The transition from semi to fully pulled waves in the noisy K-Branching Random Walk

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Abstract

We analyze a variant of the Noisy K-Branching Random Walk in which a fixed number of individuals are randomly sampled based on their fitness values. By employing a stochastic Hopf-Cole transformation, we demonstrate that the population dynamics converge to a deterministic limiting fitness wave evolving according to the solution to a free boundary problem. In scenarios where the tail of the genotypic noise is exponential, we find that the limiting wave undergoes a novel phase transition between semi and fully pulled waves, revealing a complex interaction between evolution and natural selection.

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1 Introduction

1.1 The model

We consider an asexual population of fixed size K in discrete time. The population evolves at each time step according to two sub-steps. Let $\beta > 0$ and $r \in \mathbb{N} \setminus \{0\}$.

Reproduction. Each individual produces a fixed number of offspring r. These children inherit the genotype of their parent up to an independent random fluctuation owed to the occurrence of random mutations. The child of an individual with genotype x has genotype $x + X$ where X is a random variable with prescribed densities f_X.

Selection. Following reproduction, the population consists of $N := rK$ individuals. To regulate the size of the population, we sample K individuals without replacement according to the Gibbs sampling weights $(e^{\beta x_i})_{i=1}^N$ where $(x_i)_{i=1}^N$ denotes the genotypic values of the offsprings.

The case $\beta = \infty$ is a version of the K-Branching Random Walk (K-BRW) [2, 3] in which only the K fittest offspring survive after reproduction. We will refer to the present model as the noisy K-BRW with parameter (β, r) and reproduction law f_X.

At each time $t \in \mathbb{N}$, the population is encoded as a cloud of points $\mathcal{Z}_K^t := (Z_i^t)_{i=1}^K$ on $\mathbb{R}$, and we refer to the dynamics $(\mathcal{Z}_K^t; t \geq 0)$ as a fitness wave. Standard arguments (see e.g., [4]) show the existence of a limiting rate of adaptation. That is, for a given reproduction law, there exists $c \equiv c(K, \beta, r)$ such that for almost every realisation

$$\lim_{t\to\infty} \frac{\min_{i\in[K]} Z_i^t}{t} = \lim_{t\to\infty} \frac{\max_{i\in[K]} Z_i^t}{t} \;=\; c(K, r, \beta) \quad \text{a.s..} \tag{1.1}$$

The aim of the present article is to study the case of high fertility ($r >> 1$) and show that an interesting phase transition occurs in the large population limit. We start by a brief and informal description of our results.

1.2 Large population limit

Scaling. Let F_X be the cumulative distribution of the genotypic noise and define

$$c_N := (1 - F_X)^{-1}(1/N)$$

where $(1 - F_X)^{-1}$ is the right-inverse of $1 - F_X$. For two positive sequences (a_N) and (b_N), we write $a_N \asymp b_N$ iff $\ln(a_N)/\ln(b_N) \to 1$ as $N \to \infty$. We assume the existence of a non-positive function h such that

$$\forall x > 0, \;\; \mathbb{P}\left(X > xc_N\right) \asymp N^{h(x)}.$$

To give a concrete example, let $\alpha > 0$ and let

$$f_X(x) = c(\alpha)\exp(-|x|^\alpha)$$

where $c(\alpha)$ is a renormalisation constant. A direct computation shows that

$$h(x) = -|x|^\alpha \;\text{ and }\; c_N = \log(N)^{1/\alpha}.$$

Log-profiles. Let $\beta > 0$ and $\gamma \in (0,1)$. We consider a noisy $\lfloor N^\gamma \rfloor$-BRW with Gibbs parameter β_N and fertility r_N defined as

$$\beta_N := \frac{\beta}{c_N}, \quad r_N := N/\lfloor N^\gamma \rfloor.$$

Let us briefly comment on the biological significance of γ. For a fixed value of N, a lower γ entails a higher selection pressure since only a reduced number $\lfloor N^\gamma \rfloor$ of individuals can reproduce; whereas a high γ (close to 1) corresponds to a mild selection scheme where a large proportion of children survive to the next generation. As a consequence, γ can be interpreted as capturing the selection pressure in the population. When γ is low, selection is strong; when γ is high, selection is weak.

Under this scaling of the parameters, we will now show that the diameter of the wave (distance between the left-most and the right-most particles) is of order c_N. In order to capture the dynamics, we will consider a stochastic Hopfe-Cole transformation of the system that we now describe. Recall that $\mathcal{Z}_N^t$ refers to the configuration of the fitness wave at time t. Roughly speaking, we will say that $(\mathcal{Z}_N^t)_{N\in\mathbb{N}}$ admits a limiting log-profile if and only if there exists a deterministic function $g^t \in \mathbb{R}_+ \cup \{-\infty\}$ such that the number of particles in an interval of size $c_N dx$ is around xc_N is $\asymp N^{g(x)}dx$. A more formal definition can be found in Section 2. Note that when $g(x) = -\infty$ there are no particles around xc_N. As a consequence, if the support $\mathrm{Supp}(g^t) := \{x : g^t \neq -\infty\}$ is an interval, then the diameter of the fitness wave at time t is of order c_N.

In the following, we define π as the following function

$$\pi(x) \;=\; \begin{cases} x & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases} \tag{1.2}$$

Theorem 1.1. *Assume that*

(i): the sequence of initial configurations $(\mathcal{Z}_N^0)_{N\in\mathbb{N}}$ admits a limiting log-profile g^0,

(ii): Supp(g^0) is an interval and g^0 is concave on Supp(g^0), and

(iii): h is concave on its support.

Then $(\mathcal{Z}_N^t)_{N\in\mathbb{N}}$ admits a limiting log-profile g^t satisfying property (ii). Further, (g^t) satisfies a recursive equation in terms of the following discrete "free boundary" problem

$$\forall x \in \mathbb{R},\; g^{t+1}(x) \;=\; \pi\left(1-\gamma+\sup_{z\in\mathbb{R}}(g^t(z)+h(x-z)) \;+\; \beta(x-\sigma_t)_-\right)$$
$$\text{where } \sigma_t \text{ is defined implicitly through the relation } \sup_x g^{t+1}(x) = \gamma. \tag{1.3}$$

Let us now a provide a quick heuristics for this result. If we think of $g^t(x)$ as the limiting stochastic exponent at x, a first moment argument yields that the averaged number of offspring at xc_N is

$$\asymp \int N^{g^t(z)} N^{1-\gamma} N^{h(x-z)} dz \asymp N^{\sup_z(1-\gamma+g^t(z)+h(x-z))}$$

where the integral counts the average contribution of every possible location using the fact that we have $\asymp N^{g^t}(z)$ particles at zc_N, and each of them produces about $N^{1-\gamma}$ offspring with $\asymp N^{1-\gamma+h(x-z)}$ of those offspring around xc_N. A second moment computation shows that the stochastic exponent at x converges to the deterministic function

$$r(g^t)(x) \;:=\; \pi\left(\sup_{z\in\mathbb{R}} 1-\gamma+g^t(z)+h(x-z)\right)$$

where the projector π encodes the fact that when we have on average a negative power, then with high probability there are no particles. In order to get the profile at time $t+1$, we need to sample $\lfloor N^\gamma \rfloor$ particles without replacement according to the Gibbs weight. For that purpose, we use exponential competing clocks. If $(r_i^t c_N)_{i=1}^N$ is the set of points after reproduction we endow each point with independent exponential random variables with respective rates $e^{\beta_N r_i^t}$ as the clocks. We then select the $\lfloor N^\gamma \rfloor$ particles with the earliest ring times. A first and second moment computation then yields that the limiting stochastic exponent at x satisfies

$$g^{t+1}(x) \;=\; s_{\sigma_t} \circ r(g^t)(x)$$

where $s_\sigma(v) \;:=\; \pi\left(v(x) + (\beta x+\sigma)_-\right)$ and σ_t is defined by the implicit relation

$$\sigma_t : \quad \sup_x (s_{\sigma_t} \circ r(g_t)(x)) = \gamma,$$

which (up to scaling) corresponds to the first time where $\lfloor N^\gamma \rfloor$ clocks have rung. To understand the addition of $\beta(x-\sigma)_-$ in the definition of s_σ, we show that most particles $r_i^t cN > \sigma c_N$ will have their clocks ring before time $N^{-\beta\sigma}$, while a particle $r_i^t c_N < \sigma c_N$ has probability $\asymp N^{\beta(r_i^t-\sigma)}$ that their clock will ring before time $N^{-\beta\sigma}$. The effect is that the average number of offspring around xc_N is $\asymp N^{r(g^t)(x)+\beta(x-\sigma)_-}$, and to find σ^t we solve

$$N^\gamma \asymp \int N^{\pi\left(r(g^t)(x)+\beta(x-\sigma_t)_-\right)} dz \asymp N^{\sup_x \pi\left(r(g^t)(x)+\beta(x-\sigma_t)_-\right)}.$$

1.3 Travelling wave solutions.

We say that (1.3) admits a travelling wave solution iff there exists (G^0, v) such that

$$G^0(x-\nu) = G^1(x),$$

where G^1 is defined from G^0 by applying transformation (1.3). Travelling wave solutions are always defined up to translation and provided that $\mathrm{Supp}(G^0)$ is bounded, we take the convention that the right-most boundary of the support is at 0. We show the existence of a travelling wave solution when X is Laplace distributed.

Theorem 1.2. *If X has density $f_X(x) = \frac{1}{2}\exp^{-|x|}$, then (1.3) admits a travelling wave solution (G^0, ν). Further, the travelling wave is locally stable in the sense that there exists $\delta > 0$ such that if*

$$||g_+^0 - G_+^0||_\infty \leq \delta$$

then there exists $c \in \mathbb{R}, C > 0$ and $r \in (0,1)$ such that

$$\forall t \in \mathbb{N}, \quad ||g_+^t(\cdot + \nu t + c) - G_+^0||_\infty \leq Cr^t.$$

The result of local stability is proved using the following approach. The core idea is that after only finitely many steps, the dynamics drive the system to the set of piecewise linear continuous functions with respective slopes from right to left $-1, -1+\beta, -1+2\beta$ etc. Having proved that, it remains to understand the dynamics of the set of points where a change of slope occurs. This give rise to a finite dimensional system whose local stability can be analysed.

The fully pulled/semi-pulled phase transition. Let us briefly mention an important fact about the travelling wave solution (G^0, ν). The solution obviously depends on the parameters (β, γ) where γ captures the selection pressure (high γ; low selection pressure) and β the level of noise in the selection scheme (high β; low noise). The system exhibits a phase transition from a fully to semi pulled regime that we now briefly explain. This phase transition can be first described by the rate of adaptation.

Theorem 1.3. *There exists $\gamma_c(\beta) \in [0,1]$ such that the function $\gamma \to \nu := \nu(\beta, \gamma)$*

- *is increasing on $[0, \gamma_c(\beta))$*
- *is decreasing on $(\gamma_c(\beta), 1]$.*

If $\beta < 1$, then $\gamma_c(\beta) \in (0,1)$; whereas $\gamma_c(\beta) = 0$ if $\beta \geq 1$.

To emphasize the significance of the latter result, we first note that that the rate of evolution of the particle system as defined in (1.1) should have the following asymptotic

$$\frac{c(\lfloor N^\gamma \rfloor, N/\lfloor N^\gamma \rfloor, \beta_N)}{\log(N)} \to \nu(\beta, \gamma) \quad \text{as } N \to \infty.$$

For a fixed value of N and β, we recall that increasing γ amounts to relaxing the selection pressure in the system by allowing more individuals to pass their gene to the next generation. The previous result then implies that when selection is too strong ($\gamma < \gamma_c$), relaxing selection pressure has the unintuitive effect of increasing the rate of adapatation, so that more pressure does not always translate into a higher speed of adaptation.

1.4 Conjectures and perspectives

The two main contributions of the paper are two-fold. First, we provided a general criterion on the tail of the genotypic distribution for the existence of a limiting log-profile in the noisy $[N^\gamma]$-BRW. Secondly, we prove (in the Laplace case) that the unique travelling wave is locally stable and that any perturbation is attracted at a geometric rate. This leads to two natural open questions

(Q1) What is the rate of adaptation c of the original particule system (as defined in (1.1)) ?

(Q2) Let $k \in \mathbb{N}$ and let $\Pi_{N,k}$ be the genealogical structure obtained by sampling k individuals at a given time horizon and then tracing backward in time their respective ancestral lineages. Does there exist a limiting genealogy when $N \to \infty$?

Rate of adaptation. In the Laplace case, Theorem 1.2 naturally suggests the following conjecture

$$\frac{c(\lfloor N^\gamma \rfloor, N/\lfloor N^\gamma \rfloor, \beta_N)}{\log(N)} \to \nu \quad \text{as } N \to \infty.$$

Proving such a result would presumably require to re-inforce Theorem 1.2 and show global convergence to the travelling wave solution.

Genealogical structure. In [5], a variation of the present model was considered, which was introduced by Cortines and Mallein [4] as a generalization of the exponential model of Brunet and Derrida [2, 3]. Therein, it is assumed that at each reproduction step, every individual produces an infinite number of offspring according to an exponential Poisson point process centered around the parental value. $\lfloor N^\gamma \rfloor$ individuals are then selected using a sampling scheme interpolating between truncation selection and β-Gibbs sampling.

The critical assumption of this model is the choice of the exponential reproduction law. In particular, one crucial observation (already made by Brunet and Derrida) is that, up to translation, the system reaches stationarity after a *single* step. (Compared to Theorem 1.2, the system reaches its travelling wave regime at infinite rate instead of geometric rate.) This makes the model fully integrable and allows to make precise predictions on the large-population limit and in particular, on the genealogical structure of the model. In [5], the existence of a critical $\gamma_c(\beta) \in [0, 1]$ was proved, segregating between two different limiting regimes.

- If $\gamma < \gamma_c(\mu)$, $(\Pi_{N,k})_N$ converges to a Poisson-Dirichlet coalescent with parameter $(0, \beta)$ [7, 8].
- If $\gamma > \gamma_c(\mu)$, $(\Pi_{N,k})_N$ converges after rescaling time by $c \log(N)$ (for some explicit choice $c \equiv c(\beta, \gamma)$) to the Bolthausen-Sznitman coalescent [6, 1].

We conjecture that the same transition occurs in the noisy $\lfloor N^\gamma \rfloor$-BRW. However, we emphasize that the proof in [5] heavily relies on the integrability of the exponential model and more specifically, on the property that the the system reaches stationarity after only a single step. Thus, the geometric rate of convergence in Theorem 1.2 appears to be a first important step in the direction of proving the universality of the phase transition observed in the model.

1.5 Outline of the paper

This work proceeds as follows: In section 2 we introduce the model with formal definitions and introduce the notation we will use throughout the rest of the paper. Section 3 is dedicated to proving Theorem 1.1; Theorem 3.8 provides a more general form of Theorem 1.1 where we show a similar result holds, under some conditions, when g and h are not necessarily concave, while Theorem 3.9 is a more precise reformulation of Theorem 1.1 for concave g and h. In Section 4 we examine the case when the offspring distribution is distributed as a Laplace distribution, in which case we prove a precise restatement of Theorem 1.2 (Theorem 4.15).

2 Mathematical terminology of the model

In this section we define the terminology and notation introduced for the model in Section 1. Throughout the rest of this work, for ease of reading, we will assume that N^γ and $N^{1-\gamma}$ are integers. Specifically, we will write N^γ and $N^{1-\gamma}$ for $\lfloor N^\gamma \rfloor$ and $N/\lfloor N^\gamma \rfloor$ respectively. For an integer K, we use the notation $[K] = \{1, \ldots, K\}$, and so we write

$[N^\gamma] = \{1, \dots, \lfloor N^\gamma \rfloor\}$ and $[N^{1-\gamma}] = \{1, \dots, N/\lfloor N^\gamma \rfloor\}$.

For a function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, we will say that f has a left limit at x if and only if $\lim_{z \to x^-} f(z)$ exists and is real, or if there exists $\delta > 0$ such that $f(z) = -\infty$ for all $z \in (x - \delta, x)$, in which case we say that $\lim_{z \to x^-} = -\infty$ exists. Then f is left continuous at x if and only if $\lim_{z \to x^-} f(z) = f(x)$. Similarly we define right limits and right continuity, and say that f is continuous at x if $\lim_{z \to x^-} f(z) = f(x) = \lim_{z \to x^+} f(z)$, and discontinuous otherwise. We will denote the set of discontinuity points of f by D_f. Recall that any function f which admits a left and right limit at every $x \in \mathbb{R}$ has at most countably many discontinuities.

