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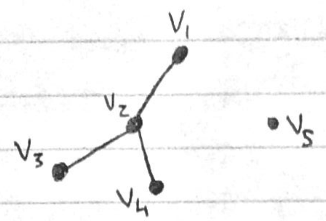
$[n] = [1, n]$
Let G be a graph on $[n]$.

Graph: $G = (V, E)$ where $E \subseteq \binom{V}{2}$



Incidence: $(v, e) \in V \times E$, $v \in e$
If (v, e) is an incidence, then we say v is incidence to e, and vice versa.

Degree: $d(v) = d_G(v)$
Number of edges incident to v

Adjacency: v is adjacent to u if they share an edge

Neighborhood: $N(v) = N_G(v)$
Set of vertices adjacent to v

Observation: $|N(v)| = d(v)$

A vertex of degree 1 is an end vertex. (v_1)
A vertex of degree 0 is an isolated vertex. (v_5)

Minimum degree: $\delta(G) = \min_{v \in V} d(v)$

Maximum degree: $\Delta(G) = \max_{v \in V} d(v)$

Average degree: $d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$

☆ A graph property is a function of a graph.

☆ We are talking about simple, undirected graph.

↳ We will specify if we aren't.

Double counting:

$$\# \text{ incidences} = 2|E| = \sum_{v \in V} d(v)$$

In particular, $\sum_{v \in V} d(v)$ is even.

↳ Hence, # vertices of odd degree is even.

Degree sequence: non-increasing list of the degrees of the vertices

$$d_1 \geq d_2 \geq \dots \geq d_n$$

yield degree sequence $(d_1, \dots, d_n)$

☆ degree sequences do not uniquely describe a graph

G_1:  G_2: 

G_1 and G_2 have the same degree sequence $(2,2,2,2,2,2)$, but they are clearly not the same.

Question: What sequences are graphic?

Graphic means it corresponds to some graph

$(3,1,1,1)$ is graphic: 

$(3,2,1)$ is not graphic, since $\Delta(G) \leq |V| - 1$

$(3,2,1,1)$ is not graphic: $3+2+1+1$ is odd (it must be even)

$(6,6,6,6,4,3,3)$ is really difficult to prove. Luckily, we have a theorem.

(2)

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Theorem:

(1) $s, t_1, \ldots, t_s, d_1, \ldots, d_n$

(2) $t_1 - 1, \ldots, t_s - 1, d_1, \ldots, d_n$

(1) is graphic $\iff$ (2) is graphic.

Proof: (2) is graphic $\Rightarrow$ (1) is graphic

Let G be a graph with $V = \{T_1, \ldots, T_s, D_1, \ldots, D_n\}$ be a graph with degree sequence (2).

Add S, connect it to $T_1, \ldots, T_n$. ✓

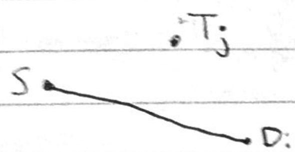
(1) is graphic $\Rightarrow$ (2) is graphic

Let G be a graph with $V = \{S, T_1, \ldots, T_s, D_1, \ldots, D_n\}$ and with degree sequence (1).

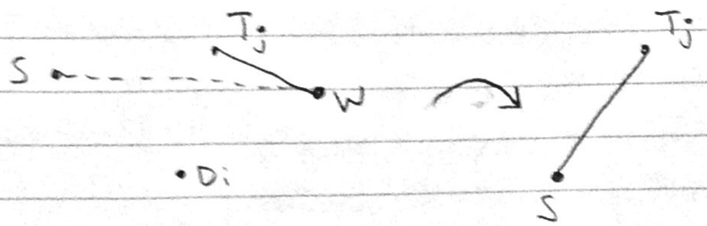
We will apply local modifications to G until $\{S, T_i\} \in E$ for $1 \leq i \leq s$.

Once we have this, we can remove S to obtain a graph with degree sequence (2).

Suppose we have



Since $d_i \leq t_j$, T_j has more neighbors in $V - \{S\}$ than D_i does. Hence we have $\{W, T_j\} \in E$ and $\{D_i, W\} \notin E$.



■

Now back to checking if $(6,6,6,6,4,3,3)$ is graphic.

$(6,6,6,6,4,3,3)$ not graphic

$(5,5,5,3,2,2)$ $\Updownarrow$

$(4,4,2,1,1)$

$(3,1,0,0)$ not graphic

A k-regular is one with exactly all degrees equal to k.

Ex. 1.1.3: Show that for each $n \geq 5$, there exists a 4-regular graph on n vertices.

Solution: $V=[n]$, $ij \in E \iff i-j \equiv -2,-1,1,$ or $2 \mod n$.

Provided that none of $-2,-1,0,1,2$ are congruent mod n, $i-2, i-1, i, i+1, i+2$ are distinct mod n and the construction works.