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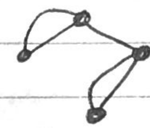
A multigraph can have multiple edges between each pair of vertices.

A loop is an edge from a vertex to itself

A pseudograph is a multigraph with loops allowed.

graph:  multigraph: 

Pseudograph: 

The degree of a vertex in a pseudograph is the # of edges incident, with loops counting twice.

Works for pseudographs

A walk is a sequence

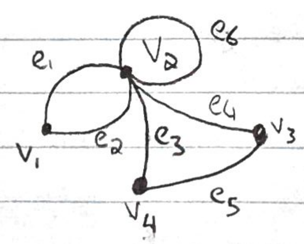
$v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$

such that $e_i = v_i v_{i+1}$.

A closed walk is a walk with the same first and last vertex.

A trail is a walk with no duplicated edges.

A circuit is a closed trail.



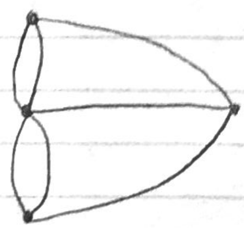
walk: $v_1 e_1 v_2 e_1 v_1 e_2 v_2 e_6 v_2$

* not a trail. not closed.

circuit: $v_1 e_1 v_2 e_6 v_2 e_2 v_1$

* not a cycle since v_2 is repeated

An Eulerian circuit is a circuit that uses all the edges.



Pseudograph modelling the bridges of Köningsberg.

<u>Theorem</u>: A connected pseudograph G has an Eulerian circuit if and only if each vertex has even degree.

<u>Proof</u>: ($\Longrightarrow$) If v occurs in a circuit k times, its degree is $2k$, unless v is the first or last vertex, in which case it has degree $2(k-1)$.

($\Longleftarrow$) Let T be the longest trail in G. $T = v_0 e_0 v_1 \ldots e_{k-1} v_k$. T must contain every edge incident to v_k, since otherwise we could extend it.

confusing

Since the number of edges incident to v_k is even, we must therefore have $v_k = v_0$. Suppose there are edges in G that are not in T. Since G is connected,

because of connectedness

there is some edge e not in T which is incident to a vertex $v_i \in T$. But, $u e v_i e_i v_{i+1} \ldots e_{k-1} v_k = v_0 v_1 e_1 \ldots e_{i-2} v_{i-1} e_{i-1} v_i$ is a trail longer than T $\rightarrow\leftarrow$

$\therefore$ All edges are in T, so T is an Eulerian circuit. $\blacksquare$

Is it easy to produce an Eulerian circuit if we know it exists?

A <u>Hamilton cycle</u> in a graph G is a cycle that contains all of the vertices in G.

②

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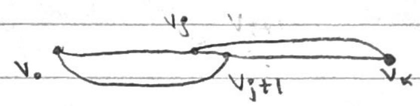
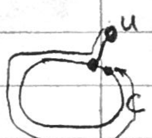
<u>Theorem:</u> If G is a graph on $n \geq 3$ vertices, and $\delta(G) \geq n/2$, then G has a hamiltonian cycle.

<u>Proof:</u> Let P be the longest path in G.
The neighborhoods of v_0 and v_k must be contained in P, otherwise, we could extend P. Let $N_1 = \{i \in [0, k-1] \mid v_i v_k \in E(G)\}$, let $N_2 = \{i \in [0, k-1] \mid v_0 v_{i+1} \in E(G)\}$.
Observe that $|N_1| + |N_2| = d(v_k) + d(v_0) \geq 2\delta(G) \geq n$.
By the pigeonhole principle, there is $j \in N_1 \cap N_2$.



$C = v_0 v_{j+1} \ldots v_k v_j \ldots v_0$ is a cycle of length $k+1$.
We claim that C is Hamilton. Suppose there are vertices not in C. Then there is a vertex $u \notin C$ with an edge to a vertex in C, since G is connected. u together with the path created by opening C is a path longer than P. $\rightarrow\leftarrow$ $\therefore$ C contains all vertices of G, so C is a Hamiltonian cycle. ∎



A <u>directed graph</u> $G = (V, E)$, $E \subseteq V^2$.
Each edge has a start or initial vertex and an end or terminal vertex.
The number of edges starting at v is its out-degree.
The number of edges ending at v is its in-degree.

A <u>directed path</u> is a sequence $v_0 v_1 \ldots v_k$ of distinct vertices such that $(v_i, v_{i+1}) \in E$ for $i \in [0, k-1]$.

A <u>directed cycle</u> is a cyclic sequence $v_0 v_1 \ldots v_k v_0$ such that $v_0 \ldots v_k$ is a path and $v_k v_0 \in E$.

If there is a path from u to v, then we say v is reachable from u.

A path is weakly connected.

A graph is weakly connected if for each pair u, v of vertices, either u is reachable from v or v is reachable from u.

A cycle is strongly connected.

A graph is strongly connected if each pair of vertices are reachable from each other.

A directed acyclic graph is called a DAG.

Note on problem 5: It is enough to show that any weakly connected graph such that $d_{in}(v) = d_{out}(v)$ for all $v \in V$ is also strongly connected.
Idea: If a graph is not strongly connected, we can partition the vertex set into nonempty V_1, V_2 such that there is no edge from V_2 to V_1. Now, look at the sum of the in- and out-degrees.