We extend the definition of concavity to functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. Notice that such a function that is concave on a compact support $\mathrm{Supp}(f) := \{x : g^t \neq -\infty\}$ is also concave on the real line; if x or y is taken outside of $\mathrm{Supp}(f)$, then for all $\alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha) f(y) = -\infty.$$

Fix a random variable X with cumulative distribution function F_X. Let $\beta > 0$ and $0 < \gamma < 1$ be real parameters, and recall that we define

$$c_N := (1 - F_X)^{-1}(1/N) := \inf\{x : 1 - F_N(x) \geq 1/N\} \qquad \text{and} \qquad \beta_N := \frac{\beta}{c_N} \log N.$$

Define the function $p_N : \mathbb{R}^2 \to \mathbb{R}_0^- \cup \{-\infty\}$ by

$$p_N(a, b) := \begin{cases} \frac{1}{\log N} \log \left(F_X(bc_N) - F_X(ac_N) \right), & a < b, \\ -\infty & a \geq b. \end{cases} \tag{2.1}$$

We assume that there exists a function $h : \mathbb{R} \to \mathbb{R}_0^- \cup \{-\infty\}$ such that for all $x \in \mathbb{R}$, h is continuous on one side of x and has a limit on the other *(h is càllàl at x; continue à l'un, limite à l'autre)* and for all $\delta > 0$,

$$p_N(x, x + \delta) \xrightarrow{N \to \infty} \sup_{z \in (x, x+\delta]} h(z) \tag{2.2}$$

uniformly as a function of x. As a consequence of (2.2), since h is bounded from above, we see that for any $x > 0$,

$$\frac{\log \mathbb{P}(X > xc_N)}{\log N} \xrightarrow{N \to \infty} \sup_{z \in (x, \infty)} h(z),$$

and if h is monotone on the interval $(0, -\infty)$, then $\mathbb{P}(X > xc_N) \asymp N^{h(x)}$. We also note that a consequence of (2.2), by letting $\delta \to \infty$, then

$$\sup_{z \in \mathbb{R}} h(z) = 0, \tag{2.3}$$

and by letting $x \to \infty$ or $x \to -\infty$ for small δ, we have

$$\lim_{z \to -\infty} h(z) = -\infty, \qquad \text{and} \qquad \lim_{z \to \infty} h(z) = -\infty. \tag{2.4}$$

2.1 Operators on point processes

For all $t \in \mathbb{N}$ let $\mathcal{Z}_N^t := (Z_{N,i}^t)_{i\in[N^\gamma]}$ denote the locations of the N particles in the population at time t. We also define the filtration $(\mathcal{F}_N^t)_{t\in\mathbb{N}}$ by letting $\mathcal{F}_N^t$ be the σ-algebra generated by $(\mathcal{Z}_N^m)_{m\le t}$.

Conditioning on $\mathcal{F}_N^t$, the locations of the next generation are constructed as follows. For all $i \in [N^\gamma]$ let $\mathcal{P}_{N,i}$ denote the point process given by

$$\mathcal{P}_{N,i} := \sum_{j=1}^{N^{1-\gamma}} \delta_{X_{ij}}, \tag{2.5}$$

where $\big(X_{i,j} \,:\, i \in [N^\gamma], j \in [N^{1-\gamma}]\big)$ are i.i.d. and distributed as X. We then define the (random) reproduction operator $\mathcal{R} : \mathbb{R}^{N^\gamma} \to \mathbb{R}^N$ acting on the particle locations:

$$\mathcal{R}(\mathcal{Z}_N^t) := \sum_{i=1}^{N^\gamma} Z_{N,i}^t + \mathcal{P}_{N,i} =: (R_{N,j}^t)_{j\in[N^\gamma]}. \tag{2.6}$$

That is, the $N^{1-\gamma}$ children of $\mathcal{Z}_{N,i}^t$ after reproduction are given by $Z_{N,i}^t + \mathcal{P}_{N,i}$, and $\mathcal{R}^t := (R_{N,j}^t)_{j\in[N]}$ denotes all the N particle locations after reproduction.

We then introduce the (random) sampling operator $\mathcal{S} : \mathbb{R}^N \to \mathbb{R}^{N^\gamma}$ acting on the locations of the children after reproduction. In our case, the operator $\mathcal{S}$ selects N^γ particles from the set $(R_{N,j}^t)_{j\in[N]}$ without replacement, where each particle has relative weight $e^{\beta_N R_{N,i}^t}$. For our purposes, it will be useful to define the operator $\mathcal{S}$ with an equivalent sampling method, which involves *exponential clocks*. Let $\mathcal{X} := (x_i)_{i\in[N]}$ be an arbitrary point process and independently for each x_i, associate an exponential random variable $\tau_i \sim \mathrm{Exp}(e^{\beta_N x_i})$. The indices of a realization of the τ_i's in increasing order are distributed as the indices of a size-biased permutation of the x_i's, where the sampling is made according to the weights $e^{\beta_N x_i}$. This fact is evident from the memoryless property of exponential random variables and the fact that the minimum of a finite number of exponential random variables is also distributed as an exponential random variable. Thus, keeping the N particles that correspond to the indices of the N smallest values of the τ_i's is equal in distribution to sampling without replacement N particles with relative weights $e^{\beta_N x_i}$.

Let $\mathcal{S}_\sigma$ be the operator that selects the x_i's whose exponential clocks have 'gone off' by time $N^{-\beta\sigma}$:

$$\mathcal{S}_\sigma(\mathcal{X}) := \{x_i \in \mathcal{X} :\ \tau_i \le N^{-\beta\sigma}\}. \tag{2.7}$$

Then $-\beta\sigma^N(\mathcal{X})$ will denote the time by which N^γ exponential clocks have gone off; that is,

$$\sigma^N(\mathcal{X}) := \frac{1}{\beta}\inf\{\sigma :\ |\mathcal{S}_\sigma(\mathcal{X})| = N^\gamma\}. \tag{2.8}$$

Since $\mathcal{S}(\mathcal{X})$ is given by the first N^γ of the x_i's whose exponential clocks have gone off, we define

$$\mathcal{S}(\mathcal{X}) := \mathcal{S}_{\sigma^N(\mathcal{X})}(\mathcal{X}). \tag{2.9}$$

For our process $\mathcal{Z}_N^t$, we perform this procedure for $\mathcal{X} := \mathcal{R}_N^t = \mathcal{R}(\mathcal{Z}_N^t)$ in each generation, so that we obtain

$$\mathcal{Z}_N^{t+1} = \mathcal{S} \circ \mathcal{R}(\mathcal{Z}_N^t) := \mathcal{S}(\mathcal{R}_N^t) =: (S_{N,i}^t)_{i\in[N^\gamma]}. \tag{2.10}$$

We study our particle process in terms of point measures describing the configuration of particles. For every $t \in \mathbb{N}$, we define

$$M_N^t := \sum_{i=1}^{N^\gamma} \delta_{Z_{N,i}^t/c_N}, \quad \text{and} \quad M_N^{R,t} := \sum_{j=1}^{N} \delta_{R_{N,j}^t/c_N}. \tag{2.11}$$

Then for all $a < b$, we define

$$M_N^t(a,b) := M_N^t((a,b]) = \#\left\{i : \, Z_{N,i}^t \in (ac_N, bc_N]\right\}. \tag{2.12}$$

and

$$M_N^{R,t}(a,b) := M_N^{R,t}((a,b]) = \#\left\{j : \, R_{N,j}^t \in (ac_N, bc_N]\right\}. \tag{2.13}$$

2.2 Operators on limiting log-profiles

We will be interested in sequences of point processes that have a *limiting log-profile* that belongs to the following classes $\mathcal{C}$ and $\mathcal{D}$ of functions.

Definition 2.1. A function $g : \mathbb{R} \to \mathbb{R}^+ \cup \{-\infty\}$ belongs to $\mathcal{D}$ if and only if

(a) $g(x)$ admits a left limit and a right limit for all $x \in \mathbb{R}$, and

(b) there exists real values $L \le U$, called the *lower edge* and *upper edge* of g respectively, such that $g(x) = -\infty$ for all $x \in \mathbb{R} \setminus [L, U]$, and for all $\delta > 0$,

$$\sup_{x \in (-\infty, L+\delta]} g(x) > 0 \quad \text{and} \quad \sup_{x \in [U-\delta, \infty)} g(x) > 0.$$

A function $g : \mathbb{R} \to \mathbb{R}^+ \cup \{-\infty\}$ belongs $\mathcal{C}$ if and only if $g \in \mathcal{D}$ and

(c) g is concave.

Remark 2.2. Note that condition (a) guarantees that g has at most countably many discontinuity points. Condition (b) guarantees that $g(x) > 0$ for some $x \in \mathbb{R}$. Note that if $L = U$, then $g(U) > 0$ while $g(x) = -\infty$ for all $x \in \mathbb{R} \setminus \{U\}$. Such a function vacuously satisfies (c) and so belongs to $\mathcal{C}$. Further note that any concave function g with bounded and non-empty support will also satisfy (a) and (b), since g will be continuous on the interior of its support.

We now formally define what it means for a sequence of point measures to have a limiting log-profile. Recall that D_g denotes the discontinuity points of g.

Definition 2.3 (limiting log-profile)**.** A sequence of (random) point measures $(M_N)_{N \in \mathbb{N}}$ is said to have a *limiting log-profile* if there exists $g \in \mathcal{D}$ such that for all $a, b \in \mathbb{R}^* \setminus D_g$ with $a < b$,

$$\frac{\log M_N((a,b])}{\log N} \xrightarrow{\mathbb{P}} \sup_{x \in (a,b]} g(x). \tag{2.14}$$

In a slight abuse of notation, we note in the above definition that if $b = \infty$, then $M_N((a,b]) := M_N((a,\infty))$. Note also that the limiting log-profile g of a sequence of point measures is unique up to the values of $g(x)$ for $x \in D_g$.

If a sequence of point processes has a limiting log-profile $g \in \mathcal{D}$, the following result is an immediate consequence.

Lemma 2.4. *Assume $(M_N)_{N\in\mathbb{N}}$ has limiting log-profile $g \in \mathcal{D}$ with lower and upper edges L, U. For any $a, b \in \mathbb{R}^* \setminus D_g$ with $a < b$, if $\sup_{x\in(a,b]} g(x) = -\infty$, then as $N \to \infty$,*

$$M_N((a,b]) \xrightarrow{\mathbb{P}} 0.$$

Furthermore, for the leftmost point $\rho_{N,L}$ and rightmost point $\rho_{N,U}$ of M_N, as $N \to \infty$,

$$\rho_{N,L} \xrightarrow{\mathbb{P}} L, \quad \text{and} \quad \rho_{N,U} \xrightarrow{\mathbb{P}} U.$$

Proof. Since $M_N((a,b])$ takes integer values, it is immediate that

$$\mathbb{P}(M_N((a,b]) > 0) = \mathbb{P}\left(\frac{\log M_N((a,b])}{\log N} \geq 0\right).$$

Thus if $\log M_N((a,b])/\log N \xrightarrow{\mathbb{P}} -\infty$, then $M_N((a,b]) \xrightarrow{\mathbb{P}} 0$.

Now we examine $\rho_{N,U}$. Using the above result and definition of $\mathcal{D}$, for any $\delta > 0$,

$$\mathbb{P}(\rho_{N,U} > U + \delta) = \mathbb{P}(M_N((U+\delta,\infty)) > 0) \xrightarrow{N\to\infty} 0. \tag{2.15}$$

Since g has countably many discontinuity points, we may always find $0 < \delta' \leq \delta$ such that g is continuous at $U - \delta'$, and so

$$\frac{\log M_N((U-\delta',\infty))}{\log N} \xrightarrow{\mathbb{P}} \sup_{x\in(U-\delta',\infty)} g(x) > 0$$

and as such,

$$\mathbb{P}(\rho_{N,U} < U - \delta) \leq \mathbb{P}(M_N((U-\delta',\infty)) = 0) \xrightarrow{N\to\infty} 0. \tag{2.16}$$

Then $\rho_{N,U} \xrightarrow{\mathbb{P}} U$ follows from (2.15) and (2.16). A symmetric argument holds for $\rho_{N,L}$. □

We define operators on $\mathcal{D}$ that mirror the operators defined in Section 2.1. Recall the function h from (2.2). For a function $g \in \mathcal{D}$, define $\bar{r}(g)$ and $r(g)$ by

$$\bar{r}(g)(x) := 1 - \gamma + \sup_{z\in\mathbb{R}}\{g(z) + h(x-z)\}, \qquad \text{and} \qquad r(g)(x) := \pi\big(\bar{r}(g)(x)\big), \tag{2.17}$$

and for any $\sigma \in \mathbb{R}$ and $r \in \mathcal{D}$, define $\bar{s}_\sigma(r)$ and $s_\sigma(r)$ by

$$\bar{s}_\sigma(r)(x) := r(x) + \beta(x-\sigma)_-, \qquad \text{and} \qquad s_\sigma(r)(x) := \pi\big(\bar{s}_\sigma(r)(x)\big). \tag{2.18}$$

If $||r||_\infty = 1$, we then define

$$\sigma^*(r) := \inf\left\{\sigma : \sup_{z\in\mathbb{R}}\{s_\sigma(r)(z)\} \leq \gamma\right\} \tag{2.19}$$

and finally, we define $\bar{s}(r)$ and $s(r)$ by

$$\bar{s}(r) := \bar{s}_{\sigma^*(r)}(r), \qquad \text{and} \qquad s(r) := s_{\sigma^*(r)}(r). \tag{2.20}$$

One of the goals of this paper is to show that if a sequence of particle systems $\mathcal{Z}_N$ admits a limiting log-profile $g \in \mathcal{C}$, then the sequence $\mathcal{S} \circ \mathcal{R}(\mathcal{Z}_N)$ admits the limiting log-profile $s \circ r(g) := s(r(g))$. To accomplish this, we first show that r and s map $\mathcal{C}$ into itself.

Proposition 2.5. *Suppose h is concave on its support. Let $\bar{r}, r$ be defined as in* (2.17) *and let $\bar{s}, s$ be defined as in* (2.20). *Then for all $g \in \mathcal{C}$ such that $||g||_\infty = \gamma$, $\bar{r}(g)$ and $\bar{s}(r(g))$ are concave,*

$$r(g),\ s(r(g)) \in \mathcal{C},$$

and $||\bar{r}(g)||_\infty, ||r(g)||_\infty = 1$ and $||\bar{s}(r(g))||_\infty, ||s(r(g))||_\infty = \gamma$.

Remark 2.6. Note that we do not include s_σ in the above proposition. If σ is large, then we may find $r \in \mathcal{C}$ such that $s_\sigma(r) = -\infty$, which fails to belong to $\mathcal{C}$.

Proof. Set $\bar{r} := \bar{r}(g)$ and $r := r(g)$. For any $x, x' \in \mathbb{R}$ and $\alpha \in [0,1]$, by using the concavity of g and h,

$$\begin{aligned}
\bar{r}(g)(\alpha x + (1-\alpha)x') &= 1 - \gamma + \sup_{z\in\mathbb{R}} \big\{ g(z) + h(\alpha x + (1-\alpha)x' - z) \big\} \\
&= 1 - \gamma + \sup_{z,z'\in\mathbb{R}} \big\{ g(\alpha z + (1-\alpha)z') + h(\alpha(x-z) + (1-\alpha)(x'-z) \big\} \\
&\geq 1 - \gamma + \alpha \sup_{z\in\mathbb{R}} \big\{ g(z) + h(x-z) \big\} + (1-\alpha) \sup_{z'\in\mathbb{R}} \big\{ g(z') + h(x'-z') \big\} \\
&= \alpha \bar{r}(x) + (1-\alpha)\bar{r}(x'),
\end{aligned}$$

and so $\bar{r}$ is concave. Since $||g||_\infty = \gamma$ and $||h||_\infty = 0$, we also have that $||\bar{r}||_\infty = 1$, while (2.4) and property (a) of Definition 2.1 guarantee that

$$\lim_{x\to-\infty} \bar{r}(x) = -\infty, \qquad \text{and} \qquad \lim_{x\to\infty} \bar{r}(x) = -\infty. \tag{2.21}$$

Since π is non-decreasing and concave, then $r = \pi(\bar{r})$ is also concave. Also, $||r||_\infty = 1$, and (2.21) guarantees the existence of lower and upper edges for r, therefore $r \in \mathcal{C}$.

Since $||r||_\infty = 1$, then $\sigma^*(r)$ exists. The function $\beta(x - \sigma^*(r))_-$ is concave, therefore, $\bar{s}(r)$ is also concave while $\bar{s}(r)(x) = -\infty$ whenever $r(x) = -\infty$. By definition, $||\bar{s}(r)||_\infty = \gamma$, and so we see that $s(r) \in \mathcal{C}$. ☐

3 Dynamics of the model as operators on limiting log-profiles

This section is divided as follows. We relegate our technical lemmas to Section 3.1. In Section 3.2, we prove that under some conditions, the operator r indeed describes the change of the limiting log-profile for point measures under the operation $\mathcal{R}$, while a similar result for s and $\mathcal{S}$ is proved in Section 3.3. The conditions are needed since though we get a quite detailed perspective of the process when looking at a logarithmic scale, there are some exceptional cases where the scale does not offer enough precision to describe a limiting log-profile after reproduction or sampling. These minor conditions, however, are always satisfied for limiting log-profiles in $\mathcal{C}$ and when h is concave. We then provide a proof of Theorem 1.1 in Section 3.4.

3.1 Technical Lemmas

Let us introduce the following notation: for sequences of random variables $(X_N)_{N\in\mathbb{N}}$ and $(Y_N)_{N\in\mathbb{N}}$ we write $X_N \lesssim_{\mathbb{P}} Y_N \in \mathbb{R}$ as $N \to \infty$, if and only if, for all $\varepsilon > 0$,

$$\mathbb{P}(X_N - Y_N > \varepsilon) \to 0,$$

as $N \to \infty$. We often use the notation with X_N or Y_N being simply a deterministic constant. In particular, if $X_N \lesssim_{\mathbb{P}} c \in \mathbb{R}$ and $X_N \gtrsim_{\mathbb{P}} c$ as $N \to \infty$, then $X_N \xrightarrow{\mathbb{P}} c$ as $N \to \infty$.

