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Hermitian/skew-Hermitian preconditioners for the indefinite Helmholtz equation

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Indefinite Helmholtz equation

$$\left(-k^2 - \nabla^2\right) u = f, \; u = 0 \text{ on } \Omega_0, \; \frac{\partial u}{\partial n} - iku = 0, \text{ on } \Gamma = \Omega - \Omega_0.$$

Equations of this type found in acoustics, elasticity, electromagnetism, geophysics, quantum.

Why is this equation hard to solve using iterative methods?

1. The operator is not positive definite.
2. The "pollution effect" requires resolution $hk \to 0$ sufficiently fast as $k \to \infty$.[a]

[a] There are ways to avoid the pollution effect but we don't discuss them here.

Solving discrete Helmholtz

Some solver strategies:

- Sweeping preconditioners (Engquist and Ying, 2011) - direct solver on part of the domain with absorbing layers, iterate to match ingoing/outgoing waves across layers: see also source transfer domain decomposition, single-layer-potential, polarised traces, double sweep.
- Overlapping and nonoverlapping Schwarz methods. (Gong et al, 2021) obtain k independent convergence rates, but patches must have width $O(1/k)$.

We propose an alternative that does not rely on direct solution on patches.

Preconditioned Krylov methods and multigrid

Krylov methods for $Ax = b$

Expand the solution in Krylov basis $\{b, Ab, A^2 b, A^3 b, \ldots\}$. Only need operation $x \mapsto Ax$ (parallelisable).

Preconditioned Krylov methods for $Ax = b$

Expand the solution in preconditioned Krylov basis $\{M^{-1}b, M^{-1}Ab, (M^{-1}A)^2 b, (M^{-1}A)^3 b, \ldots\}$. Only need operations $x \mapsto Ax$, and (parallelisable) solver for $Mz = y$.

Multigrid for $Ax = b$ (or as preconditioner)

Solves on all scales by sweeping through hierarchy of nested grids.

Multigrid for indefinite Helmholtz

Shift preconditioning

- Precondition $(-k^2 - \nabla^2)$ with $(-k^2 - i\epsilon - \nabla^2)^{-1}$.
- Would then further approximate $(-k^2 - i\epsilon - \nabla^2)^{-1}$, with one multigrid iteration for a practical method.

Gander, Graham and Spence (2015)

Need $\epsilon \lesssim k$ for k independent p-Krylov convergence.

Cocquet and Gander (2017)

Need $\epsilon \gtrsim k^2$ for k independent multigrid convergence.

The goal

The goal

To bridge the gap between $\epsilon \lesssim k$ and $\epsilon \gtrsim k^2$.

The solution

We will achieve this using inner iterations based around an $\epsilon \sim k^2$ problem.

Finite element formulation

$$\left(-k^2-\nabla^2\right)u=f,\ u=0 \text{ on } \Omega_0,\ \frac{\partial u}{\partial n}-iku=0,\ \text{ on } \Gamma=\Omega-\Omega_0.$$

Variational formulation

Find $u \in \mathring{H}^1(\Omega)$ such that
$\langle v,-k^2u\rangle+\langle\nabla v,\nabla u\rangle+\langle\!\langle v,-iku\rangle\!\rangle_\Gamma=\langle v,f\rangle,\quad \forall v\in\mathring{H}^1(\Omega).$

Galerkin finite element approximation

Find $u \in Q_h \subset \mathring{H}^1(\Omega)$ such that
$\langle v,-k^2u\rangle+\langle\nabla v,\nabla u\rangle+\langle\!\langle v,-iku\rangle\!\rangle_\Gamma=\langle v,f\rangle,\quad \forall v\in Q_h.$

Convergence analysis due to Melenk (1995).

Equivalent mixed formulation

First order formulation

$$-ik\sigma - \nabla u = 0, \quad -iku - \nabla \cdot \sigma = f/(-ik).$$

Pick $V_h \subset L^2(\Omega)^d$ such that $u \in Q_h \implies \nabla u \in V_h$.

