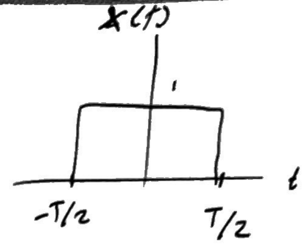
①

SES HW 8



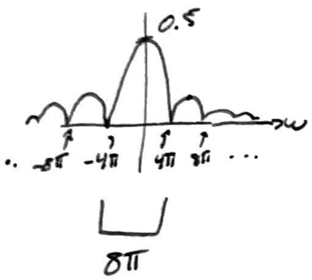
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

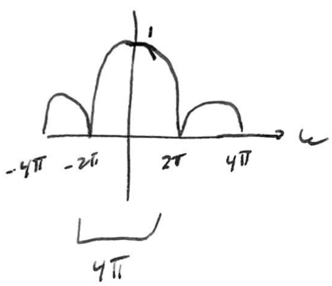
$$= \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt = \left[\frac{1}{-j\omega} e^{-j\omega t} \right]_{t=-T/2}^{t=T/2} = \frac{1}{-j\omega} e^{-j\omega \frac{T}{2}} + \frac{1}{j\omega} e^{j\omega \frac{T}{2}}$$

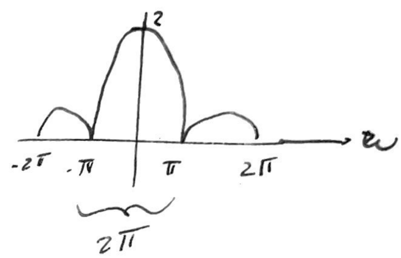
$$X(\omega) = \frac{1}{j\omega}\left(-e^{-j\omega \frac{T}{2}} + e^{j\omega\frac{T}{2}}\right) = \frac{1}{\omega} \cdot 2\sin\left(\omega \frac{T}{2}\right) = T\,\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}} = \boxed{T\,\text{sinc}\left(\omega\frac{T}{2}\right)}$$

$T = 0.5$ $\qquad$ $T = 1$ $\qquad$ $T = 2$

$\frac{1}{2}\text{sinc}\left(\frac{\omega}{4}\right)$ $\qquad$ $\text{sinc}\left(\frac{\omega}{2}\right)$ $\qquad$ $2\,\text{sinc}(\omega)$







main lobe width

taller and thinner →

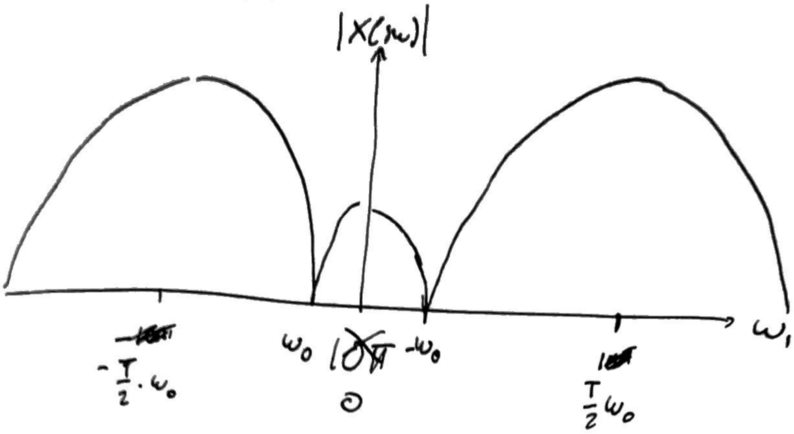
②

$$X(\omega) = \int_{-T/2}^{T/2} \cos(\omega_0 t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{tj(\omega_0-\omega)} + e^{-jt(\omega_0+\omega)}\, dt$$

$$\cancel{= \frac{1}{2}\left[\frac{1}{j(\omega_0-\omega)} e^{jt(\omega_0-\omega)} + \frac{-1}{j(\omega_0+\omega)} e^{-jt(\omega_0+\omega)}\right]_{t=-\frac{T}{2}}^{t=\frac{T}{2}} = \frac{1}{2}\left[\frac{e^{j\frac{T}{2}(\omega_0-\omega)}}{j(\omega_0-\omega)} - e^{-j}\right.}$$

$$= \frac{1}{2}\left[\frac{e^{j\frac{T}{2}(\omega_0-\omega)}}{j(\omega_0-\omega)} - \frac{e^{-j\frac{T}{2}(\omega_0+\omega)}}{j(\omega_0+\omega)} - \frac{e^{-j\frac{T}{2}(\omega_0-\omega)}}{j(\omega_0-\omega)} + \frac{e^{j\frac{T}{2}(\omega_0+\omega)}}{j(\omega_0+\omega)}\right] = \frac{\sin\left(\frac{T}{2}(\omega_0-\omega)\right)}{\omega_0-\omega} + \frac{\sin\left(\frac{T}{2}(\omega_0+\omega)\right)}{\omega_0+\omega}$$

$$X(\omega) = \frac{T}{2}\left[\text{sinc}\left(\frac{T}{2}(\omega_0-\omega)\right) + \text{sinc}\left(\frac{T}{2}(\omega_0+\omega)\right)\right] = 0.05\left[\text{sinc}(10\pi - 0.05\omega) + \text{sinc}(10\pi + 0.05\omega)\right]$$



when is $\frac{\frac{T}{2}\left(\sin\left(\frac{T}{2}(\omega_0-\omega)\right)\right)}{\frac{T}{2}(\omega_0-\omega)}$ equal to 0?