We will also use the following property. Suppose that $X_N \lesssim_{\mathbb{P}} Y_N$ as $N \to \infty$, and that

$$Y_N \xrightarrow{\mathbb{P}} Y^*$$

as $N \to \infty$ for some random (or deterministic) variable Y^*. Then we have

$$X_N \lesssim_{\mathbb{P}} Y^*,$$

as $N \to \infty$. If we assume $X_N \gtrsim_{\mathbb{P}} Y_N$ instead, then it follows that $X_N \gtrsim_{\mathbb{P}} Y^*$.

3.1.1 A Laplace's principle type Lemma

We prove a version of Laplace's principle for the type of Riemann-Stieltjes integrals that we encounter throughout this work. Recall the definition of the space of functions $\mathcal{D}$ (Definition 2.1). Recall that a function f is càllàl if and only if for every $x \in \mathbb{R}$, the function f is continuous on one side and has a limit on the other.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a càllàl function with discontinuity set D_f, and let $(M_N)_{N \in \mathbb{N}}$ be a sequence of (random) point measures on $\mathbb{R}$ which has a limiting log-profile $g \in \mathcal{D}$ with discontinuity set D_g. If $D_f \cap D_g = \emptyset$, then for all $a, b \in \overline{\mathbb{R}} \setminus (D_f \cup D_g)$, as $N \to \infty$,*

$$\frac{\log\left(\int_{(a,b]} N^{f(x)} dM_N(x)\right)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x \in (a,b]} \{f(x) + g(x)\} \in \mathbb{R} \cup \{-\infty\}.$$

Proof. Note throughout that (2.14) can be rewritten as

$$\frac{\log\left(\int_{(a,b]} dM_N(x)\right)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x \in (a,b]} g(x) \tag{3.1}$$

for $a, b \notin D_g$.

Let L and U be the lower and upper edges of g. First suppose that $a = -\infty$ and $b < L$. From Lemma 2.4, $M_N((-\infty, b]) \xrightarrow{\mathbb{P}} 0$, and so in this case,

$$\mathbb{P}\left(\int_{(-\infty,b]} N^{f(x)} dM_N(x) > 0\right) \le \mathbb{P}\left(M_N(-\infty, b) > 0\right) \to 0$$

as $N \to \infty$, and so

$$\frac{\log\left(\int_{(-\infty,b]} N^{f(x)} dM_N(x)\right)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x \in (-\infty,b]} g(x) = -\infty.$$

A similar argument holds if $a > U$ and $b = -\infty$, completing the proof in these cases.

Now suppose $a, b \in \mathbb{R} \setminus (D_f \cup D_g)$, and fix $\varepsilon > 0$. Since f is càllàl, g has left and right limits for every $x \in \mathbb{R}$, and $D_f \cap D_g = \emptyset$, then for every $x \in [a, b]$, there exists $\delta_x > 0$ such that for the open neighbourhood $B(x, \delta_x)$, one of the two holds:

(a) for all $y, z \in B(x, \delta_x)$, $|f(y)-f(z)| < \varepsilon$ and if $(y-x)(z-x) > 0$, then $|g(y)-g(z)| < \varepsilon$, or

(b): for all $y, z \in B(x, \delta_x)$, $|g(y)-g(z)| < \varepsilon$ and if $(y-x)(z-x) > 0$, then $|f(y)-f(z)| < \varepsilon$.

The condition $(y-x)(z-x) > 0$ ensures y and z are on the same side of x. Since g and f have left and right limits for all $x \in \mathbb{R}$, $D_g \cup D_f$ is countable, and so we may also specify that δ_x is chosen in such a way that in addition to the above, both f and g are continuous at $x - \delta_x$ and $x + \delta_x$. Since $[a, b]$ is compact and $\{B(x, \delta_x)\}_{x \in [a,b]}$ is an open cover of $[a, b]$, there is a finite subcover $\{B(x_i, \delta_{x_i})\}_{i=1}^K$ of $[a, b]$. Let

$$B_i := (x_i - \delta_{x_i}, x_i + \delta_{x_i}] \cap (a, b],$$

so that $\bigcup B_i = (a, b]$. We show that for all i,

$$\sup_{x \in B_i} \{f(x) + g(x)\} - 4\varepsilon \lesssim_{\mathbb{P}} \frac{\log\left(\int_{B_i} N^{f(x)} dM_N(x)\right)}{\log N} \lesssim_{\mathbb{P}} \sup_{x \in B_i} \{f(x) + g(x)\} + 2\varepsilon. \tag{3.2}$$

Let $s_i^f := \sup_{x \in B_i}\{f(x)\}$ and $s_i^g := \sup_{x \in B_i}\{g(x)\}$. By the definition of δ_{x_i} and the continuity of f and g at $b \wedge (x_i + \delta_{x_i,\varepsilon})$, we see that

$$\left| \sup_{x \in B_i} \{f(x) + g(x)\} - \left(s_i^f - s_i^g\right) \right| \leq 2\varepsilon. \tag{3.3}$$

For the upper bound of (3.2), since $a \vee (x_i - \delta_{x_i})$ and $b \wedge (x_i + \delta_{x_i})$ are continuity points of g, (3.1) guarantees that

$$\frac{\log\left(\int_{B_i} N^{f(x)} dM_N(x)\right)}{\log N} \leq \frac{\log\left(N^{s_i^f} \int_{B_i} dM_N(x)\right)}{\log N} \xrightarrow{\mathbb{P}} s_i^f + s_i^g,$$

which along with (3.3) proves the upper bound of (3.2).

For the lower bound, we consider the two cases (a) and (b) for $B(x_i, \delta_{x_i})$. Note that if $a = x_i$ or $b = x_i$, since f is continuous at a and b, we may assume $B(x_i, \delta_{x_i})$ is of type (a) in this case.

If $B(x_i, \delta_{x_i})$ is of type (a), then $s_i^f - f(y) \leq \varepsilon$ for all $y \in B_i$. By using (3.1) again,

$$\frac{\log\left(\int_{B_i} N^{f(x)} dM_N(x)\right)}{\log N} \geq \frac{\log\left(N^{s_i^f - \varepsilon} \int_{B_i} dM_N(x)\right)}{\log N} \xrightarrow{\mathbb{P}} s_i^f + s_i^g - \varepsilon,$$

which along with (3.3) completes the lower bound of (3.2) in this case.

If $B(x_i, \delta_{x_i})$ is of type (b), then $s_i^f - f(y) \leq \varepsilon$ for all $x < x_i$ in B_i or for all $x > x_i$ in B_i (this follows from the definition of $B(x_i, \delta_{x_i})$ and since f is either right continuous or left continuous at x_i). In the first case, let $y_i = a \vee (x_i - \delta_{x_i,\varepsilon})$ and $z_i \in (y_i, x_i)$ be a continuity point of g, while for the second case, let $z_i = b \wedge (x_i + \delta_{x_i,\varepsilon})$ and $y_i \in (x_i, z_i)$ be a continuity point of g. In either case, since $B(x_i, \delta_{x_i})$ is of type (b),

$$s_i^g - \sup_{x \in (y_i, z_i]} \{g(x)\} \leq \varepsilon, \tag{3.4}$$

and since $y_i, z_i \notin D_g$, from (3.1) once more,

$$\frac{\log\left(\int_{B_i} N^{f(x)} dM_N(x)\right)}{\log N} \geq \frac{\log\left(N^{s_i^f - \varepsilon} \int_{(y_i,z_i]} dM_N(x)\right)}{\log N} \xrightarrow{\mathbb{P}} s_i^f - \varepsilon + \sup_{x \in (y_i,z_i]} \{g(x)\}.$$

Substituting (3.4) and (3.3) completes the lower bound of (3.2) in this case as well.

Suppose j is such that

$$\sup_{x \in B_j} \{f(x) + g(x)\} = \max_{i=1}^{K} \left\{ \sup_{x \in B_i} \{f(x) + g(x)\} \right\}.$$

With (3.2) in hand and since there are only finitely many B_i, we see that

$$\begin{aligned}
\frac{\log\left(\int_{(a,b]} N^{f(x)} dM_N(x)\right)}{\log N} &\leq \frac{\log\left(\sum_{i=1}^K \int_{B_i} N^{f(x)} dM_N(x)\right)}{\log N} \\
&\leq \frac{\log(K) + \log\left(\max_{i=1}^K \int_{B_i} N^{f(x)} dM_N(x)\right)}{\log N} \\
&\lesssim_{\mathbb{P}} \sup_{x \in B_j} \{f(x) + g(x)\} + 2\varepsilon,
\end{aligned}$$

while

$$\frac{\log\left(\int_{(a,b]} N^{f(x)} dM_N(x)\right)}{\log N} \geq \frac{\log\left(\int_{B_j} N^{f(x)} dM_N(x)\right)}{\log N} \gtrsim_{\mathbb{P}} \sup_{x \in B_j} \{f(x) + g(x)\} - 4\varepsilon.$$

Since $\bigcup B_i = (a,b]$, we see that

$$\sup_{x \in (a,b]} \{f(x) + g(x)\} = \sup_{x \in B_j} \{f(x) + g(x)\},$$

and so letting $\varepsilon \to 0$ proves the statement of the lemma in this case.

Now consider any $a, b \in \mathbb{R}^* \setminus (D_f \cup D_g)$ such that $a < b$. Let $x_L < L$ and $x_U > U$ be continuity points of f and g. Then splitting the integrals

$$\begin{aligned}
\int_{(a,b]} N^{f(x)} dM_N(x) = \int_{(a \wedge x_L, x_L]} & N^{f(x)} dM_N(x) \\
&+ \int_{(a \vee x_L, b \wedge x_U]} N^{f(x)} dM_N(x) + \int_{(x_U, b \vee x_U]} N^{f(x)} dM_N(x)
\end{aligned}$$

and applying the previous cases proves the lemma. □

3.1.2 Concentration inequalities

We adapt classic concentration inequalities to be applied for our results. We will use the following version of McDiarmid's inequality.

Theorem 3.2 (McDiarmid). *Let $X_1, \ldots, X_n$ be independent random variables with $X_k \in [0,1]$ for all $k \in [n]$. Let $S_n := \sum_{k=1}^n X_k$ and let $\mu = \mathbb{E}[S_n]$. Then for any $\epsilon \in (0,1)$,*

$$\mathbb{P}(|S_n - \mu| > \epsilon\mu) < e^{-\frac{1}{4}\epsilon^2 \mu}.$$

Let $(l_N)_{N\in\mathbb{N}}$ and $(a_j^N)_{N,j\in\mathbb{N}}$ be sequences of random variables. For all $N \in \mathbb{N}$, let $\mathcal{F}_N$ be a σ-algebra such that $l_N, a_1^N, \ldots, a_{l_N}^N \in \mathcal{F}_N$. Let $(\xi_j^N))_{N,j\in\mathbb{N}}$ be random variables with $\xi_j^N \in [0,1]$ for all $j, N \in \mathbb{N}$. Suppose that, conditioned on $\mathcal{F}_N$, the random variables $(\xi_j^N))_{N,j\in\mathbb{N}}$ are independent, and they are distributed as $\mathrm{Ber}(a_j^N \wedge 1)$. We also let $S_{l_N} := \sum_{j=1}^{\ell_N} \xi_j^N$ and $\mathbb{E}\left[S_{l_N} \mid \mathcal{F}_N\right] =: \mu_N$ for all $N \in \mathbb{N}$.

Lemma 3.3. *(a) If there exists a constant $C_1 > 0$ such that $\frac{\log \mu_N}{\log N} \xrightarrow{\mathbb{P}} C_1$, then*

$$\frac{\log S_{l_N}}{\log N} \xrightarrow{\mathbb{P}} C_1$$

as $N \to \infty$.

(b) If there exists a constant $C_2 \in \mathbb{R}$ such that $\frac{\log \mu_N}{\log N} \lesssim_{\mathbb{P}} C_2$, then, as $N \to \infty$,

$$\frac{\log S_{l_N}}{\log N} \lesssim_{\mathbb{P}} C_2, \quad \textit{if } C_2 \geq 0,$$

and

$$S_{l_N} \xrightarrow{\mathbb{P}} 0, \quad \textit{if } C_2 < 0.$$

Proof. We first prove part (a). Take $0 < \varepsilon < 1 \wedge C_1/2$. It is enough to show that the probability of the event $\{S_{l_N} \notin [N^{C_1-2\varepsilon}, N^{C_1+2\varepsilon}]\}$ converges to zero as $N \to \infty$. Splitting this event based on the value of μ_N we get

$$\begin{aligned}\mathbb{P}\left(S_{l_N} \notin [N^{C_1-2\varepsilon}, N^{C_1+2\varepsilon}]\right) \leq \mathbb{P}&\left(S_{l_N} \notin [N^{C_1-2\varepsilon}, N^{C_1+2\varepsilon}], \mu_N \in [N^{C_1-\varepsilon}, N^{C_1+\varepsilon}]\right) \\ &+ \mathbb{P}\left(\mu_N \notin [N^{C_1-\varepsilon}, N^{C_1+\varepsilon}]\right). \quad (3.5)\end{aligned}$$

Notice that for N sufficiently large we have

$$\left\{S_{l_N} + 1 \notin [N^{C_1-2\varepsilon}, N^{C_1+2\varepsilon}], \mu_N \in [N^{C_1-\varepsilon}, N^{C_1+\varepsilon}]\right\} \subseteq \left\{|S_{l_N} - \mu_N| > \frac{\mu_N}{2}\right\}.$$

Note furthermore, that the second term on the right-hand side of (3.5) converges to zero as $N \to \infty$ by the assumption of the lemma on μ_N. Hence, continuing (3.5), and first conditioning on $\mathcal{F}_N$, and then applying Theorem 3.2, for N sufficiently large we obtain

$$\begin{aligned}\mathbb{P}\left(S_{l_N} \notin [N^{C_1-2\varepsilon}, N^{C_1+2\varepsilon}]\right) &\leq \mathbb{E}\left[\mathbb{P}\left(|S_{l_N} - \mu_N| > \frac{\mu_N}{2} \mid \mathcal{F}_N\right) \mathbb{1}_{\{\mu_N \in [N^{C_1-\varepsilon}, N^{C_1+\varepsilon}\}}\right] + o(1) \\ &\leq \mathbb{E}\left[e^{-\frac{\mu_N}{16}} \mathbb{1}_{\{\mu_N \in [N^{C_1-\varepsilon}, N^{C_1+\varepsilon}\}}\right] + o(1) \\ &\leq \mathbb{E}\left[e^{-N^{C_1-\varepsilon}/16}\right] + o(1),\end{aligned}$$

which converges to zero as $N \to \infty$, showing part (a).

We now move on to part (b). Take $0 < \varepsilon < 1 \wedge |C_2|/2$. Similarly to part (a), in this case we have

$$\mathbb{P}\left(S_{l_N} > N^{C_2+2\varepsilon}\right) \leq \mathbb{E}\left[\mathbb{P}\left(S_{l_N} > N^{C_2+2\varepsilon} \mid \mathcal{F}_N\right) \mathbb{1}_{\mu_N < N^{C_2+\varepsilon}}\right] + \mathbb{P}\left(\mu_N > N^{C_2+\varepsilon}\right)$$

By Markov's inequality and our assumption on μ_N, and recalling that $\mu_N = \mathbb{E}\left[S_{l_N} \mid \mathcal{F}_N\right]$, we arrive at

$$\mathbb{P}\left(S_{l_N} > N^{C_2+2\varepsilon}\right) \leq \mathbb{E}\left[\frac{\mu_N \mathbb{1}_{\{\mu_N < N^{C_2+\varepsilon}\}}}{N^{C_2+2\varepsilon}}\right] + o(1) \leq N^{-\varepsilon} + o(1) \to 0, \quad (3.6)$$

as $N \to \infty$. Note that (3.6) holds for any $C_2 \in \mathbb{R}$. If $C_2 \geq 0$, then this shows the first statement of part (b). If $C_2 < 0$, then the second statement follows from (3.6) and from the fact that S_{l_N} takes integer values.

□

3.2 Limiting log-profiles after reproduction

Recall the definition of $\bar{r}$ and r from (2.6). Since we are considering a single generation in the following lemma, we will drop the superscript t from M_N^t and $M_N^{R,t}$.

Lemma 3.4. *Suppose the sequence of point measures $(M_N)_{N \in \mathbb{N}}$ defined on the sequence of point processes $\mathcal{Z}_N = (Z_{N,i})_{i \in [N^\gamma]}$ has limiting log-profile $g \in \mathcal{D}$, and let $(M_N^R)_{N \in \mathbb{N}}$ be the sequence of points measures defined on the sequence of point processes $\mathcal{R}(\mathcal{Z}_N) = (R_{N,j})_{j \in [N]}$. For all $a, b \in \mathbb{R} \setminus D_{\bar{r}(g)}$, if $\sup_{x \in (a,b]} \bar{r}(g)(x) \neq 0$, then*

$$\frac{\log M_N^R(a,b)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x \in (a,b]} r(g)(x).$$

Proof. For the remainder of the proof, we will set $\bar{r} := \bar{r}(g)$ and $r := r(g)$. For a pair of points $a < b$, we start by evaluating the function

$$H_{a,b}(z) := \sup_{x \in (a-z, b-z]} h(x).$$

We note that $H_{a,b}(z)$ is continuous wherever $a - z, b - z \notin D_h$, and $H_{a,b}$ is càllàl whenever

$$b - a \notin \{z - y : \ y, z \in D_h\}. \tag{3.7}$$

For any δ, note that $H_{a-\delta,b-\delta}(z) = H_{a,b}(z+\delta)$, and since D_g and D_h have countably many discontinuities, the set $F \subset R^2$ of pairs $a < b$ such that $D_{H_{a,b}} \cap D_h \neq \emptyset$ or that fail (3.7) has empty interior.