Mixed finite element formulation

Find $(\sigma, u) \in V_h \times Q_h$ such that

$$\langle \tau, -ik\sigma - \nabla u \rangle = 0, \quad \forall \tau \in V_h$$
$$\langle v, -iku \rangle + \langle \nabla v \cdot \sigma \rangle + \langle\langle v, u \rangle\rangle_\Gamma = \langle v, f/(-ik) \rangle, \quad \forall v \in Q_h.$$

The shifted mixed formulation

Mixed finite element formulation

Find $(\sigma, u) \in V_h \times Q_h$ such that

$$\langle \tau, \delta - ik\sigma - \nabla u \rangle = 0, \quad \forall \tau \in V_h$$
$$\langle v, \delta - ik)u \rangle + \langle \nabla v \cdot \sigma \rangle + \langle\!\langle v, u \rangle\!\rangle_\Gamma = \langle v, f/(-ik) \rangle, \quad \forall v \in Q_h.$$

First we will discuss HSS iteration for the shifted problem.

HSS splitting for the shifted problem

$\langle\cdot,\cdot\rangle_H$

Define inner product $\langle\cdot,\cdot\rangle_H$ on $V_h \times Q_h$ by
$\langle(\sigma,u),(\tau,v)\rangle_H = \delta\langle\sigma,\tau\rangle + \delta\langle u,v\rangle + \langle\langle u,v\rangle\rangle_\Gamma$.

Define $S : V_h \times Q_h \to V_h \times Q_h$ by

$$\langle(\tau,v),S(\sigma,u)\rangle_H = \langle\tau,-ik\sigma-\nabla u\rangle + \langle v,-iku\rangle - \langle\nabla v,\sigma\rangle.$$

S is skew-Hermitian w.r.t. $\langle\cdot,\cdot\rangle_H$.

Shifted problem for $U=(\sigma,u)$

$(I+S)U = F, \quad \langle(\tau,v),F\rangle_H = \langle v, f/(-ik)\rangle$.

HSS iteration for the shifted problem

Solve $(I + S)U = F$ by iterating
$(1+\gamma)IU^{n+1/2} = (\gamma I - S)U^n + F, \quad (\gamma I + S)U^{n+1} = \gamma I U^{n+1/2} + F.$
Eliminate: $(\gamma I + S)U^{n+1} = \frac{\gamma-1}{\gamma+1}U^n + \frac{1+2\gamma}{1+\gamma}F.$

Error equation for $\epsilon^n = U^n - U^*$

$\epsilon^{n+1} = \frac{\gamma-1}{\gamma+1}(I + S/\gamma)^{-1}(I - S/\gamma)\epsilon^n$, so
$$\|\epsilon^{n+1}\|_H \leq \frac{\gamma-1}{\gamma+1}\underbrace{\|(I + S/\gamma)^{-1}(I - S/\gamma)\|_H}_{=1}\|\epsilon^n\|_H.$$

Convergence bound is independent of k and h.

Choice of γ for $\|\epsilon^{n+1}\|_H \leq \frac{\gamma-1}{\gamma+1}\|\epsilon^n\|_H$.

- Optimal choice $\gamma = 0$, but then we are back where we started.
- Suboptimal choice $\gamma = k$ (assume integer) leads to a shifted system suitable for multigrid.

$\gamma = k$:

$$k(\delta - i)\langle\tau, \sigma^{n+1}\rangle - \langle\tau, \nabla u^{n+1}\rangle$$
$$= \frac{k-1}{k+1}\left(k(\delta + i)\langle\tau, \sigma^n\rangle + \langle\tau, \nabla u^n\rangle\right) + \frac{2k}{k+1}F_\sigma[\tau], \quad \forall\tau \in V_h,$$
$$k(\delta - i)\langle v, u^{n+1}\rangle + \langle\nabla v, \sigma^{n+1}\rangle + k\langle\langle v, u^{n+1}\rangle\rangle_\Gamma$$
$$= \frac{k-1}{k+1}\left(k(\delta + i)\langle v, u^n\rangle - \langle\nabla v, \sigma^n\rangle + k\langle\langle v, u^n\rangle\rangle_\Gamma\right) + \frac{2k}{k+1}F_u[v], \quad \forall v \in Q_h$$

Choice of γ for $\|\epsilon^{n+1}\|_H \leq \frac{\gamma-1}{\gamma+1}\|\epsilon^n\|_H$.

