# Hermitian/skew-Hermitian splitting preconditioners for the indefinite Helmholtz equation

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# Indefinite Helmholtz equation

$$\left(-k^2-\nabla^2\right)u=f,\ u=0 \text{ on } \Omega_0,\ \frac{\partial u}{\partial n}-iku=0,\ \text{ on } \Gamma=\Omega-\Omega_0.$$

Equations of this type found in acoustics, elasticity, electromagnetism, geophysics, quantum.

**Why is this equation hard to solve using iterative methods?**

1. The operator is not positive definite.
2. The "pollution effect" requires resolution $hk \to 0$ sufficiently fast as $k \to \infty$.[a]

[a] There are ways to avoid the pollution effect but we don't discuss them here.

# Solving discrete Helmholtz

Some solver strategies:

- Sweeping preconditioners (Engquist and Ying, 2011) - direct solver on part of the domain with absorbing layers, iterate to match ingoing/outgoing waves across layers: see also source transfer domain decomposition, single-layer-potential, polarised traces, double sweep.
- Overlapping and nonoverlapping Schwarz methods. (Gong et al, 2021) obtain $k$ independent convergence rates, but patches must have width $O(1/k)$.

We propose an alternative that does not rely on direct solution on patches.

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# Preconditioned Krylov methods and multigrid

### Krylov methods for $Ax = b$

Expand the solution in Krylov basis $\{b, Ab, A^2b, A^3b, \ldots\}$. Only need operation $x \mapsto Ax$ (parallelisable).

### Preconditioned Krylov methods for $Ax = b$

Expand the solution in preconditioned Krylov basis $\{M^{-1}b, M^{-1}Ab, (M^{-1}A)^2b, (M^{-1}A)^3b, \ldots\}$. Only need operations $x \mapsto Ax$, and (parallelisable) solver for $Mz = y$.

### Multigrid for $Ax = b$ (or as preconditioner)

Solves on all scales by sweeping through hierarchy of nested grids.

# Multigrid for indefinite Helmholtz

## Shift preconditioning

- Precondition $(-k^2 - \nabla^2)$ with $(-k^2 - i\epsilon - \nabla^2)^{-1}$.
- Would then further approximate $(-k^2 - i\epsilon - \nabla^2)^{-1}$, with one multigrid iteration for a practical method.

## Gander, Graham and Spence (2015)

Need $\epsilon \lesssim k$ for $k$ independent p-Krylov convergence.

## Cocquet and Gander (2017)

Need $\epsilon \gtrsim k^2$ for $k$ independent multigrid convergence.

# The goal

## The goal

To bridge the gap between $\epsilon \lesssim k$ and $\epsilon \gtrsim k^2$.

## The solution

We will achieve this using inner iterations based around an $\epsilon \sim k^2$ problem.

# Finite element formulation

$$\left(-k^2-\nabla^2\right)u=f,\ u=0 \text{ on } \Omega_0,\ \frac{\partial u}{\partial n}-iku=0,\ \text{ on } \Gamma=\Omega-\Omega_0.$$

## Variational formulation

Find $u \in \mathring{H}^1(\Omega)$ such that

$$\langle v,-k^2u\rangle+\langle\nabla v,\nabla u\rangle+\langle\!\langle v,-iku\rangle\!\rangle_\Gamma=\langle v,f\rangle,\quad \forall v\in\mathring{H}^1(\Omega).$$

## Galerkin finite element approximation

Find $u \in Q_h \subset \mathring{H}^1(\Omega)$ such that

$$\langle v,-k^2u\rangle+\langle\nabla v,\nabla u\rangle+\langle\!\langle v,-iku\rangle\!\rangle_\Gamma=\langle v,f\rangle,\quad \forall v\in Q_h.$$

Convergence analysis due to Melenk (1995).

# Equivalent mixed formulation

## First order formulation

$-ik\sigma - \nabla u = 0, \quad -iku - \nabla\cdot\sigma = f/(-ik).$

Pick $V_h \subset L^2(\Omega)^d$ such that $u \in Q_h \implies \nabla u \in V_h$.

