributed as a generic random variable $\Delta$. Recall that $\mathbb{P}(B_n = G_i) = p_i$.

In our running example, we have $v_1 = v_2 = v_3 = 4$, $e_1 = e_2 = e_3 = 4$, and $s = s_1 = s_2 = s_3 = 8\chi + 3\rho$. It follows that

$$\mathbb{P}(\delta_1 = 0) = \frac{2\chi}{s}, \quad \mathbb{P}(\delta_1 = 1) = 2\frac{2\chi + \rho}{s}, \quad \mathbb{P}(\delta_1 = 2) = \frac{2\chi + \rho}{s};$$

the multiple 2 in $\mathbb{P}(\delta_1 = 1)$ accounts for two vertices at distance 1 from the hook of $G_1$. Since $\Delta_n$ is distributed like $\Delta$, for our running example we have

$$\mathbb{P}(\Delta_n = 0) = p_1\mathbb{P}(\delta_1 = 0) + p_2\mathbb{P}(\delta_2 = 0) + p_3\mathbb{P}(\delta_3 = 0) = \frac{8\chi}{5s}.$$

Likewise, we have

$$\mathbb{P}(\Delta_n = 1) = \frac{21\chi + 8\rho}{5s}, \quad \mathbb{P}(\Delta_n = 2) = \frac{11\chi + 7\rho}{5s}.$$

The average of $\Delta$ is

$$\mathbb{E}[\Delta] = \frac{43\chi + 22\rho}{40\chi + 15\rho},$$

and the variance is

$$\mathbb{V}\mathrm{ar}[\Delta] = \frac{751\chi^2 + 523\chi\rho + 56\rho^2}{25(8\chi + 3\rho)^2}.$$

Since the hook of $B_0$ is not absorbed in a latch of a prior graph, the depth $\Delta_0$ within $B_0$ follows a slightly different distribution from $\Delta$. The affinity of $B_0$ is $s + \rho$ and the probability of choosing any vertex $v$ within $B_0$ is $(\chi \deg(v) + \rho)/(s + \rho)$. From this, we see immdediately that $\Delta_0$ is distributed as

$$\Delta_0 \stackrel{d}{=} \mathrm{Ber}\left(\frac{s}{s + \rho}\right)\Delta, \tag{1}$$

where $\mathrm{Ber}(p)$ is a Bernoulli random variable with parameter $p$.

Notice that the total affinity of the network is given by $\tau_0 = s + \rho$ (the affinity of the first block $B_0$) and $\tau_n = sn + \tau_0 = s(n + 1) + \rho$.

## 5. Main results

We are now poised to state the main theorems. They come in two flavors: exact and asymptotic. We use $\psi_W(u) := \mathbb{E}[e^{Wu}]$ for the moment generating function of any generic random variable $W$. An exact moment generating function for $D_n$ is key to the extraction of its exact moments and serves as a starting point for its asymptotic distribution.

When choosing the $n$th latch, we can think of first choosing one of the blocks that were previously added, and then choosing a vertex within that block. The depth of the hook of the $i$th block added is $D_i$. Because of the absorption of the hook in a node of a previous network, a block contributes only according to the amount of affinity in its nonhook vertices and all edges including those connected to the hook, the only exception being the initial block as its master hook remains a viable candidate as a latch throughout all stages. Therefore, the contribution of $B_i$, for $i \geq 1$, is the depth $\Delta_i$ defined in Definition 4.1, whereas that of $B_0$ is the adjusted depth $\Delta_0$ distributed as (1). We now have the distribution

$$D_n = \begin{cases} D_0 + \Delta_0, & \text{with probability } (s+\rho)/\tau_{n-1}; \\ D_i + \Delta_i, & \text{with probability } s/\tau_{n-1}, \text{ for } i = 1, \ldots, n-1, \end{cases}$$

and $D_0 = 0$.