~~$0 = \sin(\omega_0 - \omega)$~~

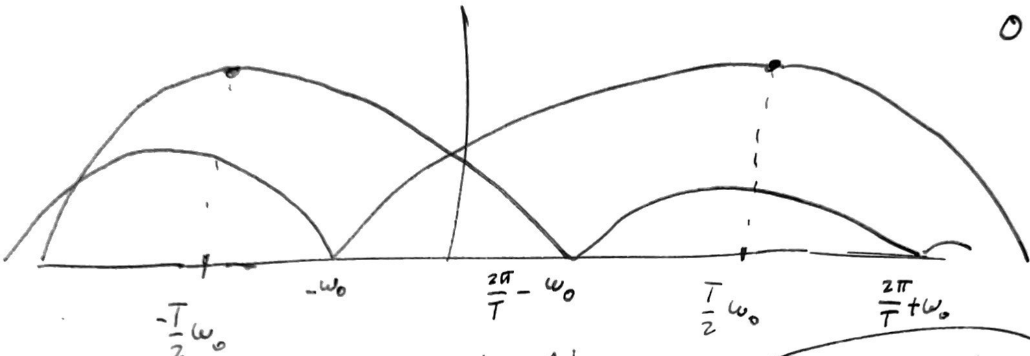
~~$\omega_0 - \omega = k\pi$~~

~~0, π, 2π ...~~

~~$\omega = \omega_0 - k\pi$~~

~~$k = 0, 1, 2 \ldots$~~

$k = -2, -1, 0, 1, 2 \ldots$



$0 = \sin\left(\frac{T}{2}(\omega_0 \pm \omega)\right)$

k = integer

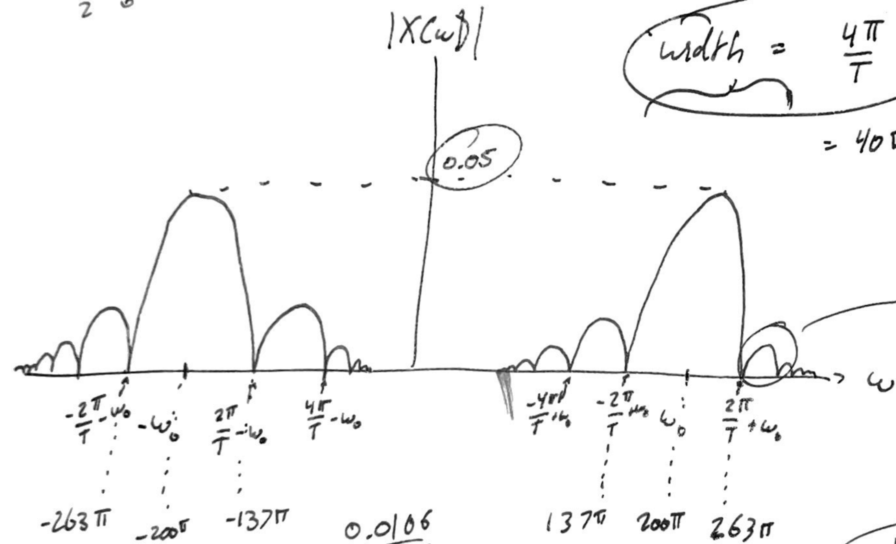
$\frac{T}{2}\omega_0 \pm \frac{T}{2}\omega = k\pi$

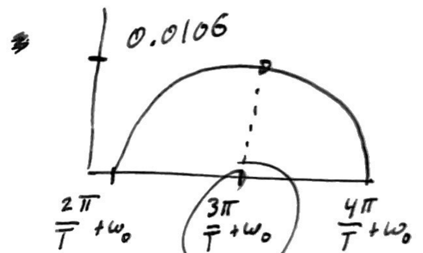
$\pm \omega \frac{T}{2} = k\pi - \frac{T}{2}\omega_0$

$\omega = \frac{2k\pi}{T} \pm \omega_0$

width = $\frac{4\pi}{T}$

$= 40\pi \approx 125.664 \ \frac{rad}{s} = 20\ Hz$





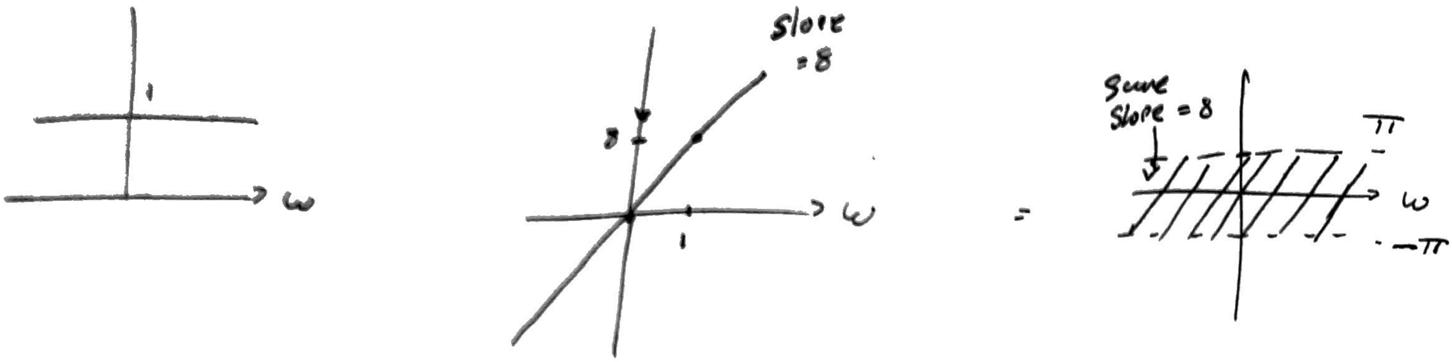
plug in ω = to $\text{sinc}\left(\frac{T}{2}(\omega_0 - \omega)\right)$

$|X(\omega)| = 0.0106$

sidelobe dB $= 20\log_{10}\left(\frac{0.0106}{0.05}\right) = \boxed{-13.465\ dB}$

(3) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-8]\, e^{-j\omega n} = e^{-j\omega 8} = \cos(8\omega) + j\sin(8\omega)$

Magnitude = 1   phase = ~~-~~ 8ω



delay in time ⟶ ~~Shift~~ shift in phase dependent on frequency

$a_k = \frac{1}{8} \sum_{n=0}^{7} x[n]\, e^{-jk\frac{2\pi}{8}n} = \frac{1}{8} \sum_{k=0}^{7} 1 \cdot e^{-jk\frac{2\pi}{8}\cdot 8} =$

always e^0

$= \frac{1}{8} \sum_{n=0}^{7} 1$ ⟶ $\boxed{a_k = \frac{1}{8} \text{ for } k = 0 \text{ to } 7}$

try $a_k = \frac{1}{N} \sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}\cdot n_0}$ Constant a_ks if $n_0 = N$ or $n_0 = Z \cdot N$

