Non-König Ideals, with Restrictions

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Studying the following definitons led to a series of findings derived from linear programming techniques using the algebraic computing tool Macaulay2.

Definition: A *hypergraph* is a pair $G = (V, E)$ where V is a set called the *verticies* of G, and E is a subset of 2^V, called the *edges* of G.

Notation: Let $V = \{x_1, ..., x_n\}$ be the set of verticies described above. Identify the polynomial ring $S = K[V] = K[x_1, ..., x_n]$ where K is a field. Thus each edge in E can be written as a squarefree monomial, $x_{i_1}...x_{i_s}$, which is an element of S.

Definition: The *edge ideal* of a hypergraph $G = (V, E)$ is

$$I(G) = (m : m \in E) \subset S$$

where m represents a squarefree monomial (corresponding to an edge). On the other hand, given a squarefree monomial ideal $I \subset S$, let

$$G(I) = (V, gens(I))$$

be the hypergraph associated with I, where $gens(I)$ is the unique set of minimal monomial generators of I.

Definition: An ideal is *tripartite* if all of its generators are of degree 3.

Definition: An ideal $I \subset S$ is *prime* if it satisfies the following two properties:

(1) If x_i and x_j are elements of S such that their product is in I, then either $x_i \in I$ or $x_j \in I$.

(2) I is not equivalent to the entire polynomial ring S.

Previous work has provided the result that every squarefree monomial ideal can be written as an intersection of prime squarefree monomial ideals. Then, clearly, each prime ideal in the intersection contains the original ideal.

Definition: The *height* of a squarefree monomial ideal I is the smallest number of variables in any prime containing I. Denote the height of I by $ht(I)$.

Definition: Two monomials are called *disjoint* if they have no common divisors in $V = \{x_1, ..., x_n\}$. Let $\beta(I)$ denote the cardinality of the largest subset of $gens(I)$ such that each pair of monomials are disjoint (where $I \subset S$ is an ideal generated by squarefree monomials).

Definition: An ideal is *König* if $ht(I) = \beta(I)$.

Using the algebraic compting tool Macaulay2, all squarefree monomial ideals were generated for a given number of generators and number of variables. Of the generated lists, all non-König ideals were found. All non-König ideals generated by 3 or 4 generators (of degree at most 3) in 5 variables were found. All tripartite non-König ideals generated by 3, 4, 5, or 7 generators in 5 variables were found.

As an example, the following is a list of all non-König ideals generated by 3 generators (of degree at most 3) in 5 varaibales (up to isomorphism).

$$(x_1x_2,\ x_1x_3,\ x_2x_3),\ (x_1x_4,\ x_2x_4,\ x_1x_2x_3),\ (x_3x_4,\ x_1x_2x_3,\ x_1x_2x_4),$$
$$(x_2x_3,\ x_1x_2x_5,\ x_1x_3x_4),\ (x_1x_2x_5,\ x_1x_3x_4,\ x_2x_3x_4)$$

The primary result from the study of the generated non-König ideals was the notion of the *Alexander dual* of the ideal - being or not being König - having significance on the conjecture of interest regarding *packed* ideals.

The research conducted in the Summer of 2016 is primarily recorded in the Macaulay2 code that will not be transcribed here, but is well-documented.