Let $\mathcal{F}_N$ denote the σ-algebra generated by the configuration $\mathcal{Z}_N$. Then by the definition of the reproduction step in (2.6) we have

$$M_R^N(a,b) = \int_{(ac_N, bc_N]} \sum_{i=1}^{N^\gamma} \sum_{j=1}^{N^{1-\gamma}} \delta_{X_{ij}+Z_{N,i}}(dx) = \sum_{i=1}^{N^\gamma} \sum_{j=1}^{N^{1-\gamma}} \mathbb{1}_{\{X_{i,j} \in (ac_N - Z_{N,i}, bc_N - Z_{N,i}]\}}. \tag{3.8}$$

Let

$$A_N(a,b) := \frac{\log(\mathbb{E}\left[M_R^N(a,b) \mid \mathcal{F}_N\right])}{\log N}.$$

We aim to prove that for $a, b \in R \setminus D_{\bar{r}}$,

$$A_N(a,b) \xrightarrow{\mathbb{P}} \sup_{x \in (a,b]} \bar{r}(x), \tag{3.9}$$

and then we will apply Lemma 3.3 to conclude the result.

First suppose $a, b \in \mathbb{R} \setminus F$. Since all $X_{i,j}$ are i.i.d., (3.8) gives us

$$\begin{aligned} A_N(a,b) &= \frac{1}{\log N} \log \left(\sum_{i=1}^{N^\gamma} N^{1-\gamma} \mathbb{P}\left(X \in (ac_N - Z_{N,i}, bc_N - Z_{N,i}] \right) \right) \\ &= \frac{1}{\log N} \log \left(\int_{\mathbb{R}} N^{1-\gamma} N^{p_N(a-z,b-z)} dM_0^N(z) \right), \end{aligned} \tag{3.10}$$

where we recall the definition of p_N from (2.1). From (2.2), we have that

$$p_N(a-z, b-z) \xrightarrow{N\to\infty} H_{a,b}(z)$$

uniformly (as a function of z). For the choice of $a < b$, the function $H_{a,b}$ is càllàl and $D_{H_{a,b}} \cap D_g = \emptyset$. Thus for any $\varepsilon > 0$, we may apply Lemma 3.1 with $H_{a,b}(z) + \varepsilon$ and $H_{a,b} - \varepsilon$, and get that

$$\sup_{z\in\mathbb{R}} \{1 - \gamma + g(z) + H_{a,b}(z) - \varepsilon\} \lesssim_{\mathbb{P}} A_N(a,b) \lesssim_{\mathbb{P}} \sup_{z\in\mathbb{R}} \{1 - \gamma + g(z) + H_{a,b}(z) + \varepsilon\}.$$

Letting $\varepsilon \to 0$ proves

$$A_N(a,b) \xrightarrow{\mathbb{P}} \sup_{z\in\mathbb{R}} \{1 - \gamma + g(z) + H_{a,b}(z)\} \tag{3.11}$$

as $N \to \infty$. Observe that by the definition of the function $H_{a,b}$ and by swapping the two suprema, we have

$$\begin{aligned} \sup_{z\in\mathbb{R}} \{1 - \gamma + g(z) + H_{a,b}(z)\} &= \sup_{z\in\mathbb{R}} \left\{ 1 - \gamma + g(z) + \sup_{x \in (a-z, b-z]} h(x) \right\} \\ &= \sup_{x\in(a,b]} \left\{ 1 - \gamma + \sup_{z\in\mathbb{R}} \{g(z) + h(x-z)\} \right\} \\ &= \sup_{x\in(a,b]} \bar{r}(x). \end{aligned}$$

Take any continuity points $a, b \in \mathbb{R} \setminus D_{\bar{r}}$. Then for any $\varepsilon > 0$, since F has empty interior, we may find $a^+ > a$ and $b^- < b$ such that

$$\left| \sup_{x\in(a^+,b^-]} \bar{r}(x) - \sup_{x\in(a,b]} \bar{r}(x) \right| < \varepsilon.$$

From (3.10), we see immediately that $A_N(a^+, b^-) \leq A_N(a,b)$, and letting $\varepsilon \to 0$, we see that $\sup_{x\in(a,b]} \bar{r}(x) \lesssim_{\mathbb{P}} A_N(a,b)$. We can prove that $A_N(a,b) \lesssim_{\mathbb{P}} \sup_{x\in(a,b]} \bar{r}(x)$ by a symmetric argument, and so we conclude that (3.9) holds in this case.

Apply Lemma 3.3 with $l_N = N$, $a_{i,j}^N = \mathbb{P}(X \in (ac_N - Z_i, bc_N - Z_i])$ for all $i \in [N^\gamma]$ and $j \in [N^{1-\gamma}]$, $\xi_{i,j} = \mathbb{1}_{\{X_{i,j} \in (ac_N - Z_i, bc_N - Z_i]\}}$, and $\mathcal{F}_N$. By (3.9) and Lemma 3.3(a) we conclude that

$$\frac{\log M_N^R(a,b)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x\in[a,b]} \bar{r}(x) = \sup_{x\in[a,b]} r(x),$$

if $\sup_{x\in[a,b]} \bar{r}(x) > 0$; and by Lemma 3.3(b) we see that

$$M_N^R(a,b) \xrightarrow{\mathbb{P}} 0,$$

if $\sup_{x\in[a,b]} \bar{r}(x) < 0$, which concludes the proof. □

3.3 Limiting log-profiles after sampling

Recall that $\mathcal{R}_N = (R_{N,j})_{j\in[N]}$ denotes the set of locations of particles after reproduction, and that we wish to sample N^γ particles $\mathcal{S}_N = (S_{N,i})_{i\in[N^\gamma]}$ from this set without replacement, where each particle $R_{N,j}$ has relative weight

$$w_{N,j} = e^{\beta_N R_{N,j}}.$$

Recall also from Section 2.1 that we perform sampling without replacement using exponential clocks: a clock associated to a particle at location $R_{N,j}$ is an exponential random variable $\tau_{N,j} \sim \mathrm{Exp}(w_{N,j})$. Then the particles corresponding to the N^γ smallest values of the $\tau_{N,j}$'s (i.e. the N^γ particles whose clocks rang the fastest) are selected.

Similarly to (2.11), we define

$$M_N^S := \sum_{i=1}^{N} \delta_{S_{N,i}/c_N}$$

and for $a < b$,

$$M_N^S(a,b) := M_S^N((a,b]) := \#\{i : S_i^N \in (ac_N, bc_N)\}. \tag{3.12}$$

We have once more dropped the superscript for generation t, but note from (2.10) that $M_N^{S,t} = M_N^{t+1}$. Recall the definitions of s_σ from (2.18), σ^* from (2.19), and s from (2.20).

Lemma 3.5. *Suppose the sequence of point measures $(M_N^R)_{N\in\mathbb{N}}$ defined on the sequence of point processes $\mathcal{R}_N = (R_{N,j})_{j\in[N]}$ has limiting log-profile $r \in \mathcal{D}$, and let $(M_N^S)_{N\in\mathbb{N}}$ be the sequence of point measures defined on the sequence of point processes $\mathcal{S}_N = (S_{N,i})_{i\in[N^\gamma]}$. If $\sigma^* := \sigma^*(r)$ is the unique value for which $||s_{\sigma^*}(r)||_\infty = \gamma$, then for all $a, b \in \mathbb{R}\setminus D_r$ such that $\sup_{x\in(a,b]} \bar{s}(r)(x) \neq 0$,*

$$\frac{\log M_N^S(a,b)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x\in(a,b]} s(r)(x).$$

In analysing the measures M_N^S we will first investigate the number of clocks that have 'gone off' by a certain time. For any $\sigma \in \mathbb{R}$, define

$$M_{N,\sigma}^S := \sum_{j=1}^{N} \delta_{R_{N,j}/c_N} \mathbb{1}_{\{\tau_{N,j} \le N^{-\beta\sigma}\}}$$

and

$$M_{N,\sigma}^S(a,b) := M_{N,\sigma}^S((a,b]) := \#\big\{j : R_i^N \in (ac_N, bc_N] \text{ and } \tau_{N,j} \le N^{-\beta\sigma}\big\}.$$

Proposition 3.6. *Suppose the sequence of point measures $(M_N^R)_{N\in\mathbb{N}}$ defined on the sequence of point processes $\mathcal{R}_N = (R_{N,j})_{j\in[N]}$ has limiting log-profile $r \in \mathcal{D}$. For all $a, b \in \mathbb{R}\setminus D_r$, if $\sup_{x\in(a,b]} \bar{s}_\sigma(r)(x) \neq 0$, then*

$$\frac{\log M_{N,\sigma}^S(a,b)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x\in(a,b]} s_\sigma(r)(x).$$

Proof. Throughout we let $\bar{s} := \bar{s}(r)$ and let $\mathcal{F}_N^R$ be the sigma algebra generated by $\mathcal{R}_N$. For a particle at location $R_{N,j} = xc_N$, we have $w_{N,j} = N^{\beta x}$, and

$$\mathbb{P}\left(\tau_{N,j} \leq N^{-\beta\sigma} \,|\, \mathcal{F}_N^R\right) = 1 - \exp(-w_{N,j}N^{-\beta\sigma}) = 1 - \exp\left(-N^{\beta(x-\sigma)}\right). \tag{3.13}$$

Since $\beta(x-\sigma)_-$ is continuous, $D_{\bar{s}_\sigma} \subseteq D_r$. For any $a, b \in \mathbb{R}^* \setminus D_r$, if we let $A_N = \mathbb{E}[M_{S,\sigma}^N(a,b)|\mathcal{F}_R^N]$, then

$$A_N = \sum_{R_i^N \in (ac_N, bc_N]} \mathbb{P}\left(\tau_{N,j} \leq N^{-\beta\sigma}|\mathcal{F}_N^R\right) = \int_{(a,b]} \left(1 - \exp\left(-N^{\beta(x-\sigma)}\right)\right) dM_N^R(x). \tag{3.14}$$

Clearly $1 - \exp(-N^{\beta(x-\sigma)}) < 1$ while a standard upper bound on $1 - e^{-x}$ implies

$$1 - \exp(-N^{\beta(x-\sigma)}) < N^{\beta(x-\sigma)},$$

and so

$$A_N \leq \int_{(a,b]} N^{\beta(x-\sigma)_-} dM_N^R(x). \tag{3.15}$$

For all $x \geq \sigma$, $(1 - \exp(-N^{\beta(x-\sigma)})) \geq 1 - e^{-1} > 1/2$, while from a second order taylor approximation, for all $x \leq -\sigma$,

$$1 - \exp\left(-N^{\beta(x-\sigma)}\right) \geq N^{\beta(x-\sigma)} - \frac{N^{2\beta(x-\sigma)}}{2} \geq \frac{1}{2}N^{\beta(x-\sigma)}.$$

Thus for any $\varepsilon > 0$ and all N such that $N^{-\varepsilon} < 1/2$,

$$A_N \geq \int_{(a,b]} N^{-\varepsilon+\beta(x-\sigma)_-} dM_N^R(x). \tag{3.16}$$

Then applying Lemma 3.1 with $\beta(x-\sigma)_-$ and $-\varepsilon + \beta(x-\sigma)_-$, (3.15) and (3.16) provide

$$\sup_{x\in(a,b]} \{r(x) + \beta(x-\sigma)_-\} - \varepsilon \lesssim_{\mathbb{P}} A_N \lesssim_{\mathbb{P}} \sup_{x\in(a,b]} \{r(x) + \beta(x-\sigma)_-\}.$$

Letting $\varepsilon \to 0$ proves

$$A_N \xrightarrow{\mathbb{P}} \sup_{x\in(a,b]} \bar{s}_\sigma(x). \tag{3.17}$$

Now we apply Lemma 3.3 with $l_N = N$, $a_j^N = (1 - \exp(-w_{N,j}N^{-\beta\sigma}))\mathbb{1}_{R_{N,j}\in(ac_N,bc_n]}$, and $\mathcal{F}_N = \mathcal{F}_N^R$. By (3.17) and Lemma 3.3(a), we conclude that

$$\frac{\log M_{N,\sigma}^S(a,b)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x\in(a,b]} \bar{s}_\sigma(x) = \sup_{x\in(a,b]} s_\sigma(x)$$

if $\sup_{x\in(a,b]} \bar{s}_\sigma(x) > 0$, and by Lemma 3.3(b), we see that

$$M_{N,\sigma}^S \xrightarrow{\mathbb{P}} 0$$

if $\sup_{x\in(a,b]} \bar{s}_\sigma(x) < 0$, which concludes the proof. $\square$

Remark 3.7. We note that we can prove a stronger statement when $0 < \beta(x+\sigma)$ From (3.13), we see that for a particle at position $R_{N,j} = xc_N$, the probability that the clock $\tau_{N,j}$ has not 'gone off' by time $N^{\beta\sigma}$ is given by

$$\mathbb{P}(\tau_{N,j} > N^{\sigma} | \mathcal{F}_N^R) = \exp(-N^{\beta(x-\sigma)}).$$

Thus, for any $\varepsilon > 0$,

$$\mathbb{P}\left(M_N^R(\sigma+\varepsilon,\infty) - M_{N,\sigma}^S(\sigma+\varepsilon,\infty) > 0 | \mathcal{F}_N^R\right) \le M_N^R(\sigma+\varepsilon,\infty)\exp(-N^{\beta\varepsilon}),$$

which tends to zero as $N \to \infty$, because $M_N^R(\sigma+\varepsilon,\infty) \le N$ and so the right-hand side tends to zero exponentially quickly. Since we have $M_{N,\sigma}^S(\sigma+\varepsilon,\infty) \le M_N^R(\sigma+\varepsilon,\infty)$ by definition, we conclude that

$$M_N^R(\sigma+\varepsilon,\infty) - M_{N,\sigma}^S(\sigma+\varepsilon,\infty) \xrightarrow{\mathbb{P}} 0.$$

Proof of Lemma 3.5. Let us write

$$\sigma^* := \sigma^*(r) \quad \text{and} \quad \sigma^N := \sigma^N(\mathcal{R}_N).$$

Then $M_{N,\sigma^N}^S(a,b) = M_N^S(a,b)$. Since σ^* is the unique value for which $||s_{\sigma^*}(r)|| = \gamma$, then for all $\varepsilon > 0$ and Proposition 3.6,

$$\frac{\log M_{N,\sigma^*-\varepsilon}^S(-\infty,\infty)}{\log N} \xrightarrow{\mathbb{P}} \sup_{x\in\mathbb{R}} s_{\sigma^*-\varepsilon}(x) > \gamma$$

and so from the definition of σ^N, as $N \to \infty$,

$$\begin{aligned}\mathbb{P}\left(\sigma^* - \varepsilon > \sigma^N\right) &\le \mathbb{P}\left(M_{N,\sigma^*-\varepsilon}^S(-\infty,\infty) \le M_{N,\sigma^N}^S(-\infty,\infty) = N^{\gamma}\right) \\ &\le \mathbb{P}\left(\frac{\log M_{N,\sigma^*-\varepsilon}^S(-\infty,\infty)}{\log N} \le \gamma\right) \to 0\end{aligned}$$

By a symmetric argument, $\mathbb{P}(\sigma^* + \varepsilon < \sigma^N) \xrightarrow{N\to\infty} 0$, and so $\sigma^N \xrightarrow{\mathbb{P}} \sigma^*$. As a consequence, for all $a, b \in \mathbb{R}$,

$$\mathbb{P}\left(\sigma^* - \varepsilon < \sigma^N < \sigma^* + \varepsilon\right) \le \mathbb{P}\left(M_{S,\sigma^*+\varepsilon}^N(a,b) \le M_{S,\sigma^N}^N(a,b) \le M_{S,\sigma^*-\varepsilon}^N(a,b)\right) \xrightarrow{\mathbb{P}} 1. \tag{3.18}$$

Note that for any $x, \sigma \in \mathbb{R}$ and any $\varepsilon > 0$, $|\bar{s}_\sigma(x) - \bar{s}_{\sigma\pm\varepsilon}(x)| \le \varepsilon$, and so it is evident that for all $a, b \in \mathbb{R} \setminus D_r$ such that $\sup_{x\in(a,b]} \bar{s}(x) \neq 0$, the function $\sigma \mapsto \sup_{x\in(a,b]} s_\sigma(x)$ is continuous around $\sigma*$. Then Lemma 3.6 and letting $\varepsilon \to 0$ in (3.18) completes the proof. □

3.4 Proof of Theorem 1.1

Recall that M_N^t is the point measure associated with $\mathcal{Z}_N^t = (Z_{N,i}^t)_{i\in[N^\gamma]}$. From Lemma 3.4 and Lemma 3.5, we are able to prove a version of Theorem 1.1 that also holds for functions h that are not concave, though we require that the conditions of Lemmas 3.4 and 3.5. For the following Theorem, let $g^0 \in \mathcal{D}$ and recursively define $\bar{r}^t := \bar{r}(g^t)$, $r^t := r(g^t)$ as well as $\sigma^t := \sigma^*(r^t)$, and define $\bar{s}^t := \bar{s}(r^t)$, $g^{t+1} := s(r^t)$ for all for $t \ge 0$.