- Optimal choice $\gamma = 0$, but then we are back where we started.
- Suboptimal choice $\gamma = k$ leads to a shifted system suitable for multigrid.

$\gamma = k$: eliminating u^{n+1}

$$k^2(\delta - i)^2\langle v, u^{n+1}\rangle + \langle \nabla v, \nabla u^{n+1}\rangle + k^2(\delta - i)\langle\langle v, u^{n+1}\rangle\rangle_\Gamma = \hat{F}[v], \qquad \forall v \in Q_h.$$

Good for multigrid

Imaginary shift: $-2i\delta k \mapsto -2i\delta k^2$.

Error equation for $\gamma = k$

$$\|\epsilon^{n+1}\|_H \leq \frac{k-1}{1+k}\|\epsilon^n\|_H.$$

k iterations reduces error by

$$\left(\frac{k-1}{k+1}\right)^{1/k} = \left(\frac{1-1/k}{1+1/k}\right)^{1/k} \leq (1-1/k)^{1/k} \leq e^{-1}.$$

HSS as a preconditioner

Define $A_\delta : V_h \times Q_h \to V_h' \times Q_h'$:

$$A_\delta(U)[V] = \langle V, U + SU \rangle_H.$$

Define $\tilde{A}_{\delta,m}^{-1} : V_h' \times Q_h' \to V_h \times Q_h$: $\tilde{A}_{\delta,n}^{-1}(F) = U^n$, where

$$\begin{aligned}(kI + S)\,U^n &= \frac{k-1}{k+1}(kI - S)\,U^{n-1} + \frac{2k}{1+k}f, \\ \langle V, f \rangle &= F[V],\ \forall V, \quad U^0 = 0.\end{aligned}$$

$$(kI + S)\,(U^n - f) = \frac{k-1}{k+1}(kI - S)\,(U^{n-1} - f)$$

so $\|\tilde{A}_{\delta,mk} A_\delta - I\|_H \le e^{-m}$.

Preconditioner convergence

Precondition $\hat{A}_\delta = \mathcal{I}_h^* A_\delta \mathcal{I}_h$ with $\hat{\hat{A}}_{\delta,m}^{-1} = \mathcal{I}_h^* \hat{\hat{A}}_{\delta,m}^{-1} \mathcal{I}_h$.

Bounds for preconditioned Krylov methods

Parameter independent convergence if $\rho(M^{-1}A - I) \leq c$ with parameter independent $c < 1$.

Kirby (2010)

$\mathcal{I}_h^* = \mathcal{I}_h^{-1}$. Therefore, $\rho(\hat{\hat{A}}_{\delta,m}\hat{A}_\delta - \hat{I}) = \rho(\tilde{A}_{\delta,m}^{-1} A_\delta - I)$.

k-independent convergence for $mk - HSS$

$\rho(\hat{\hat{A}}_{\delta,mk}\hat{A}_\delta - \hat{I}) = \rho(\tilde{A}_{\delta,mk}^{-1} A_\delta - I) \leq \|\tilde{A}_{\delta,mk} A_\delta - I\|_H \leq e^{-m}$.