## Mixed finite element formulation

Find $(\sigma, u) \in V_h \times Q_h$ such that

$$\langle \tau, -ik\sigma - \nabla u\rangle = 0, \quad \forall \tau \in V_h$$
$$\langle v, -iku\rangle + \langle \nabla v\cdot\sigma\rangle + \langle\!\langle v, u\rangle\!\rangle_\Gamma = \langle v, f/(-ik)\rangle, \quad \forall v \in Q_h.$$

# The shifted mixed formulation

**Mixed finite element formulation**

Find $(\sigma, u) \in V_h \times Q_h$ such that

$$\langle \tau, \delta - ik\sigma - \nabla u \rangle = 0, \quad \forall \tau \in V_h$$
$$\langle v, \delta - ik)u \rangle + \langle \nabla v \cdot \sigma \rangle + \langle\langle v, u \rangle\rangle_\Gamma = \langle v, f/(-ik) \rangle, \quad \forall v \in Q_h.$$

First we will discuss HSS iteration for the shifted problem.

# HSS splitting for the shifted problem

**$\langle \cdot, \cdot \rangle_H$**

Define inner product $\langle \cdot, \cdot \rangle_H$ on $V_h \times Q_h$ by $\langle (\sigma, u), (\tau, v) \rangle_H = \delta \langle \sigma, \tau \rangle + \delta \langle u, v \rangle + \langle\langle u, v \rangle\rangle_\Gamma$.

Define $S : V_h \times Q_h \to V_h \times Q_h$ by

$$\langle (\tau, v), S(\sigma, u) \rangle_H = \langle \tau, -ik\sigma - \nabla u \rangle + \langle v, -iku \rangle - \langle \nabla v, \sigma \rangle.$$

$S$ is skew-Hermitian w.r.t. $\langle \cdot, \cdot \rangle_H$.

**Shifted problem for $U = (\sigma, u)$**

$(I + S)U = F, \quad \langle (\tau, v), F \rangle_H = \langle v, f/(-ik) \rangle.$

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# HSS iteration for the shifted problem

Solve $(I+S)U = F$ by iterating
$(1+\gamma)IU^{n+1/2} = (\gamma I - S)U^n + F, \quad (\gamma I + S)U^{n+1} = \gamma I U^{n+1/2} + F.$
Eliminate: $(\gamma I + S)U^{n+1} = \frac{\gamma-1}{\gamma+1}U^n + \frac{1+2\gamma}{1+\gamma}F.$

**Error equation for $\epsilon^n = U^n - U^*$**

$\epsilon^{n+1} = \frac{\gamma-1}{\gamma+1}(I+S/\gamma)^{-1}(I-S/\gamma)\epsilon^n$, so
$\|\epsilon^{n+1}\|_H \leq \frac{\gamma-1}{\gamma+1}\underbrace{\|(I+S/\gamma)^{-1}(I-S/\gamma)\|_H}_{=1}\|\epsilon^n\|_H.$

Convergence bound is independent of $k$ and $h$.

Preconditioned HS splitting: Bai, Golub, and Pan (2004)

## Choice of $\gamma$ for $\|\epsilon^{n+1}\|_H \leq \frac{\gamma-1}{\gamma+1}\|\epsilon^n\|_H$.

- Optimal choice $\gamma = 0$, but then we are back where we started.
- Suboptimal choice $\gamma = k$ (assume integer) leads to a shifted system suitable for multigrid.

$\gamma = k$:

$$
\begin{aligned}
&k(\delta - i)\langle \tau, \sigma^{n+1}\rangle - \langle \tau, \nabla u^{n+1}\rangle \\
&= \frac{k-1}{k+1}\left(k(\delta + i)\langle \tau, \sigma^n\rangle + \langle \tau, \nabla u^n\rangle\right) + \frac{2k}{k+1}F_\sigma[\tau], \quad \forall \tau \in V_h, \\
&k(\delta - i)\langle v, u^{n+1}\rangle + \langle \nabla v, \sigma^{n+1}\rangle + k\langle\langle v, u^{n+1}\rangle\rangle_\Gamma \\
&= \frac{k-1}{k+1}\left(k(\delta + i)\langle v, u^n\rangle - \langle \nabla v, \sigma^n\rangle + k\langle\langle v, u^n\rangle\rangle_\Gamma\right) + \frac{2k}{k+1}F_u[v], \quad \forall v \in Q_h
\end{aligned}
$$

### Choice of $\gamma$ for $\|\epsilon^{n+1}\|_H \leq \frac{\gamma-1}{\gamma+1}\|\epsilon^n\|_H$.

- Optimal choice $\gamma = 0$, but then we are back where we started.
- Suboptimal choice $\gamma = k$ leads to a shifted system suitable for multigrid.