$Z \in$ integers

Circular: $a_k = \frac{1}{8} \sum_{n=0}^{7} 1\, e^0$ ⟶ $\boxed{a_k = \frac{1}{8}}$

$x[0]$
$= x[8] = 1$

```
clear;clc;
x = [0 0 0 0 0 0 0 0 1];
N = 8;
k = 0:N-1;
```

```
n = 0:N-1;
X = 1/N*sum(x(1:N).*exp(-1j.*k*2*pi/N.*n)) % x[0] to x[7] are all zero.
```

```
X =
0
```

```
x_circ = [1 0 0 0 0 0 0 0];
X_circ = 1/N*sum(x_circ(1:N).*exp(-1j.*k*2*pi/N.*n)) % x[8] = x[0] = 1, rest are
zero
```

```
X_circ =
0.1250
```

(4) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=0}^{M-1} \cos(\omega_0 n)\, e^{-j\omega n}$ $\qquad \omega_0 = \frac{2\pi}{T_0}$

$$= \sum_{n=0}^{M-1} \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{M-1} e^{jn(\omega_0-\omega)} + e^{-jn(\omega_0+\omega)}$$

$n=0 \rightarrow 1$

$$\cancel{= \sum_{n=0}^{M-1} \cos}$$

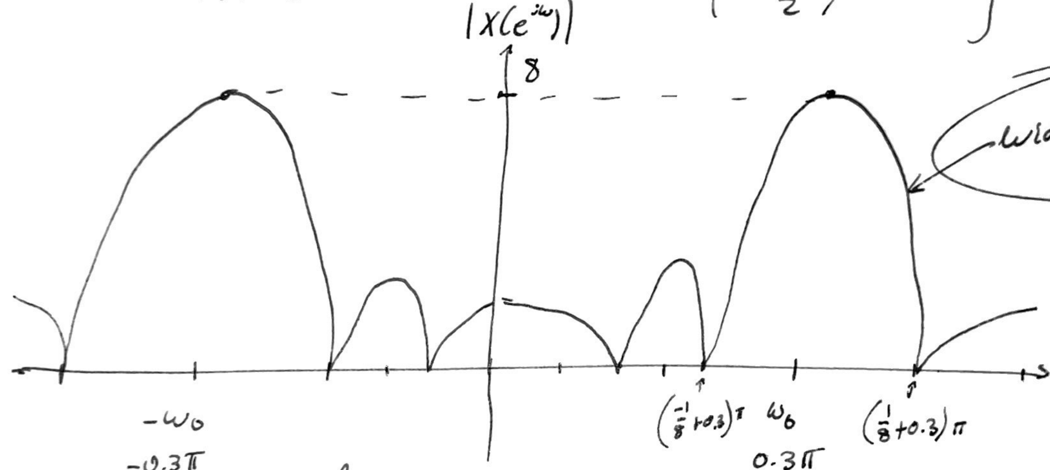
$$= \frac{1}{2}\left[\sum_{n=0}^{M-1} e^{jn(\omega_0-\omega)} + \sum_{n=0}^{M-1} e^{-jn(\omega_0+\omega)}\right] = \frac{1}{2}\left(\frac{1-e^{jM(\omega_0-\omega)}}{1-e^{j(\omega_0-\omega)}} + \frac{1-e^{-jM(\omega_0+\omega)}}{1-e^{-j(\omega_0+\omega)}}\right]$$

dirichlet kernel identity

$$= \frac{1}{2}\left[e^{j(\omega_0-\omega)\frac{M-1}{2}} \cdot \frac{\sin\left(\frac{M(\omega-\omega_0)}{2}\right)}{\sin\left(\frac{\omega-\omega_0}{2}\right)} + e^{-j(\omega_0+\omega)\frac{M-1}{2}} \cdot \frac{\sin\left(\frac{M(\omega+\omega_0)}{2}\right)}{\sin\left(\frac{\omega+\omega_0}{2}\right)}\right]$$

no effect on magnitude

repeats every $\pm 2\pi$



width $= \left(\frac{1}{4}\right)\pi$

~~$\approx 1.728 \frac{rad}{s}$~~

~~$= 0.275$ Hz~~

height: $\frac{1}{2}\left[\cancel{\frac{\sin\left(\frac{16(0)}{2}\right)}{\sin\left(\frac{0}{2}\right)}} + \cancel{\frac{\sin\left(\frac{16(-0.6\pi)}{2}\right)}{\sin\left(\frac{-0.6\pi}{2}\right)}}\right]$ undefined

$$\lim_{\omega \to \omega_0}\left[\frac{\sin\left(\frac{M(\omega-\omega_0)}{2}\right)}{\sin\left(\frac{\omega-\omega_0}{2}\right)} \cdot \frac{1}{2}\right] = 8$$

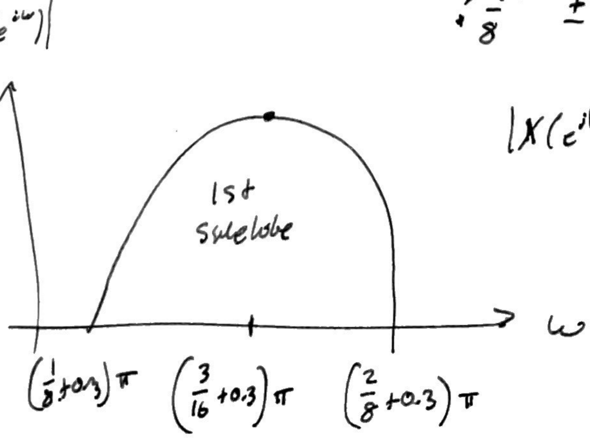
intercepts: $\sin\left(\frac{M(\omega\pm\omega_0)}{2}\right) = 0 \rightarrow \frac{M(\omega\pm\omega_0)}{2} = K\pi \rightarrow \omega = 2\frac{K}{M}\pi \pm \omega_0$ $\quad K$ = integer, non-zero