Shifted HSS preconditioning for A_0

For some $\delta > 0$, use $\tilde{A}_{\delta,mk}^{-1}$ as preconditioner for A_0.

$$\begin{aligned}
\tilde{A}_{\delta,mk}^{-1} A_0 - I &= \tilde{A}_{\delta,mk}^{-1} A_\delta - I + \tilde{A}_{\delta,mk}^{-1} A_0 - \tilde{A}_{\delta,mk}^{-1} A_\delta \\
&= (\tilde{A}_{\delta,mk}^{-1} A_\delta - I) + \tilde{A}_{\delta,mk}^{-1} A_\delta A_\delta^{-1} A_0 - \tilde{A}_{\delta,mk}^{-1} A_\delta \\
&= (\tilde{A}_{\delta,mk}^{-1} A_\delta - I) + \tilde{A}_{\delta,mk}^{-1} A_\delta \left(A_\delta^{-1} A_0 - I \right)
\end{aligned}$$

For any operator norm $\|A\| = \sup_{\|v\|>0} \|Av\|/\|v\|$,

$$\begin{aligned}
\left\|\tilde{A}_{\delta,mk}^{-1} A_0 - I\right\| &\le \left\|\tilde{A}_{\delta,mk}^{-1} A_\delta - I\right\| + \left\|\tilde{A}_{\delta,mk}^{-1} A_\delta\right\| \left\|A_\delta^{-1} A_0 - I\right\| \\
&\le \underbrace{\left\|\tilde{A}_{\delta,mk}^{-1} A_\delta - I\right\|}_{E_1} + \left(1 + \underbrace{\left\|\tilde{A}_{\delta,mk}^{-1} A_\delta - I\right\|}_{E_1}\right) \underbrace{\left\|A_\delta^{-1} A_0 - I\right\|}_{E_2}.
\end{aligned}$$

Picking a norm

$$\|\tilde{A}^{-1}_{\delta,mk}A_0 - I\| \leq \underbrace{\|\tilde{A}^{-1}_{\delta,mk}A_\delta - I\|}_{E_1} + (1 + \underbrace{\|\tilde{A}^{-1}_{\delta,mk}A_\delta - I\|}_{E_1})\underbrace{\|A_\delta^{-1}A_0 - I\|}_{E_2}.$$

But which norm?

- We have bound for $\|\tilde{A}^{-1}_{\delta,mk}A_\delta - I\|_H$.
- Gander, Graham and Spence (2015): $\|\hat{A}_\delta^{-1}\hat{A}_0 - \hat{I}\|_{l_2} \lesssim \delta$.

Solution here is to bound $\|A_\delta^{-1}A - I\|_H$.

Theorem (CJC)

$\|A_\delta^{-1}A_0 - I\| \lesssim \delta^{1/2}$.

Sketch of proof.

For $(\sigma_0, u_0) \in V_h \times Q_h$, define $(\sigma, u) = (A_\delta^{-1}A_0 - I)(\sigma_0, u_0)$, s.t.

$$\begin{aligned} \langle \tau, (\delta - ik)\sigma\rangle - \langle \tau, \nabla u\rangle &= -\delta\langle \tau, \sigma_0\rangle, \quad \forall \tau \in V_h, \\ \langle v, (\delta - ik)u\rangle + \langle\!\langle v, u\rangle\!\rangle_\Gamma + \langle \nabla v, \sigma\rangle &= -\delta\langle v, u_0\rangle, \quad \forall v \in Q_h. \end{aligned}$$

Eliminate σ,

$$\begin{aligned} &\langle v, (\delta - ik)^2 u\rangle + \langle\!\langle v, (\delta - ik)u\rangle\!\rangle_\Gamma \\ &\qquad + \langle \nabla v, \nabla u\rangle = -\delta\langle v, (\delta - ik)u_0\rangle + \delta\langle \nabla v, \sigma_0\rangle, \ \forall v \in Q_h. \end{aligned}$$

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Sketch of proof (ctd)

Main tool 1

Gander et al (2015), Theorem 2.9

Define $u \in \mathring{H}^1$ such that
$\langle v, -(\delta - ik)^2 u\rangle + \langle \nabla v, \nabla u\rangle + \langle\!\langle v, (i\delta + k)u\rangle\!\rangle_\Gamma =$
$\langle v, f\rangle + \langle\!\langle v, g\rangle\!\rangle_\Gamma$, $\forall v \in \mathring{H}^1$. Then, for δ independent of $k \geq k_0$,
$\|u\|^2_{1,k,\Omega} = |u|^2_{1,\Omega} + k^2\|u\|^2_{0,\Omega} \lesssim \left(\|f\|^2_{0,\Omega} + \|g\|^2_{0,\partial\Omega}\right)$.