$\gamma = k$: eliminating $u^{n+1}$

$$k^2(\delta-i)^2\langle v, u^{n+1}\rangle + \langle\nabla v, \nabla u^{n+1}\rangle + k^2(\delta-i)\langle\!\langle v, u^{n+1}\rangle\!\rangle_\Gamma = \hat{F}[v], \qquad \forall v \in Q_h.$$

### Good for multigrid

Imaginary shift: $-2i\delta k \mapsto -2i\delta k^2$.

# Error equation for $\gamma = k$

$$\|\epsilon^{n+1}\|_H \leq \frac{k-1}{1+k}\|\epsilon^n\|_H.$$

$k$ iterations reduces error by

$$\left(\frac{k-1}{k+1}\right)^{1/k} = \left(\frac{1-1/k}{1+1/k}\right)^{1/k} \leq (1-1/k)^{1/k} \leq e^{-1}.$$

# HSS as a preconditioner

Define $A_\delta : V_h \times Q_h \to V_h' \times Q_h'$:

$$A_\delta(U)[V] = \langle V, U + SU \rangle_H.$$

Define $\tilde{A}^{-1}_{\delta,m} : V_h' \times Q_h' \to V_h \times Q_h$: $\tilde{A}^{-1}_{\delta,n}(F) = U^n$, where

$$(kI + S)\, U^n = \frac{k-1}{k+1}\,(kI - S)\, U^{n-1} + \frac{2k}{1+k} f,$$
$$\langle V, f \rangle = F[V],\ \forall V, \quad U^0 = 0.$$

$$(kI + S)\,(U^n - f) = \frac{k-1}{k+1}\,(kI - S)\,(U^{n-1} - f)$$

so $\|\tilde{A}_{\delta,mk} A_\delta - I\|_H \le e^{-m}$.

# Preconditioner convergence

Precondition $\hat{A}_\delta = \mathcal{I}_h^* A_\delta \mathcal{I}_h$ with $\hat{\tilde{A}}^{-1}_{\delta,m} = \mathcal{I}_h^* \hat{\tilde{A}}^{-1}_{\delta,m} \mathcal{I}_h$.

## Bounds for preconditioned Krylov methods

Parameter independent convergence if $\rho(M^{-1}A - I) \leq c$ with parameter independent $c < 1$.

## Kirby (2010)

$\mathcal{I}_h^* = \mathcal{I}_h^{-1}$. Therefore, $\rho(\hat{\tilde{A}}_{\delta,m}\hat{A}_\delta - \hat{I}) = \rho(\tilde{A}^{-1}_{\delta,m} A_\delta - I)$.

## $k$-independent convergence for $mk - HSS$

$\rho(\hat{\tilde{A}}_{\delta,mk}\hat{A}_\delta - \hat{I}) = \rho(\tilde{A}^{-1}_{\delta,mk} A_\delta - I) \leq \|\tilde{A}_{\delta,mk} A_\delta - I\|_H \leq e^{-m}$.