~~except when $M=$~~

$$= \frac{K\pi}{8} \pm 0.3\pi$$

$$20\log_{10}\left(\frac{1.723}{8}\right) = \boxed{-13.34\ dB}$$

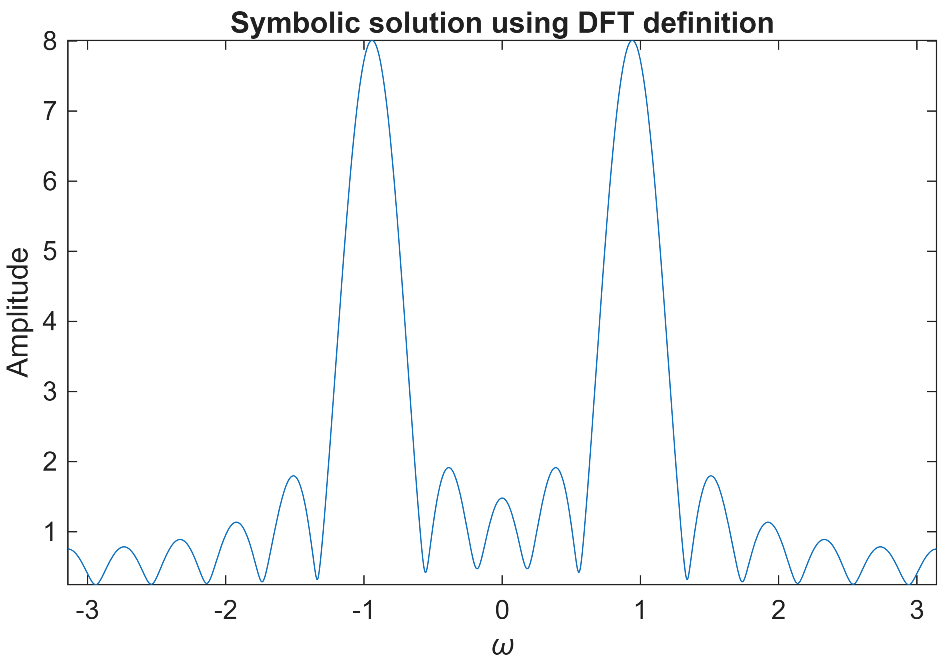
close to -13.26. I approximate the peak of the sidelobe to be at the mid point.



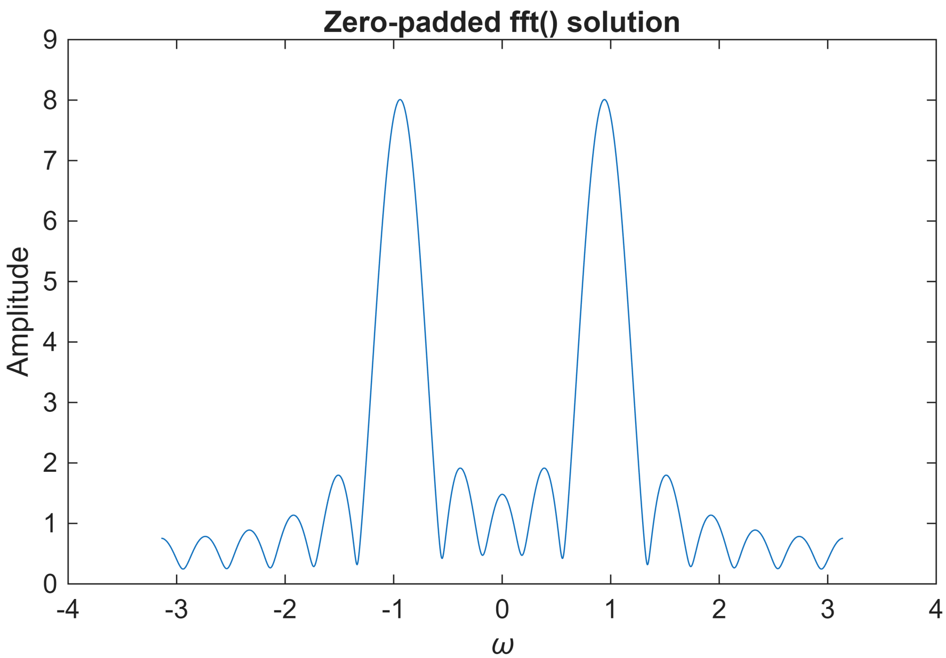
$|X(e^{j(\frac{3}{16}+0.3)\pi})| = 1.723$

```
clear; clc;
w0 = 0.3*pi;
M = 16;
n = 0:M-1;
syms w
x = cos(w0.*n);
```

```
% solve symbolically
X = sum(x(n+1).*exp(-1j.*w.*n));
fplot(abs(X), [-pi pi])
xlabel('\omega'); ylabel('Amplitude'); title('Symbolic solution using DFT
definition');
```




```
% mucho zero padding
N = 4096;
X = fft(x, N);
plot(linspace(-pi,pi, N), abs(fftshift(X)))
xlabel('\omega'); ylabel('Amplitude'); title('Zero-padded fft() solution');
```




```
[mag, idx] = max(abs(X));
Mainlobe_height = mag
```

```
Mainlobe_height =
8.0083
```

```
min_inds = find(islocalmin(abs(X)));
right = min_inds(min_inds > idx);
left = min_inds(min_inds < idx);
Mainlobe_Width = 2*pi/N*(right(1) - left(end))
```

```
Mainlobe_Width =
0.7823
```

```
[pks, ~] = findpeaks(abs(X));
pks = sort(pks,"descend");
Sidelobe_height = pks(3)
```

