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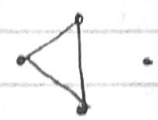
1/26/18

Graph complement:

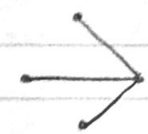
If G is a graph on V, then the complement of G, denoted $\overline{G}$, is a graph on V such that

$$uv \in E(\overline{G}) \iff uv \notin E(G).$$

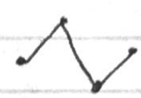
G



$\overline{G}$



H



$\overline{H}$

A disconnected graph is not connected.

Theorem: The complement of a disconnected graph is connected.

Proof: Let G be a disconnected graph. Let $u, v \in V(G)$. We'll find a path in $\overline{G}$ between u and v.

If $uv \notin E(G)$ then $uv \in E(G)$.

If $uv \in E(G)$, there must be a vertex w in another connected component from u, v. In particular, $uw, vw \notin E(G)$. Hence $uw, vw \in E(\overline{G})$. Thus uwv is a path from u to v in $\overline{G}$. ■

How few edges can a connected graph have?

P_{n-1} has n vertices and $n-1$ edges.

Theorem: A connected graph on n edges has at least $n-1$ edges.

<u>Averaging Principle</u>: A set of numbers contains a number at least ($\geq$) the average and a number at most ($\leq$) the average.

<u>Proof</u>: By induction on n. Trivial for $n=1$.

True for $n=2$.

Suppose we've shown that all connected graphs on $k<n$ vertices have at least $k-1$ edges. Let G be a graph on n vertices with fewer than $n-1$ edges.

$$d(G) = \frac{1}{n}\sum_{v\in V} d(v) = \frac{2|E(G)|}{n} < \frac{2(n-1)}{n} < 2$$

By the averaging principle there is a vertex with degree $d(v) \leq d(G) < 2$, hence 0 or 1.

If $d(v)=0$ then v is isolated and G is disconnected.

If $d(v)=1$, then remove v and its incident edge to get graph H with $n-1$ vertices and fewer than $n-2$ edges. By induction, H is disconnected, so G is disconnected (any path through v must pass through its neighbor in H). ■

How many edges can a disconnected graph have?

<u>**Theorem**</u>: A disconnected graph on n vertices has at most $\binom{n-1}{2}$ edges.

<u>Proof</u>: Let G be a disconnected graph.

Then $\bar{G}$ is connected, so $|E(\bar{G})| \geq n-1$.

Since $|E(G)| + |E(\bar{G})| = \binom{n}{2}$,

so $|E(G)| = \binom{n}{2} - |E(\bar{G})| \leq \binom{n}{2} - (n-1) = \binom{n-1}{2}$. ■

$$\binom{n}{2} - (n-1) = \frac{n(n-1)}{2} - n-1 = \frac{n(n-1)-2(n-1)}{2} = \frac{(n-2)(n-1)}{2} = \binom{n-1}{2}.$$

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②

$\delta(G)$ is minimum degree

If G is a disconnected graph on n vertices, how large can $\delta(G)$ be?

Theorem: If G is disconnected, then $\delta(G) \leq \frac{n-1}{2}$.

Pigeonhole principle: If a set of more ($>$) than kn objects is partitioned into k classes, then some class has more than n vertices.

Proof: Let G be a graph with $\delta(G) > \frac{n-1}{2}$.
Let $u, v \in V(G)$.
Case 1: $uv \in E(G)$. Then there is a path between them.
Case 2: $uv \notin E(G)$. Then $|N(u)| + |N(v)| > n-1$.
But $|V - \{u,v\}| = n-2$, so there must be a vertex $w \in N(u) \cap N(v)$, so uwv is a path. ■

Take an end vertex off of a tree and you get another tree!

How many edges does a tree have?

Theorem: A tree on n vertices has $n-1$ edges.

Proof: (by induction on n)
True for $n=1$ and $n=2$. Suppose we've shown all trees on $k<n$ vertices have $k-1$ edges.
Let T be a tree on n vertices and let P be a maximum length path in T connecting vertices u and v.
Claim: u and v are end vertices in T.
If u has a second edge, then this will either extend the path or create a cycle. $\rightarrow\leftarrow$
Remove u from T to get T', a tree on $n-1$ vertices. By induction, T' has $n-2$ edges, so T has $n-1$ edges. ■

Theorem: If G is a connected graph on n vertices with $n-1$ edges, then G is a tree.

Proof: Suppose G has a cycle. We can remove an edge from the cycle, and G will still be connected.
At least $n-1$ edges remain.
Hence, G has at least n edges. ■

Corollary: If we add an edge to a tree, we create a cycle.

Definition: A spanning subgraph is a subgraph that has all of the vertices.
A spanning tree is a spanning subgraph that is a tree.

Theorem: A maximal acyclic subgraph of a connected graph is a spanning tree. In particular, every connected graph has at least one spanning tree.

Proof: (induction on e number of edges) True for $e=1$.
Suppose we've shown it for all graphs with fewer than e edges. Let $uv \in E(G)$. Let H be the graph obtained by removing uv, and if $d(u)=1$, remove u as well.
If H is connected, then all maximal acyclic subgraphs are spanning trees. Otherwise, we have components H_1, H_2 s.t. all maximal acyclic subgraphs of H_1, H_2 are spanning trees.
Case 1: u is an end vertex. Then, any maximal acyclic subgraph must include uv. ✓
Case 2: Removing uv disconnects the graph.
Then, any maximal acyclic subgraph must include uv.
$|E| = (|V(H_1)|-1)+(V(H_2)-1)+1 = |V(G)|-1$. ✓
Case 3: Removing uv does not disconnect the graph. Any maximal acyclic subgraph that does not include uv is a spanning tree by induction.
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■