# Shifted HSS preconditioning for $A_0$

For some $\delta > 0$, use $\tilde{A}_{\delta,mk}^{-1}$ as preconditioner for $A_0$.

$$\begin{aligned}
\tilde{A}_{\delta,mk}^{-1} A_0 - I &= \tilde{A}_{\delta,mk}^{-1} A_\delta - I + \tilde{A}_{\delta,mk}^{-1} A_0 - \tilde{A}_{\delta,mk}^{-1} A_\delta \\
&= (\tilde{A}_{\delta,mk}^{-1} A_\delta - I) + \tilde{A}_{\delta,mk}^{-1} A_\delta A_\delta^{-1} A_0 - \tilde{A}_{\delta,mk}^{-1} A_\delta \\
&= (\tilde{A}_{\delta,mk}^{-1} A_\delta - I) + \tilde{A}_{\delta,mk}^{-1} A_\delta \left( A_\delta^{-1} A_0 - I \right)
\end{aligned}$$

For any operator norm $\|A\| = \sup_{\|v\|>0} \|Av\|/\|v\|$,

$$\begin{aligned}
\|\tilde{A}_{\delta,mk}^{-1} A_0 - I\| &\leq \|\tilde{A}_{\delta,mk}^{-1} A_\delta - I\| + \|\tilde{A}_{\delta,mk}^{-1} A_\delta\| \|A_\delta^{-1} A_0 - I\| \\
&\leq \underbrace{\|\tilde{A}_{\delta,mk}^{-1} A_\delta - I\|}_{E_1} + (1 + \underbrace{\|\tilde{A}_{\delta,mk}^{-1} A_\delta - I\|}_{E_1}) \underbrace{\|A_\delta^{-1} A_0 - I\|}_{E_2}.
\end{aligned}$$

# Picking a norm

$$\|\tilde{A}_{\delta,mk}^{-1}A_0 - I\| \leq \underbrace{\|\tilde{A}_{\delta,mk}^{-1}A_\delta - I\|}_{E_1} + (1 + \underbrace{\|\tilde{A}_{\delta,mk}^{-1}A_\delta - I\|}_{E_1})\underbrace{\|A_\delta^{-1}A_0 - I\|}_{E_2}.$$

## But which norm?

- We have bound for $\|\tilde{A}_{\delta,mk}^{-1}A_\delta - I\|_H$.
- Gander, Graham and Spence (2015): $\|\hat{A}_\delta^{-1}\hat{A}_0 - \hat{I}\|_{l_2} \lesssim \delta$.

Solution here is to bound $\|A_\delta^{-1}A - I\|_H$.

**Theorem (CJC)**

$\|A_\delta^{-1}A_0 - I\| \lesssim \delta^{1/2}$.

Sketch of proof.
For $(\sigma_0, u_0) \in V_h \times Q_h$, define $(\sigma, u) = (A_\delta^{-1}A_0 - I)(\sigma_0, u_0)$, s.t.

$$\begin{aligned} \langle \tau, (\delta - ik)\sigma\rangle - \langle \tau, \nabla u\rangle &= -\delta\langle \tau, \sigma_0\rangle, \quad \forall \tau \in V_h, \\ \langle v, (\delta - ik)u\rangle + \langle\!\langle v, u\rangle\!\rangle_\Gamma + \langle \nabla v, \sigma\rangle &= -\delta\langle v, u_0\rangle, \quad \forall v \in Q_h. \end{aligned}$$

Eliminate $\sigma$,

$$\begin{aligned} &\langle v, (\delta - ik)^2 u\rangle + \langle\!\langle v, (\delta - ik)u\rangle\!\rangle_\Gamma \\ &\qquad + \langle \nabla v, \nabla u\rangle = -\delta\langle v, (\delta - ik)u_0\rangle + \delta\langle \nabla v, \sigma_0\rangle, \ \forall v \in Q_h. \end{aligned}$$

# Sketch of proof (ctd)

Main tool 1

**Gander et al (2015), Theorem 2.9**

Define $u \in \mathring{H}^1$ such that
$\langle v, -(\delta - ik)^2 u\rangle + \langle \nabla v, \nabla u\rangle + \langle\!\langle v, (i\delta + k)u\rangle\!\rangle_\Gamma =$
$\langle v, f\rangle + \langle\!\langle v, g\rangle\!\rangle_\Gamma$, $\forall v \in \mathring{H}^1$. Then, for $\delta$ independent of $k \geq k_0$,
$\|u\|^2_{1,k,\Omega} = |u|^2_{1,\Omega} + k^2\|u\|^2_{0,\Omega} \lesssim \left(\|f\|^2_{0,\Omega} + \|g\|^2_{0,\partial\Omega}\right).$