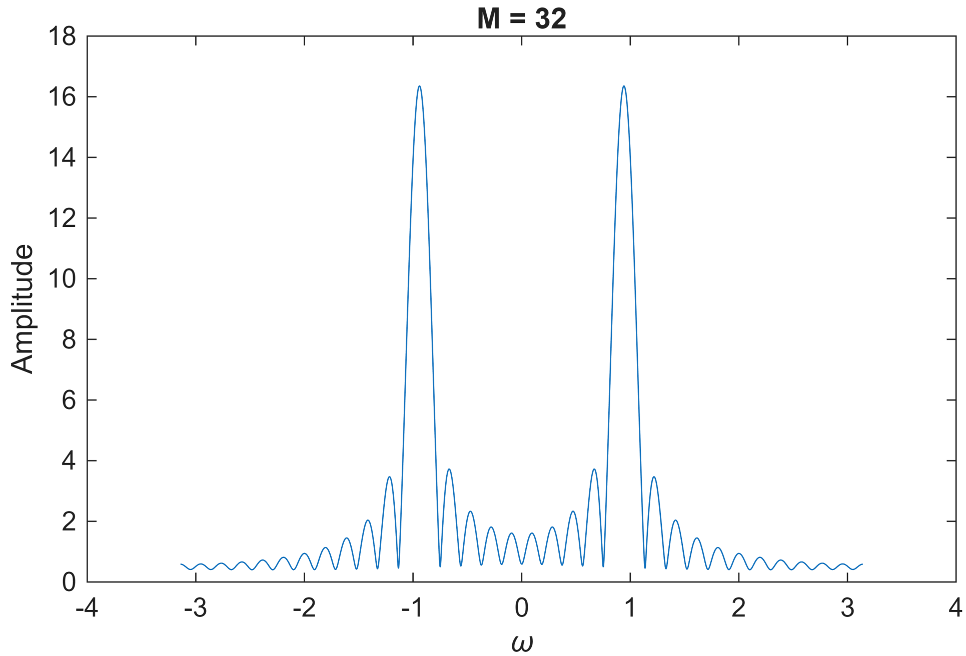
```
Sidelobe_height =
1.9149
```

```
Sidelobe_dB = 20*log10(Sidelobe_height/Mainlobe_height)
```

```
Sidelobe_dB =
-12.4278
```

```
% Double M
M = 2 * M;
n = 0:M-1;
x = cos(w0.*n);
```

```
X = fft(x, N);
plot(linspace(-pi,pi, N), abs(fftshift(X)))
xlabel('\omega'); ylabel('Amplitude'); title("M = " + num2str(M));
```




```
[mag, idx] = max(abs(X));
Mainlobe_height = mag
```

```
Mainlobe_height =
16.3534
```

```
min_inds = find(islocalmin(abs(X)));
right = min_inds(min_inds > idx);
left = min_inds(min_inds < idx);
Mainlobe_Width = 2*pi/N*(right(1) - left(end))
```

```
Mainlobe_Width =
0.3850
```

```
[pks, ~] = findpeaks(abs(X));
pks = sort(pks,"descend");
Sidelobe_height = pks(3)
```

```
Sidelobe_height =
3.7248
```

```
Sidelobe_dB = 20*log10(Sidelobe_height/Mainlobe_height)
```

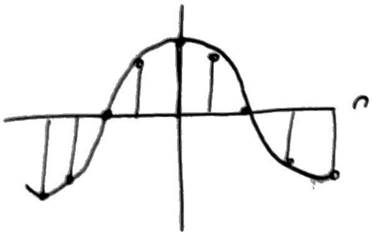
```
Sidelobe_dB =
-12.8500
```

The main lobe is approximately half as wide and twice as tall. The sidelobe doubled in height as well, so the dB value remains fairly similar.

It looks like a larger M (larger window, larger number of samples) results in a better DFT.

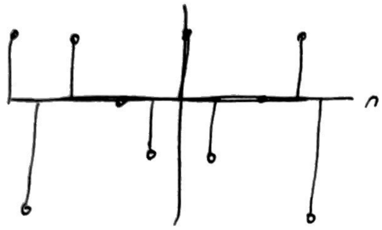
(5) $x(t) = \cos(\underset{2\pi f_0}{\omega_0} t)$ $T_s = \frac{1}{f_s}$ $x[n] = \cos(\underbrace{2\pi \frac{f_0}{f_s}}_{\omega_0} n) = \cos(\omega_0 n)$

$f_0 < 400$ Hz



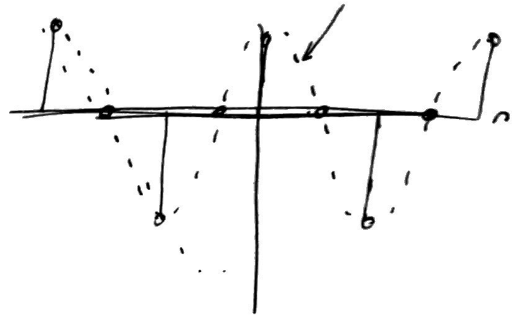
100 Hz

$f_0 > 400$ Hz



500 Hz

$f_0 > 400$ Hz Not 1000 Hz



~~800~~ 1000 Hz

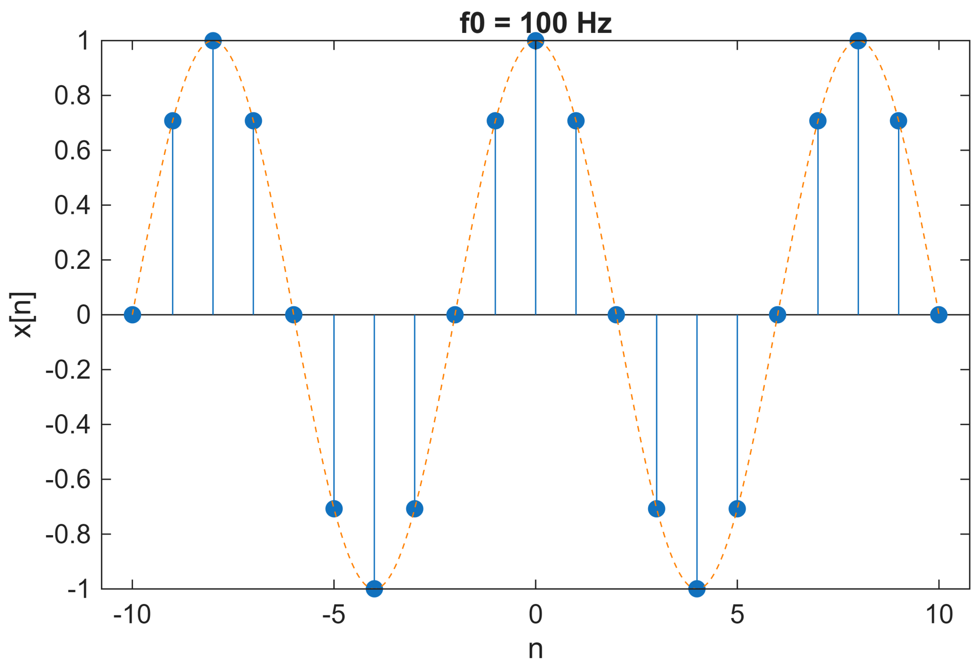
Aliasing occurring because the original 500 Hz and 1000 Hz signals cannot be seen in just the samples. Instead, we see lower frequency signals that are aliases of the original f_0.