Theorem 3.8. *Suppose that for all $t \geq 0$, we have $g^t, r^t \in \mathcal{D}$,*

$$\sup_{x \in (a,b]} \bar{r}^t \neq 0 \text{ for all } a, b \in \mathbb{R} \setminus D_{r^t}, \tag{3.19}$$

$$\sup_{x \in (a,b]} \bar{s}^t \neq 0 \text{ for all } a, b \in \mathbb{R} \setminus D_{g^{t+1}}, \tag{3.20}$$

and that σ^t is the unique value such that $||s_{\sigma^t}(r^t)|| = \gamma$. If the sequence of point measures $(M_N^0)_{N \in \mathbb{N}}$ has limiting log-profile g^0, then for all $t \in \mathbb{N}$, the sequences of point measures $(M_N^t)_{N \in \mathbb{N}}$ have limiting log-profile g^t.

Proof. Suppose the result is true for $t - 1 \geq 0$. Note that for $r^t = \pi(\bar{r}^t)$, if $x \in D_{\bar{r}^t} \setminus D_{r^t}$, then $r^t(x) = -\infty$. Let $a, b \in \mathbb{R} \setminus D_{r^t}$ with $a < b$. We define $a^- \leq a \leq a^+$ in the following way: if a is a continuity point of $\bar{r}^t$, let $a^- = a^+ = a$. If $a \in D_{\bar{r}^t}$, then by the argument above, $r^t(a) = -\infty$, and since r^t is continuous at a, we may find a^- and $a^+ < b$ such that $\bar{r}^t$ is continuous at a^- and a^+, and $r^t(x) = -\infty$ for all $x \in (a^-, a^+)$. We similarly define $b^- \leq b \leq b^+$. Then

$$\sup_{x \in (a^+, b^-]} r^t(x) = \sup_{x \in (a,b]} r^t(x) = \sup_{x \in (a^-, b^+]} r^t(x). \tag{3.21}$$

By (3.20), (3.21) and Lemma 3.4,

$$\sup_{x \in (a,b]} r^t(x) = \frac{\log M_N^{R,t}(a^+, b^-)}{\log N} \leq \frac{\log M_N^{R,t}(a, b)}{\log N} \leq \frac{\log M_N^{R,t}(a^-, b^+)}{\log N} = \sup_{x \in (a,b]} r^t(x).$$

The above holds for all $a, b \in \mathbb{R} \setminus D_{r^t}$, and so the sequence of point measures $(M_N^{R,t})$ has limiting log-profile r^t.

Let $a, b \in D_{g^{t+1}}$. Since $(\beta - \sigma^t)_-$ is continuous, then $D_{r^t} = D_{\bar{s}^t}$. By a similar argument as above, we may find $a^-, a^+ \in \mathbb{R} \setminus D_{r^t}$ and $b^-, b^+ \in \mathbb{R} \setminus D_{r^t}$ such that $a^- \leq a \leq a^+$, $b^- \leq b \leq b^+$, and

$$\sup_{x \in (a^+, b^-]} s^t(x) = \sup_{x \in (a,b]} s^t(x) = \sup_{x \in (a^-, b^+]} s^t(x). \tag{3.22}$$

Since we assumed σ^t is the unique value for which $||s_{\sigma^t}||_\infty = \gamma$, from (3.20), (3.22) and Lemma 3.5,

$$\sup_{x \in (a,b]} g^{t+1}(x) = \frac{\log M_N^{t+1}(a^+, b^-)}{\log N} \leq \frac{\log M_N^{t+1}(a, b)}{\log N} \leq \frac{\log M_N^{t+1}(a^-, b^+)}{\log N} = \sup_{x \in (a,b]} g^{t+1}(x).$$

The above holds for all $a, b \in D_{g^{t+1}}$, and so the sequence of point measues $(M_N^{t+1})_{N \in \mathbb{N}}$ has limiting log-profile g^{t+1}. Recursively applying the above argument completes the argument. □

In the case where $g^0 \in \mathcal{C}$ and h is concave, we show that the assumptions of Theorem 3.8 are always guaranteed to hold, and we are left with the following "more precise" version of Theorem 1.1.

Theorem 3.9. *Assume the sequence of point measures $(M_N^0)_{N \in \mathbb{N}}$ has limiting log-profile $g^0 \in \mathcal{C}$. If X is such that h is concave, then for all $t \in \mathbb{N}$, the sequences of point measures $(M_N^t)_{N \in \mathbb{N}}$ have limiting log-profile $g^t \in \mathcal{C}$, where g^t is recursively defined by*

$$g^{t+1} := s \circ r(g^t).$$

Proof. By proposition 2.5, for all $t \in \mathbb{N} \cup \{0\}$, we have that $r^t, g^t \in \mathcal{C} \subset \mathcal{D}$. To apply Theorem 3.8, we then only need to show that (3.20) and (3.19) hold and that σ^t is the unique value for which $||s_{\sigma^t}(r^t)||_\infty = \gamma$.

Let L_r^t and U_r^t be the lower and upper edges of r^t. By Proposition 2.5, $\bar{r}^t$ and r^t are concave, and are therefore continuous on the interior of their support, so $D_{r^t} = \{L_r^t, U_r^t\}$. By the concavity of $\bar{r}^t$, we also see that $\bar{r}^t(x) < 0$ for all $x \in \mathbb{R} \setminus [L_r^t, U_r^t]$, and since $\bar{r}^t > 0$ on (L_r^t, U_r^t) we see that

$$a, b \in \mathbb{R} \setminus \{L_r^t, U_r^t\} \Longrightarrow \sup_{x \in (a,b]} \bar{r}(x) \neq 0,$$

and so (3.20) holds.

From Proposition 2.5, $\bar{s}^t$ and g^{t+1} are concave as well, and by repeating the same argument as above, we see that $D_{g^{t+1}} = \{L^{t+1}, U^{t+1}\}$, and that

$$a, b \in \mathbb{R} \setminus \{L^{t+1}, U^{t+1}\} \Longrightarrow \sup_{x \in (a,b]} \bar{s}^t(x) \neq 0,$$

and so (3.19) holds in this case.

Suppose there exists two values $\sigma_1 < \sigma_2$ such that $||s_{\sigma_1}(r^t)||_\infty = ||s_{\sigma_2}(r^t)||_\infty = \gamma$. Then

$$\sup_{x \in (\sigma_1, \infty)} s_{\sigma_1}(r^t)(x) = \sup_{x \in (\sigma_1, \infty)} r^t(x) \leq \gamma.$$

By the concavity of $r^t(x)$ and since $||r^t||_\infty = 1 > \gamma$, then r^t is strictly decreasing for $x \in (\sigma_1, U^t)$. Therefore,

$$\sup_{x \in (\sigma_2, \infty)} s_{\sigma_2}(r^t)(x) < \gamma$$

and

$$\sup_{x \in (\sigma_1, \sigma_2)} s_{\sigma_2}(r^t)(x) = \sup_{x \in (\sigma_1, \sigma_2)} \{r^t(x) + \beta(x - \sigma_2)\} < \sup_{x \in (\sigma_1, \sigma_2)} r^t(x) + \beta(\sigma_2 - \sigma_1) < \gamma.$$

Since $s_{\sigma_2}(r^t)(x) < s_{\sigma_1}(r^t)(x)$ for all $x < \sigma_1$, we see that $||s_{\sigma_2}(r^t)|| < \gamma$, a contradiction. Therefore, $\sigma^t := \sigma^*(r^t)$ is the unique value for which $||s_{\sigma^t}(r^t)||_\infty = \gamma$.

We have then shown that all the conditions are met to apply Theorem 3.8, proving the result. □

From Theorem 3.9 and Lemma 2.4, we also immediately have the following corollary:

Corollary 3.10. *Let L^t, and U^t be the lower and upper edges of g^t. For all $t \in \mathbb{N} \cup \{0\}$, as $N \to \infty$,*

$$\min_{i \in [N^\gamma]} Z_{N,i}^t \xrightarrow{\mathbb{P}} L^t, \qquad \text{and} \qquad \max_{i \in [N^\gamma]} \xrightarrow{\mathbb{P}} U^t.$$

4 The Laplace distribution case

When particles deviate from their parent's position by an Laplace random variable, we are able to provide some results on the long term behaviour of the profile of the particles. For every β, define the parameters

$$\begin{aligned} k = k(\beta) &:= \begin{cases} \lfloor \beta^{-1} \rfloor & \beta^{-1} \notin \mathbb{Z}, \\ \beta^{-1} - 1 & \beta^{-1} \in \mathbb{Z}, \end{cases} \\ m = m(\beta) &:= \lceil 2/\beta \rceil, \\ \gamma_c = \gamma_c(\beta) &:= \frac{k(2-(k+1)\beta)}{(k+1)(2-k\beta)}. \end{aligned}$$

We refer to the case of $\gamma \in (0, \gamma_c)$ as the *fully pulled regime* and the case of $\gamma \in (\gamma_c, 1)$ as the *semi pulled regime*.

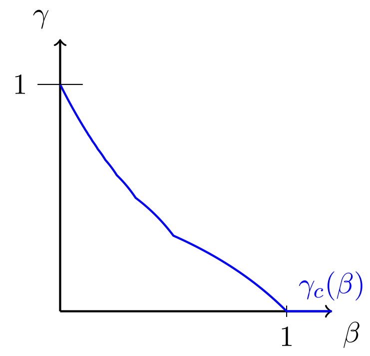

Figure 1: The value of γ_c

In each regime, we prove local convergence and stability results of the profile g^t around a translation of a function G^0. Note that $\gamma_c = 0$ for all $\beta > 1$. Define the following parameters

$$\chi = \chi(\beta, \gamma) := \begin{cases} 0 & \gamma < \gamma_c, \\ \gamma - \sum_{j=1}^{k} (1 - j\beta)(1-\gamma) & \gamma \geq \gamma_c, \end{cases} \tag{4.1}$$

$$\nu = \nu(\beta, \gamma) := \begin{cases} \gamma \left(\sum_{j=1}^{k} (1 - j\beta) \right)^{-1} & \gamma < \gamma_c, \\ 1 - \gamma & \gamma \geq \gamma_c, \end{cases} \tag{4.2}$$

and let the sequence $(y_j^0)_{j=0}^{m+1}$ be defined as $y_j^0 = 0$ and

$$y_j^0 = -\chi - (j-1)\nu$$

for $j = 1, 2, \ldots, m+1$. We define the function G^0 by

$$G^0(x) := \pi \left(-x \wedge \chi + \sum_{j=1}^{m} (1 - j\beta) \left(x \vee y_j^0 - x \vee y_{j+1}^0 \right) - (1+\beta) \left(x \vee y_{m+1}^0 - x \right) \right). \tag{4.3}$$

Note that $0 < \chi \leq \gamma$ when $\gamma_c \leq \gamma < 1$, and that $\nu < 1-\gamma$ when $0 < \gamma < \gamma_c$. The function G^0 is a continuous piecewise linear function when it is non-negative: starting from 0 and moving in the negative direction along the x-axis, G^0 consists of a line segment with slope -1 from 0 to $-\chi$, followed by successive line segments over intervals of length ν with slopes $-(1-j\beta)$ for $j = 1, 2, \ldots, m$, followed by a line segment of slope $1+\beta$ until G^0 hits the y-axis again.

Theorem 1.2 implies that under quite weak conditions, starting from any profile "close enough" to G^0 after translation, the translated profiles will convergence exponentially quickly, with respect to the L^∞-norm, to G^0. To prove Theorem 1.2 when $\beta < 1$, we proceed in three steps: Firstly, in Section 4.1, we show that starting from any g^0, after a finite number of generations, the profiles g^t are continuous and piecewise linear with the appropriate slopes $(1-j\beta), j = 0, 1, \ldots, m$. Secondly, in Section 4.2, we will describe such functions by finite-dimensional vectors $\boldsymbol{v}^t \in \mathbb{R}^k$ denoting the length of the linear segments of g^t, then describe $s \circ r$ as an operator on the corresponding vector $\boldsymbol{v}^t$, and finally use tools of matrix analysis to prove convergence of $\boldsymbol{v}^t$ with respect to the ∞-norm on $\mathbb{R}^k$ to a vector $\boldsymbol{v}$ corresponding to G^0. Thirdly, in Section 4.3, we will prove that $s \circ r$ is continuous (with respect to the L^∞-norm) around G^0, which bounds how much g^t can deviate, and then prove a sort of equivalence between the L^∞-norm applied to $g_+^t(\cdot + U^t) - G_+^0$ and the ∞-norm applied to $\boldsymbol{v}^t - \boldsymbol{v}$. The convergence of $\boldsymbol{v}^t$ to $\boldsymbol{v}$ will then imply the convergence of $g^t(\cdot + U^t)$ to G^0. The case $\beta > 1$ is handled separately at the end.

4.1 Attaining piecewise linear profiles

We proceed by simplifying the description of the operators r and s when $X \sim \mathit{Laplace}(1)$. The cumulative distribution function is given by

$$F_X(x) = \begin{cases} 1 - \frac{1}{2}e^{-x} & x \geq 0, \\ \frac{1}{2}e^{x} & x \leq 0, \end{cases}$$

and we see that $c_N \asymp \log N$, and $\beta_N = \beta$. Furthermore, we have that $h(x) = -|x|$ satisfies the convergence conditions (2.2) for p_N. Since h is concave, we can apply Theorem 1.1 if $(M_N^0)_{N\in\mathbb{N}}$ has limiting log-profile $g^0 \in \mathcal{C}$. The reproduction operator r defined in (2.17) on a function $g \in \mathcal{C}$ is given by

$$r(g)(x) := \pi\left(1 - \gamma + \sup_{z\in\mathbb{R}}\{g(z) - |x - z|\}\right). \tag{4.4}$$

Throughout this section, assume $g \in \mathcal{C}$ with lower and upper edges L and U, set $r := r(g)$ with lower and upper edges L_r and U_r, and set $\sigma^* := \sigma^*(r)$. For all $\sigma \in \mathbb{R}$, set $s_\sigma := s_\sigma(r)$ with upper edge U_σ and $s := s_{\sigma^*}$ with upper edge V. Define the value

$$\omega = \omega(g) := \sup_{z\in(-\infty,U)}\{g(z) + z\}.$$

The following lemma gathers some useful tools for the following proofs of this section.

Lemma 4.1. *The function $r(x) + x$ is non-decreasing on $(-\infty, U_r)$ and the function*

$s(x)+x$ is non-decreasing on $(-\infty, V)$. Furthermore, if $||g||_\infty = \gamma$,

$$\sigma^* \le \begin{cases} \omega - \gamma + \frac{1-\gamma}{\beta} & 0<\beta<1, \\ 1-2\gamma+\omega & \beta>1, \end{cases} \tag{4.5}$$

$$V = \begin{cases} (\omega+1-\gamma) \wedge \left(\frac{\omega+1-\gamma-\beta\sigma^*}{1-\beta}\right), & 0<\beta<1, \\ \omega+1-\gamma, & \beta>1. \end{cases} \tag{4.6}$$

Remark 4.2. Notice that $\omega \ge U$, and so from Lemma (4.5) and (4.6),

$$V - U \ge \begin{cases} (1-\gamma) \wedge \left(\frac{\beta\gamma}{1-\beta}\right) & 0<\beta<1 \\ 1-\gamma & \beta>1. \end{cases} \tag{4.7}$$

Proof. We start with proving $r(x)+x$ is non-decreasing. For any $x_1 < x_2 < U_r$, then

$$\begin{aligned} \sup_{z\in(-\infty,x_2]} \{g(z)+z-x_2\} + x_2 = & \sup_{z\in(-\infty,x_2]} \{g(z)+z-x_1\} + x_1 \\ & \ge \sup_{z\in(-\infty,x_2]} \{g(z) - |x_1 - z|\} \end{aligned}$$

while

$$\sup_{z\in(x_2,\infty)} \{g(z)+x_2-z\} + x_2 > \sup_{z\in(x_2,\infty)} \{g(z)+x_1-z\} + x_1,$$

and so

$$r(x_2)+x_2 = 1-\gamma+\sup_{z\in\mathbb{R}}\{g(z)-|x_2-z|\} \ge 1-\gamma+\sup_{z\in\mathbb{R}}\{g(z)-|x_1-z|\} = r(x_1)+x_1.$$

As for s, the claim follows from above and the fact $\beta(x-\sigma^*)_-$ is non-decreasing, since for all $x_1 < x_2 < V \le U_r$,

$$s(x_2)+x_2 = r(x_2)+x_2+\beta(x_2-\sigma^*)_- \ge r(x_1)+x_1+\beta(x_1-\sigma^*)_- = s(x_1)+x_1.$$

Next for σ^*, we know that $r(x)+x$ is non-decreasing for all $x \in (-\infty, U_r)$ and that $r(x) \le 1$. For all $z > U$, we have that $g(z) = -\infty$, and so for all $x \ge U$, from (4.4),

$$r(x) = \pi\left(1-\gamma+\sup_{z\in(-\infty,U]}\{g(z)+z-x\}\right) = \pi\left(1-\gamma+\omega-x\right). \tag{4.8}$$

We see that r is bounded above by the function $r_{\max}(x)$ defined by

$$r_{\max}(x) = \begin{cases} 1-\gamma+\omega-x & x\in(\omega-\gamma,\omega-\gamma+1), \\ 1 & x\in[L_r,\omega-\gamma], \\ -\infty & \text{otherwise.} \end{cases}$$

We can then verify that

$$\sigma^*(r_{\max}) = \begin{cases} \omega-\gamma-\frac{\gamma-1}{\beta} & 0<\beta<1, \\ 1-2\gamma+\omega & \beta>1, \end{cases}$$

and since $r \le r_{\max}$, then $\sigma^* \le \sigma^*(r_{\max})$, proving (4.5).