# Sketch of proof (ctd)

Main tool 2

**Gander et al (2015), Lemma 3.5**

Let $u_h \in Q_h$ solve
$\langle v, (\delta - ik)^2 u_h \rangle + \langle \nabla v, \nabla u \rangle$
$+\langle\langle v, (\delta - ik) u \rangle\rangle_\Gamma = \langle v, f \rangle + \langle\langle v, g \rangle\rangle_\Gamma, \quad \forall v \in Q_h.$
For "nice" $\Omega$, $\exists C_1, C_2 > 0$ (independent of $h$,$k$, and $\delta$) such that $\| u_h - u \|_{1,k,\Omega} \leq C_2 \inf_{v \in Q_h} \| u - v \|_{1,k,\Omega}$, whenever $hk\sqrt{k^2 - 2\delta k} \leq C_1$.

In particular, $\| u_h - u \|_{1,k,\Omega} \lesssim \| u \|_{1,k,\Omega}$.

# Sketch of proof (ctd)

**Technicality**

Our RHS: $-\delta\langle v, (\delta - ik)u_0\rangle + \delta\langle\nabla v, \sigma_0\rangle$
Gander et al (2015) RHS: $\langle v, f\rangle + \langle\langle v, g\rangle\rangle_\Gamma$.

Define $\phi \in Q_h$ such that
$k^2\langle\phi, v\rangle + k^2\langle\langle\phi, v\rangle\rangle_\Gamma + \langle\nabla\phi, \nabla v\rangle = \delta\langle\nabla v, \sigma_0\rangle, \quad \forall v \in Q_h$, so that
$\|\phi\|^2_{1,k,\Omega,\partial\Omega} = k^2\|\phi\|^2_{0,\Omega} + k^2\|\phi\|^2_{0,\partial\Omega} + |\phi|^2_{1,\Omega} \le \delta^2\|\sigma_0\|^2_{0,\Omega}$.
$\hat{u} = u - \phi$ has RHS
$-\delta\langle v, (\delta - ik)u_0\rangle + \left(k^2 - (\delta - ik)^2\right)\langle v, \phi\rangle + \left(k^2 - (\delta - ik)\right)\langle\langle v, \phi\rangle\rangle_\Gamma$.
Solve $H^1$ problem with this RHS, use two main tools, some triangle inequalities,
$\|u\|^2_{1,k,\Omega} \lesssim \delta^2 k^2(\|u\|^2_{0,\Omega} + \|\sigma\|^2_{0,\Omega})$.

# Sketch of proof (ctd)

Have bound for $\|u\|^2_{1,k,\Omega} = \|u\|^2_{1,k,\Omega} = |u|^2_{1,\Omega} + k^2\|u\|^2_{0,\Omega}$.
Want bound for $\|(\sigma, u)\|^2_H = \delta(\|u\|^2_{0,\Omega} + \|\sigma\|^2_{0,\Omega}) + \|u\|^2_{0,\Gamma}$.
Returning to $\sigma$-eliminated equation

$$\langle v, (\delta - ik)^2 u\rangle + \langle\langle v, (\delta - ik)u\rangle\rangle_\Gamma$$
$$+\langle \nabla v, \nabla u\rangle = -\delta\langle v, (\delta - ik)u_0\rangle + \delta\langle \nabla v, \sigma_0\rangle, \ \forall v \in Q_h.$$

Taking $v = u$ and negative imaginary parts,

$$k\|u\|^2_{0,\Gamma} \le 2\delta k\|u\|^2_{0,\Omega} + k\|u\|^2_{0,\Gamma} \lesssim k\|u\|^2_{0,\Omega} + \delta^2 k(\|u_0\|^2_{0,\Omega} + \|\sigma_0\|^2_{0,\Omega}).$$
$$\lesssim \delta^2 k(\|u_0\|^2_{0,\Omega} + \|\sigma_0\|^2_{0,\Omega}).$$

Returning to $\sigma$-equation gives

$$\|\sigma\|^2_{0,\Omega} \le \delta^2(\|\sigma_0\|^2_{0,\Omega} + \|u_0\|^2_{0,\Omega}).$$