Sketch of proof (ctd)

Main tool 2

Gander et al (2015), Lemma 3.5

Let $u_h \in Q_h$ solve
$\langle v, (\delta - ik)^2 u_h \rangle + \langle \nabla v, \nabla u \rangle$
$+\langle\langle v, (\delta - ik) u \rangle\rangle_\Gamma = \langle v, f \rangle + \langle\langle v, g \rangle\rangle_\Gamma, \quad \forall v \in Q_h.$
For "nice" Ω, $\exists C_1, C_2 > 0$ (independent of h, k, and δ) such that $\|u_h - u\|_{1,k,\Omega} \leq C_2 \inf_{v \in Q_h} \|u - v\|_{1,k,\Omega}$, whenever $hk\sqrt{k^2 - 2\delta k} \leq C_1$.

In particular, $\|u_h - u\|_{1,k,\Omega} \lesssim \|u\|_{1,k,\Omega}$.

Sketch of proof (ctd)

Technicality

Our RHS: $-\delta\langle v, (\delta - ik)u_0\rangle + \delta\langle \nabla v, \sigma_0\rangle$
Gander et al (2015) RHS: $\langle v, f\rangle + \langle\langle v, g\rangle\rangle_\Gamma$.

Define $\phi \in Q_h$ such that
$k^2\langle \phi, v\rangle + k^2\langle\langle \phi, v\rangle\rangle_\Gamma + \langle \nabla\phi, \nabla v\rangle = \delta\langle \nabla v, \sigma_0\rangle, \quad \forall v \in Q_h$, so that
$\|\phi\|^2_{1,k,\Omega,\partial\Omega} = k^2\|\phi\|^2_{0,\Omega} + k^2\|\phi\|^2_{0,\partial\Omega} + |\phi|^2_{1,\Omega} \le \delta^2\|\sigma_0\|^2_{0,\Omega}$.
$\hat{u} = u - \phi$ has RHS
$-\delta\langle v, (\delta - ik)u_0\rangle + \left(k^2 - (\delta - ik)^2\right)\langle v, \phi\rangle + \left(k^2 - (\delta - ik)\right)\langle\langle v, \phi\rangle\rangle_\Gamma$.
Solve H^1 problem with this RHS, use two main tools, some triangle inequalities,
$\|u\|^2_{1,k,\Omega} \lesssim \delta^2 k^2(\|u\|^2_{0,\Omega} + \|\sigma\|^2_{0,\Omega})$.

Sketch of proof (ctd)

Have bound for $\|u\|^2_{1,k,\Omega} = \|u\|^2_{1,k,\Omega} = |u|^2_{1,\Omega} + k^2\|u\|^2_{0,\Omega}$.
Want bound for $\|(\sigma, u)\|^2_H = \delta(\|u\|^2_{0,\Omega} + \|\sigma\|^2_{0,\Omega}) + \|u\|^2_{0,\Gamma}$.
Returning to σ-eliminated equation

$$\langle v, (\delta - ik)^2 u\rangle + \langle\!\langle v, (\delta - ik)u\rangle\!\rangle_\Gamma$$
$$+\langle \nabla v, \nabla u\rangle = -\delta\langle v, (\delta - ik)u_0\rangle + \delta\langle\nabla v, \sigma_0\rangle, \ \forall v \in Q_h.$$

