Colin Wahl
1/11/18

H is a subgraph of G if
$V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

An induced subgraph of G is a subset of the vertices of G together with all edges with both ends in the subset.

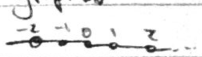
A path of length k is a sequence $x_0 x_1 \dots x_k$ of vertices with edges between subsequent vertices.

A cycle of length k is a path $x_0 \dots x_{k-1}$ of length $k-1$ together with the edge $x_0 x_{k-1}$.

Observation: A cycle is 2-regular.
A union of disjoint cycles is also 2-regular.

Theorem: A finite graph is 2-regular if and only if it is a union of disjoint cycles.

Theorem fails for infinite graphs.



Proof: Start at any vertex v, and consider a sequence of connected vertices. Since G is 2-regular, we can continue indefinitely, and since G is finite, our sequence must repeat at some point. Remove all the vertices we visited and start again. ■

1-regular graphs are matchings, i.e. unions of disjoint edges.

The complete graph on n vertices has all possible edges. (K_n)

The complete bipartite graph $K_{m,n}$ has all possible edges between a set of m vertices and a set of n vertices, and no edges inside either part.

Contains means has as a subgraph.

Theorem: Any graph G contains a path of length $\delta(G)$.

Proof: Let P be the longest path in G.

Can this be improved?

Then $N(x_k) \subseteq V(P)$, since otherwise we could extend P. So $k \geq d(x_k) \geq \delta(G)$. ■

Does there exist for each k a graph with $\delta(G) = k$ that does not contain a path longer than k?

Answer: The complete graph K_n has $\delta(K_n) = n-1$, and only n vertices, so it can't contain a path of length n.

Theorem: Any connected graph contains at least $\min(2\delta(G), |V(G)|-1)$.

Proof: Let P be the longest path in G.

If $|P| = |V(G)| - 1$, then we are done.

Otherwise, $\exists u \in V(G)$ s.t. $u \notin P$. $ux_k \notin E(G)$ and $ux_0 \notin E(G)$.

$x_0x_k \notin E(G)$ otherwise $ux_\ell \dots x_kx_0 \dots x_{\ell-1}$ would be longer than P.

We can't have a pair of edges x_kx_i and x_0x_{i+1}

Hence, the sets $N(x_0)$ and $\{x_{i+1} \mid x_i \in N(x_k)\}$ are disjoint and contained in P.

$\#$ of vertices in $P \geq \underset{x_0}{1} + \underset{|N(x_0)|}{\delta(G)} + \underset{|N(x_k)|}{\delta(G)}$

So the length of P is at least $2\delta(G)$. ■

Colin
Wahl
1/11/18

A connected graph has a path between each pair of vertices.

An acyclic graph contains no cycles.

Subgraph of forest is a forest.

A forest is an acyclic graph, and a tree is a connected acyclic graph.

Subgraph of tree is a forest.

A connected component of a graph is a maximal connected subgraph.

Observations:

1) The connected components of a forest are trees.
2) Subgraphs of forests are forests.
3) Subgraphs of trees are forests.
4) Connected subgraphs of forests are trees.