$f_{alias} = |f_0 - K f_s|$ K is smallest integer such that $f_{alias} < f_s/2$ ← Nyquist

---

```
clc; clear;
n = -10:10;
t = -10:0.01:10;
```

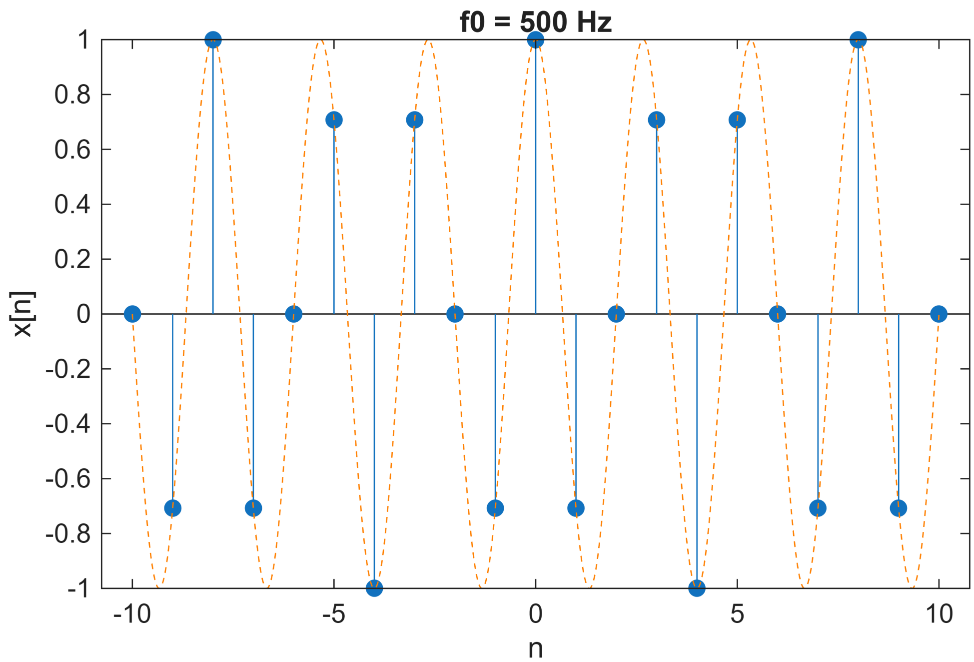
```
w0 = 2*pi*1/8;

xn = cos(w0*n);
xt = cos(w0*t);
stem(n,xn,"filled"); hold on;
plot(t,xt,'LineStyle','--','Color',[1 0.5 0 0.5]); hold off;

xlabel('n'); ylabel('x[n]'); title('f0 = 100 Hz');
```




```
f0_reconstructed = abs(w0/(2*pi)*800 - 0*800)
```

```
f0_reconstructed =
100
```

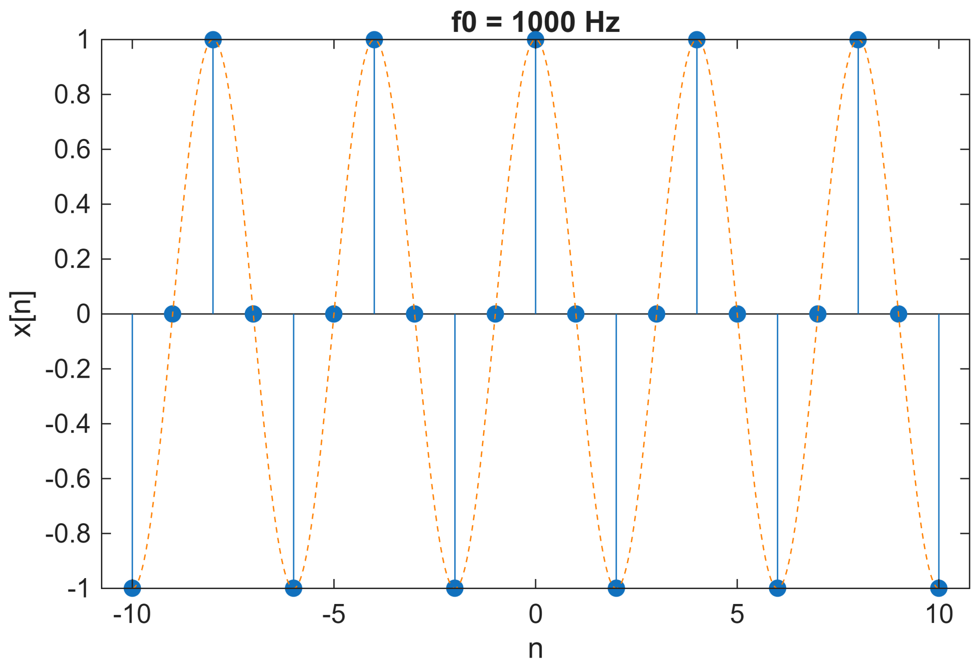
```
w0 = 2*pi*5/8;

xn = cos(w0*n);
xt = cos(w0*3/5*t);
stem(n,xn,"filled"); hold on;
plot(t,xt,'LineStyle','--','Color',[1 0.5 0 0.5]); hold off;

xlabel('n'); ylabel('x[n]'); title('f0 = 500 Hz');
```




```
f0_reconstructed = abs(w0/(2*pi)*800 - 1*800)
```

```
f0_reconstructed =
300
```

```
w0 = 2*pi*10/8;

xn = cos(w0*n);
xt = cos(w0*1/5*t);
stem(n,xn,"filled"); hold on;
plot(t,xt,'LineStyle','--','Color',[1 0.5 0 0.5]); hold off;

xlabel('n'); ylabel('x[n]'); title('f0 = 1000 Hz');
```




```
f0_reconstructed = abs(w0/(2*pi)*800 - 1*800)
```

```
f0_reconstructed =
200
```