Taking $v = u$ and negative imaginary parts,

$$k\|u\|^2_{0,\Gamma} \leq 2\delta k\|u\|^2_{0,\Omega} + k\|u\|^2_{0,\Gamma} \lesssim k\|u\|^2_{0,\Omega} + \delta^2 k(\|u_0\|^2_{0,\Omega} + \|\sigma_0\|^2_{0,\Omega}).$$
$$\lesssim \delta^2 k(\|u_0\|^2_{0,\Omega} + \|\sigma_0\|^2_{0,\Omega}).$$

Returning to σ-equation gives

$$\|\sigma\|^2_{0,\Omega} \leq \delta^2(\|\sigma_0\|^2_{0,\Omega} + \|u_0\|^2_{0,\Omega}).$$

Sketch of proof (ctd)

$$
\begin{aligned}
\|(\sigma,u)\|_H^2 &= \delta\|\sigma\|_{0,\Omega}^2 + \delta\|u\|_{0,\Omega}^2 + \|u\|_{0,\Gamma}^2, \\
&\le \delta\|\sigma\|_{0,\Omega}^2 + \frac{\delta}{k^2}\|u\|_{1,k,\Omega}^2 + \|u\|_{0,\Gamma}^2, \\
&\le \delta^3\left(\|u_0\|_{0,\Omega}^2 + \|\sigma_0\|_{0,\Omega}^2\right) + \delta^3\left(\|u_0\|_{0,\Omega}^2 + \|\sigma_0\|_{0,\Omega}^2\right) \\
&\qquad + \delta^2\left(\|u_0\|_{0,\Omega}^2 + \|\sigma_0\|_{0,\Omega}^2\right) \\
&\lesssim \delta\|(\sigma_0,u_0)\|_H^2.
\end{aligned}
$$

$$
\begin{aligned}
\|A_\delta^{-1}A_0 - I\|_H &= \sup_{0\neq(\sigma_0,u_0)\in V_h\times Q_h} \frac{\|(A_\delta^{-1}A_0 - I)(\sigma_0,u_0)\|_H}{\|(\sigma_0,u_0)\|_H} \\
&= \frac{\|(\sigma,u)\|_H}{\|(\sigma_0,u_0)\|_H} \le C\delta^{1/2}.
\end{aligned}
$$

$\square$

k independent convergence.

$$
\begin{aligned}
\|\tilde{A}_\delta^{-1}A_0 - I\|_H \leq & \|\tilde{A}_\delta^{-1}A_\delta - I\|_H \\
& + \left(\|\tilde{A}_\delta^{-1}A_\delta - I\|_H + 1\right)\|A_\delta^{-1}A_0 - I\|_H, \\
\leq & c = e^{-m} + (e^{-m} + 1)c\delta^{1/2},
\end{aligned}
$$

and $c < 1$ for m sufficiently large and δ sufficiently small.

Why is this useful?

- Practical method: approximate $(kI + S)$ using multigrid(-preconditioned Krylov).
- As k increases, h needs to decrease faster.
- But, number of inner iterations is $\mathcal{O}(k)$ independent of h.
- And (for some range of h) multigrid exhibits "weak scaling" in parallel, achieving $\mathcal{O}(1)$ wallclock time per inner iteration as h decreases (by adding more cores).
- Hence, given enough cores in the range of weak scaling, we have $\mathcal{O}(k)$ wallclock solve time.

The amount of work done per inner iteration per core stays the same, as long as the number of cores can go up.

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Primal formulation for shifted problem

Find $u \in \mathring{H}^1(\Omega)$ such that

$$\langle v, (\delta - ik)^2 u\rangle + \langle \nabla v, \nabla u\rangle + \langle\!\langle v, u\rangle\!\rangle_\Gamma = \langle v, f\rangle, \quad \forall v \in \mathring{H}^1(\Omega).$$

Multiply by i, and define

$$\langle u, v\rangle_H = 2\delta k\langle v, u\rangle + k\langle\!\langle v, u\rangle\!\rangle.$$