From (4.8), we have that see that

$$U_r = \omega - \gamma + 1. \tag{4.9}$$

When $\beta < 1$, the line $1 - \gamma + \omega - x + \beta(x - \sigma^*)$ intersects the x-axis at

$$\frac{\omega + 1 - \gamma - \beta\sigma^*}{1 - \beta}. \tag{4.10}$$

Thus, we have that V is given by the minimum of (4.9) and (4.10) in this case. When $\beta > 1$, from (4.8) and (4.5),

$$\sigma^* < \omega - \gamma + 1 = U_r,$$

thus $s(x) = r(x)$ for $x \ge \sigma^*$ and so $V = U_r$. $\square$

We make a useful observation for the reproduction operator when $g(x) + x$ is a non-decreasing function on $(-\infty, U)$.

Proposition 4.3. *If $g(x) + x$ is non-decreasing on $(-\infty, U)$, then*

$$r(x) := \begin{cases} \pi\left(1 - \gamma + \sup_{z \ge x}\{g(z) - z\} + x\right) & x < U, \\ \pi\left(1 - \gamma + \lim_{z \to U^-}\{g(z) + z\} - x\right) & x \ge U. \end{cases} \tag{4.11}$$

Proof. For all $x < U$, $\sup_{z \in (-\infty, x]}\{g(z) + z\} = g(x) + x$, and so

$$\sup_{z \in \mathbb{R}}\{g(z) - |x - z|\} = \sup_{z \ge x}\{g(z) - z + x\},$$

and applying (4.4) completes the proof in this case.

For all $z > U$, we have $g(z) = -\infty$. Since $g \in \mathcal{C}$, it is a càllàl function, and so in particular, $g(U) \le \lim_{z \to U^-} g(z)$. Using again that $g(x) + x$ is non-decreasing,

$$\sup_{z \in \mathbb{R}}\{g(z) - |x - z|\} = \sup_{z \le U}\{g(z) + z\} = \lim_{z \to U^-} g(z),$$

and applying (4.4) again completes the proof. $\square$

We define a subclass $\mathcal{C}_\ell \subset \mathcal{C}$ of potential limiting-log profiles consisting of continuous piecewise linear functions.

Definition 4.4. For fixed β, a function $g : \mathbb{R} \to \mathbb{R}^+ \cup \{-\infty\}$ belongs to $\mathcal{C}_\ell$ if and only if there exists a non-constant sequence $(x_j)_{j=0}^{m+1}$ of real numbers satisfying $x_{j+1} \le x_j$ for all $j = 0, 1, 2, \ldots, m$ such that

$$g(x) = \pi\left(\sum_{j=0}^{m}(1 - j\beta)(x \vee x_j - x \vee x_{j+1}) - (1 + \beta)(x \vee x_{m+1} - x)\right).$$

We say that g is *associated with* the sequence $(x_j)_{j=0}^{\infty}$.

A sequence associated with g may not be unique, but for ease of calculations later, we will specify that g is associated with a sequence $(x_j)_{j=0}^{m+1}$ such that $L \leq x_{m+1}$, where L is the left edge of g. It is straight forward to show that $\mathcal{C}_\ell \subset \mathcal{C}$. Notice that G^0 defined in (4.3) belongs to $\mathcal{C}_\ell$ and is associated with $(y_j^0)_{j=0}^{m+1}$. We next show how the operators r and s_σ behave on functions that belong to $\mathcal{C}_\ell$.

Lemma 4.5. *Let $g \in \mathcal{C}_\ell$ associated with $(x_j)_{j=0}^{m+1}$ and let $\sigma \in \mathbb{R}$. If $||s_\sigma||_\infty > 0$, then $s_\sigma \in \mathcal{C}_\ell$ associated with $(U_\sigma \wedge y_j)_{j=0}^{m+1}$, where*

$$\begin{aligned} y_0 &= x_0 + 1 - \gamma, \\ y_1 &= \sigma \vee x_1, \\ y_j &= (\sigma \wedge x_{j-1}) \vee x_j \text{ for } j = 2, \dots, m+1. \end{aligned} \tag{4.12}$$

Proof. We see that since $g \in \mathcal{C}$, then $g(x) + x$ is a non-decreasing function on $(-\infty, U)$, and so we may apply (4.11) for r. We also see that $\lim_{z \to U^-} \{g(z) + z\} = x_0$, and so

$$x \geq x_0 \Longrightarrow r(x) = 1 - \gamma + x_0 - x. \tag{4.13}$$

For $j = 0, \dots, m-1$, we have that $-(1 - j\beta) < 1$. By using the concavity of g, if $x \in (x_{j+1}, x_j)$, then for all $z \geq x$,

$$g(x) \geq g(z) - (1 - j\beta)(z - x) > g(z) + x - z.$$

Therefore, by using the continuity of g on (x_m, x_0) for the points $x_1, x_2, \dots, x_{m-1}$,

$$x \in (x_m, x_0) \Longrightarrow r(x) = 1 - \gamma + \sup_{z \geq x}\{g(z) + x - z\} = 1 - \gamma + g(x). \tag{4.14}$$

For $x \leq x_m$, since $(1 + \beta), -(1 - m\beta) \leq -1$, using the concavity of g again, for all $z > x$,

$$\lim_{y \to x_m^+} g(x_m) + (z - x_m) \geq g(z),$$

and so

$$x \leq x_m \Longrightarrow 1 - \gamma + \sup_{z \geq x}\{g(z) + x - z\} = 1 - \gamma + \lim_{z \to x_m^+} g(z) - (x_m - x). \tag{4.15}$$

Putting (4.13), (4.14), and (4.15) together, we see that we may write $r(x)$ as

$$\pi\left(x \vee (x_0 + 1 - \gamma) - x \vee v_0) + \sum_{j=0}^{m-1}(1 - j\beta)(x \vee x_j - x \vee x_{j+1}) - (1 + \beta)(x \vee x_m - x)\right). \tag{4.16}$$

For all $x \in (L_r, x_0 + 1 - \gamma)$, we can write

$$\begin{aligned} \beta(x - \sigma)_- = -\beta\left(\sigma \wedge (x \vee (x_0 + \gamma - 1)) - (\sigma \wedge (x \vee x_0))\right) \\ - \beta \sum_{j=0}^{m-1}\left(\sigma \wedge (x \vee x_j) - (\sigma \wedge (x \vee x_{j+1}))\right) \\ - \beta\left(\sigma \wedge (x \vee x_m) - (\sigma \wedge x)\right). \end{aligned} \tag{4.17}$$

Adding (4.16) and (4.17) and then sending all non-positive values to $-\infty$ produces s_σ associated with $(U_\sigma \wedge y_j)_{j=0}^{\infty}$ as described in (4.12). $\square$

The above lemma applies for all $g \in \mathcal{C}$, no matter the value of $||g||_\infty$ (we make use of this more general form in Lemma 4.7 below). In the next proposition, we offer formulas for σ^* and $V = U_{\sigma^*}$ when $||g||_\infty = \gamma$, which along with Lemma 4.5, provides the means to calculate the sequence associated with s directly from the sequence associated with g. We note the following expression for $||g||_\infty$ when $g \in \mathcal{C}_\ell$:

$$||g||_\infty = \sum_{j=0}^{k}(1 - j\beta)(x_j - x_{j+1}). \tag{4.18}$$

Proposition 4.6. *Let* $0 < \beta < 1$. *If* $g \in \mathcal{C}_\ell$ *is associated with* $(x_j)_{j=0}^\infty$ *and* $||g||_\infty = \gamma$, *then*

$$\sigma^* = \begin{cases} x_{k+1} + \frac{1-\gamma}{1-k\beta} & (1-k\beta)(x_k - x_{k+1}) \geq 1-\gamma, \\ x_k + \frac{1-\gamma}{\beta} - \frac{1-k\beta}{\beta}(x_k - x_{k+1}) & (1-k\beta)(x_k - x_{k+1}) < 1-\gamma, \end{cases}$$

and

$$V - U = (1-\gamma) \wedge \left(\frac{\beta(x_0 - x_k) + (1-k\beta)(x_k - x_{k+1})}{1-\beta} \right).$$

Proof. First assume $(1 - k\beta)(x_k - x_{k+1}) \geq 1 - \gamma$. From (4.16) and (4.18), we have that $r(x_{k+1}) = 1$ and

$$r(x_k) = 1 - \gamma + \sum_{j=0}^{k-1}(1 - j\beta)(x_j - x_{j+1}) = 1 - (1 - k\beta)(x_k - x_{k+1}) \leq \gamma. \tag{4.19}$$

We need to find the value of σ^* such that $||s||_\infty = \gamma$. By setting

$$\sigma^* = x_{k+1} + \frac{1-\gamma}{1-k\beta} \in (x_{k+1}, x_k], \tag{4.20}$$

from (4.19),

$$r(\sigma^*) = r(x_k) + (1 - k\beta)(x_k - \sigma^*) = \gamma.$$

Also, $s(x) = r(x) < r(\sigma^*)$ for $x > \sigma^*$ and $s(x) = \pi(r(x) + \beta(x - \sigma^*)) \leq r(\sigma^*)$ for $x \leq \sigma^*$, and so $||s||_\infty = \gamma$ as needed.

Now suppose $(1 - k\beta)(x_k - x_{k+1}) < 1 - \gamma$. From (4.16) and (4.18) once more,

$$r(x_k) = 1 - \gamma + \sum_{j=0}^{k-1}(1 - j\beta)(x_j - x_{j+1}) = 1 - (1 - k\beta)(x_k - x_{k+1}) > \gamma, \tag{4.21}$$

which implies $\sigma^* > x_k$. Then $r(x) + \beta(x - \sigma^*)_-$ is decreasing for $x \in (x_k, U_r)$, and $s(x) = \pi(r(x) + \beta(x - \sigma^*))$ is non-decreasing for $x < \sigma^*$. Therefore we need to find σ^* such that

$$\gamma = ||s||_\infty = s(x_k) = r(x_k) + \beta(x_k - \sigma^*) = 1 - (1 - k\beta)(x_k - x_{k+1}) + \beta(x_k - \sigma^*).$$

Solving for σ^* completes the first part of the lemma.

If $(1 - k\beta)(x_k - x_{k+1}) \geq 1 - \gamma$ then $\sigma^* < U$ by (4.20), and so $V = U + 1 - \gamma$.

Suppose $(1 - k\beta)(x_k - x_{k+1}) \geq 1 - \gamma$. For $g \in \mathcal{C}_\ell$, we see that

$$\omega = \sup_{z \in (-\infty, U]} \{g(z) + z\} = x_0.$$

Then

$$\frac{\omega + 1 - \gamma - \beta\sigma^*}{1 - \beta} - U = \frac{\beta(x_0 - x_k) + (1 - k\beta)(x_k - x_{k+1})}{1 - \beta}.$$

The second part of the lemma now follows from the above and (4.6). □

For the remainder of this section, we will examine the evolution of $g^t \in \mathcal{C}$. We denote the lower and upper edges of g^t by L^t and U^t, let $r^t = r(g^t)$ with edges L^t_r, U^t_r, and denote $\sigma^t = \sigma^*(r)$.

In the next lemma, we recursively apply Lemma 4.5 to g^t restricted to an interval where it is piecewise linear, and show that for all $t > t_\ell := t_\ell(\beta, \gamma)$, a value depending only on γ and β, then $g^t \in \mathcal{C}_\ell$. Recursively define the sequences $(x^t_j)_{j=0}^{m+1}$ by $x^0_j = U^0$ for all j, and for all $t \geq 1$,

$$\begin{aligned} x^t_0 &= U^t, \\ x^t_1 &= (\sigma^t \vee x^{t-1}_1) \wedge U^t, \\ x^t_j &= (\sigma^t \wedge x^{t-1}_{j-1}) \vee x^{t-1}_j \text{ for } j = 2, \ldots, m+1. \end{aligned} \tag{4.22}$$

and let $g^t_\ell \in \mathcal{C}_\ell$ be the function associated with $(x^t_j)_{j=0}^{m+1}$. Notice from (4.7) that $(x^t_0)_{t=0}^\infty$ is an increasing sequence, and by applying (4.22) recursively, we also have that $(x^t_j)_{t=0}^\infty$ is an increasing sequence for all j.

Lemma 4.7. *For all $t \geq 1$,*

$$x \geq U^0 \Longrightarrow g^t_\ell(x) = g^t(x). \tag{4.23}$$

Proof. Recalling (4.8), we have $r^0(x) = \gamma - 1 + \omega g^0 - x$ for all $x \in [U^0, U^0_r)$. Then for all $x \in [U^0, U^1)$, since $x^1_2 = U^0$,

$$g^1(x) = r^0(x) + \beta(x - \sigma^0)_- = (x \wedge x^1_0 - x \wedge x^1_1) + (1 - \beta)(x \wedge x^1_1 - x \wedge x^1_2),$$

and so (4.23) holds for $t = 1$.

Now suppose (4.23) holds for some $t \geq 1$. By Lemma 4.1, $g^t(x) + x$ is non-decreasing on $(-\infty, U^t)$, while $g^t_\ell(x) + x$ is also non-decreasing on $(-\infty, U^t)$ by definition of $g^t_\ell \in \mathcal{C}_\ell$. We note also from (4.7) that $U^t > U^0$. Thus we may apply (4.11) to both g^t and g^t_ℓ and notice that $r(g^t_\ell)(x) = r^t(x)$ for all $x \geq U^0$. As a consequence, for all $x \geq U^0$,

$$g^{t+1}(x) = s(r^t)(x) = s_{\sigma^t} \circ r(g^t_\ell)(x)$$

By applying Lemma 4.5, we see that $s_{\sigma^t} \circ r(g^t_\ell) \in \mathcal{C}$ associated with $(x^t_j)_{j=0}^{m+1}$, and so $s_{\sigma^t} \circ r(g^t_\ell) = g^{t+1}_\ell$, and (4.23) holds for $t + 1$. Then the claim that (4.23) holds for all $t \geq 1$ follows by induction. □

We now have all we need to show that starting from any $g^0 \in \mathcal{C}$, after finitely many generations, all functions g^t belong to the class $\mathcal{C}_\ell$ of piecewise linear functions.

Lemma 4.8. *Let $0 < \beta < 1$. There exists a number $t_\ell := t_\ell(\gamma, \beta)$ (depending only on γ and β) such that $g^t_\ell = g^t$ for all $t \geq t_\ell$.*

Proof. From (4.23),

$$g^t(x^t_{k+1}) = g^t_\ell(x^t_{k+1}) \geq (1 - k\beta)(x^t_0 - x^t_{k+1}). \tag{4.24}$$

By recursively applying (4.7), we have that $U^t - U^0 > \gamma(1 - k\beta)^{-1}$ for all

$$t \geq t_1 = 1 + \gamma \left((1 - k\beta) \left(\left(\frac{\beta\gamma}{1 - \beta} \right) \wedge (1 - \gamma) \right) \right)^{-1}.$$

Thus $U^0 < x^t_{k+1}$ for all $t \geq t_1$; otherwise $x^t_{k+1} = U^0$ and since $U^t = x^t_0$, from (4.24), $g^t(U^0) > \gamma$, a contradiction.

For $t \geq t_1$, then $||g^t_\ell||_\infty = \gamma$, and so from Proposition 4.6, $\sigma^t > x^t_{k+1}$. By (4.22), we therefore have that $x^{t+1}_{j+1} = x^t_j$ for all $j = k + 1, \ldots, m$. In particular, $U^0 < x^{t+1}_{k+2}$ for all $t \geq t_1$, and by letting

$$p := \min\{(k + 2)\beta - 1, 1 + \beta\} > 0,$$

from (4.23),

$$g^{t+1}(U^0) = g^{t+1}_\ell(U^0) \leq g^{t+1}_\ell(x^{t+1}_{k+2}) - p(x^{t+1}_{k+2} - U^0) = g^{t+1}(x^{t+1}_{k+2}) - p(x^{t+1}_{k+2} - U^0). \tag{4.25}$$

By rearranging (4.24) and setting $g^t(x^t_{k+1}) \leq \gamma$,

$$x^t_{k+1} \geq x^t_0 - \frac{\gamma}{1 - k\beta}. \tag{4.26}$$

By substituting (4.26) and $x^t_{k+1} = x^{t+1}_{k+2}$ into (4.25), for all $t \geq t_1$,

$$g^{t+1}(U^0) \leq \gamma - p(x^t_{k+1} - U^0) \leq \gamma - p\left(U^t - U^0 - \frac{\gamma}{1 - k\beta}\right). \tag{4.27}$$

We have already shown that $U^t - U^0 > \gamma(1 - k\beta)^{-1}$ for $t \geq t_1$, and since g^t is concave, then $g^t(x) \leq g^t(U^0)$ for all $x < U^0 < x^t_{k+1}$. By once more applying (4.7) recursively,(4.27) guarantees that for all

$$t \geq t_\ell := 1 + t_1 + \gamma \left(p \left(\left(\frac{\beta\gamma}{1 - \beta} \right) \wedge (1 - \gamma) \right) \right)^{-1},$$

we have $g^t(x) < 0$ for all $x \leq U^0$ (and so $g^t(x) = -\infty$). Therefore, from (4.23), $g^t = g^t_\ell$. $\square$

4.2 $s \circ r$ as operators on vectors

For all $t \geq t_\ell$ defined in Lemma 4.8, let $v^t_j = x^t_j - x^t_{j+1}$ and define $\boldsymbol{v}^t = (v^t_1, v^t_2, \ldots, v^t_k)$. From (4.18), we know that

$$\gamma = \sup_{x\in\mathbb{R}} g^t(x) = \sum_{j=0}^{k} (1 - j\beta)v_j, \tag{4.28}$$

and so $v^t_0 = x^t_0 - x^t_1 = \gamma - \sum_{j=1}^k (1 - j\beta)v^t_j$. Recall the definition of ν from (4.2) and let $\boldsymbol{\nu} = (\nu, \cdots, \nu) \in \mathbb{R}^k$. The main goal of this section is to show $\boldsymbol{v}^t \to \boldsymbol{\nu}$.