# Sketch of proof (ctd)

$$
\begin{aligned}
\|(\sigma,u)\|_H^2 &= \delta\|\sigma\|_{0,\Omega}^2 + \delta\|u\|_{0,\Omega}^2 + \|u\|_{0,\Gamma}^2, \\
&\le \delta\|\sigma\|_{0,\Omega}^2 + \frac{\delta}{k^2}\|u\|_{1,k,\Omega}^2 + \|u\|_{0,\Gamma}^2, \\
&\le \delta^3\left(\|u_0\|_{0,\Omega}^2 + \|\sigma_0\|_{0,\Omega}^2\right) + \delta^3\left(\|u_0\|_{0,\Omega}^2 + \|\sigma_0\|_{0,\Omega}^2\right) \\
&\qquad + \delta^2\left(\|u_0\|_{0,\Omega}^2 + \|\sigma_0\|_{0,\Omega}^2\right) \\
&\lesssim \delta\|(\sigma_0,u_0)\|_H^2.
\end{aligned}
$$

$$
\begin{aligned}
\|A_\delta^{-1}A_0 - I\|_H &= \sup_{0\neq(\sigma_0,u_0)\in V_h\times Q_h} \frac{\|(A_\delta^{-1}A_0 - I)(\sigma_0,u_0)\|_H}{\|(\sigma_0,u_0)\|_H} \\
&= \frac{\|(\sigma,u)\|_H}{\|(\sigma_0,u_0)\|_H} \le C\delta^{1/2}.
\end{aligned}
$$

$\square$

# $k$ independent convergence.

$$\begin{aligned}\|\tilde{A}_\delta^{-1}A_0 - I\|_H \leq & \|\tilde{A}_\delta^{-1}A_\delta - I\|_H \\ & + \left(\|\tilde{A}_\delta^{-1}A_\delta - I\|_H + 1\right)\|A_\delta^{-1}A_0 - I\|_H, \\ \leq & c = e^{-m} + (e^{-m}+1)c\delta^{1/2},\end{aligned}$$

and $c < 1$ for $m$ sufficiently large and $\delta$ sufficiently small.

# Why is this useful?

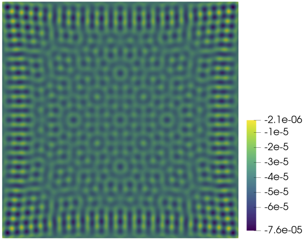
- Practical method: approximate $(kI + S)$ using multigrid(-preconditioned Krylov).
- As $k$ increases, $h$ needs to decrease faster.
- But, number of inner iterations is $\mathcal{O}(k)$ independent of $h$.
- And (for some range of $h$) multigrid exhibits "weak scaling" in parallel, achieving $\mathcal{O}(1)$ wallclock time per inner iteration as $h$ decreases (by adding more cores).
- Hence, given enough cores in the range of weak scaling, we have $\mathcal{O}(k)$ wallclock solve time.

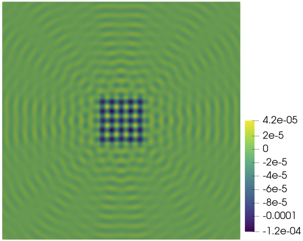
The amount of work done per inner iteration per core stays the same, as long as the number of cores can go up.

## Primal formulation for shifted problem

Find $u \in \mathring{H}^1(\Omega)$ such that
$\langle v, (\delta - ik)^2 u\rangle + \langle \nabla v, \nabla u\rangle + \langle\langle v, u\rangle\rangle_\Gamma = \langle v, f\rangle, \quad \forall v \in \mathring{H}^1(\Omega).$
Multiply by $i$, and define
$\langle u, v\rangle_H = 2\delta k\langle v, u\rangle + k\langle\langle v, u\rangle\rangle.$
Define $S_p$ such that
$\langle v, S_p u\rangle_H = i(\delta^2 - k^2)\langle v, u\rangle + i\langle \nabla v, \nabla u\rangle + \delta i\langle\langle v, u\rangle\rangle_\Gamma, \quad \forall u, v.$
Problem becomes $(I + S_p)u = if$.
HSS with $\gamma = k$ becomes (after dividing by $i$ again)