Recall the definition of $\delta_i$ from Definition 4.1. This random variable has the moment generating function

$$\psi_{\delta_i}(u) = \sum_{v \in V_i \setminus \{h_i\}} \frac{1}{s}(\chi \deg(v) + \rho)\, e^{d_i(v)u} + \frac{\chi}{s} \deg(h_i),$$

with $d_i(v)$ being the distance of $v$ from $h_i$. Going further, we express this as

$$\begin{aligned} \psi_{\delta_i}(u) &= \sum_{v \in V_i} \frac{1}{s}(\chi \deg(v) + \rho)\, e^{d_i(v)u} - \frac{\rho\, e^{d_i(h_i)u}}{s} \\ &= \sum_{v \in V_i} \frac{1}{s}(\chi \deg(v) + \rho)\, e^{d_i(v)u} - \frac{\rho}{s}. \end{aligned}$$

The distribution of $\Delta_n$ is obtained by mixing the distributions in the blocks according to their probabilities. More precisely, $\Delta_n$ has the moment generating function

$$\psi_{\Delta_n}(u) = \sum_{i=1}^{m} p_i \psi_{\delta_i}(u) = \frac{s+\rho}{s} \psi_{\Delta_0}(u) - \frac{\rho}{s}.$$

We next develop $\psi_{D_n}(u)$ in exact form.

**Lemma 5.1.**

$$\psi_{D_n}(u) = \psi_{\Delta_0}(u) \prod_{i=1}^{n-1} \frac{\rho + si + s\psi_\Delta(u)}{\rho + s(i+1)}.$$

*Proof.* In the following derivation, we use the fact that, for $i \geq 1$, the variables $D_i$ and $\Delta_i$ are independent, and that all $\Delta_i$ are identically distributed to the random variable $\Delta$. We obtain

$$\begin{aligned} \psi_{D_n}(u) &= \frac{s+\rho}{sn+\rho} \mathbb{E}[e^{(D_0+\Delta_0)u}] + \frac{s}{sn+\rho} \sum_{i=1}^{n-1} \mathbb{E}[e^{(D_i+\Delta_i)u}] \\ &= \frac{s+\rho}{sn+\rho} \psi_{\Delta_0}(u) + \frac{s\psi_\Delta(u)}{sn+\rho} \sum_{i=1}^{n-1} \psi_{D_i}(u). \end{aligned} \tag{2}$$

Differencing for $n \geq 1$, we get

$$(\rho + sn)\psi_{D_n}(u) - (\rho + s(n-1))\psi_{D_{n-1}}(u) = s\psi_\Delta(u)\psi_{D_{n-1}}(u),$$

which we arrange in the iterable form

$$\psi_{D_n}(u) = \frac{\rho + s((n-1) + \psi_\Delta(u))}{\rho + sn} \psi_{D_{n-1}}(u).$$

https://doi.org/10.1017/S0269964821000164 Published online by Cambridge University Press

Unwinding all the way back to $\psi_{D_1}(u) = \psi_{\Delta_0}(u)$ gives us

$$\psi_{D_n}(u) = \psi_{\Delta_0}(u) \prod_{i=1}^{n-1} \frac{\rho + si + s\psi_\Delta(u)}{\rho + s(i+1)},$$

for $n \geq 2$. □

The exact average depth of the latch can be found mechanically from $\psi_{D_n}(u)$ by taking derivatives. It appears in a form involving the generalized $r$th harmonic number

$$\mathcal{H}_r(y) = \frac{1}{y+1} + \cdots + \frac{1}{y+r}.$$

**Theorem 5.2.** *Let $D_n$ be the depth of the latch $v$ chosen to construct $\mathcal{G}_n$ and suppose $s = s_1 = s_2 = \cdots = s_m$. For $n \geq 1$, the exact expected value of $D_n$ is given by*

$$\mathbb{E}[D_n] = \left(\mathcal{H}_{n-1}\left(\frac{\rho}{s} + 1\right) + \frac{s}{s+\rho}\right)\mathbb{E}[\Delta].$$

*Proof.* Take the derivative of the moment generating function in Lemma 5.1, then evaluate at $u = 0$, to get

$$\begin{aligned}
\mathbb{E}[D_n] &= \psi'_{\Delta_0}(0) \prod_{i=1}^{n-1} \frac{\rho + si + s\psi_\Delta(0)}{\rho + s(i+1)} \\
&\quad + \psi_{\Delta_0}(0) \prod_{i=1}^{n-1} \frac{1}{\rho + s(i+1)} \prod_{i=1}^{n-1} (\rho + si + s\psi_\Delta(0)) \\
&\quad \times \sum_{i=1}^{n-1} \frac{s\psi'_\Delta(0)}{\rho + si + s\psi_\Delta(0)} \\
&= \mathbb{E}[\Delta_0] + \sum_{i=1}^{n-1} \frac{\mathbb{E}[\Delta]}{\rho/s + i + 1} \\
&= \left(\mathcal{H}_{n-1}\left(\frac{\rho}{s} + 1\right) + \frac{s}{\rho + s}\right)\mathbb{E}[\Delta].
\end{aligned}$$

□

From the approximate value of the generalized harmonic number, we immediately get the following result.