Question 6

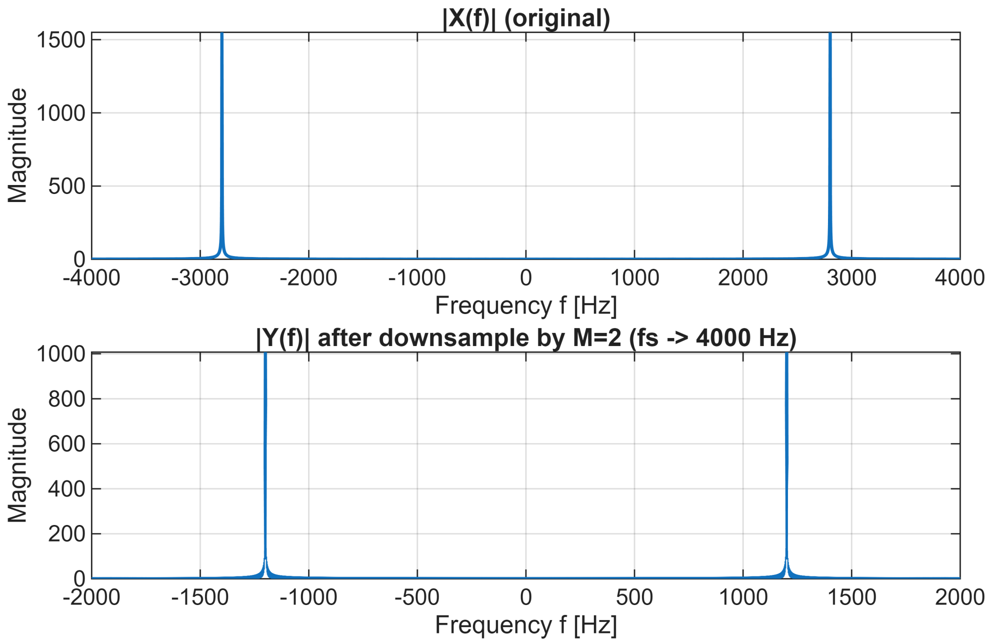
Answers at end of this script

DOWNCONVERSION

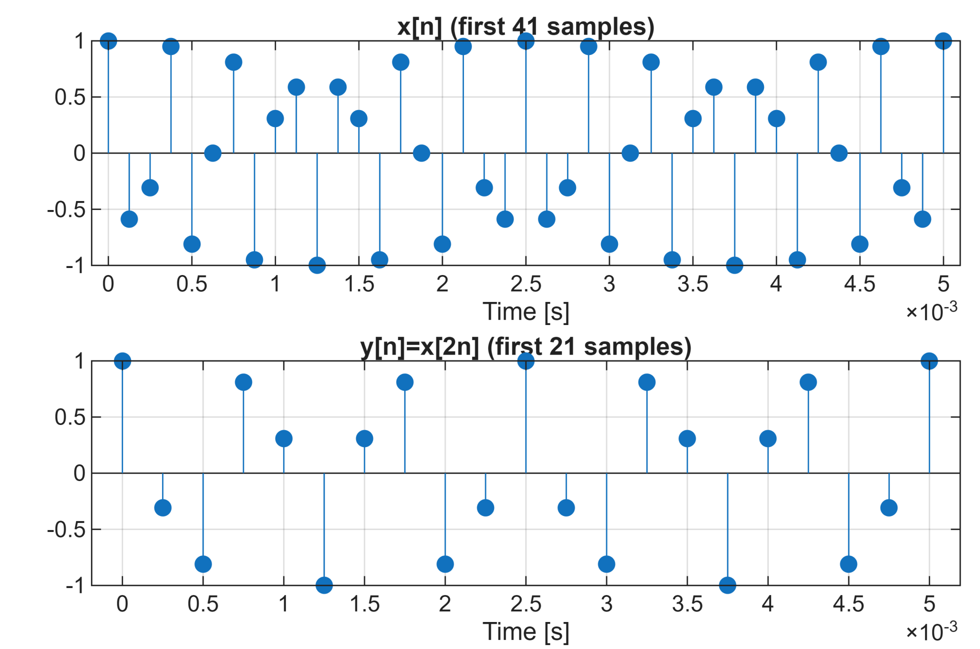
```
clear; clc;

fs = 8000; % Hz
Ts = 1/fs;
N = 4096;
t = (0:N-1)*Ts;

% Pick a tone that will alias after downsampling
f0 = 0.35*fs; % 2.8 kHz
x = cos(2*pi*f0*t);

% downsample by M=2
M = 2;
y = x(1:M:end);
fs_y = fs/M;
Ty = 1/fs_y;

% Zero padding
yZ = zeros(1,N);
yZ(1:length(y)) = y;
```

```
X = fftshift(fft(x, N));
Y = fftshift(fft(yZ, N));
fx = linspace(-fs/2, fs/2, N);
fy = linspace(-fs_y/2, fs_y/2, N);
```

```
figure('Color','w');
subplot(2,1,1); plot(fx, abs(X), 'LineWidth',1.2); grid on;
title('|X(f)| (original)'); xlabel('Frequency f [Hz]'); ylabel('Magnitude');

subplot(2,1,2); plot(fy, abs(Y), 'LineWidth',1.2); grid on;
title(sprintf('|Y(f)| after downsample by M=%d (fs -> %.0f Hz)', M, fs_y));
xlabel('Frequency f [Hz]'); ylabel('Magnitude');
```




```
% time domain view
figure('Color','w');
subplot(2,1,1);
stem(t(1:41), x(1:41), 'filled'); grid on;
title('x[n] (first 41 samples)'); xlabel('Time [s]');

subplot(2,1,2);
ty = (0:length(y)-1)*Ty;
stem(ty(1:21), y(1:21), 'filled'); grid on;
title('y[n]=x[2n] (first 21 samples)'); xlabel('Time [s]');
```