We start by examining the regime $\gamma \in (0, \gamma_c)$. Define

$$A_F = \begin{pmatrix} \frac{\beta}{1-\beta} & \frac{\beta}{1-\beta} & \cdots & \frac{\beta}{1-\beta} & \frac{1-k\beta}{1-\beta} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Lemma 4.9. *Let $0 < \beta < 1$ and $g^t \in \mathcal{C}_\ell$. If $v_0^t = 0$ and*

$$\frac{\beta}{1-\beta} \sum_{j=1}^{k-1} v_j^t + \frac{1-k\beta}{1-\beta} v_k^t \leq 1 - \gamma, \tag{4.29}$$

then

$$\boldsymbol{v}^{t+1} = A_F \boldsymbol{v}^t, \qquad \textit{and} \qquad U^{t+1} - U^t = v_1^{t+1}.$$

Proof. Since $v_0^t = 0$, then $x_0^t = x_1^t = U^t$. By (4.29),

$$(1 - k\beta)(x_k^t - x_{k+1}^t) < \frac{1-k\beta}{1-\beta} v_k^t \leq 1 - \gamma.$$

Furthermore from (4.29) and Proposition 4.6, since $x_k = U^t - \sum_{j=0}^{k-1}(x_j^t - x_{j+1}^t)$,

$$\sigma^t = U^t - \sum_{j=0}^{k-1} v_j^t + \frac{1-\gamma}{\beta} - \frac{1-k\beta}{\beta} v_k^t \geq U^t - \frac{1-\beta}{\beta}(1-\gamma) + \frac{1-\gamma}{\beta} = U^t + 1 - \gamma.$$

and

$$U^{t+1} - U^t = \frac{\beta(x_0^t - x_k^t) + (1-k\beta)(x_k^t - x_{k+1}^t)}{1-\beta} = \frac{\beta}{1-\beta} \sum_{j=1}^{k-1} v_j^t + \frac{1-k\beta}{1-\beta} v_k^t \leq 1 - \gamma.$$

Therefore, by (4.22), we have

$$\begin{aligned} x_0^{t+1} &= x_1^{t+1} = U^{t+1}, \text{ and} \\ x_j^{t+1} &= x_{j-1}^t \text{ for all } j \geq 2, \end{aligned}$$

and so $v_0^{t+1} = 0$, $v_1^{t+1} = U^{t+1} - U^t$, and $v_j^{t+1} = v_{j-1}^t$ for all $j \geq 2$, thus $\boldsymbol{v}^{t+1} = A_F \boldsymbol{v}^t$. □

So long as the operator $s \circ r$ behaves like the lemma above for all t, then we can prove that $\boldsymbol{v}^t$ and $U^{t+1} - U^t$ both converge exponentially fast to limiting values. Recall from Lemma 4.8 that $g^t \in \mathcal{C}_\ell$ for all $t \geq t_\ell$.

Lemma 4.10. *Let $0 < \beta < 1$. If $v_0^{t_\ell} = 0$ and $||\boldsymbol{v}^{t_\ell}||_\infty \leq 1-\gamma$, then there exists a constants $C_1' > 0$ and $0 < r < 1$ such that for all $t \geq t_\ell$,*

$$||\boldsymbol{v}^t - \boldsymbol{v}||_\infty < C_1' r^t, \qquad \textit{and} \qquad |(U^{t+1} - U^t) - \nu| < C_1' r^t.$$

Remark 4.11. Note that when $\gamma > \gamma_c$, then $(1-\gamma)\sum_{j=1}^{k}(1 - j\beta) < \gamma$, and so it is impossible to satisfy both the conditions $v_0^{t_\ell} = 0$ and $||\boldsymbol{v}^{t_\ell}||_\infty \leq 1 - \gamma$ of Lemma 4.10 in this case.

Proof. If $v_0^t = 0$ and $||\boldsymbol{v}^t||_\infty \leq 1-\gamma$, then

$$\frac{\beta}{1-\beta}\sum_{j=1}^{k-1} v_j^t + \frac{1-k\beta}{1-\beta} v_k^t \leq \frac{\beta(k-1)}{1-\beta}(1-\gamma) + \frac{1-k\beta}{1-\beta}(1-\gamma) = 1-\gamma.$$

Thus (4.29) is satisfied and we may apply Lemma 4.9 to find that $||\boldsymbol{v}^{t+1}||_\infty \leq 1-\gamma$ and

$$\begin{aligned}
v_0^{t+1} &= \gamma - \sum_{j=1}^{k}(1-j\beta)v_j^{t+1} \\
&= \sum_{j=1}^{k}(1-j\beta)v_j^t - \left(\beta\sum_{j=1}^{k-1} v_j^t + (1-k\beta)v_k^t + \sum_{j=1}^{k-1}(1-(j+1)\beta)v_j^t\right) \\
&= 0.
\end{aligned}$$

Therefore, if $v_0^{t_\ell} = 0$ and $||\boldsymbol{v}^{t_\ell}||_\infty \leq 1-\gamma$, by recursively applying the argument above, we see that $\boldsymbol{v}^t$ satisfy the conditions to apply Lemma 4.9 for all $t \geq t_\ell$ and so

$$\boldsymbol{v}^t = A_F^{t-t_\ell}\boldsymbol{v}^{t_\ell}.$$

Since A_F is a nonnegative matrix with row sums equal to 1, we see that A_F is a right (or row) stochastic matrix. Furthermore, it is immediate to verify that A_F is irreducible and since A_F has a positive diagonal entry, A_F is also primitive (and so aperiodic). Thus from standard stochastic matrix theory, A_F has an eigenvalue $\lambda_1 = 1$ and for all other eigenvalues λ_i, $|\lambda_i| < 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be labelled such that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_k|$. We find that

$$\boldsymbol{u} = (1-\beta, 1-2\beta, \ldots, 1-k\beta)$$

is a left eigenvector of A_F associated with λ_1, and so if $\boldsymbol{v}^{t_\ell}$ is a vector such that $\boldsymbol{u}^T\boldsymbol{v}^{t_\ell} = \gamma$, then for some limiting vector $\boldsymbol{v}$ and for all $r \in (|\lambda_2|, 1)$, there exists a constant C_r (depending on r) such that

$$||A_F^{t-t_\ell}\boldsymbol{v}^{t_\ell} - \boldsymbol{v}||_\infty < C_r r^{t-t_\ell}||\boldsymbol{v}^{t_\ell}||_\infty,$$

where $\boldsymbol{v}$ is the unique right eigenvector of A_F such that $\boldsymbol{u}^T\boldsymbol{v} = \gamma$. Since A_F has constant row sums, the eigenvector $\boldsymbol{v}$ must have constant entries. Thus by letting $\boldsymbol{v} = (v, \ldots, v)$, we may solve for v by using

$$\gamma = \boldsymbol{u}^T\boldsymbol{v} = \sum_{j=1}^{t}(1-j\beta)v,$$

and find that indeed $v = \nu$. Thus for $C_1' = C_r r^{-t_\ell}(1-\gamma)$, then

$$||\boldsymbol{v}^t - \boldsymbol{v}||_\infty < C_1' r^t.$$

The result for $U^{t+1} - U^t$ follows since $U^{t+1} - U^t = v_1^{t+1}$ and so

$$|(U^{t+1} - U^t) - \nu| \leq ||\boldsymbol{v}^{t+1} - \boldsymbol{v}||_\infty.$$

$\square$

We now examine the regime $\gamma \in (\gamma_c, 1)$. Let $\boldsymbol{e}_1 = (1, 0, 0, \dots, 0) \in \mathbb{R}^k$ and define

$$A_S = \begin{pmatrix} -1 & -1 & \cdots & -1 & \frac{-(1-k\beta)}{\beta} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Lemma 4.12. *Let $0 < \beta < 1$ and $g^t \in \mathcal{C}_\ell$. If*

$$\frac{1-\beta}{\beta}(1-\gamma) - v_0^t \leq \sum_{j=1}^{k-1} v_j^t + \frac{1-k\beta}{\beta} v_k^t \leq \frac{1-\gamma}{\beta}, \tag{4.30}$$

then

$$\boldsymbol{v}^{t+1} = A_S \boldsymbol{v}^t + \frac{1-\gamma}{\beta}\boldsymbol{e}_1, \qquad \text{and} \qquad U^{t+1} - U^t = 1-\gamma. \tag{4.31}$$

Proof. From the right inequality of (4.30),

$$(1-k\beta)(x_k^t - x_{k+1}^t) = (1-k\beta)v_k^t \leq 1-\gamma,$$

and recalling $x_k^t = U^t - \sum_{j=0}^{k-1} v_j^t$, from Proposition 4.6,

$$\sigma^t = U^t - v_0^t - \sum_{j=1}^{k-1} v_j^t - \frac{1-k\beta}{\beta} v_k^t + \frac{\gamma - 1}{\beta}. \tag{4.32}$$

From the left inequality of (4.30),

$$\frac{\beta(x_0^t - x_k^t) + (1-k\beta)(x_k^t - x_{k+1}^t)}{1-\beta} = \frac{\beta v_0^t + \beta \sum_{j=1}^{k-1} v_j^t + (1-k\beta)v_k^t}{1-\beta} \geq 1-\gamma,$$

and so from Proposition 4.6, $U^{t+1} - U^t = 1-\gamma$. Since $U^t - v_0^t = x_0^t - (x_0^t - x_1^t) = x_1^t$, using the second inequality of (4.30), we have

$$\sigma^t = x_1^t - \sum_{j=1}^{k-1} v_j^t - \frac{1-k\beta}{\beta} v_k^t + \frac{1-\gamma}{\beta} \geq x_1^t.$$

Thus applying (4.22),

$$\begin{aligned} x_0^{t+1} &= U^t + 1 - \gamma, \\ x_1^{t+1} &= \sigma^t, \\ x_2^{t+1} &= x_1^t = U^t - v_0^t, \\ x_j^{t+1} &= x_{j-1}^t \text{ for all } j \geq 3, \end{aligned}$$

and so by substituting (4.32),

$$v_1^{t+1} = x_1^t - x_2^t = -\sum_{j=1}^{k-1} v_j^t - \frac{1-k\beta}{\beta} v_k^t + \frac{1-\gamma}{\beta}$$

and $v_j^{t+1} = v_{j-1}^t$ for all $j \geq 2$. □

We wish to find conditions on $\boldsymbol{v}^{t_\ell}$ so that we may apply Lemma 4.12 repeatedly, and to find a limiting vector to such a process.

Lemma 4.13. *For $0 < \beta < 1$ such that $\beta^{-1} \notin \mathbb{N}$, if $\gamma \in (\gamma_c, 1)$, then there exists constants $\delta', C_1', C_2' >$ and $0 < r < 1$ such that if $||\boldsymbol{v}^{t_\ell} - \boldsymbol{v}||_\infty < \delta'$, then for all $t \geq t_\ell$, $U^{t+1} - U^t = 1 - \gamma$ and*

$$||\boldsymbol{v}^t - \boldsymbol{v}||_\infty < C_1' r^t, \qquad |v_0^t - \chi| < C_2' r^t.$$

Proof. We start by studying the matrix A_S. Note that A_S is a permuted polynomial companion matrix, and so by reading off the first row of A_S we see immediately that its characteristic polynomial is

$$\lambda^k + \lambda^{k-1} + \cdots + \lambda + \frac{1 - k\beta}{\beta}. \tag{4.33}$$

From the Eneström-Kakeya Theorem, if $a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial with real coefficients such that $0 \leq a_0 \leq \cdots \leq a_n$, then all of it's roots lie in the unit disk $|z| \leq 1$. Therefore, the eigenvalues λ of A_S satisfy $|\lambda| \leq 1$. Let $c = \frac{1-k\beta}{\beta}$, suppose $0 < c < 1$ (which holds when $\beta^{-1} \notin \mathbb{N}$), and assume there is an eigenvalue with $|\lambda| = 1$, i.e., $\lambda = e^{i\theta}$ for some θ. Then

$$\sum_{n=0}^{k} e^{in\theta} = \lambda^k + \lambda^{k-1} + \cdots + \lambda + c + (1 - c) = 1 - c,$$

and it can be worked out that

$$\sum_{n=0}^{k} e^{in\theta} = \frac{\sin\left(\frac{1}{2}(k+1)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} e^{\frac{ik\theta}{2}}.$$

Since $1-c$ is real, this means that in the exponent of e, $k\theta = 2K\pi$ for some integer K. Then $\sin\left(\frac{1}{2}(k+1)\theta\right) = \sin\left(K\pi + \frac{1}{2}\theta\right) = \sin\left(\frac{1}{2}\theta\right)$, which then means $\sum_{n=0}^{k} e^{in\theta} = \pm 1 \neq 1 - c$, and so $e^{i\theta}$ cannot be an eigenvalue of A_S. Therefore, $|\lambda| < 1$ for all eigenvalues of A_S, and so from Gelfand's formula,

$$\lim_{t \to \infty} ||A_S^t||_\infty^{1/t} = \rho,$$

where $0 < \rho < 1$ is the spectral radius of A_S. As a consequence, for any $\rho < r < 1$, there is a t_r such that for all $t \geq t_r$,

$$||A_S^t||_\infty^{1/t} < r \implies ||A_S^t||_\infty < r^t. \tag{4.34}$$

Since

$$\frac{1-\gamma}{\beta} - (k-1)(1-\gamma) - \frac{(1-k\beta)}{\beta}(1-\gamma) = 1 - \gamma,$$

we see that $\boldsymbol{v}$ is a fixed point of the operation, that is, $\boldsymbol{v} = A_S \boldsymbol{v} + \frac{\gamma-1}{\beta}\boldsymbol{e}_1$. For what follows, define

$$(\varepsilon_1^t, \ldots, \varepsilon_k^t) = \boldsymbol{\varepsilon}^t := \boldsymbol{v}^t - \boldsymbol{v}.$$

Since $\boldsymbol{v}$ is a fixed point, assuming $\boldsymbol{v}^{t+1} = A_S \boldsymbol{v}^t + \frac{1-\gamma}{\beta}\boldsymbol{e}_1$, then

$$\boldsymbol{\varepsilon}^{t+1} = \boldsymbol{v}^{t+1} - \boldsymbol{v} = A_S \boldsymbol{v}^t + \frac{1-\gamma}{\beta}\boldsymbol{e}_1 - \left(A_S \boldsymbol{v} + \frac{1-\gamma}{\beta}\boldsymbol{e}_1\right) = A_S \boldsymbol{\varepsilon}^t, \tag{4.35}$$

and after applying (4.35) recursively, $\boldsymbol{\varepsilon}^t = A_S^{t-t_\ell}\boldsymbol{\varepsilon}^{t_\ell}$. From (4.34),

$$S := \sup_t ||A_S^t||_\infty$$

exists, so if we find sufficient bounds on $||\boldsymbol{\varepsilon}^t||_\infty$ to apply (4.31), we can use S to find sufficient bounds on $||\boldsymbol{\varepsilon}^{t_\ell}||_\infty$ to apply (4.31) for all $t \geq t_\ell$. By rearranging (4.35), we can rewrite (4.30) as

$$0 \leq 1-\gamma-\sum_{j=1}^{k-1}\varepsilon_j^t - \frac{1-k\beta}{\beta}\varepsilon_k^t \leq v_0^t + 1-\gamma. \tag{4.36}$$

We can verify that the lower bound is satisfied if for all $j = 1, \ldots, k$,

$$\delta_1 := \frac{\beta(1-\gamma)}{1-\beta} \geq \varepsilon_j^t. \tag{4.37}$$

As for the upper bound, substituting

$$v_0^t = 1-\sum_{j=1}^{k}(1-j\beta)v_j^t = 1-\sum_{j=1}^{k}(1-j\beta)(1-\gamma+\varepsilon_j^t) \tag{4.38}$$

and subtracting $1-\gamma$ from both sides of the right inequality of (4.36) yields

$$\sum_{j=1}^{k-1}\varepsilon_j^t + \frac{1-k\beta}{\beta}\varepsilon_k^t \geq -v_0^t = -\chi + \sum_{j=1}^{k}(1-j\beta)\varepsilon_j^t,$$

where χ is defined in (4.1). The inequality above is satisfied if for all $j = 1, \ldots, k$,

$$\delta_2 := \frac{\chi}{\beta\sum_{j=1}^{k-1} j + \frac{(1-\beta)(1-k\beta)}{\beta}} \geq -\varepsilon_j^t, \tag{4.39}$$

which is positive only if $\gamma > \gamma_c$. Let

$$\delta' = \frac{\min\{\delta_1, \delta_2\}}{S}.$$

If $||\boldsymbol{\varepsilon}^{t_\ell}||_\infty \leq \delta'$, then for all $t \geq t_\ell$,

$$||\boldsymbol{\varepsilon}^t||_\infty \leq ||A_S^t||_\infty ||\boldsymbol{\varepsilon}^{t_\ell}||_\infty \leq \min\{\delta_1, \delta_2\},$$

both (4.37) and (4.39) are satisfied, and (4.31) can be applied. From (4.34), for all $t \geq t_r + t_\ell$,

$$||\boldsymbol{\varepsilon}^t||_\infty = ||A_S^{t-t_\ell}\boldsymbol{\varepsilon}^{t_\ell}||_\infty < r^{t-t_\ell}||\boldsymbol{\varepsilon}^{t_\ell}||_\infty \leq \delta' r^{t-t_\ell}.$$

Since $||\boldsymbol{\varepsilon}^t||_\infty$ is bounded for all $t \geq t_\ell$, there exists a constant C_1' such that

$$||\boldsymbol{v}^t - \boldsymbol{v}||_\infty = ||\boldsymbol{\varepsilon}^t||_\infty < C_1' r^t.$$

From (4.38), with $C_2' = C_1'\sum_{j=1}^{k}(1-j\beta)$,

$$|v_0^t - \chi| = \left|\chi - \sum_{j=1}^{k}(1-j\beta)\varepsilon_j^t - \chi\right| \leq \sum_{j=1}^{k}(1-j\beta)||\boldsymbol{\varepsilon}^t||_\infty \leq C_2' r^t.$$

$\square$

Remark 4.14. If $\beta \in \mathbb{Z}$, then $\frac{1-k\beta}{\beta} = 1$, and $\lambda^k + \cdots + \lambda + 1$ has as its roots all the $(k+1)'th$ roots of unity except 1, and so $|\lambda| = 1$ for all eigenvalues. In fact in this case, we may quickly verify that $A_S^{k+1} = I$, and so as $t \to \infty$, we see that $\boldsymbol{\varepsilon}^t$ will simply rotate through $\boldsymbol{\varepsilon}^{t_\ell}, \ldots, \boldsymbol{\varepsilon}^{t_\ell+k}$. Thus g^t does not have a convergence result in this case.