**Corollary 5.3.** *As $n \to \infty$, we have*

$$\mathbb{E}[D_n] \sim \mathbb{E}[\Delta] \ln n.$$

**Theorem 5.4.** *Let $D_n$ be the depth of the latch $v$ chosen to construct $\mathcal{G}_n$. If $s_1 = s_2 = \cdots = s_m$, then*

$$\frac{D_n - \mathbb{E}[\Delta] \ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}(0, \mathbb{V}ar[\Delta] + \mathbb{E}^2[\Delta]).$$

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*Proof.* We use Lemma 5.1 to develop the moment generating function of $(D_n - \mathbb{E}[\Delta] \ln n)/\sqrt{\ln n}$. By Stirling's approximation to the ratio of Gamma functions, as $n \to \infty$, we have

$$\begin{aligned}
\psi_{D_n}\left(\frac{u}{\sqrt{\ln n}}\right) &= \psi_{\Delta_0}\left(\frac{u}{\sqrt{\ln n}}\right) \frac{\Gamma(\rho/s + \psi_\Delta(u/\sqrt{\ln n}) + n)\Gamma(\rho/s + 2)}{\Gamma(\rho/s + 1 + n)\Gamma(\rho/s + 1 + \psi_\Delta(u/\sqrt{\ln n}))} \\
&\sim \frac{\Gamma(\rho/s + \psi_\Delta(u/\sqrt{\ln n}) + n)}{\Gamma(\rho/s + 1 + n)} \\
&\sim n^{\psi_\Delta(u/\sqrt{\ln n}) - 1} \\
&= \exp\left(\left(\psi_\Delta\left(\frac{u}{\sqrt{\ln n}}\right) - 1\right) \ln n\right).
\end{aligned}$$

We use the asymptotic relation

$$\psi_\Delta\left(\frac{u}{\sqrt{\ln n}}\right) = 1 + \frac{u\,\mathbb{E}[\Delta]}{\sqrt{\ln n}} + \frac{u^2(\mathbb{V}\mathrm{ar}[\Delta] + \mathbb{E}^2[\Delta])}{2\ln n} + O\left(\frac{1}{\ln^{3/2} n}\right),$$

to get

$$\begin{aligned}
\psi_{D_n}\left(\frac{u}{\sqrt{\ln n}}\right) e^{-(\mathbb{E}[\Delta]\ln n/\sqrt{\ln n})u} &\sim \exp\left(\left(\frac{u^2(\mathbb{V}\mathrm{ar}[\Delta] + \mathbb{E}^2[\Delta])}{2\ln n}\right) \ln n\right) \\
&\to \exp\left(\frac{u^2}{2}(\mathbb{V}\mathrm{ar}[\Delta] + \mathbb{E}^2[\Delta])\right).
\end{aligned}$$

The last expression is the moment generating function of a centered normal distribution with variance $\mathbb{V}\mathrm{ar}[\Delta] + \mathbb{E}^2[\Delta]$. According to Lévy's continuity theorem (see [6, Thm. 9.1], for example), convergence of moment generating functions implies the convergence in distribution stated in the theorem. □

## 6. The uninformed visit

A visitor to the network may be uninformed about the affinity mechanism that constructed it. In the absence of such prior information, the visitor may decide to explore the network according to a uniform choice of a node, in which all the nodes in the network are equally likely.

When choosing a node uniformly at random, we can once again think of first choosing a block and then a node within that block. If the node is in $B_i$, its depth in the network is $D_i$ plus the depth of a uniformly random node in $B_i$. The latter is an $O(1)$ random variable. Let $B$ be the random block chosen and suppose that all blocks have the same number of vertices, that is, $\nu_i = \nu_1 = \nu_2 = \cdots = \nu_m$. From the vantage point of uniform randomness, the probability of choosing the node from block $B_i$, with $i \geq 1$, is given by

$$\mathbb{P}(B = B_i) = \frac{\nu - 1}{(\nu - 1)n + \nu} \sim \frac{1}{n}.$$

The probability of $B_0$ is special:

$$\mathbb{P}(B = B_0) = \frac{\nu}{(\nu - 1)n + \nu} = O\left(\frac{1}{n}\right).$$