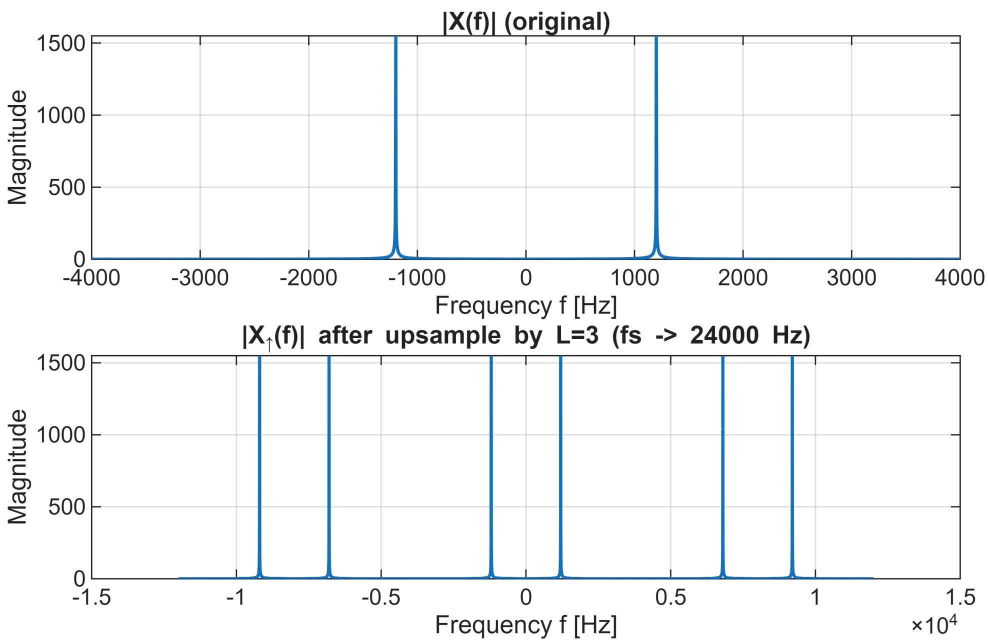
UPSAMPLING

```
clear; clc;

fs = 8000;
Ts = 1/fs;
N = 4096;
t = (0:N-1)*Ts;
```

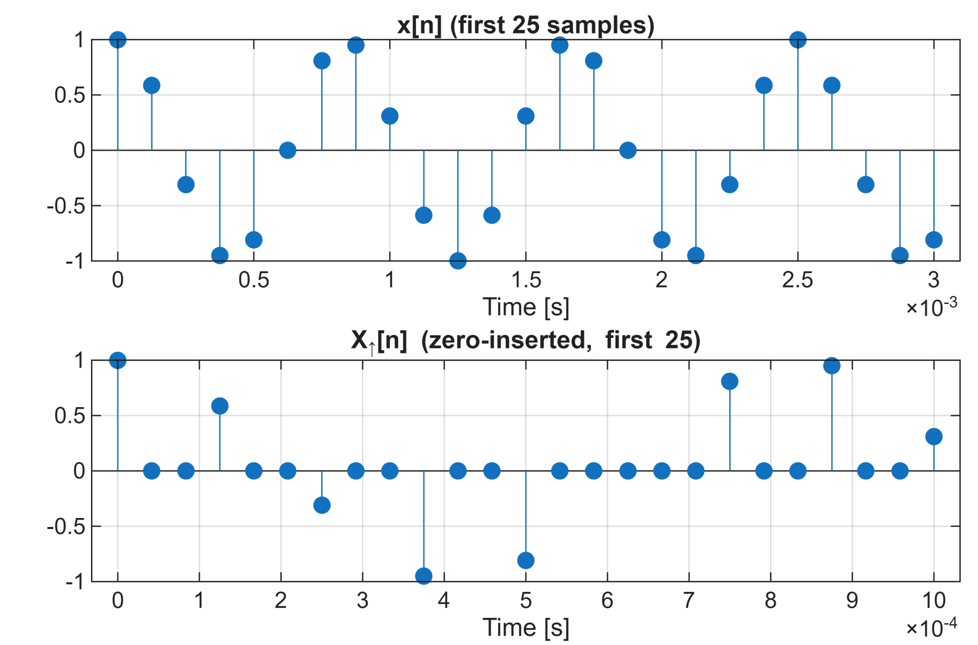
```
% Tone that is clear after upsampling
f0 = 0.15*fs; % 1200 Hz for fs = 8 kHz
x = cos(2*pi*f0*t);

% Upsample
L = 3;
xu = zeros(1, L*length(x));
xu(1:L:end) = x;

fs_u = L*fs;
Tu = 1/fs_u;
Nu = L*N;
```

```
X = fftshift(fft(x,N));
Xu = fftshift(fft(xu, Nu));
fx = linspace(-fs/2, fs/2, N);
fu = linspace(-fs_u/2, fs_u/2, Nu);
```

```
figure('Color','w');
subplot(2,1,1); plot(fx, abs(X), 'LineWidth',1.2); grid on;
title('|X(f)| (original)'); xlabel('Frequency f [Hz]'); ylabel('Magnitude');

subplot(2,1,2); plot(fu, abs(Xu), 'LineWidth',1.2); grid on;
title(sprintf('|X_{\\uparrow}(f)| after upsample by L=%d (fs -> %.0f Hz)', L,
fs_u));
xlabel('Frequency f [Hz]'); ylabel('Magnitude');
```




```
% time domain view
figure('Color','w');
subplot(2,1,1);
stem(t(1:25), x(1:25), 'filled'); grid on;
title('x[n] (first 25 samples)'); xlabel('Time [s]');

subplot(2,1,2);
tu = (0:length(xu)-1)*Tu;
stem(tu(1:25), xu(1:25), 'filled'); grid on;
title('X_{\uparrow}[n] (zero-inserted, first 25)'); xlabel('Time [s]');
```



ANSWERS

Downsampling

- The new apparent tone frequency is at 1200 Hz
- |Y(f)| looks compressed/overlapped near the base of the tone peak because original tone has folded into the upper and lower frequencies, due to the decrease in sampling frequency when downsampling.
- The bandlimit on |X(f)| to avoid aliasing for M=2 is less than 1/4 the original sample frequency. For original fs = 8000 Hz, the highest frequency |X(f)| is 2000 Hz. When downsampling occurs and the new sampling frequency is 4000 Hz, no aliasing will occur as |X(f)| fuffils the Nyquist-Shannon sampling theorem.

Upsampling

- 3 images appear, although one is the same as the original |X(f)| at +/- 1.2 kHz.
- The other two images are +/- 8000 Hz from 0 Hz, which was the original sampling frequency fs. Each image also contains 2 tones, each one at +/- 8.0 kHz +/- 1.2 kHz.
- A low-pass filter with cutoff frequency of ~1.5 kHz would attenuate the outer images, leaving only the original.