$$\left(-2\delta k^2 i + \delta^2 - k^2\right)\langle v, u^{n+1}\rangle + (-k^2 i + \delta)\langle\langle v, u^{n+1}\rangle\rangle_\Gamma + \langle \nabla v, \nabla u^{n+1}\rangle$$
$$= \frac{k-1}{k+1}\left((-2\delta k^2 i - \delta^2 + k^2)\langle v, u^n\rangle + (-k^2 i - \delta)\langle\langle v, u^n\rangle\rangle_\Gamma - \langle \nabla v, \nabla u^n\rangle\right)$$
$$+ \frac{2k}{k+1}\langle v, f\rangle, \quad \forall v \in Q_h.$$




- Experiments performed using Firedrake (`firedrake-project.org`)
- Parallel iterative solvers enabled by PETSc
- All experiments with $\delta = 1$ for $\tilde{A}_\delta^{-1}$.
- HHS iteration: direct solution, then algebraic multigrid

| $k$ | Mixed | | | Primal | | |
|---|---|---|---|---|---|---|
| | $m = k^{1/2}$ | $m = k$ | $m = k^{3/2}$ | $m = k^{1/2}$ | $m = k$ | $m = k^{3/2}$ |
| 10 | 18 | 9 | 7 | 17 | 9 | 7 |
| 20 | 29 | 10 | 7 | 30 | 10 | 8 |
| 40 | 52 | 10 | 7 | 53 | 10 | 7 |
| 80 | 104 | 10 | 6 | 103 | 10 | 7 |

Number of GMRES iterations to solve the uniform source case when $A_\delta$ is replaced with $m$ HSS iterations, and each HSS iteration is solved directly. (mesh refinement is $k^{3/2}$.)

| $k$ | Mixed | | | Primal | | |
|---|---|---|---|---|---|---|
| | $m = k^{1/2}$ | $m = k$ | $m = k^{3/2}$ | $m = k^{1/2}$ | $m = k$ | $m = k^{3/2}$ |
| 10 | 17 | 9 | 7 | 15 | 6 | 6 |
| 20 | 27 | 10 | 7 | 26 | 6 | 6 |
| 40 | 55 | 10 | 7 | 56 | 7 | 7 |
| 80 | 112 | 10 | 6 | 113 | 9 | 7 |

Number of GMRES iterations to solve the box source case when $A_\delta$ is replaced with $m$ HSS iterations, and each HSS iteration is solved directly.

| $k$ | Mixed | Primal |
|---|---|---|
| 10 | 9 | 9 |
| 20 | 10 | 10 |
| 40 | 10 | 10 |
| 80 | 10 | 10 |
| 160 | 10 | 10 |
| 200 | 10 | 10 |

Number of FGMRES iterations to solve the uniform source case when $A_\delta$ is replaced with $k$ HSS iterations, and each HSS iteration is replaced with $n \leq r = 15$ multigrid sweeps. (mesh refinement is $k^{3/2}$.)

| $k$ | Mixed | Primal |
|---|---|---|
| 10 | 9 | 9 |
| 20 | 10 | 10 |
| 40 | 10 | 10 |
| 80 | 10 | 11 |
| 160 | 11 | 10 |
| 200 | 11 | 10 |

Number of FGMRES iterations to solve the box source case when $A_\delta$ is replaced with $k$ HSS iterations, and each HSS iteration is replaced with 15 multigrid sweeps. (mesh refinement is $k^{3/2}$.)

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# Summary

- Precondition mixed or primal indefinite Helmholtz problem with $\mathcal{O}(k)$ iterations of pHSS applied to the $\delta$-shifted problem, with suboptimal HSS parameter $\gamma = k$.
- The inner problem is compatible with multigrid, and hence is weak scalable for on parallel computers.
- Proved $k$- and $h$-independent Krylov iteration counts but in a nonstandard norm $\|(\sigma, u)\|_H$.
- Theoretical results confirmed with numerical experiments.