4.3 Proof of Theorem 1.2

With all the tools needed, we can prove the following slightly more precise version of Theorem 1.2:

Theorem 4.15. *For every $0 < \beta$ and $0 < \gamma < 1$ such that $\beta^{-1} \notin \mathbb{Z}$ whenever $\gamma_c < \gamma$, there exists a $\delta > 0$, a positive constant $r < 1$, and constants $C_1, C_2 > 0$ (depending only on β and γ) such that for all $g^0 \in \mathcal{C}$, if*

$$||g_+^0(\cdot + U^0) - G_+^0||_\infty < \delta,$$

then for all $t \geq 1$,

$$||g_+^t(\cdot + U^t) - G_+^0||_\infty < Cr^t$$

and

$$|(U^t - U^{t-1}) - \nu| < C_2 r^t.$$

We start by proving that G^0 as defined in (4.3) is a traveling wave solution to the operation $s \circ r$.

Theorem 4.16. *Let G^0 be as defined in* (4.3) *and ν be as defined in* (4.2). *Then*

$$s \circ r(G^0)(x) = G^0(x - \nu).$$

Proof. The result is a consequence of Proposition 4.6 and Lemma 4.5.

First look at $0 < \gamma < \gamma_c$. Then $\chi = 0$ and $\nu < 1 - \gamma$, and so from Lemma 4.6

$$\sigma^* = -(k-1)\nu + \frac{1-\gamma}{\beta} - \frac{1-k\beta}{\beta}\nu = \frac{1-\gamma}{\beta} - \frac{1-\beta}{\beta}\nu \geq 1 - \gamma$$

and

$$U^1 = \frac{\beta(k-1)\nu + (1-k\beta)\nu}{1-\beta} = \nu.$$

Thus by applying Lemma 4.5, we see that

$$\begin{aligned} y_0^1 &= U^1 = \nu = y_0^0 + \nu, \\ y_1^1 &= U^\ell = \nu = y_1^0 + \nu, \\ y_j^1 &= x_{j-1}^\ell = y_j^0 + \nu \text{ for all } j \geq 2, \end{aligned}$$

and so $G^1(x) = G^0(x - \nu)$ for all $x \in \mathbb{R}$.

Now we consider $\gamma_c < \gamma < 1$. Then $\nu = 1 - \gamma$ and from Proposition 4.6,

$$\sigma = -\chi - (k-1)\nu + \frac{1-\gamma}{\beta} - \frac{1-k\beta}{\beta}\nu = -\chi + \nu < 1 - \gamma$$

and

$$U^1 = 1 - \gamma < \nu + \frac{\beta}{1-\beta}\chi = \frac{\beta(\chi + (k-1)\nu) + (1-k\beta)\nu}{1-\beta}.$$

By applying Lemma 4.5, we see that

$$\begin{aligned} y_0^1 &= U^1 = 1 - \gamma = y_0^0 + \nu, \\ y_1^1 &= \sigma = -\chi + \nu = y_1^0 + \nu, \\ y_j^1 &= y_{j-1}^0 = x_j^\ell + \nu \text{ for all } j \geq 2, \end{aligned}$$

and so once again, $G^1(x) = G^0(x - \nu)$ for all $x \in \mathbb{R}$. □

We next prove that the operator $(s \circ r)_+$ is continuous around G^0_+ with respect to the L^∞-norm.

Lemma 4.17. *There exists a constant η such that for all $t \geq 0$,*

$$||g_+^{t+1}(\cdot + U^{t+1}) - G_+^0||_\infty \leq \eta ||g_+^t(\cdot + U^t) - G_+^0||_\infty.$$

Proof. Let $\xi = ||g_+^t(x - U^t) - G_+^0||_\infty$ and assume without loss of generality that $U^t = 0$. Then for all x,

$$\left| 0 \vee \left(\sup_{z \in \infty} \{g^t(z) - |x - z|\} \right) - 0 \vee \left(\sup_{z \in \mathbb{R}} \{G^0(z) - |x - z|\} \right) \right| \leq \xi, \tag{4.40}$$

and so $||r_+^t - (r(G^0))_+||_\infty \leq \xi$. It is then evident that for any σ and x,

$$\left| \left(r^t(x) + \beta(x - \sigma)_-\right)_+ - \left(r(G^0)(x) + \beta(x - \sigma)_-\right)_+ \right| \leq \xi,$$

and as a consequence, for $\sigma^* := \sigma^*(r(G_0))$, $|\sigma^t - \sigma^*| \leq \xi$. Altogether, for all $x \in \mathbb{R}$,

$$\begin{aligned}
&\left| g_+^{t+1}(x) - s \circ r(G^0)_+(x) \right| \\
&\qquad = \left| \left(r^t(x) + 0 \wedge (\beta x + \sigma^t)\right)_+ - \left(r(G^0)(x) + 0 \wedge (\beta x + \sigma^*)\right)_+ \right| \\
&\qquad \leq ||r_+^t(x) - (r(G^0)(x))_+||_\infty + |\sigma^t - \sigma^*| \\
&\qquad \leq 2\xi.
\end{aligned}$$

From (4.40), we have in particular that

$$|\omega^t - \omega(G^0)| = \left| \sup_{z \in (-\infty, 0)} \{g^t(z) + z\} - \sup_{z \in (-\infty, 0)} \{G^0(z) + z\} \right| \leq \xi,$$

and so

$$\left| (\omega^t + 1 - \gamma) \wedge \left(\frac{\omega^t + 1 - \gamma - \beta\sigma^t}{1 - \beta} \right) - (\omega(G^0) + 1 - \gamma) \wedge \left(\frac{\omega(G^0) + 1 - \gamma - \beta\sigma^*}{1 - \beta} \right) \right|$$

is bounded above by

$$\frac{|\omega^t - \omega(G^0)| + |\omega^t - \omega^\ell|}{1 - \beta} = \frac{2\xi}{1 - \beta}.$$

and so from Lemma 4.1 (c),

$$|U^{t+1} - \nu| \leq \frac{2\xi}{1 - \beta}.$$

Since G^0 is piecewise linear, it is clearly C-Lipschitz for some C. Thus for all x, by using Proposition 4.16,

$$\begin{aligned}
&|g_+^{t+1}(x + U^{t+1}) - G_+^0(x)| \\
&\qquad \leq |g_+^{t+1}(x) - G_+^0(x - \nu)| + |G_+^0(x - \nu) - G_+^0(x - U^{t+1})| \\
&\qquad\qquad \leq |g_+^{t+1}(x) - (s \circ r(G^0)(x))_+| + C|U^{t+1} - \nu| \leq 2\xi + \frac{2C\xi}{1 - \beta}.
\end{aligned}$$

Therefore,

$$||g_+^{t+1}(\cdot + U^{t+1}) - G_+^0||_\infty \leq \frac{2(1 - \beta + C)}{1 - \beta} ||g_+^t - 0 \vee G^0||_\infty.$$

∎

Proof of Theorem 4.15. We start with the case $\beta > 1$, in which case $1-\beta < 0$. Therefore, from Lemma 4.7 and (4.22), we have for all t that

$$x_1^t = U^0 \vee (U^t - 1), \tag{4.41}$$

otherwise $\sup_{x\in\mathbb{R}} g^t(x) < 1$. From (4.7), we have that $U^{t+1} - U^t \geq 1-\gamma$ for all t, and so for all t large enough, $U^t - U^0 > 1$, which from (4.41) implies $x_1^t = U^t - 1 = x_0^t - 1$. For all $t \geq 1$, since $g^t(x) + x$ is non-decreasing (by Lemma 4.1(a)), we have that $\omega^t = \sup_{z\in(-\infty,U^t)}\{g(z)+z\} = U^t$, and so from Lemma 4.1(c), $U^{t+1} - U^t = 1-\gamma$. Thus by recursively applying (4.22), we see that for any $j > 1$ and all t large enough, $x_j^t = x_{j-1}^t - (1-\gamma)$, completing the proof in this case.

Now we turn to $\beta < 1$. If $\beta < \beta_c$, let $\delta' = 1-\gamma-\nu$, while if $\beta_c < \beta < 1$ and $\beta^{-1} \notin \mathbb{Z}$, let δ' be the value from Lemma 4.13. Define

$$\delta := \frac{\beta\delta'}{4\eta^{t_\ell}}.$$

If $||g_+{}^0(\cdot + U^0) - G_+^0||_\infty < \delta$, then by recursively applying Lemma 4.17, we have

$$||g_+^{t_\ell}(\cdot + U^{t_\ell}) - G_+^0||_\infty < \frac{\beta\delta'}{4}. \tag{4.42}$$

Let $t \geq t_\ell$ and suppose without loss of generality that $U^t = 0$ (and so $x_t = 0$). Let $||\boldsymbol{v}^t - \boldsymbol{v}||_\infty = \lambda$, and suppose this value is reached for $|v_i^t - v_i| = \lambda$. Then

$$\lambda = |v_i^t - v_i| = |(x_i^t - x_{i+1}^t) - (x_i - x_{i+1})| \leq |x_i^t - x_i| + |x_{i+1}^t - x_{i+1}|.$$

Suppose that $|x_i^t - x_i| \geq \lambda/2$ (a similar argument holds if $|x_{i+1}^t - x_{i+1}| \geq \lambda/2$), and assume that $x_i < x_i^t$ (a symmetric argument holds for $x_i^t < x_i$). By Lemma 4.7, for all $x < x_i^t$,

$$g^t(x) \leq g^t(x_i^t) + (1-i\beta)(x_i^t - x)$$

and by definition of G^0, for all $x_i \leq x \leq x_i^t$,

$$G^0(x) \geq G^0(x_i^t) + (1-(i-1)\beta)(x_i^t - x).$$

If $g^t(x_i^t) - g(x_i^t) \leq \beta m/4$, then

$$\begin{aligned} G^0(x_i) - g^t(x_i) &\geq G^0(x_i^t) + (1-(i-1)\beta)\frac{\lambda}{2} - (g^t(x_i^t) + (1-i\beta)\frac{\lambda}{2}) \\ &= G^0(x_i^t) - g^t(x_i^t) + \beta\frac{\lambda}{2} \geq \beta\frac{\lambda}{4}. \end{aligned}$$

In either case, we have shown that

$$||g_+^t(\cdot + U^t) - G_+^0||_\infty \geq \frac{\beta||\boldsymbol{v}^t - \boldsymbol{v}||_\infty}{4}.$$

In particular, from (4.42),

$$||\boldsymbol{v}^{t_\ell} - \boldsymbol{v}||_\infty \leq \frac{4}{\beta}||g_+^{t_\ell}(\cdot + U^{t_\ell}) - G_+^0||_\infty < \delta',$$

thus we may apply Lemma 4.10 if $0 < \gamma < \gamma_c$ or Lemma 4.13 if $\gamma_c < \gamma < 1$. In both cases, for all $t \geq t_\ell$, $||\boldsymbol{v}^t - \boldsymbol{v}||_\infty < C_1' r^t$. Both lemmas imply $v_j^{t+1} = v_{j-1}^t$ for $t \geq t_\ell$ and $j = 2, \ldots, k$, but from (4.12), this extends to $j = k+1, \ldots$ (even if these values are zero). Let k_ℓ be the smallest value such that

$$\sum_{j=k+1}^{k+k_\ell} (j\beta - 1)\nu > 1,$$

and let $\varepsilon > 0$ be a value small enough such that

$$\sum_{j=k+1}^{k+k_\ell} (j\beta - 1)(\nu - \varepsilon) > 1.$$

If t is large enough such that $C_1' r^{t-k_\ell} < \varepsilon$, since $|v_j^t - \nu| < C_1' r^{t-k_\ell}$ for $j = 1, \ldots, k + k_\ell$, we have that

$$\sum_{j=0}^{k+k_\ell} (1 - j\beta) v_j^t < 0,$$

and since $g^t \in \mathcal{C}_\ell$ associated with $(x_j^t)_{j=0}^\infty$, $g^t(x) = -\infty$ for all $x < x_{t+k_\ell+1}^t$. Therefore we may write

$$0 \vee g^t(x) = 0 \vee \sum_{j=0}^{k+k_\ell} (1 - j\beta)(x \vee x_j^t - x \vee x_{j+1}^t).$$

If we assume that $U^t = 0$, then by setting C_2' as the value from Lemma 4.13 for $\gamma_c < \gamma < 1$ and $C_2' = 0$ for $0 < \gamma < \gamma_c$,

$$\begin{aligned}
|g_+^t(x + U^t) - G_+^0(x)| &\leq \left| \sum_{j=0}^{k+k_\ell} (1 - j\beta) \Big((x \vee x_j^t - x \vee x_{j+1}^t) - (x \vee x_j^\ell - x \vee x_{j+1}^\ell) \Big) \right| \\
&\leq \sum_{j=0}^{k+k_\ell} |1 - j\beta| |(x_j^t - x_{j+1}^t) - (x_j^\ell - x_{j+1}^\ell)| \\
&= |v_0^t - \chi| + \sum_{j=1}^{k+k_\ell} |1 - j\beta| |v_j^t - v_j| \\
&\leq C_2' r^t + \sum_{j=1}^{k+k_\ell} |1 - j\beta| C_1' r^{t-k_\ell}.
\end{aligned}$$

Letting $C_1 = C_2' + \sum_{j=1}^{k+k_\ell} |1 - j\beta| C_1' r^{-k_\ell}$, then for all t large enough, we have that

$$||g_+^t(\cdot + U^t) - G_+^0||_\infty < C_1 r^t.$$

The result for $|\nu^t - \nu| = |(U^t - U^{t-1}) - \nu|$ is a result of Lemma (4.10) or Lemma (4.13). $\square$

References

[1] Eric Bolthausen and Alain-Sol Sznitman, On ruelle's probability cascades and an abstract cavity model, *Communications in Mathematical Physics,* 197(2):247–276, 1998.

[2] Éric Brunet and Bernard Derrida, Shift in the velocity of a front due to a cut-off, *Phys. Rev. E*, 56:2597–2604, 1997.

[3] Éric Brunet, Bernard Derrida, Alfred H. Mueller, and Stéphane Munier, Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization, *Phys. Rev. E.* 76(4):041104, 2007.

[4] Aser Cortines and Bastien Mallein, A N-branching random walk with random selection, *ALEA Lat. Am. J. Probab. Math. Stat.*, 92:117–137, 2017.

[5] Emmanuel Schertzer and Alejandro H. Wences, Relative vs absolute fitness in a population genetics model. How stronger selection may promote genetic diversity, Preprint arXiv:2301.07762, 2023.

[6] Coalescents with multiple collisions, *Ann of Prob,* 27(4):1870–1902, 1999.

[7] Jim Pitman and Marc Yor, The two-parameter poisson-dirichlet distribution derived from a stable subordinator, *Ann. of Prob.*, 25(2):855–900, 1997.

[8] Coalescent processes obtained from supercritical galton-watson processes. *Stoch. Proc. Their Appl.*, 106(1):107–139, 2003.