**Theorem 6.1.** *Let $\widetilde{D}_n$ be the depth of a vertex chosen uniformly at random in the graph $\mathcal{G}_n$. If $s_1 = s_2 = \cdots = s_m$ and $\nu = \nu_1 = \nu_2 = \cdots = \nu_m$, then we have*

$$\frac{\widetilde{D}_n - \mathbb{E}[\Delta]\ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}(0, \mathbb{V}ar[\Delta] + \mathbb{E}^2[\Delta]).$$

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*Proof.* Let $B$ be the block in which the chosen node falls. We compute

$$\begin{aligned}\psi_{\widetilde{D}_n}(u) &= \sum_{i=0}^{n} \mathbb{E}[e^{\widetilde{D}_n u} \mid B = B_i]\mathbb{P}(B = B_i)\\ &= O\left(\frac{1}{n}\right) + \frac{\nu - 1}{(\nu - 1)n + \nu} \sum_{i=1}^{n} \mathbb{E}[e^{(D_i + O(1))u}].\end{aligned}$$

The $O(1)$ in the exponent is the distance of a node chosen uniformly at random from the collection of building blocks. This random distance is independent of $D_i$. Upon changing the scale of $u$, we find by independence

$$\begin{aligned}\psi_{\widetilde{D}_n}\left(\frac{u}{\sqrt{\ln n}}\right) &\sim \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[e^{(D_i + O(1))(u/\sqrt{\ln n})}]\\ &\sim \frac{1}{n}\sum_{i=1}^{n} \psi_{D_i}\left(\frac{u}{\sqrt{\ln n}}\right).\end{aligned}$$

From (2), the moment generating function of $D_{n+1}/\sqrt{\ln n}$ can be written as

$$\begin{aligned}\psi_{D_{n+1}}\left(\frac{u}{\sqrt{\ln n}}\right) &= \frac{s+\rho}{sn+\rho}\psi_{\Delta_0}\left(\frac{u}{\sqrt{\ln n}}\right) + \frac{s\psi_{\Delta}(\frac{u}{\sqrt{\ln n}})}{sn+\rho}\sum_{i=1}^{n}\psi_{D_i}\left(\frac{u}{\sqrt{\ln n}}\right)\\ &\sim \frac{1}{n}\sum_{i=1}^{n}\psi_{D_i}\left(\frac{u}{\sqrt{\ln n}}\right).\end{aligned}$$

Therefore, the moment generating functions of $\widetilde{D}_n/\sqrt{\ln n}$ and $D_{n+1}/\sqrt{\ln n}$ have the same asymptotic behavior, and the convergence stated in the theorem follows from Theorem 5.4. □

## 7. Concluding remarks

To get a feel for what is involved in the central limit theorem and see the various nuances of the network parameters, we give some concrete forms relating to the running example. Under pure uniform attachment ($\chi = 0$), we have

$$\frac{D_n - \frac{22}{15}\ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}\left(0, \frac{12}{5}\right),$$

whereas for preferential attachment ($\rho = 0$), we have

$$\frac{D_n - \frac{43}{40}\ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}\left(0, \frac{13}{8}\right).$$

A case of mixed uniform and preferential attachments with $\chi = 2.4$ and $\rho = 5$ gives rise to the central limit theorem

$$\frac{D_n - \frac{1066}{855}\ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}\left(0, \frac{112}{57}\right).$$

Observe that the asymptotic average depth in the case of uniform attachment is higher than that of preferential attachment, because attaching preferentially "favors" older vertices in the network, and thus has the effect of "lifting" the joining blocks closer to the master hook. Of course, the asymptotic average depth in the mixed case is somewhere in between uniform and preferential attachments.

The class of random graphs we study in the article lives in the "small world," a phrase commonly used nowadays to refer to networks with average inter-node distances of order $\log n$. The author of [7] reports on a variety of networks arising in nature and technology that exhibit small world behavior. An important application appears in [1]. For a wider scope on small-world network, see [2,11].

**Acknowledgments.** The work of the first author is partly supported by the Knut and Alice Wallenberg Foundation, the Ragnar Söderbergs Foundation, and the Swedish Research Council. The second author wishes to thank Northeastern University for hosting his sabbatical visit. In particular, he thanks Albert-László Barabási for providing the opportunity.

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**Cite this article:** Desmarais C and Mahmoud H (2022). Depths in hooking networks. *Probability in the Engineering and Informational Sciences* **36**, 941–949. https://doi.org/10.1017/S0269964821000164