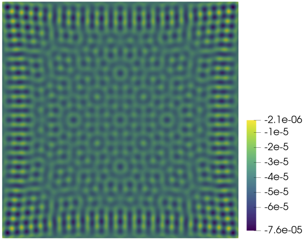
Define S_p such that

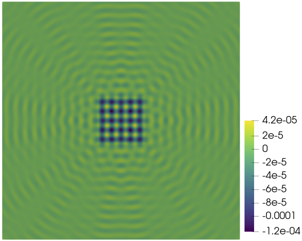
$$\langle v, S_p u\rangle_H = i(\delta^2 - k^2)\langle v, u\rangle + i\langle \nabla v, \nabla u\rangle + \delta i\langle\!\langle v, u\rangle\!\rangle_\Gamma, \quad \forall u, v.$$

Problem becomes $(I + S_p)u = if$.

HSS with $\gamma = k$ becomes (after dividing by i again)

$$\begin{aligned}\left(-2\delta k^2 i + \delta^2 - k^2\right)\langle v, u^{n+1}\rangle + (-k^2 i + \delta)\langle\!\langle v, u^{n+1}\rangle\!\rangle_\Gamma + \langle \nabla v, \nabla u^{n+1}\rangle \\ = \frac{k-1}{k+1}\left((-2\delta k^2 i - \delta^2 + k^2)\langle v, u^n\rangle + (-k^2 i - \delta)\langle\!\langle v, u^n\rangle\!\rangle_\Gamma - \langle \nabla v, \nabla u^n\rangle\right) \\ + \frac{2k}{k+1}\langle v, f\rangle, \quad \forall v \in Q_h.\end{aligned}$$





- Experiments performed using Firedrake (`firedrake-project.org`)
- Parallel iterative solvers enabled by PETSc
- All experiments with $\delta = 1$ for $\tilde{A}_\delta^{-1}$.
- HHS iteration: direct solution, then algebraic multigrid

k	Mixed			Primal		
	$m = k^{1/2}$	$m = k$	$m = k^{3/2}$	$m = k^{1/2}$	$m = k$	$m = k^{3/2}$
10	18	9	7	17	9	7
20	29	10	7	30	10	8
40	52	10	7	53	10	7
80	104	10	6	103	10	7

Number of GMRES iterations to solve the uniform source case when A_δ is replaced with m HSS iterations, and each HSS iteration is solved directly. (mesh refinement is $k^{3/2}$.)

k	Mixed			Primal		
	$m = k^{1/2}$	$m = k$	$m = k^{3/2}$	$m = k^{1/2}$	$m = k$	$m = k^{3/2}$
10	17	9	7	15	6	6
20	27	10	7	26	6	6
40	55	10	7	56	7	7
80	112	10	6	113	9	7

Number of GMRES iterations to solve the box source case when A_δ is replaced with m HSS iterations, and each HSS iteration is solved directly.

k	Mixed	Primal
10	9	9
20	10	10
40	10	10
80	10	10
160	10	10
200	10	10

Number of FGMRES iterations to solve the uniform source case when A_δ is replaced with k HSS iterations, and each HSS iteration is replaced with $n \leq r = 15$ multigrid sweeps. (mesh refinement is $k^{3/2}$.)

k	Mixed	Primal
10	9	9
20	10	10
40	10	10
80	10	11
160	11	10
200	11	10

Number of FGMRES iterations to solve the box source case when A_δ is replaced with k HSS iterations, and each HSS iteration is replaced with 15 multigrid sweeps. (mesh refinement is $k^{3/2}$.)

Summary

- Precondition mixed or primal indefinite Helmholtz problem with $\mathcal{O}(k)$ iterations of pHSS applied to the δ-shifted problem, with suboptimal HSS parameter $\gamma = k$.
- The inner problem is compatible with multigrid, and hence is weak scalable for on parallel computers.
- Proved k- and h-independent Krylov iteration counts but in a nonstandard norm $\|(\sigma, u)\|_H$.
- Theoretical results confirmed with numerical experiments.